CONCERNING THE SUMMATION OF INNUMERABLE PROGRESSIONS.

by L. Euler.

§ I. It is apparent, in the previous discourse concerning transcendental progressions that I treated, that the general terms of these progressions are of much broader extent than it was possible to consider; and that among other things to which most can be adapted, they are excellent to use in finding the sums of innumerable progressions. As indeed innumerable progressions in the above discourse refer to general terms, which transcend ordinary algebra, thus I adapt the same method here in order to find the terms corresponding to the sums of progressions, for which ordinary algebra is insufficient in the summing of such indefinite progressions.

§2. A certain indefinite progression is said to be summed, if a formula is given containing the indefinite number n, which sets out the sum of all the terms of that progression, as many terms are taken as there are units in n, thus, for example if n is put equal to 10, this formula shows the sum of the ten terms beginning from the first of the numbers. This formula is called the *summation term* [i. e. the term representing the n^{th} partial sum] expressing the sum of this progression and it likewise is the term of a general progression, any term of which is equal to the whole sum of the terms of that progression, and showing how many units are contained within it.

[Thus, the summatory or summation terms in turn form a progression, the n^{th} term of which is the n^{th} partial sum of the initial progression.]

§3. Since all progressions are expressed by general terms, this is the question concerning the summation of progressions, that the summatory term can be found from the general term of the progression. And indeed when that has been attained, so that just as often as the general tem is rational function of the index n and the exponents are positive whole numbers, then the summatory term can always be found. But when the exponents of n are negative, except in a few exceptional cases, no one hitherto has given the summatory terms. The reason for this difficulty is that generally the summatory terms cannot be expressed algebraically, but require such forms as may be contained in their quadratures.

[Thus, the n^{th} partial sum involves an integral.]

§4. This form is assumed as the general term of a certain progression:

$$\int \frac{1-x^n}{1-x} dx\,;$$

which clearly integrated, thus so that it becomes equal to zero if $x = 0$, and on putting $x = 1$, the term of order n is given. The progression, which is formed in this manner from that, is this:

$$1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} \text{ etc.},$$

hence the general term of this is the formula assumed :

$$\int \frac{1-x^n}{1-x} dx$$

Now indeed this summatory series found is that of the harmonic progression :

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \text{ etc.},$$

of which the general term is $\frac{1}{n}$. On account of which the summatory term of the [initial] progression is $\int \frac{1-x^n}{1-x}dx$, which is the general term of this summatory progression.

§5. Since the general term of the progression :

$$1, 1+\tfrac{1}{2}, 1+\tfrac{1}{2}+\tfrac{1}{3} \text{ etc.}$$

is

$$\int \frac{1-x^n}{1-x}dx,$$

from this term it is possible for that progression to be interpolated, or for any mean term to be found; so if the term is required of which the index is $\frac{1}{2}$, then it is required to integrate

$$\frac{1-\sqrt{x}}{1-x}dx \text{ or } \frac{dx}{1+\sqrt{x}},$$

the integral of which is :

$$2\sqrt{x}-2l(1+\sqrt{x});$$

as it becomes 0 if $x = 0$; when $x = 1$ the term with order $\frac{1}{2}$ is equal to $2 - 2l2$. Hence, since the general term with order $n + 1$ exceeds the term with order n by the fraction $\frac{1}{n+1}$, [as this rule still applies] then the term with order n $1\frac{1}{2}$ is equal to $2\frac{2}{3}-2l2$, and the term with order n $2\frac{1}{2}$ is equal to $2+\frac{2}{3}+\frac{2}{5}-2l2$ etc. Therefore the interpolated [summatory] series will be :

$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	etc
$2-2l2$	1	$2+\frac{2}{3}-2l2$	$1+\frac{1}{2}$	$2+\frac{2}{3}+\frac{2}{3}-2l2$	

§6. I have embraced the more general argument in this way, and I have assumed the formula

$$\int \frac{1-P^n}{1-P}dx,$$

where P denotes any function of x. For this integral as always must be taken so that thus on putting $x = 0$, the total becomes equal to zero. Finally I do not as before put $x = 1$, for in order that the expression is of wider use, I put $x = k$. In this way the resulting expression is the term of order n of some progression, and the general term of this takes the form :

$$\int \frac{1-P^n}{1-P}dx$$

Now the progression itself is this :

$$k, k+\int Pdx, k+\int Pdx+\int P^2dx \text{ etc.};$$

where in the integrals $\int Pdx, \int P^2dx$ etc., I can now put k in place of x [as the upper limit of the integration.]

§7. The progression found, if any term is subtracted [from the sequence of summatory terms], is given by this :

$$k, \int Pdx, \int P^2dx, \int P^3dx \text{ etc.},$$

the general term of which is :

$$\int P^{n-1}dx.$$

The summatory term of this is equal to the general term of the preceding progression, which is served by this formula $\int \frac{1-P^n}{1-P}dx$.

Let $P = x^\alpha : a^\alpha$; the progression of this is given by :

$$k, \frac{k^{\alpha+1}}{(\alpha+1)a^\alpha}, \frac{k^{2\alpha+1}}{(2\alpha+1)a^{2\alpha}} \text{ etc .};$$

the general term by :

$$\frac{k^{(n-1)\alpha+1}}{(1+(n-1)\alpha)a^{(n-1)\alpha}};$$

and the summatory term of this progression is:

$$\int \frac{a^{n\alpha}-x^{n\alpha}}{(a^\alpha - x^\alpha)a^{n\alpha-\alpha}}dx.$$

§8. Hence the general summatory term is found for all progressions, of which the terms are fractions, and of these the numerators constitute a geometric progression, and the denominators an arithmetic progression. Now as it is easily applied to all cases, this particular progression is taken :

$$\frac{b}{c}, \frac{b^{i+1}}{c+e}, \frac{b^{2i+1}}{c+2e}, \frac{b^{3i+1}}{c+3e} \text{ etc .},$$

the general term of which is :

$$\frac{b^{(n-1)i+1}}{c+(n-1)e};$$

this can be compared with that for the general term :

$$\frac{k^{(n-1)\alpha+1}}{(1+(n-1)\alpha)a^{(n-1)\alpha}} \text{ or } \frac{ck^{(n-1)\alpha+1}}{(c+(n-1)\alpha c)a^{(n-1)\alpha}};$$

then

$$\alpha = \frac{e}{c} \text{ and } \frac{ck^{(n-1)\frac{e}{c}+1}}{a^{(n-1)\frac{e}{c}}} = b^{(n-1)i+1}$$

and

$$a = \left(\frac{ck^{(n-1)\frac{e}{c}+1}}{b^{(n-1)i+1}}\right)^{\frac{c}{(n-1)e}} = \left(\frac{ck}{b}\right)^{\frac{c}{(n-1)e}} \frac{k}{b^{ci:e}}$$

Here, so that a does not depend on n (for a must be a constant quantity), it is required that $\frac{ck}{b}$ = 1; hence $k=\frac{b}{c}$ and $a=\frac{b^{\frac{e-ci}{e}}}{c}$. On account of which the summatory term is :

$$\int\frac{b^{\frac{ne-nci}{c}}-c^{\frac{ne}{c}}x^{\frac{ne}{c}}}{b^{\frac{(n-1)(e-ci)}{c}}(b^{\frac{e-ci}{c}}-c^{\frac{e}{c}}x^{\frac{e}{c}})}dx$$

Which thus must be integrated, so that it becomes zero if x = 0; then it is necessary now to put $x=\frac{b}{c}$.

§9. From the known sum of the indefinite progression the sum of the infinite progression can be obtained, on putting $n=\infty$. Indeed the summatory term found is seen to be adapted in this case as in any other. Now I have another method of summing an infinite series to be investigated, which appears to be of the widest application. Let the series be

$$\frac{b}{c},\frac{b^{i+1}}{c+e},\frac{b^{2i+1}}{c+2e},\frac{b^{3i+1}}{c+3e}\text{ etc}.$$

The number of terms is put as n and the sum of these is taken as A. The number n is increased by one; the sum A is increased by the term of order n + 1, which is $\frac{b^{ni+1}}{c+ne}$. Now if n and A can be considered as fluent quantities, since n is taken as if infinitely greater than 1, the differences of these dn and dA are increased by 1 and $\frac{b^{ni+1}}{c+ne}$. Hence the equation is produced :

$$dA=\frac{b^{ni+1}dn}{c+ne}.$$

Which integrated gives the equation between the sum A and the number of terms n.

