CONCERNING TRANSCENDENTAL PROGRESSIONS,

OR THE GENERAL TERMS OF WHICH ARE UNABLE TO BE GIVEN ALGEBRAICALLY.

By L. Euler. [p.36]

§ I.

Since recently the celebrated Goldbach communicated a paper to the Society [*Concerning the general terms of series*, Comment.acad.sc.Petrop.3. (1728), 1732, pp. 164 - 173] in which an instance of these series arose, I shall inquire about a certain general expression which gives all the terms of this progression [formed from the terms of $n!$]

$$1+1.2+1.2.3+1.2.3.4+etc.\ ,$$

happening to come upon the following expression for consideration,

$$\frac{1.2^n}{1+n}\cdot\frac{2^{1-n}.3^n}{2+n}\cdot\frac{3^{1-n}.4^n}{3+n}\cdot\frac{4^{1-n}.5^n}{4+n}\ \text{etc.},$$

which shows the n^{th} term of the said progression, as that continued to infinity finally can be combined with geometry. Indeed this series is in no case interrupted, whether n be a whole or fractional number, but any term generally can only be found by putting approximations in place, except for the cases $n = 0$ and $n = 1$, in which case the general term actually becomes equal to 1. Putting $n = 2$, gives

$$\frac{2.2}{1.3}\cdot\frac{3.3}{2.4}\cdot\frac{4.4}{3.5}\cdot\frac{5.5}{4.6}\cdot\text{etc.} = \text{second term 2.}$$

If $n = 3$, there we have

$$\frac{2.2.2}{1.1.4}\cdot\frac{3.3.3}{2.2.5}\cdot\frac{4.4.4}{3.3.6}\cdot\frac{5.5.5}{4.4.7}\cdot\text{etc.} = \text{third term 6.}$$

[Note that only a few numbers in these products do not cancel, but the whole infinite product must be considered for this to happen, whereas for $n = 0$ and $n = 1$ there is no need for an infinite product.]

§ 2. Moreover this expression is seen not to be of any use in the finding of the terms, yet it is uncommonly well adapted for the interpolation of this series, or for terms with fractional indices,. But I have decided not to explain more about this here, since more [examples] of a suitable kind occur below to be effected in the same way. I introduce here only that which concerns the general term, as those properties that follow then do so in a natural way. I have looked for the term of which the index is $n=\frac{1}{2}$, or which is equally placed between the first term 1 and that preceding which is likewise 1. Moreover on putting $n=\frac{1}{2}$, I have pursued this series

$$\sqrt{\frac{2.4}{3.3}\cdot\frac{4.6}{5.5}\cdot\frac{6.8}{7.7}\cdot\frac{8.10}{9.9}.\text{etc}}$$

which expresses the term sought. But this series was at once seen by me to be similar to that I remembered seeing in the works of Wallis for the area of the circle. For Wallis had found that the area of the circle is to the square of the diameter as

$$2.4.4.6.6.8.8.10.\text{etc. to } 3.3.5.5.7.7.9.9.\ \text{etc.}$$

[J. Wallis (1616 - 1703), *Arithmetica infinitorum* : 'A new method of enquiring into the quadrature of curves, and other more difficult mathematical problems', Oxford. p. 182; Mathematical Works, Book I, Oxford 1695, p. 355, esp. p. 469. (in Latin)]
Therefore if the diameter should be equal to 1, then the area of the circle is equal to

$$\frac{2.4}{3.3}\cdot\frac{4.6}{5.5}\cdot\frac{6.8}{7.7}\cdot\frac{8.10}{9.9}.\text{etc}$$

It is possible to concluded therefore from this, that the term with the index $\frac{1}{2}$ is equal to the square root of the area of the circle of which the diameter is equal to 1.
[Thus, Wallis' product gives $\frac{\pi}{2}$, while the expression considered by Euler is $\frac{\sqrt{\pi}}{2}$; the area of the circle A = $\frac{\pi}{4}$, for which $\sqrt{A}$ is Euler's mean expression, as he asserts.]

§ 3. Before, I was inclined to think of the general term of the series 1, 2, 6, 24, etc, that if it were not given algebraically, then it would still be given by exponentials. But after I understood that certain intermediate terms depended on the quadrature of the circle, I knew that neither algebraic nor exponential quantities were suitable for expressing the general term. For the general term of this expression thus has to be prepared, [p.38] so that it must be understood as an algebraic quantity as well as from the quadrature of the circle, or perhaps depending on some other quadrature which is not present in any formula, either algebraic or exponential.

§ 4. Moreover since I could examine formulas given between differential quantities of this kind, which could be integrated in certain cases with certainty, and then algebraic quantities be presented; however in other cases such quantities cannot be integrated, and then quantities depending on the quadrature of curves can be shown; perhaps formulas of this kind call to mind progressions and other general terms similar to this that can easily be adapted. Now progressions, which require such general terms, which are unable to be given algebraically, I call *transcendental* ; as with geometry, all that which surpasses the strengths of common algebra is usually called transcendental.

§ 5. I have therefore studied this question of how differential formulas can be adapted to express especially the general terms of progressions. But the general term is a formula, quantities are advanced either by constant amounts as well as by some other variable amount such as n, which shows the order of the term or the exponent, so that, if the third term is desired, it is required to put 3 in place of n. But in a differential formula it is required that a certain variable is present. For which it has been decided not to use n, [p.39] since its variation does not pertain to integration, but after the formula had been integrated or the integral has been put in place, then at last n is of use in forming the progression. Therefore in the differential formula it is required to put in place a certain variable quantity x , but for which after integrating an equivalent term is put in place in the progression; and what arises properly is the term, the index of which is n.

§ 6. In order that these points can be grasped more clearly, I say that $\int pdx$ is the general term of the following progression, to be elicited from this integral in the following way; moreover p denotes some function of x and of a constant, in which n itself must still be present. The integration pdx is performed and increased by such a quantity, that on putting $x = 0$ the whole integral vanishes ; then x is put equal to a certain known quantity. With which done, on finding the integral only quantities pertaining to the progression remain, and that expresses the term of which the index is equal to n. Or the general term has been determined properly from the integral in the said manner. If indeed that can be obtained, then there is no need for the differential formula, as the progression formed hence has an algebraic general term; otherwise the situation arises, that the integration can only be performed on putting certain values in place of n.

§ 7. Therefore I have assumed several differential formulas of this kind that can only be integrated [p.40] if a positive integer is put in place of n, in order that the principal terms of the series become algebraic, and thus I have formed progressions. Thus the general terms of these are displayed, and from which each quadrature of the intermediate term can be defined. Indeed here I will not run through several formulas of this kind, for I will treat only a single somewhat general formula, which clearly extends widely enough to all progressions, any terms of which have been composed with constancy from a number of factors adapted from the depending index ; which factors are fractions, the numerators and denominators of which are in some arithmetical progression, such as

$$\frac{2}{3}+\frac{2.4}{3.5}+\frac{2.4.6}{3.5.7}+\frac{2.4.6.8}{3.5.7.9}+\text{etc.}$$

§ 8. Let this be the proposed formula :

$$\int x^{e}dx(1-x)^{n}$$

substituting in turn the general terms, which on integration thus in order that it becomes equal to 0 if $x = 0$, and then on putting $x = 1$ the term with order n of the progression thus arises. Hence we see that some such progression is needed : it is

$$(1-x)^{n}=1-\frac{n}{1}x+\frac{n(n-1)}{1.2}x^{2}-\frac{n(n-1)(n-2)}{1.2.3}x^{3}+\text{etc.}$$

and hence

$$x^{e}dx(1-x)^{n}=x^{e}dx-\frac{n}{1}x^{e+1}dx+\frac{n(n-1)}{1.2}x^{e+2}dx-\frac{n(n-1)(n-2)}{1.2.3}x^{e+3}dx+\text{etc.}$$

Whereby

$$\int x^{e}dx(1-x)^{n}=\frac{x^{e+1}}{e+1}-\frac{nx^{e+2}}{1.(e+2)}+\frac{n(n-1)x^{e+3}}{1.2.(e+3)}-\frac{n(n-1)(n-2)x^{e+4}}{1.2.3.(e+4)}+\text{etc.}$$

Put $x = 1$, since there is no need to add a constant, and there is obtained : [p.41]

$$\frac{1}{e+1}-\frac{n}{1.(e+2)}+\frac{n(n-1)}{1.2.(e+3)}-\frac{n(n-1)(n-2)}{1.2.3.(e+4)}+\text{etc.}$$

the general term of the series required to be found. Which is such that, if $n = 0$, a term equal to $\frac{1}{e+1}$ is produced; if $n = 1$, the term is equal to $\frac{1}{(e+1)(e+2)}$; if $n = 2$, the term is $\frac{1.2}{(e+1)(e+2)(e+3)}$; if $n = 3$, the term produced is equal to $\frac{1.2.3}{(e+1)(e+2)(e+3)(e+4)}$; and the rule by which these terms are progressing is clear.

§ 9. Hence I have pursued this progression :

$$\frac{1.}{(e+1)(e+2)}+\frac{1.2}{(e+1)(e+2)(e+3)}+\frac{1.2.3}{(e+1)(e+2)(e+3)(e+4)}+\text{etc.},$$

the general term of which is :

$$\int x^e dx(1-x)^n.$$

Now the terms of order n are of this form :

$$\frac{1.2.3.4....n}{(e+1)(e+2).....(e+n+1)}.$$

Indeed this form is sufficient for finding the terms of integer indices, but if the indices should not be integers, then the terms cannot be found from this form Moreover for finding these approximately this series is of use :

$$\frac{1}{e+1}-\frac{n}{1.(e+2)}+\frac{n(n-1)}{1.2.(e+3)}-\frac{n(n-1)(n-2)}{1.2.3.(e+4)}+\text{etc.}$$

