Concerning Rectifiable Algebraic Curves with Reciprocal Algebraic Trajectories [e23].

Author Leonhard Euler.

Although it is certainly easy to give innumerable algebraic curves which are able to be rectified, whether they be arranged either by their evolutes or by caustic curves of algebraic curvatures ; yet if we may consider the arrangements of the curves, the rarest of these occur with these, which are allowed to be rectified. In lines of the second order, which may be agreed from conic sections, none is of this kind; in the third order it is agreed only two rectifiable curves are to be had. Moreover since for a number of years at first I seem to have been occupied in finding algebraic reciprocal trajectories, I had acquired the method of the celebrated *Johanne Bernoulli* who first set out rectifiable curves with great care, so that I might make use of these proposed. Also I have discovered a rectifiable curve in the sixth order, which provided me with an algebraic trajectory of the fourth order, and with that I satisfied the question then to be resolved, with the simpler reciprocal algebraic equations shown. Also I came upon the general equations for many rectifiable curves, when that was put in place all the simpler rectifiable curves were elicited. Now traversing this again I have deduced the most certain general equation containing in that all these rectifiable curves. Indeed many universal quantities are contained in that, according to which any may be substituted, and a rectifiable curve is produced. Let z be a certain variable quantity, of which the differential may be put constant, and P and Q shall be at least some algebraic functions of this variable z. If now a curve may be constructed in this manner so that its abscissa, which may be put equal to x, shall become

$$x = P + \frac{dQ\left(dP^2 - dQ^2\right)}{dQddP - dPddQ},$$

and I call the applied line

$$y = \frac{\left(dP^2 - dQ^2\right)^{\frac{3}{2}}}{dQddP - dPddQ}.$$

The length of this curve called s is equal to $Q + \frac{dQ\left(dP^2 - dQ^2\right)}{dQddP - dPddQ}$. Therefore whatever values shall be attributed to the letters P and Q, the curve will always be rectifiable and algebraic. I judge there is no need to give a demonstration of this, indeed any, if the differentials were taken of the coordinates x and y, and of the curve s, $dx^2 + dy^2 = ds^2$ shall become known without too much labour, as long as the work does not involve too many artifices.

Certainly this form appears the most widely used, but at this point I have another much more general, indeed the following is the most general, as before, but with capital letters, some functions L, M, and N of the variable x, if there may be assumed :

$$x = L + \frac{\left(dL^2 + dM^2 - dN^2\right)\left(dLdN - dM\right)\left(dL^2 + dM^2\right)\sqrt{\left(dL^2 + dM^2 - dN^2\right)}}{dLdNddL + dMdNddM - dL^2 ddN - dM^2 ddN + \left(dLddM - dMddL\right)\sqrt{\left(dL^2 + dM^2 - dN^2\right)}}$$

$$y = M + \frac{\left(dL^2 + dM^2 - dN^2\right)\left(dMdN + dL\sqrt{\left(dL^2 + dM^2 - dN^2\right)}\right)}{dLdNddL + dMdNddM - dL^2 ddN - dM^2 ddN + \left(dLddM - dMddL\right)\sqrt{\left(dL^2 + dM^2 - dN^2\right)}}$$

the length of the corresponding curve will be:

$$s = N + \frac{\left(dL^2 + dM^2 - dN^2\right)\left(dL^2 + dM^2\right)}{dLdNddL + dMdNddM - dL^2 ddN - dM^2 ddN + \left(dLddM - dMddL\right)\sqrt{\left(dL^2 + dM^2 - dN^2\right)}}.$$

These formulae may be changed into the preceding ones, if there may be put $M = 0$, these ones therefore are contained in these formulas.

It is easily understood, if the letters *L*, *M* and *N* not only signify algebraic quantities, but also transcending ones as well, evidently both algebraic as well as transcending formulas to be contained in these same formulas. Indeed since in these same formulas in no manner are the functions *L*, *M* and *N* depending on each other, which would not be reducible by the same formulas.

