Concerning the sums of series of reciprocals.

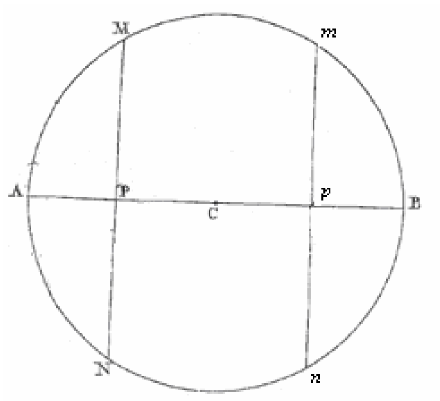
Leonhard Euler.

§. 1.

Now the series of the reciprocals of the powers of the natural numbers have been explored and investigated to such an extent that it hardly seems likely that it would be possible to find anything new about them. Indeed whichever of the sums of the series have been considered, and nearly all those have involved an investigation of this kind of series, still no suitable way has been found for expressing them. Now I also, as more often than not, have treated a number of methods of summation, and I have diligently pursued these series, yet I have never understood any other method by which I could define the sum of these, other than as an approximation, or perhaps I could reduce the sums to the quadrature of curves of quickest descent; I have set out these methods here in the preceding read dissertations [*i.e.* in the Commentaries of the weekly seminars at the St. Petersburg institute]. But here I discuss series of fractions, the numerators of which are 1, and the denominators are either squares, cubes, or any other powers of the natural numbers : series of this kind are $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}$, likewise $1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\text{etc.}$ and similar higher powers, the general term of which is expressed in the form $\frac{1}{x^n}$.

§.2. Moreover recently in an unexpected manner, I have been able to deduce an expression for the entire sum of this series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.}$, which depends on the quadrature of the circle, thus, as if truly the sum of this series is obtained, then likewise the quadrature of the circle follows. For I have found that the sum of this series is a sixth part of the square of the periphery of the circle of which the diameter is 1; or with the sum of this series put equal to s, it has the ratio $\sqrt{6s}$ to1 of the periphery to the diameter. Moreover I will soon establish that the sum of this series is approximately 1. 64493406684822643 64, and from six times this number, if the square root is extracted, the number 3.141592653589793238 is indeed produced, expressing the circumference of the circle the diameter of which is 1. Again, by following the same steps again by which I have pursued this sum, the sum of this series beginning $1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\frac{1}{625}+\text{etc.}$, can also be grasped from the quadrature of the circle. Now the sum of this series multiplied by 90 gives the fourth power of the circumference of the circle of which the diameter is 1. And by similar reasoning also the sums can be determined of the following series, in which the exponents of the powers are even numbers.

§.3. In which therefore, I explain most conveniently in an orderly manner the whole argument of how I have arrived at this sum, making use of the circle. In the circle



AMBNA described with centre C and radius AC or BC = 1, I consider any arc AM, the sine of which is MP and the cosine truly CP. Now with the arc AM put equal to *s*, with the sine PM = *y*, and with the cosine CP = *x*, by the well-enough known method, so the sine *y* and the cosine *x* can be defined for the given arc s by series, it is indeed, as can be considered,

$y = s - \frac{s^3}{1.2.3} + \frac{s^5}{1.2.3.4.5} - \frac{s^7}{1.2.3.4.5.6.7} + \text{etc.}$ and

$x = 1 - \frac{s^2}{1.2} + \frac{s^4}{1.2.3.4} - \frac{s^6}{1.2.3.4.5.6} + \text{etc.}$

Clearly from a consideration of these equations, the sums of the above series of reciprocals can be reached ; indeed each of these equations is directed almost towards the same target, and on this account it suffices only for the one to be treated in the manner that I now present.

§.4. Hence the first equation $y = s - \frac{s^3}{1.2.3} + \frac{s^5}{1.2.3.4.5} - \frac{s^7}{1.2.3.4.5.6.7} + \text{etc.}$ expressed the relation between the arc and the sine. Whereby from this equation, from the given arc the sine of this arc can be determined, as from the given sine, the arc of this can be determined. Moreover, I consider the sine *y* as given, and I investigate how the arc *s* ought to be elicited from the given sine *y*. Now here before all else it is to be observed, that innumerable arcs correspond to the same sine *y*, hence the proposed equation for the sine must present innumerable arcs. If indeed s is considered as the unknown in that equation then it has infinite dimensions, and thus it is not to be wondered at, if that equation should contain innumerable simple factors, any of which put equal to zero must give a suitable value for *s*.

§.5. Moreover since, if all the factors of this equation are to become known, then all the roots of this also or the values of *s* are to be known, and thus in turn if all the values of *s* can be assigned, then also all the factors of this can be found. Moreover, so that I can indicate both the roots and the factors better in this equation, I change the proposed equation into this form : $0 = 1 - \frac{s}{y} + \frac{s^3}{1.2.3.y} - \frac{s^5}{1.2.3.4.5.y} + \text{etc.}$ Now if all the roots of this equation, or all the arcs which have the same sine *y*, are *A, B, C, D, E*, etc. then the factors also are all these quantities, $1 - \frac{s}{A}, 1 - \frac{s}{B}, 1 - \frac{s}{C}, 1 - \frac{s}{D},$ etc. On account of which,

$$0 = 1 - \frac{s}{y} + \frac{s^3}{1.2.3.y} - \frac{s^5}{1.2.3.4.5.y} + \text{etc.} = (1 - \frac{s}{A})\,(1 - \frac{s}{B})\,(1 - \frac{s}{C})\,(1 - \frac{s}{D}) \text{ etc.}$$

§.6. Moreover, from the nature and resolution of the equations it is agreed that the coefficient of the terms in which *s* is present, or $\frac{1}{y}$, is equal to the sum of all the coefficients of *s* in the factors or, $\frac{1}{y}=\frac{1}{A}+\frac{1}{B}+\frac{1}{C}+\frac{1}{D}+\text{etc.}$ Then in turn the coefficient of s^2 is equal to 0, on account of this term being missing in the equation, is equal to the sum of the factors taken two terms at a time of the series $\frac{1}{A},\frac{1}{B},\frac{1}{C},\frac{1}{D},$ etc. Again, $-\frac{1}{1.2.3.y}$ is the sum of three [four in the original ms.] terms of the series $\frac{1}{A},\frac{1}{B},\frac{1}{C},\frac{1}{D},$ etc. In a like manner the sum of the factors taken four terms at a time of the same series is equal to zero, and $+\frac{1}{1.2.3.4.5.y}$ is the sum of the factors taken five terms at a time of the series, and henceforth.

