The Solution of a Singular Case Concerning Tautochronism.

Author

Leonhard Euler

§.1. Within the passing of three years, the most illustrious *Bernoulli* had proposed a method for finding innumerable tautochronous curves *in vacuo*, where he made mention of an an elegant problem, the solution of which I am going to present in this paper. Indeed this problem was considered to be the most difficult from that time, and on that account I was spending little time on finding a solution. Truly after I had investigated the flow of fluids with more diligence, I came upon a method for solving all the problems of this kind, which also led me to the solution of this problem.

§.2. Moreover, this is the problem: *For a given curve ANB at B, to adjoin* (Fig.1) *a curve BMC with this property, so that all the descents due to gravity beginning at some point along the curve BMC as far as to the lowest point A shall be made in equal times.*

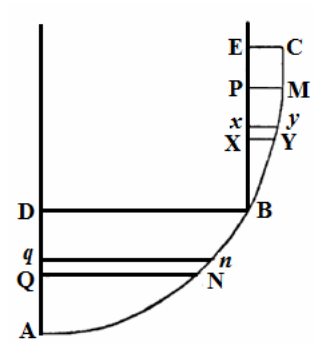


Fig. 1

Therefore it is required to find the curve *BMC*, from this condition, as assumed on the curve *BMC* for any point *M* as it pleases, the descent time along *MBNA* shall be constant, and it shall not depend on the location of the point *M*. Or the descent time along *MBNA* must be equal to the descent time along the given curve *BNA* ; which is the case with the point *M* beginning at *B*.

§.3. Therefore the body shall fall from the point *M*, and we seek the descent time along the arc *MB* and the arc *BMA*. With the vertical line drawn *BP*, there shall be put BP = a , therefore which letter, since the position of the point *M* is defined, cannot be present in the expression of the time to fall through *MBA*. The height *AD* of the given curve shall be $= c$. Some applied line *QN* and *XY* shall be assumed in each part of the curve and for these the nearest applied lines *qn* and *xy*. The following are called :
$AQ = t,\ AN = r$ and $BX = x,\ BY = s;$ of which the equation between *t* and *r* is given, and the equation between *x* and *s* shall be desired. The speed, which the body will have at *N*, is

$$\sqrt{(a+c-t)} = \sqrt{(PB+DQ)}.$$

And thus the time, in which the arc *AN* is absolved, is est

$$\int \frac{dr}{\sqrt{(a+c-t)}}.$$

Which integral thus must be taken, so that it shall become equal to zero, if the time shall be taken to be $t=0$.

§.4. Following which, if there may be put $t=c$, the time will be given by the integral of the given curve *BNA*, which therefore will be set out by a formula composed from *a* and with constants. I have computed several special cases, and I have considered the descent time along the curve *BNA* from the initial position at *M*, always to be able to be set out by the following series :

$$k-\alpha a-\beta a^2-\gamma a^3-\delta a^4-\text{etc.}-\zeta\sqrt{a}\ -\eta a\sqrt{a}-\vartheta a^2\sqrt{a}-\text{etc.},$$

of which the coefficients $\alpha,\ \beta$ etc. and *k* will be able to be determined in any special case. This time therefore, added to the descent time along *MB*, must be constant: and so that it is necessary that the sum of all the letters *a* affected may themselves be removed.

§.5. For the descent time along the curve *MB* requiring to be found, the speed at $Y=\sqrt{(a-x)}$, and the element of the time $=ds:\sqrt{(a-x)}$. The integral of this, thus to be assumed, so that it shall become $=0$ if $x=0$ will give the descent time along YB, in which therefore if there shall be put $x=a$, the descent time along *MB* will be produced, which since in the first place since the constant quantity must be independent from *a*. If the point *M* shall fall at the point *B*, *i. e*. if *a* shall vanish, the whole descent time will be the descent time along the curve *BMA*, which from the above formula emerges $=k$. On this account the remaining descent time along the curve *MBNA* must be $=k$. Hence the time along *MB* must become

$$=\alpha a+\beta a^2+\gamma a^3+\text{etc.}+\zeta\sqrt{a}+\eta a\sqrt{a}+\vartheta a^2\sqrt{a}+\text{etc.}$$

§.6 So that this shall be the case, I assume the following equation for the curve :

$$ds=Adx\sqrt{x}+\ Bxdx\sqrt{x}+Cx^2dx\sqrt{x}+etc+Edx+Fxdx+Gx^2dx+etc.$$

Therefore the descent time along the arc *M* B will be

$$=\int\frac{Adx\sqrt{x}}{\sqrt{(a-x)}}+\int\frac{Bxdx\sqrt{x}}{\sqrt{(a-x)}}+\int\frac{Cx^2dx\sqrt{x}}{\sqrt{(a-x)}}+\text{etc.}+\int\frac{Exdx}{\sqrt{(a-x)}}+\int\frac{Fxdx}{\sqrt{(a-x)}}+\int\frac{Gx^2dx}{\sqrt{(a-x)}}dx+\text{etc.}$$

clearly, if with these integrals taken, so that they shall become $=0$, if $x=0$ and $x=a$ everywhere. Therefore the coefficients *A*, *B*, *C* , etc. shall be determined so that

$$\int \frac{Adx\sqrt{x}}{\sqrt{(a-x)}} = \alpha a;\ \int \frac{Bxdx\sqrt{x}}{\sqrt{(a-x)}} = \beta a^2;\ \int \frac{Cx^2dx\sqrt{x}}{\sqrt{(a-x)}} = \gamma a^3 \text{ etc.}$$

$$\text{and } \int \frac{Edx}{\sqrt{(a-x)}} = \zeta\sqrt{a};\ \int \frac{Fxdx}{\sqrt{(a-x)}} = \eta a\sqrt{a};\ \int \frac{Gx^2dx}{\sqrt{(a-x)}}dx = \theta a^2\sqrt{a} \text{ etc.}$$

Truly I have assumed that same value for *ds*, so that the letters A, B, C, etc. may be determined, not depending on a .