In a similar manner, if with the most general functions *N* remaining, thus yet, so that $\frac{dN}{dz}$ shall be an algebraic function, truly *L* and *M* shall denote algebraic functions, truly in short all the curves in the given formulas are understood to be algebraic; moreover these will be rectifiable, but if *N* were not an algebraic function, but transcending or from the quadrature of a curve depending on this kind, the resulting curve will not be rectifiable, but its rectification will depend on the quadrature of this curve. Therefore in this manner also that most celebrated problem is solved, causing a great disturbance between geometers, postulating a method for the rectification of algebraic curves requiring to be reduced, of which the solutions by the two most celebrated men Jacob Hermann and Johann Bernoulio are in the *Acta Lipsiensia*. Truly from these formulas the problem thus is resolved ; the curves shall be, of which the quadrature for the rectification of algebraic curves is required to be reduced, the coordinates *t* and *v*, of which each shall be an algebraic function of *z*. There may be taken

$$N = R + a\int vdt,$$

where *R* also designates some algebraic function of *z*. With this in place, the formulas treated will give all the algebraic curves, the rectification of which from the quadrature of the proposed curve, clearly will depend on $a\int vdt$, if indeed *L* and *M* shall be assumed, so that now it is to be reminded, they are

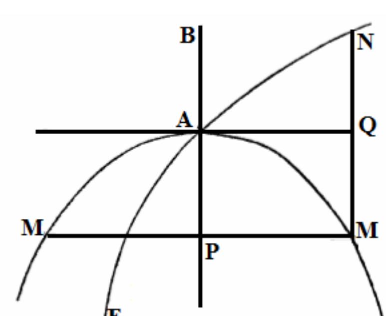

Fig. 1

to be algebraic functions of *z*. Other uses of these formulas, which I have put in place to set out here, consider the discovery of reciprocal trajectories ; indeed from these the most universal equation containing all the reciprocal trajectories is found within it, which thus may be put together easily, so that all the algebras shall be produced from these alone.

But this discovery depends on the theorem by Bernoulli, whereby from the rectification of curves the reciprocal trajectories are constructed having a given diameter, in this manner (Fig.1) : *MAM* is a curve of this kind having the diameter *AP*, and the vertex *A*, at which the tangent *AQ* is perpendicular to the diameter *AP*, and the vertex *A*. Through the singular points *M* of this curve are drawn right lines parallel to the diameter, and from these *MN* are taken equal to the arcs *AM*. Through the individual points *M* of this curve right lines are drawn parallel to the diameter and through these the right lines *MN* are drawn parallel to the diameter, and for the same, *MN* are taken equal to the arcs *AM*, the points *N* will constitute the reciprocal trajectory curve *NAE*, of which the axis of conversion is that same diameter *PAB*. If the abscissa *Q* of this curve and the applied line *QN*, there will be $AQ = PM$ and $QN = AM - AP.$ On account of which if the coordinates *AP* and *PM* of the curve *MAM* may be assumed to be the same, which before were called *x* and *y*, at once the equation will be had for the reciprocal trajectory *EAN*. Truly since not all the curves can be located on the position *MAM* , in the equations treated above the letters *L*, *M* and *N* must be restricted in some manner, so that only curves shall be allowed adapted to the principle. Since all the lines *AP*, *PM* and *AM* shall be functions of *z*, thus these shall be determined in *z*, so that with *z* taken positive the root of the curve *MAM* shall be positive, but with *z* negative, so that the negative root shall be produced. It is required according to this, so that *AP*, since in each case it shall remain the same, shall be an even function of *z* or a function, which remains unchanged, even if *z* shall become negative. But it is required that *PM* and *AM* to be odd functions of *z*, that is which shall be made negative with *z* made negative, that is which shall become negative with *z* changed into $-z$. On account of which *L* must be an even function of *z*, *M* and *N* truly odd functions. For with these in place the abscissa will be equal to an even function, and truly the applied lines odd functions for the curve itself. For *dL* will be an odd function, *ddL* an even function, *dM* and *dN* even functions and *ddM* and *ddN* odd functions. From which it shall be understood the lines *AP*, *PM* and *AM* to be required to be having this property. Therefore there will become :