§.7. But with the smallest arc *AM* = *A* put in place, the sine of this is *PM* = *y*, and the half-circumference of the circle is put equal to *p*, then we have
$A, p-A, 2p+A, 3p-A, 4p+A, 5p-A, 6p+A$ etc., and likewise
$-p-A, -2p+A, -3p-A, -4p+A, -5p-A,$ etc.,
[*i. e.* the places where the sine is zero, both in a clockwise and in an anticlockwise sense.] taken together are all the arcs having the same sine *y*. Therefore since before we assumed the series $\frac{1}{A},\frac{1}{B},\frac{1}{C},\frac{1}{D},$ etc. that is changed into this :
$\frac{1}{A},\frac{1}{p-A},\frac{1}{-p-A},\frac{1}{2p+A},\frac{1}{-2p+A},\frac{1}{3p-A},\frac{1}{-3p-A},\frac{1}{4p+A},\frac{1}{-4p+A}$ etc.
Hence the sum of all these terms is equal to $\frac{1}{y}$; but the sum of the factors taken two at a time of this series is equal to 0 ; the sum of the factors from the three terms is equal to $\frac{-1}{1.2.3.y}$, the sum of the factors from four terms is equal to 0; the sum of the factors from five terms is equal to $\frac{+1}{1.2.3.4.5.y}$; the sum of the factors from six terms is equal to 0. And thus henceforth.

§.8. But if any series is had $a+b+c+d+e+f+\text{etc.}$ the sum of this is α, the sum of the factors from the terms taken two at a time is equal to β; the sum of the factors from the terms taken three at a time is equal to γ; the sum of the factors taken four at a time is δ, etc. then the sum of the squares of the individual terms , that is
$a^2+b^2+c^2+d^2+e^2+f^2+\text{etc.}=\alpha^2-2\beta$;
now the sum of the cubes $a^3+b^3+c^3+d^3+e^3+f^3+\text{etc.}=\alpha^3-3\alpha\beta+3\gamma$; the sum of the fourth powers $\alpha^4-4\alpha^2\beta+4\alpha\gamma+2\beta^2-4\delta.$
Moreover so that it is appears clearer, as these formulae proceed, we put the sum of the terms *a, b, c, d*, etc. equal to *P*, the sum of the squares equal to *Q*, the sum of the fourth powers *S*, the sum of the fifth powers equal to *T*, the sum of the sixth powers equal to *V,* etc. With these in place we have :

$$P=\alpha;\ Q=P\alpha-2\beta;\ R=Q\alpha-P\beta+3\gamma;\ S=R\alpha-Q\beta+P\gamma+4\delta;\ T=S\alpha-R\beta+Q\gamma-P\delta+5\varepsilon;\ \text{etc}$$

§.9. Therefore when in our case $\frac{1}{A},\frac{1}{p-A},\frac{1}{-p-A},\frac{1}{2p+A},\frac{1}{-2p+A},\frac{1}{3p-A},\frac{1}{-3p-A}$ etc.,
the sum of all the terms of the series or α is equal to $\frac{1}{y}$; the sum of the factors from the terms taken two at a time or β is equal to 0, and of those beyond :
$\gamma=\frac{-1}{1.2.3.y};\ \delta=0;\ \varepsilon=\frac{+1}{1.2.3.4.5.y};\ \varsigma=0;$ etc The sum of these terms $P=\frac{1}{y};$ the sum of the squares of these terms $Q=\frac{P}{y}=\frac{1}{y^2};$ the sum of the cubes of these terms $R=\frac{Q}{y}-\frac{1}{1.2.y};$ the sum of the fourth powers $S=\frac{R}{y}-\frac{P}{1.2.3y}$. And thus again,

$$T=\frac{S}{y}-\frac{Q}{1.2.3.y}+\frac{1}{1.2.3.4.y};\ V=\frac{T}{y}-\frac{R}{1.2.3.y}+\frac{P}{1.2.3.4.5.y};$$

$$W=\frac{V}{y}-\frac{S}{1.2.3.y}+\frac{Q}{1.2.3.4.5.y}-\frac{1}{1.2.3.4.5.6.y}.$$

From which rule, the sums of the remaining higher powers can be easily determined.

§.10. Now we put the sine $PM = y$ equal to the radius in order that $y = 1$, and the fourth part of the circumference is the smallest arc A of this sine for which the sine is 1, equal to $\frac{1}{2}p$, or with q denoting the quarter of the circumference, then $A = q$ and $p = 2q$. Hence the above series is changed into this :

$$\frac{1}{q},\frac{1}{q},\frac{-1}{3q},-\frac{1}{3q},\frac{+1}{5q},+\frac{1}{5q},-\frac{1}{7q},-\frac{1}{7q},+\frac{1}{9q},+\frac{1}{9q},\text{etc.}$$

with two equal terms being present. Therefore the sum of these terms, which is :
$\frac{2}{q}(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\text{etc.})$ is equal to P which is equal to 1. Hence this series arises :
$(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\text{etc.})=\frac{q}{2}=\frac{p}{4}$. Therefore the fourth part of this series is equal to the semi-circumference of the circle, of which the radius is 1, or to the whole circumference of the circle of which the diameter is 1. And this is that series advanced now some time ago by *Leibniz* , in which he defined the quadrature of the circle. From which the power of this method, if perhaps one was not entirely sure of it, is manifestly supported; thus as with the remaining series, which are to be derived by this method, these are not to be doubted.

