# CONCERNING THE SUMMATION OF INNUMERABLE PROGRESSIONS

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§ I. In the past year, I have proposed a method of summing innumerable progressions [see E20], which not only extended to the series themselves having an algebraic sum, but also to establishing sums depending on the quadrature of curves, and which cannot be summed algebraically. Thereupon, by using a synthetic method, I have indeed searched for the sum of series that can be expressed from assumed general formulas. And in this way, I have come upon many more general series to which I can assign the sum. Therefore for some proposed progression requiring to be summed, whereas [before] it was necessary for it to be compared to see whether it was contained in any of these [summation formulas]. Moreover, I have been able to increase the number of these series indefinitely, and more often therefore series have occurred to me, which even if they could not be summed from the given general terms, yet these could still be summed by this method. I will therefore communicate this analytical method, by which it is rather easy to bring out the sum of any proposed series, if indeed it is possible to be found at all, and from which one can elicit the nature of the sum of the terms of the series. This method appears to be of the widest application; for not only [can it be applied to] all of the series for which the whole sum has been elicited by diverse means, but also it can be shown how an indefinite number of other sums can be found by a similar and easy operation.

§2. If it is equally easy to find the general summatory term from a given general term and conversely, then this should be of the greatest help in the summation of the series. Indeed is possible to give an equation between the summatory term and the general term, but it is agreed we are not helped much by that equation for an infinite number of terms. But yet from the [integral] sign there arises a shortened method for establishing the sums of algebraic progressions. Let the general term be $t$ in some progression, or that with exponent $n$, and the summatory term, or the sum of all the terms of the series from the first as far as to $t$ is equal to $s$; then

$$t=\frac{ds}{1dn}-\frac{dds}{1.2dn^2}+\frac{d^3s}{1.2.3dn^3}-\frac{d^4s}{1.2.3.4dn^4}+\text{etc.},$$

[This formula is seen to be related to a backwards Taylor expansion of the function $s$ as a function of $n$, with unit step size, about the point $n$.]
in which equation $dn$ has been made constant. Moreover, it is possible to change the above equation into this :

$$s=\int tdn+\alpha t+\frac{\beta dt}{dn}+\frac{\gamma ddt}{dn^2}+\frac{\delta d^3t}{dn^3}+\text{etc.},$$

in which the coefficients $\alpha, \beta, \gamma,$ etc. have the following values :

$$\alpha = \frac{1}{2},$$
$$\beta = \frac{\alpha}{2} - \frac{1}{6},$$
$$\gamma = \frac{\beta}{2} - \frac{\alpha}{6} + \frac{1}{24},$$
$$\delta = \frac{\gamma}{2} - \frac{\beta}{6} + \frac{\alpha}{24} - \frac{1}{120},$$
$$\varepsilon = \frac{\delta}{2} - \frac{\gamma}{6} + \frac{\beta}{24} - \frac{\alpha}{120} + \frac{1}{720},$$
etc.

Moreover, the equation becomes :

$$s = \int tdn + \frac{t}{2} + \frac{dt}{12dn} - \frac{ddt}{720dn^2} + \frac{d^3t}{304240dn^3} - \text{etc.}$$

Therefore as often as $t$ has a value of this kind, so that the series $s$ presented is either interrupted somewhere or made summable, then with the aid of this equation $s$ is found from $t$. E. g. let

$$t = n^2 + 2n;$$

then

$$dt = 2ndn + 2dn,\ ddt = 2dn^2, d^3t = 0 \ \text{etc.}$$

Hence the sum becomes :

$$s = \int \left(n^2 + 2n\right)dn + \frac{n^2+2n}{2} + \frac{2n+2}{12} = \frac{n^3}{3} + \frac{3n^2}{2} + \frac{7n}{6} = \frac{2n^3+9n^2+7n}{6}.$$

§3. Moreover the method that I am about to set out here thus has the task to perform that a proposed progression is either reduced by certain operations to another simpler form that can be summed, or it is reduced to itself again; as in each way it agrees with the sum of the progression proposed. The operations, that I employ in these transformations, are either common, such as addition or subtraction, etc, or are taken from higher analysis, such as differentiation and integration. Indeed those [terms] from other series are not useful, unless the sum of these is known and can be assigned algebraically. Now for these also, the algebraic sums of progressions are not found to have sums depending on the quadrature of curves. Moreover all the series to which this method can be applied, include among themselves the geometric progression, and they have a form of this kind:

$$\alpha x^a + \beta x^{a+b} + \gamma x^{a+2b} + \delta x^{a+3b} + \text{etc.}$$

So that is not an impediment, any progression that does not have this form is not continued.

§4. As I shall begin from the simplest, let the proposed progression be geometric :

$$x^a + x^{a+b} + x^{a+2b} + x^{a+3b} \ldots\ldots\ldots + x^{a+(n-1)b},$$

in which the furthest term is that for which the index is $n$; and this is to be noted in the following, that the final term is always that with the index $n$, so that there is no need to ascribe the indices ; and hence also I will always show the sum as far as to the term of index $n$. The sum of the proposed progression is put as $s$; then

$$s = x^a + x^{a+b} + x^{a+2b} + x^{a+3b} \ldots\ldots + x^{a+(n-1)b};$$

then it becomes

$$s - x^a = x^{a+b} + x^{a+2b} + x^{a+3b} \ldots\ldots + x^{a+(n-1)b};$$

and to each side is added $x^{a+nb}$ and divided by $x^b$; there is produced

$$\frac{s - x^a + x^{a+nb}}{x^b} = x^{a+b} + x^{a+2b} + x^{a+3b} \ldots\ldots + x^{a+(n-1)b} = s.$$

Therefore we have the equation :

$$s - x^a + x^{a+nb} = sx^b,$$

from which it is found :

$$s = \frac{x^a - x^{a+nb}}{1 - x^b},$$

which is the sum of the proposed geometric progression. Hence this is the example in which the proposed progression is changed into itself. If $x$ should be a fraction less than one and $n$ an infinitely large number, then $x^{a+nb} = 0$ and

$$s = \frac{x^a}{1 - x^b}$$

provides the sum of the geometric progression

$$x^a + x^{a+b} + x^{a+2b} + \text{etc.}$$

continued to infinity. If we should have $x = 1$, it is apparent that the sum $s = n$; now this appears more difficult from the equation

$$s = \frac{x^a - x^{a+nb}}{1 - x^b},$$

because the numerator and the denominator are evanescent. Now in order that this value is found, there is put in place $x = 1 - \omega$ with $\omega$ denoting an infinitely small quantity; then

$$x^a = 1 - a\omega,\ x^{a+nb} = 1 - (a + nb)\omega \text{ and } x^b = 1 - b\omega.$$

Hence the sum becomes :

$$s = \frac{nb\omega}{b\omega} = n.$$

Now it is clear, if the general term of the series should be $\alpha x^{a+(n-1)b}$, then the summatory term becomes :

$$s = \frac{\alpha x^a - \alpha x^{a+nb}}{1 - x^b}.$$

§5. Now let this progression be proposed :

$$x^a + 2x^{a+b} + 3x^{a+2b} + 4x^{a+3b} \ldots\ldots + nx^{a+(n-1)b},$$

the sum of which is put as $s$. Then we have

$$s - x^a = 2x^{a+b} + 3x^{a+2b} + \ldots\ldots + nx^{a+(n-1)b};$$

and the following term is added $(n+1)x^{a+nb}$ and divided by $x^b$; then

$$\frac{s-x^a+(n+1)x^{a+nb}}{x^b}=2x^a+3x^{a+b}+........+(n+1)x^{a+(n-1)b}.$$

From this series there is subtracted the first series, clearly that proposed; there is produced :

$$\frac{s-x^a+(n+1)x^{a+nb}}{x^b}-s=x^a+x^{a+b}+x^{a+2b}+........+x^{a+(n-1)b}=\frac{x^a-x^{a+nb}}{1-x^b}.$$

From this equation there is found :

$$s=\frac{x^a-(n+1)x^{a+nb}}{1-x^b}+\frac{x^{a+b}-x^{a+(n+1)b}}{\left(1-x^b\right)^2}=\frac{x^a-(n+1)x^{a+nb}+nx^{a+(n+1)b}}{\left(1-x^b\right)^2}=\frac{x^a-x^{a+nb}}{\left(1-x^b\right)^2}-\frac{nx^{a+nb}}{1-x^b},$$

which is the summatory term corresponding to the general term $nx^{a+(n-1)b}$. If it is the case that $x < 1$ and on putting $n = \propto$, then there is produced the sum of the proposed series continued to infinity :

$$=\frac{x^a}{\left(1-x^b\right)^2}.$$

Moreover if $x$ is made equal to 1, there must be produced the sum of the progression :

$$1+2+3+4+.........n;$$

now here the same difficulty arises as before with the numerator and denominator vanishing; therefore again I put $x=1-\omega$; then [Euler's $\omega$ is reminiscent of Newton's $o$] :

$$1-x^b=b\omega,$$

$$x^a=1-a\omega+\frac{a(a-1)\omega^2}{2},$$

$$x^{a+nb}=1-(a+nb)\omega+\frac{(a+nb)(a+nb-1)\omega^2}{2},$$

and

$$x^{a+(n+1)b}=1-(a+(n+1)b)\omega+\frac{(a+(n+1)b)(a+(n+1)b-1)\omega^2}{2}$$

and the sum becomes :

$$s=\frac{\left(n^2b^2+nb^2\right)\omega^2}{2b^2\omega^2}=\frac{n(n+1)}{2}.$$

Further, if the general term is $\beta nx^{a+(n-1)b}$, then the summatory term is

$$\frac{\beta x^a-\beta x^{a+nb}}{\left(1-x^b\right)^2}-\frac{\beta nx^{a+nb}}{1-x^b}.$$

§6. The summatory terms can be found in a like manner, if the general terms are

$$n^2 x^{a+(n-1)b},\ n^3 x^{a+(n-1)b} \text{ etc.};$$

for the summation is always reduced to a summation of a series of lower degree. From which it is understood by this reason that it is possible to find generally the summatory term corresponding to the general term

$$(\alpha+\beta n+\gamma n^2+\text{etc.})\, x^{a+(n-1)b}.$$

But I will not linger any longer in solving these, because they have been known for some time now. Thus I only brought this forward so that the strength of a method through the use of only common operations could be made apparent. Therefore I progress beyond, and I will investigate which series can be reduced to a sum with the help of differentiation and integration. Indeed in the first place also algebraic progressions are taken and treated in the same manner and the sums found are not different from that now given; but yet the finding of these by these operations is shown to be easier and shorter. On this account I begin again from these.

§7. Let the progression to be summed be

$$x+2x^2+3x^3+4x^4+.......+nx^n;$$

this is put equal to *s*, it is divided by *x* and multiplied by *dx*; then

$$\frac{sdx}{x}=dx+2xdx+3x^2dx+.......+nx^{n-1}dx$$

and with the integrals taken this equation is found

$$\int\frac{sdx}{x}=x+x^2+x^3+.......+x^n=\frac{x-x^{n+1}}{1-x}.$$

Therefore from the equation :

$$\int\frac{sdx}{x}=\frac{x-x^{n+1}}{1-x}$$

on differentiation *s* is found. For the equation becomes:

$$\frac{sdx}{x}=\frac{dx-(n+1)x^n dx+nx^{n+1}dx}{(1-x)^2},$$

thus there is produced :

$$s=\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2},$$

as before in §5, if in place of *a* and *b* there is written 1.

From this it can be understood, how the sum of the progression can be found :

$$ax^{\alpha}+(a+b)x^{\alpha+\beta}+(a+2b)x^{\alpha+2\beta}+.........+(a+(n-1)b)x^{\alpha+(n-1)\beta}.$$

For $s$ is put in place for this sum sought, and it is multiplied by $x^{\pi}dy$ ; then

$$x^{\pi}sdy=ax^{\alpha+\pi}dy+(a+b)x^{\alpha+\beta+\pi}dy+(a+2b)x^{\alpha+2\beta+\pi}dy+...+(a+(n-1)b)x^{\alpha+(n-1)\beta+\pi}dy.$$

Now on making

$$x^{\alpha+\pi}=y^{a-1} \text{ and } x^{\alpha+\beta+\pi}=y^{a+b-1};$$

then

$$x^{\beta}=y^{b} \text{ and } x=y^{b:\beta}.$$

And hence the equation becomes :

$$x^{\alpha+\pi}=y^{(\alpha+\pi)b:\beta}=\mathrm{y}^{a\text{-}1}.$$

Hence we have :

$$\pi=\frac{\beta a-\alpha b-\beta}{b}$$

and

$$x^{\alpha+(n-1)\beta+\pi}=y^{a+(n-1)b-1}.$$

With these in place, then

$$x^{\frac{\beta a-\alpha b-\beta}{b}}sdy=ay^{a-1}dy+(a+b)y^{a+b-1}dy+.....+(a+(n-1)b)y^{a+(n-1)b-1}dy$$

and with the integrals taken,

$$\int x^{\frac{\beta a-\alpha b-\beta}{b}}sdy=y^{a}+y^{a+b}+.....+y^{a+(n-1)b}=\frac{y^{a}-y^{a+nb}}{1-y^{b}}.$$

Now because $y^{b}=x^{\beta}$, then

$$y=x^{\frac{\beta}{b}} \text{ and } dy=\frac{\beta}{b}x^{\frac{\beta-b}{b}}dx$$

and with these substituted,

$$\frac{\beta}{b}\int x^{\frac{\beta a-\alpha b-\beta}{b}}sdx=\frac{x^{\frac{a\beta}{b}}-x^{\frac{a\beta+nb\beta}{b}}}{1-x^{\beta}}.$$

[Hence, by a further differentiation, finally $s$ can be found.]
This same equation can be found more easily in this manner without changing the variable $x$. The proposed progression is multiplied by $px^{\pi}dx$ ; then

$$px^{\pi}sdx=pax^{\alpha+\pi}dx+...+p(a+(n-1)b)x^{\alpha+(n-1)\beta+\pi}dx.$$

Thus $p$ and $\pi$ can thus be determined, in order that

$$\alpha+(n-1)\beta+\pi=p(a+(n-1)b)-1$$

or

$$\alpha+\pi+(n-1)\beta=ap+(n-1)bp-1.$$

From which, because $p$ and $\pi$ cannot depend on $n$, the two equations emerge :

$$\beta = bp \text{ and } \alpha + \pi = ap - 1,$$

hence there is produced:

$$p = \frac{\beta}{b} \text{ and } \pi = \frac{a\beta - \alpha b - b}{b}.$$

With these substituted and with the integrals taken as before, there eventuates,

$$\frac{\beta}{b}\int x^{\frac{a\beta-\alpha b-\beta}{b}} sdx = x^{\frac{a\beta}{b}} + x^{\frac{a\beta+b\beta}{b}} + \ldots + x^{\frac{a\beta+(n-1)b\beta}{b}} = \frac{x^{\frac{a\beta}{b}} - x^{\frac{a\beta+nb\beta}{b}}}{1-x^{\beta}}.$$

§8. Let the term with order $n$ of the proposed progression be this

$$(an+b)(cn+e)x^{\alpha+(n-1)\beta};$$

the summatory term of this is put as $s$; then

$$s = (a+b)(c+e)x^{\alpha} + (2a+b)(2c+e)x^{\alpha+\beta} + \ldots(an+b)(cn+e)x^{\alpha+(n-1)\beta};$$

this is multiplied by $px^{\pi}dx$ ; the series becomes

$$psx^{\pi}dx = p(a+b)(c+e)x^{\alpha+\pi} + \ldots + p(an+b)(cn+e)x^{\alpha+(n-1)\beta+\pi}dx;$$

Let

$$pcn + pe = \alpha + n\beta - \beta + \pi + 1;$$

then it must be the case that :

$$p = \frac{\beta}{c} \text{ and } \pi = \frac{\beta e + \beta c - \alpha c - c}{c}.$$

Hence with the integrals taken then :

$$\frac{\beta}{c}\int x^{\pi} sdx = (a+b)x^{\alpha+\pi+1} + \ldots + (an+b)x^{\alpha+(n-1)\beta+\pi+1}.$$

The equation is multiplied again by $qx^{\rho}dx$ ; then

$$\frac{\beta}{c}qx^{\rho}dx\int x^{\pi} sdx = q(a+b)x^{\alpha+\pi+\rho+1} + \ldots + q(an+b)x^{\alpha+(n-1)\beta+\pi+\rho+1}dx$$

and put in place :

$$anq + bq = \alpha + n\beta - \beta + \pi + \rho + 2;$$

hence :

$$q = \frac{\beta}{a} \text{ and } \rho = \frac{\beta b - \alpha a + \beta a - \pi a - 2a}{a} = \frac{\beta bc - ac - \beta ae}{ac}.$$

And with the integrals taken there arises :

$$\frac{\beta^2}{ac}\int x^{\rho}dx\int x^{\pi}sdx = x^{\alpha+\pi+\rho+2}+\ldots+x^{\alpha+(n-1)\beta+\pi+\rho+2} = \frac{x^{\alpha+\pi+\rho+2}-x^{\alpha+n\beta+\pi+\rho+2}}{1-x^{\beta}}$$

or this equation :

$$\frac{\beta^2}{ac}\int x^{\frac{\beta bc-\alpha c-\beta ae}{ac}}dx\int x^{\frac{\beta e+\beta c-\alpha c-e}{c}}sdx = \frac{x^{\frac{\beta(a+b)}{a}}-x^{\frac{\beta(a+b+na)}{a}}}{1-x^{\beta}} = x^{\frac{\beta(a+b)}{a}}\frac{1-x^{n\beta}}{1-x^{\beta}}.$$

The operation is put in place in a similar manner, if there are more than two factors in the general term, from which likewise there are as many integral signs produced, as there are factors in the coefficient of the general term.