If $\int x^e dx(1-x)^n$ is multiplied by $e + n + 1$, the progression is obtained of which the term of order n has this form :

$$\frac{1.2.3.4....n}{(e+1)(e+2).....(e+n)},$$

now the general term of this is :

$$(e+n+1)\int x^e dx(1-x)^n.$$

Here it can be observed that the progression can always become algebraic, when a positive number is put in place of e. For example on putting $e = 2$; the n^{th} term of the progression is [with an equal number of terms in the numerator and denominator] :

$$\frac{1.2.3.4....n}{3.4.5.....(n+2)} \text{ or } \frac{1.2}{(n+1)(n+2)}.$$

Because this also indicates the general term of the series itself, which is :

$$(n+3)\int xxdx(1-x)^n.$$

Now the integral of this is : [p.42]

$$\left(C-\frac{(1\text{-}x)^{n+1}}{n+1}+\frac{2(1\text{-}x)^{n+2}}{n+2}-\frac{(1\text{-}x)^{n+3}}{n+3}\right)(n+3);$$

in order that it becomes equal to zero, if $x = 0$, then

$$C=\frac{1}{n+1}-\frac{2}{n+2}+\frac{1}{n+3}.$$

Putting $x = 1$; the general term is :

$$\frac{n+3}{n+1}-\frac{2(n+3)}{n+2}+1=\frac{2}{(n+1)(n+2)}.$$

§ 10. In order therefore that we may arrive at a transcendental progression, *e* is put equal to the fraction $\frac{f}{g}$. Then the term of order *n* of the progression is :

$$\frac{1.2.3.4....n}{(f+g)(f+2g).....(f+ng)}g^n,\ \text{or}\ \frac{g.2g.3g.4g....ng}{(f+g)(f+2g).....(f+ng)}.$$

Now the general term is equal to :

$$\frac{(f+(n+1)g)}{g}\int x^{\frac{f}{g}}xdx(1-x)^n.$$

Which if it is divided by g^n, will be the general term for the progression :

$$\frac{1}{f+g}+\frac{1.2}{(f+g)(f+2g)}+\frac{1.2.3}{(f+g)(f+2g)(f+3g)}\ \text{etc.},$$

and the term of this for order *n* is

$$\frac{1.2.3...n}{(f+g)(f+2g)....(f+ng)}\ .$$

Therefore the general term of this progression is :

$$\frac{(f+(n+1)g)}{g^{n+1}}\int x^{\frac{f}{g}}dx(1-x)^n.$$

Where if the fraction $\frac{f}{g}$ is not equal to a whole number, or if *f* to *g* is not in a multiple ratio, then the progression is transcendental, and the intermediate terms will depend on quadratures.

§ 11. I introduce a certain example for you to see involving the mean, so that the use of the general term becomes clearer. In the previous paragraph initially, for a progression put $f = 1$, and $g = 2$, then the term of order n is equal to $\frac{2.4.6.8......2n}{3.5.7.9......(2n+1)}$, now this progression is :

$$\frac{2}{3}+\frac{2.4}{3.5}+\frac{2.4.6}{3.5.7}+\text{etc},$$

and the general term of this is hence $\frac{2n+3}{2}\int dx(1-x)^n\sqrt{x}$ The term of this is sought for which the index $n=\frac{1}{2}$, [p.43] and the term sought is equal to $2\int dx\sqrt{(x-xx)}$. Which as it must signify an element of the area of the circle, it is evident that the term sought is the square of the circle, of which the diameter is equal to 1.
[The circle has its centre at $(\frac{1}{2},0)$ and radius $\frac{1}{2}$]

Again this series is proposed :

$$1+\frac{r}{1}+\frac{r(r-1)}{1.2}+\frac{r(r-1)(r-2)}{1.2.3}+\text{etc.},$$

which is the coefficient of the binomial raised to the r^th^ power. The term with order *n* is therefore

$$\frac{r(r\text{-}1)(r-2).....(r-n+2)}{1.2.3......(n-1)}\,\text{etc.}$$

In the preceding section this expression is found :

$$\frac{1.2.3...n}{(f+g)(f+2g)....(f+ng)}\ .$$

This can be inverted so that it can be compared with the above expression, so that it becomes :

$$\frac{(f+g)(f+2g)....(f+ng)}{1.2.3...n}\ ;$$

this expression is multiplied by $\frac{n}{f+ng}$ and then it is equal to :

$$\frac{(f+g)(f+2g)....(f+(n-1)g)}{1.2.3...(n-1)}\ ;$$

therefore it is required that $f + g = r$ and $f + 2g = r - 1$, hence we have $g = -1$ and $f = r + 1$. The general term can then be treated in the same way :

$$\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}dx(1-x)^n.$$

For the proposed progression

$$1+\frac{r}{1}+\frac{r(r\text{-}1)}{1.2}+\text{etc.}$$

It produces this general term [on inversion] :

$$\frac{n(\text{-}1)^{n+1}}{(r-n)(r-n+1)\int x^{\text{-}r\text{-}1}dx(1\text{-}x)^n}\ .$$

[For $\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}dx(1-x)^n$ becomes

$$\frac{r+1-(n+1)}{(\text{-}1)^{n+1}}\int x^{\frac{r+1}{-1}}dx(1-x)^n=\frac{r-n}{(\text{-}1)^{n+1}}\int x^{-r-1}dx(1-x)^n=\left[\frac{r(r\text{-}1)(r-2).....(r-n+2)}{1.2.3......(n-1)}\times\frac{(r\text{-}n+1)}{n}\right]^{-1}$$

and hence the result follows.]
Let $r = 2$; the general term of this progression 1, 2, 1, 0, 0, 0 etc is :

$$\frac{n(\text{-}1)^{n+1}}{(2-n)(3-n)\int x^{\text{-}3}dx(1\text{-}x)^n}\ .$$

But here, this case and others it must be noted, in which $e + 1$ is a negative number, that it is not possible to be deduced from the general term, since then the integral is not equal to zero if $x = 0$. Now for these : it is appropriate that

$$\int x^e dx(1-x)^n$$

is integrated in a special way; for after the integration an infinite constant must be added.[p.44] Now when $e + 1$ is a positive number, as put in place in § 8, there is no need for the addition of a constant. But with the progression considered, the term of this with the order n is the following :

$$\frac{r(r-1)(r-2)(r-3)}{1.2.3.....(n\text{-}1)},$$

those terms of the exponent n can be changed into this form :

$$\frac{r(r-1).....1}{(1.2.3.....(n\text{-}1))(1.2......(\mathrm{r}\text{-}n+1))}$$

But by § 14 we have :

$$r(r-1).....1 = \int dx(-lx)^{r}$$

and

$$1.2.3.....(n-1) = \int dx(-lx)^{n-1}$$

and

$$1.2.3.....(r - n+1) = \int dx(-lx)^{r-n+1} .$$

On account of which for the progression treated there:

$$1+\frac{r}{1}+\frac{r(r\text{-}1)}{1.2}+\frac{r(r\text{-}1)(r-2)}{1.2.3}+\text{etc.}$$

this general term is had :

$$\frac{\int dx(-lx)^{r}}{\int dx(-lx)^{n-1}\int dx(-lx)^{r-n+1}} .$$

If we have $r = 2$, the general term is :

$$\frac{2}{\int dx(-lx)^{n-1}\int dx(-lx)^{3-n}}$$

to which this progression corresponds :

1, 2, 1, 0, 0 etc.;

and if the term of the index $\frac{3}{2}$ is sought, then this is :

$$\frac{2}{\int dx(-lx)^{\frac{1}{2}}\int dx(-lx)^{\frac{3}{2}}} .$$

Therefore for a said area of the circle A, of which the diameter is equal to 1, since

$$\int dx(-lx)^{\frac{1}{2}} = \sqrt{A} \text{ and } \int dx(-lx)^{\frac{3}{2}} = \tfrac{3}{2}\sqrt{A},$$

the mean term inserted between the two first terms of the progression 1, 2, 1, 0, 0, 0 etc. is of this form $\frac{4}{3A}$, that is $\frac{5}{3}$ as an approximation.

§ 12. Now I move on to the progression, for which I said in the beginning :

$$1+1.2+1.2.3+\text{etc.},$$

and in which the term of order n is 1.2.3.4......n. This progression is contained in our general term, but the general term must be derived in a special way. Up to the present it is clear that the general term to be had, if the term of order n is :[p.45]

$$\frac{1.2.3...n}{(f+g)(f+2g)....(f+ng)}\ ,$$

which, if there is put f = 1 and g = 0, changes into 1.2.3.....n, and of this the general term is sought ; hence these values are substituted into the general term :

$$\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}dx(1-x)^n$$

in place of f and g; then the general term sought is :

$$\int\frac{x^{\frac{1}{0}}xdx(1-x)^n}{0^{n+1}}\ .$$

Now what the value of this expression shall be, I investigate in the following manner.

§ 13. From the agreement, by which the general terms of this kind must be adapted to be used, it is understood that other functions can be proposed in place of x, provided these are such that they are equal to zero if x = 0, and = 1, if x = 1. Indeed if functions of this kind are substituted in place of x, the general term is likewise satisfied as before. Therefore $x^{\frac{g}{f+g}}$ is put in place of x and consequently $\frac{g}{f+g}x^{\frac{-f}{g+f}}$ in place of dx, with which done there is had :

$$\frac{f+(n+1)g}{g^{n+1}}\int\frac{g}{f+g}dx(1-x^{\frac{g}{f+g}})^n$$

Thus here on putting f = 1 and g = 0; there is had :

$$\int\frac{dx(1-x^0)^n}{0^n}.$$

But as $x^0=1$, we have this case, in which the numerator and the denominator are evanescent, $(1-x^0)^n$ and 0^n. Therefore by the known rule, we seek the value of the fraction $\frac{1-x^0}{0}$. [p.46] That which it becomes is found by finding the value of the fraction $\frac{1-x^z}{z}$, when z is evanescent; therefore the numerator and the denominator are differentiated with only the variable in place [*i. e.* de L'Hospital's rule]; there is had $\frac{-x^z dzlx}{dz}$ or $-x^z lx$; if now there is put z = 0, then there is produced $-lx$. Thus,

$$\frac{1-x^0}{0}=-lx.$$

[Thus, though Euler wrote down expressions where one is instructed to divide by zero, this is not what he had in mind, as he was thinking about quantities that approached zero, as he mentioned in the Introduction to the *Mechanica*, and in which he used the same technique throughout. Clearly he knew very well what he was doing, but an appropriate notation that explicitly stated that a limiting value was being taken had not occurred to him, or he did not perhaps consider that one was necessary.]