§10. We put

$$l(c+ne)=z;$$

then

$$\frac{edn}{c+ne}=dz$$

and $c+ne=g^z$ with g denoting the number, of which the logarithm is 1. Hence

$$n=\frac{g^z-c}{e}\text{ and }b^{ni+1}=b^{\frac{g^z i-ci+e}{e}}=b^{\frac{e-ci}{e}}b^{\frac{g^z i}{e}},$$

consequently

$$dA=\frac{b^{\frac{e-ci}{e}}}{e}b^{\frac{g^z i}{e}}dz.$$

Indeed this equation does not thus admit to being integrated in general unless the integration is performed through a series expansion. If now we put i = 0, in order that the series arises :

$$\frac{b}{c}+\frac{b}{c+e}+\frac{b}{c+2e}+\text{etc}.,$$

the equation is obtained :

$$dA=\frac{b}{e}dz\text{ and }A=\frac{b}{e}(z+lC)=\frac{b}{e}lC(c+ne).$$

Indeed the constant C is not determined, but yet the equation serves to define the difference between two sums; so that if another number of terms is m and the sum B; then

$$B = \frac{b}{e} lC(c+me).$$

Hence

$$B - A = \frac{b}{e} l \frac{c+me}{c+ne} = \frac{b}{e} l \frac{m}{n},$$

since m and n are infinite.

§11. Keeping $i = 0$ and the progression becomes this :

$$\frac{b}{c}, \frac{b}{c+e}, \frac{b}{c+2e}, \frac{b}{c+3e} \text{ etc ;}$$

the general term of this is :

$$\frac{b}{c+(n-1)e}.$$

Moreover, the summatory term is

$$\int \frac{b^{\frac{ne}{c}} - c^{\frac{ne}{c}} x^{\frac{ne}{c}}}{b^{\frac{(n-1)e}{c}} (b^{\frac{e}{c}} - c^{\frac{e}{c}} x^{\frac{e}{c}})} dx.$$

Another progression is taken :

$$\frac{b}{c}, \frac{b}{c+f}, \frac{b}{c+2f}, \frac{b}{c+3f} \text{ etc .,}$$

and the general term of this is :

$$\frac{b}{c+(n-1)f}$$

and the summatory term :

$$\int \frac{b^{\frac{nf}{c}} - c^{\frac{nf}{c}} x^{\frac{nf}{c}}}{b^{\frac{(n-1)f}{c}} (b^{\frac{f}{c}} - c^{\frac{f}{c}} x^{\frac{f}{c}})} dx,$$

in which integration it is required to put : $x = k = \frac{b}{e}$. These two progressions are added, clearly the first with the first, the second with the second, and thus henceforth; this progression arises :

$$\frac{2b}{c}, \frac{2bc+b(e+f)}{(c+e)(c+f)}, \frac{2bc+2b(e+f)}{(c+2e)(c+2f)}, \text{ etc .,}$$

the general term of which is :

$$\frac{2bc+(n-1)b(e+f)}{(c+(n-1)e)(c+(n-1)f)}.$$

Now the summatory term is :

$$\int dx \left(\frac{b^{\frac{ne}{c}} - c^{\frac{ne}{c}} x^{\frac{ne}{c}}}{b^{\frac{(n-1)e}{c}} (b^{\frac{e}{c}} - c^{\frac{e}{c}} x^{\frac{e}{c}})} + \frac{b^{\frac{nf}{c}} - c^{\frac{nf}{c}} x^{\frac{nf}{c}}}{b^{\frac{(n-1)f}{c}} (b^{\frac{f}{c}} - c^{\frac{f}{c}} x^{\frac{f}{c}})} \right).$$

§12. In a like manner, but more general, for the general term, in which the denominator n has two dimensions, the summation term is found, if a p- multiple of this is added to a q- multiple of that summation term. In this way a progression is obtained, the general term of which is :

$$\frac{pb}{c+(n-1)e}+\frac{qb}{c+(n-1)f}=\frac{(p+q)bc+(n-1)b(pf+qe)}{(c+(n-1)e)(c+(n-1)f)}.$$

Moreover the summation term corresponding to this general term is :

$$\int\frac{pdx}{b^{\frac{(n-1)e}{c}}}\left(\frac{b^{\frac{ne}{c}}-c^{\frac{ne}{c}}x^{\frac{ne}{c}}}{b^{\frac{e}{c}}-c^{\frac{e}{c}}x^{\frac{e}{c}}}\right)+\int\frac{qdx}{b^{\frac{(n-1)f}{c}}}\left(\frac{b^{\frac{nf}{c}}-c^{\frac{nf}{c}}x^{\frac{nf}{c}}}{b^{\frac{f}{c}}-c^{\frac{f}{c}}x^{\frac{f}{c}}}\right)=$$

$$\int dx\left(\begin{array}{l}\dfrac{pb^{\frac{n(e+f)}{c}}-pb^{\frac{n(e+f)-f}{c}}c^{\frac{f}{c}}x^{\frac{f}{c}}-pb^{\frac{nf}{c}}c^{\frac{ne}{c}}x^{\frac{ne}{c}}+pb^{\frac{(n-1)f}{c}}c^{\frac{ne+f}{c}}x^{\frac{ne+f}{c}}}{b^{\frac{(n-1)(e+f)}{c}}\left(b^{\frac{e}{c}}-c^{\frac{e}{c}}x^{\frac{e}{c}}\right)\left(b^{\frac{f}{c}}-c^{\frac{f}{c}}x^{\frac{f}{c}}\right)}\\[2ex]+\dfrac{qb^{\frac{n(e+f)}{c}}-qb^{\frac{n(e+f)-e}{c}}c^{\frac{e}{c}}x^{\frac{e}{c}}-qb^{\frac{ne}{c}}c^{\frac{nf}{c}}x^{\frac{nf}{c}}+qb^{\frac{(n-1)e}{c}}c^{\frac{nf+e}{c}}x^{\frac{nf+e}{c}}}{b^{\frac{(n-1)(e+f)}{c}}\left(b^{\frac{e}{c}}-c^{\frac{e}{c}}x^{\frac{e}{c}}\right)\left(b^{\frac{f}{c}}-c^{\frac{f}{c}}x^{\frac{f}{c}}\right)}\end{array}\right).$$

Put $b = 1$, for in this way nothing is removed from the generality, and the general term becomes :

$$\frac{(p+q)c+(n-1)(pf+qe)}{(c+(n-1)e)(c+(n-1)f)}.$$

Let $cx = y$; then

$$dx=\frac{dy}{c}.$$

And the summation term becomes equal to :

$$\int\frac{dy}{c}\left(\frac{p+q-py^{\frac{f}{c}}-qy^{\frac{e}{c}}-py^{\frac{ne}{c}}-qy^{\frac{nf}{c}}+py^{\frac{ne+f}{c}}+qy^{\frac{nf+e}{c}}}{\left(1-y^{\frac{e}{c}}\right)\left(1-y^{\frac{f}{c}}\right)}\right),$$

in which with the integrated formula so that thus on putting $y = 0$ this too becomes equal to zero, and then it is required to put $y = 1$.

