The Solution of the Differential Equation

$$ax^n dx = dy + y^2 dx.$$

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§1.

Recently, I communicated to the Society an example of the solution of a certain differential equation, in which not only could the indeterminates not be separated from each other, but also I demonstrated from that construction that a separation of this kind could never be shown. Indeed, my method of construction given there differs from the normal : but yet anyone who reads that paper understands that by no means has anything been left out. Now not having the time then to extend this method further, and to see if it could be applied to free up other equations, and since I could not arrive at a final [general] equation from the given construction [at that time], I was unable moreover to elicit the construction in turn from the given equation. [p.232] But consequently I have considered this matter more carefully, and in a certain way what I have done now has answered my prayers, as thus by inverting the equation by the method now presented, I can find the construction of the proposed equation.

§2. Therefore, I have selected at once the equation $ax^n dx = dy + y^2 dx$ as posing the greatest difficulty [on which to demonstrate the method], that the celebrated *Count Riccati* first proposed to the Geometers for examination, and nobody could construct solutions to this problem, except for certain values of the letter *n*. Now, I have happily overcome all the difficulties with the benefit of this method of mine, and I have found the general solution of this equation, in which nothing at all is left to be desired. Moreover, not only does this single method supply the solution, but also much more. Therefore I see this method deserving to be considered as so outstanding, since a common method of construction has been pointed out in the solution of all differential equations, in which other methods used have been frustrated.

§3. As in the above dissertation that I used with the arc of the ellipse, in constructing the solution of this equation :

$$dy + \frac{y^2 dx}{x} = \frac{x dx}{x^2 - 1},$$

thus for the proposed equation, another curve is needed to be put in place of the ellipse. In order that I can find such a curve, I put in place the most general element of this kind, equal to $PRdz$, in which P and R are functions of z, such as acted upon by the same operations as above for the elliptical element, lead to the proposed equation. Again I put in place a certain series, that arises to be examined : [p.233]

$$R = 1 + AgQ + ABg^2Q^2 + ABCg^3Q^3 + ABCDg^4Q^4 + \text{etc.}$$

in which series Q is a certain function of *z, g* is a given [length of a] line or taken as if a parameter of the curve; and *A, B, C, D*, etc. are constant coefficients. Putting [this general element] in place,

$$PRdz = dZ\,;$$

hence, [on integrating] :

$$Z = \int Pdx + Ag\int PQdz + ABg^2\int PQ^2dz + ABCg^3\int PQ^3dz + \text{etc.}$$

§4. Thus moreover, P and Q may depend on each other so that all these integrals can be reduced to [multiples of] $\int Pdz$ [with some constant amount added]. Hence let it be the case that :

$$\int PQdz = \alpha\int Pdz + \mathrm{O1};\ \int PQ^2dz = \alpha\beta\int Pdz + \mathrm{O2};\ \int PQ^3dz = \alpha\beta\gamma\int Pdz + \mathrm{O3};\ \text{etc.}$$

Here O1,O2, O3 etc denote algebraic quantities. After the integration has been completed in this manner, [the upper limit of the integral] z is put equal to h : moreover h is such a quantity, which put in place of *z,* makes all these algebraic quantities O1, O2, O3 etc. vanish, and then the integral becomes $\int Pdz = H$, in short it becomes equal to a constant amount. Therefore from these, on integrating and putting $z = h$, the integration then becomes :

$$Z = H(1 + A\alpha g + AB\alpha\beta g^2 + ABC\alpha\beta\gamma g^3 + \text{etc.})$$

Thus, with the parameter g made the variable, an infinite number of values of Z are obtained from the infinite number of values of *g*, and a curve can be constructed from the given element $PRdz$, in which, if the abscissae are designated by the letter *g*, then the lengths of the applied lines are equal to Z.

§5. Thus, the sum of the series

$$1 + A\alpha g + AB\alpha\beta g^2 + ABC\alpha\beta\gamma g^3 + \text{etc.}$$

will be constructed in this manner, although perhaps the sum cannot be determined in a straight forwards manner from the consideration of these terms. Moreover, I use my own method for finding the sum of series in the resolution of the equations to be reduced, that I explained some time ago [E20], in order that I can obtain an equation, the resolution of which depends on the sum of this series. [p.234] For it is evident that however complicated this resulting equation has become, yet this construction can soon bring the sum into view. Therefore now nothing other has to be done, except that the quantities of the kinds *A, B, C*, etc. and α, β, γ, etc. are to be put into place, in order that by finding the sum of this series the resolution of this equation can be described:

$$ax^n dx = dy + y^2 dx\,.$$

Now this has to be done so that the sum of the series

$$1 + A\alpha g + AB\alpha\beta g^2 + ABC\alpha\beta\gamma g^3 + \text{etc.}$$

can be returned, since otherwise the value of R cannot be known, and hence the whole construction is useless. On account of which, one cannot accept arbitrary values in place of *A, B, C*, etc., for these are such that return this summable series.

§6. Therefore it should be apparent that the series

$$1 + A\alpha g + AB\alpha\beta g^2 + ABC\alpha\beta\gamma g^3 + \text{etc.}$$

must be of the kind just mentioned, so that the summation of this series leads to the resolution of the equation :

$$ax^n dx = dy + y^2 dx \;;$$

[Initially,] I resolve this equation itself in a series, and in order that this can be done more conveniently, I put

$$y = \frac{dt}{tdx},$$

and by taking dx constant [*i. e*. the free variable], the equation becomes

$$ax^n dx = \frac{ddt}{tdx} \text{ or } ax^n t dx^2 = ddt.$$

[The reader can observe that this is a sort of integral transform, as $t = e^{\int ydx}$.]
Now in the customary manner, I substitute this series in place of t :

$$1 + \mathfrak{A}\, x^{n+2} + \mathfrak{B}\, x^{2n+4} + \mathfrak{C}\, x^{3n+6} + \text{etc.}$$

and the equation becomes [in the original document, Gothic script is used for these constants, whereas I have used Old English here as an alternative, as Gothic is not readily available]:

$$ddt = (n+1)(n+2)\,\mathfrak{A}\, x^n dx^2 + (2n+3)(2n+4)\,\mathfrak{B}\, x^{2n+2} dx^2 + (3n+5)(3n+6)\,\mathfrak{C}\, x^{3n+4} dx^2$$
+ etc.

For the sum of this series must therefore be equal to $ax^n t dx^2$, or to this series :

$$ax^n dx^2 + \mathfrak{A}\, a\, x^{2n+2} dx^2 + \mathfrak{B}\, bx^{3n+4} dx^2 + \text{etc.} \;;$$

therefore, I make the homogeneous terms [*i. e.* the corresponding powers] determined by the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. equal to these taken arbitrary, and this makes :

$$\mathfrak{A} = \frac{a}{(n+1)(n+2)},\; \mathfrak{B} = \frac{\mathfrak{A}a}{(2n+3)(2n+4)},\; \mathfrak{C} = \frac{\mathfrak{B}a}{(3n+5)(3n+6)}, \text{ etc.}$$

[p.235] Putting $ax^{n+2} = f$ for the sake of brevity, the equation for t thus becomes :

$$t = 1 + \frac{f}{(n+1)(n+2)} + \frac{f^2}{(n+1)(n+2)(2n+3)(2n+4)} + \frac{f^3}{(n+1)(n+2)(2n+3)(2n+4)(3n+5)(3n+6)} + \text{etc.}$$

Hence [as $t = e^{\int ydx}$] the summation of this series depends on the solution of the proposed equation

$$ax^n dx = dy + y^2 dx .$$

On account of which, if the above series can be transformed into this one :

$$1 + A\alpha g + AB\alpha\beta g^2 + ABC\alpha\beta\gamma g^3 + \text{etc.},$$

then likewise the construction of the proposed equation is obtained.

§7. But in order that this series, which is clearly quite general, is made a little more restricted, and the determination of the arbitrary letters is then more easily done, I put these values into the formula *PRdz* originally assumed,

$$P = \frac{1}{(1+bz^{\mu})^{\nu}}, \text{ and } Q = \frac{z^{\mu}}{1+bz^{\mu}} .$$

Hence the integrals become :

$$\int Pdz = \int \frac{dz}{(1+bz^{\mu})^{\nu}}; \int PQdz = \int \frac{z^{\mu}dz}{(1+bz^{\mu})^{\nu+1}}; \text{ and } \int PQ^2dz = \int \frac{z^{2\mu}dz}{(1+bz^{\mu})^{\nu+2}}, \text{ etc.}$$

Moreover, all these integrals can be reduced to the first integral:

$$\int \frac{dz}{(1+bz^{\mu})^{\nu}} :$$

for generally the integral is given by* :

$$\int \frac{z^{\theta\mu}dz}{(1+bz^{\mu})^{\nu+\theta}} = \frac{(\theta-1)\mu+1}{b\mu(\nu+\theta-1)} . \int \frac{z^{(\theta-1)\mu}dz}{(1+bz^{\mu})^{\nu+\theta-1}} - \frac{1}{b\mu(\nu+\theta-1)} . \frac{z}{(1+bz^{\mu})^{\nu+\theta-1}} .$$

[* There is a typo' error in this, as the correct formula on integrating by parts is :

$$\int \frac{z^{\theta\mu}dz}{(1+bz^{\mu})^{\nu+\theta}} = \frac{(\theta-1)\mu+1}{b\mu(\nu+\theta-1)} . \int \frac{z^{(\theta-1)\mu}dz}{(1+bz^{\mu})^{\nu+\theta-1}} - \frac{1}{b\mu(\nu+\theta-1)} . \frac{z^{(\theta-1)\mu+1}}{(1+bz^{\mu})^{\nu+\theta-1}}$$, on making use of $z^{\theta\mu}dz = \frac{1}{b\mu} z^{(\theta-1)\mu} d(bz^{\mu})$; note that θ is a counting index with integer values.]