§. 7. The integration of this expression $\frac{Adx\sqrt{x}}{\sqrt{(a-x)}}$ depends on the quadrature of the circle; But if it may be integrated with the aid of imaginary logarithms, as ought to be the case, and there shall be put $x = a$, there will be produced

$$\int_0^a \frac{Adx\sqrt{x}}{\sqrt{(a-x)}} = \tfrac{1}{2}\mathrm{A}a\sqrt{-1}\cdot l-1,$$

which must be equal to αa if $\mathrm{A} = \frac{2\alpha}{1.\sqrt{-1}.l-1}$. In a similar manner, the integral $\frac{Bxdx\sqrt{x}}{\sqrt{(a-x)}}$ will give $\frac{1}{2}.\frac{3}{4}\mathrm{B}.a^2\sqrt{-x}.l-x = \beta a^2$, therefore there shall become : $B = \frac{2.4.\beta}{1.3\sqrt{-1}.l-1}$.

And again there will be produced: $C = \frac{2.4.6.\gamma}{1.3.5\sqrt{-1}.l-1}$, and $D = \frac{2.4.6.8.\delta}{1.3.5.7\sqrt{-1}.l-1}$, etc.

Therefore with $\alpha,\ \beta,\ \chi,\ \delta$ etc. given, which may be found from the known curve *BNA*, the coefficients A, B, C, D, etc for the known curve may be determined.

§. 8. For the other part, which is rational, there must become $\int \frac{Edx}{\sqrt{(a-x)}} = \zeta\sqrt{a}$; but there shall become $\int \frac{Edx}{\sqrt{(a-x)}} = 2E\sqrt{a}$, from which $E = \frac{\zeta}{2}$ is produced. From which

$$\int \frac{Fxdx}{\sqrt{(a-x)}} = \tfrac{2}{3}.2Fa.\sqrt{a},$$

and that equation must become equal to this $\eta a\surd a$, therefore there must become $F=\frac{3}{2}\frac{\eta}{2}$, likewise $G=\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{\theta}{2}$; and $H=\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{\iota}{2}$; and so forth thus. Therefore by this reasoning A, B, C, D, etc. will become known, and the equation for the curve sought:

$$ds=Adx\surd x+Bxdx\surd x+Cx^2dx\surd x+etc+Edx+Fxdx+Gx^2dx+etc.$$

Which nevertheless may be continued indefinitely, yet it can still happen, so that often its sum shall be able to be defined, and thus a finite equation shall be found for the curve.

§. 9. *ANB* shall be a right line inclined to the horizontal, thus so that there shall be

$$AN:AQ=n:1 \text{ or } r=nt \text{ and } dr=ndt.$$

From which there shall become:

$$\int\frac{dr}{\sqrt{(a+c-t)}}=\int\frac{ndt}{\sqrt{(a+c-t)}}=\text{Const.}-2n\sqrt{(a+c-t)}.$$

Truly this constant is

$$2n\sqrt{(a+c)},$$

now there may be put $t=c$, the time produces the descent through *BA*

$$=2n\surd c+\frac{1\cdot2na}{2\surd c}-\frac{1\cdot1\cdot2na^2}{4\cdot2c\surd c}+\frac{1\cdot1\cdot3\cdot2na^3}{8\cdot2\cdot3c^2\surd c}-\frac{1\cdot1\cdot3\cdot5\cdot2na^4}{16\cdot2\cdot3\cdot4c^3\surd c}+\text{etc.}-2n\surd a, \text{ with}$$

$\sqrt{(a+c)}$ resolved into a series. The form of which shall be compared with this :

$$k-\alpha a-\beta a^2-\gamma a^3-\delta a^4-\text{etc.}-\zeta\surd a-\eta a\surd a-\text{etc.},$$

which will produce :

$$k=2n\surd c,\quad \alpha=-\frac{1\cdot2n}{2\surd c},\quad \beta=\frac{1\cdot1\cdot2n}{4\cdot2\cdot c\surd c},$$
$$\gamma=-\frac{1\cdot1\cdot3\cdot2n}{8\cdot2\cdot3\cdot c^2\surd c},\quad \delta=\frac{1\cdot1\cdot3\cdot5\cdot2n}{16\cdot2\cdot3\cdot4\cdot c^3\surd c}\text{ etc.}$$

and

$$\zeta=2n,\ \eta=0,\ \vartheta=0,\ \iota=0,\ \text{etc.}$$

§. 10. A, B, C, etc. will be produced from these known values, as follows:

$$A=-\frac{2n}{\sqrt{-c}\cdot l-1},\quad B=\frac{2n}{3c\sqrt{-c}\cdot l-1},\quad C=-\frac{2n}{5c^2\sqrt{-c}\cdot l-1},$$
$$D=\frac{2n}{7c^3\sqrt{-c}\cdot l-1}\ \text{etc.}\quad E=n,\quad F=0,\quad G=0\ \text{etc.}$$

Therefore this equation is found for the curve sought BMC :

$$ds=-\frac{2ndx\sqrt{x}}{1\cdot\sqrt{-c}\cdot l-1}+\frac{2nxdx\sqrt{x}}{3c\sqrt{-c}\cdot l-1}-\frac{2nx^2dx\sqrt{x}}{5c^2\sqrt{-c}\cdot l-1}+\text{etc.}+ndx,$$

of which the integral is :

$$s=nx-\frac{4nx^{\frac{3}{2}}}{\sqrt{-c}\cdot l-1}\left(\frac{1}{1\cdot 3}-\frac{x}{3\cdot 5\cdot c}+\frac{x^2}{5\cdot 7\cdot c^2}-\frac{x^3}{7\cdot 9\cdot c^3}+\text{etc.}\right).$$

But the differential equation may be readily changed into a finite expression, moreover this is :

$$ds=ndx-\frac{2ndx}{1\cdot\sqrt{-c}\cdot l-1}\left(\sqrt{x}-\frac{x\sqrt{x}}{3c}+\frac{x^2\sqrt{x}}{5c^2}-\frac{x^3\sqrt{x}}{7c^3}+\text{etc.}\right).$$