§13. Now, this general term is assumed :

$$\frac{\alpha+\beta n}{\gamma+\delta n+\varepsilon nn}.$$

Which compared with :

$$\frac{(p+q)c+(n-1)(pf+qe)}{(c+(n-1)e)(c+(n-1)f)}$$

gives

$c=\sqrt{(\gamma+\delta+\varepsilon)},\ e=\frac{\delta+2\varepsilon+\sqrt{(\delta\delta-4\gamma\varepsilon)}}{2\sqrt{(\gamma+\delta+\varepsilon)}},\ f=\frac{\delta+2\varepsilon-\sqrt{(\delta\delta-4\gamma\varepsilon)}}{2\sqrt{(\gamma+\delta+\varepsilon)}},$

$$p=\frac{\alpha\delta-\beta\delta+2\alpha\varepsilon-2\beta\gamma+(\alpha+\beta)\sqrt{(\delta\delta-4\gamma\varepsilon)}}{2\sqrt{(\gamma+\delta+\varepsilon)(\delta\delta-4\gamma\varepsilon)}}$$

and

$$q=\frac{\beta\delta-\alpha\delta+2\beta\gamma-2\alpha\varepsilon+(\alpha+\beta)\sqrt{(\delta\delta-4\gamma\varepsilon)}}{2\sqrt{(\gamma+\delta+\varepsilon)(\delta\delta-4\gamma\varepsilon)}}.$$

With these terms substituted into the summation term, the summation term of these terms is produced:

$$\frac{\alpha+\beta}{\gamma+\delta+\varepsilon},\ \frac{\alpha+2\beta}{\gamma+2n+4\varepsilon},\ \frac{\alpha+3\beta}{\gamma+3\delta+9\varepsilon}\ \text{etc.},$$

the general term of this is :

$$\frac{\alpha+\beta n}{\gamma+\delta n+\varepsilon nn}.$$

§13a. [Section number has been repeated] In the same way, if the general term *n* has more than two dimensions, then the summation term arising from so many simple progressions combined together must have the same number of dimensions *n*, as has occurred likewise in the case of two dimensions. But yet for this [following] reason, a general series cannot be arrived, that is considered to be formed from general terms in this way. For whenever the denominator $\gamma+\delta n+\varepsilon n^2+\xi n^3+\eta n^4+$ etc. has two or more simple equal factors [in the previous general term], then the progression cannot be resolved into so many simple progressions, and therefore the summatory term cannot be found.

§14. On account of this I present another method which does not exclude these cases. Let a certain simple progression be :

$$\frac{1}{a},\ \frac{1}{a+b},\ \frac{1}{a+2b}\ \text{etc.}$$

the general term of which is :

$$\frac{1}{a+(n-1)b}$$

The summation term of this progression is [see §11] :

$$\int\frac{1-a^{\frac{nb}{a}}x^{\frac{nb}{a}}}{1-a^{\frac{b}{a}}x^{\frac{b}{a}}}dx,$$

or, on putting $ax=y$, this becomes :

$$\int\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}\frac{dy}{a},$$

in which on integrating it is necessary to put $y=1$. This is multiplied by $y^{\alpha}dy$ and the sum of this is made equal to

$$\int y^{\alpha}dy\int\frac{dy}{a}.\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}$$

the summation term of the following progression is treated in following way:

$$\frac{a}{a.\beta a},\ \frac{a}{(a+b)(\beta a+b)},\ \frac{a}{(a+2b)(\beta a+2b)}\ \text{etc.}$$

thus on putting for brevity β in place of $\alpha+2$.
[The general term of the G.P. of the first series is integrated without taking the limit, this is multiplied by y^{α} and integrated again, at which point we set $y = 1$.]
The general term of this progression is :

$$\frac{a}{(a+(n-1)b)(\beta a+(n-1)b)} \text{ or } \frac{a}{b^2n^2+(ab+\beta ab-2bb)n+(a-b)(\beta a-b)}.$$

§15. We can take a general progression of this kind, which can be more easily adapted to any case; let the general term of this be :

$$\frac{1}{a+(n-1)b+\frac{(n-1)(n-2)}{2}c}.$$

Since this must be compared with the above general term, this gives :

$$a=\frac{(2b-c)^2-4ac+(2b-c)\sqrt{((2b-c)^2-8ac)}}{4c},$$
$$b=\frac{2b-c+\sqrt{((2b-c)^2-8ac)}}{4},$$
$$\beta=\frac{2b-c-\sqrt{((2b-c)^2-8ac)}}{2b-c+\sqrt{((2b-c)^2-8ac)}}.$$

These values are substituted in place of *a*, *b* and α (now $\alpha=\beta-2$) in

$$\int y^{\alpha}dy\int\frac{dy}{a}.\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}},$$

and the summation term of the proposed progression is produced:

$$\frac{1}{a},\ \frac{1}{a+b},\ \frac{1}{a+2b+c},\ \frac{1}{a+3b+3c}\ \text{etc.}$$

§16. It is possible to progress further in this manner :

$$\int\frac{dy}{a}.\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}$$

is multiplied by $y^{\alpha-2}dy$ and the integral produced :

$$\int y^{\alpha-2}dy\int\frac{dy}{a}.\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}$$

is again multiplied by $y^{\beta-\alpha-1}$, and the integral of this is produced

$$\int y^{\beta-\alpha-1}dy\int y^{\alpha-2}dy\int \frac{dy}{a}.\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}$$

is the summation term of this progression [on setting $y = 1$]:

$$\frac{a^2}{a.\alpha a.\beta a},\ \frac{a^2}{(a+b)(\alpha a+b)(\beta a+b)},\ \frac{a^2}{(a+2b)(\alpha a+2b)(\beta a+2b)}\ \text{etc.},$$

the general term of which is :

$$\frac{a^2}{(a+(n-1)b)(\alpha a+(n-1)b)(\beta a+(n-1)b)};$$

similarly

$$\int y^{\gamma-\beta-1}dy\int y^{\beta-\alpha-1}dy\int y^{\alpha-2}dy\int \frac{dy}{a}.\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}$$

is the summation term of the progression, of which the general term is :

$$\frac{a^3}{(a+(n-1)b)(\alpha a+(n-1)b)(\beta a+(n-1)b)(\gamma a+(n-1)b)}.$$

Therefore in this manner all the progressions can be arrived at, the terms of which are fractions present with constant numerators and denominators, but with the denominators making some algebraic progression.

§17. If the sums of progressions of this kind continued to infinity are wanted, it is necessary to put n equal to infinity. Finally with this in place the member of the summation term

$$\int \frac{dy}{a}.\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}},$$

is clearly changed into this :

$$\int \frac{dy}{a\left(1-y^{\frac{b}{a}}\right)}.$$

Indeed since y is always less than 1 except in the final case, in which $y = 1$, $y^{\frac{nb}{a}}$ besides 1 becomes zero and thus $1-y^{\frac{nb}{a}}$ changes into 1. Hence the sum of this series

$$\frac{a}{a.\alpha a}+\frac{a}{(a+b)(\alpha a+b)}+\frac{a}{(a+2b)(\alpha a+2b)}+\ \text{etc. to infinity}$$

will be

$$\int y^{\alpha-2}\int \frac{dy}{a\left(1-y^{\frac{b}{a}}\right)}$$

and of this progression,

$$\frac{a^2}{a.\alpha a.\beta a},\ \frac{a^2}{(a+b)(\alpha a+b)(\beta a+b)},\ \frac{a^2}{(a+2b)(\alpha a+2b)(\beta a+2b)}\ \text{etc.}$$

the sum is

$$\int y^{\beta-\alpha-1}\int y^{\alpha-2}\int\frac{dy}{a\left(1-y^{\frac{b}{a}}\right)}$$

and thus with all the rest.