§ 14. Therefore since the ratio shall be

$$\frac{1-x^0}{0} = -lx,$$

then

$$\frac{(1-x^0)^n}{0^n} = (-lx)^n$$

and therefore the general term sought $\int \frac{dx(1-x^0)^n}{0^n}$ has been changed into $\int dx(-lx)^n$. The value of this can be found by quadrature. On this account, the general term of the progression

1, 2, 6, 24, 120, 720 etc.

is

$$\int dx(-lx)^n,$$

being put to use in the same way as was anticipated above. Moreover thus being the general term of the proposed progression, from that too it is recognised that the terms of which the indices of the numbers are positive, are actually present. For example, let $n = 3$; then

$$\int dx(-lx)^3 = \int -dx(lx)^3 = -x(lx)^3 + 3x(lx)^2 - 6xlx + 6x;$$

there is no need to add a constant, since on making $x = 0$ everything vanishes; therefore put $x = 1$; since $l1 = 0$, all the affected terms of the logarithms vanish and there remains 6, which is the third term. [p.47]

§ 15. Now indeed this is a most painstaking method of finding the terms of a series, clearly without doubt these are obtained much easier from a continued progression, of which the indices are whole numbers. Yet now it is extremely suitable for finding terms of fractional indices, clearly which hitherto indeed can only be defined by the most labourous method. If there is put $n = \frac{1}{2}$, the corresponding term is had $\int dx\sqrt{-lx}$, the value of which is given by quadrature. But in the beginning I have shown that [§11] this term is equal to the square root of the square [of area A] from the circle, of which the diameter is 1. Hence indeed the same cannot be concluded on account of the failure of analysis ; but below the method follows the intermediate terms to be reduced to the quadratures of algebraic curves. From which by this comparison nothing perhaps can be derived to further analysis.

§ 16. The general term of the progression, of which the n^{th} term is shown by :

$$\frac{1.2.3.4....n}{(f+g)(f+2g).....(f+ng)},$$

by § 10 is equal to :

$$\frac{f+(n+1)g}{g^{n+1}} \int x^{\frac{f}{g}} dx(1-x)^n .$$

But if the term of order n were

1.2.3.....n,

then the general term is

$$\int dx(-lx)^n .$$

If this formula is substituted in place of 1.2.3.....*n*, there is had : [p.48]

$$\frac{\int dx(-lx)^n}{(f+g)(f+2g).....(f+ng)} = \frac{f+(n+1)g}{g^{n+1}} \int x^{\frac{f}{g}} dx(1-x)^n .$$

From which there is brought about :

$$(f+g)(f+2g).....(f+ng) = \frac{g^{n+1}\int dx(-lx)^n}{(f+(n+1)g)\int x^{\frac{f}{g}} dx(1-x)^n} .$$

Which expression therefore is the general term of this general progression :

$f+g,\ (f+2g)(f+g),\ (f+g)(f+2g)(f+3g)$ etc.

Therefore for every progression of this kind, with the help of the general term all the terms are defined for whatever index. Which follows below from the reduction of $\int dx(-lx)^n$ to well known integrals or of algebraic curves, thus also they have this use.

§ 17. Let $f+g=1$ and $f+2g=3$, then $g=2$ et $f=-1$. Hence this particular progression arises :

1, 1.3, 1.3.5, 1.3.5.7 etc.

Therefore the general term of this is :

$$\frac{2^{n+1}\int dx(-lx)^n}{(2n+1)\int x^{-\frac{1}{2}} dx(1-x)^n} .$$

Though here the exponent of x is negative, yet that inconvenience, from what has been said above, here does not have a place, since it is less than unity. Put $n=\frac{1}{2}$, in order that the term with order $\frac{1}{2}$ can be found; this is equal to :

$$\frac{2^{\frac{3}{2}}\int dx\sqrt{-lx}}{2\int x^{-\frac{1}{2}} dx\sqrt{(1-x)}} = \frac{\sqrt{2}\int dx\sqrt{-lx}}{\int \frac{dx-xdx}{\sqrt{(x-xx)}}} .$$

But by § 15 it is agreed that $\int dx\sqrt{-lx}$ gives the square root from the area of the circle, of which the diameter is 1; [p.49] let the periphery of this circle be p; then the area is $\frac{1}{4}p$,and thus $\int dx\sqrt{-lx}$ gives $\frac{1}{2}\sqrt{p}$.

Finally

$$\int \frac{dx-xdx}{\sqrt{(x-xx)}} = \int \frac{dx}{2\sqrt{(x-xx)}} + \sqrt{(x-xx)} ;$$

but $\int \frac{dx}{2\sqrt{(x-xx)}}$ gives the arc of the circle, of which the versed sine is x. Thus with x put equal to 1 there comes about $\frac{1}{2}p$. On account of which the term sought is equal to $\sqrt{\frac{2}{p}}$.

§ 18. Since the general term of the progression, of which the term with the order *n* is indicated by

$$(f+g)(f+2g).....(f+ng),$$

by § 16 is equal to

$$\frac{g^{n+1}\int dx(-lx)^n}{(f+(n+1)g)\int x^{\frac{f}{g}}dx(1-x)^n},$$

likewise, if the term with order *n* should be

$$(h+k)(h+2k).....(h+nk),$$

then the general term is given by

$$\frac{k^{n+1}\int dx(-lx)^n}{(h+(n+1)k)\int x^{\frac{h}{k}}dx(1-x)^n}.$$

That progression is divided by this, clearly the first by the first, the second by the second, and thus again; there is arrived at a new progression, of which the term with order *n* is :

$$\frac{(f+g)(f+2g).....(f+ng)}{(h+k)(h+2k).....(h+nk)}.$$

And the general term of this progression composed from these two progressions is :

$$\frac{g^{n+1}(h+(n+1)k)\int x^{\frac{h}{k}}dx(1-x)^n}{k^{n+1}(f+(n+1)g)\int x^{\frac{f}{h}}dx(1-x)^n}.$$

Which is devoid of the logarithmic integral $\int dx(-lx)^n$.

§ 19. In all the general terms of this kind this is to be noted the most, that it is not required to put constant numbers in place of [p.50] *f, g, h, k*, but any of those terms that depend on *n* that can be assumed. For in integration these letters and *n* are all handled in the same way as constants. Let the term with order *n* here be

$$(f+g)(f+2g).....(f+ng);$$

put $g = 1$, but $f=\frac{nn-n}{2}$. Since the progression itself is

$$f+g\,,(f+g)(f+2g),(f+g)(f+2g)(f+3g)\text{ etc.,}$$

and 1 is put in place of *g* everywhere; then that becomes

$$f+1,(f+1)(f+2),(f+1)(f+2)(f+3)\ \text{ etc.}$$

But in place of *f* in the first term there must be written 0, in the second 1, in the third 3, in the fourth 6 and thus henceforth; this progression is produced :

1, 2.3, 4.5.6, 7.8.9.10 etc.,

therefore the general term of this is :

$$\frac{2\int dx(-lx)^n}{(nn+n+2))\int x^{\frac{nn-n}{2}}dx(1-x)^n}=\frac{2\int dx(-lx)^n}{(nn+n+2))\int dx(x^{\frac{n-1}{2}}-x^{\frac{n+1}{2}})^n}.$$

§ 20. Now I move on to these progressions, thus to that one compended from the definition of the intermediate terms of this progression

1, 2, 6, 24, 120, etc.,

that I have obtained. For that appears to be of greater extend than that progression alone, since the general term of this

$$\int dx(-lx)^n$$

can also appear in the general terms of an infinitude of other progressions. I assume this general term :

$$\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}dx(1-x)^n,$$

to which this term with order n corresponds : [p.51]

$$\frac{1.2.3.4....n}{(f+g)(f+2g).....(f+ng)}.$$

Here I put $f = n$, $g = 1$; then the general term is :

$$(2n+1)\int x^n dx(1-x)^n \text{ or } (2n+1)\int dx(x-xx)^n$$

and the form of this for order n :

$$\frac{1.2.3.4....n}{(n+1)(n+2)(n+3)....2n}.$$

Now this progression itself

$$\frac{1}{2},\frac{1.2}{3.4},\frac{1.2.3}{4.5.6} \text{ etc.}$$

or this [equivalent form arises :]

$$\frac{1.1}{1.2},\frac{1.2.1.2}{1.2.3.4},\frac{1.2.3.1.2.3}{1.2.3.4.5.6} \text{ etc.}$$

In the numerators are the squares of the progression 1, 2, 6, 24, etc., now between two nearest denominators an equidistant denominator is easily found. Let the term in the progression 1, 2, 6, 24, etc., of which the index is $\frac{1}{2}$ be equal to A; then the term of this progression with order $\frac{1}{2}=\frac{AA}{1}$.

[The denominators of the first two terms in the series, corresponding to the zero$^{\text{th}}$ and the first terms, are both 1 : recall that 0! is defined to equal 1. The term with order

$\frac{1}{2}$ lies between these two terms; note however that A is no longer the area of the circle considered above!]

§ 21. In the general term

$$(2n+1)\int x^n dx(1-x)^n$$

there is put $n=\frac{1}{2}$; the term of this exponent is equal to:

$$2\int dx\sqrt{(x-xx)}=\frac{AA}{1},$$

hence

$$A=\sqrt{1.2\int dx\sqrt{(x-xx)}}$$

which is equal to the term of the progression 1, 2, 6, 24 etc., of which the index is $\frac{1}{2}$, which hence, as elicited from that, is the square root from the circle of diameter 1. Now the term of this progression with order $\frac{3}{2}$ is called A; in the assumed expression this is equal to the corresponding :

$$\frac{AA}{1.2.3}=4\int dx(x-xx)^{\frac{3}{2}},$$

hence

$$A=\sqrt{1.2.3.4\int dx(x-xx)^{\frac{3}{2}}}\ .$$

In a like manner the term with order $\frac{5}{2}$ is equal to :

$$\sqrt{1.2.3.4.5.6\int dx(x-xx)^{\frac{5}{2}}}$$

From which generally I conclude that the term with order $\frac{p}{2}$ to be equal to :

$$\sqrt{1.2.3.4.5.6....(p+1)\int dx(x-xx)^{\frac{p}{2}}}\ .$$

Therefore in this way all the terms of the progression 1, 2, 6, 24 etc. can be found, of which the indices are fractions with the denominator present being 2. [p.52]

§ 22. Again in the general term

$$\frac{(f+(n+1)g)}{g}\int x^{\frac{f}{g}}xdx(1-x)^n$$

I put $f=2n$ with g remaining equal to 1; there is produced :

$$(3n+1)\int dx(xx-x^3)^n$$

the general term of this progression :

$$\frac{1}{3},\frac{1.2}{5.6},\frac{1.2.3}{7.8.9}\ \text{etc.}$$

This is multiplied by the preceding $(2n+1)\int dx(x-xx)^n$; there is produced :

$$(2n+1)(3n+1)\int dx(x-xx)^n \int dx(x^2-x^3)^n .$$

Which gives this progression:

$$\frac{1.1.1}{1.2.3}, \frac{1.2.1.2.1.2}{1.2.3.4.5.6} \text{ etc.}$$

where the numerators are the cubes of the corresponding terms of the progression 1, 2, 6, 24, etc. The term with order $\frac{1}{3}$ of this progression shall be A; the corresponding of that :

$$\frac{A^3}{1} = 2(\frac{2}{3}+1)\int dx(x-xx)^{\frac{1}{3}} \int dx(xx-x^3)^{\frac{1}{3}},$$

hence the term with order $\frac{1}{3}$ is :

$$\sqrt[3]{1.2\frac{5}{3}\int dx(x-xx)^{\frac{1}{3}} \int dx(xx-x^3)^{\frac{1}{3}}};$$

similarly the term with order $\frac{2}{3}$:

$$\sqrt[3]{1.2.3.\frac{7}{3}\int dx(x-xx)^{\frac{2}{3}} \int dx(xx-x^3)^{\frac{2}{3}}}.$$

