CONCERNING THE CONSTRUCTION

OF DIFFERENTIAL EQUATIONS WITHOUT THE SEPARATION

OF THE INDETERMINATE VARIABLES.

Leonhard Euler.

§.l. Thus the separation of indeterminate quantities in differential equations is to be desired so much, since with that found, the construction of the equation shall proceed at once from that, and I believe to be understood well enough with training put in place in these matters. Besides the integration of the differential equation, if indeed it succeeds, and with the optimum separation of the indeterminate magnitudes established. Indeed innumerable equations are given, the integrals of which can be found without the separation of the variables, just as the celebrated Johan Bernoulli has shown by a method of this kind in the commentary of our Book I, page 167; yet all these equations have been prepared thus, so that either the separation of the indeterminates shall be apparent at once, or perhaps will be found easily from the integration itself. Likewise truly, the account of the construction which Analysts have used to the present, indeed are also all of this kind, so that the indeterminate quantities are unable to be separated by any other way. Hence so far I judge the construction of a differential equation cannot be shown in any other way , of which the separation of the variables shall elude all the forces.

§.2. But recently, on being occupied in the rectification of the ellipse, unexpectedly, I came across a differential equation, with the aid of which, I was able to resolve the rectification of the ellipse, not only without the separation of the variables; but neither is it possible to be found in that manner. Truly this was the equation that I came upon: $dy+\frac{y^2dx}{x}=\frac{xdx}{x^2-1}$. Almost similar to the Riccati equatiom, and perhaps as equally difficult to be separated as this: $dy+y^2dx=x^2dx.$ At first the case appeared to be extremely difficult; but with the construction viewed with more care, I understood readily that not only was the separation of the indeterminates not possible, but also if this separation were to succeed by another method, it would appear to be even more absurd; so that clearly the comparison of the perimeters of dissimilar ellipses appears to me to defy all analysis. But that construction itself is extremily easy, indeed it is preferred to be performed from the elongation of an infinite ellipse having a common axis in each direction with the squares being added together.

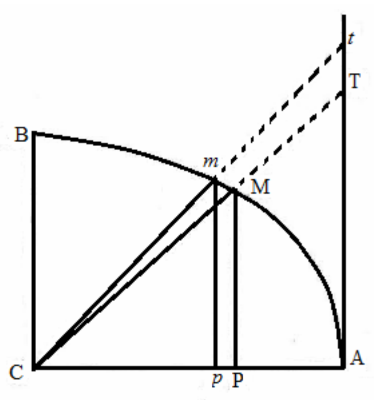

fig. 1.

§. 3. Therefore I shall propose the whole thing, so that I reach that ACB shall be an ellipse quadrant, of which C shall be the centre C, truly AC and BC the semi-axes. There shall be placed $\mathrm{AC}=a$ and $\mathrm{BC}=b,$ and from A the indefinite tangent AT shall be drawn, and to that some line

CT shall be drawn from the centre C, cutting the arc $\mathrm{AM} = s$, and the perpendicular shall be dropped from M to AC, and there shall be called $\mathrm{CP} = x$, from the nature of the ellipse there will be $\mathrm{PM} = \frac{b\sqrt{\left(a^2-b^2\right)}}{a}$; and on account of the ratio $\mathrm{CP} : \mathrm{PM} = \mathrm{CA} : \mathrm{AT}$ there will be had $tx = b\sqrt{\left(a^2-x^2\right)}$ or $x = \frac{ab}{\sqrt{(bb+tt)}}$) .

The element M*m* of the arc AM may be taken, and *mp*, C*t* may be drawn, equal in the first approximations to MP, CT; M*m* will become

$$ds = \frac{-dx\sqrt{\left(a^4-\left(a^2-b^2\right)x^2\right)}}{a\sqrt{\left(a^2-b^2\right)}} \text{ and } \mathrm{T}t = dt. \text{ But since } x = \frac{ab}{\sqrt{\left(b^2+t^2\right)}}, \text{ there will be}$$

$$dx = \frac{-abtdt}{\left(b^2+t^2\right)^{\frac{3}{2}}}, \text{ and } \sqrt{\left(a^2-x^2\right)} = \frac{at}{\left(b^2+t^2\right)}, \text{ and also } \sqrt{\left(a^4-\left(a^2-b^2\right)x^2\right)} = \frac{a\sqrt{\left(b^4+a^2t^2\right)}}{\sqrt{\left(b^2+t^2\right)}}.$$