§18. Let $b = a$, so that $\frac{b}{a}=1$; then

$$\int\frac{dy}{a(1-y)}=A-\tfrac{1}{a}l(1-y).$$

Since on putting $y = 0$ the whole integral must be equal to 0, then $A = 0$ and thus

$$\int\frac{dy}{a(1-y)}=-\tfrac{1}{a}l(1-y).$$

This is multiplied by $y^{\alpha-2}dy$; there is obtained :

$$-\frac{y^{\alpha-2}dy}{a}l(1-y).$$

In order that the integral of this can be found, put $1 - y = z$; then $y = 1 - z$; therefore the integrand is obtained :

$$-\frac{(1-z)^{\alpha-2}dz}{a}lz$$

$$=\left(1-\frac{\alpha-2}{1}z+\frac{(\alpha-2)(\alpha-3)}{1.2}z^2-\frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.2.3}z^3\right)\frac{dz}{a}lz.$$

Now since

$$\int z^{\eta}dzlz=C-\frac{z^{\eta+1}}{(\eta+1)^2}+\frac{z^{\eta+1}lz}{\eta+1},$$

the integral of this is this series:

$$\tfrac{1}{a}\left(C-z+zlz+\frac{\alpha-2}{1.4}z^2-\frac{(\alpha-2)}{1.2}z^2lz-\frac{(\alpha-2)(\alpha-3)}{1.2.9}z^3+\frac{(\alpha-2)(\alpha-3)}{1.2.3}z^3lz+\text{etc}\right).$$

This integral, if $y = 0$ or $z = 1$, must become $= 0$; on account of this, then

$$C=1-\frac{\alpha-2}{1.4}+\frac{(\alpha-2)(\alpha-3)}{1.2.9}-\frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.2.3.16}+\text{etc.}$$

§19. It is evident from this integral, that as often as α is a whole number greater than one, then the integral of this term is always a finite number and thus the sum of the progression is defined. But yet even if the number of the terms is infinite, the sum of the proposed series is given by another infinite series, which now is generally more convergent than that proposed and thus is extremely useful in determining the sum.

§20. Let the sum of the progression continued to infinity be :

$$\int-\frac{y^{\alpha-2}}{a}dyl(1-y);$$

since here on placing $b = a$, then the progression is :

$$\frac{1}{\alpha a}+\frac{1}{2(\alpha+1)a}+\frac{1}{3(\alpha+2)a}+\frac{1}{4(\alpha+3)a}+\text{etc.}$$

The sum of this is obtained, if in that integral there is put $y = 1$, but by making $y = 1 - z$, that integral becomes :

$$\frac{1}{a}\left(\begin{array}{l}1-\frac{\alpha-2}{1.4}+\frac{(\alpha-2)(\alpha-3)}{1.2.9}-\text{etc.}-z+\frac{\alpha-2}{1.4}z^2-\frac{(\alpha-2)(\alpha-3)}{1.2.9}z^3+\text{etc.}\\+zlz-\frac{(\alpha-2)}{1.2}z^2lz+\frac{(\alpha-2)(\alpha-3)}{1.2.3}z^3lz-\text{etc.}\end{array}\right).$$

If now on making $y = 1$ or $z = 0$, then the sum of the series

$$\frac{1}{\alpha a}+\frac{1}{2(\alpha+1)a}+\frac{1}{3(\alpha+2)a}+\text{etc.}$$

is equal to the sum of this series :

$$\frac{1}{a}\left(\frac{1}{a}-\frac{\alpha-2}{1.4.a}-\frac{(\alpha-2)(\alpha-3)}{1.2.9.a}+\text{etc}\right).,$$

or the sum of this:

$$\frac{1}{\alpha}+\frac{1}{2(\alpha+1)}+\frac{1}{3(\alpha+2)}+\text{etc.}$$

is equal to the sum of this :

$$1-\frac{\alpha-2}{1.4}+\frac{(\alpha-2)(\alpha-3)}{1.2.9}-\text{etc.}$$

§21. Besides I have another way for finding strongly convergent series, the sum of which is equal to the proposed sum.

$$\int -y^{\alpha-2}dyl(1-y)$$

is thus equal to the integral, so that on making the integral equal to zero, if $y = 0$, for this series :

$$\begin{array}{l}1-\frac{\alpha-2}{1.4}+\frac{(\alpha-2)(\alpha-3)}{1.2.9}-\frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.2.9.16}+\text{etc.}\\-z+\frac{\alpha-2}{1.4}z^2-\frac{(\alpha-2)(\alpha-3)}{1.2.9}z^3+\frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.2.9.16}z^4-\text{etc}\\+zlz-\frac{(\alpha-2)}{1.2}z^2lz+\frac{(\alpha-2)(\alpha-3)}{1.2.3}z^3lz-\frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.2.3.4}z^4lz+\text{etc}\end{array}$$

with $z = 1 - y$; but since

$$-l(1-y)=y+\frac{y^2}{2}+\frac{y^3}{3}+\frac{y^4}{4}+\text{etc.},$$

then

$$\int -y^{\alpha-2}dyl(1-y)=\frac{y^\alpha}{\alpha}+\frac{y^{\alpha+1}}{2(\alpha+1)}+\frac{y^{\alpha+2}}{3(\alpha+2)}+\text{etc.}$$

This series, if α is a positive number, is equal to that, whatever y shall be; and thus the sums of many series of the same kind can be found, of which the one with the help of the other is more easily summed.

§22. I will make this clear with an example. Let $\alpha = 1$; there is had the sum of following three series equal to this :

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}$$
$$- z - \frac{1}{4}zz - \frac{1}{9}z^3 - \frac{1}{16}z^4 - \text{etc.}$$
$$+ zlz + \frac{1}{2}z^2 lz + \frac{1}{3}z^3 lz + \text{etc.}$$
$$= \frac{y}{1} + \frac{yy}{4} + \frac{y^3}{9} + \text{etc.}$$

Now since

$$z + \frac{1}{2}zz + \frac{1}{3}z^3 + \text{etc.} = -l(1-z) = -ly,$$

then

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.} = \frac{y+z}{1} + \frac{y^2+z^2}{4} + \frac{y^3+z^3}{9} + \frac{y^4+z^4}{16} + \text{etc.} + lylz;$$

then here $y + z = 1$, and it is clear that such numbers can be taken in place of y or z in order that the series converges maximally. Now this comes about, when $y = z$ or each is equal to $\frac{1}{2}$, and then in this case :

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.} = 1 + \frac{1}{8} + \frac{1}{36} + \frac{1}{128} + \frac{1}{400} + \text{etc.} + (l2)^2.$$

In this way the sum of the progression

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}$$

can almost be found; for then

$$l2 = \frac{1}{1.2} + \frac{1}{2.4} + \frac{1}{3.8} + \frac{1}{4.16} + \text{etc.}$$

The sum of the progression

$$1 + \frac{1}{8} + \frac{1}{36} + \text{etc.}$$

is then almost equal to 1,164481 and $(l2)^2 = 0{,}480453$; hence the sum of the series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}$$

is equal to 1,644934 as an approximation. But for which, if it is wished to determine the sum of this series by adding a number of terms from the beginning, then more than a thousand terms must be added, so that our number found may be come upon. [Here surely we have an indication of Euler's extraordinary mental powers.]

§23. Therefore from these one is allowed to perceive a method, by means of which the summation term for progressions of any kind may be found, the terms of which are fractions, and the denominators consist of algebraic progressions of some kind. Equally, as we have considered this thing, the numerators must be constant quantities; but this method can be extended without difficulty to these progressions also, in which the progression numerators also make some algebraic form. Therefore this method can be adapted to all progressions, of which the general terms are able to be set out algebraically, and with the help of this the summation terms can be found. Yet to be excepted are the cases in which the general term is irrational.

DE SUMMATIONE INNUMERABILIUM PROGRESSIONUM

Auct. L. Eulero.

§ I. Quae in praecedente dissertatione de progressionibus transcententibus earum terminus generalibus tradidi, multo latius patent, quam videri possent; et inter alia quam plurima, ad quae accommodari possunt, eximius earum potest esse usus in inveniendis summis innumerabilium progressionum. Quaemadmodum enim in superiore dissertatione innumerae progressiones ad terminos generales sunt revocatae, quae communem algebram transcendunt, ita hic eandem methodum accommodabo ad terminos summatorios inveniendos progressionum, ad quas indefinite summandas communis algebra non sufficit.

§2. Progressio quaepiam summari dicitur indefinite, si detur formula numerum indefinitum *n* continens, quae exponat summam tot terminorum illius progressionis, quot *n* comprehendit unitates, ita ut, si ponatur v.gr. $n = 10$, ea formula exhibeat summam decem terminorum a primo numeratorum. Formula haec vocatur *terminus summatorius* illius progressionis atque est simul terminus generalis progressionis, cuius terminus quicunque aequatur summae tot terminorum illius progressionis, quot eius exponens in se continet unitates.