§.11. Now we take up the case of finding the terms when $y = 1$, the squares, and this series is produced $+\frac{1}{q^2}+\frac{1}{q^2}+\frac{1}{9q^2}+\frac{1}{9q^2}+\frac{1}{25q^2}+\frac{1}{25q^2}+\text{etc.}$, the sum of which is $\frac{2}{q^2}(\frac{1}{1}+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\text{etc.})$, which hence is equal to $Q=P=1$. From which it follows that the sum of this series $1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\text{etc.}$ is equal to $\frac{q^2}{2}=\frac{p^2}{8}$; with p denoting the whole circumference of the circle, the diameter of which is equal to 1. But the sum of this series $1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\text{etc.}$ depends on the sum of the series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+etc.$ since this

series is a quarter part of that whole series. Hence the sum of this whole series is equal to the sum of that series found and a third of it.
[As Q = sum odd squares + Q/4 or Q = 4/3 × sum odd squares.]
On account of which $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+etc.=\frac{p^2}{6}$, and thus the sum of this series multiplied by 6 is equal to the square of the circumference of the circle the diameter of which is 1; which is that proposition that I mentioned in the beginning.

§.12. Therefore in the case when $y = 1$, on letting $P = 1$ and $Q = 1$, it follows that the remaining letters R, S, T, V, etc., are as follows:
$R=\frac{1}{2}; S=\frac{1}{3}; T=\frac{5}{24}; V=\frac{2}{15}; W=\frac{61}{720}; X=\frac{17}{325};$ etc. But since for the sum of the cubes, $R=\frac{1}{2}$, then $\frac{2}{q^3}(1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}\text{ - etc.})=\frac{1}{2}$. Whereby the equation becomes :
$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}\text{ - etc.}=\frac{q^3}{4}=\frac{p^3}{32}$. Thus the sum of this series multiplied by 32 gives the cube of the circumference of the circle of which the diameter is 1. In a similar manner the sum of the fourth powers, which is $\frac{2}{q^4}(1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\text{etc.})$ must be equal to $\frac{1}{3}$, and thus $1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\text{etc.}=\frac{q^4}{6}=\frac{p^4}{96}$. Now this series multiplied by $\frac{16}{15}$ is $\frac{p^4}{90}$ of that series; or to the series $1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\text{etc.}$ whereby this series is equal to $\frac{p^4}{90}$; or for the $1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\text{etc.}$, the sum multiplied by 90 gives the fourth power of the circumference of the circle of which the diameter is 1.

§.13. In a similar way the sum of the higher powers can be found; moreover as it follows that $1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{9^5}\text{ - etc.}=\frac{5q^5}{48}=\frac{5p^5}{1536};$
then $1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}+\text{etc.}=\frac{q^6}{16}=\frac{p^6}{960}$.
Now from the sum of this series, likewise the sum of this series
$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\text{etc.}$ becomes known, which is equal to $\frac{p^6}{945}$. Again for the seventh powers, there is $1-\frac{1}{3^7}+\frac{1}{5^7}-\frac{1}{7^7}+\frac{1}{9^7}\text{ - etc.}=\frac{61q^7}{1440}=\frac{61p^7}{184320}$ and for the powers of eight
$1+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\frac{1}{9^8}+\text{etc.}=\frac{q^8}{630}=\frac{17p^8}{161280};$ hence it is deduced that
$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\frac{1}{6^8}+\text{etc.}=\frac{p^8}{9450}$. Moreover it is to be observed that in these series with the odd powers of the exponent with the alternating sign, that for the even powers the signs are now the same; and because of this, since in the general series, $1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\text{etc.}$ the sum can only be shown in the cases in which n is an even number. Besides also it is to be observed that if we find the terms of the series

$1, 1, \frac{1}{2}, \frac{1}{3}, \frac{5}{24}, \frac{2}{15}, \frac{61}{720}, \frac{17}{315}$ etc. which are the values of the letters *P, Q, R, S,* etc., then the general sum can be assigned, while from that itself the quadrature of the circle can be shown.

§.14. In these we have put the sine *PM* equal to the radius, hence we may see what series are produced if other values are presented to *y*. Therefore, let $y = \frac{1}{\sqrt{2}}$, to which sine the smallest corresponding arc is $\frac{1}{4}p$. Hence with $A = \frac{1}{4}p$, the series of the simplest terms or of the first powers is thus $\frac{4}{p} + \frac{4}{3p} - \frac{4}{5p} - \frac{4}{7p} + \frac{4}{9p} + \frac{4}{11p} - \text{etc.}$, and the sum *P* of this series is equal to $\frac{1}{y} = \sqrt{2}$. Hence there is obtained :

$\frac{p}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} - \frac{1}{9} - \frac{1}{11} - \frac{1}{13} - \frac{1}{15}\text{etc.}$, which series differs from that of *Leibniz* only by reason of the sign, and advanced by *Newton* now a long time ago. Now the sum of the squares of these terms, clearly $\frac{16}{p^2}(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \text{etc.})$ is itself equal to Q = 2. Hence there is $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \text{etc.} = \frac{p^2}{8}$, as was found before.

§.15. If we make $y = \frac{3}{\sqrt{2}}$, the smallest arc of this sine corresponds to 60^0, and thus $A = \frac{1}{3}p$. Therefore in this case the series of terms is produced :

$\frac{3}{p} + \frac{3}{2p} - \frac{3}{4p} - \frac{3}{5p} + \frac{3}{7p} + \frac{3}{8p} - \text{etc.}$, and the sum of which terms is equal to $\frac{1}{y} = \frac{2}{\sqrt{3}}$. Therefore it is found that $\frac{2p}{3\sqrt{3}} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \text{etc.}$ Now the sum of the squares of these terms is equal to $\frac{1}{y^2} = \frac{4}{3}$; thus it follows to be the case that

$\frac{4p^2}{27} = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{25} + \frac{1}{49} + \frac{1}{64} + \text{etc.}$ in which series the successive third terms are missing. But this series also depends on that series $1 + \frac{1}{4} + \frac{2}{9} + \frac{3}{16}\text{etc.}$, the sum of which was found to be equal to $\frac{p^2}{6}$; for if this series is diminished by its ninth part then the above series is produced, thus the sum of this must be equal to $\frac{p^2}{6}(1 - \frac{1}{9}) = \frac{4pp}{27}$. In a similar manner, if other sines are assumed, then other series are produced, as of the first power and then of the squares and of higher powers, and the sums of which involve the quadrature of the circle.