And the term with order $\frac{4}{3}$:

$$\sqrt[3]{1.2.3.5.\frac{11}{3}\int dx(x-xx)^{\frac{4}{3}} \int dx(xx-x^3)^{\frac{4}{3}}},$$

and generally the term with order $\frac{p}{3}$ is :

$$\sqrt[3]{1.2....p.\frac{2p+3}{3}\int dx(x-xx)^{\frac{p}{3}} \int dx(xx-x^3)^{\frac{p}{3}}}.$$

§ 23. If we want further progressions on putting $f = 3n$, the general term is required :

$$(4n+1)\int dx(x^3-x^4)^n$$

multiplied by the preceding, thus there is had :

$$(2n+1)(3n+1)(4n+1)\int dx(x-xx)^n \int dx(x^2-x^3)^n \int dx(x^3-x^4)^n ,$$

which is for this series :

$$\frac{1.1.1.1}{1.2.3.4}, \frac{1.2.1.2.1.2.1.2}{1.2.3.4.5.6.7.8} \text{ etc.}$$

From which the terms of the progression 1, 2, 6, 24 etc. can be defined of which the indices are fractions having the denominators 4. Clearly the term, of which the index is $\frac{p}{4}$, can be found : [p.52]

$$\sqrt[4]{1.2....p.(\frac{2p}{4}+1)(\frac{3p}{4}+1)(p+1)\times\int dx(x-xx)^{\frac{p}{4}} \int dx(xx-x^3)^{\frac{p}{4}} \int dx(x^3-x^4)^{\frac{p}{4}}}.$$

Hence generally it is possible to conclude that the term with order $\frac{p}{q}$ is

$$\sqrt[q]{1.2....p.(\frac{2p}{q}+1)(\frac{3p}{q}+1)(\frac{4p}{q}+1)......(p+1)\times}$$
$$\sqrt[q]{\int dx(x-xx)^{\frac{p}{q}}\int dx(xx-x^3)^{\frac{p}{q}}\int dx(x^3-x^4)^{\frac{p}{q}}.......\int dx(x^{q-1}-x^q)^{\frac{p}{q}}}.$$

Therefore from this formula the terms of any fractional indices can be found from the quadrature of algebraic curves ; moreover for that there the term 1.2.3.4..... p is required, the index of which is the numerator of the proposed fraction.

§ 24. In the same way it is allowed to go on to more composite progressions by assuming more composite general terms, but I will not pursue these longer terms. The integration signs can also be multiplied, so that the general term is :

$$\int qdx\int pdx\,;$$

clearly the integral of pdx has to be multiplied by qdx and what results has to be integrated anew, that which finally gives the term of the series on making $x = 1$. Moreover in which integration, in order that it is determined, it is required that a constant be added to be effective, so that on putting $x = 0$ the integral likewise becomes equal to 0. [Double integrals rather than repeated single integrations.]

The general terms are to be treated similarly, in which several integration signs are contained, as

$$\int rdx\int qdx\int pdx\,.$$

But yet always in place of p, q, r etc. such functions are to be taken, so that, as long as n is a positive integer, the terms are produced with the least algebra. [p.53]

§ 25. Let the general term be :

$$\int \frac{dx}{x}\int x^e xdx(1-x)^n\,;$$

this converted into a series gives :

$$\frac{x^{e+1}}{(e+1)^2}-\frac{nx^{e+2}}{1.(e+2)^2}+\frac{n(n\text{-}1)x^{e+3}}{1.2.(e+3)^2}-\text{etc.}$$

On putting $x = 1$, for this series the term of order n is obtained:

$$\frac{1}{(e+1)^2}-\frac{n}{1.(e+2)^2}+\frac{n(n\text{-}1)}{1.2.(e+3)^2}-\text{etc.}$$

Now the progression itself, starting from the term for which the index n is zero, begins

$$\frac{1}{(e+1)^2},\frac{(e+2)^2-(e+1)^2}{(e+2)^2(e+1)^2},\frac{(e+3)^2(e+2)^2-2(e+3)^2(e+1)^2+(e+2)^2(e+1)^2}{(e+3)^2(e+2)^2(e+1)^2},$$
$$\frac{(e+4)^2(e+3)^2(e+2)^2-3(e+4)^2(e+3)^2(e+1)^2+3(e+4)^2(e+2)^2(e+1)^2\text{-}(e+3)^2(e+2)^2(e+1)^2}{(e+4)^2(e+3)^2(e+2)^2(e+1)^2},$$

etc.

The rule of this progression is clear and does not need an explanation. Let $e = 0$; then

$$\int dx(1-x)^n=\frac{1\text{-}(1-x)^{n+1}}{n+1}\,;$$

hence the general term is

$$\int \frac{dx\text{-}dx(1-x)^{n+1}}{(n+1)x},$$

now this progression is :

$$\frac{1}{1},\ \frac{4\text{-}1}{4.1},\ \frac{9.4\text{-}2.9\text{+}4.1}{9.4.1},\ \frac{16.9.4\text{-}3.16.9.1\text{+}3.16.4.1\text{-}9.4.1}{16.9.4.1}\text{ etc.}$$

The differences of this make the progression :

$$\frac{\text{-}1}{4.1},\ \frac{\text{-}9\text{+}4}{9.4.1},\ \frac{\text{-}16.9\text{+}2.16.4\text{-}9.4}{16.9.4.1}\text{ etc.}$$

§ 26. In this discourse therefore that which I had especially intended, I have pursued, clearly so that I could find the general terms of all progressions, the individual terms of which [p.55] were made from factors progressing in arithmetical progression, in each the number of the factors depends in any way on the indices of the terms. But though here the number of factors is always put equal to the index, yet if it is wished for this to depend in another way, there is no difficulty in doing this. The index has been denoted by the letter n; if now it is required for which, so that the number of factors is $\frac{nn+n}{2}$, there is no need for another operation, except that everywhere in place of n there is substituted $\frac{nn+n}{2}$.

§ 27. I add besides in place of some final flourish, that which is more curious than useful. It is known that by $d^n x$ to be understood the differential of order n of x and $d^n p$, if p denotes some function of x and dx is put constant, to be homogeneous with $d^n x$; but always, when n is a positive whole number, the ratio, that $d^n p$ has to dx^n, is able to be expressed algebraically; for if $n = 2$ and $p = x^3$, then $d^2(x^3)$ to dx^2 as $6x$ to 1. Now it is enquired, if n is a fractional number, what then becomes of the ratio. The difficulty in these cases is easily understood, for if n is a positive whole number, then d^n is found from continued differentiation; but such a way is not apparent, if n is a fractional number. But yet with the help of interpolation of the progression, which I have set forth in this dissertation, a way of doing this can be established. [p.56]

§ 28. Let the ratio be required to be found between $d^n(z^e)$ and dz^n with dz put constant, or the value is required of the fraction $\frac{d^n(z^e)}{dz^n}$. In the first place we may consider what the values of this are if n is a whole number, as afterwards generally it is possible to come to a conclusion. Let $n = 1$, then the value of this is

$$ez^{e-1} = \frac{1.2.3....e}{1.2.3.....(e-1)} z^{e-1};$$

with e expressed in this way, so that these later ratios are treated more easily by referring them to this.

If $n = 2$, then the value is

$$e(e-1)z^{e-1} = \frac{1.2.3....e}{1.2.3.....(e-2)} z^{e-2}.$$

If $n = 3$, there is had :

$$e(e-1)(e-2)z^{e-1} = \frac{1.2.3....e}{1.2.3.....(e-3)} z^{e-3}.$$

Hence I infer generally, whatever n shall be, to be always :

$$\frac{dn(z^e)}{dz^n} = \frac{1.2.3....e}{1.2.3.....(e-n)} z^{e-n} .$$

But by § 14

$$1.2.3.....e = \int dx(-lx)^e \text{ and } 1.2.3.....(e-n) = \int dx(-lx)^{e-n} .$$

Whereby there is obtained:

$$\frac{d^n(z^e)}{dz^n} = \frac{\int dx(-lx)^e}{\int dx(-lx)^{e-n}}$$

or

$$d^n(z^e) = z^{e-n} dz^n \frac{\int dx(-lx)^e}{\int dx(-lx)^{e-n}} .$$

Here dz is put constant, and both $\int dx(-lx)^e$ and $\int dx(-lx)^{e-n}$ thus must be integrated, as was done in advance above, and then it is necessary to put $x = 1$.

§ 29. It is not necessary to show how this can be elicited ; that will be apparent on placing some whole positive number in place of n. But what is sought, what is $d^{\frac{1}{2}}z$, if dz is constant. Therefore $e = 1$ and $n = \frac{1}{2}$. Thus there is obtained : [p.57]

$$d^{\frac{1}{2}}z = \frac{\int dx(-lx)}{\int dx\sqrt{-lx}} \sqrt{z}dz .$$

But

$$\int dx(-lx) = 1$$

and the said area of the circle A, of which the diameter is 1 is

$$\int dx\sqrt{-lx} = \sqrt{A} ,$$

thus

$$d^{\frac{1}{2}}z = \sqrt{\frac{zdz}{A}} .$$

Therefore this is the proposed equation for some curve : $yd^{\frac{1}{2}}z = z\sqrt{dy}$,
where dz is put constant, and what the curve shall be is sought. Since $d^{\frac{1}{2}}z = \sqrt{\frac{zdz}{A}}$,
that equation changes into this : $y\sqrt{\frac{zdz}{A}} = z\sqrt{dy}$,
which squared gives :

$$\frac{yydz}{A} = zdy ;$$

and thus there is found

$$\frac{1}{A} lz = c - \frac{1}{y}$$

or

$$ylz = cAy - A,$$

which is the equation for the curve sought.

DE PROGRESSIONIBUS TRANSCENDENTIBUS, SEV QUARUM TERMINI GENERALES ALGEBRAICE DARI NEQUEVENT.

Auct. L. Eulero. [p.36]

§ I.