§3. Cum progressiones quaeque exponantur terminis generalibus, quaestio de summandis progressionibus est haec, ut ex termino generali terminus summatorius inveniatur. Et quidem iam eo est perventum, ut, quoties terminus generalis est functio rationalis ipsius indicis *n* et exponentes sunt numeri integri affirmativi, semper terminus summatorius invenire queat. Quando autem exponentes ipsius n sunt negativi, nisi excipiantur pauci casus, nemo adhuc terminos summatorios dedit. Ratio huius difficultatis est, quod tum termini summatori plerumque algebraice exprimi nequeant, sed tales requirant formas, quae quadraturas in se contineant.

§4. Assumatur haec forma

$$\int \frac{1-x^n}{1-x}dx$$

tanquam terminus generalis cuiusdam progressionis; quae scilicet integrata, ita ut fiat $= 0$, si $x = 0$, positoque $x = 1$ daret terminum ordine *n*. Progressio , quae hoc modo ex ea formatur, erit haec

$$1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} \text{ etc.},$$

cuius ergo terminus generalis est formula assumta

$$\int \frac{1-x^n}{1-x}dx$$

Series vero haec inventa summatoria est progressionis harmonicqae

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \text{ etc.},$$

cuius terminus generalis est $\frac{1}{n}$. Quamobrem huius progressionis terminus summtorius

erit $\int \frac{1-x^n}{1-x}dx$, qui illius est terminus generalis

§5. Cum terminus generalis progressionis

$$1,1+\tfrac{1}{2},1+\tfrac{1}{2}+\tfrac{1}{3}\text{ etc.}$$

sit

$$\int \tfrac{1-x^n}{1-x}dx,$$

poterit ex hoc ea progressio interpolari seu quilibet terminus medius inveniri; ut si requiratur terminus, cuius index est $\tfrac{1}{2}$, oportebit integrari

$$\tfrac{1-\sqrt{x}}{1-x}dx \text{ vel } \tfrac{dx}{1+\sqrt{x}},$$

cuius integrale est

$$2\sqrt{x}-2l(1+\sqrt{x});$$

quod cum fiat = 0, si x = 0, ponatur $x = 1$; erit terminus ordine $\tfrac{1}{2} = 2 - 2l2$. Deinde, quia generaliter terminus ordine $n + 1$ terminum ordine n superat fractione $\tfrac{1}{n+1}$, erit terminus ordine $1\tfrac{1}{2} = 2\tfrac{2}{3} - 2l2$ et terminus ordine $2\tfrac{1}{2} = 2 + \tfrac{2}{3} + \tfrac{2}{5} - 2l2$ etc. Series igitur interpola erit

$\tfrac{1}{2}$	1	$1\tfrac{1}{2}$	2	$2\tfrac{1}{2}$	etc
$2-2l2$	1	$2+\tfrac{2}{3}-2l2$	$1+\tfrac{1}{2}$	$2+\tfrac{2}{3}+\tfrac{2}{3}-2l2$	

§6. Ad hunc modum rem generalius complexus sum et assumsi formulam

$$\int \tfrac{1-P^n}{1-P}dx$$

ubi P denotat functionem quamcunque ipsius x. Integrale hoc, ut semper, ita debet accipi, ut posito $x = 0$ id totum fiat = 0. Deinde hoc facto non, ut ante, pono $x = 1$, sed, ut latius pateat, pono $x = k$. Forma hoc modo resultans erit terminus ordine n progressionis cuiusdam, cuius terminus generalis est forma assumta

$$\int \tfrac{1-P^n}{1-P}dx$$

Progressio vero ipsa haec erit

$$k, k+\int Pdx, k+\int Pdx+\int P^2dx\ \text{ etc.};$$

ubi in integralibus $\int Pdx, \int P^2dx$ etc. loco x iam positum esse k pono.

§7. Progressio inventa, si quivis terminus a sequente subtrahatur, praebebit hanc

$$k, \int Pdx, \int P^2dx, \int P^3dx \text{ etc.},$$

cuius terminus generalis est

$$\int P^{n-1}dx.$$

Huiusque terminus summatorius aequalis est termino generali praecedentis progressionis, quem expeditat haec formula $\int \frac{1-P^n}{1-P}\,dx$.

Sit $P = x^n : a^n$; erit progressionis huius

$$k,\ \frac{k^{\alpha+1}}{(\alpha+1)a^{\alpha}},\ \frac{k^{2\alpha+1}}{(2\alpha+1)a^{2\alpha}} \text{ etc}$$

terminus generalis

$$\frac{k^{(n-1)\alpha+1}}{(1+(n-1)\alpha)a^{(n-1)\alpha}}$$

atque terminus summatorius hic

$$\int \frac{a^{n\alpha}-x^{n\alpha}}{(a^{\alpha}-x^{\alpha})a^{n\alpha-\alpha}}dx.$$

§8. Inventus ergo est terminus summatorius pro omnibus progressionibus, quorum termini sunt fractiones harumque numeratores progressionem geometricam, denominatores vero arithmeticam constituunt. Ut vero facilius ad omnes casus accommodari possit, sumatur haec progressio

$$\frac{b}{c},\ \frac{b^{i+1}}{c+e},\ \frac{b^{2i+1}}{c+2e},\ \frac{b^{3i+1}}{c+3e} \text{ etc.},$$

cuius terminus generalis est

$$\frac{b^{(n-1)i+1}}{c+(n-1)e};$$

comparetur hic cum illo

$$\frac{k^{(n-1)\alpha+1}}{(1+(n-1)\alpha)a^{(n-1)\alpha}} \text{ vel } \frac{ck^{(n-1)\alpha+1}}{(c+(n-1)\alpha c)a^{(n-1)\alpha}};$$

erit

$$\alpha = \frac{e}{c} \text{ et } \frac{ck^{(n-1)\frac{e}{c}+1}}{a^{(n-1)\frac{e}{c}}} = b^{(n-1)i+1}$$

atque

$$a = \left(\frac{ck^{(n-1)\frac{e}{c}+1}}{b^{(n-1)i+1}}\right)^{\frac{c}{(n-1)e}} = \left(\frac{ck}{b}\right)^{\frac{c}{(n-1)e}} \frac{k}{b^{ci:e}}$$

Hic, ne a pendeat ab n (debet enim a esse constans quantitas), oportet, ut $\frac{ck}{b}$ sit = 1; erit ergo $k = \frac{b}{c}$ atque $a = \frac{b^{\frac{e-ci}{e}}}{c}$. Quocirca terminus summatorius est

$$\int \frac{b^{\frac{ne-nci}{c}} - c^{\frac{ne}{c}} x^{\frac{ne}{c}}}{b^{\frac{(n-1)(e-ci)}{c}} \left(b^{\frac{e-ci}{c}} - c^{\frac{e}{c}} x^{\frac{e}{c}}\right)} dx$$

Quae ita debet integrari, ut fiat = 0, si x = 0; tum vero ponere oportet $x = \frac{b}{c}$.

§9. Cognita summa progressionis indefinita habebitur summa progressionis in infinitum, si ponatur $n = \infty$. Terminus quidem summatorius inventus non magis ad hunc casum quam ad aliumquemque accommodatus videtur. Est mihi vero alia methodus summas serierum infinitarum investigandi, quae latissime patet. Sit series

$$\frac{b}{c}, \frac{b^{i+1}}{c+e}, \frac{b^{2i+1}}{c+2e}, \frac{b^{3i+1}}{c+3e} \text{ etc}$$

Ponatur numerus terminorum n et summa eorum A. Augeatur numerus n unitate; augebitur summa A termino ordino $n + 1$, que est $\frac{b^{ni+1}}{c+ne}$. Si nunc n et A tanquam quantitaties fluentes considerentur, quia n est quasi infinities maior quam 1, erunt earum differentialia dn et dA inter se ut augmenta 1 et $\frac{b^{ni+1}}{c+ne}$. Unde prodit aequatio

$$dA = \frac{b^{ni+1}dn}{c+ne}.$$

Quae integrata dabit aequationem inter summam A et numerum terminorum n.