On this account :

$$\int \frac{z^{\mu}dz}{(1+bz^{\mu})^{\nu+1}} = \frac{1}{b\mu\nu} \int \frac{dz}{(1+bz^{\mu})^{\nu}} - \frac{1.z}{b\mu\nu(1+bz^{\mu})^{\nu}} ,$$

and

$$\int \frac{z^{2\mu}dz}{(1+bz^{\mu})^{\nu+2}} = \frac{1(\mu+1)}{b^2\mu^2\nu(\nu+1)} \int \frac{dz}{(1+bz^{\mu})^{\nu}} - \frac{(\mu+1).z}{b^2\mu^2\nu(\nu+1)(1+bz^{\mu})^{\nu}} - \frac{1.z^{\mu+1}}{b\mu(\nu+1)(1+bz^{\mu})^{\nu+1}} \text{ etc.}$$

[p.236]
Therefore *h* must be a quantity of this kind that by substituting it in place of *z* makes [the algebraic part in the general integration by parts]

$$\frac{z^{\theta\mu+1}}{(1+bz^{\mu})^{\nu+\theta}} = 0.$$

Now it is not possible for *h* = 0, since then many like quantities $\int \frac{dz}{(1+bz^{\mu})^{\nu}}$ vanish.

Therefore from the comparisons taken between these reductions with those above, the letters $\alpha, \beta, \gamma, \delta$ etc are determined. Clearly these become :

$$\alpha = \frac{1}{b\mu\nu},\ \beta = \frac{\mu+1}{b\mu(\nu+1)},\ \gamma = \frac{2\mu+1}{b\mu(\nu+2)}, \text{ etc.}$$

§8. With the letters $\alpha, \beta, \gamma,$ etc defined in this way, I consider the others A, B, C, etc. any of which I see to be of this form $\frac{1}{\sigma\tau}$ [to correspond to the equivalent expansion for *t*], clearly it is required to have one divided by a factor made from two factors. Moreover since it is required to sum the like series :

$$1 + A\alpha g + AB\alpha\beta g^2 + ABC\alpha\beta\gamma g^3 + \text{etc.},$$

I put [in place a very general expansion of the required form:]

$$A = \frac{1}{\pi(\pi+\rho)},\ B = \frac{1}{(\pi+2\rho)(\pi+3\rho)},\ C = \frac{1}{(\pi+4\rho)(\pi+5\rho)} \text{ etc.}$$

and then the series can be summed with the help of my general method of summing series. In the first place, for the sake of brevity, I put [in R in §3]

$$gQ = q^2,$$

then the series becomes :

$$R = 1 + \frac{q^2}{\pi(\pi+\rho)} + \frac{q^4}{\pi(\pi+\rho)(\pi+2\rho)(\pi+3\rho)} + \text{etc.},$$

and on putting R – 1 = S,
then

$$S = \frac{q^2}{\pi(\pi+\rho)} + \frac{q^4}{\pi(\pi+\rho)(\pi+2\rho)(\pi+3\rho)} + \text{etc.}$$

Now I multiply everywhere by $\rho q^{\frac{\pi-\rho}{\rho}}$ and with the derivative taken, there arises :

$$\frac{\rho d q^{\frac{\pi-\rho}{\rho}} S}{dq} = \frac{q^{\frac{\pi}{\rho}}}{\pi} + \frac{q^{\frac{\pi+2\rho}{\rho}}}{\pi(\pi+\rho)(\pi+2\rho)} + \text{etc.}$$

Thus, I multiply this series by ρ, and with the derivative taken again on putting dq constant [*i.e.* the independent variable], this equation is produced :

$$\frac{\rho^2 dd(q^{\frac{\pi-\rho}{\rho}} S)}{dq^2} = q^{\frac{\pi-\rho}{\rho}} + \frac{q^{\frac{\pi+\rho}{\rho}}}{\pi(\pi+\rho)} + \text{etc.} = q^{\frac{\pi-\rho}{\rho}} + q^{\frac{\pi-\rho}{\rho}}\left(\frac{q^2}{\pi(\pi+\rho)} + \frac{q^4}{\pi(\pi+\rho)(\pi+2\rho)(\pi+3\rho)} + \text{etc.}\right)$$

[p.237].

We now have, on restoring S in place of the series $\frac{q^2}{\pi(\pi+\rho)}$ + etc., this equation:

$$\rho^2 dd(q^{\frac{\pi-\rho}{\rho}} S) = q^{\frac{\pi-\rho}{\rho}} dq^2 + q^{\frac{\pi-\rho}{\rho}} S dq^2.$$

Again for the sake of brevity, I put:

$$q^{\frac{\pi-\rho}{\rho}} S = T,$$

and the equation becomes :

$$\rho^2 ddT = q^{\frac{\pi-\rho}{\rho}} dq^2 + T dq^2.$$

[Hence, on integrating this equation, a formula is found for the sum R in §3.]

§9. Towards integrating this equation, I put $T = rs$, which gives :

$$ddT = rdds + 2drds + sddr,$$

with which put in place, there is obtained :

$$\rho^2 rdds + 2\rho drds + \rho^2 sddr = q^{\frac{\pi-\rho}{\rho}} dq^2 + rsdq^2$$

which can be in split up into two equations :

$$\rho^2 dds = sdq^2,$$

which on multiplying by *ds* gives this equation :

$$\rho^2 dsdds = sdsdq^2,$$

and the integral of this is :

$$\rho^2 ds^2 = s^2 dq^2,$$

or this :

$$\rho ds = sdq,$$

which gives, on integrating again :

$$\rho l\, s = q \text{ and } s = c^{\frac{q}{\rho}},$$

with *c* denoting the number of which the logarithm is one. Thus with *s* found, I take the other equation :

$$2\rho^2 drds + \rho^2 sddr = q^{\frac{\pi-\rho}{\rho}} dq^2,$$

which with the value found $c^{\frac{q}{\rho}}$ put in place of s, becomes this equation :

$$2\rho c^{\frac{q}{\rho}} dqdr + \rho^2 c^{\frac{q}{\rho}} ddr = q^{\frac{\pi-\rho}{\rho}} dq^2.$$

Putting

$$dr = vdq,$$

then this gives :

$$ddr = dvdq,$$

and the equation is changed into this simpler differential :

$$2\rho c^{\frac{q}{\rho}} vdq + \rho^2 c^{\frac{q}{\rho}} dv = q^{\frac{\pi-\rho}{\rho}} dq,$$

which I multiply by $c^{\frac{q}{\rho}}$, in order that this equation is produced :

$$2\rho c^{\frac{2q}{\rho}} vdq + \rho^2 c^{\frac{2q}{\rho}} dv = c^{\frac{q}{\rho}} q^{\frac{\pi-\rho}{\rho}} dq,$$

of which the integral is :

$$\rho^2 c^{\frac{2q}{\rho}} v = \int c^{\frac{q}{\rho}} q^{\frac{\pi-\rho}{\rho}} dq.$$

[p.238] Therefore, this becomes :

$$v = \frac{1}{\rho^2} c^{\frac{-2q}{\rho}} \int c^{\frac{q}{\rho}} q^{\frac{\pi-\rho}{\rho}} dq,$$

and

$$\int vdq, \text{ or } r = \frac{1}{\rho^2} \int c^{\frac{-2q}{\rho}} dq \int c^{\frac{q}{\rho}} q^{\frac{\pi-\rho}{\rho}} dq.$$

Therefore there arises :

$$rs = T = \frac{1}{\rho^2} c^{\frac{q}{\rho}} \int c^{\frac{-2q}{\rho}} dq \int c^{\frac{q}{\rho}} q^{\frac{\pi-\rho}{\rho}} dq,$$

and

$$S = \frac{1}{\rho^2} c^{\frac{q}{\rho}} q^{\frac{\pi-\rho}{\rho}} \int c^{\frac{-2q}{\rho}} dq \int c^{\frac{q}{\rho}} q^{\frac{\pi-\rho}{\rho}} dq.$$

[Thus, R is given by a repeated integration involving the unknown parameters π and ρ.]

§10. Because a two-fold integration is involved in this form found, it is to be noted that these have to be prepared so that S as well as $\frac{dS}{dq}$ are made equal to zero on putting $q = 0$, as is apparent from the series to which S is equal. In the end there is obtained from these the series :

$$R = 1 + \frac{1}{\rho^2} c^{\frac{q}{\rho}} q^{\frac{\pi-\rho}{\rho}} \int c^{\frac{-2q}{\rho}} dq \int c^{\frac{q}{\rho}} q^{\frac{\pi-\rho}{\rho}} dq.$$

Now

$$q = \sqrt{gQ}\,,$$

and since

$$Q = \frac{z^\mu}{1+bz^\mu}\,,$$

then

$$q = \sqrt{\frac{gz^\mu}{1+bz^\mu}}\,.$$

Therefore, from these there is given [eventually the solution for Z]:

$$\int PRdz \text{ or } \int \frac{Rz^\mu}{(1+bz^\mu)^\nu}\,.$$

Whereby if the letters $\pi, \rho, \mu,$ and ν are given values in terms of n, then the solution of the proposed equation $ax^n dx = dy + y^2 dx$ is made available.