Cum nuper occasione eorum, quae Cel. Goldbach de seriebus cum Societate communicaverit [De terminis generalibus serierum, Comment.acad.sc.Petrop.3. (1728), 1732, pp. 164 - 173], in expressionem quandam generalem inquirerem, quae huius Progressionis

$$1+1.2+1.2.3+1.2.3.4+etc.$$

terminos omnes daret; incidi considerans, quod ea in infinitum continuata tandem cum geometrica confundatur in sequentem expressionem,

$$\frac{1.2^n}{1+n}.\frac{2^{1-n}.3^n}{2+n}.\frac{3^{1-n}.4^n}{3+n}.\frac{4^{1-n}.5^n}{4+n}, \text{ etc}$$

quae dictae progressionis terminum ordine n exponit. Ea quidem in nullo casu abrumpitur; neque si n est numerus integer neque si fractus, sed ad quemuis terminum inveniendum tantummodo approximationes suppeditat, nisi excipiantur casus $n = 0$, et $n = 1$, quibus ea actu abit in 1. Ponatur $n = 2$, habebitur

$$\frac{2.2}{1.3},\frac{3.3}{2.4},\frac{4.4}{3.5},\frac{5.5}{4.6}.\text{etc.} = \text{termino secundo 2.}$$

Si $n = 3$, habebitur

$$\frac{2.2.2}{1.1.4},\frac{3.3.3}{2.2.5},\frac{4.4.4}{3.3.6},\frac{5.5.5}{4.4.7},\text{etc.} = \text{termino tertio 6.}$$

§ 2. Quanquam autem haec expressio nullum usum habere videatur in inventione [p.37] terminorum, tamen ad interpolationem eius seriei seu ad terminos, quorum indices sunt numeri fracti, egregie accommodari potest. De hoc autem hic explicare non constitui, cum infra magis idonei modi occurrant ad idem efficiendum. Id tantum de isto termino generali afferam, quod ad ea, quae sequuntur, quasi manuducat. Quaesivi terminum, cuius index $n=\frac{1}{2}$ seu qui aequaliter interiacet inter primum,1, et praecedentem, qui itidem est 1. Posito autem $n=\frac{1}{2}$ assecutus sum seriem

$$\sqrt{\frac{2.4}{3.3}.\frac{4.6}{5.5}.\frac{6.8}{7.7}.\frac{8.10}{9.9}.\text{etc}}$$

quae terminum quaesitum exprimit. Haec autem series similis mihi statim visa est eius, quam in Wallisii operibus pro area circuli vidisse memineram. Invenit enim Wallisius
[J. Wallis (1616 - 1703), Arithmetica infinitorum sive nova methodus inquirendi in curvilineorum quadratura, aliaque difficiliora Matheseos problemata, Oxoniae. p. 182; Opera mathematica, t. I, Oxoniae 1695, p. 355, imprimis p. 469.]
circulum esse ad quadratum diametri ut

$$2.4.4.6.6.8.8.10.\text{ etc. ad } 3.3.5.5.7.7.9.9.\text{ etc.}$$

Si igitur fuerit diameter = 1, erit circuli area

$$=\frac{2.4}{3.3}\cdot\frac{4.6}{5.5}\cdot\frac{6.8}{7.7}\cdot\frac{8.10}{9.9}.\text{etc}$$

Ex huius igitur cum mea convenientia concludere licet terminum indicis $\frac{1}{2}$ esse aequalis radici quadratae ex circulo cuius diameter = 1.

§ 3. Arbitratus eram ante seriei 1, 2, 6, 24, etc terminum generalem, si non algebraicum, tamen exponentialem dari. Sed postquam intellexissem terminos quosdam intermedios a quadratura circuli pendere, neque algebraicas neque exponentiales quantitates ad eum exprimendum idoneas esse cognovi. Terminus enim generalis eius progressionis ita debet esse comparatus, [p.38] ut tum quantitates algebraicas tum a quadratura circuli tum forte ab aliis quoque quadraturis pendentes comprehendat; id quod in nullum formulam nec algebraicam nec exponentialem competit.

§ 4. Cum autem considerassem dari inter quantitates differentiales eiusmodi formulas, quae certis in casibus integrationem admittant et tum quantitates algebraicas praebeant, in aliis vero non admittant et tum quantitates a quadraturis curvarsum pendentes exhibeant, animum subiit huiusmodi forte formulas ad progressionis memoratae aliarumque eius similium terminos generales suppeditandos aptas esse. Progressiones vero, quae tales requirunt terminos generales, qui algebraice dari nequent, voco transcendentes. quemadmodum Geometriae omne id, quod vires communis Algebrae superat, transcendens appellare solent.

§ 5. Id ergo meditatus sum, quomodo formulas differentiales ad progressionum terminos generales exprimendos accommodari maxime conveniat. Terminus autem generalis est formula, quam ingrediuntur tum quantitates constantes tum alia quaepiam non constans ut n, quae ordinem terminorum seu indicem exponet, ut, si tertius terminus desideretur, oporteat loco n ponere 3. Sed in formula differentiali quantitatem quandam variabilem inesse oportet. Pro qua non consultum est adhibere n, [p.39] cum eius variabilitas non ad integrationem pertineat, sed postquam ea formula integrata est vel integrata esse ponitur, tum demum ad progressionem formandam inserviat. In formula igitur differentiali insit oportet quantitas quaedam variabilis x , quae autem post integrationem alii ad progressionem spectanti aequalis ponenda est; et quod oritur proprie est terminus, cuius index est n.

§ 6. Ut haec clarius concipiantur, dico $\int pdx$ esse terminum generalem progressionis sequinti modo ex eo eruendae; denotet autem p functionem quamcunque ipsius x et constantium, in quarum numero adhuc ipsum n haberi debet. Concipiatur pdx integatum talique constante auctum, ut facto $x = 0$ totum integrale evanescat; tum ponatur x aequale quantitati cuidam cognitae. Quo facto in invento integrali nonnisi quantitates ad progressionem pertinentes supererunt, et id exprimet terminum,cuius index = n. Seu integrale dicto modo determinatum erit proprie terminus generalis. Si quidem id haberi potest, non opus est formula differentiali, sed progressio inde formata habebit terminum generalem algebraicum; secus res se habet, si integratio non succedit nisi certis numeris loco n substitutis.

§ 7. Assumsi igitur plures huiusmodi formulas differentiales integrationem non admittentes, [p.40] nisi si ponatur loco n numerus integer affirmativus, ut seriei termini principales fiant algebraici, et inde progressiones formavi. Earum itaqua termini generales in promtu erunt, et a quanam quadratura quique eius termini intermedii pendeant, definire licebit. Hic quidem non plures eiusmodi formulas percurram, sed unicam tantum aliquanto generaliorem pertractabo, quae valde late patet et ad omnes progressiones, quarum quilibet termini sunt facta constantia ex numero factorum ab indice pendente,accommodari potest; quique factores sunt fractiones, quarum numeratores et denominatores in progressione quacunque arithmetica progrediuntur, ut

$$\frac{2}{3}+\frac{2.4}{3.5}+\frac{2.4.6}{3.5.7}+\frac{2.4.6.8}{3.5.7.9}+\text{etc.}$$

§ 8. Sit proposita haec formula

$$\int x^e dx(1-x)^n$$

vicem termini generalis subiens, quae integrata ita, ut fiat = 0, si sit $x = 0$, et tum posito $x = 1$ det terminum ordine n progressionis inde ortae. Videamus ergo, qualem ea suppeditet progressionem. Est

$$(1-x)^n = 1-\frac{n}{1}x+\frac{n(n-1)}{1.2}x^2-\frac{n(n-1)(n-2)}{1.2.3}x^3+\text{etc.}$$

et propterea

$$x^e dx(1-x)^n = x^e dx-\frac{n}{1}x^{e+1}dx+\frac{n(n-1)}{1.2}x^{e+2}dx-\frac{n(n-1)(n-2)}{1.2.3}x^{e+3}dx+\text{etc.}$$

Quare

$$\int x^e dx(1-x)^n = \frac{x^{e+1}}{e+1}-\frac{nx^{e+2}}{1.(e+2)}+\frac{n(n-1)x^{e+3}}{1.2.(e+3)}-\frac{n(n-1)(n-2)x^{e+4}}{1.2.3.(e+4)}+\text{etc.}$$

Ponatur $x = 1$, quia constantis additione non est opus, et habebitur [p.41]

$$\frac{1}{e+1}-\frac{n}{1.(e+2)}+\frac{n(n-1)}{1.2.(e+3)}-\frac{n(n-1)(n-2)}{1.2.3.(e+4)}+\text{etc.}$$

terminus generalis seriei inveniendae. Quae talis erit, ut , si $n = 0$, prodeat terminus = $\frac{1}{e+1}$; si $n = 1$, terminus = $\frac{1}{(e+1)(e+2)}$; si $n = 2$, terminus $\frac{1.2}{(e+1)(e+2)(e+3)}$; si $n = 3$, prodeat terminus = $\frac{1.2.3}{(e+1)(e+2)(e+3)(e+4)}$; lex, qua hi termini progrediuntur, manifesta est.