§9. If the general term of the progression to be summed is

$$\frac{x^{\alpha+(n-1)\beta}}{an+b},$$

only in this case the operation differs from before, since this must be solved by differentiation, because there the work was completed by integration. Therefore let the summation term sought be $s$; then

$$s=\frac{x^{\alpha}}{a+b}+\ldots..+\frac{x^{\alpha+(n-1)\beta}}{an+b}$$

and

$$px^{\pi}s=\frac{px^{\alpha+\pi}}{a+b}+\ldots..+\frac{px^{\alpha+(n-1)\beta+\pi}}{an+b}.$$

The differential are taken; there is produced :

$$px^{\pi}ds+p\pi x^{\pi-1}sdx=\frac{p(\alpha+\pi)x^{\alpha+\pi-1}dx}{a+b}+\ldots..+\frac{p(\alpha+n\beta-\beta+\pi)x^{\alpha+(n-1)\beta+\pi-1}}{an+b}$$

Let

$$p\pi+pn\beta-p\beta+p\pi=an+b;$$

then

$$p=\frac{a}{\beta}\ \text{et}\ \pi=\beta-\alpha+\frac{b\beta}{a}.$$

Hence

$$\frac{ax^{\beta-\alpha+\frac{b\beta}{a}}ds+(a\beta-a\alpha+b\beta)x^{\beta-\alpha+\frac{b\beta}{a}-1}sdx}{\beta dx}=x^{\frac{a\beta+b\beta-a}{a}}+\ldots.+x^{\frac{na\beta+b\beta-a}{a}}$$
$$=x^{\frac{a\beta+b\beta-a}{a}}\frac{1-x^{n\beta}}{1-x^{\beta}},$$

or

$$\frac{a}{\beta}x^{\frac{a\beta-a\alpha+b\beta}{a}}s=\int x^{\frac{a\beta+b\beta-a}{a}}dx\frac{1-x^{n\beta}}{1-x^{\beta}},$$

or

$$s=\frac{\beta}{a}x^{\frac{a\beta-a\beta-b\beta}{a}}\int x^{\frac{a\beta+b\beta-a}{a}}dx\frac{1-x^{n\beta}}{1-x^{\beta}}.$$

In this equation the integral thus must be taken, so that on putting $x = 0$ the integral itself vanishes. If the sum of the series continued to infinity is wished, on putting $n = \infty$ , then

$$s = \frac{\beta}{a} x^{\frac{a\beta - a\beta - b\beta}{a}} \int \frac{x^{\frac{a\beta + b\beta - a}{a}}}{1 - x^{\beta}} dx.$$

If we let $x = 1$, as differentials are present in the [integral] expression for the sum $s$, then it is not possible to put $x = 1$ yet, but after the integration then we can put $x = 1$. As still in the same way, such numbers can be put in place of $\alpha$ and $\beta$; therefore let $\alpha = \beta = 1$. Then

$$s = \frac{x^{\alpha}}{a+b} + \ldots.. + \frac{x^{\alpha+(n-1)\beta}}{na+b} = \frac{\beta}{a} x^{\frac{a\beta - a\beta - b\beta}{a}} \int x^{\frac{b}{a}} dx \frac{1 - x^{n}}{1 - x}.$$

And it must be after integration that $x$ is put equal to 1. As cited at the beginning of the first dissertation on summation I have set out.

§ 10. Let the proposed progression, of which the general term of order $n$ be

$$\frac{x^n}{(an+b)(cn+e)};$$

with this assumed, there is so much gained by taking $x^n$ in place of $x^{\alpha+(n-1)\beta}$, as then this power can easily be changed into that in the business of calculation. Let the summatory term be $s$, then

$$px^{\pi} s = \frac{px^{\pi+1}}{(a+b)(c+e)} + \ldots.. + \frac{px^{\pi+n}}{(an+b)(cn+e)}$$

and thus

$$\frac{diff.\,px^{\pi} s}{dx} = \frac{p(\pi+1)x^{\pi}}{(a+b)(c+e)} + \ldots.. + \frac{p(\pi+n)x^{\pi+n-1}}{(an+b)(cn+e)}.$$

Let

$$p\pi + pn = an + b;$$

then

$$p = a \text{ and } \pi = \frac{b}{a}.$$

Hence there is obtained :

$$\frac{ad(x^{\frac{b}{a}} s)}{dx} = \frac{x^{\frac{b}{a}}}{c+e} + \ldots + \frac{x^{\frac{b}{a}+n-1}}{cn+e}.$$

This is multiplied again by $px^{\pi}$ ; then

$$\frac{apx^{\pi} ad(x^{\frac{b}{a}} s)}{dx} = \frac{px^{\frac{b}{a}+\pi}}{c+e} + \ldots + \frac{px^{\frac{b}{a}+n+\pi-1}}{cn+e}$$

Hence there is produced :

$$\frac{apd(x^{\pi} d(x^{\frac{b}{a}} s))}{dx^2} = \frac{p(\frac{b}{a}+\pi)x^{\frac{b}{a}+\pi-1}}{c+e} + \ldots + \frac{p(\frac{b}{a}+n+\pi-1)x^{\frac{b}{a}+n+\pi-2}}{cn+e}.$$

There arises :

$$\frac{pb}{a}+pn+p\pi-p=cn+e\,;$$

then

$$p=c \text{ et } \pi=1-\frac{b}{a}+\frac{e}{c}.$$

With these in place this equation emerges :

$$\frac{acd(x^{1-\frac{b}{a}+\frac{e}{c}}d(x^{\frac{b}{a}}s))}{dx^2}=x^{\frac{e}{c}}+\ldots+x^{\frac{e}{c}+n-1}=x^{\frac{e}{c}}\frac{1-x^n}{1-x}.$$

Again integrations are taken ; then

$$\frac{acx^{1-\frac{b}{a}+\frac{e}{c}}d(x^{\frac{b}{a}}s)}{dx}=\int x^{\frac{e}{c}}dx\frac{1-x^n}{1-x}$$

and hence :

$$s=\frac{1}{acx^{\frac{b}{a}}}\int x^{\frac{b}{a}-\frac{e}{c}-1}dx\int x^{\frac{e}{c}}dx\frac{1-x^n}{1-x}=\frac{x^{\frac{b}{a}-\frac{e}{c}}\int x^{\frac{e}{c}}dx\frac{1-x^n}{1-x}-\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}}{(bc-ae)x^{\frac{b}{a}}}.$$

This case is to be noted, if $bc = ae$, because it makes $s=\frac{0}{0}$. But the nearby first form

$$s=\frac{1}{acx^{\frac{b}{a}}}\int\frac{dx}{x}\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x},$$

is changed into [on integrating by parts]:

$$s=\frac{lx\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}-\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}lx}{acx^{\frac{b}{a}}}.$$

Here the case arises, if the denominators $(an+b)(cn+e)$ are squares or certain multiples of these [i. e. $a = c$ and $b = e$ or $a = kc$, $b = ke$.] If $x = 1$, this substitution, as before can finally be made, after the integration into the quantities under the integral sign, as in the end $x$ can at once be put equal to 1. Hence the equation arises:

$$s=\frac{\int\left(x^{\frac{e}{c}}-x^{\frac{b}{a}}\right)dx\frac{1-x^n}{1-x}}{bc-ae}.$$

From which it is apparent, if $x^{\frac{e}{c}}-x^{\frac{b}{a}}$ can be divided by $1-x$, then the progression is algebraic. But in the case in which $bc = ae$, there arises $lx = 0$ on putting $x = 1$.
On account of which [in the above equation]

$$s=-\frac{\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}lx}{ac}.$$

§11. It is understood in the same manner, if $n$ has 3 or more dimensions in the denominator, that it is required to show how the sum ought to be found, and thus to shed light on the operation without the help of too many examples. Let this be the proposed progression, the general term of which is

$$\frac{x^n}{(an+b)(cn+e)(fn+g)};$$

let the sum of this progression be $s$. This progression handled in the same way as in the preceding §, and gives after two differentiations (§9) :

$$\frac{acd\left(x^{1-\frac{b}{a}+\frac{e}{c}}d\left(x^{\frac{b}{a}}s\right)\right)}{dx^2}=\frac{x^{\frac{e}{c}}}{f+g}+\ldots+\frac{x^{\frac{e}{c}+n-1}}{nf+g}=\frac{1}{f}x^{\frac{e}{c}-\frac{g}{f}-1}\int x^{\frac{g}{f}}dx\frac{1-x^n}{1-x};$$

the integrals are taken; then

$$\frac{acfx^{1-\frac{b}{a}+\frac{e}{c}}d\left(x^{\frac{e}{c}}s\right)}{dx}=\int x^{\frac{e}{c}-\frac{g}{f}-1}dx\int x^{\frac{g}{f}}dx\frac{1-x^n}{1-x}$$

and again :

$$acfx^{\frac{b}{a}}s=\int x^{\frac{b}{a}-\frac{e}{c}-1}dx\int x^{\frac{e}{c}-\frac{g}{f}-1}dx\int x^{\frac{g}{f}}dx\frac{1-x^n}{1-x}$$

and thus :

$$s=\frac{1}{acfx^{\frac{b}{a}}}\int x^{\frac{b}{a}-\frac{e}{c}-1}dx\int x^{\frac{e}{c}-\frac{g}{f}-1}dx\int x^{\frac{g}{f}}dx\frac{1-x^n}{1-x}.$$

No more integral signs are in placed in turn, and this form can be changed into the following :

$$s=\frac{fx^{-\frac{g}{f}}}{(bf-ag)(ef-cg)}\int x^{\frac{g}{f}}dx\frac{1-x^n}{1-x}+\frac{cx^{-\frac{e}{c}}}{(bc-ae)(cg-ef)}\int x^{\frac{e}{c}}dx\frac{1-x^n}{1-x}+\frac{ax^{-\frac{b}{a}}}{(ae-bc)(ag-bf)}\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}.$$

From this likewise it is apparent that if there are more factors in the general term, then the form of the sum shall have the same number. For let the general term be

$$\frac{x^n}{(an+b)(cn+e)(fn+g)(hn+k)};$$

then the summatory term is :

$$s=\frac{1}{acfhx^{\frac{b}{a}}}\int x^{\frac{b}{a}-\frac{e}{c}-1}dx\int x^{\frac{e}{c}-\frac{g}{f}-1}dx\int x^{\frac{g}{f}-\frac{k}{h}-1}dx\int x^{\frac{k}{h}}dx\frac{1-x^n}{1-x}$$

$$=\frac{ax^{-\frac{b}{a}}\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}}{(ae-bc)(ag-bf)(ak-bh)}+\frac{cx^{-\frac{e}{c}}\int x^{\frac{e}{c}}dx\frac{1-x^n}{1-x}}{(bc-ae)(cg-ef)(ck-eh)}$$

$$+\frac{fx^{-\frac{a}{f}}\int x^{\frac{g}{f}}dx\frac{1-x^n}{1-x}}{(bf-ag)(ef-cg)(fk-gh)}+\frac{hx^{-\frac{k}{h}}\int x^{\frac{k}{h}}dx\frac{1-x^n}{1-x}}{(bh-ak)(eh-ck)(gh-fk)}.$$

If the sum is desired in the case in which $x = 1$, then for the general term :

$$\frac{1}{(an+b)(cn+e)(fn+g)}$$

the summatory term is

$$s=\frac{\int dx\frac{1-x^n}{1-x}((aef-bcf)x^{\frac{g}{f}}+(bcf-acg)x^{\frac{e}{c}}+(acg-aef)x^{\frac{b}{a}})}{(ae-bc)(ag-bf)(cg-ef)}.$$

Therefore as often as the quantity multiplied in $dx\frac{1-x^n}{1-x}$ can be divided by $1-x$, then the proposed progression has an algebraic sum. This follows if $\frac{b}{a}-\frac{e}{c}$ and $\frac{e}{c}-\frac{g}{f}$ are whole numbers. In addition it should also be noted that all the progressions of this kind are either summable algebraically or by logarithms – whether they depend on real or imaginary numbers; nor can any other square of this kind be expressed in a progression of this kind.

§12. But these formulas are applied with difficulty to these cases in which the factors in the denominator are equal, here we are pleased to treat these by a special case. Thus let the general term of the progression to be summed be

$$\frac{x^n}{(an+b)^3}$$

and the summatory term is $s$; then

$$s=\frac{1}{a^3x^{\frac{b}{a}}}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x},$$

in which it follows from §11, where $c=f=a$ and $e=g=b$; the above form is changed into this :

$$s=\frac{\frac{1}{2}(lx)^2\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}-lx\int x^{\frac{b}{a}}dxlx\frac{1-x^n}{1-x}+\frac{1}{2}\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}(lx)^2}{a^3x^{\frac{b}{a}}}$$

But if the general term is :

$$\frac{x^n}{(an+b)^4},$$

then

$$s=\frac{\left[(lx)^3\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}-3(lx)^2\int x^{\frac{b}{a}}dxlx\frac{1-x^n}{1-x}+3lx\int x^{\frac{b}{a}}dx(lx)^2\frac{1-x^n}{1-x}-\int x^{\frac{b}{a}}dx(lx)^3\frac{1-x^n}{1-x}\right]}{6a^4x^{\frac{b}{a}}}$$

From these it appears, how $s$ is progressing for the remaining powers ; indeed generally, if the term is

$$\frac{x^n}{(an+b)^m}$$

then the sum is given by :

$$s=\frac{\left[(lx)^{m-1}\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}-\frac{m-1}{1}(lx)^{m-2}\int x^{\frac{b}{a}}dxlx\frac{1-x^n}{1-x}+\frac{m-1}{1}.\frac{m-2}{1}(lx)^{m-3}\int x^{\frac{b}{a}}dx(lx)^2\frac{1-x^n}{1-x}-\text{etc.}\right]}{1.2.3....(m-1)a^m x^{\frac{b}{a}}}$$

These values become much simpler if there is put $x = 1$; for then $lx = 0$. Indeed to the general term $\frac{1}{(an+b)^2}$ there corresponds this summatory term :

$$s=\frac{\int x^{\frac{b}{a}}dxl\frac{1}{x}\frac{1-x^n}{1-x}}{1.a^2},$$

to the general term $\frac{1}{(an+b)^3}$, this

$$s=\frac{\int x^{\frac{b}{a}}dx\left(l\frac{1}{x}\right)^2\frac{1-x^n}{1-x}}{1.2a^3}$$

and to the general term $\frac{1}{(an+b)^m}$, this

$$s=\frac{\int x^{\frac{b}{a}}dx\left(l\frac{1}{x}\right)^{m-1}\frac{1-x^n}{1-x}}{1.2.3......(m-1)a^m}=\frac{\int x^{\frac{b}{a}}dx\left(l\frac{1}{x}\right)^{m-1}\frac{1-x^n}{1-x}}{a^m\int dx\left(l\frac{1}{x}\right)^{m-1}},$$

which integrals must thus be taken, so that on putting $x = 0$ the whole sum vanishes; whereby then it is necessary to put $x = 1$ and the resulting quantity is now the sum. Again it is to be noted, if the sum is desired of the progression continued to infinity, it is necessary only to write everywhere $\frac{1}{1-x}$ in place of $\frac{1-x^n}{1-x}$.

§ 13. Now two classes of progressions have been handled, of which the one had the general term $Ax^n$, and now the other $\frac{x^n}{A}$ with $A$ denoting some constant algebraic quantity depending on $n$ and constants, yet thus, so that $n$ does not have any exponents except positive integers. From these there arises a third class having $\frac{Ax^n}{B}$ for the general term, where $A$ and $B$ describe the same kind of algebraic quantities. Such a progression is also reduced to a geometric progression with the numerator $A$ being removed with the help of integration, and the denominator $B$ with the help of differentiation, as had been done in each of these separately. Let the general term of the progression to be summed be

$$\frac{(\alpha n+\beta)x^n}{an+b},$$

the summatory term of which is put as $s$; then

$$s=\frac{(\alpha+\beta)x}{a+b}+.....+\frac{(\alpha n+\beta)x^n}{an+b}.$$

This equation is multiplied by $px^\pi$; then

$$px^\pi s=\frac{p(\alpha+\beta)x^{\pi+1}}{a+b}+.....+\frac{p(\alpha n+\beta)x^{n+\pi}}{an+b};$$

the differentials are taken; then

$$pd(x^{\pi}s)=\frac{p(\pi+1)(\alpha+\beta)x^{\pi}dx}{a+b}+.....+\frac{p(n+\pi)(\alpha n+\beta)x^{n+\pi-1}dx}{an+b};$$

there arises

$$pn+p\pi=an+b;$$

hence

$$p=a \text{ and } \pi=\frac{b}{a}.$$

Hence

$$pd(x^{\frac{b}{a}}s)=(\alpha+\beta)x^{\frac{b}{a}}dx+.....+(\alpha n+\beta)x^{\frac{b}{a}+n-1}dx.$$

Again the equation is multiplied by $px^{\pi}$ ; then

$$apx^{n}pd(x^{\frac{b}{a}}s)=p(\alpha+\beta)x^{\frac{b}{a}+\pi}dx+.....+p(\alpha n+\beta)x^{\frac{b}{a}+\pi+n-1}dx$$

The integrals are taken; there is obtained:

$$ap\int x^{n}pd(x^{\frac{b}{a}}s)=\frac{p(\alpha+\beta)x^{\frac{b}{a}+\pi+1}}{b+a\pi+a}+.....+\frac{p(\alpha n+\beta)x^{\frac{b}{a}+\pi+n}}{b+a\pi+an}.$$

Let

$$a\alpha pn+a\beta p=an+a\pi+b$$

then

$$p=\frac{1}{\alpha} \text{ and } \pi=\frac{\beta}{\alpha}-\frac{b}{a}.$$

On this account

$$\frac{a}{\alpha}\int x^{\frac{\beta}{\alpha}-\frac{b}{a}}d(x^{\frac{b}{a}}s)=x^{\frac{b}{a}+1}+.....+x^{\frac{b}{a}+n}=x^{\frac{\beta}{\alpha}+1}\frac{1-x^{n}}{1-x}.$$

From this equation there is produced :

$$s=\frac{\alpha\int x^{\frac{\beta}{\alpha}-\frac{b}{a}}d\left(x^{\frac{\beta+\alpha}{a}}\frac{1-x^{n}}{1-x}\right)}{ax^{\frac{b}{a}}}.$$

If the general term is :

$$\frac{(\alpha n+\beta)(\gamma n+\delta)x^{n}}{an+b}$$

and of which the summatory term is put as $s$, there is produced from the same absolute operations in whichever way :

$$\frac{a}{\alpha}\int x^{\frac{\beta}{\alpha}-\frac{b}{a}}d(x^{\frac{b}{a}}s)=(\gamma+\delta)x^{\frac{b}{a}+1}+.....+(\gamma n+\delta)x^{\frac{b}{a}+n}$$

again this is multiplied by $px^{\pi}dx$ and the integrals are taken ; there is produced :

$$\frac{ap}{\alpha}\int x^{n}dx\int x^{\frac{\beta}{\alpha}-\frac{b}{a}}d(x^{\frac{b}{a}}s)=\frac{\alpha p(\gamma+\delta)x^{\frac{b}{a}+\pi+2}}{\beta+\alpha\pi+2\alpha}+.....+\frac{\alpha p(\gamma n+\delta)x^{\frac{b}{a}+\pi+n+1}}{\beta+\alpha\pi+\alpha n+\alpha}.$$

Let

$$a\gamma pn + a\delta p = \alpha + \beta + \alpha\pi + \alpha n;$$

then

$$p = \frac{1}{\gamma} \text{ and } \pi = \frac{\delta}{\gamma} - \frac{\beta}{\alpha} - 1.$$

Hence

$$\frac{a}{\alpha\gamma}\int x^{\frac{\delta}{\gamma}-\frac{\beta}{\alpha}-1}dx\int x^{\frac{\beta}{\alpha}-\frac{b}{a}}d(x^{\frac{b}{a}}s) = x^{\frac{\delta}{\gamma}+1} + \ldots.. + x^{\frac{\delta}{\gamma}+n} = x^{\frac{\delta}{\gamma}+1}\frac{1-x^n}{1-x}.$$

Whereby

$$s = \frac{\alpha\gamma\int x^{\frac{b}{a}-\frac{\beta}{\alpha}}d(x^{\frac{\beta}{a}-\frac{\delta}{\gamma}+1})d(x^{\frac{\delta}{\gamma}+1}\frac{1-x^n}{1-x})}{ax^{\frac{b}{a}}dx}.$$

But I will not tarry any longer in the summation of progressions of this kind ; for it suffices to have treated the method by which all these can be summed. Yet meanwhile and that still prevails, as I said in § 11, clearly all the progressions of this kind can be summed either algebraically or a sum formed that depends on either real or imaginary logarithms [relating to the inverse trig. function].