§11. With this done I revert to the original proposition [see §4], and I recover the series :

$$1 + A\alpha g + AB\alpha\beta g^2 + ABC\alpha\beta\gamma g^3 + \text{etc.}$$

which, with the selected values of $A, \alpha, B, \beta, C, \gamma,$ etc. put in place, is changed into this series :

$$1 + \frac{g}{b\mu\nu\pi(\pi+\rho)} + \frac{(\mu+1)g^2}{b^2\mu^2\nu(\nu+1)\pi(\pi+\rho)(\pi+2\rho)(\pi+3\rho)} + \text{etc.},$$

for which the rule [of formation] is this :
The term corresponding to the index $\theta+1$ divided by the term with the index θ is given by :

$$\frac{g(1+(\theta-1)\mu)}{b\mu(\nu+\theta-1)(\pi+(2\theta-2)\rho)(\pi+(2\theta-1)\rho)}\,.$$

Now, as in the series in §6., from the proposed equation we have elicited that the quotient of like terms with index $\theta+1$ divided by the term with index θ is equal to:

$$\frac{f}{(\theta n+2\theta-1)(\theta n+2\theta)}\,.$$

Therefore as these two series are congruent, it is required that these two quotients are equal to each other. Thus, in the first place put

$$\frac{g}{b} = f \text{ or } g = bf,$$

and following this, it is the case that [on absorbing μ]:

$$\frac{1}{(\theta n+2\theta-1)(\theta n+2\theta)} = \frac{\theta\mu-\mu+1}{(\mu\nu+\mu\theta-\mu)(\pi+2\theta\rho-2\rho)(\pi+2\theta\rho-\rho)}.$$

Thus if this equation is arranged according to the powers of θ, and if the coefficients of each power of θ are put equal to zero, four equations are produced from which $\mu, \nu, \pi,$ and ρ are determined in terms of n. [p.239] Nor indeed is there only one solution given, for there are four different solutions which are relevant to our intentions. The first gives :

$$\mu = \frac{2n+4}{3n+4}, \nu = 1, \pi = n+1, \text{ and } \rho = \frac{n+2}{2}.$$

The second gives

$$\mu = \frac{2n+4}{n}, \nu = 1, \pi = \frac{n}{2}, \text{ and } \rho = \frac{n+2}{2}.$$

The third gives

$$\mu = 2, \nu = \frac{n+1}{n+2}, \pi = \frac{n+2}{2}, \text{ et } \rho = \frac{n+2}{2}.$$

The fourth gives [corrected in *O.O.*]

$$\mu = \frac{2}{3}, \nu = \frac{n+1}{n+2}, \pi = n+2, \text{ and } \rho = \frac{n+2}{2}.$$

§12. Of these four solutions not each can be used freely, but must be selected in one way or another for various cases of the exponent n. A judgement is deduced from the condition recalled from §7, since $\frac{z^{\theta\mu+1}}{(1+bz^{\mu})^{\nu+\theta}}$ must vanish on putting $z = h$. This indeed is the case if $z = 0$, but when in addition another situation is required, it is readily apparent that none eventuate, unless $h = \infty$. Therefore for whatever the case of n above, the solution is selected such that $\frac{z^{\theta\mu+1}}{(1+bz^{\mu})^{\nu+\theta}}$ is made equal to zero on making $z = \infty$. Moreover, here the number θ denotes some positive number including zero, on account of which ν cannot be a negative number, since otherwise the binomial $1+bz^{\mu}$ arises in the numerator. But μ can signify a positive as well as a negative number, from which two cases of these need to be considered, according as μ is either positive or negative. At first let μ be the positive number $= +\lambda$, and it is evident in order that $\frac{z^{\lambda\theta+1}}{(1+bz^{\lambda})^{\nu+\theta}}$ becomes $= 0$, on putting $z = \infty$, then it is required that the maximum exponent z in the denominator, [p.240] which is $\lambda(\nu+\theta)$ is greater than the exponent of the same z in the numerator, which is $\lambda\theta+1$. Therefore the inequality becomes $\lambda\nu > 1$. But moreover, if the number μ is negative or put equal to $= -\lambda$, then

$$\frac{z^{-\lambda\theta+1}}{(1+bz^{-\lambda})^{\nu+\theta}} = \frac{z^{\lambda\nu+1}}{(z^{\lambda}+b)^{\nu+\theta}},$$

which quantity in order that it becomes = 0 on putting $z = \infty$ has to satisfy

$$\lambda\nu + \lambda\theta > \lambda\nu + 1, \text{ or } \lambda\theta > 1,$$

and in that case θ cannot be made equal to zero. According to which μ can never become a negative number. Therefore in the first solution, since $\nu = 1$, as often as λ *i. e.* $\frac{2n+4}{3n+4}$ is a positive number, likewise it must always be a number greater than one, therefore these cases are excepted in which $\frac{2n+4}{3n+4}$ is equal to or less than one. Therefore, unless n is contained between the limits 0 and $-\frac{4}{3}$, then the first solution cannot be used. In the second solution, since again $\nu = 1$, likewise the cases in which λ or $\frac{2n+4}{n}$ is one or less than one are excepted. Therefore this solution always has a place, only with these cases excepted, when n is contained between the boundaries – 4 and 0. For the third solution, since μ now is a positive number, clearly equal to 2, it is only required that $\frac{2n+2}{n+2}$ is a number greater than one. Therefore we are always able to use this case unless n is contained between the bounds – 2 et 0; therefore whenever the second case is in place, so also the third condition can be used. Finally in the fourth case, since μ is also a positive number, clearly $\frac{2}{3}$, it is required that $\frac{2n+2}{3n+6}$ is a number greater than one, which happens as often as n is contained between the limits -2 and - 4 [The last three numerical expressions corrected in *O. O.*]. Therefore in these cases it is agreed that the fourth solution to be used. From which it is observed in turn from the comparison, that this kind of equation proposed can be constructed, [p.241] unless n is contained between the narrow limits $-\frac{4}{3}$ and -2.

§13. Moreover, in order that this whole matter can be more clearly understood, I will apply what have been treated this far to a particular case, which is $n = 2$, so that this equation

$$ax^2 dx = dy + y^2 dx$$

can thus be solved. For this example, I select the third solution, and it becomes therefore :

$$\mu = 2, \nu = \frac{3}{4}, \pi = \rho = 2.$$

With these values substituted, there is obtained :

$$S = \frac{1}{4} c^{\frac{q}{2}} \int c^{-q} dq \int c^{\frac{q}{2}} dq.$$

Now truly,

$$\int c^{\frac{q}{2}} dq = 2c^{\frac{q}{2}} + i,$$

hence

$$\int c^{-q} dq \int c^{\frac{q}{2}} dq = \int 2c^{\frac{-q}{2}} dq + i \int c^{-q} dq = -c^{\frac{-q}{2}} - ic^{-q} + k.$$

Consequently there is produced [there is an error in this integration, as the first term is out by a factor of 4; however , we continue with the original; the corrected version can be found in the O. O. Vol. 22, First Series, p.29]:

$$S = \frac{k}{4}c^{\frac{q}{2}} - \frac{i}{4}c^{\frac{-q}{2}} - \frac{1}{4}.$$

Since now on putting $q = 0$, S [s in the original] must vanish, and this equation is obtained :

$$\frac{k}{4} - \frac{i}{4} - \frac{1}{4} = 0, \text{ or } k = 1 + i.$$

Again since $\frac{dS}{dq}$ must be = 0 if $q = 0$, it comes about that $i + k = 0$. And for this case :

$$dS = \frac{k}{8}c^{\frac{q}{2}}dq + \frac{i}{8}c^{\frac{-q}{2}}dq,$$

and therefore by making $q = 0$, this makes

$$\frac{dS}{dq} = \frac{k}{8} + \frac{i}{8} = 0.$$

Therefore from these conditions, it is found that

$$i = -\frac{1}{2}, \text{ and } k = \frac{1}{2}:$$

on account of which,

$$S = \frac{c^{\frac{q}{2}} + c^{\frac{-q}{2}}}{8} - \frac{1}{4}, \text{ and } R = \frac{3}{4} + \frac{c^{\frac{q}{2}} + c^{\frac{-q}{2}}}{8}.$$

Now since $\mu = 2$ and $g = bf$, and $q = \sqrt{\frac{bfz^2}{1+bz^2}}$, and thus

$$R = \frac{3}{4} + \frac{c^{\frac{1}{2}\sqrt{\frac{bfz^2}{1+bz^2}}} + c^{\frac{-1}{2}\sqrt{\frac{bfz^2}{1+bz^2}}}}{8}.$$

Consequently, it is found that :

$$\int PRdz = \frac{3}{4}\int \frac{dz}{(1+bz^2)^{\frac{3}{4}}} + \frac{1}{8}\int \frac{dz(c^{\frac{1}{2}\sqrt{\frac{bfz^2}{1+bz^2}}} + c^{-\frac{1}{2}\sqrt{\frac{bfz^2}{1+bz^2}}})}{(1+bz^2)^{\frac{3}{4}}}.$$

[p.242] Which integral is thus taken, so that on putting $z = 0$ it becomes equal to zero, and on putting $z = \infty$, there is produced a quantity which can be seen as a function of f. Then f is made the variable, and in place of this is put ax^4, and this function divided by H is equal to t (see §6). And having found t, then $y = \frac{dt}{tdx}$, which is now the value of y from the proposed equation:

$$ax^2 dx = dy + y^2 dx.$$

[Note that there is an error in the coefficients of S in this solution that has been left, as the principle is the same. See the above note for the corrected formulae.]