§ 9. Hanc ergo assecutus sum progressionem

$$\frac{1.}{(e+1)(e+2)}+\frac{1.2}{(e+1)(e+2)(e+3)}+\frac{1.2.3}{(e+1)(e+2)(e+3)(e+4)}+\text{etc.},$$

cuius terminus generalis est

$$\int x^e dx(1-x)^n.$$

Termini vero ordine n ipsius haec erit forma

$$\frac{1.2.3.4....n}{(e+1)(e+2).....(e+n+1)}.$$

Haec quidem forma sufficit ad terminos indicum integrorum inveniendos, sed si indices non fuerint integri, ex ea ipsi termini inveniri nequeunt. Iis autem proximis inveniendis inservit haec series

$$\frac{1}{e+1}-\frac{n}{1.(e+2)}+\frac{n(n-1)}{1.2.(e+3)}-\frac{n(n-1)(n-2)}{1.2.3.(e+4)}+\text{etc.}$$

Si $\int x^e dx(1-x)^n$ multiplicetur per $e + n + 1$, habebitur progressio, cuius terminus ordine n hanc formam habet

$$\frac{1.2.3.4....n}{(e+1)(e+2).....(e+n)},$$

cuius igitur verus terminus generalis erit

$$(e+n+1)\int x^e dx(1-x)^n .$$

Hic observandum est progressionem semper fieri algebraicam, quando loco e assumatur numerus affirmativus. Ponatur e.g. $e = 2$; progressionis terminus n^{mus} erit

$$\frac{1.2.3.4....n}{3.4.5.....(n+2)} \text{ seu } \frac{1.2}{(n+1)(n+2)}.$$

Id quod ipse terminus generalis quoque indicat, qui erit

$$(n+3)\int xxdx(1-x)^n.$$

Nam eius integrale est [p.42]

$$\left(C-\frac{(1-x)^{n+1}}{n+1}+\frac{2(1-x)^{n+2}}{n+2}-\frac{(1-x)^{n+3}}{n+3}\right)(n+3);$$

ut hoc fiat = 0, si $x = 0$, erit

$$C=\frac{1}{n+1}-\frac{2}{n+2}+\frac{1}{n+3}.$$

Ponatur $x = 1$; erit terminus generalis

$$\frac{n+3}{n+1}-\frac{2(n+3)}{n+2}+1=\frac{2}{(n+1)(n+2)}.$$

§ 10. Ut igitur progressiones transcendentes adipiscamur, ponatur e aequale fractioni $\frac{f}{g}$. Erit progressionis terminus ordine n

$$\frac{1.2.3.4....n}{(f+g)(f+2g).....(f+ng)}g^n, \text{ sive } \frac{g.2g.3g.4g....ng}{(f+g)(f+2g).....(f+ng)}.$$

Terminus vero generalis erit =

$$\frac{(f+(n+1)g)}{g}\int x^{\frac{f}{g}}xdx(1-x)^n.$$

Qui si dividatur per g^n, erit pro progressione

$$\frac{1}{f+g}+\frac{1.2}{(f+g)(f+2g)}+\frac{1.2.3}{(f+g)(f+2g)(f+3g)}\text{ etc.},$$

cuius terminus ordine n est

$$\frac{1.2.3...n}{(f+g)(f+2g)....(f+ng)}.$$

Eius progressionis igitur terminus generalis erit

$$\frac{(f+(n+1)g)}{g^{n+1}}\int x^{\frac{f}{g}}dx(1-x)^n.$$

Ubi si fractio $\frac{f}{g}$ non sit numero integro aequalis, seu si *f* ad *g* non habuerit rationem multiplicem, progressio erit transendens, et termini intermedii a quadraturis pendebunt.

§ 11. Exemplum quoddam in medium afferam, ut usus termini generalis clarius ob oculos ponatur. Sit in paragraphi praecedentis progressione priore *f* = 1, *g* = 2, erit terminus ordine $n = \frac{2.4.6.8......2n}{3.5.7.9......(2n+1)}$, progressio vero ipsa haec

$$\frac{2}{3} + \frac{2.4}{3.5} + \frac{2.4.6}{3.5.7} + \text{etc},$$

cuius terminus generalis ergo erit $\frac{2n+3}{2}\int dx(1-x)^n\sqrt{x}$. Quaeritur terminus cuius index $n = \frac{1}{2}$, [p.43] et habebitur terminus quaesitus = $2\int dx\sqrt{(x-xx)}$. Quod cum significet elementum areae circulis, perspicuum est terminum quaesitum esse aream circuli, cuius diameter = 1. Proposita porro sit haec series

$$1 + \frac{r}{1} + \frac{r(r\text{-}1)}{1.2} + \frac{r(r\text{-}1)(r-2)}{1.2.3} + \text{etc.},$$

quae est coefficientium binomii ad potestatem *r* elevati. Terminus ordine *n* est ergo

$$\frac{r(r\text{-}1)(r-2).....(r-n+2)}{1.2.3......(n-1)}\text{etc.}$$

In § praecedente habetur hic

$$\frac{1.2.3...n}{(f+g)(f+2g)....(f+ng)}.$$

Hic, ut cum illo comparetur invertendus est, ut habeatur

$$\frac{(f+g)(f+2g)....(f+ng)}{1.2.3...n};$$

multiplicetur hic per $\frac{n}{f+ng}$ et erit is =

$$\frac{(f+g)(f+2g)....(f+(n-1)g)}{1.2.3...(n-1)};$$

oportet igitur esse *f* + *g* = *r* et *f* + 2*g* = *r* – 1, unde fiet *g* = –1 et *f* = *r* + 1. Eodem modo tractetur terminus generalis

$$\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}dx(1-x)^n.$$

Prodibit pro progressione proposita

$$1 + \frac{r}{1} + \frac{r(r\text{-}1)}{1.2} + \text{etc.}$$

hic terminus generalis

$$\frac{n(\text{-}1)^{n+1}}{(r-n)(r-n+1)\int \mathrm{x}^{\text{-}r\text{-}1}dx(1\text{-}x)^n}.$$

Sit *r* = 2; erit huius progressionis 1, 2, 1, 0, 0, 0 etc terminus generalis

$$\frac{n(\text{-}1)^{n+1}}{(2-n)(3-n)\int \mathrm{x}^{\text{-}3}dx(1\text{-}x)^n}.$$

Hic autem notari debet hunc casum et alios, quibus $e+1$ fit numerus negativus, non posse ex generali deduci, quia tunc integrale non sit = 0, si $x = 0$. Pro his vero

$$\int x^e dx(1-x)^n$$

peculari modo integrari convenit; post integrationem enim constans infinita est adiicienda. [p.44] Quando vero $e+1$ est numerus affirmativus, ut posui § 8, constantis additione non est opus. Considerata autem progressione, cuius terminus ordine n erat sequens

$$\frac{r(r-1)(r-2)(r-3)}{1.2.3.....(n\text{-}1)},$$

transmutari potest illa termini exponentis n forma in hanc

$$\frac{r(r-1).....1}{(1.2.3.....(n\text{-}1))(1.2......(r\text{-}n+1))}$$

Sed per § 14 est

$$r(r-1).....1 = \int dx(-lx)^r$$

et

$$1.2.3.....(n-1) = \int dx(-lx)^{n-1}$$

et

$$1.2.3.....(r-n+1) = \int dx(-lx)^{r-n+1}.$$

Quamobrem ibi tractatae progressionis

$$1+\frac{r}{1}+\frac{r(r\text{-}1)}{1.2}+\frac{r(r\text{-}1)(r-2)}{1.2.3}+\text{etc.}$$

hic habetur terminus generalis

$$\frac{\int dx(-lx)^r}{\int dx(-lx)^{n-1}\int dx(-lx)^{r-n+1}}.$$

Si fuerit $r = 2$, erit terminus generalis

$$\frac{2}{\int dx(-lx)^{n-1}\int dx(-lx)^{3-n}}$$

cui respondet haec progressio

1, 2, 1, 0, 0 etc.;

ut si quaeratur terminus indicis $\frac{3}{2}$, erit is

$$\frac{2}{\int dx(-lx)^{\frac{1}{2}}\int dx(-lx)^{\frac{3}{2}}}.$$

Dicta ergo area A circuli , cuius diameter est = 1, quia est

$$\int dx(-lx)^{\frac{1}{2}} = \sqrt{A} \text{ et } \int dx(-lx)^{\frac{3}{2}} = \tfrac{3}{2}\sqrt{A},$$

erit terminus medium interiacens inter duos primos terminos progressionis

1, 2, 1, 0, 0, 0 etc. huius formae $\frac{4}{3A}$, hoc est $\frac{5}{3}$ quam proxime.

§ 12. Progredior nunc ad progressionem, de qua initio dixi,

$$1+1.2+1.2.3+\text{etc.},$$

et in qua terminus ordine *n* est 1.2.3.4......*n*. Continetur haec progressio in generali nostra, sed terminus generalis peculiari modo inde derivari debet. Hactenus scilicet terminum generalem habui, si terminus ordine *n* est [p.45]

$$\frac{1.2.3...n}{(f+g)(f+2g)....(f+ng)}\ ,$$

qui, si ponatur $f = 1$ et $g = 0$, abit in 1.2.3.....*n*, cuius terminus generalis quaeritur; substituantur ergo in termino generali

$$\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}dx(1-x)^n$$

hi valores loco *f* et *g*; erit terminus generalis quaesitus

$$\int\frac{x^{\frac{1}{0}}xdx(1-x)^n}{0^{n+1}}.$$

Qui vero huius expressionis sit valor, sequenti modo investigo.

§ 13. Ex conditione, qua huiusmodi termini generales usui accommodari debent, intelligitur loco x alias functiones ipsius x posse subrogari, dummodo eae tales fuerint, ut sint = 0, si $x = 0$, et = 1, si $x = 1$. Huiusmodi enim functiones si loco *x* substituantur, terminus generalis perinde satisfaciet ac ante. Ponatur igitur $x^{\frac{g}{f+g}}$ loco *x* et consequenter $\frac{g}{f+g}x^{\frac{-f}{g+f}}$ loco *dx*, quo facto habebitur

$$\frac{f+(n+1)g}{g^{n+1}}\int\frac{g}{f+g}dx(1-x^{\frac{g}{f+g}})^n$$

Iam hic ponatur $f = 1$ et $g = 0$; habebitur

$$\int\frac{dx(1-x^0)^n}{0^n}.$$

Cum autem sit $x^0 = 1$, habemus hic casum, quo numerator et denominator evanescunt, $(1-x^0)^n$ et 0^n. Per regulam igitur cognitam quaeramus valorem fractionis [p.46] $\frac{1-x^0}{0}$. Id quod fiet quaerendo valorem fractionis $\frac{1-x^z}{z}$ tum, cum *z* evanescit; differentietur igitur et numerator et denominator sola *z* variabili posita; habebitur $\frac{-x^z dzlx}{dz}$ seu $-x^z lx$; si iam ponatur z = 0, prodibit $-lx$. Est itaque

$$\frac{1-x^0}{0} = -lx.$$

§ 14. Cum igitur sit

$$\frac{1-x^0}{0} = -lx,$$

erit

$$\frac{(1-x^0)^n}{0^n} = (-lx)^n$$

et propterea terminus generalis quaesitus $\int\frac{dx(1-x^0)^n}{0^n}$ transmutatus est in $\int dx(-lx)^n$. Cuius valor inveniri per quadratus potest. Quamobrem huius progressionis

1, 2, 6, 24, 120, 720 etc.

terminus generalis est $\int dx(-lx)^n$, eodem modo adhibendus, quo supra praeceptum est. Hunc autem esse terminum generalem progressionis propositae ex eo quoque cogniscitur, quod terminos, quorum indices numeri integri affirmativi, revera praebeat. Sit v.g. $n = 3$; erit

$$\int dx(-lx)^3 = \int -dx(lx)^3 = -x(lx)^3 + 3x(lx)^2 - 6xlx + 6x;$$

constantis additione opus non est, cum facto $x = 0$ omnia evanescant; ponatur igitur $x = 1$; quia $l1 = 0$, omnes termini logarithmis affecti evanescent et restabit 6, qui est terminus tertius. [p.47]

§ 15. Verum quidem est hanc methodum terminorum istius seriei inveniendorum nimis esse operosam, eorum nimirum, quorum indices sunt numeri integri, qui utique facilius continuanda progressione obtinentur. Verum tamen ad terminos indicum fractorum inveniendos perquam est idonea, quippe qui adhuc ne operosissima quidem methodo definiri potuerunt. Si ponatur $x = \frac{1}{2}$, habebitur respondes terminus $\int dx\sqrt{-lx}$, cuius valor per quadraturas datur. Sed initio [§11] ostendi hunc terminum esse aequalem radici quadratae ex circulo, cuius diameter est 1. Hinc quidem idem concludere non licet ob defectum analysis; infra autem sequetur methodus eosdem terminos intermedios ad algebraicarum curvarum quadraturas reducendi. Ex cuius cum hac comparatione forte nonnihil ad amplificationem analysis derivari poterit.