§14. Now I progress to another kind of progression, the general terms of which cannot be expressed algebraically, but which belong to the class of hypergeometric series. A series of this kind is :

$$(\alpha+\beta)x + (\alpha+\beta)(2\alpha+\beta)x^2 + \ldots.. + (\alpha+\beta)(2\alpha+\beta)\ldots(\alpha n+\beta)x^n.$$

The sum of this series is put as $s$ and it is multiplied by $px^{\pi}$ ; then

$$px^{\pi}s = p(\alpha+\beta)x^{\pi+1} + \ldots.. p(\alpha+\beta)(2\alpha+\beta)\ldots(\alpha n+\beta)x^{n+\pi}$$

and the integral of this multiplied by $dx$ :

$$p\int x^{\pi}sdx = \frac{p(\alpha+\beta)x^{\pi+2}}{\pi+2} + \ldots.. \frac{p(\alpha+\beta)(2\alpha+\beta)\ldots(\alpha n+\beta)x^{n+\pi+1}}{n+\pi+1}$$

Let

$$p\alpha n + p\beta = n + \pi + 1;$$

then

$$p = \frac{1}{\alpha} \text{ et } \pi = \frac{\beta}{\alpha} - 1,$$

and thus there is produced :

$$\frac{1}{\alpha}\int x^{\frac{\beta}{\alpha}-1}sdx = x^{\frac{\beta}{\alpha}+1} + (\alpha+\beta)x^{\frac{\beta}{\alpha}+2} + \ldots.. + (\alpha+\beta)(2\alpha+\beta)\ldots(\alpha(n-1)+\beta)x^{\frac{\beta}{\alpha}+n}.$$

This is divided by $x^{\frac{\beta}{\alpha}+1}$; there is obtained :

$$\frac{\int x^{\frac{\beta}{\alpha}-1}sdx}{\alpha x^{\frac{\beta}{\alpha}+1}}-1=(\alpha+\beta)x+.....+(\alpha+\beta)(2\alpha+\beta)....(\alpha(n-1)+\beta)x^{n-1}.$$

Which is the proposed progression with the final term truncated. Therefore this expression is :

$$\frac{\int x^{\frac{\beta}{\alpha}-1}sdx}{\alpha x^{\frac{\beta}{\alpha}+1}}-1=s-(\alpha+\beta)x+.....+(\alpha+\beta)(2\alpha+\beta)....(\alpha n+\beta)x^{n}=s-Ax^{n}.$$

But I have already set out forms for this expression presented in another dissertation concerning the general terms of transcendental progressions [See E19 in this series of translations, especially § 16.], from which, if it pleases, a finite value can be chosen in place $A$. Hence we have :

$$\int x^{\frac{\beta}{\alpha}-1}sdx=\alpha x^{\frac{\beta}{\alpha}+1}+\alpha x^{\frac{\beta}{\alpha}+1}s-\alpha Ax^{\frac{\beta}{\alpha}+n+1}$$

and

$$x^{\frac{\beta}{\alpha}-1}sdx=(\alpha+\beta)x^{\frac{\beta}{\alpha}}dx+(\alpha+\beta)x^{\frac{\beta}{\alpha}}sdx+\alpha x^{\frac{\beta}{\alpha}+1}ds-(\alpha+\beta+\alpha n)Ax^{\frac{\beta}{\alpha}+n}dx$$

or

$$sdx=(\alpha+\beta)xdx+(\alpha+\beta)xsdx+\alpha x^{2}ds-(\alpha+\beta+\alpha n)Ax^{n+1}dx.$$

From this equation the value of $s$ itself can be elicited and gives the sum of the proposed equation. For this can also be accomplished, so that the factors in the following term are increased not by one, but by two or more. There are always two new factors added, so that this progression is produced :

$$(\alpha+\beta)x+(\alpha+\beta)(2\alpha+\beta)(3\alpha+\beta)x^{2}+....$$
$$+(\alpha+\beta)(2\alpha+\beta).....(\alpha(2n-1)+\beta)x^{n}.$$

The sum of this progression is called $s$; then

$$p\int x^{\pi}sdx=\frac{p(\alpha+\beta)x^{\pi+2}}{\pi+2}+.....+\frac{p(\alpha+\beta)(2\alpha+\beta)....(\alpha(2n-1)+\beta)x^{n+\pi+1}}{n+\pi+1}.$$

Let

$$2p\alpha n-p\alpha+p\beta=n+\pi+1;$$

then

$$p=\frac{1}{2\alpha}\text{ et }\pi=\frac{\beta-3\alpha}{2\alpha}.$$

Thus

$$\frac{\int x^{\frac{\beta-3\alpha}{2\alpha}}sdx}{2\alpha}=x^{\frac{\beta+\alpha}{2\alpha}}+.....+(\alpha+\beta)(2\alpha+\beta)....(\alpha(2n-2)+\beta)x^{\frac{n+\beta-\alpha}{s\alpha}}.$$

And again :

$$\frac{p\int x^{\pi}dx\int x^{\frac{\beta-3\alpha}{2\alpha}}sdx}{2\alpha}=\frac{2\alpha px^{\frac{\beta+\alpha}{2\alpha}+\pi}}{\beta+3\alpha+2\alpha\pi}+.....+\frac{2\alpha p(\alpha+\beta)....(\alpha(2n-2)+\beta)x^{n+\pi+\frac{\beta+\alpha}{s\alpha}}}{\beta+\alpha+2\alpha n+2\alpha\pi}.$$

Let

$$4p\alpha^2 n - 4p\alpha^2 + 2p\alpha\beta = 2\alpha n + 2\alpha\pi + \alpha + \beta\ ;$$

then

$$p = \frac{1}{2\alpha} \text{ and } \pi = \frac{\beta - 2\alpha}{2\alpha} - \frac{\alpha+\beta}{2\alpha} = -\frac{3}{2},$$

consequently

$$\frac{\int x^{-\frac{3}{2}} dx \int x^{\frac{\beta-3\alpha}{2\alpha}} sdx}{4\alpha^2 x^{\frac{\beta}{2\alpha}}} - \frac{1}{\beta} = (\alpha+\beta)x + \ldots + (\alpha+\beta)(2\alpha+\beta)..(\alpha(2n-3)+\beta)x^{n-1}$$

$$= s - Ax^n.$$

Put

$$A = (\alpha+\beta)(2\alpha+\beta)\ldots + (\alpha(2n-1)+\beta).$$

From which equation *s* becomes known.

§ 15. The method of operation must be set up in a similar way if three or more factors are added in the coefficients of the following terms from the start. From what has been established, the whole sum always has to be joined with the resulting equation under the integral sign, as many times as there are factors, and by which each following term is increased. Thus for the progression :

$$(\alpha+\beta)x + \ldots.. + (\alpha+\beta)(2\alpha+\beta)\ldots(\alpha(3n-2)+\beta)x^n$$

the sum *s* is determined from this equation :

$$\frac{\int x^{-\frac{4}{3}} dx \int x^{-\frac{4}{3}} dx \int x^{\frac{\beta-5\alpha}{3\alpha}} sdx}{27\alpha^3 x^{\frac{\beta-\alpha}{3\alpha}}} - \frac{1}{\beta(\beta-\alpha)} = s - (\alpha+\beta)\ldots(\alpha(3n-2)+\beta)x^n.$$

From which, as an introduction to the following case can be made, it is noted that $\frac{1}{\beta(\beta-\alpha)}$ is the term of the proposed progression before the first, or that for which the index is 0. If the factors which are multiplied by the powers of *x*, do not constitute an arithmetical progression, but an algebraic expression of a higher order, an operation must likewise be established; for let the proposed progression be:

$$(\alpha+\beta)(\gamma+\delta)x + \ldots.. + (\alpha+\beta)(2\alpha+\beta)\ldots(\alpha n+\beta)(\gamma+\delta)(2\gamma+\delta)\ldots(\gamma n+\delta)x^n\ ;$$

and the sum of this is put as *s*; then

$$p\int x^{\pi} sdx = \frac{p(\alpha+\beta)(\gamma+\delta)x^{\pi+2}}{\pi+2} + \ldots.. + \frac{p(\alpha+\beta)\ldots(\alpha n+\beta)(\gamma+\delta)\ldots\ldots(\gamma+\delta)(\gamma n+\delta)x^{n+\pi+1}}{n+\pi+1}.$$

There is put

$$p\gamma n + p\delta = n + \pi + 1;$$

then

$$p = \frac{1}{\gamma} \text{ and } \pi = \frac{\gamma-\delta}{\gamma}.$$

Hence,

$$\frac{\int x^{\frac{\delta-\gamma}{\gamma}}sdx}{\gamma}=(\alpha+\beta)x^{\frac{\delta+\gamma}{\gamma}}+.....+(\alpha+\beta)....(\alpha n+\beta)(\gamma+\delta)......(\gamma+\delta)(\gamma(n-1)+\delta)x^{n+\frac{\delta}{\gamma}}.$$

Now again :

$$\frac{p\int x^{\pi}dx\int x^{\frac{\delta-\gamma}{\gamma}}sdx}{\gamma}=\frac{\gamma p(\alpha+\beta)x^{\frac{\delta+2\gamma}{\gamma}+\pi}}{\delta+2\gamma+\pi\gamma}+.....+\frac{\gamma p(\alpha+\beta)....(\alpha n+\beta)(\gamma+\delta)......(\gamma+\delta)(\gamma(n-1)+\delta)x^{n+\pi+\frac{\delta+\gamma}{\gamma}}}{\delta n+\pi\gamma+\delta+\gamma}.$$

Let

$$p\alpha\gamma n+p\beta\gamma=\gamma n+n\gamma+\delta+\gamma\ ;$$

then

$$p=\frac{1}{\alpha}\text{ and }\pi=\frac{\beta}{\alpha}-\frac{\delta}{\gamma}-1=\frac{\beta\gamma-\alpha\delta-\alpha\gamma}{\alpha\gamma}.$$

Hence

$$\frac{\int x^{\frac{\beta\gamma-\alpha\delta-\alpha\gamma}{\alpha\gamma}}dx\int x^{\frac{\delta-\gamma}{\gamma}}sdx}{\alpha\gamma}=x^{\frac{\beta+\alpha}{\alpha}}+.....+(\alpha+\beta)....(\alpha(n-1)+\beta)(\gamma+\delta)......(\gamma+\delta)(\gamma(n-1)+\delta)x^{\frac{\beta}{\alpha}+n}.$$

Consequently

$$\frac{\int x^{\frac{\beta\gamma-\alpha\delta-\alpha\gamma}{\alpha\gamma}}dx\int x^{\frac{\delta-\gamma}{\gamma}}sdx}{\alpha\gamma x^{\frac{\beta+\alpha}{\alpha}}}-1=s-ABx^{n},$$

on putting

$$A=(\alpha+\beta)....(\alpha n+\beta)\text{ et }B=(\gamma+\delta)......(\gamma+\delta)(\gamma n+\delta).$$

This is the case, if the general term made by the factors is :

$$(\alpha n+\beta)(\gamma n+\delta)\text{ or }\alpha\gamma n^{2}(\alpha\delta+\beta\gamma)n+\beta\delta\ .$$

Hence all progressions of the second order are understood under this form. Moreover the above formula, from which $s$ is determined, is changed into this :

$$\frac{\int x^{\frac{\delta-\gamma}{\gamma}}sdx}{(\beta\gamma-\alpha\delta)x^{\frac{\delta+\gamma}{\gamma}}}+\frac{\int x^{\frac{\beta-\alpha}{\alpha}}sdx}{(\alpha\delta-\beta\gamma)x^{\frac{\beta+\alpha}{\alpha}}}=1+s-ABx^{n}.$$

From which the following form can be more easily understood.

§ 16. I will now consider reciprocals of these series, in which the powers of $x$ are divided by that, by which before they were multiplied. Therefore let this be the series to be summed :

$$\frac{x}{\alpha+\beta}+\frac{x^2}{(\alpha+\beta)(2\alpha+\beta)}+\ldots\ldots+\frac{x^n}{(\alpha+\beta)(2\alpha+\beta)\ldots(\alpha n+\beta)};$$

of which the sum is put as $s$. then

$$\frac{pd\left(x^{\pi}s\right)}{dx}=\frac{p(\pi+1)x^{\pi}}{(\alpha+\beta)}+\frac{p(\pi+2)x^{\pi+1}}{(2\alpha+\beta)}+\ldots\ldots+\frac{p(\pi+n)x^{\pi+n-1}}{(\alpha+\beta)\ldots(\alpha n+\beta)}.$$

Let

$$pn+p\pi=\alpha n+\beta;$$

then

$$p=\alpha \text{ and } \pi=\frac{\beta}{\alpha}.$$

On this account,

$$\frac{pd(x^{\frac{\beta}{\alpha}}s)}{dx}=x^{\frac{\beta}{\alpha}}+\frac{x^{\frac{\beta}{\alpha}+1}}{(\alpha+\beta)}+\ldots\ldots+\frac{x^{\frac{\beta}{\alpha}+n-1}}{(\alpha+\beta)(2\alpha+\beta)\ldots(\alpha(n-1)+\beta)}$$

and hence

$$\frac{pd(x^{\frac{\beta}{\alpha}}s)}{x^{\frac{\beta}{\alpha}}dx}=1+s-\frac{x^n}{A}$$

on putting, as before,

$$A=(\alpha+\beta)\ldots(\alpha n+\beta).$$

This gives the extricated equation:

$$\alpha x^{\frac{\beta}{\alpha}}ds+\beta x^{\frac{\beta-\alpha}{\alpha}}sdx=x^{\frac{\beta}{\alpha}}dx+x^{\frac{\beta}{\alpha}}sdx-\frac{x^{\frac{\beta}{\alpha}+n}dx}{A},$$

which divided by $x^{\frac{\beta}{\alpha}-1}$ changes into :

$$\alpha xds+\beta sdx=xdx+xsdx-\frac{x^{n+1}dx}{A}$$

or

$$ds+\frac{\beta sdx}{\alpha x}-\frac{sdx}{\alpha}=\frac{dx}{\alpha}-\frac{x^n dx}{A\alpha}.$$

This equation is multiplied by $c^{-\frac{x}{\alpha}}x^{\frac{\beta}{\alpha}}$, where $c$ is the number of which the logarithm is 1; with that done and integrated there is produced :

$$c^{-\frac{x}{\alpha}}x^{\frac{\beta}{\alpha}}s=\frac{1}{\alpha}\int c^{-\frac{x}{\alpha}}x^{\frac{\beta}{\alpha}}dx(1-\frac{x^n}{A})$$

and

$$s=\frac{1}{\alpha}c^{\frac{x}{\alpha}}x^{\frac{-\beta}{\alpha}}\int c^{-\frac{x}{\alpha}}x^{\frac{\beta}{\alpha}}dx(1-\frac{x^n}{A}).$$

The sum of this progression continued to infinity therefore is :

$$\frac{1}{\alpha}c^{\frac{x}{\alpha}}x^{\frac{-\beta}{\alpha}}\int c^{-\frac{x}{\alpha}}x^{\frac{\beta}{\alpha}}dx=\beta(\beta-\alpha)(\beta-2\alpha)\alpha c^{\frac{x}{\alpha}}x^{\frac{-\beta}{\alpha}}$$
$$-1-\frac{\beta}{x}-\frac{\beta(\beta-\alpha)}{x^2}-\ldots\ldots-\frac{\beta(\beta-\alpha)\ldots\ldots\alpha}{x^{\frac{\beta}{\alpha}}}.$$

If $\beta=0$, then the sum becomes [on integrating the exponential]:

$$=c^{\frac{x}{\alpha}}-1.$$

But if $\beta=\alpha$, then the sum is :

$$=\frac{\alpha c^{\frac{x}{\alpha}}}{x}-1-\frac{\alpha}{x}.$$

Now if there is put $\beta=2\alpha$, then the sum of the series is :

$$=\frac{2\alpha^2c^{\frac{x}{\alpha}}}{x^2}-1-\frac{2\alpha}{x}-\frac{2\alpha^2}{x^2}$$

and thus henceforth. From which it is understood, as many times as $\beta$ is a multiple of $\alpha$, the sum of the series can be shown from a finite and whole expression. But if $\frac{\beta}{\alpha}$ is not a proper fraction, then it is not possible to find a formula.

§ 17. With each term increased by two factors, this progression is obtained :

$$\frac{x}{\alpha+\beta}+\frac{x^2}{(\alpha+\beta)..(3\alpha+\beta)}+\frac{x^3}{(\alpha+\beta)..(5\alpha+\beta)}+\ldots..+\frac{x^n}{(\alpha+\beta)\ldots(\alpha(2n-1)+\beta)}.$$

The sum of which is put as *s*; then

$$\frac{pd(x^\pi s)}{dx}=\frac{p(\pi+1)x^\pi}{(\alpha+\beta)}+\frac{p(\pi+2)x^{\pi+1}}{(\alpha+\beta)\ldots(3\alpha+\beta)}+\ldots..+\frac{p(\pi+n)x^{\pi+n-1}}{(\alpha+\beta)\ldots(\alpha(2n-1)+\beta)}.$$

Let

$$pn+p\pi=2\alpha n-\alpha+\beta;$$

then

$$p=2\alpha \text{ and } \pi=\frac{\beta-\alpha}{2\alpha},$$

and on this account :

$$\frac{2\alpha d(x^{\frac{\beta-\alpha}{2\alpha}}s)}{dx}=x^{\frac{\beta-\alpha}{2\alpha}}+\frac{x^{\frac{\beta+\alpha}{2\alpha}}}{(\alpha+\beta)(2\alpha+\beta)}+\ldots..+\frac{x^{\frac{\beta-3\alpha}{\alpha}+n}}{(\alpha+\beta)\ldots(\alpha(2n-2)+\beta)}.$$

And again:

$$\frac{2\alpha pd(x^\pi d(x^{\frac{\beta-\alpha}{2\alpha}}s))}{dx^2}=\frac{p(\beta-\alpha+2\alpha\pi)}{2\alpha}x^{\frac{\beta-3\alpha}{2\alpha}+\pi}+\ldots..+\frac{p(\beta-3\alpha+2\alpha n+2\alpha\pi)x^{\frac{\beta-5\alpha}{\alpha}+n+\pi}}{2\alpha(\alpha+\beta)(2\alpha+\beta)\ldots(\alpha(2n-2)+\beta)}.$$

Let

$$p\beta-3p\alpha+2p\alpha n+2p\alpha\pi=4\alpha^2n-4\alpha^2+2\alpha\beta;$$

then

$$p = 2\alpha \text{ and } \pi = \tfrac{1}{2}.$$

Thus there is produced :

$$\frac{4\alpha^2 d(x^{\frac{1}{2}} d(x^{\frac{\beta-\alpha}{2\alpha}} s))}{dx^2} = \beta x^{\frac{\beta-2\alpha}{2\alpha}} + \ldots\ldots + \frac{x^{\frac{\beta-4\alpha}{2\alpha}+n}}{(\alpha+\beta)\ldots(\alpha(2n-3)+\beta)}$$
$$= \beta x^{\frac{\beta-2\alpha}{2\alpha}} + x^{\frac{\beta-2\alpha}{2\alpha}} s - \frac{x^{\frac{\beta-2\alpha}{2\alpha}+n}}{(\alpha+\beta)\ldots(\alpha(2n-1)+\beta)}.$$