Which series expresses the arc of a circle, of which the tangent is $\sqrt{x}$: with the radius put to be $\sqrt{c}$, on this account there will become :

$$ds=ndx-\frac{ndx}{l-1}l\frac{\sqrt{c}+\sqrt{-x}}{\sqrt{c}-\sqrt{-x}}.$$

§. 11. This equation found :

$$ds=ndx-\frac{ndx}{l-1}l\frac{\sqrt{c}+\sqrt{-x}}{\sqrt{c}-\sqrt{-x}}.$$

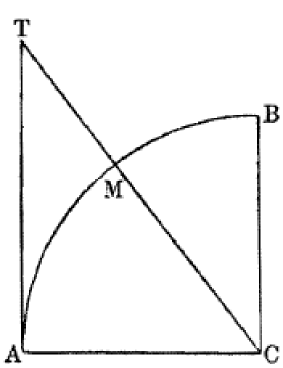


Fig. 2

can be integrated, and this equation is produced after the integration:

$$s=nx-\frac{2n\sqrt{cx}}{\sqrt{-1}\cdot l-1}-\frac{n(c+x)}{l-1}l\frac{\sqrt{c}+\sqrt{-x}}{\sqrt{c}-\sqrt{-x}}.$$

This value of s itself is constructed from the rectification of the circle in the following manner. The quadrant of a circle s shall be put in place, the radius of which $AC = c$, the tangent $AT = \sqrt{cx}$, and the secant TMC, which will become

$$s = \frac{n \cdot AB \cdot AT^2 + n \cdot AT \cdot AC^2 - n \cdot AM \cdot CT^2}{AC \cdot AB}.$$

Indeed, from the nature of the circle, there becomes

$$\mathrm{AB} = \frac{cl-1}{2\sqrt{-1}}$$

and

$$AM = \frac{c}{2\sqrt{-1}} l \frac{\sqrt{c} + \sqrt{-x}}{\sqrt{c} - \sqrt{-x}},$$

from which the given construction follows easily.

§.12. From the equation

$$ds = ndx - \frac{ndx}{l-1} l \frac{\sqrt{c} + \sqrt{-x}}{\sqrt{c} - \sqrt{-x}}$$

it will be apparent that $ds < ndx$, except in the case $x = 0$, where there becomes $ds = ndx$; indeed the curves *AB* and *BC* will have always a common tangent at *B*. From which it is clear the curve sought to be concave towards the axis *BP*, and evidently for the curve to be rising at *C*, so that its tangent shall be vertical, and the curve to have a cusp at that same point C. Therefore the height of this curve *BE* will be found, if there may be put $ds = dx$ in the equation. Hence in our case, where the given curve is a right line, x will give teh altitudine *BE* from the equation

$$(n-1)l-1 = nl \frac{\sqrt{c} + \sqrt{-x}}{\sqrt{c} - \sqrt{-x}}$$

or from this

$$-1 = \left(\frac{\sqrt{c} + \sqrt{-x}}{\sqrt{c} - \sqrt{-x}} \right)^{\frac{n}{n-1}},$$

or

$$\left(\sqrt{c} + \sqrt{-x}\right)^{\frac{n}{n-1}} + \left(\sqrt{c} - \sqrt{-x}\right)^{\frac{n}{n-1}} = 0.$$

In additon the arc may be taken

$$AM = \frac{n-1}{n} AB,$$

and with the tangent *AT* drawn, the quantity $\frac{AT^2}{AC}$ will be the height of the curve sought.

§.13. If the differential equation found shall be differentiated again with *dx* constant this equation will be produced

$$dds = \frac{ndx^2 \sqrt{c}}{(c+x)\sqrt{-x \cdot l - 1}},$$

which with the ratio of the periphery to the diameter put $\pi : 1$ agrees with this

$$dds = -\frac{ndx^2 \sqrt{c}}{\pi(c+x)\sqrt{x}}.$$

From this equation the cae is easily understood, where $c = 0$ and $n = \infty$, yet still, so that there shall be $n\sqrt{c} = \sqrt{b}$. This happens, if the given right line is indefinitely small and it makes a small finite angle with the horizontal, thus so that nevertheless the time of descent shall be finite $= 2\sqrt{b}$. Therefore there will be $AM = AB$, and therefore the tangent *AT* shall be infinite with respect to the radius *c*, will become $c + x$ at *x*, and therefore the curve sought will have this equation :

$$dds = -\frac{dx^2 \sqrt{b}}{\pi x \sqrt{x}}$$

or

$$ds = \frac{2dx\sqrt{b}}{\pi\sqrt{x}}$$

and $s = \frac{4\sqrt{bx}}{\pi}$. On this account the curve sought will be a cycloid, as the nature of the matter requires.

§.14. If the given curve had this equation

$$dr = ht^n dt,$$

the element of the time will be

$$\frac{ht^n dt}{\sqrt{(a+c-t)}},$$

there may be put $a + c = f$ and $f - t = z^2$, there will become $t = f - z^2$ and

$$t^{n} = f^{n} - \frac{n}{1} f^{n-1}z^{2} + \frac{n(n-1)}{1\cdot 2} f^{n-2}z^{4} - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} f^{n-3}z^{6} + \text{etc.}$$

and

$$\frac{hdt}{\sqrt{(f-t)}} = -2hdz.$$

Hence there will be produced :

$$\int \frac{ht^{n}dt}{\sqrt{(a+c-t)}} = \text{Const.} - 2hf^{n}z + \frac{2hn}{1\cdot 3} f^{n-1}z^{3} - \frac{2hn(n-1)}{1\cdot 3\cdot 5} f^{n-2}z^{5}$$
$$+ \frac{2hn(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 7} f^{n-3}z^{7} - \text{etc.}$$

So that on making $t=0$ or $z=\sqrt{f}$ must vanish, there will become

$$\text{Const.} = 2hf^{n}\sqrt{f}\left(1 - \frac{n}{1\cdot 3} + \frac{n(n-1)}{1\cdot 2\cdot 5} - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 7} + \text{etc.}\right).$$
$$+ \frac{2hn(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 7} f^{n-3}z^{7} - \text{etc.}$$

the descent time along *BA* will be produced by

$$= 2hf^{n}\sqrt{f}\left(1 - \frac{n}{1\cdot 3} + \frac{n(n-1)}{1\cdot 2\cdot 5} - \frac{n(n-1)(n-2)}{1\cdot 3\cdot 5\cdot 7} + \text{etc}\right)$$
$$-2h\left(f^{n}\sqrt{a} - \frac{n}{1\cdot 3} f^{n-1}a\sqrt{a} + \frac{n(n-1)}{1\cdot 2\cdot 5} f^{n-2}a^{2}\sqrt{a} - \text{etc.}\right).$$