§14. The construction of the general equation results in no more difficulty

$$ax^n dx = dy + y^2 dx,$$

provided n is not contained between the limits 0 and – 2. For we are able to use the third solution, in which

$$\mu = 2, \nu = \frac{n+1}{n+2}, \pi = \rho = \frac{n+2}{2}.$$

Therefore the sum S becomes :

$$S = \frac{1}{\rho^2} c^{\frac{q}{\rho}} \int c^{\frac{-2q}{\rho}} dq \int c^{\frac{q}{\rho}} dq.$$

With the integration set out in a like manner to that above, it is found that :

$$S = \frac{k}{\rho^2} c^{\frac{q}{\rho}} - \frac{i}{\rho^2} c^{\frac{-q}{\rho}} - \frac{1}{\rho^2},$$

[The correct value of S is given by : $S = \frac{k}{\rho^2} c^{\frac{q}{\rho}} - \frac{i}{2\rho} c^{\frac{-q}{\rho}} - 1.$]

where i and k from these equations are to be defined by :
$k = 1 + i,$ and $k + i = 0$, that is as before $i = -\frac{1}{2},$ and $k = \frac{1}{2}$.
Wherefore:

$$S = \frac{1}{2\rho^2} c^{\frac{q}{\rho}} + \frac{1}{2\rho^2} c^{\frac{-q}{\rho}} - \frac{1}{\rho^2}, \text{ and } R = 1 - \frac{1}{\rho^2} + \frac{1}{2\rho^2} c^{\frac{q}{\rho}} + \frac{i}{2\rho^2} c^{\frac{-q}{\rho}}$$

or with the value $\frac{n+1}{2}$ put in place of ρ, there is obtained :

$$R = \frac{n(n+4) + 2c^{\frac{2q}{n+2}} + 2c^{\frac{-2q}{n+2}}}{(n+2)^2}.$$

Now as before,

$$q = \sqrt{\frac{bfz^2}{1+bz^2}},$$

but

$$Pdz = \frac{dz}{(1+bz^2)^{\frac{n+1}{n+2}}}.$$

On account of which [Z is equal to] :

$$\int PRdz = \frac{1}{(n+2)^2} \int \frac{dz}{(1+bz^2)^{\frac{n+1}{n+2}}} (n(n+4) + 2c^{\frac{2q}{n+2}} + 2c^{\frac{-2q}{n+2}})$$

where q is put in place of $\sqrt{\frac{bfz^2}{1+bz^2}}$. [p.243] The integral of $PRdz$ is thus taken, so that it is vanishes on putting $z = 0$, and it again vanishes on integrating on putting $z = \propto$, and let H denote that which arises, if only $\int \frac{dz}{(1+bz^2)^{\frac{n+1}{n+2}}}$ is integrated in this manner so that it becomes zero on putting $z = 0$, and afterwards on putting $z = \propto$. Hence there is produced the integral of $PRdz$ taken in the prescribed manner, since we have put Z (§4) a function of f. Moreover that can be put equal to the quantity H, which is to multiply this series $1 + A\alpha g + AB\beta g^2 + \text{etc.}$, and which series is changed into the following :

$$1 + \frac{f}{(n+1)(n+2)} + \frac{f^2}{(n+1)(n+2)(2n+3)(2n+4)} + \text{etc.}$$

and the sum of this is t, see. §6., where f designates ax^{n+2}. Therefore the co–ordinate of the curve becomes $Z = Ht$, in which H is a constant quantity, since f is not present in that, and thus neither x. Thus there arises $t = \frac{Z}{H}$, but $y = \frac{dt}{tdx}$: hence for the proposed equation $ax^n dx = dy + y^2 dx$ there is produced $y = \frac{dZ}{Zdx}$ [in place of the original expression $y = \frac{dt}{tdx}$]. Therefore in the solution of that equation we have this rule : this formula is integrated

$$\frac{1}{(n+2)^2}\frac{dz}{(1+bz^2)^{\frac{n+1}{n+2}}}(n(n+4)+2c^{\frac{2}{n+2}\sqrt{\frac{bfz^2}{(1+bz^2)}}}+2c^{\frac{-2}{n+2}\sqrt{\frac{bfz^2}{(1+bz^2)}}})$$

thus so that it vanishes on putting $z = 0$. Then z is made infinite, and ax^{n+2} is put in place of f . That which comes about is Z, from which it is known that $y = \frac{dZ}{Zdx}$. If this gives trouble, since after integration we must set $z = \infty$, $\frac{u}{1-u}$ is substituted in place of z, and after the integration putting $u = 1$, from which being done the same value for Z is produced, as before. Moreover, whenever an analytical value cannot be obtained for the expression Z, when that formula is still not integrable, [p.244] the value of Z can be constructed from quadrature or rectification. [Again note that there is an error in the coefficients of S that we have not changed.]

§15. But nevertheless in this construction these cases are excluded in which n is contained between the limits -2 and 0, yet no less this general solution can be obtained. For because, if the equation can be resolved in the case $n = m$, the solution can also be had in the case $n = -m - 4$, as this agrees with these which have been found for the separate cases ; it is evident that if m is a number contained between the limits 0 and – 2, to be $-m - 4$ contained between the terms – 2 et – 4, and thus is contained in ours solution. On account of which if there occurs a case in which n is contained between 0 and – 2, this is at once reduced to the other by the said theorem, since it is contained between – 2 et – 4, and the solution of this equation is in place.

§16. In the differential formula in §13. I observe from what has been found, that as often as $\frac{n+1}{n+2}$ is of the form $k + \frac{1}{2}$, where k denotes a positive whole number, then the formula in the integral can be integrated, and this can be shown on account of the value of Z. Therefore in these cases the value of y can be defined and the equation integrated. Moreover then, this becomes $n = \frac{-4k}{2k-1}$, hence as often as n has such a form, then equation $ax^n dx = dy + y^2 dx$ can be integrated. Hence because then if the case $n = \frac{-m}{m+1}$ or $n = -m = 4$ can be reduced to the case $n = m$, the equation is also integrable if $n = \frac{-4k}{2k+1}$ or $\frac{-4k-4}{2k+1}$, with k denoting a positive whole number. [p.245] And these are themselves the cases which are integrable or separable, from the other cases now elicited, which can be seen in our *Commentariis* A. 1726.

§17. Moreover that the equation is integrable, whenever $\frac{n+1}{n+2} = k + \frac{1}{2}$ [= *v*, cases 3 and 4 above], can be shown in this manner. I put

$$\frac{bz^2}{1+bz^2} = u^2;$$

then

$$z = \frac{u}{\sqrt{b(1-u^2)}}; 1 + bz^2 = \frac{1}{1-u^2},$$

and thus

$$dz = \frac{du}{(1-u^2)^{\frac{3}{2}}\sqrt{b}}.$$

Therefore the integrand becomes :

$$\frac{dz}{(1+bz^2)^{\frac{n+1}{n+2}}} = \frac{du}{\sqrt{b}}(1-u^2)^{k\text{-}1}.$$

On account of this the formula to be integrated in §14, $PRdz$, is transformed into this :

$$\frac{1}{(n+2)^2\sqrt{b}}(n(n+4)du(1-u^2)^{k\text{-}1} + 2c^{\frac{2u\sqrt{f}}{n+2}}du(1-u^2)^{k\text{-}1} + 2c^{\frac{-2u\sqrt{f}}{n+2}}du(1-u^2)^{k\text{-}1}),$$

which expression itself is readily seen to be integrable, whenever *k* is a positive whole number. And hence I consider that no small amount of excellence should be agreed upon concerning this method of mine, that is so easy and clear, so that also all the cases that admit to integration or separation can be shown at a single glance.

§18. For the sake of an example I take here $k = 1$, then $n = -4$, it is agreed to use this case, which is the easiest of these which can be separated. Therefore the formula to be integrated goes into this :

$$\frac{1}{2\sqrt{b}}(c^{-u\sqrt{f}}du - c^{u\sqrt{f}}du),$$

and the integral of this is :

$$\frac{1}{2\sqrt{bf}}(c^{u\sqrt{f}} - c^{-u\sqrt{f}}).$$

[Euler has reverted to the correct formula for the integral of the exponential function!] I do not add a constant since on putting $z = 0$, or what amounts to the same, $u = 0$, the whole integral now vanishes. Now make $z = \infty$ or in our case $u = 1$ and putting ax^{-2} in place of *f* , there is obtained :

$$Z = \frac{x}{2\sqrt{ab}}(c^{\frac{\sqrt{a}}{x}} - c^{-\frac{\sqrt{a}}{x}}).$$

With this found, it now becomes as has been shown, that $y = \frac{dZ}{Zdx}$. [p.246] Therefore by differentiating *Z* and the differentials divided by *Zdx*, there is produced:

$$y = \frac{1}{x} - \frac{\sqrt{a}}{x^2}\left(\frac{c^{\frac{2\sqrt{a}}{x}}+1}{c^{\frac{2\sqrt{a}}{x}}-1}\right) \text{ or } \frac{2\sqrt{a}}{x} = l\left(\frac{xxy-x-\sqrt{a}}{xxy-x+\sqrt{a}}\right)$$

which equation is the integral of this equation :

$$ax^{-4}dx = dy + y^2 dx .$$

And in a like manner for the remaining cases, which admit to separation, the integral equations can be found.