$$AQ = M + \frac{\left(dL^2 + dM^2 - dN^2\right)\left(dMdN + dL\sqrt{\left(dL^2 + dM^2 - dN^2\right)}\right)}{dLdNddL + dMdNddM - dL^2ddN - dM^2ddN + \left(dLddM - dMddL\right)\sqrt{\left(dL^2 + dM^2 - dN^2\right)}}.$$

And the applied line of the reciprocal trajectory EAN will become

$$QN = N - L$$
$$+\frac{\left(dL^2 + dM^2 - dN^2\right)\left(dL^2 + dM^2 - dLdN + dM\sqrt{\left(dL^2 + dM^2 - dN^2\right)}\right)}{dLdNddL + dMdNddM - dL^2ddN - dM^2ddN + \left(dLddM - dMddL\right)\sqrt{\left(dL^2 + dM^2 - dN^2\right)}}.$$

All the reciprocal trajectories emanate from this construction, if in place of *L*, *M* and *N* not only algebraic functions may be substituted, but also transcendental ones. Truly all these will be changed into reciprocal algebraic trajectories, if these quantities were algebraic functions, and indeed so that *L* is required to be an even function and *M* and *N* to be odd functions.

Again it is required to be noted,

$$\sqrt{\left(dL^2+dM^2-dN^2\right)}$$

must be an odd function of *z*, whatever the nature of *dL*. Therefore there may be put

$$\sqrt{\left(dL^2+dM^2-dN^2\right)}=dL-dS,$$

where *dS* is an odd function of *z* and thus *S* to be an even function; there will become:

$$\frac{dN^2-dM^2+dS^2}{2dS}=dL,$$

from which *dL* will be an odd function, as required, and

$$\sqrt{\left(dL^2+dM^2-dN^2\right)}=\frac{dN^2-dM^2-dS^2}{2dS}.$$

Therefore there will become:

$$L=\int\frac{dN^2-dM^2}{dS}+\tfrac{1}{2}dS.$$

On account of which, so that the reciprocal trajectory shall become algebraic, $\dfrac{dN^2-dM^2}{dS}$ to be required to be integrable. Moreover the abscissa will be

$$\mathrm{A}Q=M+\frac{\left(dN^2-dM^2-dS^2\right)\left(\left(dN+dM\right)^2-dS^2\right)}{4dS\left(dSddM+dSddN-dNddS-dMddS\right)}$$

and

$$QN=N-\tfrac{1}{2}\int\frac{dN^2-dM^2}{dS}-\tfrac{1}{2}S+\frac{\left(dN^2-dM^2-dS^2\right)\left(dS^2+\left(dM+dN\right)^2\right)}{4dS\left(dSddM+dSddN-dNddS-dMddS\right)}.$$

Moreover as long as $\sqrt{\left(dL^2+dM^2-dN^2\right)}$ remains a surd quantity, there is no need for a particular determination, for it is the square root especially to be determined from an even function not to be a square both for even function as well as and odd function.

DE CURVIS RECTIFICABILIBUS ALGEBRAICIS ATQUE TRAIECTORIIS RECIPROCIS ALGEGRAICIS [E23].

Auct. Leonh. Eulero.