§.15. For the sake of brevity we shall put

$$1 - \frac{n}{1\cdot 3} + \frac{n(n-1)}{1\cdot 2\cdot 5} - \text{etc.} = p,$$

and with $a+c$ substituted in place of f the descent time along *BA*

$$= 2hpc^{n+\frac{1}{2}} + \frac{n+\frac{1}{2}}{1} 2hpc^{n-\frac{1}{2}} a + \frac{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}{1\cdot 2} 2hpc^{n-\frac{3}{2}} a^2 + \text{etc.}$$
$$-2hc^n \sqrt{a} - \frac{2n}{3}\cdot 2hc^{n-1} a\sqrt{a} - \frac{2n(2n-2)}{3\cdot 5}\cdot 2hc^{n-2} a^2 \sqrt{a}$$
$$-\frac{2n(2n-2)(2n-4)}{3\cdot 5\cdot 7}\cdot 2hc^{n-3} a^3 \sqrt{a} - \text{etc.}$$

This form compared with the form in paragraph 4 gives

$$k - \alpha a - \beta a^2 - \gamma a^3 - \text{etc.} - \zeta\sqrt{a} - \eta a\sqrt{a} - \vartheta a^2\sqrt{a} - \text{etc.}$$

there will become:

$$k = 2hpc^{n+\frac{1}{2}},\ \alpha = -\frac{\left(n+\frac{1}{2}\right)}{1} 2hpc^{n-\frac{1}{2}},\ \beta = -\frac{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}{1\cdot 2} 2hpc^{n-\frac{3}{2}},$$
$$\gamma = -\frac{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)}{1\cdot 2\cdot 3} 2hpc^{n-\frac{5}{2}}, \quad \text{etc.}$$

and

$$\zeta = 2hc^n, \quad \eta = \frac{2n}{3} 2hc^{n-1},$$
$$\vartheta = \frac{2n(2n-2)}{3\cdot 5} 2hc^{n-2}, \quad \iota = \frac{2n(2n-2)(2n-4)}{3\cdot 5\cdot 7} 2hc^{n-3} \quad \text{etc.}$$

§.16. Therefore the letters A, B, C etc. will be found, as follows

$$A = -\frac{2h(2n+1)\,pc^{n-\frac{1}{2}}}{\sqrt{-1}\cdot l-1},$$

and

$$B = -\frac{2h(2n+1)(2n-1)\,pc^{n-\frac{3}{2}}}{1\cdot 3\sqrt{-1}\cdot l-1},$$
$$C = -\frac{2h(2n+1)(2n-1)(2n-3)\,pc^{n-\frac{5}{2}}}{1\cdot 3\cdot 5\sqrt{-1}\cdot l-1} \quad \text{etc.}$$

Atque

$$E = hc^n,\ F = \frac{hnc}{1},$$
$$G = \frac{hn(n-1)}{1\cdot 2}c^{n-2}, \quad H = \frac{hn(n-1)(n-2)}{1\cdot 2\cdot 3}c^{n-3} \text{ etc.}$$

And therefore for the curve sought this equation is found :

$$ds = -\frac{2hpdx\sqrt{x}}{\sqrt{-1}\cdot l - 1}\left(\frac{2n+1}{1}c^{n-\frac{1}{2}} + \frac{(2n+1)(2n-1)}{1\cdot 3}c^{n-\frac{3}{2}}x + \frac{(2n+1)(2n-1)(2n-3)}{1\cdot 3\cdot 5}c^{n-\frac{5}{2}}x^2 + \text{etc.}\right)$$

$$+hdx\left(c^n + \frac{n}{1}c^{n-1}x + \frac{n(n-1)}{1\cdot 2}c^{n-2}x^2 + \text{etc.}\right)$$
$$= h(c+x)^n\, dx - \frac{2hpcdx\sqrt{x}}{\sqrt{-1}\cdot l - 1}\left(\begin{array}{l}\frac{2n+1}{1\cdot\sqrt{c}} + \frac{(2n+1)(2n-1)x}{1\cdot 3\cdot c\sqrt{c}} + \frac{(2n+1)(2n-1)(2n-3)x^2}{1\cdot 3\cdot 5\cdot c^2\sqrt{c}} \\ + \text{ etc.}\end{array}\right).$$

§.17. Although here for the given curve only the equation $dr = ht^n dt$ has been assumed, yet all the curves in short can be accomodated by this example. Indeed that same given curve may be set out by the equation

$$dr = At^{\alpha}dt + Bt^{\beta}dt + Ct^{\gamma}dt + \text{etc.}$$

Then the equation may be sought in the first place from this equation only, $dr = At^{\alpha}dt$ and the resulting equation shall be $ds = Pdx$. Then the equation shall be taken $dr = Bt^{\beta}dt$, and the resulting equation $ds = Qdx$. Similalrly from the equations

$$dr = Ct^{\gamma}dt,\ dr = Dt^{\delta}dt \text{ etc.}$$

these same equations shall arise:

$$ds = Rdx,\ ds = Sdx \text{ etc.,}$$

and this equation for the curve sought will be :

$$ds = (P + Q + R + S + \text{etc.})dx,$$

clearly if the given curve shall satisfy the equation

$$dr = At^{\alpha}dt + Bt^{\beta}dt + Ct^{\gamma}dt + \text{etc.}$$

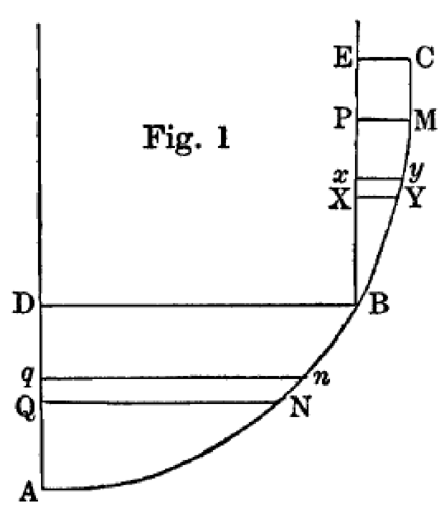
§.18. Also it is evident from the equation in §.16, if there were $n = -\frac{1}{2}$ or $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$ etc., the series to be halted and at once to have a finite equation ; if $dr = ht^{-\frac{1}{2}}dt,$ the given curve evidently shall be a cycloid, fo which $n = -\frac{1}{2}$ and $ds = \frac{hdx}{\sqrt{(c+x)}}.$ From which it is understood the above curve added on together with the given lower part continued without doubt to constitute a cycloid.