The operation is to be put in place in a similar way, if each term increases by more factors in the denominator. But also it appears not to be satisfactory how the equation for that sum to be reached can be determined, if the progression made from the factors in the denominators that is not arithmetic but is an algebraic expression of some higher order. Clearly any factor made from simple factors can be resolved, as has been done in § 15, where the general term of the factors is $(\alpha n+\beta)(\gamma n+\delta)$, under which all the equations of the second order are embraced. But even this is not a help, if the following method of working should be wished : For let the proposed progression be :

$$\frac{x}{1} + \frac{x^2}{1.7} + \frac{x^3}{1.7.17} + \frac{x^4}{1.7.17.31} + \ldots\ldots + \frac{x^n}{1.7\ldots(2n^2-1)}$$

of which the sum is put as $s$; then

$$\frac{pd(x^\pi s)}{dx} = p(\pi+1)x^\pi + \ldots\ldots + \frac{p(\pi+n)x^{n+\pi-1}}{1.7\ldots(2n^2-1)}.$$

And again :

$$\frac{pd(x^\rho(x^\pi s))}{dx^2} = p(\pi+1)(\pi+\rho)x^{\pi+\rho-1} + \ldots\ldots + \frac{p(\pi+n)(n+\pi+\rho-1)(\pi+n)x^{n+\pi+\rho-2}}{1.7\ldots(2n^2-1)}.$$

Let

$$pn^2 + 2p\pi n + p\rho n - pn + p\pi^2 + p\pi\rho - p\pi = 2n^2 - 1;$$

then

$$p = 2,\ 4\pi + 2\rho - 2 = 0 \text{ seu } \rho = 1 - 2\pi$$

and

$$-2\pi^2 = -1 \text{ seu } \pi = \sqrt{\tfrac{1}{2}} \text{ et } \rho = 1 - \sqrt{2}.$$

Whereby there is obtained :

$$\frac{2d(x^{1-\sqrt{2}}(x^{\sqrt{\frac{1}{2}}} s))}{dx^2} = x^{\sqrt{\frac{1}{2}}} + \ldots\ldots + \frac{x^{\frac{2n-2-\sqrt{2}}{2}}}{1.7\ldots(2n^2-4n+1)}$$
$$= x^{\frac{-\sqrt{2}}{2}} + x^{\frac{-\sqrt{2}}{2}}\left(s - \frac{x^n}{1.7\ldots(2n^2-1)}\right).$$

Now if the sum of this series continued to infinity then is found from this equation :

$$\left(2-2\sqrt{2}\right)x^{-\sqrt{2}}d(x^{\sqrt{\frac{1}{2}}}s)+2x^{1-\sqrt{2}}dd(x^{\sqrt{\frac{1}{2}}}s)=\left(2-2\sqrt{2}\right)x^{\frac{-\sqrt{2}}{2}}dxds+\left(\sqrt{2}-2\right)x^{\frac{-2-\sqrt{2}}{2}}sdx^2$$
$$+2x^{\frac{2-\sqrt{2}}{2}}dds+2\sqrt{2}x^{\frac{-\sqrt{2}}{2}}dxds+\left(1-\sqrt{2}\right)x^{\frac{-2-\sqrt{2}}{2}}sdx^2$$
$$=x^{\frac{-\sqrt{2}}{2}}dx^2+x^{\frac{-\sqrt{2}}{2}}sdx^2=2x^{\frac{-\sqrt{2}}{2}}dsdx-x^{\frac{-2-\sqrt{2}}{2}}sdx^2+2x^{\frac{2-\sqrt{2}}{2}}dds$$

or

$$2xdds-\frac{sdx^2}{x}+2dsdx=dx^2+sdx^2,$$

from which equation all irrationality vanishes.

§ 18. If the factors of the denominators constitute a progression of powers, then I will investigate sums of progressions of this kind. Let the proposed progression be :

$$\frac{x}{(\alpha+\beta)^2}+\frac{x^2}{(\alpha+\beta)^2(2\alpha+\beta)^2}+.....+\frac{x^n}{(\alpha+\beta)^2....(n\alpha+\beta)^2};$$

and the sum is put as $s$; then

$$\frac{pd(x^{\pi}s)}{dx}=\frac{p(\pi+1)x^{\pi}}{(\alpha+\beta)^2}+.....+\frac{p(\pi+n)x^{n+\pi-1}}{(\alpha+\beta)^2....(n\alpha+\beta)^2};$$

let

$$p\pi+pn=\alpha n+\beta;$$

then

$$p=\alpha \text{ and } \pi=\frac{\beta}{\alpha}.$$

Therefore

$$\frac{\alpha d(x^{\frac{\beta}{\alpha}}s)}{dx}=\frac{x^{\frac{\beta}{\alpha}}}{\alpha+\beta}+.....+\frac{x^{\frac{\beta}{\alpha}+n-1}}{(\alpha+\beta)^2....(n\alpha+\beta)}.$$

Again,

$$\frac{\alpha pd(x^{\pi}d(x^{\frac{\beta}{\alpha}}s))}{dx^2}=\frac{p(\beta+\alpha\pi)x^{\frac{\beta}{\alpha}+\pi-1}}{\alpha(\alpha+\beta)}+.....+\frac{p(\beta+\alpha\pi+\alpha n-\alpha)x^{\frac{\beta}{\alpha}+\pi+n-2}}{\alpha(\alpha+\beta)^2....(\alpha n+\beta)}.$$

Let

$$p\alpha n+p\beta+p\alpha\pi-p\alpha=\alpha^2 n+\alpha\beta,$$

then

$$p=\alpha \text{ et } \pi=\frac{\alpha}{\alpha}=1.$$

Thus

$$\frac{\alpha^2 d(xd(x^{\frac{\beta}{\alpha}}s))}{dx^2}=x^{\frac{\beta}{\alpha}}+.....+\frac{x^{\frac{\beta}{\alpha}+n-1}}{(\alpha+\beta)^2....(\alpha(n-1)+\beta)^2}$$
$$=x^{\frac{\beta}{\alpha}}+x^{\frac{\beta}{\alpha}}(s-\frac{x^n}{(\alpha+\beta)^2....(\alpha n+\beta)^2}).$$

And the sum of the progression to infinity can be determined from the equation :

$$\frac{\alpha^2 d(xd(x^{\frac{\beta}{\alpha}}s))}{x^{\frac{\beta}{\alpha}}dx^2} = 1+s.$$

Similarly if the factors were cubes, the sum $s$ of the progression

$$\frac{x}{(\alpha+\beta)^3} + \frac{x^2}{(\alpha+\beta)^3(2\alpha+\beta)^3} + \text{etc. to infinity}$$

can be found from this equation :

$$\frac{\alpha^3 \mathrm{d}(xd(xd(x^{\frac{\beta}{\alpha}}s)))}{x^{\frac{\beta}{\alpha}}dx^3} = 1+s.$$

And so on for what follows.

§ 19. Now let the coefficients of the powers of $x$ be fractions, of which the numerators as well as the denominators are made from a certain number of factors for any index of the term by a constant increase. Thus let the proposed progression be this:

$$\frac{a+b}{\alpha+\beta}x + \frac{(a+b)(2a+b)}{(\alpha+\beta)(2\alpha+\beta)}x^2 + \ldots.. + \frac{(a+b)\ldots.(an+b)}{(\alpha+\beta)\ldots.(\alpha n+\beta)}x^n;$$

and the sum of this is put as $s$; then

$$p\int x^{\pi} sdx = \frac{p(a+b)}{(\pi+2)(\alpha+\beta)}x^{\pi+2} + \ldots.. + \frac{p(a+b)\ldots.(an+b)}{(\pi+n+1)(\alpha+\beta)\ldots.(\alpha n+\beta)}x^{\pi+n+1}.$$

Let

$$\alpha pn + bp = \pi + n + 1;$$

then

$$p = \frac{1}{a} \text{ and } \pi = \frac{b-a}{a}$$

and thus

$$\frac{\int x^{\frac{b-a}{a}} sdx}{a} = \frac{x^{\frac{b+a}{a}}}{\alpha+\beta} + \ldots.. + \frac{(a+b)\ldots.(a(n-1)+b)}{(\alpha+\beta)\ldots.(\alpha n+\beta)}x^{\frac{b}{a}+n}.$$

And again

$$\frac{pd(x^{\pi}\int x^{\frac{b-a}{a}} sdx)}{adx} = \frac{p(b+a+a\pi)}{a(\alpha+\beta)}x^{\frac{b}{a}+\pi} + \ldots.. + \frac{p(b+an+a\pi)(a+b)\ldots.(a(n-1)+b)}{a(\alpha+\beta)\ldots.(\alpha n+\beta)}x^{\frac{b}{a}+\pi+n-1}.$$

Let

$$bp + apn + ap\pi = a\alpha n + a\beta;$$

then

$$p = \alpha \text{ and } \pi = \frac{\beta}{\alpha} - \frac{b}{a},$$

thus

$$\frac{\alpha d(x^{\frac{\beta}{\alpha}-\frac{b}{a}}\int x^{\frac{b-a}{a}} sdx)}{adx} = x^{\frac{b}{a}} + \ldots\ldots + \frac{(a+b)\ldots(a(n-1)+b)}{(\alpha+\beta)\ldots(\alpha(n-1)+\beta)} x^{\frac{b}{a}+n-1} = x^{\frac{b}{a}} + x^{\frac{b}{a}}\left(s - \frac{(a+b)\ldots(a(n-1)+b)}{(\alpha+\beta)\ldots(\alpha(n-1)+\beta)} x^{n}\right).$$

It is possible to determine $s$ from this equation. If the sum of the progression to infinity is desired, then

$$\frac{\alpha d(x^{\frac{\beta}{\alpha}-\frac{b}{a}}\int x^{\frac{b-a}{a}} sdx)}{adx} = x^{\frac{b}{a}} + x^{\frac{b}{a}} s$$

or

$$\frac{\alpha}{a}(\frac{\beta}{\alpha} - \frac{b}{a})x^{\frac{\beta}{\alpha}-\frac{b}{a}-1}\int x^{\frac{b-a}{a}} sdx + \frac{\alpha}{a} x^{\frac{\beta}{\alpha}-1} s = x^{\frac{\beta}{\alpha}} + x^{\frac{\beta}{\alpha}} s,$$

which becomes this :

$$(\frac{\beta}{a} - \frac{\alpha b}{aa})x^{-\frac{b}{a}}\int x^{\frac{b-a}{a}} sdx + \frac{\alpha}{a} s = x + xs$$

or this :

$$(\frac{\beta}{a} - \frac{\alpha b}{a^2})\int x^{\frac{b-a}{a}} sdx + \frac{\alpha}{a} s = x^{\frac{b+a}{a}} + x^{\frac{b+a}{a}} s.$$

This differentiated gives :

$$(\frac{\beta}{a} - \frac{\alpha b}{a^2})x^{\frac{b-a}{a}} sdx + \frac{\alpha}{a} x^{\frac{b}{a}} ds + \frac{\alpha b}{a^2} x^{\frac{b-a}{a}} sdx = \frac{b+a}{a} x^{\frac{b}{a}} dx + x^{\frac{b+a}{a}} ds + \frac{b+a}{a} x^{\frac{b}{a}} sdx,$$

which is reduced to this :

$$\frac{\beta}{a} sdx + \frac{\alpha}{a} xds = \frac{b+a}{a} xdx + x^2 ds + \frac{b+a}{a} xsdx$$

or

$$ds + \frac{\beta sdx-(b+a)xsdx}{\alpha x-ax^2} = \frac{(b+a)xdx}{\alpha x-ax^2}.$$

This equation is multiplied by :

$$c^{\int \frac{\beta dx-(b+a)xdx}{\alpha x-ax^2}} \text{ or by } x^{\frac{\beta}{\alpha}}(\alpha - ax)x^{\frac{b}{a}-\frac{\beta}{\alpha}+1}$$

then

$$x^{\frac{\beta}{\alpha}}(\alpha\text{-}ax)x^{\frac{b}{a}-\frac{\beta}{\alpha}+1} s = (b+a)\int x^{\frac{\beta}{\alpha}}(\alpha\text{-}ax)^{\frac{b}{a}-\frac{\beta}{\alpha}} dx$$

and

$$s = \frac{(b+a)\int x^{\frac{\beta}{\alpha}}(\alpha\text{-}ax)^{\frac{b}{a}-\frac{\beta}{\alpha}} dx}{x^{\frac{\beta}{\alpha}}(\alpha\text{-}ax)x^{\frac{b}{a}-\frac{\beta}{\alpha}+1}}.$$

Therefore an algebraic sum can be assigned, if either $\frac{\beta}{\alpha}$ or $\frac{b}{a} - \frac{\beta}{\alpha}$ is a positive whole number.

§ 20. If the progression is composed from coefficients of $x$ of this kind and both algebraic, first the algebraic coefficients must be removed by differentiation and integration, as has been done there, and then the resulting progression is to be treated in the explained manner. As let the proposed progression be :

$$\frac{1x}{1}+\frac{3x^2}{1.2}+\frac{3x^3}{1.2.3}+\ldots\ldots+\frac{(2n-1)x^n}{1.2.3\ldots.n};$$

and the sum of this is put as *s*; then

$$p\int x^{\pi}sdx=\frac{1.px^{\pi+2}}{(\pi+2)1}+\ldots\ldots+\frac{(2n-1)px^{\pi+n+1}}{(\pi+n+1)1.2.3\ldots.n}.$$

Let

$$2np-p=\pi+n+1;$$

then

$$p=\frac{1}{2}\text{ and }\pi=-\frac{3}{2},$$

from which

$$\frac{\int x^{-\frac{3}{2}}sdx}{2}=\frac{x^{\frac{1}{2}}}{1}+\frac{x^{\frac{3}{2}}}{1.2}+\ldots\ldots+\frac{x^{n-\frac{1}{2}}}{1.2.3\ldots.n}.$$

This is multiplied by $x^{\frac{1}{2}}$; then

$$\frac{x^{\frac{1}{2}}\int x^{-\frac{3}{2}}sdx}{2}=\frac{x}{1}+\frac{x^2}{1.2}+\ldots\ldots+\frac{x^n}{1.2.3\ldots.n},$$

hence

$$\frac{d(x^{\frac{1}{2}}\int x^{-\frac{3}{2}}sdx)}{2dx}=1+\frac{x}{1}+\frac{x^2}{1.2}+\ldots\ldots+\frac{x^{n-1}}{1.2.3\ldots(n-1)},$$

$$=1+\frac{x^{\frac{1}{2}}\int x^{-\frac{3}{2}}sdx}{2}-\frac{x^n}{1.2.3\ldots.n},$$

from which equation *s* can be found. Moreover,

$$\frac{\int x^{-\frac{3}{2}}sdx}{4x^{\frac{1}{2}}}+\frac{s}{2x}=1+\frac{x^{\frac{1}{2}}\int x^{-\frac{3}{2}}sdx}{2}-\frac{x^n}{1.2.3\ldots.n}.$$

There is put in place 1.2.3.....$n=A$; again

$$(1-2x)\int x^{-\frac{3}{2}}sdx=4x^{\frac{1}{2}}-\frac{2s}{x^{\frac{1}{2}}}-\frac{4x^{n+\frac{1}{2}}}{A}.$$

The sum of the progression proposed continued to infinity can now be defined from this equation

$$\int x^{-\frac{3}{2}}sdx=\frac{4x-2s}{(1-2x)\sqrt{x}},$$

which differentiated gives

$$\frac{sdx}{x\sqrt{x}}=\frac{2xdx+4xxdx+sdx-6sxdx-2xds+4x^2ds}{(1-2x)^2x\sqrt{x}}$$

or

$$xdx+2x^2dx-sxdx-2sx^2dx-xds+2x^2ds=0,$$

which is reduced to this:

$$ds+\frac{sdx(1+2x)}{1-2x}=\frac{dx(1+2x)}{1-2x}.$$

Which multiplied by per $\frac{c^{-x}}{1-2x}$ becomes integrable; moreover it gives :

$$\frac{c^{-x}s}{1-2x} = \int \frac{c^{-x}dx(1+2x)}{(1-2x)^2} = \frac{c^{-x}}{1-2x} - 1$$

and hence

$$s = 1 - c^{x}(1+2x).$$

Whereby if it is the case that $x = \frac{1}{2}$, then $s$ = 1. And thus :

$$1 = \frac{1}{1.2} + \frac{3}{1.2.4} + \frac{5}{1.2.3.8} + \frac{7}{1.2.3.4.16} + \text{etc. to infinity.}$$

§ 21. From these it is apparent, to which progressions requiring to be summed the method set out in this exposition can be extended : clearly to all these progressions which are understood by this general term $\frac{AP}{BQ}x^{\alpha n+\beta}$, where $A$ and $B$ designate terms of the order $n$ of some algebraic progression, and $P$ has been made from the terms of any algebraic expression $\gamma n+\delta$, and likewise Q has similarly been made from the terms $\varepsilon n+\zeta$, also of any algebraic progression. Moreover in general the sums of these kind of progressions found have been set out in three ways. For either the sum has been produced in a straightforward way using algebra, or there has been assigned a certain quadrature upon which the sum depends. Or in the third case an equation is found, of which the variable quantities $s$ and $x$ within cannot be separated from each other, as often it cannot be agreed whether the progression has an algebraic sum or depend on the quadrature of a curve. Now although this method extends so widely, yet innumerable progressions can occur not summable by this method, of which indeed either by no other method can the sum be assigned , as of this

$$1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \ldots\ldots + \frac{1}{2^n - 1},$$

or of which also the sums agree, as of this

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \text{etc.}$$

with the general term being $\frac{1}{a^{\alpha}-1}$, in which $a$ and $\alpha$ denote any integers except being equal to 1 shown by the Most Celebrated Goldbach, [See E 72, Theorem I.] But since the general term of this particular said progression cannot be shown, it is no wonder that it cannot be summed by this method.

# DE SUMMATIONE INNUMERABILIUM PROGRESSIONUM

*Auct. L. Eulero.*

§ I. Proposui anno praeterito [Vide E20] methodum innumeras progressiones summandi, quae non solum se ad series algebraicam summam habentes extendit, sed earum etiam, quae algebraice summari nequeunt, summas a quadraturis curvarum pendentes exhibet. Synthetica tum usus sum methodo; generalibus enim assumtis formulis quaesivi series, quarum summae iis formulis expremerentur. Hocque modo plurimas series generales adeptus sum, quarum summas poteram assignare. Proposita igitur quipiam progressione summanda necesse erat eam cum illis formulis comparare et indagare, num in aliqua earum contineatur. Potuissem autem numerum earum generalum serierum in infinitum multiplicare et propterea saepius mihi series occurrerunt, quae, etiamsi in datis generalibus non comprehenderentur, ipsa tamen methodo poterant summari. Quo igitur facilius magisque in promtu sit seriei cuiuscunque propositae summam, si quidem fieri potest, invenire, communicabo hic methodum analyticam, qua ex ipsius seriei natura terminum summatorium eruere licet. Latissime ea patet; non solum enim omnium earum serierum, quarum summa tot diversis modis iam sunt erutae, sed infinatarum aliarum summas simili et facili operatione invenire docet.