Quanquam admodum facile est innumeras dare curvas algebraicas, quae rectificari possunt, quaerendis vel evolutis vel causticis curvarum algebraicarum; tamen si ordines curvarum consideremus, rarissime in iis occurrunt, quae rectificationem admittant. In ordine linearum secundo, qui ex sectionibus conicis constat, nulla est huiusmodi; in tertio duae habentur rectificabiles quantum quidem constat. Cum autem ante aliquot annos in inveniendis traiectoriis reciprocis algebraicis primum essem, methodum Celeb. *Ioh. Bernoulli* primum sequutus diligentissimme curvas rectificabiles acquirebam, ut iis ad propositum uterer. Detexi etiam in ordine sexto curvam rectificabilem, quae mihi praebebat traiectoriam algebraicam ordinis quarti, eaque satisfeci quaestioni tum agitatae, de exhibendis simplicioribus reciprocis algebraicis. Inveni quoque multas aequationes generales curvas rectificabiles, ex quibus in promtu erat omnes curvas rectificabiles simpliciores eruere. Haec iterum nunc perlustrans deductus sum ad generalissimam quandam aequationem curvas rectificabiles omnes in se continentem. Insunt enim in ea plures quantitates universales, pro quibus quicquid substituatur, curva recificabilis prodit. Sit quantitas quaedam variabilis *z*, cuius differentiale ponatur constans, sintque P et Q huius variabilis $dx^2+dy^2=ds^2 z$ functiones quaecunque salte algebraicae. Si iam hoc modo construatur curva Sit quantitas quaedam variabilis *z*, cuius differentiale ponatur constans, sintque *P* et *Q* huius variabilis *z* functiones quaecunque saltem algebraicae. Si iam hoc modo construatur curva ut eius abscissa, quae ponatur *x*, sumatur aequalis

$$P+\frac{dQ\left(dP^2-dQ^2\right)}{dQddP-dPddQ}$$

et applicataa quam voco

$$y=\frac{\left(dP^2-dQ^2\right)^{\frac{3}{2}}}{dQddP-dPddQ}.$$

Erit huius curvae longitudo *s* appellata aequalis $Q+\frac{dQ\left(dP^2-dQ^2\right)}{dQddP-dPddQ}$. Semper igitur quiccunque valores literis P et Q tribuantur, curva erit rectificabilis et algebraica. Demonstrationem huius dare non opus esse iudico, cuilibet enim, si sumferit differentialia coordinatarum *x* et *y* et curvae *s*, innotescet esse $dx^2+dy^2=ds^2$ labore tantum est opus nullo autem artificio.

Haec forma quidem latissime patet, sed habeo tamen aliam adhuc multo generaliorem, imo generalissimam sequentem. Designantibus ut ante litteris maiusculis *L*, *M*, et *N* functiones quascunque variabilis *x*, si sumantur

$$x = L + \frac{\left(dL^2 + dM^2 - dN^2\right)\left(dLdN - dM\right)\left(dL^2 + dM^2\right)\sqrt{\left(dL^2 + dM^2 - dN^2\right)}}{dLdNddL + dMdNddM - dL^2ddN - dM^2ddN + \left(dLddM - dMddL\right)\sqrt{\left(dL^2 + dM^2 - dN^2\right)}}$$

$$y = M + \frac{\left(dL^2 + dM^2 - dN^2\right)\left(dMdN + dL\sqrt{\left(dL^2 + dM^2 - dN^2\right)}\right)}{dLdNddL + dMdNddM - dL^2ddN - dM^2ddN + \left(dLddM - dMddL\right)\sqrt{\left(dL^2 + dM^2 - dN^2\right)}}$$

erit longitudo curvae respondentis

$$s = N + \frac{\left(dL^2 + dM^2 - dN^2\right)\left(dL^2 + dM^2\right)}{dLdNddL + dMdNddM - dL^2ddN - dM^2ddN + \left(dLddM - dMddL\right)\sqrt{\left(dL^2 + dM^2 - dN^2\right)}}.$$

Formulae hae in praecedentes mutantumr , si ponatur $M = 0$, sunt igitur illae in his contentae.