CONSTRUCTIO AEQUATIONIS DIFFERENTIALIS $ax^n dx = dy + y^2 dx$.

Auctore

Leonh. Eulero.

§1.

Communicavi nuper cum Societate specimen constructionis aequationis cuiusdam differentialis, in qua solum indeterminatas a se invicem separere non potueram, sed etiam monstraveram ex ipsa constructione huiusmodi separationem omnino non posse exhiberi. Differt quidem meus ibi datus construendi modus ab usitatis : attamen iis nequaquam illum esse postponendum quilibet intelliget, qui hanc schedam inspexerit. Neque vero tum temporis hanc methodum ulterius extendere, atque ad alias aequationes accommodare licuit, quia ex posita constructione ad aequationem demum perveneram, non autem vicissim data aequatione constructionem eruere potueram. [p.232] At deinceps hanc rem diligentius contemplatus essem, voti mei compos quodammodo sum factus, ita ut hanc methodum invertere, atque propositae aequationis constructionem invenire potuerim.

§2. Selegi igitur statim ad periculum faciendum hanc maxime agitatam aequationem $ax^n dx = dy + y^2 dx$, quam Clar. *Comes Riccati* primum Geometris examinandam proposiut, nemo vero eius constructionem, nisi pro certis litterae *n* valoribus , dedit. Meae vero methodi beneficio omnes difficultates feliciter superavi, atque universalem huius aequationis constructionem inveni, in qua nihil omnino desiderari quaeat. Non solum autem unicam haec methodus suppeditat constructionem, sed plures, immo etiam innumerabiles. Merito igitur mihi videor isti methodo tantam praestantiam adscribere, ut as omnes aequationes differentiales construendas, in quibus aliae methodi frustra funt adhibitae, viam fit commonstratura.

§3. Quemadmodum in superiore Dissertatione arcu Elliptico sum usus, ad constructionem huius aequationis $dy + \frac{y^2 dx}{x} = \frac{xdx}{x^2-1}$, ita pro aequatione proposita alia opus erit curva, loco Ellipsis substituenda. Quam ut inveniam pono universalissime eius elementum = $PRdz$, in quo *P* et *R* sunt functiones ipsius *z* tales, quae iisdem factis operationibus, ut supra in elemento elliptico, deducant ad aequationem propositam. Pono porro, ut series quaedam in considerationem veniat, [p.233]
$R = x + AgQ + ABg^2Q^2 + ABCg^3Q^3 + ABCDg^4Q^4 + \text{etc.}$ in qua serie est *Q* functio quaedam ipsius *z, g* linea data seu quasi parameter curvae, *A, B, C, D*, etc. coefficientes

constantes. Ponatur $PRdz = dZ$; erit ergo
$Z = \int Pdx + Ag\int PQdz + ABg^2\int PQ^2dz + ABCg^3\int PQ^3dz + \text{etc.}$

§4. Ita autem P et Q a se invicem pendeant, ut omnia haec integralia possint ad $\int Pdz$ reduci. Sit ergo
$\int PQdz = \alpha\int Pdz + \mathrm{O}\,1;\ \int PQ^2dz = \alpha\beta\int Pdz + \mathrm{O}\,2;\ \int PQ^3dz = \alpha\beta\gamma\int Pdz + \mathrm{O}\,3;$ etc.
Denotant hic O 1,O 2, O3 etc quantitates algebraicas. Post peractam hoc modo integratiomem ponatur $z = h$: est autem h talis quantitas, quae loco *z* substituta faciat omnes eas quantitates algebraicas O 1,O 2, O3 etc. evanescere, atque tum fiat $\int Pdz = H$ quantitati prorsus constanti. Ex his igitur, facto post integrationem $z = h$, erit
$Z = H(1 + A\alpha g + AB\alpha\beta g^2 + ABC\alpha\beta\gamma g^3 + \text{etc.})$ Facta iam parametro *g* variabili obtinebuntur infiniti valores ipsius *Z* pro infinitis ipsius *g*, atque ex dato elemento *PRdz* poterit construi curva, in qua, si abscissae designentur littera *g*, applicatae sunt = *Z*.

§5. Hoc itaque modo poterit construi summa seriei
$1 + A\alpha g + AB\alpha\beta g^2 + ABC\alpha\beta\gamma g^3 + \text{etc.}$ quamvis forte ex sui ipsius consideratione summa prorsus non possit determinari. Utor autem ad summam huius seriei investigandam methodo mea summae serierum inventionem ad resolutionem aequationum reducendi, quam anno praeterito exposui, ut nanciscar aequationem, eius resoluto a seriei illius summa pendeat. [p.234] Perspicuum enim est, utcunque haec aequatio resultans fuerit perplexa, eius tamen constructionem in promtu futuram. Nunc igitur nihil aliud est faciendum, nisi ut quantitates *A, B, C*, etc. et α, β, γ, etc. efficiantur eiusmodi, ut summae seriei istius inventio ad resolutionem huius aequationis $ax^n dx = dy + y^2 dx$ deducatur. Hoc vero loco id est efficendum ut series
$1 + A\alpha g + AB\alpha\beta g^2 + ABC\alpha\beta\gamma g^3 + \text{etc.}$ possit in summam redigi, quia alias valor ipsius *R* non esset cognitus, et proinde integra constructio inutilis. Quamobrem non licebit loco *A, B, C*, etc. valores quosvis pro arbitrio accipere, sed tales, quae hanc seriem summabilem reddant.

§6. Quo igitur appareat, cuiusmodi esse debeat series
$1 + A\alpha g + AB\alpha\beta g^2 + ABC\alpha\beta\gamma g^3 + \text{etc.}$ ut eius summatio perducatur ad resolutionem aequationis $ax^n dx = dy + y^2 dx$; hanc ipsam aequationem in seriem resoluo. Quod ut commodius effici possit, pono $y = \frac{dt}{tdx}$, sumtoque *dx* constante erit $ax^n dx = \frac{ddt}{tdx}$ seu $ax^n tdx^2 = ddt$. Nunc more consueto substituo loco *t* hanc seriem
$1 + \mathfrak{A}x^{n+2} + \mathfrak{B}x^{2n+4} + \mathfrak{C}x^{3n+6} + \text{etc.}$ erit
$ddt = (n+1)(n+2)\mathfrak{A}x^n dx^2 + (2n+3)(2n+4)\mathfrak{B}x^{2n+2}dx^2 + (3n+5)(3n+6)\mathfrak{C}x^{3n+4}dx^2 + \text{etc.}$ Huic igitur seriei aequalis esse debet $ax^n tdx^2$, seu ista series

$ax^n dx^2 + \mathfrak{A}\, a\, x^{2n+2} dx^2 + \mathfrak{B}\, b x^{3n+4} dx^2$ + etc. ; propterea aequales facio terminos homogeneos determinandis litteris $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. pro arbitrio assumtis, fietque $\mathfrak{A} = \frac{a}{(n+1)(n+2)}$, $\mathfrak{B} = \frac{\mathfrak{A}a}{(2n+3)(2n+4)}$, $\mathfrak{C} = \frac{\mathfrak{B}a}{(3n+5)(3n+6)}$, etc. [p.235] Ponatur $ax^{n+2} = f$ brevitatis gratia, erit

$$t = 1 + \frac{f}{(n+1)(n+2)} + \frac{f^2}{(n+1)(n+2)(2n+3)(2n+4)} + \frac{f^3}{(n+1)(n+2)(2n+3)(2n+4)(3n+5)(3n+6)} + \text{etc.}$$

Huius ergo seriei summato pendet a constructione aequationis propositae $ax^n dx = dy + y^2 dx$. Quamobrem si series $1 + A\alpha g + AB\alpha\beta g^2 + ABC\alpha\beta\gamma g^3 + \text{etc.}$ in eam possit transmutari, habebitur simul constructio aequationis propositae.