From these there becomes $ds = \frac{btd\sqrt{\left(b^4+a^2t^2\right)}}{\left(b^2+t^2\right)^{\frac{3}{2}}}$. For the integral requiring to be found anyhow by series, I put $a^2 = (n+1)b^2$, which shall produce $ds = \frac{b^2dt\sqrt{\left(\left(b^2+t^2\right)+nt^2\right)}}{\left(b^2+t^2\right)^{\frac{3}{2}}}$, and the upper binomial shall be irrational, of which the lower member is b^2+t^2, and the other simple term nt^2. Now with $\sqrt{\left(\left(b^2+t^2\right)+nt^2\right)}$ expanded by the known procedure in the series $\left(b^2+t^2\right) + \frac{\mathrm{A}nt^2}{\left(b^2+t^2\right)^{\frac{1}{2}}} + \frac{\mathrm{B}n^2t^4}{\left(b^2+t^2\right)^{\frac{3}{2}}} + \frac{\mathrm{C}n^3t^6}{\left(b^2+t^2\right)^{\frac{5}{2}}} + \text{etc.}$, in which for the sake of brevity to be $\mathrm{A} = \frac{1}{2}$, $\mathrm{B} = \frac{-1\cdot 1}{2\cdot 4}$, $\mathrm{C} = \frac{-1\cdot 1\cdot 3}{2\cdot 4\cdot 6}$, $\mathrm{D} = \frac{-1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8} + \text{etc.}$ Therefore there is had :

$$ds = \frac{b^2dt}{b^2+t^2} + \frac{\mathrm{A}b^2nt^2dt}{\left(b^2+t^2\right)^2} + \frac{\mathrm{B}b^2n^2t^4dt}{\left(b^2+t^2\right)^3} + \frac{\mathrm{C}b^2n^3t^6dt}{\left(b^2+t^2\right)^4} + \text{etc.}$$

and the whole elliptic arc *s* will be the sum of this series.

§. 4. Here it is required to be observed the integration of these terms to be able to be reduced to the integration of the first term $\int\frac{bbdt}{bb+tt}$, truly $\int\frac{bbdt}{bb+tt}$ gives the arc of the circle of radius *b*, the tangent of which is *t*. On this account I will integrate the individual terms according to this circular arc, as follows:

$$\int\frac{bbt^2dt}{(bb+tt)^2}=\tfrac{1}{2}\int\frac{b^2dt}{bb+tt}-\tfrac{1}{2}\frac{b^2t}{bb+tt};\ \int\frac{b^2t^4dt}{(b^2+t^2)^3}=\tfrac{1}{2}\cdot\tfrac{3}{4}\int\frac{b^2dt}{bb+tt}-\tfrac{1}{2}\cdot\tfrac{3}{4}\frac{b^2t}{bb+tt}-\tfrac{1}{4}\frac{b^2t^3}{(bb+tt)^2};$$

$$\int\frac{b^2t^6dt}{(b^2+t^2)^4}=\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\int\frac{b^2dt}{bb+tt}-\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{b^2t}{bb+tt}-\tfrac{1\cdot5}{4\cdot6}\cdot\frac{b^2t^3}{(bb+tt)^2}-\tfrac{1}{6}\cdot\frac{b^2t^5}{(bb+tt)^3},$$

from which the rule for the integration of the remaining terms becomes clear enough.

§. 5. If the fourth part AMB of the perimeter of the ellipse shall be required, it will be required to make t infinite, and with this done all the algebraic terms in the above integrals vanish.
Truly the arc of the circle $\int\frac{bbdt}{bb+tt}$ on putting $t=\infty$ will give the fourth part of the circular periphery, of which the radius or BC, is b, which we will designate by the letter e. Therefore there will become:

$$\int\frac{b^2dt}{bb+tt}=e,\ \int\frac{b^2t^2dt}{(bb+tt)^2}=\frac{1\cdot e}{2},\ \int\frac{b^2t^4dt}{(bb+tt)^3}=\frac{1\cdot3\cdot e}{2\cdot4},\ \int\frac{b^2t^6dt}{(bb+tt)^4}=\frac{1\cdot3\cdot5\cdot e}{2\cdot4\cdot6},\ \text{etc.}$$

Therefore the fourth part of the perimeter of the ellipse AMB will be produced

$$=e\left(1+\frac{\mathrm{A}n}{2}+\frac{1\cdot3}{2\cdot4}\mathrm{B}n^2+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\mathrm{C}n^3+\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\mathrm{D}n^4+\text{etc.}\right).$$

And with the values substituted in place of A, B, C, D, etc., there will be had

$$\mathrm{AMB}=e\left(1+\frac{1\cdot n}{2\cdot2}-\frac{1\cdot1\cdot3\cdot n^2}{2\cdot2\cdot4\cdot4}+\frac{1\cdot1\cdot3\cdot3\cdot5n^3}{2\cdot2\cdot4\cdot4\cdot6\cdot6}-\frac{1\cdot1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot n^4}{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8}+\text{etc.}\right)$$