Solutio Singularis Casus Circa Tautochronismum.

Leonhard Euler

§.1. Cum ante annos tres Clariss. *Bernoulli* methodum innumeras curvas tautochronas in vacuo inveniendi proponeret, mentionem fecit problematis non parum elegantis, cuius solutionem hac scheda daturus sum. Difficillimum quidem eo tempore videbatur hoc problema, et propterea parum studii ad id soluendum impendebam. Postmodum vero cum diligentius in tautochronas pro fluidis inquisivisssem, universalem detexi methodum problemata huiusmodi omnia solvendi , quae etiam me ad solutionem problematis illius manuduxit.

§.2. Problema autem hoc est: *Datae curvae ANB in* B (Fig.1) *adiungere curvam BMC eius proprietatis, ut omnes descensus gravis alicubi in curva BMC incipiens usque ad imum punctum A fiant temporibus aequalibus.*
Oportet ergo inveniri curvam *BMC,* ex hac conditione, ut sumpto in curva *BMC* pr~~o~~ lubitu



pnncto *M* tempus descensus per *MBNA* sit constans, neque pendeat a loco puncti *M*. Seu tempus descensus per *MBNA* aequale esse debet tempori descensus per curvam datam *BNA* ; qui est casus incidente puncto *M* in *B*.

§.3. Descendat ergo corpus ex puncto *M*, et quaeramus descensus tempus per arcum MB et *BMA*. Ducta verticali *BP*, ponatur BP = a , quae igitur littera, quia locum puncti *M* definit, in expressione temporis per *MBA* inesse non potest. Curvae datae altitudo AD sit $= c.$ Assumantur in utraque curva applicatae quaecunque *QN* et *XY* iisque proximae *qn* et *xy*. Dicantur $AQ = t,\ AN = r$ et $BX = x,\ BY = s;$ quarum inter *t* et *r* aequatio est dato, inter *x* et *s* desideratur. Celeritas, quam corpus in *N* habebit, est

$$\sqrt{(a+c-t)} = \sqrt{(PB+DQ)}.$$

Adeoque tempus, quo arcus *AN* absolvitur est

$$\int \frac{dr}{\sqrt{(a+c-t)}}.$$

Quod integrale ita debet accipi, ut fiat $=0,$ si sit $t = 0.$

§.4. Deinceps, si ponatur $t = c,$ habebitur tempus per integram curvam datam *BNA*, quod igitur erit expositum formula ex *a* et constantibus composita. Nonnullos computavi casus speciales, et vidi tempus descensus per curvam *BNA* initio descensus posito in *M*, semper exponi posse sequente serie

$$k - \alpha a - \beta a^2 - \gamma a^3 - \delta a^4 - \text{etc.} - \zeta\sqrt{a} - \eta a\sqrt{a} - \vartheta a^2\sqrt{a} - \text{etc.},$$

cuius in quolibet casu speciali coefficientes $\alpha,\ \beta$ etc. et k poterunt determinari. Hoc tempus igitur, additum ad descensus tempus per *MB*, constans esse debet: atque ut summa omnes termini littera *a* affecti sese tollant, necesse est.

§.5. Ad tempus descensus per curvam *MB* inveniendum, est celeritas in $Y = \sqrt{(a-x)}$ et elementum temporis $= ds : \sqrt{(a-x)}.$ Huius integrale ita assutum, ut fiat $=0$ si $x = 0$ dabit tempus desccndis per *YB* , in quo ergo si ponatur $x = a$ prodibit descensus tempus per *MB*, quod cum priore constantem quantitatem ab *a* liberam conficere debet. Si punctum *M* incidit in punctum *B*, *i*. e. si *a* evanescit, integrum tempus descensus erit tempus descensus per curvam *BMA*, quod ex superiore formula evadit $= k$. Hanc ob rem etiam tempus descensus per *MBNA* debet esse $= k$. Proinde tempus per *MB* debebit esse

$$= \alpha a + \beta a^2 + \gamma a^3 + \text{etc.} + \zeta\sqrt{a} + \eta a\sqrt{a} + \vartheta a^2\sqrt{a} + \text{etc.}$$

§.6 Hoc ut fiat assumo pro curva quaesita sequentem aequatiom :

$$ds = Adx\sqrt{x} + Bxdx\sqrt{x} + Cx^2dx\sqrt{x} + etc + Edx + Fxdx + Gx^2dx + etc.$$

Tempus ergo descensus per arcum M B erit

$$= \int\frac{Adx\sqrt{x}}{\sqrt{(a-x)}} + \int\frac{Bxdx\sqrt{x}}{\sqrt{(a-x)}} + \int\frac{Cx^2dx\sqrt{x}}{\sqrt{(a-x)}} + \text{etc.} + \int\frac{Exdx}{\sqrt{(a-x)}} + \int\frac{Fxdx}{\sqrt{(a-x)}} + \int\frac{Gx^2dx}{\sqrt{(a-x)}}dx + \text{etc.}$$

scilicet si integralibus his ita sumtis ut fiant $=0$ si $x=0$ ibique $x=a$. Determinentur ergo coefficientes *A, B, C* , etc. ut sint

$$\int\frac{Adx\sqrt{x}}{\sqrt{(a-x)}} = \alpha a;\ \int\frac{Bxdx\sqrt{x}}{\sqrt{(a-x)}} = \beta a^2;\ \int\frac{Cx^2dx\sqrt{x}}{\sqrt{(a-x)}} = \gamma a^3 \text{ etc.}$$

$$\text{et } \int\frac{Edx}{\sqrt{(a-x)}} = \zeta\sqrt{a};\ \int\frac{Fxdx}{\sqrt{(a-x)}} = \eta a\sqrt{a};\ \int\frac{Gx^2dx}{\sqrt{(a-x)}}dx = \theta a^2\sqrt{a} \text{ etc.}$$

Assumsi vero istum loco ds valorem, ut litterae A, B, C, etc. non ab a pendentes determinatur .