§7. Sed quo haec series, quippe quae nimis est generalis, aliquanto magis restringatur, et determinatio litterarum arbitrariarum facilior efficiatur, pono in formula *PRdz* initio assumta $P = \frac{1}{(1+bz^\mu)^\nu}$, et $Q = \frac{z^\mu}{1+bz^\mu}$. Erit ergo

$$\int Pdz = \int \frac{dz}{(1+bz^\mu)^\nu}, \text{ et } \int PQdz = \int \frac{z^\mu dz}{(1+bz^\mu)^{\nu+1}}, \text{ et } \int PQ^2 dz = \int \frac{z^{2\mu} dz}{(1+bz^\mu)^{\nu+2}} \text{ etc.}$$

Possunt autem haec omnia integralia ad primum $\int \frac{dz}{(1+bz^\mu)^\nu}$ reduci : est enim generaliter

$$\int \frac{z^{\theta\mu} dz}{(1+bz^\mu)^{\nu+\theta}} = \frac{(\theta-1)\mu+1}{b\mu(\nu+\theta-1)} . \int \frac{z^{(\theta-1)\mu} dz}{(1+bz^\mu)^{\nu+\theta-1}} - \frac{1}{b\mu(\nu+\theta-1)} . \frac{z}{(1+bz^\mu)^{\nu+\theta-1}}.$$

Hanc ob rem erit

$$\int \frac{z^\mu dz}{(1+bz^\mu)^{\nu+1}} = \frac{1}{b\mu\nu} \int \frac{dz}{(1+bz^\mu)^\nu} - \frac{1.z}{b\mu\nu(1+bz^\mu)^\nu},$$

[p.236]

$$\text{et } \int \frac{z^{2\mu} dz}{(1+bz^\mu)^{\nu+2}} = \frac{1(\mu+1)}{b^2\mu^2\nu(\nu+1)} \int \frac{dz}{(1+bz^\mu)^\nu} - \frac{(\mu+1).z}{b^2\mu^2\nu(\nu+1)(1+bz^\mu)^\nu} - \frac{1.z^{\mu+1}}{b\mu(\nu+1)(1+bz^\mu)^{\nu+1}} \text{ etc.}$$

Debebit ergo $h = 0$, ut loco *z* substituta faciat $\frac{z^{\theta\mu+1}}{(1+bz^\mu)^{\nu+\theta}} = 0.$ Non vero poterit esse *h* = 0, quia tum plerumque simul quantitas $\int \frac{dz}{(1+bz^\mu)^\nu}$ evanesceret. Comparitis iam his reductionibus cum supra assumitis, determinantur litterae $\alpha, \beta, \gamma, \delta$ etc. Erit scilicet $\alpha = \frac{1}{b\mu\nu}$, $\beta = \frac{\mu+1}{b\mu(\nu+1)}$, $\gamma = \frac{2\mu+1}{b\mu(\nu+2)}$, etc.

§8. Definitis hoc modo litteris $\alpha, \beta, \gamma,$ etc considero alteras A, B, C, etc. quarum quamlibet video huiusmodi formam $\frac{1}{\sigma\tau}$, scilicet unitatem divisam per factum ex duobus factoribus, habere oportere. Quo autem simul series $1 + A\alpha g + AB\alpha\beta g^2 + ABC\alpha\beta\gamma g^3 + \text{etc.}$ possit summari, facio $A = \frac{1}{\pi(\pi+\rho)}$, $B = \frac{1}{(\pi+2\rho)(\pi+3\rho)}$, $C = \frac{1}{(\pi+4\rho)(\pi+5\rho)}$ etc. atque tum series ope methodi meae universalis series summandi poterit summari. Pono primo brevitas gratia $gQ = q^2$, erit R = 1 + $\frac{q^2}{\pi(\pi+\rho)}$ + $\frac{q^4}{\pi(\pi+\rho)(\pi+2\rho)(\pi+3\rho)}$ + etc., facioque R – 1 = S, erit S = $\frac{q^2}{\pi(\pi+\rho)} + \frac{q^4}{\pi(\pi+\rho)(\pi+2\rho)(\pi+3\rho)}$ + etc. Multiplico nunc ubique per $\rho q^{\frac{\pi-\rho}{\rho}}$ sumoque differentalia, erit $\frac{\rho dq^{\frac{\pi-\rho}{\rho}} S}{dq} = \frac{q^{\frac{\pi}{\rho}}}{\pi} + \frac{q^{\frac{\pi+2\rho}{\rho}}}{\pi(\pi+\rho)(\pi+2\rho)}$ + etc. Iam per ρ multiplico sumoque denuo differentialia ponendo dq constante, prodibit $\frac{\rho^2 dd(q^{\frac{\pi-\rho}{\rho}} S)}{dq^2} = q^{\frac{\pi-\rho}{\rho}} + \frac{q^{\frac{\pi+\rho}{\rho}}}{\pi(\pi+\rho)}$ + etc. $= q^{\frac{\pi-\rho}{\rho}} + q^{\frac{\pi-\rho}{\rho}} \left(\frac{q^2}{\pi(\pi+\rho)} + \frac{q^4}{\pi(\pi+\rho)(\pi+2\rho)(\pi+3\rho)}\right.$ + etc.) [p.237] Habebimus ergo restituto S loco seriei $\frac{q^2}{\pi(\pi+\rho)}$ + etc. , hanc aequationem $\rho^2 dd(q^{\frac{\pi-\rho}{\rho}} S) = q^{\frac{\pi-\rho}{\rho}} dq^2 + q^{\frac{\pi-\rho}{\rho}} S dq^2$. Pono porro brevitatis gratia $q^{\frac{\pi-\rho}{\rho}} S = T$, erit $\rho^2 ddT = q^{\frac{\pi-\rho}{\rho}} dq^2 + T dq^2$.

§9. Ad hanc aequationem integrandam pono $T = rs$, erit $ddT = rdds + 2drds + sddr$, quibus substitutis habetur $\rho^2 rdds + 2\rho drds + \rho^2 sddr = q^{\frac{\pi-\rho}{\rho}} dq^2 + rsdq^2$ quae in duas aequationes discerpatur, $\rho^2 dds = sdq^2$, quae per ds multiplicata dat hanc $\rho^2 dsdds = sdsdq^2$, cuius integralis est $\rho^2 ds^2 = s^2 dq^2$, sive haec $\rho ds = sdq$, quae denuo integrata dat $\rho l\, s = q$ atque $s = c^{\frac{q}{\rho}}$ denotante c numerum, cuius logarithmus est 1. Invento itaque s assumo alteram aequationem $2\rho^2 drds + \rho^2 sddr = q^{\frac{\pi-\rho}{\rho}} dq^2$, quae substituto loco s valore invento $c^{\frac{q}{\rho}}$ abit in istam $2\rho c^{\frac{q}{\rho}} dqdr + \rho^2 c^{\frac{q}{\rho}} ddr = q^{\frac{\pi-\rho}{\rho}} dq^2$. Ponatur $dr = vdq$, erit $ddr = dvdq$ atque aequatio mutabitur in hanc simpliciter differentialem $2\rho c^{\frac{q}{\rho}} vdq + \rho^2 c^{\frac{q}{\rho}} dv = q^{\frac{\pi-\rho}{\rho}} dq$, quam multiplico per $c^{\frac{q}{\rho}}$, ut prodeat $2\rho c^{\frac{2q}{\rho}} vdq + \rho^2 c^{\frac{2q}{\rho}} dv = c^{\frac{q}{\rho}} q^{\frac{\pi-\rho}{\rho}} dq$, cuius integralis est $\rho^2 c^{\frac{2q}{\rho}} v = \int c^{\frac{q}{\rho}} q^{\frac{\pi-\rho}{\rho}} dq$. [p.238] Fit igitur $v = \frac{1}{\rho^2} c^{\frac{-2q}{\rho}} \int c^{\frac{q}{\rho}} q^{\frac{\pi-\rho}{\rho}} dq$, et $\int vdq$, seu $r = \frac{1}{\rho^2} \int c^{\frac{-2q}{\rho}} dq \int c^{\frac{q}{\rho}} q^{\frac{\pi-\rho}{\rho}} dq$. Erit ergo $rs = T = \frac{1}{\rho^2} c^{\frac{q}{\rho}} \int c^{\frac{-2q}{\rho}} dq \int c^{\frac{q}{\rho}} q^{\frac{\pi-\rho}{\rho}} dq$, et $S = \frac{1}{\rho^2} c^{\frac{q}{\rho}} q^{\frac{\pi-\rho}{\rho}} \int c^{\frac{-2q}{\rho}} dq \int c^{\frac{q}{\rho}} q^{\frac{\pi-\rho}{\rho}} dq$.

§10. Quoniam in hac forma inventa duplex involuitur integratio, notandum est eas ita institui debere ut tam S quam $\frac{dS}{dq}$ fiant = 0, posito q = 0, quemadmodum ex seriei, cui S est aequale, apparet. His observatis habetur tandem $R = 1 + \frac{1}{\rho^2} c^{\frac{q}{\rho}} q^{\frac{\pi-\rho}{\rho}} \int c^{\frac{-2q}{\rho}} dq \int c^{\frac{q}{\rho}} q^{\frac{\pi-\rho}{\rho}} dq$. Est vero $q = \sqrt{g}Q$, atque ob $Q = \frac{z^\mu}{1+bz^\mu}$, erit $q = \sqrt{\frac{gz^\mu}{1+bz^\mu}}$. Dabitur igitur ex his $\int PRdz$ seu $\int \frac{Rz^\mu}{(1+bz^\mu)^\nu}$. Quare si litteris

$\pi, \rho, \mu,$ et ν tribuantur debiti valores in n, in promtu erit aequationis propositae $ax^n dx = dy + y^2 dx$ constructio.