§.16. But if y is put equal to 0, it is not possible to assign series of this kind any more. on account of placing y in the denominator, or the initial series divided by y. But the series are able to be deduced in another way, while which series are of the form
$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\text{etc.}$, if n is an even number : as the sums of these series are found, I can deduce separately from these the case in which $y = 0$. Now with $y = 0$ the fundamental equation is turned into this equation :

$$0=s-\frac{s^3}{1.2.3}+\frac{s^5}{1.2.3.4.5}-\frac{s^7}{1.2.3.4.5.6.7}+\text{etc.},$$

and the roots of this equation give all the arcs, of which the sine is equal to zero. But the smallest root is $s = 0$, whereby the equation divided by s shows all the remaining arcs of which the sine is equal to zero, which arcs are hence produced by the roots of this equation : $0=1-\frac{s^2}{1.2.3}+\frac{s^4}{1.2.3.4.5}-\frac{s^6}{1.2.3.4.5.6.7}+\text{etc.}$

Now the arcs of these of which the sine is equal to zero are :
$p,-p,2p,-2p,3p,-3p$ etc . of which each second term is negative, that which the equation also indicates on account of only having even dimensions of s. Whereby the divisors of this equation are :

$$1-\frac{s}{p},1+\frac{s}{p},1-\frac{s}{2p},1+\frac{s}{2p},\text{etc.},$$

and with the pairs of these divisors joined together, there is the equation :

$$1-\frac{s^2}{1.2.3}+\frac{s^4}{1.2.3.4.5}-\frac{s^6}{1.2.3.4.5.6.7}+\text{etc.}=(1-\frac{s^2}{p^2})(1-\frac{s^2}{4p^2})(1-\frac{5s^2}{9p^2})(1-\frac{s^2}{16p^2})\text{ etc.}$$

§.17. Now it is evident from the nature of the equations, that the coefficient of ss or $\frac{1}{1.2.3}$ is equal to $\frac{1}{p^2}+\frac{1}{4p^2}+\frac{1}{9p^2}+\frac{1}{16p^2}+\text{etc.}$ Now the sum of the factors from the two terms of this series is equal to $\frac{1}{1.2.3.4.5}$; and the sum of the factors from the three terms is equal to $\frac{1}{1.2.3.4.5.6.7}$ etc. On account of this, just as in §.8
$\alpha=\frac{1}{1.2.3};\beta=\frac{1}{1.2.3.4.5};\gamma=\frac{1}{1.2.3.4.5.6.7};$ etc. and with the sum of the terms also put equal to P : $\frac{1}{p^2}+\frac{1}{4p^2}+\frac{1}{9p^2}+\frac{1}{16p^2}+\text{etc.}=P,$ and with the sum of the squares of the terms equal to Q; the sum of the cubes equal to R; the sum of the fourth powers equal to S, etc. , by §.8 we have : $P=\alpha=\frac{1}{1.2.3}=\frac{1}{6}$; $Q=P\alpha-2\beta=\frac{1}{90}$; $R=Q\alpha-P\beta+3\gamma=\frac{1}{945}$;

$$S=R\alpha-Q\beta+P\gamma-4\delta=\frac{1}{9450}\ T=S\alpha-R\beta+Q\gamma-P\delta+5\varepsilon=\frac{1}{93555};$$

$$V=T\alpha-S\beta+R\gamma-Q\delta+P\varepsilon-6\varsigma=\frac{691}{6825\bullet 93555}\text{ etc.}$$

§.18. Hence from these the following sums can be derived :

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\text{etc.}=\frac{p^2}{6}=P$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\text{etc.}=\frac{p^4}{90}=Q$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\text{etc.}=\frac{p^6}{945}=R$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\text{etc.}=\frac{p^8}{9450}=S$$

$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\frac{1}{5^{10}}+\text{etc.}=\frac{p^{10}}{93555}=T$$

$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\frac{1}{5^{12}}+\text{etc.}=\frac{691p^{12}}{6821\bullet 93555}=V.$$

which series from the given law but yet with much labour can be extended to the higher powers. Moreover, with a particular series divided by the preceding series, this sequence of equations arises :

$p^2=6P=\frac{15Q}{P}=\frac{21R}{2Q}=\frac{10S}{R}=\frac{99T}{10S}=\frac{6825V}{691T}$ etc. from which individual expressions the square of the perimeter of the circle of which the diameter is 1 is equal.

§.19. Moreover when the sums of these series, even if now they can be easily shown as above, yet they are not able to bring much help to expressing the perimeter of the circle on account of the root of the square, that has to be extracted; from the former series we can elicit expressions, which are equal to the periphery *p* itself . Moreover, these are produced as follows :

$$p = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.})$$

$$p = 2\left(\frac{1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\text{etc.}}{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\text{ etc.}}\right)$$

$$p = 4\left(\frac{1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+\text{etc.}}{1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\text{etc.}}\right)$$

$$p = 8\left(\frac{1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\frac{1}{11^4}+\text{etc.}}{1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+\text{etc.}}\right)$$

$$p = \frac{16}{5}\left(\frac{1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{9^5}-\frac{1}{11^5}+\text{etc.}}{1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\frac{1}{11^4}+\text{etc.}}\right)$$

$$p = \frac{25}{8}\left(\frac{1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}+\frac{1}{11^6}+\text{etc.}}{1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{9^5}-\frac{1}{11^5}+\text{etc.}}\right)$$

$$p = \frac{192}{61}\left(\frac{1-\frac{1}{3^7}+\frac{1}{5^7}-\frac{1}{7^7}+\frac{1}{9^7}-\frac{1}{11^7}+\text{etc.}}{1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}+\frac{1}{11^6}+\text{etc.}}\right).$$

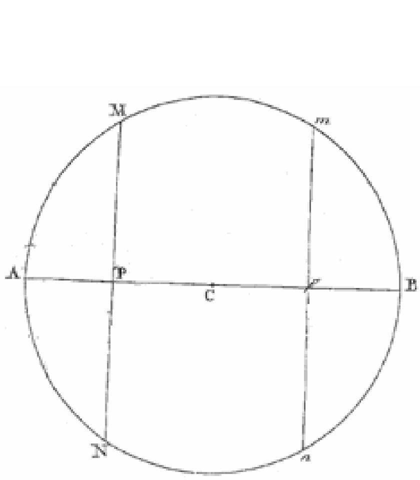
De summis serierum reciprocarum.

Auctore Leonh. Eulero.

§. 1.