§. 7. Integratio huius $\frac{Adx\sqrt{x}}{\sqrt{(a-x)}}$ pendet *a* quadratura circuli; At si ope logarithmorum imaginariorum integretur ut decet, atque ponantur $x=a$ prodibit $\frac{1}{2}\mathrm{A}a\sqrt{-x}.l-x$ quod aequale esse debet αa sit ergo $\mathrm{A} = \frac{2\alpha}{1.\sqrt{-1}.l-1}$. Simili modo, $\frac{Bxdx\sqrt{x}}{\sqrt{(a-x)}}$ integratum dabit $\frac{1}{2}.\frac{3}{4}\mathrm{B}.a^2\sqrt{-x}.l-x = \beta a^2$, sit igitur $B = \frac{2.4.\beta}{1.3\sqrt{-1}.l-1}$.

Atque porro prodit $C = \frac{2.4.6.\gamma}{1.3.5\sqrt{-1}.l-1}$, et $D = \frac{2.4.6.8.\delta}{1.3.5.7\sqrt{-1}.l-1}$, etc.

Datis ergo $\alpha,\ \beta,\ \chi,\ \delta$ etc. quae ex curva *BNA* nota inveniuntur, determinantur coefficientes pro curva quaesita A, B, C, D, etc.

§. 8. Pro altera parte, quae est rationalis, esse debet $\int \frac{Edx}{\sqrt{(a-x)}} = \zeta \surd a$; sit autem $\int \frac{Edx}{\sqrt{(a-x)}} = 2E\surd a$, ex quo prodi $E = \frac{\zeta}{2}$. Deinde $\int \frac{Fxdx}{\sqrt{(a-x)}} = \frac{2}{3}.2Fa.\surd a$, idque aequari debet huic $\eta a\surd a$, reperitur ergo $F = \frac{3}{2}\frac{\eta}{2}$, similiter $G = \frac{3}{2}\cdot\frac{5}{4}\cdot\frac{\theta}{2}$; atque $H = \frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{\iota}{2}$; et ita porro. Hac igitur ratione determinatis A, B, C, D, etc. cognita erit aequatio pro curva quaesita

$$ds = Adx\surd x + Bxdx\surd x + Cx^2dx\surd x + etc + Edx + Fxdx + Gx^2dx + etc.$$

Quae quanquam in infinitum plerumque continuetur, tamen fieri potest, ut saepe eius summa possit definire sicque inveniatur aequatio finita pro curva quaesita.

§. 9. Sit *ANB* linea recta ad horizontem inclinata, ita ut sit

$$AN : AQ = n : 1 \text{ seu } r = nt \text{ et } dr = ndt.$$

Ex quo sit

$$\int \frac{dr}{\sqrt{(a+c-t)}} = \int \frac{ndt}{\sqrt{(a+c-t)}} = \text{Const.} - 2n\sqrt{(a+c-t)}.$$

Constans vero haec est

$$2n\sqrt{(a+c)}$$

ponatur iam $t = c$, prodit tempus descensus per *BA*

$$= 2n\surd c + \frac{1\cdot 2na}{2\surd c} - \frac{1\cdot 1\cdot 2na^2}{4\cdot 2c\surd c} + \frac{1\cdot 1\cdot 3\cdot 2na^2}{8\cdot 2\cdot 3c^2\surd c} - \frac{1\cdot 1\cdot 3\cdot 5\cdot 2na^4}{16\cdot 2\cdot 3\cdot 4c^3\surd c} + \text{etc.} - 2n\surd a,$$

in seriem $\sqrt{(a+c)}$ resoluta. Comparetur haec forma cum hac

$$k - \alpha a - \beta a^2 - \gamma a^3 - \delta a^4 - \text{etc.} - \zeta\surd a - \eta a\surd a - \text{etc.},$$

prodibit

$$k = 2n\surd c, \quad \alpha = -\frac{1\cdot 2n}{2n\surd c}, \quad \beta = \frac{1\cdot 1\cdot 2n}{4\cdot 2\cdot c\surd c},$$
$$\gamma = -\frac{1\cdot 1\cdot 3\cdot 2n}{8\cdot 2\cdot 3\cdot c^2\surd c}, \quad \delta = \frac{1\cdot 1\cdot 3\cdot 5\cdot 2n}{16\cdot 2\cdot 3\cdot 4\cdot c^2\surd c} \text{ etc.}$$

et

$$\zeta = 2n,\ \ \eta = 0,\ \ \vartheta = 0,\ \ \iota = 0,\ \text{etc.}$$

§. 10. Cognitis his valoribus prodibunt A, B, C, etc., ut sequuntur:

$$A = -\frac{2n}{\surd - c \cdot l - 1},\ \ B = \frac{2n}{3c\surd - c \cdot l - 1},\ \ C = -\frac{2n}{5c^2 \surd - c \cdot l - 1},$$
$$D = \frac{2n}{7c^3 \surd - c \cdot l - 1}\ \text{etc.}\quad E = n,\quad F = 0,\quad G = 0\ \text{etc.}$$

Pro curva igitur quaesita *BMC* invenitur ista aequatio

$$ds = -\frac{2ndx\surd x}{1 \cdot \surd - c \cdot l - 1} + \frac{2nxdx\surd x}{3c\surd - c \cdot l - 1} - \frac{2nx^2 dx\surd x}{5c^2 \surd - c \cdot l - 1} + \text{etc.} + ndx,$$

cuius integralis haec est

$$s = nx - \frac{4nx^{\frac{3}{2}}}{\surd - c \cdot l - 1}\left(\frac{1}{1 \cdot 3} - \frac{x}{3 \cdot 5 \cdot c} + \frac{x^2}{5 \cdot 7 \cdot c^2} - \frac{x^3}{7 \cdot 9 \cdot c^3} + \text{etc.}\right).$$

Facilius autem erit aequationem differentialem in expressionem finitam transmutare, est autem ea haec

$$ds = ndx - \frac{2ndx}{1 \cdot \surd - c \cdot l - 1}\left(\surd x - \frac{x\surd x}{3c} + \frac{x^2 \surd x}{5c^2} - \frac{x^3 \surd x}{7c^3} + \text{etc.}\right).$$