§11. Hoc facto revertor ad propositum, atque resumo seriem $1 + A\alpha g + AB\alpha\beta g^2 + ABC\alpha\beta\gamma g^3 + \text{etc.}$ quae positis loco $A, \alpha, B, \beta, C, \gamma,$ etc. electis valoribus transmutatur in hanc $1 + \frac{g}{b\mu\nu\pi(\pi+\rho)} + \frac{(\mu+1)g^2}{b^2\mu^2\nu(\nu+1)\pi(\pi+\rho)(\pi+2\rho)(\pi+3\rho)} + \text{etc.}$, cuius haec est lex, ut terminus indicis $\theta+1$ divisus per terminum indicis θ fit $\frac{g(1+(\theta\text{-}1)\mu)}{b\mu(\nu+\theta-1)(\pi+(2\theta-2)\rho)(\pi+(2\theta-1)\rho)}$. In serie vero quam §6. ex aequatione proposita elicuimus, est similis quotus termini indicis $\theta+1$ per terminum indicis θ divisi $\frac{f}{(\theta n+2\theta-1)(\theta n+2\theta)}$. Quo igitur hae duae series congruant, oportet ut hi duo quoti inter se aequales. Fiat ergo primo $\frac{g}{b} = f$ seu $g = bf$, hoc posito debebit esse $\frac{1}{(\theta n+2\theta-1)(\theta n+2\theta)} = \frac{\theta\mu\text{-}\mu+1}{(\mu\nu+\mu\theta-\mu)(\pi+2\theta\rho-2\rho)(\pi+2\theta\rho-\rho)}$. Unde si aequatio secundum dimensiones ipsius θ ordinetur, et coefficientes cuiusque ipsius θ potentiae ponantur = 0, prodibunt quatuor aequationes ex quibus $\mu, \nu, \pi,$ et ρ determinabuntur in n. [p.239] Neque vero unica datur soluto, sed sunt quatuor diversae quae ad nosturm institutum pertinent. Prima dat $\mu = \frac{2n+4}{3n+4}, \nu = 1, \pi = n+1,$ et $\rho = \frac{n+2}{2}$.

Secunda dat $\mu = \frac{2n+4}{n}, \nu = 1, \pi = \frac{n}{2},$ et $\rho = \frac{n+2}{2}$.

Tertia dat $\mu = 2, \nu = \frac{n+1}{n+2}, \pi = \frac{n+2}{2},$ et $\rho = \frac{n+2}{2}$.

Quadrata dat $\mu = \frac{1}{3}, \nu = \frac{n+1}{n+2}, \pi = \frac{n+2}{\sqrt{2}},$ et $\rho = \frac{n+2}{\sqrt{2}}$.

§12. Quatuor harum solutionum non unaquaeque pro lubitu potest adhiberi, sed pro variis casibus exponentis n alia atque alia eligi debet. Quae diiudicatio deducenda est ista conditione §7 memorata, quod $\frac{z^{\theta\mu+1}}{(1+bz^{\mu})^{\nu+\theta}}$ evanescere debeat facto $z = h$. Fit hoc quidem si $z = 0$, sed cum praeter hunc alius requiratur, facile apparet, id non evenire posse, nisi ponatur $h = \infty$. In quolibet igitur casu ipsius n talis eligenda est solutio, ut $\frac{z^{\theta\mu+1}}{(1+bz^{\mu})^{\nu+\theta}}$ fiat = 0 posito $z = \infty$. Denotat hic autem θ numerum quemcunque integrum affirmativum non excepta cyphra, quamobrem et ν nunquam esse poterit numerus negativus, quia alioquin binomium $1+z^{\mu}$ in numeratorem veniret. At μ tam affirmativum quam negativum numerum significare potest, ex quo duplex existit huius rei consideratio, prout fuerit μ vel affirmativus numerus vel negativus. Sit primo μ numerus affirmativus $= +\lambda$, perspicuum est quo $\frac{z^{\lambda\theta+1}}{(1+bz^{\lambda})^{\nu+\theta}}$ fiat = 0, posito $z = \infty$, oportere maximum ipsius z exponentem in denominatore, [p.240] qui est $\nu+\lambda\theta$ maiorem esse eiusdem z exponente in numeratore, qui est $\lambda\theta+1$. Erit igitur

$\lambda\nu > 1$. Sin autem fuerit μ numerus negativus seu $= -\lambda$, fiet $\frac{z^{-\lambda\theta+1}}{(1+bz^{-\lambda})^{\nu+\theta}} = \frac{z^{\lambda\nu+1}}{(z^{\lambda}+b)^{\nu+\theta}}$, quae quantitas ut fiat = 0 posito $z = \infty$ debebit esse $\lambda\theta + \lambda\theta > \lambda\nu + 1$, seu $\lambda\theta > 1$, idquod in casu $\theta = 0$ fieri nequit. Quocirca μ nunquam esse poterit numerus negativus. In prima igitur solutione, quia est $\nu = 1$, quoties fuerit λ *i. e.* $\frac{2n+4}{3n+4}$ numerus positius, toties simul esse debebit numerus unitate maior, excipiuntur igitur esse ii casus quibus $\frac{2n+4}{3n+4}$ est 1 vel unitate minor. Nisi ergo *n* contineatur intra hos limites 0 et - $\frac{4}{3}$ prima solutio adhiberi nequit. In secunda solutione, quia iterum est $\nu = 1$, similiter excipiuntur casus, quibus λ seu $\frac{2n+4}{n}$ est unitas seu unitate minor. Semper igitur haec solutio locum habebit, his tantum exceptis casibus, quando *n* continetur intra hos limites – 4 et 0. Pro tertia solutione, quia μ iam est numerus positivus nempe = 2 debebit tantum $\frac{2n+2}{n+2}$ esse numerus unitate maior. Hac igitur semper uti poterimus nisi *n* contineatur intra hos limites – 2 et 0; quoties ergo secunda locum habet, toties et tertia poterit usurpari. In quarta denique solutione quia μ quoque est numerus affirmativus, scilicet $\frac{1}{3}$, requiretur ut $\frac{n+1}{3n+6}$ sit numerus unitate maior, id quod accidit, quoties n continetur intra hos limites -2 et - $\frac{1}{2}$. In his igitur casibus quarta solutione uti conveniet. Ex quibus invicem comparatis perspicitur, semper hoc modo aequationis propositae constructionem exhiberi posse, [p.241] nisi *n* contineatur intra hos angustus limites $\frac{-4}{3}$ et -2.

§13. Quo autem totum hoc negotium evidentius percipiatur, accommodabo, quae hactenus tradita sunt, ad casum particularem, quo est n = 2, ut itaque construenda sit haec aequatio $ax^2 dx = dy + y^2 dx$. Pro hoc casu eligo solutionem tertiam, eritque propterea $\mu = 2, \nu = \frac{3}{4}, \pi = \rho = 2.$ His valoribus substitutis habebitur $S = \frac{1}{4} c^{\frac{q}{2}} \int c^{-q} dq \int c^{\frac{q}{2}} dq.$ Est vero $\int c^{\frac{q}{2}} dq = 2c^{\frac{q}{2}} + i$, ergo $\int c^{-q} dq \int c^{\frac{q}{2}} dq = \int 2c^{\frac{-q}{2}} dq + i \int c^{-q} dq = -c^{\frac{-q}{2}} - ic^{-q} + k$. Consequenter prodit $S = \frac{k}{4} c^{\frac{q}{2}} - \frac{i}{4} c^{\frac{-q}{2}} - \frac{1}{4}$. Quia iam posito $q = 0$ debet evanescere *s*, habebitur ista aequatio $\frac{k}{4} - \frac{i}{4} - \frac{1}{4} = 0$, seu $k = 1 + i$. Porre cum $\frac{dS}{dq}$ debeat esse = 0 si $q = 0$, proveniet $i + k = 0$. Namque est $dS = \frac{k}{8} c^{\frac{q}{2}} dq + \frac{i}{8} c^{\frac{-q}{2}} dq$, et idcirco facto $q = 0$, fit $\frac{dS}{dq} = \frac{k}{8} + \frac{i}{8} = 0$. Ex his igitur conditionibus invenientur $i = -\frac{1}{2}$, et $k = \frac{1}{2}$: quamobrem erit $S = \frac{c^{\frac{q}{2}} + c^{\frac{-q}{2}}}{8} - \frac{1}{4}$, atque $R = \frac{3}{4} + \frac{c^{\frac{q}{2}} + c^{\frac{-q}{2}}}{8}$. Quoniam vero est $\mu = 2$ et $g = bf$, erit $q = \sqrt{\frac{bfz^2}{1+bz^2}}$, adeoque

$R = \frac{3}{4} + \frac{c^{\frac{1}{2}\sqrt{\frac{bfz^2}{1+bz^2}}} + c^{\frac{-1}{2}\sqrt{\frac{bfz^2}{1+bz^2}}}}{8}$. Consequenter reperitur

$\int PRdz = \frac{3}{4}\int \frac{dz}{(1+bz^2)^{\frac{3}{4}}} + \frac{1}{8}\int \frac{dz(c^{\frac{1}{2}\sqrt{\frac{bfz^2}{1+bz^2}}} + c^{\frac{-1}{2}\sqrt{\frac{bfz^2}{1+bz^2}}})}{(1+bz^2)^{\frac{3}{4}}}$. [p.242] Quod integrale ita capiatur, ut posito $z = 0$ ipsum fiat = 0, quo facto ponatur $z = \propto$, et prodibit quantitas, quae ut functio ipsius f potest spectati. Fiat deinde f variabilis, eiusque loco ponatur ax^4, erit ista functio per H divisa = t (vid. §6). Atque invento hoc t erit $y = \frac{dt}{tdx}$, qui est verus valor ipsius y ex aequatione proposita $ax^2dx = dy + y^2dx$.