§. 6. This series will become strongly convergent, if n is very small, or $\frac{a^2-b^2}{b^2}$ to which it eventuates, whenever the ellipse is exceedingly close to being circular; and therefore in this case the perimeter of the ellipse is found easily. Truly when n is such a minimal quantity, $a=b+\omega$, with ω denoting such a mimimal quantity, there will become $n=\frac{2\omega}{b}$, and $\mathrm{AMB}=e\left(1+\frac{\omega}{2b}\right)q\cdot p.$

Truly when there becomes $a = 0$, the point A coincides with the point C, and there arises $\text{AMB} = \text{BC} = b$; truly in this case there will become $n = -1$, therefore there will be had
$$\frac{b}{e} = 1 - \frac{1}{2\cdot 2} - \frac{1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \text{etc.}$$

Therefore the sum of this series expresses the ratio of the radius to the fourth part of the periphery in the circle.

§.7. Therefore, whatever value the letter *n* may have had in series §.5, the sum of the series will be able to be assigned always to help in the rectification of the ellipse, of which the major axis to the minor axis has the ratio $\surd(n+1)$ to 1. Thus since this ratio shall itself be had, also by using my method the resolution of the problem to be reduced to the summation of the series, which I have shown recently, so that I might investigate, from which equations the summation of the series might depend. But so that this method shall be easier to use, I put $n = -x^2$, and this series shall be required to be summed:

$1 - \frac{1\cdot x^2}{2\cdot 2} - \frac{1\cdot 1\cdot 3\cdot x^4}{2\cdot 2\cdot 4\cdot 4} + \frac{1\cdot 1\cdot 3\cdot 3\cdot 5x^6}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \text{etc.}$, the sum of which I put to be *s*.
Therefore by differentiation there will become:

$$\frac{ds}{dx} = -\frac{1\cdot x}{2} - \frac{1\cdot 1\cdot 3\cdot x^3}{2\cdot 2\cdot 4} - \frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot x^5}{2\cdot 2\cdot 4\cdot 4\cdot 6} - \text{etc.}$$

Now by multiplying again by *x*, and the sum of the differentials taken with *dx* put constant, there will become:

$$\frac{d\cdot xds}{dx^2} = -1\cdot x - \frac{1\cdot 1\cdot 3\cdot x^3}{2\cdot 2} - \frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot x^5}{2\cdot 2\cdot 4\cdot 4} - \text{etc.}$$

And again finally on dividing by *x*, and indeed multiplying by *dx*, I take the integrals, there will become
$$\int \frac{d\cdot xds}{xdx} = -x - \frac{1\cdot 1\cdot x^3}{2\cdot 2} - \frac{1\cdot 1\cdot 3\cdot 3\cdot x^5}{2\cdot 2\cdot 4\cdot 4} - \text{etc.}$$

Finally again by multiplying by *dx* , and indeed by dividing by x^3, and I take the integrals, there will become

$$\int \frac{dx}{x^3}\int \frac{d\cdot xds}{x} = \frac{1}{x} - \frac{1\cdot x}{2\cdot 2} - \frac{1\cdot 1\cdot 3\cdot x^3}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot x^5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \text{etc.}$$

Truly this series initially is divided by x, the sum of which therefore is $\frac{s}{x}$. On account of which we have this equation: $\int\frac{dx}{x^3}\int\frac{d\cdot xds}{x}=\frac{s}{x}$, which with the differential taken will be changed into this $x^2ds-sxdx=\int\frac{d\cdot xds}{x}$. This differentiated again will produce $x^2dds+xdxds-sdx^2=\frac{d\ xds}{x}=dds+\frac{dxds}{x}$. The resolution of this equation therefore will depend on the summation of the proposed series, which since it shall be obtained from the rectification of the ellipse, the construction of the equation also will be obtained.