§10. Ponatur

$$l(c+ne) = z;$$

erit

$$\frac{edn}{c+ne} = dz$$

atque $c + ne = g^z$ denotante g numerum,cuius logarithmus est 1. Est ergo

$$n = \frac{g^z - c}{e} \text{ et } b^{ni+1} = b^{\frac{g^z i - ci + e}{e}} = b^{\frac{e-ci}{e}} b^{\frac{g^z i}{e}},$$

consequenter

$$dA = \frac{b^{\frac{e-ci}{e}}}{e} b^{\frac{g^z i}{e}} dz.$$

Haec quidem aequatio ita generaliter instituta integrationem nisi per series non admittit. Si vero ponatur $i = 0$, ut prodeat series

$$\frac{b}{c} + \frac{b}{c+e} + \frac{b}{c+2e} + \text{etc.},$$

habebitur aequatio

$$dA = \frac{b}{e}dz \text{ et } A = \frac{b}{e}(z + lC) = \frac{b}{e}lC(c+ne).$$

Constans quidem C non determinatur, sed tamen aequatio ad definiendam differentiam inter duas summas inservit; ut sit alius numerus terminorum m et summa B; erit

$$B = \frac{b}{e}lC(c+me).$$

Ergo

$$B - A = \frac{b}{e}l\frac{c+me}{c+ne} = \frac{b}{e}l\frac{m}{n},$$

quia m et n sunt infinita.

§11. Maneat $i = 0$ et progressio erit haec

$$\frac{b}{c}, \frac{b}{c+e}, \frac{b}{c+2e}, \frac{b}{c+3e} \text{ etc}$$

cuius terminus generalis est

$$\frac{b}{c+(n-1)e}.$$

Terminus autem summatorius est

$$\int \frac{b^{\frac{ne}{c}} - c^{\frac{ne}{c}} x^{\frac{ne}{c}}}{b^{\frac{(n-1)e}{c}}(b^{\frac{e}{c}} - c^{\frac{e}{c}} x^{\frac{e}{c}})} dx.$$

Sumatur alia progressio

$$\frac{b}{c},\ \frac{b}{c+f},\ \frac{b}{c+2f},\ \frac{b}{c+3f}\ \text{ etc .,}$$

cuius terminus generalis est

$$\frac{b}{c+(n-1)f}$$

et summatorius

$$\int \frac{b^{\frac{nf}{c}}-c^{\frac{nf}{c}}x^{\frac{nf}{c}}}{b^{\frac{(n-1)f}{c}}(b^{\frac{f}{c}}-c^{\frac{f}{c}}x^{\frac{f}{c}})}dx\,,$$

in quo integrato itidem ponere oportet $x=k=\frac{b}{e}$. Addantur hae duae progressiones, scilicet terminus primus primo, secundus secundo, et ita porro; probibit haec progressio

$$\frac{2b}{c},\ \frac{2bc+b(e+f)}{(c+e)(c+f)},\ \frac{2bc+2b(e+f)}{(c+2e)(c+2f)},\ \text{ etc .,}$$

cuius terminus generalis est

$$\frac{2bc+(n-1)b(e+f)}{(c+(n-1)e)(c+(n-1)f)}\,.$$

Terminus vero summatorius erit

$$\int dx\left(\frac{b^{\frac{ne}{c}}-c^{\frac{ne}{c}}x^{\frac{ne}{c}}}{b^{\frac{(n-1)e}{c}}(b^{\frac{e}{c}}-c^{\frac{e}{c}}x^{\frac{e}{c}})}+\frac{b^{\frac{nf}{c}}-c^{\frac{nf}{c}}x^{\frac{nf}{c}}}{b^{\frac{(n-1)f}{c}}(b^{\frac{f}{c}}-c^{\frac{f}{c}}x^{\frac{f}{c}})}\right).$$

§12. Simili modo, sed universalius. pro termino generali, in cuius denominatore *n* duas tenet dimensiones, invenitur terminus summatorius, si illius progressionis *p*-cuplum addatur ad *q*-cuplum huius. Obtinebitur hoc modo progressio, cuius terminus generalis est

$$\frac{pb}{c+(n-1)e}+\frac{qb}{c+(n-1)f}=\frac{(p+q)bc+(n-1)b(pf+qe)}{(c+(n-1)e)(c+(n-1)f)}\,.$$

Terminus autem summatorium huic termino generali respondens erit

$$\int\frac{pdx}{b^{\frac{(n-1)e}{c}}}\left(\frac{b^{\frac{ne}{c}}-c^{\frac{ne}{c}}x^{\frac{ne}{c}}}{b^{\frac{e}{c}}-c^{\frac{e}{c}}x^{\frac{e}{c}}}\right)+\int\frac{qdx}{b^{\frac{(n-1)f}{c}}}\left(\frac{b^{\frac{nf}{c}}-c^{\frac{nf}{c}}x^{\frac{nf}{c}}}{b^{\frac{f}{c}}-c^{\frac{f}{c}}x^{\frac{f}{c}}}\right)=$$

$$\int dx\left(\begin{array}{l}\dfrac{pb^{\frac{n(e+f)}{c}}-pb^{\frac{n(e+f)-f}{c}}c^{\frac{f}{c}}x^{\frac{f}{c}}-pb^{\frac{nf}{c}}c^{\frac{ne}{c}}x^{\frac{ne}{c}}+pb^{\frac{(n-1)f}{c}}c^{\frac{ne+f}{c}}x^{\frac{ne+f}{c}}}{b^{\frac{(n-1)(e+f)}{c}}\left(b^{\frac{e}{c}}-c^{\frac{e}{c}}x^{\frac{e}{c}}\right)\left(b^{\frac{f}{c}}-c^{\frac{f}{c}}x^{\frac{f}{c}}\right)}\\ +\dfrac{qb^{\frac{n(e+f)}{c}}-qb^{\frac{n(e+f)-e}{c}}c^{\frac{e}{c}}x^{\frac{e}{c}}-qb^{\frac{ne}{c}}c^{\frac{nf}{c}}x^{\frac{nf}{c}}+qb^{\frac{(n-1)e}{c}}c^{\frac{nf+e}{c}}x^{\frac{nf+e}{c}}}{b^{\frac{(n-1)(e+f)}{c}}\left(b^{\frac{e}{c}}-c^{\frac{e}{c}}x^{\frac{e}{c}}\right)\left(b^{\frac{f}{c}}-c^{\frac{f}{c}}x^{\frac{f}{c}}\right)}\end{array}\right).$$

Ponatur *b* = 1, hoc enim modo universalitat nihil decedit, eritque terminus generalis

$$\frac{(p+q)c+(n-1)(pf+qe)}{(c+(n-1)e)(c+(n-1)f)}\,.$$

Sit $cx = y$; erit

$$dx = \frac{dy}{c}.$$

Atque terminus summatorius habetur =

$$\int \frac{dy}{c}\left(\frac{p+q-py^{\frac{f}{c}}-qy^{\frac{e}{c}}-py^{\frac{ne}{c}}-qy^{\frac{nf}{c}}+py^{\frac{ne+f}{c}}+qy^{\frac{nf+e}{c}}}{\left(1-y^{\frac{e}{c}}\right)\left(1-y^{\frac{f}{c}}\right)}\right),$$

in qua formula integrata, ita ut posito $y = 0$ ea quoque fiat = 0, oportet ponere $y = 1$.