§2. Si aeque esset facile dato termino generali invenire summatorium ac inverse ex summatorio generalem, maximum hoc esset subsidium in summatione serierum. Potest quidem inter terminum summatorium et generalem dari aequatio, at quia ex infinitas constat terminis, ex ea non multum adiuvamur. Attamen insigne inde nascitur compendium ad progressionum algebraicarum summas exhibendas. Sit terminus generalis seu is, cuius exponens est $n$, in progressione quacunque $t$ et terminus summatorius seu summa omnium terminorum a primo usque ad $t = s$; erit

$$t=\frac{ds}{1dn}-\frac{dds}{1.2dn^2}+\frac{d^3s}{1.2.3dn^3}-\frac{d^4s}{1.2.3.4dn^4}+\text{etc.},$$

in qua aequatione positum est $dn$ constans. Transmutari autem haec aequatio potest in hanc

$$s=\int tdn+\alpha t+\frac{\beta dt}{dn}+\frac{\gamma ddt}{dn^2}+\frac{\delta d^3t}{dn^3}+\text{etc.},$$

in qua coefficientes $\alpha, \beta, \gamma,$ etc. sequentes habent valores :

$$\alpha=\frac{1}{2},$$
$$\beta=\frac{\alpha}{2}-\frac{1}{6},$$
$$\gamma=\frac{\beta}{2}-\frac{\alpha}{6}+\frac{1}{24},$$
$$\delta=\frac{\gamma}{2}-\frac{\beta}{6}+\frac{\alpha}{24}-\frac{1}{120},$$
$$\varepsilon=\frac{\delta}{2}-\frac{\gamma}{6}+\frac{\beta}{24}-\frac{\alpha}{120}+\frac{1}{720},$$
etc.

Fiet autem

$$s = \int tdn + \frac{t}{2} + \frac{dt}{12dn} - \frac{ddt}{720dn^2} + \frac{d^3t}{304240dn^3} - \text{etc.}$$

Quoties igitur $t$ eiusmodi habet valorem, ut series $s$ praebens vel alicubi abrumpatur vel fiat summabilis, tum ope huius aequationis reperietur $s$ ex $t$. E. g. sit

$$t = n^2 + 2n;$$

erit

$$dt = 2ndn + 2dn,\ ddt = 2dn^2, d^3t = 0 \text{ etc.}$$

Erit ergo

$$s = \int \left(n^2 + 2n\right)dn + \frac{n^2+2n}{2} + \frac{2n+2}{12} = \frac{n^3}{3} + \frac{3n^2}{2} + \frac{7n}{6} = \frac{2n^3+9n^2+7n}{6}.$$

§3. Methodus autem, quam hic sum expositurus, ita se habet, ut progressio proposita certis quibusdam operationibus vel ad aliam simpliciorem, quae summari potest, vel iterum ad se ipsam reducatur; utroque enim modo summa progressionis proposit constabit. Operationes, quibus in hisce transformationibus utor, sunt vel vulgares, ut additio, subtractio etc, vel ex altiori analysi sumtae, ut differentatio et integratio. Illa quidem aliis seriebus non inserviunt, nisi quarum summatio iam est cognita et algebraice assignari potest. His vero etiam progressionum summas algebraicas non habentium summae a curvarum quadraturis pendentes reperiuntur. Omnes autem series ad quas haec methodus accommodari potest, in se complectuntur progressionem geometricam et huiusmodi habent formam

$$\alpha x^a + \beta x^{a+b} + \gamma x^{a+2b} + \delta x^{a+3b} + \text{etc.}$$

Id quod non impedit, quominus progressio quaecunque in hac forma contineatur.

§4. Ut a simplicissimus incipiam, sit progressio proposita geometrica

$$x^a + x^{a+b} + x^{a+2b} + x^{a+3b} \ldots\ldots + x^{a+(n-1)b},$$

in qua extremus terminus est is, cuius index est $n$; atque hoc in sequentibus semper notetur terminum ultimum esse eum, cuius index est n, ne opus habeam indices adscribere; et proinde etiam semper summam usque ad terminum indicis n exhibebo. Ponatur summa progressionis propositae s; erit

$$s = x^a + x^{a+b} + x^{a+2b} + x^{a+3b} \ldots\ldots + x^{a+(n-1)b};$$

tunc fiet

$$s - x^a = x^{a+b} + x^{a+2b} + x^{a+3b} \ldots\ldots + x^{a+(n-1)b};$$

addatur utrinque $x^{a+nb}$ et dividatur per $x^b$; prodibit

$$\frac{s-x^a+x^{a+nb}}{x^b} = x^{a+b} + x^{a+2b} + x^{a+3b} \ldots\ldots + x^{a+(n-1)b} = s.$$

Habemus igitur aequationem

$$s - x^a + x^{a+nb} = sx^b,$$

ex qua invenitur

$$s = \frac{x^a - x^{a+nb}}{1-x^b},$$

quae est summa progressionis geometricae propositae. Est ergo hoc exemplum, quo progressio proposita in se ipsam transmutatur. Si fuerit $x$ fractio unitate minor et $n$ numerus infinite magnus, erit $x^{a+nb} = 0$ atque

$$s = \frac{x^a}{1-x^b}$$

summam praebebit progressionis geometricae

$$x^a + x^{a+b} + x^{a+2b} + \text{etc.}$$

in infinitum continuatae. Si fuerit $x = 1$, patet esse $s = n$; id vero difficilius apparet ex aequatione

$$s = \frac{x^a - x^{a+nb}}{1-x^b},$$

quia numerator et denominator evanescunt. Ut vero valor hoc inveniatur, ponatur $x = 1-\omega$ denotante $\omega$ quantitatem infinite parvam; erit

$$x^a = 1 - a\omega,\ x^{a+nb} = 1-(a+nb)\omega \text{ et } x^b = 1 - b\omega.$$

Hincque fit

$$s = \frac{nb\omega}{b\omega} = n.$$

Apparet etiam, si terminus generalis seriei fuerit $\alpha x^{a+(n-1)b}$, fore terminum summatorium

$$s = \frac{\alpha x^a - \alpha x^{a+nb}}{1-x^b}.$$

§5. Sit nunc proposita ista progressio

$$s - x^a = 2x^{a+b} + 3x^{a+2b} + \ldots\ldots.. + nx^{a+(n-1)b};$$

cuius summa ponatur $s$. Erit

$$s - x^a = 2x^{a+b} + 3x^{a+2b} + x^{a+3b} \ldots\ldots.. + nx^{a+(n-1)b};$$

addatur sequens terminus $(n+1)x^{a+nb}$ et dividatur per $x^b$;erit

$$\frac{s - x^a + (n+1)x^{a+nb}}{x^b} = 2x^a + 3x^{a+b} + x^{a+2b} + \ldots\ldots.. + (n+1)x^{a+(n-1)b}.$$

Subtrahatur ab hac serie prior, scilicet ipsa proposita; prodibit

$$\frac{s - x^a + (n+1)x^{a+nb}}{x^b} - s = x^a + x^{a+b} + x^{a+2b} + \ldots\ldots.. + x^{a+(n-1)b} = \frac{x^a - x^{a+nb}}{1-x^b}.$$

Ex hac invenitur

$$s = \frac{x^a - (n+1)x^{a+nb}}{1-x^b} + \frac{x^{a+b} - x^{a+(n+1)b}}{(1-x^b)^2} = \frac{x^a - (n+1)x^{a+nb} + nx^{a+(n+1)b}}{(1-x^b)^2} = \frac{x^a - x^{a+nb}}{(1-x^b)^2} - \frac{nx^{a+nb}}{1-x^b},$$

qui est terminus summatorius respondens termino generali $nx^{a+(n-1)b}$. Si fuerit $x < 1$ et ponatur $n = \infty$, prodibit seriei propositae in infinitum continuatae summa

$$= \frac{x^a}{(1-x^b)^2}.$$

Si autem fiat $x = 1$, prodire debet summa progressionis

$$1 + 2 + 3 + 4 + \ldots\ldots\ldots n;$$

hic vero eadem quae ante oritur difficultas numeratore et denominatore evanescentibus; pono igitur eterum $x = 1 - \omega$; erit

$$1 - x^b = b\omega,$$
$$x^a = 1 - a\omega + \frac{a(a-1)\omega^2}{2},$$
$$x^{a+nb} = 1 - (a+nb)\omega + \frac{(a+nb)(a+nb-1)\omega^2}{2},$$

et

$$x^{a+(n+1)b} = 1 - (a+(n+1)b)\omega + \frac{(a+(n+1)b)(a+(n+1)b-1)\omega^2}{2}$$

sitque

$$s = \frac{(n^2b^2 + nb^2)\omega^2}{2b^2\omega^2} = \frac{n(n+1)}{2}.$$

Praeterea, si terminus generalis sit $\beta n x^{a+(n-1)b}$, erit terminus summatorius

$$\frac{\beta x^a - \beta x^{a+nb}}{(1-x^b)^2} - \frac{\beta n x^{a+nb}}{1-x^b}.$$

§6. Simili modo invenientur termini summatorii, si termini generales sint

$$n^2 x^{a+(n-1)b},\ n^3 x^{a+(n-1)b} \text{ etc.};$$

semper enim summatio reducitur ad summationem seriei gradus inferioris. Ex quo intelligitur hac ratione inveniri posse generaliter terminum summatorium respondentem termino generali

$$(\alpha + \beta n + \gamma n^2 + \text{etc.})\, x^{a+(n-1)b}.$$

In his autem absolvendis longius non immoror, quia iam dudum satis sunt cognita. Ideo haec tantum attuli, ut methodi vis etiam per vulgares operationes patescat. Progredior igitur ultra et, quaenam series ope differentiationis et integrationis in summam redigi queant, investigabo. Primo quidem etiam progressiones algebraicae modo tractatae summantur et summae inveniuntur a iam datis non differentes; attamen earum inventio per has operationes videtur facilior et brevior. Hanc ob rem ab his iterum incipio.

§7. Sit progressio summanda

$$x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots + nx^n;$$

ponatur ea = $s$, dividatur per $x$ et multiplicetur per $dx$; erit

$$\frac{sdx}{x} = dx + 2xdx + 3x^2dx + \ldots\ldots + nx^{n-1}dx$$

sumtisque integralis habetur

$$\int \frac{sdx}{x} = x + x^2 + x^3 + \ldots\ldots + x^n = \frac{x - x^{n+1}}{1-x}.$$

Ex aequatione igitur

$$\int \frac{sdx}{x} = \frac{x - x^{n+1}}{1-x}$$

differentia invenietur $s$. Erit enim

$$\frac{sdx}{x} = \frac{dx - (n+1)x^n dx + nx^{n+1}dx}{(1-x)^2},$$

unde prodit

$$s = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2},$$

ut ante §5, si ibi loco $a$ et $b$ scribatur 1.

Ex hoc intelligi potest, quomodo progressionis

$$ax^{\alpha} + (a+b)x^{\alpha+\beta} + (a+2b)x^{\alpha+2\beta} + \ldots\ldots\ldots + (a+(n-1)b)x^{\alpha+(n-1)\beta}$$

summa sit invenienda. Ponatur enim haec summa quaesita s et multiplicetur per $x^{\pi}dy$; erit

$$x^{\pi}sdy = ax^{\alpha+\pi}dy + (a+b)x^{\alpha+\beta+\pi}dy + (a+2b)x^{\alpha+2\beta+\pi}dy + \ldots + (a+(n-1)b)x^{\alpha+(n-1)\beta+\pi}dy.$$

Fiat iam

$$x^{\alpha+\pi} = y^{a-1} \text{ and } x^{\alpha+\beta+\pi} = y^{a+b-1};$$

erit

$$x^{\beta} = y^{b} \text{ et } x = y^{b:\beta}.$$

Hincque fiet

$$x^{\alpha+\pi} = y^{(\alpha+\pi)b:\beta} = \mathrm{y}^{a\text{-}1}.$$

Ergo erit

$$\pi = \frac{\beta a - \alpha b - \beta}{b}$$

atque

$$x^{\alpha+(n-1)\beta+\pi} = y^{a+(n-1)b-1}.$$

His positis erit

$$x^{\frac{\beta a - \alpha b - \beta}{b}}sdy = ay^{a-1}dy + (a+b)y^{a+b-1}dy + \ldots\ldots + (a+(n-1)b)y^{a+(n-1)b-1}dy$$

sumtisque integralibus

$$\int x^{\frac{\beta a - \alpha b - \beta}{b}}sdy = y^{a} + y^{a+b} + \ldots\ldots + y^{a+(n-1)b} = \frac{y^{a} - y^{a+nb}}{1-y^{b}}.$$

Quia vero est $y^{b} = x^{\beta}$, erit

$$y = x^{\frac{\beta}{b}} \text{ et d}y = \frac{\beta}{b}x^{\frac{\beta-b}{b}}dx$$

hisque substitutis

$$\frac{\beta}{b}\int x^{\frac{\beta a-\alpha b-\beta}{b}} sdy = \frac{x^{\frac{a\beta}{b}} - x^{\frac{a\beta+nb\beta}{b}}}{1-x^{\beta}}.$$

Haec eadem aequatio potest facilius sine permutatione variabilis $x$ inveniri hoc modo. Multiplicetur progressio proposita per $px^{\pi}dx$; erit

$$px^{\pi}sdx = pax^{\alpha+\pi}dx + \ldots + p(a+(n-1)b)x^{\alpha+(n-1)\beta+\pi}dx.$$

Determinentur $p$ et $\pi$ ita, ut sit

$$\alpha+(n-1)\beta+\pi = p(a+(n-1)b)-1$$

seu

$$\alpha+\pi+(n-1)\beta = ap+(n-1)bp-1.$$

Ex qua, quia $p$ et $\pi$ ab $n$ pendere nequeunt, duae resurgunt aequationes

$$\beta = bp \text{ et } \alpha+\pi = ap-1,$$

unde prodit

$$p = \frac{\beta}{b} \text{ et } \pi = \frac{a\beta-\alpha b-b}{b}.$$

His substitutis et integralibus sumtis proveniet, ut ante,

$$\frac{\beta}{b}\int x^{\frac{a\beta-\alpha b-\beta}{b}} sdx = x^{\frac{a\beta}{b}} + x^{\frac{a\beta+b\beta}{b}} + \ldots + x^{\frac{a\beta+(n-1)b\beta}{b}} = \frac{x^{\frac{a\beta}{b}} - x^{\frac{a\beta+nb\beta}{b}}}{1-x^{\beta}}.$$

§8. Sit progressionis propositae terminus ordine $n$ hic

$$(an+b)(cn+e)x^{\alpha+(n-1)\beta};$$

ponatur huius terminus summatorius $s$; erit

$$s = (a+b)(c+e)x^{\alpha} + (2a+b)(2c+e)x^{\alpha+\beta} + \ldots (an+b)(cn+e)x^{\alpha+(n-1)\beta};$$

multiplicetur per $px^{\pi}dx$; fiet

$$psx^{\pi}dx = p(a+b)(c+e)x^{\alpha+\pi} + \ldots + p(an+b)(cn+e)x^{\alpha+(n-1)\beta+\pi}dx;$$

Sit

$$pcn+pe = \alpha+n\beta-\beta+\pi+1;$$

debebit esse

$$p = \frac{\beta}{b} \text{ et } \pi = \frac{\beta e+\beta c-\alpha c-c}{c}.$$

Ergo sumtis integralibus erit

$$\frac{\beta}{c}\int x^{\pi}sdx = (a+b)x^{\alpha+\pi+1} + \ldots + (an+b)x^{\alpha+(n-1)\beta+\pi+1}.$$

Multiplicetur denuo per $qx^{\rho}dx$; erit

$$\frac{\beta}{c}qx^{\rho}dx\int x^{\pi}sdx = q(a+b)x^{\alpha+\pi+\rho+1} + \ldots + q(an+b)x^{\alpha+(n-1)\beta+\pi+\rho+1}dx$$

fiatque

$$anq + bq = \alpha + n\beta - \beta + \pi + \rho + 2;$$

hinc erit

$$q = \frac{\beta}{a} \text{ and } \rho = \frac{\beta b - \alpha a + \beta a - \pi a - 2a}{a} = \frac{\beta bc - ac - \beta ae}{ac}.$$

Sumtisque integralibus proveniet

$$\frac{\beta^2}{ac}\int x^{\rho} dx \int x^{\pi} s dx = x^{\alpha+\pi+\rho+2} + \ldots + x^{\alpha+(n-1)\beta+\pi+\rho+2} = \frac{x^{\alpha+\pi+\rho+2} - x^{\alpha+n\beta+\pi+\rho+2}}{1-x^{\beta}}$$

seu haec aequatio

$$\frac{\beta^2}{ac}\int x^{\frac{\beta bc-\alpha c-\beta ae}{ac}} dx \int x^{\frac{\beta e+\beta c-\alpha c-e}{c}} s dx = \frac{x^{\frac{\beta(a+b)}{a}} - x^{\frac{\beta(a+b+na)}{a}}}{1-x^{\beta}} = x^{\frac{\beta(a+b)}{a}} \frac{1-x^{n\beta}}{1-x^{\beta}}.$$

Simili modo operatio est instituenda, si plures duobus factores fuerint in termino generali, ex quo simul tot prodire signa integralia, quot sunt factores in coefficiente termini generalis.

§9. Si fuerit progressionis summandae terminus generalis

$$\frac{x^{\alpha+(n-1)\beta}}{an+b},$$

operatio a priori in hoc tantum differt, quod hic differentiatione absolvi debeat, quod ibi integralibus sumendis perficiebatur. Sit igitur terminus summatorius quaesitus $s$; erit

$$s = \frac{x^{\alpha}}{a+b} + \ldots\ldots + \frac{x^{\alpha+(n-1)\beta}}{an+b}$$

atque

$$px^{\pi}s = \frac{px^{\alpha+\pi}}{a+b} + \ldots\ldots + \frac{px^{\alpha+(n-1)\beta+\pi}}{an+b}.$$

Summatur differentalia; prodibit

$$px^{\pi}ds + p\pi x^{\pi-1}sdx = \frac{p(\alpha+\pi)x^{\alpha+\pi-1}dx}{a+b} + \ldots\ldots + \frac{p(\alpha+n\beta-\beta+\pi)x^{\alpha+(n-1)\beta+\pi-1}}{an+b}$$

Fiat

$$p\pi + pn\beta - p\beta + p\pi = an + b;$$

erit

$$p = \frac{a}{\beta} \text{ et } \pi = \beta - \alpha + \frac{b\beta}{a}.$$

Ergo

$$\frac{ax^{\beta-\alpha+\frac{b\beta}{a}}ds + (a\beta - a\alpha + b\beta)x^{\beta-\alpha+\frac{b\beta}{a}-1}sdx}{\beta dx} = x^{\frac{a\beta+b\beta-a}{a}} + \ldots. + x^{\frac{na\beta+b\beta-a}{a}}$$

$$x^{\frac{a\beta+b\beta-a}{a}} \frac{1-x^{n\beta}}{1-x^{\beta}}$$

seu

$$\frac{a}{\beta}x^{\frac{a\beta-a\alpha+b\beta}{a}}s=\int x^{\frac{a\beta+b\beta-a}{a}}dx\frac{1-x^{n\beta}}{1-x^{\beta}}$$

vel

$$s=\frac{\beta}{a}x^{\frac{a\beta-a\beta-b\beta}{a}}\int x^{\frac{a\beta+b\beta-a}{a}}dx\frac{1-x^{n\beta}}{1-x^{\beta}}.$$

In hac formula integrale ita debet accipi, ut posito $x = 0$ ipsum evanescat. Si desideretur summa seriei propositae in infinitum continuatae, fiet $n=\infty$ et

$$s=\frac{\beta}{a}x^{\frac{a\beta-a\beta-b\beta}{a}}\int\frac{x^{\frac{a\beta+b\beta-a}{a}}}{1-x^{\beta}}dx.$$

Si sit $x = 1$, in expressione quidem summae $s$, quia differentialia insunt, non potest poni $x = 1$, sed post integrationem fiat $x = 1$. Attamen perinde est, quales numeri loco $\alpha$ et $\beta$ substituantur; sit igitur $\alpha=\beta=1$. Erit

$$s=\frac{x^{\alpha}}{a+b}+.....+\frac{x^{\alpha+(n-1)\beta}}{na+b}=\frac{\beta}{a}x^{\frac{a\beta-a\beta-b\beta}{a}}\int x^{\frac{b}{a}}dx\frac{1-x^{n}}{1-x}.$$

Atque post integrationem fieri debet $x = 1$. Quemadmodum in dissertatione de summationibus initio citata inveneram.