Facile perspicitur, si litterae *L, M* et *N* non solum significent quantitates algebraicas, sed etiam transcendentes quasque, in istis formulis omnes prorsus contineri curvas tam agebraicas quam transcendentes. Quia enim hae functiones *L, M* et *N* nullo modo a se invicem sunt pendentes, quae non ad praescriptas formulas esset reducibilis.

Simili modo, si manente *N* functions universalissima, ita tamen, ut $\frac{dN}{dz}$ sit functio algebraicae in formulis datis comprehenduntur; erunt autem eae rectificabiles, si *N* non fuerit functio algebraica, L vero et M denotent functiones algebraicas, omnes prorsus curvae algebraicae in formulas datis comprehenduntur ; erunt autem eae rectificabiles, si N assumatur functio algebraica, at vero si non fuerit functio algebraica, sed transcendens seu a quadratura curvae cuiusdem pendens, curva resultans non erit rectificabilis, sed eius rectificatio pendebit a quadratura eius curvae. Hoc igitur modo solvitur etiam celebre illud problema multum inter Geometras agitatum, postulans methodum quadraturas curvarum ad rectificationes curvarum algebraicarum reducendi, cuius solutiones duae datae sunt in Actis Lipsiensibus a Viris Celeberrimis Iacoba Hermanno et Iohanni Bernoulio. Ex istis vero formulis ita solvetur ; sint curvae, cuius quadratura ad rectificationem curvae algebraicae est reducenda, coordinatae *t* et *v*, quarum utraque sit functio algebraica ipsius *z*. Sumatur

$$N = R + a\int vdt,$$

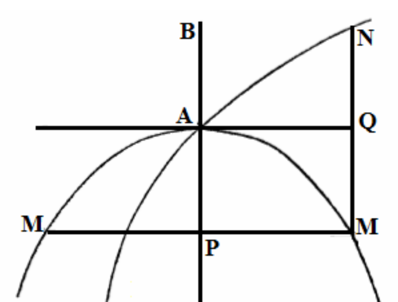

Fig. 1

ubi *R* etiam designet functionem ipsius *z* algebraicam quamcunque. Hoc posito dabunt formulae omnes curvas algebraicas, quarum rectificatio a quadratura curvae propositae, scilicet a $a\int vdt,$ pendet, siquidem *L* et *M* assumantur, ut iam est monitum, functiones algebraicae ipsius *z*. Alter usus harum formularum, quem hic exponere constitui, respicit ad inventionem traiectoriarum reciprocarum; reperitur enim ex iis equatio universalissima omnes traiectorias reciprocarum in se complectens, quae etiam facillime ita restringitur, ut algebraicas tantum easque omnes praebeat.

Nititur autum haec inventio theoremate Bernoulliano, quo ex rectificatione curvarum diametrum habentium construuntur traiectoriae reciprocae, hoc modo (Fig.1) : *MAM* ist curva huiusmodi diametrum habens *AP*, et verticem *A*, in quo tangens *AQ* est perpendicuaris ad diametrum *AP*, et verticem *A*. Per huius curvae singula puncta *M* ducantur rectae diametro parallelae in iisque sumantur *MN* aequales arcubus *AM*, constituent puncta *N* curvam *NAE* traiectoriam reciprocam, cuius axis conversionis est ipsa diameter *PAB*. Huius curvae si sumatur abscissa *AQ* et applicataa QN, erit $AQ = PM$ et $QN = AM - AP.$ Quamobrem si curvae *MAM* coordinatae *AP* et *PM* eadem sumantur, quae ante vocatae erant *x* et *y*, haebitur statim aequatio pro traiectoria reciproca *EAN*. Quam vero non omnis curva in locum *MAM* collocari potest, in aequationibus supra traditis litterae *L, M* et *N* quodammodo restringi debent, ut tantum curvas ad institutum accommodatas praebeant. Cum omnes lineae *AP*, *PM* et *AM* sint functiones ipsius *z*, ita eae in *z* determinentur, ut sumto *z* affirmativo prodeat curvae *MAM* ramus dexter, at posito *z* negativo, ut prodeat ramus sinister. Ad hoc requiritur, ut *AP*, quia in utroque casu eadem manet, sit functio par ipsius *z* seu functio, quae immutata manet, etiamsi *z* fiat negativum. At *PM* et *AM* esse oportert functiones ipsius *z* impares, id est quae fiant negativae mutato *z* impares, id est quae fiant negativae mutato *z* in -*z*. Quamobrem esse debet *L* functio par ipsius *z*, *M* vero et *N* functiones impares. His enim positis abscissa aequabitur fuctioni pari, applicataa vero et ipsa curva functionibus imparibus. Nam *dL* erit functio impar, *ddL* functio par, *dM* et *dN* functiones pares atque *ddM* et *ddN* functiones impares. Ex quo perspicietur lineas *AP*, *PM* et *AM* requisitam habituras esse proprietatem. Erit igitur