Tantopere iam pertractatae et investigatae sunt series reciprocae potestatum numerorum naturalium, ut vix probabile videatur de iis novi quicquam invenire posse. Quicunque enim de summis serierum meditati sunt, ii fere omnes quoque in summas huiusmodi serierum inquisiverunt, neque tamen ulla methodo eas idoneo modo exprimere potuerunt. Ego etiam iam saepius, cum varias summandi methodos tradidissem, has series diligenter sum persecutus, neque tamen quicquam aliud sum assecutus, nisi ut earum summam vel proxime veram definiverim vel ad quadraturas curvarum maxime transcendentium reduxerim; quorum illud in differtatione proxime praelecta, hoc vero in praecedentibus praestiti. Loquor hic autem de seriebus fractionum, quarum numeratores sunt 1, denominatores vero vel quadrata, vel cubi, vel aliae dignitates numerorum naturalem : cuius modi sunt $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+etc.$, item $1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+etc.$ atque similes superiorum potestatum, quarum termini generales continentur in hac forma $\frac{1}{x^n}$.

§.2. Deductus sum autem nuper omnino inopinato ad elegantem sumae huius seriei $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+etc.$ expressionem, quae a circuli quadratura pendet, ita, ut si hius seriei vera summa habetetur, inde simul circuli quadratura sequeratur. Inveni enim summae huius seriei sextuplum aequale esse quadrato peripheriae circuli, cuius diameter est 1; seu posita istius seriei summa = s, tenebit $\sqrt{6s}$ ad 1 rationem peripheriae ad diametrum. Huius autem seriei summam nuper ostendi proxime esse 1, 6449340668482264364, ex cuius numeri sextuplo, si extrahatur radix quadrata, reipsa prodit numerus 3,141592653589793238 exprimens circuli peripheriam, cuius diameter est 1. Iisdam porro vestigiis quibus hanc summam sum consecutus, incedens huius seriei $1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256}+\frac{1}{625}+etc.$ summam quoque a quadratura circuli pendere deprehendi. Summa nempe eius per 90 multiplicata dat biquadratum peripheriae circuli, cuius diameter est 1. Atque simili ratione etiam sequentium serierum, in quibus exponentes dignitatum sunt numeres pares, summas determinare potui.



§.3. Quo igitur, quemadmodum haec sum adeptus, commodissime ostendam, totam rem, quo ipse usus sum, ordine exponam. In circulo AMBNA centro C radio AC vel BC = 1 descripto contemplatus sum arcum quemcunque AM, cuius sinus est MP, cosinus vero CP. Posito nunc arc AM = s, sinu PM = y, et cosinu CP = x, per methodum iam satis cognitam tam sinus y cosinusx ex dato arcu s per series possunt definiri, est enim, uti passim videre licet $y = s - \frac{s^3}{1.2.3} + \frac{s^5}{1.2.3.4.5} - \frac{s^7}{1.2.3.4.5.6.7} + \text{etc.}$ atque $x = 1 - \frac{s^2}{1.2} + \frac{s^4}{1.2.3.4} - \frac{s^6}{1.2.3.4.5.6} + \text{etc.}$ Ex harum scilicit aequationem consideratione ad summas supra memoratarum serierum reciprocarum perveni; quarum aequationum quidem utraque ad eundem fere scopum dirigitur, et hanc ob rem sufficiet alteram tantum eo, quem sum expositurus, modo tractasse.

§.4. Aequatio ergo prior $y = s - \frac{s^3}{1.2.3} + \frac{s^5}{1.2.3.4.5} - \frac{s^7}{1.2.3.4.5.6.7} + \text{etc.}$ exprimit relationem inter arcum et sinum. Quare ex ea tam ex dato arcu eius sinus, quam ex dato sinu eius arcus determinari poterit. Considero autem sinum y tanquam datum, et investigo, quemadmodum arcum s ex y erui oporteat. Hic vero ante omnia animadvertendum est, eidem sinui y innumerabiles arcus respondere, quos ergo innumerabiles arcus aequatio proposita praebere debebit. Si quidem in ista aequatio s tanquam incognita spectetur, ea infinitas habet dimensiones, ideoque mirum non est, si ista aequatio innumeros contineat factores simplices, quorum quisque nihilo aequalis positus, idoneum pro s valorem dare debet.

§.5. Quemadmodum autem, si omnes factores huius aequationis cogniti essent, omnes quoque radices illius seu valores cogniti essent, omnes quoque radices illius valores s innotescerent, ita vicissim si omnes valores ipsius s assingari poterunt, tum quoque ipsi factores omnes habebuntur. Quo autem eo melius tam de radicibus quam de factoribus iudicare queam, transmuto aequationem propositam in hanc formam :
$0 = 1 - \frac{s}{y} - \frac{s^3}{1.2.3.y} + \frac{s^5}{1.2.3.4.5.y} + \text{etc.}$ Si nunc omnes radices huius aequationis seu omnes arcus, quorum idem est sinus y, fuerint *A, B, C, D, E*, etc. tum factores quoque erunt omnes istae quantitates, $1 - \frac{s}{A}, 1 - \frac{s}{B}, 1 - \frac{s}{C}, 1 - \frac{s}{D}$, etc. Quamobrem erit

$$0 = 1 - \frac{s}{y} - \frac{s^3}{1.2.3.y} + \frac{s^5}{1.2.3.4.5.y} + \text{etc.} = (1 - \tfrac{s}{A})\,(1 - \tfrac{s}{B})\,(1 - \tfrac{s}{C})\,(1 - \tfrac{s}{D}) \text{ etc.}$$

§.6. Ex natura autem et resolutione aequationum constat, esse coefficentem termini, in quo inest s, seu $\frac{1}{y}$ aequalem summae omnium coefficientium ipsius s in factoribus seu $\frac{1}{y} = \frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D} + \text{etc.}$ Deinde est coefficiens ipsius s^2, qui est = 0, ob hunc terminum in aequatione deficientem, aequalis aggregatio factorum ex binis terminis seriei, $\frac{1}{A}, \frac{1}{B}, \frac{1}{C}, \frac{1}{D},$ etc. Porro erit $-\frac{1}{1.2.3.y}$ aequale aggregato factorum ex quaternis terminis eiusdem seriei $\frac{1}{A}, \frac{1}{B}, \frac{1}{C}, \frac{1}{D},$ etc. Similique modo erit 0 = aggregato factorum ex quaternis terminis eiusdem seriei, et $+\frac{1}{1.2.3.4.5.y} =$ aggregato factorum ex quinis terminis istius seriei, et ita porro.