Quae series exprimit arcum circuli, cuius tangens est $\surd x$: posito radio $\surd c$, hanc ob rem erit

$$ds = ndx - \frac{ndx}{l - 1} l \frac{\surd c + \surd - x}{\surd c - \surd - x}.$$

11. Aequatio haec inventa

$$ds = ndx - \frac{ndx}{l - 1} l \frac{\surd c + \surd - x}{\surd c - \surd - x}.$$

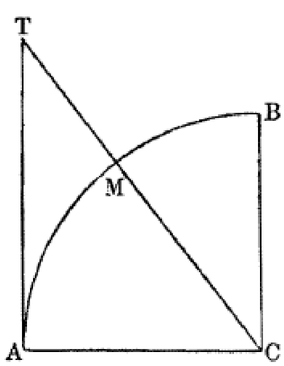


Fig. 2

potest integrari, proditque post integrationem haec aequatio

$$s = nx - \frac{2n\surd cx}{\surd - 1 \cdot l - 1} - \frac{n(c + x)}{l - 1} l \frac{\surd c + \surd - x}{\surd c - \surd - x}.$$

Hic ipsius valor *s* ope rectificationis circuli construitur sequente modo. Fiat circuli quadrans, cuius radius $AC = c$, ducatur tangens $AT = \surd cx$, et secans *TMC*, erit

$$s = \frac{n \cdot AB \cdot AT^2 + n \cdot AT \cdot AC^2 - n \cdot AM \cdot CT^2}{AC \cdot AB}.$$

Namque ex natura circuli est

$$\mathrm{AB} = \frac{cl-1}{2\sqrt{-1}}$$

et

$$AM = \frac{c}{2\sqrt{-1}} l \frac{\sqrt{c} + \sqrt{-x}}{\sqrt{c} - \sqrt{-x}},$$

unde dat constructio facile sequitur.

§.12. Ex aequatione

$$ds = ndx - \frac{ndx}{l-1} l \frac{\sqrt{c} + \sqrt{-x}}{\sqrt{c} - \sqrt{-x}}$$

apparet esse $ds < ndx$, nisi in casu $x = 0$, quo est $ds = ndx$; habebunt enim curvae *AB* et *BC* semper in *B* tangentem communem. Ex quo apparet curvam quaesitam esse concavam versus axem *BP* , atque eousque scilicet in *C* ascendere, quoad eius tangens fiat verticalis, in eoque puncto *C* curvam habere cuspidem. Altitudo igitur huius curvae . *BE* invenietur, si in aequatione ponatur $ds = dx$. In nostro ergo casu, quo curva data est linea recta, dabit *x* altitudinem *BE* ex aequatione

$$(n-1)l-1 = nl \frac{\sqrt{c} + \sqrt{-x}}{\sqrt{c} - \sqrt{-x}}$$

seu hac

$$-1 = \left(\frac{\sqrt{c} + \sqrt{-x}}{\sqrt{c} - \sqrt{-x}} \right)^{\frac{n}{n-1}}$$

sive hac

$$\left(\sqrt{c} + \sqrt{-x}\right)^{\frac{n}{n-1}} + \left(\sqrt{c} - \sqrt{-x}\right)^{\frac{n}{n-1}} = 0.$$

Vel etiam sumatur arcus

$$AM = \frac{n-1}{n} AB,$$

et ducta eius tangente *AT* erit altitudo $\dfrac{AT^2}{AC}$ curvae quaesitae.

§.13. Si aequatio differentialis inventa denuo differentietur posito *dx* constante prodibit aequatio haec

$$dds = \frac{ndx^2\sqrt{c}}{(c+x)\sqrt{-x\cdot l-1}},$$

quae posita ratione peripheriae ad diametrum $\pi:1$ congruit cum hac

$$dds = -\frac{ndx^2\sqrt{c}}{\pi(c+x)\sqrt{x}}.$$

Ex hac aequatione casus, quo $c=0$ et $n=\infty$, ita tamen, ut sit $n\sqrt{c}=\sqrt{b}$, facile cognoscitur. Evenit hoc, si recta data est infinite parva et angulum finite parvum cum horizonte constituit, ita vt tempus descensus per eam tamen sit finitum nimirum $=2\sqrt{b}$. Erit igitur $AM = AB$, ideoque tangens *AT* infinita respectu radii *c*, abibit ergo $c+x$ in x, atque curva quaesita hanc habebit aequationem

$$dds = -\frac{dx^2\sqrt{b}}{\pi x\sqrt{x}}$$

seu

$$ds = \frac{2dx\sqrt{b}}{\pi\sqrt{x}}$$

atque $s = \dfrac{4\sqrt{bx}}{\pi}$. Hanc ob rem curva quaesita erit cyclois, ut natura rei requiret.

§.14. Si curva data hanc habuerit aequatione

$$dr = ht^n dt,$$

erit elementum temporis

$$\frac{ht^n dt}{\sqrt{(a+c-t)}},$$

ponatur $a+c=f$ et $f-t=z^2$, erit $t=f-z^2$ et

$$t^n = f^n - \frac{n}{1}f^{n-1}z^2 + \frac{n(n-1)}{1\cdot 2}f^{n-2}z^4 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}f^{n-3}z^6 + \text{etc.}$$

et

$$\frac{hdt}{\sqrt{(f-t)}} = -2hdz.$$

Hinc prodibit

$$\int \frac{ht^n dt}{\sqrt{(a+c-t)}} = Const. - 2hf^n z + \frac{2hn}{1\cdot 3} f^{n-1} z^3 - \frac{2hn(n-1)}{1\cdot 2\cdot 5} f^{n-2} z^5 + \frac{2hn(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 7} f^{n-3} z^7 - \text{etc.}$$

Quod cum facto $t=0$ seu $z=\sqrt{a}$, prodibit tempus descensus per *BA*

$$= 2hf^n \sqrt{f}\left(1 - \frac{n}{1\cdot 3} + \frac{n(n-1)}{1\cdot 2\cdot 5} - \frac{n(n-1)(n-2)}{1\cdot 3\cdot 5\cdot 7} + \text{etc}\right)$$
$$-2h\left(f^n \sqrt{a} - \frac{n}{1\cdot 3} f^{n-1} a\sqrt{a} + \frac{n(n-1)}{1\cdot 2\cdot 5} f^{n-2} a^2 \sqrt{a} - \text{etc.}\right).$$