§ 10. Sit proposita progressio, cuius terminus ordine $n$ est

$$\frac{x^{n}}{(an+b)(cn+e)};$$

assumto hic tantum $x^n$ loco $x^{\alpha+(n-1)\beta}$ tum compendii ergo, tum quia haec potentia in illam facili negotio potest transmutari. Sit terminus summatorius $s$, erit

$$px^{\pi}s=\frac{px^{\pi+1}}{(a+b)(c+e)}+.....+\frac{px^{\pi+n}}{(an+b)(cn+e)}$$

adeoque

$$\frac{diff.px^{\pi}s}{dx}=\frac{p(\pi+1)x^{\pi}}{(a+b)(c+e)}+.....+\frac{p(\pi+n)x^{\pi+n-1}}{(an+b)(cn+e)}.$$

Fiat

$$p\pi+pn=an+b;$$

erit

$$p=a \text{ et } \pi=\frac{b}{a}.$$

Ergo habetur

$$\frac{ad(x^{\frac{b}{a}}s)}{dx}=\frac{x^{\frac{b}{a}}}{c+e}+\ldots+\frac{x^{\frac{b}{a}+n-1}}{cn+e}.$$

Multiplicetur denuo per $px^{\pi}$; erit

$$\frac{apx^{\pi}ad(x^{\frac{b}{a}}s)}{dx}=\frac{px^{\frac{b}{a}+\pi}}{c+e}+\ldots+\frac{px^{\frac{b}{a}+n+\pi-1}}{cn+e}$$

Hincque prodit

$$\frac{apd(x^{\pi}d(x^{\frac{b}{a}}s))}{dx^2}=\frac{p\left(\frac{b}{a}+\pi\right)x^{\frac{b}{a}+\pi-1}}{c+e}+\ldots+\frac{p\left(\frac{b}{a}+n+\pi-1\right)x^{\frac{b}{a}+n+\pi-2}}{cn+e}.$$

Fiat

$$\frac{pb}{a}+pn+p\pi-p=cn+e;$$

erit

$$p=c \text{ et } \pi=1-\frac{b}{a}+\frac{e}{c}.$$

His substitutis emergit ista aequatio

$$\frac{acd(x^{1-\frac{b}{a}+\frac{e}{c}}d(x^{\frac{b}{a}}s))}{dx^2}=x^{\frac{e}{c}}+\ldots+x^{\frac{e}{c}+n-1}=x^{\frac{e}{c}}\frac{1-x^n}{1-x}.$$

Sumantur iterum integralia; erit

$$\frac{acx^{1-\frac{b}{a}+\frac{e}{c}}d(x^{\frac{b}{a}}s)}{dx}=\int x^{\frac{e}{c}}dx\frac{1-x^n}{1-x}$$

hincque

$$s=\frac{1}{acx^{\frac{b}{a}}}\int x^{\frac{b}{a}-\frac{e}{c}-1}dx\int x^{\frac{e}{c}}dx\frac{1-x^n}{1-x}=\frac{x^{\frac{b}{a}-\frac{e}{c}}\int x^{\frac{e}{c}}dx\frac{1-x^n}{1-x}-\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}}{(bc-ae)x^{\frac{b}{a}}}.$$

Casus hic notandus est, si $bc=ae$, quo fit $s=\frac{0}{0}$. Erit autem iuxta priorem formam

$$s=\frac{1}{acx^{\frac{b}{a}}}\int\frac{dx}{x}\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x},$$

quae mutatur in hanc

$$s=\frac{lx\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}-\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}lx}{acx^{\frac{b}{a}}}.$$

Casus hic accidit, si denominatores $(an+b)(cn+e)$ fuerint quadrata vel horum quaedam multipla. Si fuerit $x=1$, haec substitutio, ut ante, demum post integrationem fieri debet in quantitatibus signa integralia prae se habentibus, at in finitis statim fieri potest $x=1$. Erit ergo

$$s=\frac{\int\left(x^{\frac{e}{c}}-x^{\frac{b}{a}}\right)dx\frac{1-x^n}{1-x}}{bc-ae}.$$

Ex quo apparet, si $x^{\frac{e}{c}}-x^{\frac{b}{a}}$ potest dividi per $1-x$, summam progressionis esse algebraicam. At casu, quo $bc=ae$, fiet $lx=0$, si scilicet sit $x=1$.
Quacirca erit

$$s=-\frac{\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}lx}{ac}.$$

§11. Simili modo intelligitur, si $n$ in denominatore 3 pluresve dimensiones habeat, quomodo summam invenire oporteat, ita ut opus non sit pluribus exemplis operationem illustrare. Sit progressionio proposita haec, cuius terminus generalis est

$$\frac{x^n}{(an+b)(cn+e)(fn+g)};$$

summa huius sit s. Haec progressio eodem quo praecendente § modo tractata dabit post duas differentiationes (§9)

$$s=\frac{1}{acfhx^{\frac{b}{a}}}\int x^{\frac{b}{a}-\frac{e}{c}-1}dx\int x^{\frac{e}{c}-\frac{g}{f}-1}dx\int x^{\frac{g}{f}-\frac{k}{h}-1}dx\int x^{\frac{k}{h}}dx\frac{1-x^n}{1-x};$$

sumantur integralia; erit

$$\frac{acfx^{1-\frac{b}{a}+\frac{e}{c}}d(x^{\frac{e}{c}}s)}{dx}=\int x^{\frac{e}{c}-\frac{g}{f}-1}dx\int x^{\frac{g}{f}}dx\frac{1-x^n}{1-x}$$

et denuo

$$acfx^{\frac{b}{a}}s=\int x^{\frac{b}{a}-\frac{e}{c}-1}dx\int x^{\frac{e}{c}-\frac{g}{f}-1}dx\int x^{\frac{g}{f}}dx\frac{1-x^n}{1-x}$$

adeoque

$$s=\frac{1}{acfx^{\frac{b}{a}}}\int x^{\frac{b}{a}-\frac{e}{c}-1}dx\int x^{\frac{e}{c}-\frac{g}{f}-1}dx\int x^{\frac{g}{f}}dx\frac{1-x^n}{1-x}.$$

Ne plura signa integralia post se invicem sint posita, haec forma in sequentem transmutari potest

$$s=\frac{fx^{-\frac{g}{f}}}{(bf-ag)(ef-cg)}\int x^{\frac{g}{f}}dx\frac{1-x^n}{1-x}+\frac{cx^{-\frac{e}{c}}}{(bc-ae)(cg-ef)}\int x^{\frac{e}{c}}dx\frac{1-x^n}{1-x}+\frac{ax^{-\frac{b}{a}}}{(ae-bc)(ag-bf)}\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}.$$

Ex hoc simul apparet, si plures fuerint factores in termino generali, quam formam habitura sit summa. Sit enim terminus generalis

$$\frac{x^n}{(an+b)(cn+e)(fn+g)(hn+k)};$$

erit terminus summatorius

$$s=\frac{1}{acfhx^{\frac{b}{a}}}\int x^{\frac{b}{a}-\frac{e}{c}-1}dx\int x^{\frac{e}{c}-\frac{g}{f}-1}dx\int x^{\frac{g}{f}-\frac{k}{h}-1}dx\int x^{\frac{k}{h}}dx\frac{1-x^n}{1-x}$$

$$=\frac{ax^{-\frac{b}{a}}\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}}{(ae-bc)(ag-bf)(ak-bh)}+\frac{cx^{-\frac{e}{c}}\int x^{\frac{e}{c}}dx\frac{1-x^n}{1-x}}{(bc-ae)(cg-ef)(ck-eh)}$$

$$+\frac{fx^{-\frac{a}{f}}\int x^{\frac{g}{f}}dx\frac{1-x^n}{1-x}}{(bf-ag)(ef-cg)(fk-gh)}+\frac{hx^{-\frac{k}{h}}\int x^{\frac{k}{h}}dx\frac{1-x^n}{1-x}}{(bh-ak)(eh-ck)(gh-fk)}.$$

Si desideretur summa casu, quo $x = 1$, erit pro termino generali

$$\frac{1}{(an+b)(cn+e)(fn+g)}$$

terminus summatorius

$$s=\frac{\int dx\frac{1-x^n}{1-x}((aef-bcf)x^{\frac{g}{f}}+(bcf-acg)x^{\frac{e}{c}}+(acg-aef)x^{\frac{b}{a}})}{(ae-bc)(ag-bf)(cg-ef)}.$$

Quoties igitur quantitas in $dx\frac{1-x^n}{1-x}$ ducta dividi potest per $1-x$, tunc progressio proposita algebraicam habet summam. Accidit hoc, si $\frac{b}{a}-\frac{e}{c}$ et $\frac{e}{c}-\frac{g}{f}$ sunt numeri integri. Praeterea hoc etiam est notandum omnes huiusmodi progressiones vel algebraice esse summabiles vel a logarithmis sive realibus sive imaginariis pendere neque ullam aliam quadraturam huiusmodi progressione posse exprimi.

§12. At cum difficile sit has formulas ad eos casus accommodare, quibus denominatorum factores sunt aeqales, libet hic hos casus in specie tractare. Sit itaque progressionis summandae terminus generalis

$$\frac{x^n}{(an+b)^3}$$

et summatorius $s$; erit

$$s=\frac{1}{a^3x^{\frac{b}{a}}}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x},$$

id quod sequitur ex §11, ubi sit $c=f=a$ et $e=g=b$; haec forma transmutata abit in hanc

$$s=\frac{\frac{1}{2}(lx)^2\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}-lx\int x^{\frac{b}{a}}dxlx\frac{1-x^n}{1-x}+\frac{1}{2}\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}(lx)^2}{a^3x^{\frac{b}{a}}}$$

Si autem fuerit terminus generalis

$$\frac{x^n}{(an+b)^4},$$

erit

$$s=\frac{\left[(lx)^3\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}-3(lx)^2\int x^{\frac{b}{a}}dxlx\frac{1-x^n}{1-x}+3lx\int x^{\frac{b}{a}}dx(lx)^2\frac{1-x^n}{1-x}-\int x^{\frac{b}{a}}dx(lx)^3\frac{1-x^n}{1-x}\right]}{6a^4x^{\frac{b}{a}}}$$

Ex his apparet, quomodo pro reliquis potentiis valor ipsius s progrediatur; generalite enim, si terminus est

$$\frac{x^n}{(an+b)^m}$$

erit summa

$$s = \frac{\left[(lx)^{m-1}\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x} - \frac{m-1}{1}(lx)^{m-2}\int x^{\frac{b}{a}}dxlx\frac{1-x^n}{1-x} + \frac{m-1}{1}.\frac{m-2}{1}(lx)^{m-3}\int x^{\frac{b}{a}}dx(lx)^2\frac{1-x^n}{1-x} - \text{etc.}\right]}{1.2.3....(m-1)a^m x^{\frac{b}{a}}}$$

Valores hi multo fiunt simpliores, si ponatur $x = 1$; erit enim $lx = 0$. Termino generali enim $\frac{1}{(an+b)^2}$ respondet hic summatorius

$$s = \frac{\int x^{\frac{b}{a}}dxl\frac{1}{x}\frac{1-x^n}{1-x}}{1.a^2},$$

termino generali $\frac{1}{(an+b)^3}$ hic

$$s = \frac{\int x^{\frac{b}{a}}dx\left(l\frac{1}{x}\right)^2\frac{1-x^n}{1-x}}{1.2a^3}$$

atque termino generali $\frac{1}{(an+b)^m}$ hic

$$s = \frac{\int x^{\frac{b}{a}}dx\left(l\frac{1}{x}\right)^{m-1}\frac{1-x^n}{1-x}}{1.2.3......(m-1)a^m} = \frac{\int x^{\frac{b}{a}}dx\left(l\frac{1}{x}\right)^{m-1}\frac{1-x^n}{1-x}}{a^m\int dx\left(l\frac{1}{x}\right)^{m-1}},$$

quae integralia ita debent accipi, ut posito $x = 0$ tota summa evanescat; tum autem poni debet $x = 1$ et quantitas resultans vera erit summa. Porro notetur, si summa desideretur in infinitum continuatae progressionis, ubique tantum scribi debere $\frac{1}{1-x}$ loco $\frac{1-x^n}{1-x}$.

§ 13. Duae iam pertractatae sunt progressionum classes, quarum illa habebat terminum generalem $Ax^n$, haec vero $\frac{x^n}{A}$ denotante $A$ quantitatem algebraicam ex $n$ et constantibus constantem, ita tamen, ut $n$ non habeat alios exponentes nisi integros affirmativos. Ex his oritur tertia classis pro termino generali habens $\frac{Ax^n}{B}$, ubi $A$ et $B$ eiusdem modi quantities algebraicas designant. Talis progressio reducitur etiam ad progressionem geometricam tollendo numeratorem $A$ ope integrationis et denominatorem $B$ ope differentationis, quemadmodem in utraque pertractata seorsim factum est. Sit progressionis summandae terminus generalis

$$\frac{(\alpha n+\beta)x^n}{an+b},$$

huius terminus summatorius ponatur $s$; erit

$$s = \frac{(\alpha+\beta)x}{a+b} + ..... + \frac{(\alpha n+\beta)x^n}{an+b}.$$

Multiplicetur haec aequatio per $px^{\pi}$; erit

$$px^{\pi}s = \frac{p(\alpha+\beta)x^{\pi+1}}{a+b} + ..... + \frac{p(\alpha n+\beta)x^{n+\pi}}{an+b};$$

sumantur differentialia; erit

$$pd(x^{\pi}s) = \frac{p(\pi+1)(\alpha+\beta)x^{\pi}dx}{a+b} + \ldots\ldots + \frac{p(n+\pi)(\alpha n+\beta)x^{n+\pi-1}dx}{an+b};$$

fiat

$$pn + p\pi = an + b;$$

erit

$$p = a \text{ et } \pi = \frac{b}{a}.$$

Ergo est

$$pd(x^{\frac{b}{a}}s) = (\alpha+\beta)x^{\frac{b}{a}}dx + \ldots\ldots + (\alpha n+\beta)x^{\frac{b}{a}+n-1}dx.$$

Multiplicatur denuo per $px^{\pi}$; erit

$$apx^{n}pd(x^{\frac{b}{a}}s) = p(\alpha+\beta)x^{\frac{b}{a}+\pi}dx + \ldots\ldots + p(\alpha n+\beta)x^{\frac{b}{a}+\pi+n-1}dx$$

Sumantur integralia; habebitur

$$ap\int x^{n}pd(x^{\frac{b}{a}}s) = \frac{p(\alpha+\beta)x^{\frac{b}{a}+\pi+1}}{b+a\pi+a} + \ldots\ldots + \frac{p(\alpha n+\beta)x^{\frac{b}{a}+\pi+n}}{b+a\pi+an}.$$

Fiat

$$a\alpha pn + a\beta p = an + a\pi + b$$

erit

$$p = \frac{1}{\alpha} \text{ et } \pi = \frac{\beta}{\alpha} - \frac{b}{a}.$$

Propterea est

$$\frac{a}{\alpha}\int x^{\frac{\beta}{\alpha}-\frac{b}{a}}d(x^{\frac{b}{a}}s) = x^{\frac{b}{a}+1} + \ldots\ldots + x^{\frac{b}{a}+n} = x^{\frac{\beta}{\alpha}+1}\frac{1-x^{n}}{1-x}.$$

Ex hac equatione prodit

$$s = \frac{\alpha\int x^{\frac{\beta}{\alpha}-\frac{b}{a}}d\left(x^{\frac{\beta+\alpha}{a}}\frac{1-x^{n}}{1-x}\right)}{ax^{\frac{b}{a}}}.$$

Si fuerit terminus generalis

$$\frac{(\alpha n+\beta)(\gamma n+\delta)x^{n}}{an+b}$$

huiusque summatorius ponatur $s$, prodibit iisdem quibus modo absolutis operationibus

$$\frac{a}{\alpha}\int x^{\frac{\beta}{\alpha}-\frac{b}{a}}d(x^{\frac{b}{a}}s) = (\gamma+\delta)x^{\frac{b}{a}+1} + \ldots\ldots + (\gamma n+\delta)x^{\frac{b}{a}+n}$$

multiplicatur iterum per $px^{\pi}dx$ et summantur integralia; prodit

$$\frac{ap}{\alpha}\int x^{n}dx\int x^{\frac{\beta}{\alpha}-\frac{b}{a}}d(x^{\frac{b}{a}}s) = \frac{\alpha p(\gamma+\delta)x^{\frac{b}{a}+\pi+2}}{\beta+\alpha\pi+2\alpha} + \ldots\ldots + \frac{\alpha p(\gamma n+\delta)x^{\frac{b}{a}+\pi+n+1}}{\beta+\alpha\pi+\alpha n+\alpha}.$$

Fiat

$$a\gamma pn + a\delta p = \alpha + \beta + \alpha\pi + \alpha n;$$

erit

$$p = \frac{1}{\gamma} \text{ et } \pi = \frac{\delta}{\gamma} - \frac{\beta}{\alpha} - 1.$$

Ergo

$$\frac{a}{\alpha\gamma}\int x^{\frac{\delta}{\gamma}-\frac{\beta}{\alpha}-1}dx\int x^{\frac{\beta}{\alpha}-\frac{b}{a}}d\left(x^{\frac{b}{a}}s\right) = x^{\frac{\delta}{\gamma}+1} + \ldots\ldots + x^{\frac{\delta}{\gamma}+n} = x^{\frac{\delta}{\gamma}+1}\frac{1-x^n}{1-x}.$$

Quare

$$s = \frac{\alpha\gamma\int x^{\frac{b}{a}-\frac{\beta}{\alpha}}d(x^{\frac{\beta}{a}-\frac{\delta}{\gamma}+1})d(x^{\frac{\delta}{\gamma}+1}\frac{1-x^n}{1-x})}{ax^{\frac{b}{a}}dx}.$$

Sed huiusmodi progressionis summandis diutius non immoror; sufficit enim methodum tradidisse, qua omnes summari possint. Interim tamen et id valet, quod § 11 dixi, omnes scilicet huiusmodi progressiones vel algebraice posse summari vel summam a logarithmis sive realibus sive imaginariis pendere.