§.7. Posito autem mimimo arcu $AM = A$, cuius sinus est $PM = y$, et semiperipheria circuli $= p$, erunt $A, p-A, 2p+A, 3p-A, 4p+A, 5p-A, 6p+A$ etc.
item $-p-A, -2p+A, -3p-A, -4p+A, -5p-A,$ etc. omnes arcus, quorum sinus est idem y. Quam igitur ante assumsimus seriem $\frac{1}{A}, \frac{1}{B}, \frac{1}{C}, \frac{1}{D},$ etc. ea transmutatur in hanc $\frac{1}{A}, \frac{1}{p-A}, \frac{1}{-p-A}, \frac{1}{2p+A}, \frac{1}{-2p+A}, \frac{1}{3p-A}, \frac{1}{-3p-A}, \frac{1}{4p+A}, \frac{1}{-4p+A}$ etc. Horum ergo omnium terminorum summa est $= \frac{1}{y}$; summa autem factorum ex binis terminis huius seriei est aequalis 0; summa factorum ex ternis = $\frac{-1}{1.2.3.y}$, summa factorum ex quaternis = 0; summa factorum ex quinis = $\frac{+1}{1.2.3.4.5.y}$; summa factorum ex senis = 0. Atque ita porro.

§.8. Si autem habeatur series quaecunque $a+b+c+d+e+f+$etc. cuius summa sit α, summa factorum ex binis terminis = β; summa factorum ex ternis = γ; summa factorum ex quaternis = δ, etc. erit summa quadratorum singulorum terminorum, hoc est $a^2+b^2+c^2+d^2+e^2+f^2+\text{etc.}=\alpha^2-2\beta$; summa vero cuborum $a^3+b^3+c^3+d^3+e^3+f^3+\text{etc.} = \alpha^3-3\alpha\beta+3\gamma$; summa biquadratorum = $\alpha^4-4\alpha^2\beta+4\alpha\gamma+2\beta^2-4\delta.$ Quo autem clarius appareat, quomodo hae formulae progrediantur, ponamus ipsorum terminorum *a, b, c, d*, etc. summam esse = *P*, summam quadratorum = *Q*, summa biquadratorum = *S*, summam potestatum quintarum = *T*, summam sextarum = *V* etc. His positis erit
$P=\alpha;\ Q=P\alpha-2\beta;\ R=Q\alpha-P\beta+3\gamma;\ S=R\alpha-Q\beta+P\gamma+4\delta;\ T=S\alpha-R\beta+Q\gamma-P\delta+5\varepsilon;$ etc

§.9. Cum igitur in nostro casu seriei
$\frac{1}{A}, \frac{1}{p-A}, \frac{1}{-p-A}, \frac{1}{2p+A}, \frac{1}{-2p+A}, \frac{1}{3p-A}, \frac{1}{-3p-A}$ etc. summa omnium terminorum seu α sit $= \frac{1}{y}$; summa factorum ex binis seu β = 0, atque ulterius
$\gamma=\frac{-1}{1.2.3.y};\ \delta=0;\ \varepsilon=\frac{+1}{1.2.3.4.5.y};\ \varsigma=0;$ etc erit summa ipsorum illorum terminorum $P=\frac{1}{y};$ summa quadratorum illorum terminorum $Q=\frac{P}{y}=\frac{1}{y^2};$ summa cuborum illorum terminorum $R=\frac{Q}{y}-\frac{1}{1.2.y}$;summa biquadratorum $S=\frac{R}{y}-\frac{P}{1.2.3y}.$ Atque porro
$T=\frac{S}{y}-\frac{Q}{1.2.3.y}+\frac{1}{1.2.3.4.y};\ V=\frac{T}{y}-\frac{R}{1.2.3.y}+\frac{P}{1.2.3.4.5.y};$
$W=\frac{V}{y}-\frac{S}{1.2.3.y}+\frac{Q}{1.2.3.4.5.y}-\frac{1}{1.2.3.4.5.6.y}.$ Ex qua lege facile reliquarum altiorum potestatum summae determinantur.

§.10. Ponamus nunc sinum $PM = y$ aequalem radio, ut sit $y = 1$, erit minimus arcus A cuius sinus est 1 quarta peripheria pars, = $\frac{1}{2}p$, seu denotante q quartam peripheriae partem erit $A = q$ et $p = 2q$. Superior ergo series abibit in istam $\frac{1}{q}, \frac{1}{q}, \frac{-1}{3q}, -\frac{1}{3q}, \frac{+1}{5q}, +\frac{1}{5q}, -\frac{1}{7q}, -\frac{1}{7q}, +\frac{1}{9q}, +\frac{1}{9q}$, etc. binis terminis existentibus aequalibus. Horum ergo terminorum summa, quae est $\frac{2}{q}(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\text{etc.})$ aequalis est ipsi $P = 1$. Hinc igitur oritur

$(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\text{etc.}) = \frac{q}{2} = \frac{p}{4}$. Huius ergo seriei quadruplum aequatur semiperipheriae circuli, cuius radius est 1, seu toti peripheriae circuli, cuius diameter est 1. Atque haec est ipsa series a *Leibnitio* iam pridem prolata, qua circuli quadraturam definiuit. Ex quo magnum huius methodi, si cui forte ea non satis certa videatur, firmamentum elucit; ita ut de reliquis, quae ex hac methodo derivabantur, omnino non liceat dubitari.