§.15. Ponamus brevitatis causa

$$1 - \frac{n}{1\cdot 3} + \frac{n(n-1)}{1\cdot 2\cdot 5} - \text{etc.} = p,$$

erit substituto $a+c$ loco f descensus per *BA*

$$= 2hpc^{n+\frac{1}{2}} + \frac{n+\frac{1}{2}}{1} 2hpc^{n-\frac{1}{2}} a + \frac{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}{1\cdot 2} 2hpc^{n-\frac{3}{2}} a^2 + \text{etc.}$$
$$-2hc^n \sqrt{a} - \frac{2n}{3}\cdot 2hc^{n-1} a\sqrt{a} - \frac{2n(2n-2)}{3\cdot 5}\cdot 2hc^{n-2} a^2 \sqrt{a}$$
$$- \frac{2n(2n-2)(2n-4)}{3\cdot 5\cdot 7}\cdot 2hc^{n-3} a^3 \sqrt{a} - \text{etc.}$$

Haec forma comparata cum forma paragrapho 4 data

$$k - \alpha a - \beta a^2 - \gamma a^3 - \text{etc.} = -\zeta\sqrt{} - \eta a\sqrt{a} - \vartheta a^2 \sqrt{a} - \text{etc.}$$

erit

$$k = 2hpc^{n+\frac{1}{2}},\ \alpha = -\frac{\left(n+\frac{1}{2}\right)}{1} 2hpc^{n-\frac{1}{2}},\ \beta = -\frac{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)}{1\cdot 2} 2hpc^{n-\frac{3}{2}},$$
$$\gamma = -\frac{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)}{1\cdot 2\cdot 3} 2hpc^{n-\frac{5}{2}}, \quad \text{etc.}$$

atque

$$\zeta = 2hc^{n}, \quad \eta = \frac{2n}{3}2hc^{n-1},$$
$$\vartheta = \frac{2n(2n-2)}{3\cdot 5}2hc^{n-2}, \quad \iota = \frac{2n(2n-2)(2n-4)}{3\cdot 5\cdot 7}2hc^{n-3} \text{ etc.}$$

§.16. Invenientur igitur litterae A, B, C etc., ut sequitur

$$A = -\frac{2h(2n+1)pc^{n-\frac{1}{2}}}{\sqrt{-1}\cdot l-1},$$

atque

$$B = -\frac{2h(2n+1)(2n-1)pc^{n-\frac{1}{2}}}{1\cdot 3\sqrt{-1}\cdot l-1},$$
$$C = -\frac{2h(2n+1)(2n-1)(2n-3)pc^{n-\frac{5}{2}}}{1\cdot 3\cdot 5\sqrt{-1}\cdot l-1} \text{ etc.}$$

Atque

$$E = hc^{n},\ F = \frac{hnc}{1},$$
$$G = \frac{hn(n-1)}{1\cdot 2}c^{n-2}, \quad H = \frac{hn(n-1)(n-2)}{1\cdot 2\cdot 3}c^{n-2} \text{ etc.}$$

Pro curvas itaque quaesita reperitur ista aequatio

$$ds = -\frac{2hpdx\sqrt{x}}{\sqrt{-1}\cdot l-1}\left(\frac{2n+1}{1}c^{n-\frac{1}{2}} + \frac{(2n+1)(2n-1)}{1\cdot 3}c^{n-\frac{3}{2}}x + \frac{(2n+1)(2n-1)(2n-3)}{1\cdot 3\cdot 5}c^{n-\frac{5}{2}}x^{2} + \text{etc.}\right)$$

$$+hdx\left(c^{n} + \frac{n}{1}c^{n-1}x + \frac{n(n-1)}{1\cdot 2}c^{n-2}x^{2} + \text{etc.}\right)$$
$$= h(c+x)^{n}\,dx - \frac{2hpdx\sqrt{x}}{\sqrt{-1}\cdot l-1}\left(\begin{array}{l}\frac{2n+1}{1\cdot\sqrt{c}} + \frac{(2n+1)(2n-1)}{1\cdot 3\cdot c\sqrt{c}} + \frac{(2n+1)(2n-1)(2n-3)}{1\cdot 3\cdot 5\cdot c^{2}\sqrt{c}}\\ +\text{ etc.}\end{array}\right).$$

§.17. Quanquam hic pro curva data haec tantum aequatio $dr = ht^{n}dt$ est assumta, tamen ad omnes prorsus curvas exemplum hoc accommodari potest. Sit enimcurva data ista exposita aequatione

$$dr = At^{\alpha}dt + Bt^{\beta}dt + Ct^{\gamma}dt + \text{etc.}$$

Tum quaeratur aequatio pro curva quaesita primo ex hac tantum aequatione $dr = At^{\alpha} dt$ et sit aequatio resultans $ds = Pdx$. Deinde sumatur aequatio $dr = Bt^{\beta} dt$ et sit aequatio resultans $ds = Qdx$. Similiter ex aequationibus

$$dr = Ct^{\gamma} dt,\ dr = Dt^{\delta} dt \text{ etc.}$$

emergant istae

$$ds = Rdx,\ ds = Sdx \text{ etc.},$$

erit aequatio pro curva quaesita haec

$$ds = \left(P + Q + R + S + \text{etc.}\right) dx,$$

si scilicet curva data habuerit aequationem

$$dr = At^{\alpha} dt + Bt^{\beta} dt + Ct^{\gamma} dt + \text{etc.}$$

§.18. Apparet etiam ex aequatione §.16, si fuerit $n = -\frac{1}{2}$ vel $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$ etc., seriem abrumpi atque statim haberi aequationem finitam ; sit $dr = ht^{-\frac{1}{2}} dt$, curva scilicet data cyclois, erit $n = -\frac{1}{2}$ atque $ds = \frac{hdx}{\sqrt{(c+x)}}$. Ex quo cognoscitur curvam superiorem annexam cum data inferiore eandem curvam continuam nimirum cycoidem constituere.