§14. Non difficilior evadit constructio aequationis generalis $ax^n dx = dy + y^2 dx$, dummodo n non contineatur intra hos limites 0 et – 2. Uti enim poterimus solutione tertia, in qua fit $\mu = 2, \nu = \frac{n+1}{n+2}, \pi = \rho = \frac{n+2}{2}$. Erit igitur $S = \frac{1}{\rho^2} c^{\frac{q}{\rho}} \int c^{\frac{-2q}{\rho}} dq \int c^{\frac{q}{\rho}} dq.$

Integratione simili quo supra modo instituta, reperitur $S = \frac{k}{\rho^2} c^{\frac{q}{\rho}} - \frac{i}{\rho^2} c^{\frac{-q}{\rho}} - \frac{1}{\rho^2}$, ubi i et k ex his aequationibus debent definiri $k = 1 + i,$ et $k + i = 0$, est ergo ut ante $i = -\frac{1}{2},$ et $k = \frac{1}{2}$. Quapropter est $S = \frac{1}{2\rho^2} c^{\frac{q}{\rho}} + \frac{i}{2\rho^2} c^{\frac{-q}{\rho}} - \frac{1}{\rho^2}$, atque $R = 1 - \frac{1}{\rho^2} + \frac{1}{2\rho^2} c^{\frac{q}{\rho}} + \frac{i}{2\rho^2} c^{\frac{-q}{\rho}}$ vel posito loco ρ valore $\frac{n+1}{2}$ habebitur $R = \frac{n(n+4) + 2c^{\frac{2q}{n+2}} + 2c^{\frac{-2q}{n+2}}}{(n+2)^2}$. Est vero ut ante $q = \sqrt{\frac{bfz^2}{1+bz^2}}$, at $Pdz = \frac{dz}{(1+bz^2)^{\frac{n+1}{n+2}}}$. Quamobrem erit $\int PRdz = \frac{1}{(n+2)^2} \int \frac{dz}{(1+bz^2)^{\frac{n+1}{n+2}}} (n(n+4) + 2c^{\frac{2q}{n+2}} + 2c^{\frac{-2q}{n+2}})$ ubi loco $\sqrt{\frac{bfz^2}{1+bz^2}}$ relinquo q.

[p.243] Integrale huius $PRdz$ ita capiatur, ut posito $z = 0$, ipsum evanescat, quo facto ponatur $z = \propto$, denotetque H id quod provenit, si tantum $\int \frac{dz}{(1+bz^2)^{\frac{n+1}{n+2}}}$ hoc modo integretur ut fiat = 0 posito $z = 0$, et postmodum ponatur $z = \propto$. Tum ergo erit integrale ipsius $PRdz$ praescripto modo acceptum, quod posuimus Z §4 functio ipsius f. Aequale id autem erat positum quantitati H, in hanc seriem $1 + A\alpha g + AB\beta g^2 +$ etc. multiplicatae, quae series in sequentem est transmutata

$1 + \frac{f}{(n+1)(n+2)} + \frac{f^2}{(n+1)(n+2)(2n+3)(2n+4)} +$ etc. cuius summa est t, vid. §6. ubi f designat ax^{n+2}. Erit ergo $Z = Ht$, inquo H est quantitas constans, quia in ea non inest f, adeoque nec x. Provenit itaque $t = \frac{Z}{H}$, at est $y = \frac{dt}{tdx}$: ergo pro aequatione proposita $ax^n dx = dy + y^2 dx$ prodibit $y = \frac{dZ}{Zdx}$ Ad illam igitur aequationem construendam habemus hanc regulam : Integretur haec formula

$\frac{1}{(n+2)^2}\frac{dz}{(1+bz^2)^{\frac{n+1}{n+2}}}(n(n+4)+2c^{\frac{2}{n+2}\sqrt{\frac{bfz^2}{(1+bz^2)}}}+2c^{\frac{-2}{n+2}\sqrt{\frac{bfz^2}{(1+bz^2)}}})$ ita ut evanescat facto z = 0. Tum ponatur z infinitum, et loco f substitutuatur ax^{n+2}. Id quod provenit sit Z, quo cognito erit $y=\frac{dZ}{Zdx}$. Si quem offendat, quod post integrato debeat $z=\propto$, is loco z substituat $\frac{u}{1-u}$ et post integrationem ponatur u = 1, quo facto pro Z idem prodibit valor, qui ante. Quamvis autem analytica pro Z expressio obtineri non potest, quando formula illa non est integrabilis tamen, [p.244] per quadraturas vel rectificationes valor ipsius Z contrui poterit.

§15. Quanquam autem in hac constructione ii casu excluduntur, in quibus n continetur intra limites –2 et 0, nihil tamen minus haec solutio pro universali est habenda. Nam quia, si aequatio potest resolvi in casu $n = m$, resolutio quoque habetur in casu $n = -m - 4$, ut constat, ex iis, quae de casibus separabilibus sunt detecta; perspicuum est, si m sit numerus intra limites 0 et – 2 contentus, fore $-m - 4$ intra terminos – 2 et – 4 comprehensum, adeoque in solutione nostra contineri. Quamobrem si occurrat casus, quo n contineatur intra 0 et – 2, hic statim reducatur ad alium per dictum theorema, qui intra – 2 et – 4 contineatur, huiusque constructio erit in promtu.

§16. In formula differentiali §13. eruta observo, quoties habuerit $\frac{n+1}{n+2}$ huiusmodi formam $k+\frac{1}{2}$, ubi k numerum integrum affirmatium denotat, integram formulam posse integrari, et hanc ob rem valorem ipsius Z re ipsa exhiberi. His igitur in casibus valor ipsius y quoque poterit definiri et aequatio integrari. Fiet tum autem $n=\frac{-4k}{2k-1}$, quoties ergo n talem habuerit formam, aequatio $ax^n dx = dy + y^2 dx$ integrationem admittet. Deinde quia casus, si $n=\frac{-m}{m+1}$ vel $n = -m =$ 4 reduci potest ad casum $n = m$, integrabilis etiam erit aequatio si $n=\frac{-4k}{2k+1}$ vel $\frac{-4k-4}{2k+1}$ denotante k numerum integrum affirmativum. [p.245] Atque hi sunt illi ipsi casus integrabiles vel separabiles, ab aliis iam eruti, ubi videre licet in nostris Commentariis A. 1726.

§17. Esse autem aequationem integrabilem, quoties sit $\frac{n+1}{n+2}=k+\frac{1}{2}$ hoc loco modo ostendo. Pono $\frac{bz^2}{1+bz^2}=u^2$; erit $z=\frac{u}{\sqrt{b(1-u^2)}}$; $1+bz^2=\frac{1}{1-u^2}$, ideoque $dz=\frac{du}{(1-u^2)^{\frac{3}{2}}\sqrt{b}}$.
Fiet igitur $\frac{dz}{(1+bz^2)^{\frac{n+1}{n+2}}}=\frac{du}{\sqrt{b}}(1-u^2)^{k-1}$. Hanc ob rem formula §14 integranda transformabitur in hanc

$\frac{1}{(n+2)^2\sqrt{b}}(n(n+4)du(1-u^2)^{k-1}+2c^{\frac{2u\sqrt{f}}{n+2}}du(1-u^2)^{k-1}+2c^{\frac{-2u\sqrt{f}}{n+2}}du(1-u^2)^{k-1})$, quae ut facile perspicitur re ipsa integrari potest, quoties k fuerit numerus integer affirmativus. Atque hinc non parum praestantiae accedere arbitro huic meae methodo, quod tam sit facilis et perspicua, ut casus etiam omnes qui reipsa integrationem vel separationem admittunt, uno obtutu ostendat.

§18. Exempli gratia assumo $k = 1$, erit $n = -4$, qui casus, uti constat, est facillimus eorum, qui separationem admittunt. Formula igitur integranda abibit in hanc $\frac{1}{2\sqrt{b}}(c^{-u\sqrt{f}}du - c^{u\sqrt{f}}du)$, cuius integralis est $\frac{1}{2\sqrt{bf}}(c^{u\sqrt{f}} - c^{-u\sqrt{f}})$. Constantem non adiicio quia posito $z = 0$, seu quod eodem recidit $u = 0$. totum integrale iam evanescit. Fiat nunc $z = \infty$ seu in nostro casu $u = 1$ et ponatur ax^{-2} loco f habebitur $Z = \frac{x}{2\sqrt{ab}}(c^{\frac{\sqrt{a}}{x}} - c^{-\frac{\sqrt{a}}{x}})$. Hoc invento, erit ut iam est ostensum $y = \frac{dZ}{Zdx}$. [p.246]

Differentiato igitur Z et differentiali per Zdx diviso prodibit $y = \frac{1}{x} - \frac{\sqrt{a}}{x^2}\left(\frac{c^{\frac{2\sqrt{a}}{x}}+1}{c^{\frac{2\sqrt{a}}{x}}-1}\right)$ sive $\frac{2\sqrt{a}}{x} = l\left(\frac{xxy-x-\sqrt{a}}{xxy-x+\sqrt{a}}\right)$ quae aequatio est integralis huius differentialis $ax^{-4}dx = dy + y^2 dx$.

Atque simili modo pro reliquis casibus, qui separationem admittunt, aequationes integrales inveniuntur.