$$AQ = M + \frac{\left(dL^2 + dM^2 - dN^2\right)\left(dMdN + dL\sqrt{\left(dL^2 + dM^2 - dN^2\right)}\right)}{dLdNddL + dMdNddM - dL^2ddN - dM^2ddN + \left(dLddM - dMddL\right)\sqrt{\left(dL^2 + dM^2 - dN^2\right)}}.$$

Atque traiectoriae reciprocae EAN applicata erit

$$QN = N - L$$
$$+\frac{\left(dL^2 + dM^2 - dN^2\right)\left(dL^2 + dM^2 - dLdN + dM\sqrt{\left(dL^2 + dM^2 - dN^2\right)}\right)}{dLdNddL + dMdNddM - dL^2ddN - dM^2ddN + \left(dLddM - dMddL\right)\sqrt{\left(dL^2 + dM^2 - dN^2\right)}}.$$

Ex hac constructione fluent omnes traiectoriae reciprocae, si loco *L*, *M* et *N* substituantur functiones non solum algebraicae, sed etiam transcendentes. Algebraicae vero traiectoriae reciprocae omnes habibuntur, si eae quantitates fuerint functiones algebraicae, et quidem ut requiritur *L* functio par atque *M* et *N* functiones impares.

Notandum porro est etiam

$$\sqrt{\left(dL^2 + dM^2 - dN^2\right)}$$

esse debere functionem ipsius z imparem, quemadmodum *dL*. Ponatur

$$\sqrt{\left(dL^2+dM^2-dN^2\right)}=dL-dS,$$

ubi *dS* est functio impar ipsius *z* adeoque *S* functio par; erit

$$\frac{dN^2-dM^2+dS^2}{2dS}=dL,$$

ex quo erit *dL* functio impar, ut requiritur, et

$$\sqrt{\left(dL^2+dM^2-dN^2\right)}=\frac{dN^2-dM^2-dS^2}{2dS}.$$

Erit igitur

$$L=\int\frac{dN^2-dM^2}{dS}+\tfrac{1}{2}dS.$$

Quamobrem, ut traiectoria reciproca fiat algebraica, oportet $\frac{dN^2-dM^2}{dS}$ esse integrabile. Erit autem abscissa

$$\mathrm{A}Q=M+\frac{\left(dN^2-dM^2-dS^2\right)\left(\left(dN+dM\right)^2-dS^2\right)}{4dS\left(dSddM+dSddN-dNddS-dMddS\right)}$$

et

$$QN=N-\tfrac{1}{2}\int\frac{dN^2-dM^2}{dS}-\tfrac{1}{2}S+\frac{\left(dN^2-dM^2-dS^2\right)\left(dS^2+\left(dM+dN\right)^2\right)}{4dS\left(dSddM+dSddN-dNddS-dMddS\right)}.$$

Quamdiu autem $\sqrt{\left(dL^2+dM^2-dN^2\right)}$ manet quantitas surda, non opus est peculiari determinatione, est enim radix quadrata ex functione pari non quadrato tam functio par quam impar.