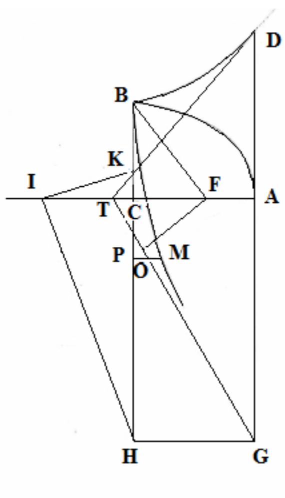

Fig. 2

§. 8. Since in this equation s maintains one dimension everywhere, that can be reduced by the method I have inserted by my method in Book III for a simpler differential equation, by substituting $s=c^{\int pdx}$, where c specifies the number the log.of which is 1. With this in place there will become $ds=c^{\int pdx}pdx$ and $dds=c^{\int pdx}\left(dpdx+ppdx^2\right)$, and the equation found will be transformed into this :
$x^2dp+x^2p^2dx+pxdx-dx=dp+ppdx+\frac{pdx}{x}$, which divided by $xx-1$ is changed into this equation:
$$dp+ppdx+\frac{pdx}{x}=\frac{dx}{xx-1}.$$
In order to be bringing about this to be simpler, I put $p=\frac{y}{x}$, and there arises $dy+\frac{yydx}{x}=\frac{xdx}{xx-1}$.
However neither do I see how to separate the variables, nor do I consider a construction leading from that.

§. 9. So that the construction of this equation shall be deduced from the precedings, I put that semi-axis AC equal to γ, which before I had denoted by the letter a, because it must be considered as a variable; and q corresponding to the fourth part of the perimeter; there will become $-xx=n=\frac{r^2-b^2}{b^2}$, and $x=\frac{\surd\left(b^2-r^2\right)}{b}$. Again there will be $q=es$, truly there becomes $s=c^{\int pdx}=c^{\int\frac{ydx}{x}}$, on account of which there will be had $q=ec^{\int\frac{ydx}{x}}$, and $lq-le=\int\frac{ydx}{x}$, thus $y=\frac{xdq}{qdx}=\frac{\left(\gamma^2-b^2\right)dq}{qrdr}$. But lest, when γ is greater than b, it may become irrational, with the value $-n$ restored in place of xx, this equation will be had

$2dy + \frac{y^2 dn}{n} = \frac{dn}{n+1}$, which is constructed on taking $n = \frac{r^2 - b^2}{b^2}$ and $y = \frac{\left(r^2 - b^2\right)dy}{qrdr}$, or now with n found, on taking $y = \frac{2ndq}{qdn}$. Hence the following construction arises : from the described elliptic quadrant BCA, of which the centre is at C and the constant semi -axis BC is put $= \pi$, here I put 1 in place of b, so that the homogeneity shall be able to be maintained more easily. Therefore the semi-axis will be $\mathrm{AC} = r$, and from A the normal AD shall be erected equal to the elliptic arc AB, B shall be some point on the curve the construction of which has been established. Therefore in this there will become $\mathrm{AD} = q$. F shall be the focus of this ellipse, there will become $\mathrm{CF} = \sqrt{\left(r^2 - 1\right)}$ and the normal FP shall be drawn to BF, there will become $\mathrm{EP} = r^2 - 1 = n.$ Here it may be observed, when there shall become $\mathrm{AC} < \mathrm{BC}$ and the focus F lies on BC, the value n to become negative, and must be taken from the other side of the point C towards B. Then the tangent DT of the curve BD may be drawn at D. Then the tangent DT of the curve BD at D, will become, there will become $\mathrm{AT} = \frac{qdr}{dq}$, and with AP joined, from T the right line TG shall be drawn cutting AP normally, if there is a need, to be produced to O and crossing DA at G, and on account of the similar triangle PCA and TAG, $\mathrm{AG} = \frac{rqdr}{\left(r^2 - 1\right)dq}$. AG itself shall be taken equal to CH and with $\mathrm{CI} = \mathrm{CB} = \mathrm{I},$ ad ductam HI erigatur perpendicularis IK , there will become $\mathrm{CK} = \frac{\left(r^2 - 1\right)dq}{rqdr} = y.$ For this line CK shall be made equal to PM, and M shall be on the curve sought BM, indeed it is a property of this curve, so that, calling CP, n and PM , y, there shall become $2dy + \frac{y^2 dn}{n} = \frac{dn}{n+1}$.

SPECIMEN DE CONSTRUCTIONE AEQUATIONUM DIFFERENTIALIUM SINE INDETERMINATARUM SEPARATIONE.

AUCTORE
Leonh. Eulero.