§13. Assumatur iam terminus generalis hic

$$\frac{\alpha+\beta n}{\gamma+\delta n+\varepsilon nn}.$$

Qui comparatus cum

$$\frac{(p+q)c+(n-1)(pf+qe)}{(c+(n-1)e)(c+(n-1)f)}$$

dabit

$$c = \sqrt{(\gamma+\delta+\varepsilon)},\ e = \frac{\delta+2\varepsilon+\sqrt{(\delta\delta-4\gamma\varepsilon)}}{2\sqrt{(\gamma+\delta+\varepsilon)}},\ f = \frac{\delta+2\varepsilon-\sqrt{(\delta\delta-4\gamma\varepsilon)}}{2\sqrt{(\gamma+\delta+\varepsilon)}},$$

$$p = \frac{\alpha\delta-\beta\delta+2\alpha\varepsilon-2\beta\gamma+(\alpha+\beta)\sqrt{(\delta\delta-4\gamma\varepsilon)}}{2\sqrt{(\gamma+\delta+\varepsilon)(\delta\delta-4\gamma\varepsilon)}}$$

atque

$$q = \frac{\beta\delta-\alpha\delta+2\beta\gamma-2\alpha\varepsilon+(\alpha+\beta)\sqrt{(\delta\delta-4\gamma\varepsilon)}}{2\sqrt{(\gamma+\delta+\varepsilon)(\delta\delta-4\gamma\varepsilon)}}.$$

His in termino summatorio substitutis prodibit terminus summatorius huius progressionis

$$\frac{\alpha+\beta}{\gamma+\delta+\varepsilon},\ \frac{\alpha+2\beta}{\gamma+2n+4\varepsilon},\ \frac{\alpha+3\beta}{\gamma+3\delta+9\varepsilon}\ \text{etc.},$$

cuius terminus generalis est

$$\frac{\alpha+\beta n}{\gamma+\delta n+\varepsilon nn}.$$

§13*a*. Eodem modo, si in termino generali *n* plures duabus dimesiones habuerit, eruetur terminus summatorius combinandis tot progressionibus simplicibus, quot dimensiones *n* habere debet, quemadmodum idem in casu duarum dimensionun factum est. Attamen hac ratione non ad quasvis, quae in huiusmodi terminis generalibus contineri videntur, series perveniri potest. Nam quoties denominator $\gamma+\delta n+\varepsilon n^2+\xi n^3+\eta n^4+\text{etc.}$ duos pluresve habet factores simplices aequales, tum progressio in tot simplices progressiones resolvi nequit neque igitur eius terminus summatorius inveniri.

§14. Hanc ob rem aliam tradam methodum, quae hos casus non excludat. Sit progressio quaedam simplex

$$\frac{1}{a},\ \frac{1}{a+b},\ \frac{1}{a+2b} \text{ etc.}$$

cuius terminus generalis est

$$\frac{1}{a+(n-1)b}$$

Huius terminus summatorius erit

$$\int \frac{1-a^{\frac{nb}{a}}x^{\frac{nb}{a}}}{1-a^{\frac{b}{a}}x^{\frac{b}{a}}}dx,$$

vel ponatur $ax = y$; erit is

$$\int \frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}\frac{dy}{a},$$

in quo integrato poni oportet $y = 1$. Multiplicetur hic in $y^{\alpha}dy$ et summa huius facti

$$\int y^{\alpha}dy\int \frac{dy}{a}.\frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}$$

erit secundum modum descriptum tractata terminus summatorius hius progressionis

$$\frac{a}{a.\beta a},\ \frac{a}{(a+b)(\beta a+b)},\ \frac{a}{(a+2b)(\beta a+2b)} \text{ etc.}$$

positio brevitatis ergo β loco $\alpha+2$. Huius progressionis terminus generalis est

$$\frac{a}{(a+(n-1)b)(\beta a+(n-1)b)} \text{ vel } \frac{a}{b^2n^2+(ab+\beta ab-2bb)n+(a-b)(\beta a-b)}.$$

§15. Assumamus progressionem generalem huius generis, quae facilius ad casus quovis adaptatur; sit eius terminus generalis

$$\frac{1}{a+(n-1)b+\frac{(n-1)(n-2)}{2}c}.$$

Hic cum illo termino generali comparatus dabit

$$a=\frac{(2b-c)^2-4ac+(2b-c)\sqrt{\left((2b-c)^2-8ac\right)}}{4c},$$

$$b=\frac{2b-c+\sqrt{\left((2b-c)^2-8ac\right)}}{4},$$

$$\beta=\frac{2b-c-\sqrt{\left((2b-c)^2-8ac\right)}}{2b-c+\sqrt{\left((2b-c)^2-8ac\right)}}.$$

Hi valores si substituantur loco *a*, *b* et α (est vero $\alpha=\beta-2$) in

$$\int y^{\alpha} dy \int \frac{dy}{a} \cdot \frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}$$

prodibit terminus summatorius progressionis propositae

$$\frac{1}{a},\ \frac{1}{a+b},\ \frac{1}{a+2b+c},\ \frac{1}{a+3b+3c} \text{ etc.}$$

§16. Hoc modo ulterius progredi licet; multiplicetur

$$\int \frac{dy}{a} \cdot \frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}$$

in $y^{\alpha-2}dy$ et facti integrale

$$\int y^{\alpha-2} dy \int \frac{dy}{a} \cdot \frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}$$

denuo in $y^{\beta-\alpha-1}$ huiusque producti integrale

$$\int y^{\beta-\alpha-1} dy \int y^{\alpha-2} dy \int \frac{dy}{a} \cdot \frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}$$

erit terminus summatorius progressionis huius

$$\frac{a^2}{a.\alpha a.\beta a},\ \frac{a^2}{(a+b)(\alpha a+b)(\beta a+b)},\ \frac{a^2}{(a+2b)(\alpha a+2b)(\beta a+2b)} \text{ etc.},$$

cuius terminus generalis est

$$\frac{a^2}{(a+(n-1)b)(\alpha a+(n-1)b)(\beta a+(n-1)b)};$$

similiter

$$\int y^{\gamma-\beta-1} dy \int y^{\beta-\alpha-1} dy \int y^{\alpha-2} dy \int \frac{dy}{a} \cdot \frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}}$$

est terminus summatorius progressionis, cuius terminus generalis est

$$\frac{a^3}{(a+(n-1)b)(\alpha a+(n-1)b)(\beta a+(n-1)b)(\gamma a+(n-1)b)}.$$

Hoc igitur modo omnes progressiones pervenitur, quarum termini sunt fractiones numeratoribus existentibus numeris constantibus, denominatoribus autem constituentibus progressionem algebraicam.

§17. Si summae huiusmodi progressionum in infinitum continuatarum desiderentur, oportet ponere n = infinito. Hoc posito postremum cuiusque termini summatorii membrum, scilicet

$$\int \frac{dy}{a} \cdot \frac{1-y^{\frac{nb}{a}}}{1-y^{\frac{b}{a}}},$$

transmutabitur in hoc

$$\int \frac{dy}{a\left(1-y^{\frac{b}{a}}\right)}.$$

Quia enim y semper est minus quam 1 praeter casum ultimum, quo fit y = 1, evanescet $y^{\frac{nb}{a}}$ prae 1 atque ideo $1-y^{\frac{nb}{a}}$ abibit in 1. Propterea huius seriei

$$\frac{a}{a.\alpha a}+\frac{a}{(a+b)(\alpha a+b)}+\frac{a}{(a+2b)(\alpha a+2b)}+\text{ etc. in infinitum}$$

summa erit

$$\int y^{\alpha-2}\int \frac{dy}{a\left(1-y^{\frac{b}{a}}\right)}$$

et huius

$$\frac{a^2}{a.\alpha a.\beta a},\ \frac{a^2}{(a+b)(\alpha a+b)(\beta a+b)},\ \frac{a^2}{(a+2b)(\alpha a+2b)(\beta a+2b)}\text{ etc.}$$

summa erit

$$\int y^{\beta-\alpha-1}\int y^{\alpha-2}\int \frac{dy}{a\left(1-y^{\frac{b}{a}}\right)}$$

et ita de reliquis omnibus.