§ 16. Progressionis, cuius terminus ordine n indicatur per

$$\frac{1.2.3.4....n}{(f+g)(f+2g).....(f+ng)},$$

terminus generalis est per § 10

$$\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}dx(1-x)^n .$$

Si autem terminus ordine n fuerit

$$1.2.3.....n,$$

tum est terminus generalis

$$\int dx(-lx)^n .$$

Quae formula si loco $1.2.3.....n$ substituatur, habebitur [p.48]

$$\frac{\int dx(-lx)^n}{(f+g)(f+2g).....(f+ng)} = \frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}dx(1-x)^n.$$

Ex quo efficitur

$$(f+g)(f+2g).....(f+ng) = \frac{g^{n+1}\int dx(-lx)^n}{(f+(n+1)g)\int x^{\frac{f}{g}}dx(1-x)^n}.$$

Quae expressio igitur est terminus generalis huius generalis progressionis

$f+g,\ (f+2g)(f+g),\ (f+g)(f+2g)(f+3g)$ etc.

Huiusmodi igitur progressionum omnium ope termini generalis omnes termini cuiuscunque indicis definiuntur. Quae infra sequentur de reductione $\int dx(-lx)^n$ ad quadraturas notiores seu curvarum algebraicarum, etiam hic usum habebunt.

§ 17. Sit $f + g = 1$ et $f + 2g = 3$, erit $g = 2$ et $f = -1$. Unde orietur haec progressio particularis

1, 1.3, 1.3.5, 1.3.5.7 etc.

Cuius igitur terminus generalis est

$$\frac{2^{n+1}\int dx(-lx)^n}{(2n+1)\int x^{-\frac{1}{2}}dx(1-x)^n}.$$

Quanquam hic exponens ipsius x sit negativus, tamen id incommodum, de quo supra dictum, hic locum non habet, cum sit unitate minor. Ponatur $n = \frac{1}{2}$, ut inveniatur terminus ordine $\frac{1}{2}$; erit is =

$$\frac{2^{\frac{3}{2}}\int dx\sqrt{-lx}}{2\int x^{-\frac{1}{2}}dx\sqrt{(1-x)}} = \frac{\sqrt{2}\int dx\sqrt{-lx}}{\int \frac{dx-xdx}{\sqrt{(x-xx)}}}.$$

Per § 15 autem constat dare $\int dx\sqrt{-lx}$ radicem quadratam ex circulo, cuius diameter = 1; [p.49] sit peripheria eius circuli p; erit area $\frac{1}{4}p$ adeoque $\int dx\sqrt{-lx}$ dat $\frac{1}{2}\sqrt{p}$. Deinde

$$\int \frac{dx-xdx}{\sqrt{(x-xx)}} = \int \frac{dx}{2\sqrt{(x-xx)}} + \sqrt{(x-xx)};$$

sed $\int \frac{dx}{2\sqrt{(x-xx)}}$ dat arcum circuli, cuius sinus versus est x. Posito itaque $x = 1$ proveniat $\frac{1}{2}p$. Quamobrem terminus quaesitus erit = $\sqrt{\frac{2}{p}}$.

§ 18. Cum progressionis, cuius terminus ordine n indicatur per

$$(f+g)(f+2g).....(f+ng),$$

terminus generalis per § 16 sit =

$$\frac{g^{n+1}\int dx(-lx)^n}{(f+(n+1)g)\int x^{\frac{f}{g}}dx(1-x)^n},$$

similiter, si terminus ordine n fuerit

$$(h+k)(h+2k).....(h+nk),$$

erit terminus generalis

$$\frac{k^{n+1}\int dx(-lx)^n}{(h+(n+1)k)\int x^{\frac{h}{k}}dx(1-x)^n}.$$

Dividatur illa progressio per hanc, nempe terminus primus per primam, secundus per secundum et ita porro; devenietur ad novam progressionem, cuius terminus ordine n erit

$$\frac{(f+g)(f+2g).....(f+ng)}{(h+k)(h+2k).....(h+nk)}.$$

Et terminus generalis huius progressionis ex illis duobus compositus erit

$$\frac{g^{n+1}(h+(n+1)k)\int x^{\frac{h}{k}}dx(1-x)^n}{k^{n+1}(f+(n+1)g)\int x^{\frac{f}{h}}dx(1-x)^n}.$$

Qui vacuus est ab integrali logarithmico $\int dx(-lx)^n$.

§ 19. In omnibus huiusmodi terminis generalibus hoc maxime notandum est non [p.50] quidem loco *f, g, h, k* numeros constantes poni oportere, sed eos quomodocunque ab *n* pendentes quoque assumi posse. In integratione enim eae litterae perinde atque n tractantur, omnes tanquam constantes. Sit terminus ordine *n* hic

$$(f+g)(f+2g).....(f+ng);$$

ponatur $g = 1$, sed $f = \frac{nn-n}{2}$. Quia progressio ipsa est

$$f+g\,,(f+g)(f+2g),(f+g)(f+2g)(f+3g)\text{ etc.},$$

ponatur ubique 1 loco *g*; erit ea

$$f+1,(f+1)(f+2),(f+1)(f+2)(f+3)\text{ etc.}$$

Sed loco *f* scribi debet in termino primo 0, in secundo 1, in tertio 3, in quarto 6 et ita porro; prodibit haec progressio

1, 2.3, 4.5.6, 7.8.9.10 etc.,

cuius igitur terminus generalis

$$\frac{2\int dx(-lx)^n}{(nn+n+2))\int x^{\frac{nn-n}{2}}dx(1-x)^n}=\frac{2\int dx(-lx)^n}{(nn+n+2))\int dx(x^{\frac{n-1}{2}}-x^{\frac{n+1}{2}})^n}.$$

§ 20. Accedo nunc ad eas progressiones, unde compendium illud definiendis terminis intermediis huius progressionis

1, 2, 6, 24, 120, etc.,

nactus sum. Id enim latius patet quam ad hanc solam progressionem, quoniam eius terminus generalis

$$\int dx(-lx)^n$$

etiam in infinitarum aliarum progressionum terminos generales ingreditur. Assumo hunc terminum generalem

$$\frac{f+(n+1)g}{g^{n+1}}\int x^{\frac{f}{g}}dx(1-x)^n,$$

cui respondet terminus ordine *n* hic [p.51]

$$\frac{1.2.3.4....n}{(f+g)(f+2g).....(f+ng)}.$$

Pono hic $f = n$, $g = 1$; erit terminus generalis

$$(2n+1)\int x^n dx(1-x)^n \text{ vel } (2n+1)\int dx(x-xx)^n$$

et forma eius ordine n

$$\frac{1.2.3.4....n}{(n+1)(n+2)(n+3)....2n}.$$

Progressio vero ipsa haec

$$\frac{1}{2}, \frac{1.2}{3.4}, \frac{1.2.3}{4.5.6} \text{ etc.}$$

vel haec

$$\frac{1.1}{1.2}, \frac{1.2.1.2}{1.2.3.4}, \frac{1.2.3.1.2.3}{1.2.3.4.5.6} \text{ etc.}$$

In qua numeratores sunt quadrata progressionis 1, 2, 6, 24, etc., inter denominatores vero duos proximos aequidistans facile invenitur. Sit in progressione 1, 2, 6, 24, etc. terminus, cuius index $\frac{1}{2}$, A; erit progressionis illius terminus ordine $\frac{1}{2} = \frac{AA}{1}$.

§ 21. Ponatur in termino generali

$$(2n+1)\int x^n dx(1-x)^n$$

$n = \frac{1}{2}$; erit terminus huius exponentis =

$$2\int dx\sqrt{(x-xx)} = \frac{AA}{1},$$

unde

$$A = \sqrt{1.2\int dx\sqrt{(x-xx)}}$$

= termino progressionis 1, 2, 6, 24 etc., cuius index est $\frac{1}{2}$, qui ergo, ut ex eo elucet, est radix quadrata ex circulo diametri 1. Dicatur nunc terminus huius progressionis ordine $\frac{3}{2}$ A; erit respondens in assumta progressione =

$$\frac{AA}{1.2.3} = 4\int dx(x-xx)^{\frac{3}{2}},$$

ergo

$$A = \sqrt{1.2.3.4\int dx(x-xx)^{\frac{3}{2}}}.$$

Simili modo invenietur terminus ordine $\frac{5}{2}$ =

$$\sqrt{1.2.3.4.5.6\int dx(x-xx)^{\frac{5}{2}}}$$

Ex quibus generaliter concludo terminum ordine $\frac{p}{2}$ fore =

$$\sqrt{1.2.3.4.5.6....(p+1)\int dx(x-xx)^{\frac{p}{2}}}.$$

Hoc igitur modo inveniuntur omnes termini progressionis 1, 2, 6, 24 etc., quorum indices sunt fractiones denominatore existente 2. [p.52]