§.l. Indeterminarum separationem in aequationibus differenialibus ideo tam sollicite desiderari, quod ex ea inventa aequationis constructio sponte fluat, cuique in his rebus exercitato sitis perspectum esse arbitror. Integratio praeterea aequationum differentialum, siquidem succedit, optime indeterminatis separandis instituitur. Quanquam enim innumerabiles dantur aequationes, quarum integrales sine huiusmodi separatione inveniri possunt, cuiusmodi methodum exhibuit Celeb. Iob. Bernoulli in Comme nostrorum Tom. 1. pag. 167; tamen eae aequationes omnes ita sunt comparatae ut vel per se obvia fit indeterminatarum separatio, vel saltem ex ipsa integratione facile derivetur. Similis vero est etiam ratio constructionum, quibus adhuc usi sunt Analystae, sunt enim omnes huiusmodi, ut aequationis, si nullo alio modo indeterminatae a se invicem separari possunt, separatio tamen ex ipsa constructione proficiscatur. Hanc ob rem nullam adhuc exhiberi posse existimo aequationem differentialem construibilem, cuius separatio omnes vires eluderet.

§.2. Nuper, autem in ellipsi rectificanda occupatus inopinato incidi in aequationem differentialem, quam ope, rectificationis ellipsis construere poteram, neque tamen indeterminatarum separatio nequidem ex ipso construendi modo inveniri poterit. Aequatio vero quam obtinui erat haec $dy+\frac{y^2dx}{x}=\frac{xdx}{x^2-1}$. Riccatianae sere similis, et forte ad separandum aeque difficilis ac haec $dy+y^2dx=x^2dx.$ Casus mihi primum vehementer videbatur; at constructione attentius perspecta facile intellexi ex ea non solum separationem indeterminatarum non posse deduci, sed etiam, si alio modo separatio haec sussederet, multo maiora, sequutura esse absurda; comparationem scilicet perimetrorum ellipsium dissimilium, quae, ut mihi quidem videtur, omnem analysin superat. Constructio autem ipsa perquam est facilis, perficitur enim elongatione infinitarum ellipsium alterutrum axem communem habentium, et hanc obrem consueto per quadraturas construendi modo longe est praeferenda.

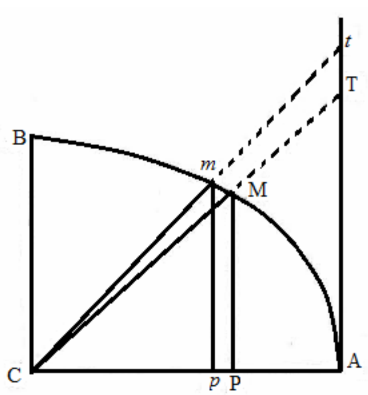

fig. 1.

§. 3. Proponam igitur totam rem, prout ad perveni. Sit ACB quadrans ellipticus, cuius centrum C, semi-axes vero AC et BC. Ponantur $\mathrm{AC}=a$ et $\mathrm{BC}=b,$ et ex A ducatur tangens indefinita AT , ad eamque ex centro C secans quaecunque CT, abscindens arcum $\mathrm{AM}=s$, voceturque $\mathrm{AT}=t$. Demisso ex M in AC perpendiculo vocetur $\mathrm{CP}=x$, erit ex natura ellipsis $\mathrm{PM}=\frac{b\sqrt{\left(a^2-b^2\right)}}{a};$ atque ob analogiam

$\mathrm{CP}:\mathrm{PM}=\mathrm{CA}:\mathrm{AT}$ habebitur $tx=b\sqrt{\left(a^2-x^2\right)}$ seu $x=\dfrac{ab}{\sqrt{\left(bb+tt\right)}}$) .

Sumatur arcus AM elementum M*m*,ducanturque *mp*, C*t* prioribus MP, CT proximae; erit M*m*, $ds=\dfrac{-dx\sqrt{\left(a^4-\left(a^2-b^2\right)x^2\right)}}{a\sqrt{\left(a^2-b^2\right)}}$ et T*t* = *dt*. Quia autem est $x=\dfrac{ab}{\sqrt{\left(b^2+t^2\right)}}$; erit

$dx=\dfrac{-abtdt}{\left(b^2+t^2\right)^{\frac{3}{2}}}$, et $\sqrt{\left(a^2-x^2\right)}=\dfrac{at}{\left(b^2+t^2\right)}$, et $\sqrt{\left(a^4-\left(a^2-b^2\right)x^2\right)}=\dfrac{a\sqrt{\left(b^4+a^2t^2\right)}}{\sqrt{\left(b^2+t^2\right)}}$.