§.11. Sumamus nunc inventorum terminorum pro casu quo $y = 1$, quadrata, prodibitque haec series $+\frac{1}{q^2}+\frac{1}{q^2}+\frac{1}{9q^2}+\frac{1}{9q^2}+\frac{1}{25q^2}+\frac{1}{25q^2}+\text{etc.}$ cuius summa est $\frac{2}{q^2}(\frac{1}{1}+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\text{etc.})$, quae ergo aequalis esse debet ipsi $Q = P = 1$. Ex quo sequitur huius seriei $1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\text{etc.}$ summam esse = $\frac{q^2}{2} = \frac{p^2}{8}$; denotante p totam circuli peripheriam, cuius diameter est = 1. Summa autem huius seriei $1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\text{etc.}$ pendet a summa seriei $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+etc.$ quia haec quarta sui parte minuta illam dat. Est ergo summa huius seriei aequalis summa illius sui triente. Quamobrem erit $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}+etc. = \frac{p^2}{6}$, ideoque huius seriei summa per 6 multiplicata aequalis est quadrato peripheriae circuli cuius diameter est 1; quae est ipsa propositio cuius initio mentionem feci.

§.12. Cum igitur casu quo y = 1, sit P = 1 et Q = 1, erunt reliquarum litterarum R, S, T, V, etc. ut sequitur : $R = \frac{1}{2}; S = \frac{1}{3}; T = \frac{5}{24}; V = \frac{2}{15}; W = \frac{61}{720}; X = \frac{17}{325};$ etc. Cum autem summa cuborum ipsi $R = \frac{1}{2}$ sit aequalis, erit $\frac{2}{q^3}(1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\text{etc.}) = \frac{1}{2}$. Quare erit $1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\text{etc.} = \frac{q^3}{4} = \frac{p^3}{32}$. Huius ideo seriei summa per 32 multiplicat dat cubum peripheriae circuli cuius diameter est 1. Simili modo summa biquadratorum, quae est $\frac{2}{q^4}(1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\text{etc.})$ aequalis esse debet $\frac{1}{3}$, ideoque erit

$1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\text{etc.} = \frac{q^4}{6} = \frac{p^4}{96}$. Est vero haec series per $\frac{16}{15}$ multiplicata ista series aequalis est $\frac{p^4}{90}$; seu seriei $1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\text{etc.}$ quare ista series aequalis est

$\frac{p^4}{90}$; seu seriei $1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\text{etc.}$ summa per 90 multiplicata dat biquadratum peripheriae circuli cuius diameter est 1.

§.13. Simili modo invenientur summae superiorum potestatum; prodibit autem ut sequitur $1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{9^5}-\text{etc.}=\frac{5q^5}{48}=\frac{5p^5}{1536};$ atque $1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}+\text{etc.}=\frac{q^6}{16}=\frac{p^6}{960}$. Inventa vero huius seriei summa, cognoscetur simul summa huius seriei $1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\text{etc.}$ quae erit = $\frac{p^6}{945}$. Porro pro potestatibus septimus erit $1-\frac{1}{3^7}+\frac{1}{5^7}-\frac{1}{7^7}+\frac{1}{9^7}-\text{etc.}=\frac{61q^7}{1440}=\frac{61p^7}{184320}$ ac pro octavis $1+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\frac{1}{9^8}+\text{etc.}=\frac{q^8}{630}=\frac{17p^8}{161280};$ unde deducitur $1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\frac{1}{6^8}+\text{etc.}=\frac{p^8}{9450}$. Observandum autem est de his seriebus in potentiis exponentium imparium signa terminoru alternari, pro potestatibus paribus vero esse aequalia; hocque in cause est, quod huius generalis seriei $1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\text{etc.}$ iis tantum casibus summa possit exhiberi, quibus *n* est numerus par. Praeterea quoque notandum est, si seriei $1,1,\frac{1}{2},\frac{1}{3},\frac{5}{24},\frac{2}{15},\frac{61}{720},\frac{17}{315}\text{ etc.}$ quos valores pro litteris *P, Q, R, S,* etc. invenimus, terminus generalis posset assignari, tum eo ipso quadraturam circuli exhibitum iri.

§.14. In his posuimus sinum *PM* aequalem radio, videamus ergo quales series prodeant, si ipsi *y* alii valores tribuantur. Sit igitur $y=\frac{1}{\sqrt{2}}$, cui sinui minimus arcus respondens est $\frac{1}{4}p.$ Posito ergo $A=\frac{1}{4}p$ erit series terminorum simplicium seu primae potestatis ista $\frac{4}{p}+\frac{4}{3p}-\frac{4}{5p}-\frac{4}{7p}+\frac{4}{9p}+\frac{4}{11p}-\text{etc.}$ cuius seriei summa *P* aequalis est $\frac{1}{y}=\sqrt{2}$. Habibitur ergo $\frac{p}{2\sqrt{2}}=1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}-\frac{1}{9}-\frac{1}{11}-\frac{1}{13}-\frac{1}{15}\text{etc.}$ quae series tantum ratione signorum a Leibnitiana differt, et a *Newtono* iam dudum est prolata. Summa vero quadratorum illorum terminorum nempe $\frac{16}{p^2}(1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\text{etc.})$ aequalis est ipsi Q = 2. Erit ergo $1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\text{etc.}=\frac{p^2}{8}$, uti ante est inventum.

§.15. Si fiat $y=\frac{3}{\sqrt{2}}$ erit minimus arcus huic sinui respondens 60^0, ideoque $A=\frac{1}{3}p$. Hoc ergo casu sequens prodibit series terminorum $\frac{3}{p}+\frac{3}{2p}-\frac{3}{4p}-\frac{3}{5p}+\frac{3}{7p}+\frac{3}{8p}-\text{etc.}$ quorum terminorum summa aequalis est ipsi $\frac{1}{y}=\frac{2}{\sqrt{3}}$. Habitur ergo

$\frac{2p}{3\sqrt{3}} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \text{etc.}$ Summa vero quadratorum illorum terminorum est = $\frac{1}{y^2} = \frac{4}{3}$; unde sequitur fore $\frac{4p^2}{27} = 1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{25} + \frac{1}{49} + \frac{1}{64} + \text{etc.}$ in qua serie desunt termini ternario constantes. Pendet autem haec series quoque ab ista $1 + \frac{1}{4} + \frac{2}{9} + \frac{3}{16}\,\text{etc.}$ cuius summa erat inventa = $\frac{p^2}{6}$; nam si haec series sui parte nona minuatur prodit ipsa superior series, cuius ideo summa debet esse = $\frac{p^2}{6}(1 - \frac{1}{9}) = \frac{4pp}{27}$. Simili modo si alii assumatur sinus, aliae prodibunt series, tam simplicium, quam terminorum quadratorum altiorumque potestatum, quarum summae quadraturum circuli involuent.