§14. Progredior nunc ad aliud progressionum genus, quarum termini generales algebraice exprimi non possunt, sed quae ad classem serierum hypergeometricarum pertinent. Huiusmodi series est

$$(\alpha+\beta)x + (\alpha+\beta)(2\alpha+\beta)x^2 + \ldots\ldots + (\alpha+\beta)(2\alpha+\beta)\ldots(\alpha n+\beta)x^n.$$

Ponatur huius summa $s$ et multiplicatur per $px^{\pi}$; erit

$$px^{\pi}s = p(\alpha+\beta)x^{\pi+1} + \ldots\ldots p(\alpha+\beta)(2\alpha+\beta)\ldots(\alpha n+\beta)x^{n+\pi}$$

et huius in $dx$ ductae integralis

$$p\int x^{\pi}sdx = \frac{p(\alpha+\beta)x^{\pi+2}}{\pi+2} + \ldots\ldots \frac{p(\alpha+\beta)(2\alpha+\beta)\ldots(\alpha n+\beta)x^{n+\pi+1}}{n+\pi+1}$$

Fiat

$$p\alpha n + p\beta = n + \pi + 1;$$

erit

$$p = \frac{1}{\alpha} \text{ et } \pi = \frac{\beta}{\alpha} - 1,$$

unde prodit

$$\frac{1}{\alpha}\int x^{\frac{\beta}{\alpha}-1}sdx = x^{\frac{\beta}{\alpha}+1} + (\alpha+\beta)x^{\frac{\beta}{\alpha}+2} + \ldots\ldots + (\alpha+\beta)(2\alpha+\beta)\ldots(\alpha(n-1)+\beta)x^{\frac{\beta}{\alpha}+n}.$$

Dividatur per $x^{\frac{\beta}{\alpha}+1}$; habebitur

$$\frac{\int x^{\frac{\beta}{\alpha}-1} sdx}{\alpha x^{\frac{\beta}{\alpha}+1}} - 1 = (\alpha+\beta)x + \ldots\ldots + (\alpha+\beta)(2\alpha+\beta)\ldots(\alpha(n-1)+\beta)x^{n-1}.$$

Quae est ipsa progressio proposita truncata termino ultimo. Erit igitur

$$\frac{\int x^{\frac{\beta}{\alpha}-1} sdx}{\alpha x^{\frac{\beta}{\alpha}+1}} - 1 = s - (\alpha+\beta)x + \ldots\ldots + (\alpha+\beta)(2\alpha+\beta)\ldots(\alpha n+\beta)x^{n} = s - Ax^{n}.$$

Huiusmodi autem formas finita expressione exposui in alia iam pralecta dissertatione [Vide Commentatione 19 huius voluminis, imprimis § 16.] de terminis generalibus progressionum transcendentalium, ex qua, si libet, finitus valor loco *A* desumi potest. Erit ergo

$$\int x^{\frac{\beta}{\alpha}-1} sdx = \alpha x^{\frac{\beta}{\alpha}+1} + \alpha x^{\frac{\beta}{\alpha}+1} s - \alpha A x^{\frac{\beta}{\alpha}+n+1}$$

atque

$$x^{\frac{\beta}{\alpha}-1} sdx = (\alpha+\beta)x^{\frac{\beta}{\alpha}}dx + (\alpha+\beta)x^{\frac{\beta}{\alpha}}sdx + \alpha x^{\frac{\beta}{\alpha}+1}ds - (\alpha+\beta+\alpha n)Ax^{\frac{\beta}{\alpha}+n}dx$$

seu

$$sdx = (\alpha+\beta)xdx + (\alpha+\beta)xsdx + \alpha x^{2}ds - (\alpha+\beta+\alpha n)Ax^{n+1}dx.$$

Ex qua aequatione valor ipsius *s* erutus dabit summam progressionis propositae. Fieri etiam potest, ut factores in termino sequente non uno tantum, sed duobus pluribusve augeantur. Accedant semper duo de novo, ut prodeat ista progressio

$$\begin{gathered}(\alpha+\beta)x + (\alpha+\beta)(2\alpha+\beta)(3\alpha+\beta)x^{2} + \ldots\\ + (\alpha+\beta)(2\alpha+\beta)\ldots\ldots(\alpha(2n-1)+\beta)x^{n}.\end{gathered}$$

Huius summa vocetur *s*; erit

$$p\int x^{\pi} sdx = \frac{p(\alpha+\beta)x^{\pi+2}}{\pi+2} + \ldots\ldots + \frac{p(\alpha+\beta)(2\alpha+\beta)\ldots(\alpha(2n-1)+\beta)x^{n+\pi+1}}{n+\pi+1}.$$

Fiat

$$2p\alpha n - p\alpha + p\beta = n + \pi + 1;$$

erit

$$p = \frac{1}{2\alpha} \text{ et } \pi = \frac{\beta-3\alpha}{2\alpha}.$$

Unde

$$\frac{p\int x^{\pi}dx \int x^{\frac{\beta-3\alpha}{2\alpha}} sdx}{2\alpha} = \frac{2\alpha p x^{\frac{\beta+\alpha}{2\alpha}+\pi}}{\beta+3\alpha+2\alpha\pi} + \ldots\ldots + \frac{2\alpha p(\alpha+\beta)\ldots(\alpha(2n-2)+\beta)x^{n+\pi+\frac{\beta+\alpha}{s\alpha}}}{\beta+\alpha+2\alpha n+2\alpha\pi}.$$

Atque iterum

$$\frac{p\int x^{\pi}dx \int x^{\frac{\beta-3\alpha}{2\alpha}} sdx}{2\alpha} = \frac{2\alpha p x^{\frac{\beta+\alpha}{2\alpha}+\pi}}{\beta+3\alpha+2\alpha\pi} + \ldots\ldots + \frac{2\alpha p(\alpha+\beta)\ldots(\alpha(2n-2)+\beta)x^{n+\pi+\frac{\beta+\alpha}{s\alpha}}}{\beta+\alpha+2\alpha n+2\alpha\pi}.$$

Fiat

$$4p\alpha^{2}n - 4p\alpha^{2} + 2p\alpha\beta = 2\alpha n + 2\alpha\pi + \alpha + \beta;$$

erit

$$p=\frac{1}{2\alpha}\text{ et }\pi=\frac{\beta-2\alpha}{2\alpha}-\frac{\alpha+\beta}{2\alpha}=-\frac{3}{2},$$

consequenter

$$\frac{\int x^{-\frac{3}{2}}dx\int x^{\frac{\beta-3\alpha}{2\alpha}}sdx}{4\alpha^2x^{\frac{\beta}{2\alpha}}}-\frac{1}{\beta}=(\alpha+\beta)x+\ldots+(\alpha+\beta)(2\alpha+\beta)..(\alpha(2n-3)+\beta)x^{n-1}$$

$$=s-Ax^n.$$

Posito

$$A=(\alpha+\beta)(2\alpha+\beta)....+(\alpha(2n-1)+\beta).$$

Ex qua aequatione $s$ innotescit.

§ 15. Simili modo operationem institui oportet, si in coefficiente termini sequentis tres pluresve factores de novo accedunt. De quo notandum est tot semper in aequatione resultante signa integralia sibi invicem esse iuncta, quot sunt factores, quibus sequens quisque terminus augetur. Ita progressionis

$$(\alpha+\beta)x+.....+(\alpha+\beta)(2\alpha+\beta)....(\alpha(3n-2)+\beta)x^n$$

summa $s$ determinabitur ex hac aequatione

$$\frac{\int x^{-\frac{4}{3}}dx\int x^{-\frac{4}{3}}dx\int x^{\frac{\beta-5\alpha}{3\alpha}}sdx}{27\alpha^3x^{\frac{\beta-\alpha}{3\alpha}}}-\frac{1}{\beta(\beta-\alpha)}=s-(\alpha+\beta)....(\alpha(3n-2)+\beta)x^n.$$

Ex qua, ut inductio ad sequentes casus fieri possit, notandum est $\frac{1}{\beta(\beta-\alpha)}$ esse terminum progressionis propositae ante primum seu eum, cuius index est 0. Si factores, qui in potentias ipsius $x$ ducantur, non constituant progressionem arithmeticam, sed aliam algebraicam alterios ordinis, operatio similiter debet institui; ut sit progressio proposita

$$(\alpha+\beta)(\gamma+\delta)x+.....+(\alpha+\beta)(2\alpha+\beta)....(\alpha n+\beta)(\gamma+\delta)(2\gamma+\delta)...(\gamma n+\delta)x^n;$$

ponatur huius summa $s$; erit

$$p\int x^{\pi}sdx=\frac{p(\alpha+\beta)(\gamma+\delta)x^{\pi+2}}{\pi+2}+.....+\frac{p(\alpha+\beta)....(\alpha n+\beta)(\gamma+\delta)......(\gamma+\delta)(\gamma n+\delta)x^{n+\pi+1}}{n+\pi+1}.$$

Ponatur

$$p\gamma n+p\delta=n+\pi+1;$$

erit

$$p=\frac{1}{\gamma}\text{ et }\pi=\frac{\gamma-\delta}{\gamma}.$$

Ergo

$$\frac{\int x^{\frac{\delta-\gamma}{\gamma}} sdx}{\gamma} = (\alpha+\beta)x^{\frac{\delta+\gamma}{\gamma}} + \ldots\ldots + (\alpha+\beta)\ldots(\alpha n+\beta)(\gamma+\delta)\ldots\ldots(\gamma+\delta)(\gamma(n-1)+\delta)x^{n+\frac{\delta}{\gamma}}.$$

Porro erit

$$\frac{p\int x^{\pi} dx \int x^{\frac{\delta-\gamma}{\gamma}} sdx}{\gamma} = \frac{\gamma p(\alpha+\beta)x^{\frac{\delta+2\gamma}{\gamma}+\pi}}{\delta+2\gamma+\pi\gamma} + \ldots\ldots + \frac{\gamma p(\alpha+\beta)\ldots(\alpha n+\beta)(\gamma+\delta)\ldots\ldots(\gamma+\delta)(\gamma(n-1)+\delta)x^{n+\pi+\frac{\delta+\gamma}{\gamma}}}{\delta n+\pi\gamma+\delta+\gamma}.$$

Fiat

$$p\alpha\gamma n + p\beta\gamma = \gamma n + n\gamma + \delta + \gamma\,;$$

erit

$$p = \frac{1}{\alpha} \text{ et } \pi = \frac{\beta}{\alpha} - \frac{\delta}{\gamma} - 1 = \frac{\beta\gamma - \alpha\delta - \alpha\gamma}{\alpha\gamma}.$$

Ergo

$$\frac{\int x^{\frac{\beta\gamma-\alpha\delta-\alpha\gamma}{\alpha\gamma}} dx \int x^{\frac{\delta-\gamma}{\gamma}} sdx}{\alpha\gamma} = x^{\frac{\beta+\alpha}{\alpha}} + \ldots\ldots + (\alpha+\beta)\ldots(\alpha(n-1)+\beta)(\gamma+\delta)\ldots\ldots(\gamma+\delta)(\gamma(n-1)+\delta)x^{\frac{\beta}{\alpha}+n}.$$

Consequenter

$$\frac{\int x^{\frac{\beta\gamma-\alpha\delta-\alpha\gamma}{\alpha\gamma}} dx \int x^{\frac{\delta-\gamma}{\gamma}} sdx}{\alpha\gamma x^{\frac{\beta+\alpha}{\alpha}}} - 1 = s - ABx^{n}$$

posito

$$A = (\alpha+\beta)\ldots(\alpha n+\beta) \text{ et } B = (\gamma+\delta)\ldots\ldots(\gamma+\delta)(\gamma n+\delta).$$

Hic est casus, si progressionis, quam factores conficiunt, terminus generalis est

$$(\alpha n+\beta)(\gamma n+\delta) \text{seu } \alpha\gamma n^{2}(\alpha\delta+\beta\gamma)n + \beta\delta.$$

Comprehenduntur ergo sub hac forma omnes progressiones ordinis secundi. Superior autem formula, ex qua $s$ determinatur, transmutatur in hanc

$$\frac{\int x^{\frac{\delta-\gamma}{\gamma}} sdx}{(\beta\gamma-\alpha\delta)x^{\frac{\delta+\gamma}{\gamma}}} + \frac{\int x^{\frac{\beta-\alpha}{\alpha}} sdx}{(\alpha\delta-\beta\gamma)x^{\frac{\beta+\alpha}{\alpha}}} = 1 + s - ABx^{n}.$$

Ex qua facilius forma sequentium intelligi potest.

§ 16. Considerabo nunc harum serierum reciprocas, in quibus potentiae ipsius $x$ sunt divisae per id, per quod ante erant multiplicatae. Sit igitur series summanda haec

$$\frac{x}{\alpha+\beta}+\frac{x^2}{(\alpha+\beta)(2\alpha+\beta)}+\ldots\ldots+\frac{x^n}{(\alpha+\beta)(2\alpha+\beta)\ldots(\alpha n+\beta)};$$

huius summa ponatur $s$. Erit

$$\frac{pd\left(x^{\pi}s\right)}{dx}=\frac{p(\pi+1)x^{\pi}}{(\alpha+\beta)}+\frac{p(\pi+2)x^{\pi+1}}{(2\alpha+\beta)}+\ldots\ldots+\frac{p(\pi+n)x^{\pi+n-1}}{(\alpha+\beta)\ldots(\alpha n+\beta)}.$$

Fiat

$$pn+p\pi=\alpha n+\beta;$$

erit

$$p=\alpha \text{ et } \pi=\frac{\beta}{\alpha}.$$

Quamobrem erit

$$\frac{pd(x^{\frac{\beta}{\alpha}}s)}{dx}=x^{\frac{\beta}{\alpha}}+\frac{x^{\frac{\beta}{\alpha}+1}}{(\alpha+\beta)}+\ldots\ldots+\frac{x^{\frac{\beta}{\alpha}+n-1}}{(\alpha+\beta)(2\alpha+\beta)\ldots(\alpha(n-1)+\beta)}$$

et propterea

$$\frac{pd(x^{\frac{\beta}{\alpha}}s)}{x^{\frac{\beta}{\alpha}}dx}=1+s-\frac{x^n}{A}$$

posito, ut ante,

$$A=(\alpha+\beta)\ldots(\alpha n+\beta).$$

Aequatio haec evoluta dabit

$$\alpha x^{\frac{\beta}{\alpha}}ds+\beta x^{\frac{\beta-\alpha}{\alpha}}sdx=x^{\frac{\beta}{\alpha}}dx+x^{\frac{\beta}{\alpha}}sdx-\frac{x^{\frac{\beta}{\alpha}+n}dx}{A},$$

quae divisa per $x^{\frac{\beta}{\alpha}-1}$ transit in

$$\alpha xds+\beta sdx=xdx+xsdx-\frac{x^{n+1}dx}{A}$$

seu

$$ds+\frac{\beta sdx}{\alpha x}-\frac{sdx}{\alpha}=\frac{dx}{\alpha}-\frac{x^n dx}{A\alpha}.$$

Multiplicetur haec aequatio per $c^{-\frac{x}{\alpha}}x^{\frac{\beta}{\alpha}}$, ubi $c$ est numerus, cuius logarithmus est 1; fiet ea integrabilis prodibitque

$$c^{-\frac{x}{\alpha}}x^{\frac{\beta}{\alpha}}s=\frac{1}{\alpha}\int c^{-\frac{x}{\alpha}}x^{\frac{\beta}{\alpha}}dx(1-\frac{x^n}{A})$$

atque

$$s=\frac{1}{\alpha}c^{\frac{x}{\alpha}}x^{\frac{-\beta}{\alpha}}\int c^{-\frac{x}{\alpha}}x^{\frac{\beta}{\alpha}}dx(1-\frac{x^n}{A}).$$

Huius progressionis in infinitum continuatae summa igitur erit

$$\frac{1}{\alpha}c^{\frac{x}{\alpha}}x^{\frac{-\beta}{\alpha}}\int c^{-\frac{x}{\alpha}}x^{\frac{\beta}{\alpha}}dx=\beta(\beta-\alpha)(\beta-2\alpha)\alpha c^{\frac{x}{\alpha}}x^{\frac{-\beta}{\alpha}}$$
$$-1-\frac{\beta}{x}-\frac{\beta(\beta-\alpha)}{x^2}-\ldots\ldots-\frac{\beta(\beta-\alpha)\ldots\ldots\alpha}{x^{\frac{\beta}{\alpha}}}.$$

Si fuerit $\beta=0$, erit summa

$$=c^{\frac{x}{\alpha}}-1.$$

Sin sit $\beta=\alpha$, erit summa

$$=\frac{\alpha c^{\frac{x}{\alpha}}}{x}-1-\frac{\alpha}{x}.$$

Si vero ponatur $\beta=2\alpha$, summa seriei erit

$$=\frac{2\alpha^2c^{\frac{x}{\alpha}}}{x^2}-1-\frac{2\alpha}{x}-\frac{2\alpha^2}{x^2}$$

et ita porro. Ex quo intelligitur, quoties $\beta$ sit multiplum ipsius $\alpha$, summam seriei finita et integrata expressione exhiberi posse. Si autem $\frac{\beta}{\alpha}$ evadat fractio, formula inventa non potest.