Ex his conficitur

$ds=\dfrac{btd\sqrt{\left(b^4+a^2t^2\right)}}{\left(b^2+t^2\right)^{\frac{3}{2}}}$. Ad cuius integrale per seriem satem inveniendem pono

$a^2=\left(n+1\right)b^2$, quae prodeat $ds=\dfrac{b^2dt\sqrt{\left(\left(b^2+t^2\right)+nt^2\right)}}{\left(b^2+t^2\right)^{\frac{3}{2}}}$,

superiusque irrationale sit binomium, cuius alterum membrum est b^2+t^2, alterumque simplex terminus nt^2. Resolvo nunc $\sqrt{\left(\left(b^2+t^2\right)+nt^2\right)}$ per canonem notum in seriem

$\left(b^2+t^2\right)+\dfrac{\mathrm{A}nt^2}{\left(b^2+t^2\right)^{\frac{1}{2}}}+\dfrac{\mathrm{B}n^2t^4}{\left(b^2+t^2\right)^{\frac{3}{2}}}+\dfrac{\mathrm{C}n^3t^6}{\left(b^2+t^2\right)^{\frac{5}{2}}}+\text{etc.}$ in qua brevitatis gratia esse

$\mathrm{A}=\frac{1}{2}$, $\mathrm{B}=\frac{-1\cdot1}{2\cdot4}$, $\mathrm{C}=\frac{-1\cdot1\cdot3}{2\cdot4\cdot6}$, $\mathrm{D}=\frac{-1\cdot1\cdot3\cdot5}{2\cdot4\cdot6\cdot8}+\text{etc.}$ Habebitur ergo

$$ds=\frac{b^2dt}{b^2+t^2}+\frac{\mathrm{A}b^2nt^2dt}{\left(b^2+t^2\right)^2}+\frac{\mathrm{B}b^2n^2t^4dt}{\left(b^2+t^2\right)^3}+\frac{\mathrm{C}b^2n^3t^6dt}{\left(b^2+t^2\right)^4}+\text{etc.}$$

et integer arcus ellipticus *s* erit integrale huius seriei.

§. 4. Notandum hic est singulorum horum terminorum integrationem ad primi termini $\int\dfrac{bbdt}{bb+tt}$ posse reduci, dat vero $\int\dfrac{bbdt}{bb+tt}$ arcum circuli radii *b* cuius tangens est *t*. Hanc ob rem singulos terminos assumto hoc circulari arcu integrabo, ut sequitur:

$$\int\frac{bbdt}{\left(bb+tt\right)^2}=\tfrac{1}{2}\int\frac{b^2t}{bb+tt}-\tfrac{1}{2}\frac{b^2t}{bb+tt};\quad\int\frac{b^2t^4dt}{\left(b^2+t^2\right)^3}=\tfrac{1}{2}\cdot\tfrac{3}{4}\int\frac{b^2t}{bb+tt}-\tfrac{1}{2}\cdot\tfrac{3}{4}\frac{b^2t}{bb+tt}-\tfrac{1}{4}\frac{b^2t^3}{\left(bb+tt\right)^2};$$

$$\int\frac{b^2t^6dt}{\left(b^2+t^2\right)^4}=\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\int\frac{b^2t}{bb+tt}-\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{b^2t}{bb+tt}-\tfrac{1\cdot5}{4\cdot6}\cdot\frac{b^2t^3}{\left(bb+tt\right)^2}-\tfrac{1}{6}\cdot\frac{b^2t^5}{\left(bb+tt\right)^3},$$

ex quibus lex integralium reliquorum terminorum iam satis apparet.

§. 5. Si quarta perimetri elliptici pars AMB requiratur, oportet facere t infinitum, hocque facto omnes termini algebraici in superioribus integralibus evanescunt.
Arcus circularis vero $\int \frac{bbdt}{bb+tt}$ posito $t=\infty$ dabit quartam peripheriae circuli partem, cuius radius est b seu BC, quam designabimus littera e. Erit propterea

$$\int \frac{b^2dt}{bb+tt} = e,\ \int \frac{b^2t^2dt}{(bb+tt)^2} = \frac{1\cdot e}{2},\ \int \frac{b^2t^4dt}{(bb+tt)^3} = \frac{1\cdot 3\cdot e}{2\cdot 4},\ \int \frac{b^2t^6dt}{(bb+tt)^4} = \frac{1\cdot 3\cdot 5\cdot e}{2\cdot 4\cdot 6},\ \text{etc.}$$

Prodibit igitur quarta perimetri elliptici pars AMB

$$= e\left(1+\frac{\mathrm{A}n}{2}+\frac{1\cdot 3}{2\cdot 4}\mathrm{B}n^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\mathrm{C}n^3+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\mathrm{D}n^4+\text{etc.}\right).$$

Atque substitutis loco A, B, C, D, etc., valoribus debitis habebitur

$$\mathrm{AMB} = e\left(1+\frac{1\cdot n}{2\cdot 2}-\frac{1\cdot 1\cdot 3\cdot n^2}{2\cdot 2\cdot 4\cdot 4}+\frac{1\cdot 1\cdot 3\cdot 3\cdot 5n^3}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}-\frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot n^4}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8}+\text{etc.}\right)$$