§18. Sit $b = a$, ut fiat $\frac{b}{a}=1$; erit

$$\int \frac{dy}{a(1-y)}=A-\frac{1}{a}l(1-y).$$

Quia posito $y = 0$ totum integrale fieri debet = 0, erit $A = 0$ adeoque

$$\int \frac{dy}{a(1-y)}=-\frac{1}{a}l(1-y).$$

Multiplicetur hoc in $y^{\alpha-2}dy$; habebitur

$$-\frac{y^{\alpha-2}dy}{a}l(1-y).$$

Huius integrale ut inveniatur, ponatur $1 - y = z$; erit $y = 1 - z$; habebitur igitur integrandum

$$-\frac{(1-z)^{\alpha-2}dz}{a}lz$$
$$=\left(1-\frac{\alpha-2}{1}z+\frac{(\alpha-2)(\alpha-3)}{1.2}z^2-\frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.2.3}z^3\right)\frac{dz}{a}lz.$$

Quia vero

$$\int z^{\eta}dzlz=C-\frac{z^{\eta+1}}{(\eta+1)^2}+\frac{z^{\eta+1}lz}{\eta+1},$$

erit illius integrale haec series

$$\frac{1}{a}\left(C-z+zlz+\frac{\alpha-2}{1.4}z^2-\frac{(\alpha-2)}{1.2}z^2lz-\frac{(\alpha-2)(\alpha-3)}{1.2.9}z^3+\frac{(\alpha-2)(\alpha-3)}{1.2.3}z^3lz+\text{etc}\right).$$

Hoc integrale, si fiat $y = 0$ seu $z = 1$, debet fieri = 0; hanc ob rem erit

$$C=1-\frac{\alpha-2}{1.4}+\frac{(\alpha-2)(\alpha-3)}{1.2.9}-\frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.2.3.16}+\text{etc.}$$

§19. Perspicuum est ex hoc integrali, quoties α sit numerus integer unitate maior, tum semper integralis eius terminorum numerum fore finitum atque ideo summam progressionis definiri. Attamen etiamsi terminorum numerus sit infinitus, summa propositae seriei dabitur per aliam seriem infinitam, quae vero plerumque magis convergit quam proposita atque ideo perquam est utilis ad summam determinandam.

§20. Sit summa progressionis in infinitum continuatae

$$\int -\frac{y^{\alpha-2}}{a}dyl(1-y);$$

quia hic est positum $b = a$, erit progressio ipsa

$$\frac{1}{\alpha a}+\frac{1}{2(\alpha+1)a}+\frac{1}{3(\alpha+2)a}+\frac{1}{4(\alpha+3)a}+\text{etc.}$$

Huius summa habetur, si in illo integrali ponitur $y = 1$, sed facto $y = 1 - z$ est integrale illud

$$\frac{1}{a}\left(\begin{array}{l}1-\frac{\alpha-2}{1.4}+\frac{(\alpha-2)(\alpha-3)}{1.2.9}-\text{etc.}-z+\frac{\alpha-2}{1.4}z^2-\frac{(\alpha-2)(\alpha-3)}{1.2.9}z^3+\text{etc.}\\+zlz-\frac{(\alpha-2)}{1.2}z^2lz+\frac{(\alpha-2)(\alpha-3)}{1.2.3}z^3lz-\text{etc.}\end{array}\right).$$

Si iam fiat $y = 1$ vel $z = 0$, erit summa seriei

$$\frac{1}{\alpha a}+\frac{1}{2(\alpha+1)a}+\frac{1}{3(\alpha+2)a}+\text{etc.}$$

aequalis summae huius seriei

$$\frac{1}{a}\left(\frac{1}{a}-\frac{\alpha-2}{1.4.a}-\frac{(\alpha-2)(\alpha-3)}{1.2.9.a}+\text{etc}\right).,$$

vel summa huius

$$\frac{1}{\alpha}+\frac{1}{2(\alpha+1)}+\frac{1}{3(\alpha+2)}+\text{etc.}$$

aequalis summae huius

$$1-\frac{\alpha-2}{1.4}+\frac{(\alpha-2)(\alpha-3)}{1.2.9}-\text{etc.}$$

§21. Praeterea alium habeo modum series valde convergentes inveniendi, quarum summa aequalis sit seriei propositae.

$$\int -y^{\alpha-2}dyl(1-y)$$

aequatur ita integratum, ut fiat = 0, si $y = 0$, huic seriei

$$\begin{array}{l}1-\frac{\alpha-2}{1.4}+\frac{(\alpha-2)(\alpha-3)}{1.2.9}-\frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.2.9.16}+\text{etc.}\\-z+\frac{\alpha-2}{1.4}z^2-\frac{(\alpha-2)(\alpha-3)}{1.2.9}z^3+\frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.2.9.16}z^4-\text{etc}\\+zlz-\frac{(\alpha-2)}{1.2}z^2lz+\frac{(\alpha-2)(\alpha-3)}{1.2.3}z^3lz-\frac{(\alpha-2)(\alpha-3)(\alpha-4)}{1.2.3.4}z^4lz+\text{etc}\end{array}$$

existente $z = 1 - y$; sed cum sit

$$-l(1-y) = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \text{etc.},$$

erit

$$\int -y^{\alpha-2}dyl(1-y) = \frac{y^\alpha}{\alpha} + \frac{y^{\alpha+1}}{2(\alpha+1)} + \frac{y^{\alpha+2}}{3(\alpha+2)} + \text{etc.}$$

Haec series, si α est numerus affirmativus, est aequalis illi, quicquid sit y; et ita multis modis series eiusdem summae reperiuntur, quarum altera alterius ope facilius summatur.

§22. Exemplo rem illustrabo. Sit $\alpha = 1$; habebitur trium sequentium serierum summa aequalis huic

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}$$
$$- z - \frac{1}{4}zz - \frac{1}{9}z^3 - \frac{1}{16}z^4 - \text{etc.}$$
$$+ zlz + \frac{1}{2}z^2lz + \frac{1}{3}z^3lz + \text{etc.}$$
$$= \frac{y}{1} + \frac{yy}{4} + \frac{y^3}{9} + \text{etc.}$$

Quia vero est

$$z + \frac{1}{2}zz + \frac{1}{3}z^3 + \text{etc.} = -l(1-z) = -ly,$$

erit

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.} = \frac{y+z}{1} + \frac{y^2+z^2}{4} + \frac{y^3+z^3}{9} + \frac{y^4+z^4}{16} + \text{etc.} + lylz;$$

est hic $y + z = 1$, et manifestum est tales loco y vel z numeros assumi posse ut series maxime convergat. Id vero evenit, quando $y = z$ vel utrumque $= \frac{1}{2}$, eritque hoc casu

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.} = 1 + \frac{1}{8} + \frac{1}{36} + \frac{1}{128} + \frac{1}{400} + \text{etc.} + (l2)^2.$$

Hoc modo summa progressionis

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}$$

valde prope haberi potest; est enim

$$l2 = \frac{1}{1.2} + \frac{1}{2.4} + \frac{1}{3.8} + \frac{1}{4.16} + \text{etc.}$$

Summa progressionis

$$1 + \frac{1}{8} + \frac{1}{36} + \text{etc.}$$

est quam proxime = 1,164481 et $(l2)^2 = 0{,}480453$; ergo sum seriei

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}$$

est = 1,644934 quam proxime. Si quis autem huius seriei summam addendis aliquot terminis initialibus determinare voluerit, plus quam mille terminos addere deberet, quo nostrum inventum numerum reperiret.

§23. Ex his igitur methodum percipere licet, quomodo cuiuslibet progressionis, cuius termini sunt fractiones, quarum denominatores constituunt progressionem quamcunque algebraicam, terminum summatorium inveniri oporteat. Equidem, ut hic rem consideravimus, numeratores deberent esse quantitates constantes; sed non difficulter haec methodus extendetur ad eas quoque progressionis, in quibus numeratores progressionem etiam quamcunque algebraicam faciunt. Propterea haec methodus ad omnes progressiones, quarum termini generales algebraice possunt exponi, accommodari potest eiusque ope termini sumatorii inveniri. Excipiendi tamen sunt casus, quibus terminus generalis irrationalis est.