§ 17. Crescat terminus quisque duobus factoribus; habebitur progressio haec

$$\frac{x}{\alpha+\beta}+\frac{x^2}{(\alpha+\beta)..(3\alpha+\beta)}+\frac{x^3}{(\alpha+\beta)..(5\alpha+\beta)}+\ldots..+\frac{x^n}{(\alpha+\beta)\ldots(\alpha(2n-1)+\beta)}.$$

Cuius summa ponatur $s$; erit

$$\frac{pd(x^\pi s)}{dx}=\frac{p(\pi+1)x^\pi}{(\alpha+\beta)}+\frac{p(\pi+2)x^{\pi+1}}{(\alpha+\beta)\ldots(3\alpha+\beta)}+\ldots..+\frac{p(\pi+n)x^{\pi+n-1}}{(\alpha+\beta)\ldots(\alpha(2n-1)+\beta)}.$$

Sit

$$pn+p\pi=2\alpha n-\alpha+\beta;$$

erit

$$p=2\alpha \text{ et } \pi=\frac{\beta-\alpha}{2\alpha},$$

idcirco

$$\frac{2\alpha d(x^{\frac{\beta-\alpha}{2\alpha}}s)}{dx}=x^{\frac{\beta-\alpha}{2\alpha}}+\frac{x^{\frac{\beta+\alpha}{2\alpha}}}{(\alpha+\beta)(2\alpha+\beta)}+\ldots..+\frac{x^{\frac{\beta-3\alpha}{\alpha}+n}}{(\alpha+\beta)\ldots(\alpha(2n-2)+\beta)}.$$

Atque iterum

$$\frac{2\alpha pd(x^\pi d(x^{\frac{\beta-\alpha}{2\alpha}}s))}{dx^2}=\frac{p(\beta-\alpha+2\alpha\pi)}{2\alpha}x^{\frac{\beta-3\alpha}{2\alpha}+\pi}+\ldots..+\frac{p(\beta-3\alpha+2\alpha n+2\alpha\pi)x^{\frac{\beta-5\alpha}{\alpha}+n+\pi}}{2\alpha(\alpha+\beta)(2\alpha+\beta)\ldots(\alpha(2n-2)+\beta)}.$$

Fiat

$$p\beta-3p\alpha+2p\alpha n+2p\alpha\pi=4\alpha^2n-4\alpha^2+2\alpha\beta;$$

erit

$$p = 2\alpha \text{ et } \pi = \tfrac{1}{2}.$$

Unde prodit

$$\frac{4\alpha^2 d(x^{\frac{1}{2}} d(x^{\frac{\beta-\alpha}{2\alpha}} s))}{dx^2} = \beta x^{\frac{\beta-2\alpha}{2\alpha}} + \ldots\ldots + \frac{x^{\frac{\beta-4\alpha}{2\alpha}+n}}{(\alpha+\beta)\ldots(\alpha(2n-3)+\beta)}$$
$$= \beta x^{\frac{\beta-2\alpha}{2\alpha}} + x^{\frac{\beta-2\alpha}{2\alpha}} s - \frac{x^{\frac{\beta-2\alpha}{2\alpha}+n}}{(\alpha+\beta)\ldots(\alpha(2n-1)+\beta)}.$$

Simili modo operatio est instituenda, si terminus quisque pluribus factoribus in denominatore crescat. Nec non satis apparet, si progressio, quam factores denominatorum constituunt, non fuerit arithmetica, sed algebraica alterioris ordinins, quomodo ad aequationem, ex qua summa determinatur, perveniri oporteat. Scilicet quilibet factor in factores simplices est resolvendus, ut § 15 factum est, ubi terminus generalibus factorum est $(\alpha n+\beta)(\gamma n+\delta)$, qui omnes aequationes ordinis secundi sub se complectitur. At ne hoc quidem opus est, si sequenti modo operari libuerit. Ut proposita sit progressio

$$\frac{x}{1} + \frac{x^2}{1.7} + \frac{x^3}{1.7.17} + \frac{x^4}{1.7.17.31} + \ldots\ldots + \frac{x^n}{1.7\ldots(2n^2-1)}$$

summa huius ponatur $s$; erit

$$\frac{pd(x^\pi s)}{dx} = p(\pi+1)x^\pi + \ldots\ldots + \frac{p(\pi+n)x^{n+\pi-1}}{1.7\ldots(2n^2-1)}.$$

Atque denuo

$$\frac{pd(x^\rho(x^\pi s))}{dx^2} = p(\pi+1)(\pi+\rho)x^{\pi+\rho-1} + \ldots\ldots + \frac{p(\pi+n)(n+\pi+\rho-1)(\pi+n)x^{n+\pi+\rho-2}}{1.7\ldots(2n^2-1)}.$$

Fiat

$$pn^2 + 2p\pi n + p\rho n - pn + p\pi^2 + p\pi\rho - p\pi = 2n^2 - 1;$$

erit

$$p = 2,\ 4\pi + 2\rho - 2 = 0 \text{ seu } \rho = 1 - 2\pi$$

atque

$$-2\pi^2 = -1 \text{ seu } \pi = \sqrt{\tfrac{1}{2}} \text{ et } \rho = 1 - \sqrt{2}.$$

Quare habebitur

$$\frac{2d(x^{1-\sqrt{2}}(x^{\sqrt{\frac{1}{2}}} s))}{dx^2} = x^{\sqrt{\frac{1}{2}}} + \ldots\ldots + \frac{x^{\frac{2n-2-\sqrt{2}}{2}}}{1.7\ldots(2n^2-4n+1)}$$
$$= x^{\frac{-\sqrt{2}}{2}} + x^{\frac{-\sqrt{2}}{2}}\left(s - \frac{x^n}{1.7\ldots(2n^2-1)}\right).$$

Summa vero huius seriei infinitum continuatae invenietur ex hac aequatione

$$\left(2-2\sqrt{2}\right)x^{-\sqrt{2}}d(x^{\sqrt{\frac{1}{2}}}s)+2x^{1-\sqrt{2}}dd(x^{\sqrt{\frac{1}{2}}}s)=\left(2-2\sqrt{2}\right)x^{\frac{-\sqrt{2}}{2}}dxds+\left(\sqrt{2}-2\right)x^{\frac{-2-\sqrt{2}}{2}}sdx^2$$
$$+2x^{\frac{2-\sqrt{2}}{2}}dds+2\sqrt{2}x^{\frac{-\sqrt{2}}{2}}dxds+\left(1-\sqrt{2}\right)x^{\frac{-2-\sqrt{2}}{2}}sdx^2$$
$$=x^{\frac{-\sqrt{2}}{2}}dx^2+x^{\frac{-\sqrt{2}}{2}}sdx^2=2x^{\frac{-\sqrt{2}}{2}}dsdx-x^{\frac{-2-\sqrt{2}}{2}}sdx^2+2x^{\frac{2-\sqrt{2}}{2}}dds$$

seu

$$2xdds-\frac{sdx^2}{x}+2dsdx=dx^2+sdx^2,$$

ex qua aequatione irrationalia omnia evanuere.

§ 18. Si factores denominatorum constituant progressionem potentiarum, huiusmodi progressionum summas investigabo. Ut sit progressio proposita

$$\frac{x}{(\alpha+\beta)^2}+\frac{x^2}{(\alpha+\beta)^2(2\alpha+\beta)^2}+.....+\frac{x^n}{(\alpha+\beta)^2....(n\alpha+\beta)^2};$$

ponatur summa $s$; erit

$$\frac{pd(x^{\pi}s)}{dx}=\frac{p(\pi+1)x^{\pi}}{(\alpha+\beta)^2}+.....+\frac{p(\pi+n)x^{n+\pi-1}}{(\alpha+\beta)^2....(n\alpha+\beta)^2};$$

fiat

$$p\pi+pn=\alpha n+\beta;$$

erit

$$p=\alpha \text{ et } \pi=\frac{\beta}{\alpha}.$$

Propterea

$$\frac{\alpha d(x^{\frac{\beta}{\alpha}}s)}{dx}=\frac{x^{\frac{\beta}{\alpha}}}{\alpha+\beta}+.....+\frac{x^{\frac{\beta}{\alpha}+n-1}}{(\alpha+\beta)^2....(n\alpha+\beta)}.$$

Porro

$$\frac{\alpha pd(x^{\pi}d(x^{\frac{\beta}{\alpha}}s))}{dx^2}=\frac{p(\beta+\alpha\pi)x^{\frac{\beta}{\alpha}+\pi-1}}{\alpha(\alpha+\beta)}+.....+\frac{p(\beta+\alpha\pi+\alpha n-\alpha)x^{\frac{\beta}{\alpha}+\pi+n-2}}{\alpha(\alpha+\beta)^2....(\alpha n+\beta)}.$$

Fiat

$$p\alpha n+p\beta+p\alpha\pi-p\alpha=\alpha^2 n+\alpha\beta,$$

erit

$$p=\alpha \text{ et } \pi=\frac{\alpha}{\alpha}=1.$$

Unde est

$$\frac{\alpha^2 d(xd(x^{\frac{\beta}{\alpha}}s))}{dx^2}=x^{\frac{\beta}{\alpha}}+.....+\frac{x^{\frac{\beta}{\alpha}+n-1}}{(\alpha+\beta)^2....(\alpha(n-1)+\beta)^2}$$
$$=x^{\frac{\beta}{\alpha}}+x^{\frac{\beta}{\alpha}}(s-\frac{x^n}{(\alpha+\beta)^2....(\alpha n+\beta)^2}).$$

Et summa progressionis in infiniturm determinabitur aequatione

$$\frac{\alpha^2 d(xd(x^{\frac{\beta}{\alpha}}s))}{x^{\frac{\beta}{\alpha}}dx^2}=1+s.$$

Similiter si factores fuerint cubi, summa progressionis

$$\frac{x}{(\alpha+\beta)^3}+\frac{x^2}{(\alpha+\beta)^3(2\alpha+\beta)^3}+\text{etc. in infinitum}$$

$s$ invenietur ex hac aequatione

$$\frac{\alpha^3 \mathrm{d}(xd(xd(x^{\frac{\beta}{\alpha}}s)))}{x^{\frac{\beta}{\alpha}}dx^3}=1+s.$$

Atque ita porro pro sequentibus.

§ 19. Sint nunc coefficientes potentiarum ipsius $x$ fractiones, quarum tam numeratores quam denominatores sint facta ex certo factorum numero pro indice cuiusque termini crescente constantia. Ita sit progressio proposita haec

$$\frac{a+b}{\alpha+\beta}x+\frac{(a+b)(2a+b)}{(\alpha+\beta)(2\alpha+\beta)}x^2+.....+\frac{(a+b)....(an+b)}{(\alpha+\beta)....(\alpha n+\beta)}x^n;$$

huius summa ponatur $s$; erit

$$p\int x^{\pi}sdx=\frac{p(a+b)}{(\pi+2)(\alpha+\beta)}x^{\pi+2}+.....+\frac{p(a+b)....(an+b)}{(\pi+n+1)(\alpha+\beta)....(\alpha n+\beta)}x^{\pi+n+1}.$$

Fiat

$$\alpha pn+bp=\pi+n+1;$$

erit

$$p=\frac{1}{a}\text{ et }\pi=\frac{b-a}{a}$$

adeoque

$$\frac{\int x^{\frac{b-a}{a}}sdx}{a}=\frac{x^{\frac{b+a}{a}}}{\alpha+\beta}+.....+\frac{(a+b)....(a(n-1)+b)}{(\alpha+\beta)....(\alpha n+\beta)}x^{\frac{b}{a}+n}.$$

Et denuo

$$\frac{pd(x^{\pi}\int x^{\frac{b-a}{a}}sdx)}{adx}=\frac{p(b+a+a\pi)}{a(\alpha+\beta)}x^{\frac{b}{a}+\pi}+.....+\frac{p(b+an+a\pi)(a+b)....(a(n-1)+b)}{a(\alpha+\beta)....(\alpha n+\beta)}x^{\frac{b}{a}+\pi+n-1}.$$

Fiat

$$bp+apn+ap\pi=a\alpha n+a\beta;$$

erit

$$p=\alpha\text{ et }\pi=\frac{\beta}{\alpha}-\frac{b}{a},$$

unde erit

$$\frac{\alpha d(x^{\frac{\beta}{\alpha}-\frac{b}{a}}\int x^{\frac{b-a}{a}}sdx)}{adx}=x^{\frac{b}{a}}+.....+\frac{(a+b)....(a(n-1)+b)}{(\alpha+\beta)....(\alpha(n-1)+\beta)}x^{\frac{b}{a}+n-1}=x^{\frac{b}{a}}+x^{\frac{b}{a}}\left(s-\frac{(a+b)....(a(n-1)+b)}{(\alpha+\beta)....(\alpha(n-1)+\beta)}x^{n}\right).$$

Ex qua aequatione $s$ determinare licet. Si summa progressionis propositae in infinitum desideretur, erit

$$\frac{\alpha d(x^{\frac{\beta}{\alpha}-\frac{b}{a}}\int x^{\frac{b-a}{a}}sdx)}{adx}=x^{\frac{b}{a}}+x^{\frac{b}{a}}s$$

seu

$$\frac{\alpha}{a}(\frac{\beta}{\alpha}-\frac{b}{a})x^{\frac{\beta}{\alpha}-\frac{b}{a}-1}\int x^{\frac{b-a}{a}}sdx+\frac{\alpha}{a}x^{\frac{\beta}{\alpha}-1}s=x^{\frac{\beta}{\alpha}}+x^{\frac{\beta}{\alpha}}s,$$

quae abit in hanc

$$(\frac{\beta}{a}-\frac{\alpha b}{aa})x^{-\frac{b}{a}}\int x^{\frac{b-a}{a}}sdx+\frac{\alpha}{a}s=x+xs$$

vel hanc

$$(\frac{\beta}{a}-\frac{\alpha b}{a^2})\int x^{\frac{b-a}{a}}sdx+\frac{\alpha}{a}s=x^{\frac{b+a}{a}}+x^{\frac{b+a}{a}}s.$$

Haec differentia dat

$$(\frac{\beta}{a}-\frac{\alpha b}{a^2})x^{\frac{b-a}{a}}sdx+\frac{\alpha}{a}x^{\frac{b}{a}}ds+\frac{\alpha b}{a^2}x^{\frac{b-a}{a}}sdx=\frac{b+a}{a}x^{\frac{b}{a}}dx+x^{\frac{b+a}{a}}ds+\frac{b+a}{a}x^{\frac{b}{a}}sdx,$$

quae reducitur ad hanc

$$\frac{\beta}{a}sdx+\frac{\alpha}{a}xds=\frac{b+a}{a}xdx+x^2ds+\frac{b+a}{a}xsdx$$

seu

$$ds+\frac{\beta sdx-(b+a)xsdx}{ax-ax^2}=\frac{(b+a)xdx}{ax-ax^2}.$$

Multiplicatur haec aequatio per

$$c^{\int\frac{\beta dx-(b+a)xdx}{ax-ax^2}}\text{ vel per } x^{\frac{\beta}{\alpha}}(\alpha-ax)x^{\frac{b}{a}-\frac{\beta}{\alpha}+1}$$

erit

$$x^{\frac{\beta}{\alpha}}(\alpha\text{-}ax)x^{\frac{b}{a}-\frac{\beta}{\alpha}+1}s=(b+a)\int x^{\frac{\beta}{\alpha}}(\alpha\text{-}ax)^{\frac{b}{a}-\frac{\beta}{\alpha}}dx$$

atque

$$s=\frac{(b+a)\int x^{\frac{\beta}{\alpha}}(\alpha\text{-}ax)^{\frac{b}{a}-\frac{\beta}{\alpha}}dx}{x^{\frac{\beta}{\alpha}}(\alpha\text{-}ax)x^{\frac{b}{a}-\frac{\beta}{\alpha}+1}}.$$

Summa igitur algebraice poterit assignari, si vel $\frac{\beta}{\alpha}$ vel $\frac{b}{a}-\frac{\beta}{\alpha}$ fuerit numerus integer affirmativus.

§ 20. Si progressio fuerit ex huiusmodi ipsius $x$ coefficientibus et algebraicis composita, prima coefficentes algebraici differentiatione et integratione debent tolli, ut ibi est factum, et tum progressio resultans modo hic exposito tractari. Ut sit progressio proposita

$$\frac{1x}{1}+\frac{3x^2}{1.2}+\frac{3x^3}{1.2.3}+\ldots\ldots+\frac{(2n-1)x^n}{1.2.3\ldots n};$$

summa huius ponatur $s$; erit

$$p\int x^{\pi} sdx = \frac{1.px^{\pi+2}}{(\pi+2)1} + \ldots\ldots + \frac{(2n-1)px^{\pi+n+1}}{(\pi+n+1)1.2.3....n}.$$

Fiat

$$2np - p = \pi + n + 1;$$

erit

$$p = \frac{1}{2} \text{ et } \pi = -\frac{3}{2},$$

ex quo erit

$$\frac{\int x^{-\frac{3}{2}} sdx}{2} = \frac{x^{\frac{1}{2}}}{1} + \frac{x^{\frac{3}{2}}}{1.2} + \ldots\ldots + \frac{x^{n-\frac{1}{2}}}{1.2.3....n}.$$

Multiplicatur per $x^{\frac{1}{2}}$; erit

$$\frac{x^{\frac{1}{2}}\int x^{-\frac{3}{2}} sdx}{2} = \frac{x}{1} + \frac{x^2}{1.2} + \ldots\ldots + \frac{x^n}{1.2.3....n},$$

ergo

$$\frac{d(x^{\frac{1}{2}}\int x^{-\frac{3}{2}} sdx)}{2dx} = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \ldots\ldots + \frac{x^{n-1}}{1.2.3....(n-1)},$$

$$= 1 + \frac{x^{\frac{1}{2}}\int x^{-\frac{3}{2}} sdx}{2} - \frac{x^n}{1.2.3....n},$$

ex qua aequatione $s$ invenietur. Erit autem

$$\frac{\int x^{-\frac{3}{2}} sdx}{4x^{\frac{1}{2}}} + \frac{s}{2x} = 1 + \frac{x^{\frac{1}{2}}\int x^{-\frac{3}{2}} sdx}{2} - \frac{x^n}{1.2.3....n}.$$

Ponatur 1.2.3.....$n = A$; erit porro

$$(1-2x)\int x^{-\frac{3}{2}} sdx = 4x^{\frac{1}{2}} - \frac{2s}{x^{\frac{1}{2}}} - \frac{4x^{n+\frac{1}{2}}}{A}.$$

Summa progressionis propositae in infinitum continuatae vero definietur ex ista aequatione

$$\int x^{-\frac{3}{2}} sdx = \frac{4x-2s}{(1-2x)\sqrt{x}},$$

quae differentiata dat

$$\frac{sdx}{x\sqrt{x}} = \frac{2xdx+4xxdx+sdx-6sxdx-2xds+4x^2ds}{(1-2x)^2 x\sqrt{x}}$$

seu

$$xdx + 2x^2dx - sxdx - 2sx^2dx - xds + 2x^2ds = 0,$$

quae reducitur ad hanc

$$ds + \frac{sdx(1+2x)}{1-2x} = \frac{dx(1+2x)}{1-2x}.$$

Quae multiplicata per $\frac{c^{-x}}{1-2x}$ fit integrabilis; prodit autem

$$\frac{c^{-x}s}{1-2x} = \int \frac{c^{-x}dx(1+2x)}{(1-2x)^2} = \frac{c^{-x}}{1-2x} - 1$$

atque hinc

$$s = 1 - c^x(1 + 2x).$$

Quare si fuerit $x = \frac{1}{2}$, erit $s$ = 1. Adeoque

$$1 = \frac{1}{1.2} + \frac{3}{1.2.4} + \frac{5}{1.2.3.8} + \frac{7}{1.2.3.4.16} + \text{etc. in infin.}$$

§ 21. Ex his apparet, ad quas progressiones summandas methodus hac dissertatione exposita se extendat: scilicet ad omnes eas progressiones, quae comprehenduntur hoc termino generali $\frac{AP}{BQ}x^{\alpha n+\beta}$, ubi $A$ et $B$ designant terminos ordinis $n$ quarumcunque progressionum algebraicarum, et $P$ est factum ex $\gamma n + \delta$ terminis progressionis cuiscunque algebraicae, itemque Q est simile factum ex $\varepsilon n + \zeta$ terminis etiam cuiuscunque progressionis algebraicae. Omnino autem summae huiusmodi progressionum tribus modis expositae invenientur. Vel enim prodit summa prorsus algebraica, vel assignatur quadratura quaepiam, a qua summa pendet. Vel tertio aequatio reperitur, cuius variabiles quantitates $s$ et $x$ penitus non possunt a se invicem separari, ut saltem non constet, utrum progressio summam habeat algebraicam an a cuius curvae quadratura pendeat. Quamvis vero haec methodus tam late pateat, tamen innumerae occurrere possunt progressiones per eam non summabiles, quarum quidem vel nullo alio modo summae assignari possunt, ut huius

$$1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \ldots\ldots + \frac{1}{2^n - 1},$$

vel quarum summae etiam constant, ut huius

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \text{etc.}$$

termino generali existente $\frac{1}{a^\alpha - 1}$, in quo $a$ et $\alpha$ numerus quoscunque integros praeter unitatem denotant, cuius summam esse = 1 demonstravit Celeberrimus Goldbachius, [Vide Commentationem 72 hius voluminis, Theorema I.] Quia autem eius terminus generalis proprie sic dictus non potest exhiberi, mirum non est eam hac methodo non posse summari.