§. 6. Haec series, si n est valde paruum seu $\frac{a^2-b^2}{b^2}$ id quod evenit, quoties ellipsis admodum propinqua est circulo, vehementer convergit; hocque casu igitur facile ellipsis perimeter invenitur. Quando vero n est quantitas, quam minima, seu $a=b+\omega$, denotante ω quantitatem quam minimam , erit $n=\frac{2\omega}{b}$, et $\mathrm{AMB}=e\left(1+\frac{\omega}{2b}\right)q\cdot p.$
Quando vero fit $a=0$, incidit punctum A in C, et evadit $\mathrm{AMB}=\mathrm{BC}=b$; hoc vero si $n=-1$, habebitur igitur $\frac{b}{e}=1-\frac{1}{2\cdot 2}-\frac{1\cdot 1\cdot 3}{2\cdot 2\cdot 4\cdot 4}-\frac{1\cdot 1\cdot 3\cdot 3\cdot 5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}-\text{etc.}$
Summa huius seriei ergo exprimit ratio rationem radii ad quartam peripheriae partem in circulo.

§.7. Quemcumque igitur habeat valorem littera n in serie §.5. inventa, summa seriei semper poterit assignari ope rectificationis ellipsis, cuius axis maior se habet ad minorem, ut $\sqrt{(n+1)}$ ad 1. Hoc cum ita se habeat, usus quoque methodo mea summationes serierum ad resolutionem aequationum reducendi, quam nuper exhibui, ut investigarem, a cuius aequationis resolutione sumatio inventae seriei pendeat. Quo autem haec methodus facilius possit adhiberi pono $n=-x^2$, eritque summanda ista series

$$1-\frac{1\cdot x^2}{2\cdot 2}-\frac{1\cdot 1\cdot 3\cdot x^4}{2\cdot 2\cdot 4\cdot 4}+\frac{1\cdot 1\cdot 3\cdot 3\cdot 5x^6}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}-\text{etc.},$$ huius igitur summam pono s.

Erit ergo differentiando $\frac{ds}{dx} = -\frac{1\cdot x}{2} - \frac{1\cdot 1\cdot 3\cdot x^3}{2\cdot 2\cdot 4} - \frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot x^5}{2\cdot 2\cdot 4\cdot 4\cdot 6} - \text{etc.}$ Iam denuo per multiplico, sumoque differentialia posito dx constante, erit

$\frac{d\cdot xds}{dx^2} = -1\cdot x - \frac{1\cdot 1\cdot 3\cdot x^3}{2\cdot 2} - \frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot x^5}{2\cdot 2\cdot 4\cdot 4} - \text{etc.}$ Porro divido ubique x, contraque per dx multiplico, sumoque integralia, erit $\int \frac{d\cdot xds}{xdx} = -x - \frac{1\cdot 1\cdot x^3}{2\cdot 2} - \frac{1\cdot 1\cdot 3\cdot 3\cdot x^5}{2\cdot 2\cdot 4\cdot 4} - \text{etc.}$
Denique iterum per dx multiplico, divido vero per x^3, sumo integralia, erit

$$\int \frac{dx}{x^3}\int \frac{d\cdot xds}{x} = \frac{1}{x} - \frac{1\cdot x}{2\cdot 2} - \frac{1\cdot 1\cdot 3\cdot x^3}{2\cdot 2\cdot 4\cdot 4} - \frac{1\cdot 1\cdot 3\cdot 3\cdot 5\cdot x^5}{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6} - \text{etc.}$$

Haec vero series est ipsa initialis per x divisa eius igitur summa est $\frac{s}{x}$. Quodcirca habemus hanc aequationem $\int \frac{dx}{x^3}\int \frac{d\cdot xds}{x} = \frac{s}{x}$, quae sumtis differentialibus abit in hanc $x^2 ds - sxdx = \int \frac{d\cdot xds}{x}$. Differentietur haec denuo prodibit $x^2 dds + xdxds - sdx^2 = \frac{d\ xds}{x} = dds + \frac{dxds}{x}$. Huius acquationis resolutio igitur pendet a summatione seriei propositae, quae cum per rectificatinem ellipsis habeatur, aequationis constructio quoque dabitur.