§.16. At si ponatur $y = 0$, huiusmodi series non amplius assignari poterunt, propter y in denominatorem positum, seu aequationem inialem per y divisam. Alio autem modo series inde deduci poterunt, quae cum sint ipsae series $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \text{etc.}$ si n est numerus par : quaemadmodum harum serierum summae sint inveniendae, seorsum ex hoc casu quo $y = 0$ deducam. Posito vero $y = 0$ ipsa aequatio fundamentalis abit in hanc $0 = s - \frac{s^3}{1.2.3} + \frac{s^5}{1.2.3.4.5} - \frac{s^7}{1.2.3.4.5.6.7} + \text{etc.}$ cuius aequationis radices dant omnes arcus, quorum sinus est = 0. Est autem una minimaque radix $s = 0$, quare aequatio per s divisa exhibebit reliquos arcus omnes, quorum sinus est = 0, qui arcus proinde erunt radices huius aequationis $0 = 1 - \frac{s^2}{1.2.3} + \frac{s^4}{1.2.3.4.5} - \frac{s^6}{1.2.3.4.5.6.7} + \text{etc.}$ Ipsi vero arcus quorum sinus est = 0 sunt $p, -p, 2p, -2p, 3p, -3p$ etc . quorum binorum alter alterius est negativus, id quod quoque ipsa aequatio propter dimensiones ipsius s tantum pares indicat. Quare divisores illius aequationis erunt $1 - \frac{s}{p}, 1 + \frac{s}{p}, 1 - \frac{s}{2p}, 1 + \frac{s}{2p}$, etc. atque coniungendis binis horum divisorum erit

$$1 - \frac{s^2}{1.2.3} + \frac{s^4}{1.2.3.4.5} - \frac{s^6}{1.2.3.4.5.6.7} + \text{etc.} = (1 - \frac{s^2}{p^2})(1 - \frac{s^2}{4p^2})(1 - \frac{5s^2}{9p^2})(1 - \frac{s^2}{16p^2})\ \text{etc.}$$

§.17. Manifestum iam est ex natura aequationum, fore coefficientem ipsius ss seu $\frac{1}{1.2.3}$ aequalem $\frac{1}{p^2} + \frac{1}{4p^2} + \frac{1}{9p^2} + \frac{1}{16p^2} + \text{etc.}$ Summa vero factorum ex binis terminis huius seriei erit $= \frac{1}{1.2.3.4.5}$; summaque factorum ex ternis $= \frac{1}{1.2.3.4.5.6.7}$ etc. Hanc ob rem erit iuxta §.8 $\alpha = \frac{1}{1.2.3}$; $\beta = \frac{1}{1.2.3.4.5}$; $\gamma = \frac{1}{1.2.3.4.5.6.7}$; etc. atque posita quoque summa terminorum $\frac{1}{p^2} + \frac{1}{4p^2} + \frac{1}{9p^2} + \frac{1}{16p^2} + \text{etc.} = P$, et summa quadratorum eorundem terminorum = Q; summa cuborum = R; summa biquadratorum = S, etc. , erit per §.8

$$P = \alpha = \frac{1}{1.2.3} = \frac{1}{6};\ Q = P\alpha - 2\beta = \frac{1}{90};\ R = Q\alpha - P\beta + 3\gamma = \frac{1}{945};$$

$S = R\alpha - Q\beta + P\gamma - 4\delta = \frac{1}{9450}$ $T = S\alpha - R\beta + Q\gamma - P\delta + 5\varepsilon = \frac{1}{93555}$;

$V = T\alpha - S\beta + R\gamma - Q\delta + P\varepsilon - 6\varsigma = \frac{691}{6825 \cdot 93555}$ etc.

§.18. Ex his ergo derivantur summae sequentes :

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} = \frac{p^2}{6} = P$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} = \frac{p^4}{90} = Q$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \text{etc.} = \frac{p^6}{945} = R$$

$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \text{etc.} = \frac{p^8}{9450} = S$$

$$1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \frac{1}{5^{10}} + \text{etc.} = \frac{p^{10}}{93555} = T$$

$$1 + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{4^{12}} + \frac{1}{5^{12}} + \text{etc.} = \frac{691p^{12}}{6821 \cdot 93555} = V.$$

quae series ex data lege attamen multo laboread altiores potestates produci possunt. Dividendis autem singulis seriebus per praecedentes orientur sequentes aequationes :

$p^2 = 6P = \frac{15Q}{P} = \frac{21R}{2Q} = \frac{10S}{R} = \frac{99T}{10S} = \frac{6825V}{691T}$ etc. quibus expressionibus singulis quadratum peripheriae cuius diameter est 1, aequatur.

§.19. Cum autem harum serierum summae etiamsi vero proxime facile exhiberi possent, tamen non multum adiumenti afferre queant ad peripheriam circuli vero proxime exprimendam propter radicem quadratam, quae extrahi deberet; ex prioribus seriebus elicemus expressiones, quae ipsi peripheriae *p* sint aequales. Prodibit autem vi sequitur :

$$p = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.})$$

$$p = 2\left(\frac{1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\text{etc.}}{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\text{ etc.}}\right)$$

$$p = 4\left(\frac{1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+\text{etc.}}{1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\text{etc.}}\right)$$

$$p = 8\left(\frac{1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\frac{1}{11^4}+\text{etc.}}{1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+\text{etc.}}\right)$$

$$p = \frac{16}{5}\left(\frac{1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{9^5}-\frac{1}{11^5}+\text{etc.}}{1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\frac{1}{11^4}+\text{etc.}}\right)$$

$$p = \frac{25}{8}\left(\frac{1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}+\frac{1}{11^6}+\text{etc.}}{1-\frac{1}{3^5}+\frac{1}{5^5}-\frac{1}{7^5}+\frac{1}{9^5}-\frac{1}{11^5}+\text{etc.}}\right)$$

$$p = \frac{192}{61}\left(\frac{1-\frac{1}{3^7}+\frac{1}{5^7}-\frac{1}{7^7}+\frac{1}{9^7}-\frac{1}{11^7}+\text{etc.}}{1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\frac{1}{9^6}+\frac{1}{11^6}+\text{etc.}}\right).$$