§. 8. Cum in ista aequatione s ubique unam tenet dimensionem, reduci ea poterit per methodum meam Tom. III. Comme insertam ad aequationem simpliciter differentialem, facta substitutio $s = c^{\int pdx}$, ubi c denotat numerum, cuius log. est 1. Hoc positio erit $ds = c^{\int pdx} pdx$ et $dds = c^{\int pdx}\left(dpdx + ppdx^2\right)$, atque aequatio inventa transformabitur in hanc $x^2 dp + x^2 p^2 dx + pxdx - dx = dp + ppdx + \frac{pdx}{x}$, quae divisa per $xx - 1$ mutatur in istam $dp + ppdx + \frac{pdx}{x} = \frac{dx}{xx-1}$.
Ad hanc simpliciorem efficiendam pono $p = \frac{y}{x}$, et proveniet $dy + \frac{yydx}{x} = \frac{xdx}{xx-1}$.
Quae quomodo separari possit neque perspicio, neque constructioni consideratio eo perducit.

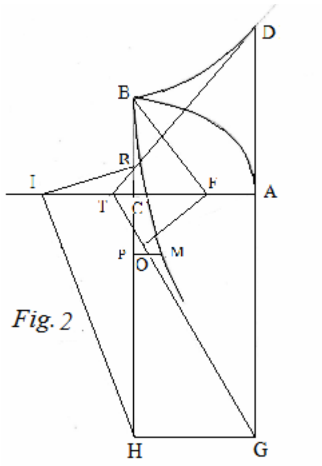

Fig. 2

§. 9. Quo ipsa constructio huius aequationis ex praecedentibus deducatur, pono illam axis semissem AC, quem ante littera *a* denotavi, aequalem γ, quia ut variabilis debet considerari; et quartam perimetri ellipsis partem respondentem *q*; erit

$$-xx = n = \frac{r^2 - b^2}{b^2}, \text{ et } x = \frac{\sqrt{\left(b^2 - r^2\right)}}{b}. \text{ Porro}$$

erit $q = es$, est vero $s = c^{\int pdx} = c^{\int \frac{ydx}{x}}$,

quocirca habebitur $q = ec^{\int \frac{ydx}{x}}$, et $lq - le = \int \frac{ydx}{x}$,

adeoque $y = \frac{xdq}{qdx} = \frac{\left(\gamma^2 - b^2\right)dq}{qrdr}$. Ne autem, quando γ maior est quam *b*, irrationalia proveniant, restituo loco *xx* valorem $-n$, erit $\frac{dx}{x} = \frac{dm}{2n}$, et $\frac{xdx}{xx-1} = \frac{dn}{2(n+1)}$. His substitutis habebitur ista aequatio

$2dy + \frac{y^2dn}{n} = \frac{dn}{n+1}$, quae construetur sumendis $n = \frac{r^2 - b^2}{b^2}$ et $y = \frac{\left(r^2 - b^2\right)dy}{qrdr}$, seu iam invento *n*, sumendis $y = \frac{2ndq}{qdn}$. Hinc sequens nascitur constructio : descripto quadrante elliptico sequens nascitur constructio: descripto quadrante elliptico BCA, cuius centrum in C et semi -axis BC constans est puta $= \pi$, pono hic 1 loco *b*, quo facilius homogeneitas possit servari. Erit ergo semi-axis $\mathrm{AC} = r$, ex A erigatur normalis AD $=$ arcui elliptico AB, erit punctum B in curva aliqua BD, cuius constructio hoc modo est in promtu. In ea igitur erit $\mathrm{AD} = q.$ Sit F huius ellipsis focus, erit $\mathrm{CF} = \sqrt{\left(r^2 - 1\right)}$ et ad BF ducatur normals FP, erit $\mathrm{EP} = r^2 - 1 = n.$ Notetur hic, quando sit $\mathrm{AC} < \mathrm{BC}$ et focus F in BC incidit, valorem *n* fieri negatiuum, et ex altera parte puncti C versus B accipi oportere. Deinceps ducatur tangens DT curvae BD in D, erit $\mathrm{AT} = \frac{qdr}{dq}$, et iuncta AP , ex T ducatur recta TG normaliter secans AP, si opus est, productam in O et DA productae occurrens

in G, erit ob similia triangula PCA et TAG, $\mathrm{AG} = \frac{rqdr}{\left(r^2 - 1\right)dq}$. Ipsi AG aequalis capiatur CH et sumta $\mathrm{CI} = \mathrm{CB} = \mathrm{I},$ ad ductam HI erigatur perpendicularis IK erit

$\mathrm{CK} = \frac{\left(r^2 - 1\right)dq}{rqdr} = y.$ Huic CK fiat aequalis PM, eritque M in curva quaesita BM, huius enim curvae haec est proprietas, ut, dictis CP, *n* et PM ,*y*, fit $2dy + \frac{y^2dn}{n} = \frac{dn}{n+1}.$