§ 22. Porro in termino generali

$$\frac{(f+(n+1)g)}{g}\int x^{\frac{f}{g}}xdx(1-x)^{n}$$

pono $f = 2n$ manente $g = 1$; prodibit

$$(3n+1)\int dx(xx-x^{3})^{n}$$

terminus generalis huius progressionis

$$\frac{1}{3},\frac{1.2}{5.6},\frac{1.2.3}{7.8.9}\text{ etc.}$$

Multiplicatur ille per praecedentem $(2n+1)\int dx(x-xx)^{n}$; prodibit

$$(2n+1)(3n+1)\int dx(x-xx)^{n}\int dx(x^{2}-x^{3})^{n} .$$

Qui dabit hanc progressionem

$$\frac{1.1.1}{1.2.3},\frac{1.2.1.2.1.2}{1.2.3.4.5.6}\text{ etc.}$$

ubi numeratores sunt cubi terminorum respondentium progressionis 1, 2, 6, 24, etc. Huius progressionis terminus ordine $\frac{1}{3}$ sit A; erit respondens illius

$$\frac{\mathrm{A}^{3}}{1}=2(\tfrac{2}{3}+1)\int dx(x-xx)^{\frac{1}{3}}\int dx(xx-x^{3})^{\frac{1}{3}},$$

ergo terminus ordine $\frac{1}{3}$ est

$$\sqrt[3]{1.2\tfrac{5}{3}\int dx(x-xx)^{\frac{1}{3}}\int dx(xx-x^{3})^{\frac{1}{3}}} ;$$

similiter terminus ordine $\frac{2}{3}$

$$\sqrt[3]{1.2.3.\tfrac{7}{3}\int dx(x-xx)^{\frac{2}{3}}\int dx(xx-x^{3})^{\frac{2}{3}}} .$$

Atque terminus ordine $\frac{4}{3}$

$$\sqrt[3]{1.2.3.5.\tfrac{11}{3}\int dx(x-xx)^{\frac{4}{3}}\int dx(xx-x^{3})^{\frac{4}{3}}} ,$$

et generaliter terminus ordine $\frac{p}{3}$ est

$$\sqrt[3]{1.2....p.\tfrac{2p+3}{3}\int dx(x-xx)^{\frac{p}{3}}\int dx(xx-x^{3})^{\frac{p}{3}}} .$$

§ 23. Si ulterius progredi velimus ponendo $f = 3n$, oportebit terminum generalem

$$(4n+1)\int dx(x^{3}-x^{4})^{n}$$

in praecendentes multiplicare, unde habetur

$$(2n+1)(3n+1)(4n+1)\int dx(x-xx)^{n}\int dx(x^{2}-x^{3})^{n}\int dx(x^{3}-x^{4})^{n} ,$$

qui est pro hac serie

$$\frac{1.1.1.1}{1.2.3.4},\frac{1.2.1.2.1.2.1.2}{1.2.3.4.5.6.7.8}\text{ etc.}$$

Ex qua definientur termini progressionis 1, 2, 6, 24 etc., quorum indices sunt fractiones denominatorem 4 habentes. Nimirum terminus, cuius index est $\frac{p}{4}$, invenietur [p.52]

$$\sqrt[4]{1.2....p.(\frac{2p}{4}+1)(\frac{3p}{4}+1)(p+1)\times\int dx(x-xx)^{\frac{p}{4}}\int dx(xx-x^3)^{\frac{p}{4}}\int dx(x^3-x^4)^{\frac{p}{4}}}\,.$$

Hinc generaliter concludere licet terminum ordine $\frac{p}{q}$ esse

$$\sqrt[q]{1.2....p.(\frac{2p}{q}+1)(\frac{3p}{q}+1)(\frac{4p}{q}+1)......(p+1)\times}$$
$$\sqrt[q]{\int dx(x-xx)^{\frac{p}{q}}\int dx(xx-x^3)^{\frac{p}{q}}\int dx(x^3-x^4)^{\frac{p}{q}}.......\int dx(x^{q-1}-x^q)^{\frac{p}{q}}}\,.$$

Ex hac igitur formula termini cuiuscunque indicis fracti inveniuntur per quadraturas curvarum algebraicarum; ad id autem requiritur 1.2.3.4..... *p*, terminus, cuius index est numerator fractionis propositae.

§ 24. Eodem modo ulterius progredi licet ad progressiones magis compositas assumendis terminis generalibus magis compositis, sed ea longius non persequor. Possunt etiam signa integralia multiplicari, ut terminus generalis sit

$$\int qdx\int pdx\,;$$

nimirum integrale ipsius *pdx* debet multiplicari per *qdx* et quod resultat denuo integrari, id quod demum dabit facto $x = 1$ terminum seriei. In utraque autem integratione, ut sit determinata, oportet addenda constante efficere, ut posito $x = 0$ integrale fiat itidem = 0.

Similiter tractandi sunt termini generales, qui pluribus signis integralibus continentur, ut

$$\int rdx\int qdx\int pdx\,.$$

Attamen semper loco *p, q, r* etc. tales sunt sumendae functiones, ut, quoties *n* fuerit numerus integer affirmativus, prodeant termini ad minimum algebraici. [p.53]

§ 25. Sit terminus generalis

$$\int\frac{dx}{x}\int x^e xdx(1-x)^n\,;$$

hic in seriem conversus dat

$$\frac{x^{e+1}}{(e+1)^2}-\frac{nx^{e+2}}{1.(e+2)^2}+\frac{n(n\text{-}1)x^{e+3}}{1.2.(e+3)^2}-\text{etc.}$$

Posito $x = 1$ habebitur terminus ordine *n* per hanc seriem

$$\frac{1}{(e+1)^2}-\frac{n}{1.(e+2)^2}+\frac{n(n\text{-}1)}{1.2.(e+3)^2}-\text{etc.}$$

Progressio vero ipsa haec erit a termino, cuius index est 0, incipiens

$$\frac{1}{(e+1)^2}, \frac{(e+2)^2-(e+1)^2}{(e+2)^2(e+1)^2}, \frac{(e+3)^2(e+2)^2-2(e+3)^2(e+1)^2+(e+2)^2(e+1)^2}{(e+3)^2(e+2)^2(e+1)^2},$$

$$\frac{(e+4)^2(e+3)^2(e+2)^2-3(e+4)^2(e+3)^2(e+1)^2+3(e+4)^2(e+2)^2(e+1)^2\text{-}(e+3)^2(e+2)^2(e+1)^2}{(e+4)^2(e+3)^2(e+2)^2(e+1)^2},$$

etc.

Lex huius progressionis manifesta est et non indiget explicatione. Sit e = 0; erit

$$\int dx(1-x)^n = \frac{1\text{-}(1-x)^{n+1}}{n+1};$$

ergo terminus generalis est

$$\int \frac{dx\text{-}dx(1-x)^{n+1}}{(n+1)x},$$

progressio vero haec erit

$$\frac{1}{1}, \frac{4\text{-}1}{4.1}, \frac{9.4\text{-}2.9\text{+}4.1}{9.4.1}, \frac{16.9.4\text{-}3.16.9.1\text{+}3.16.4.1\text{-}9.4.1}{16.9.4.1} \text{ etc.}$$

Huius differentiae hanc constituent progressionem

$$\frac{\text{-}1}{4.1}, \frac{\text{-}9\text{+}4}{9.4.1}, \frac{\text{-}16.9\text{+}2.16.4\text{-}9.4}{16.9.4.1} \text{ etc.}$$

§ 26. In hac dissertatione ergo id, quod praecipue intendi, assecutus sum, nempe ut [p.55] terminos generales invenirem omnium progressionum, quarum singuli termini sunt facta ex factoribus in progressione arithmetica progredientibus, in quibusque numerus factorum ut libuerit ab indicibus terminorum pendeat. Quanquam autem hic semper numerus factorum indici aequalis positus sit, tamen, si is alio modo inde pendens desideretur, res nihil habet difficultatis. Index denotatus est littera n; si iam quis requirat, ut numerus factorum sit $\frac{nn+n}{2}$, alia operatione opus non est, nisi ut ubique loco n substituar $\frac{nn+n}{2}$.

§ 27. Coronidis loco adhuc aliquid, curiosum id quidem magis quam utile, adiungam. Notum est per $d^n x$ intelligi differentiale ordinis n ipsius x et $d^n p$, si p denotet functionem quampiam ipsius x ponaturque dx constans, esse homogeneum cum dx^n; semper autem, quando n est numerus integer affirmativus, ratio, quam habet $d^n p$ ad dx^n, algebraice potest exprimi; ut si $n = 2$ et $p = x^3$, erit $d^2(x^3)$ ad dx^2 ut $6x$ ad 1. Quaeritur nunc, si n sit numerus fractus, qualis tum futura sit ratio. Difficultas in his casibus facile intelligitur; nam si n est numerus integer affirmativus, d^n continuata differentiatione invenitur; talis autem via non patet, si n est numerus fractus. Sed tamen ope interpolationum progressionum, de quibus in hac dissertatione explacavi, rem expedire licebit. [p.56]

§ 28. Sit invenienda ratio inter $d^n(z^e)$ et dz^n posita dz constante, seu requiritur valor fractionis $\frac{d^n(z^e)}{dz^n}$. Videamus primo, qui sint eius valores, si n est numerus integer, ut postmodum generaliter illatio fieri possit. Sit $n = 1$, erit eius valor

$$ez^{e-1} = \frac{1.2.3....e}{1.2.3.....(e-1)} z^{e-1};$$

hoc modo e exprimo, ut facilius postea ea, quae tradita sunt, huc referantur.

Si $n = 2$, erit valor $e(e-1)z^{e-1} = \frac{1.2.3....e}{1.2.3.....(e-2)} z^{e-2}$, Si $n = 3$, habebitur

$$e(e-1)(e-2)z^{e-1} = \frac{1.2.3....e}{1.2.3.....(e-3)} z^{e-3}.$$

Hinc generaliter infero, quicquid sit *n*, fore semper

$$\frac{dn(z^e)}{dz^n} = \frac{1.2.3....e}{1.2.3.....(e-n)} z^{e-n}.$$

Est autem per § 14

$$1.2.3.....e = \int dx(-lx)^e \text{ et } 1.2.3.....(e-n) = \int dx(-lx)^{e-n}.$$

Quare habetur

$$\frac{d^n(z^e)}{dz^n} = \frac{\int dx(-lx)^e}{\int dx(-lx)^{e-n}}$$

vel

$$d^n(z^e) = z^{e-n} dz^n \frac{\int dx(-lx)^e}{\int dx(-lx)^{e-n}}.$$

Ponitur hic *dz* constans et $\int dx(-lx)^e$ ut et $\int dx(-lx)^{e-n}$ ita debent integrari, ut supra praeceptum est, et tum ponere oportet $x = 1$.

§ 29. Non necesse est, quomodo verum eliciatur, ostendere; apparebit id ponendo loco *n* numerum integrum affirmativum quemcunque. Quaeratur autem, quid sit $d^{\frac{1}{2}}z$, si sit dz constans. Erit ergo $e = 1$ et $n = \frac{1}{2}$. Habebitur itaque [p.57]

$$d^{\frac{1}{2}}z = \frac{\int dx(-lx)}{\int dx\sqrt{-lx}} \sqrt{z}dz.$$

Est autem

$$\int dx(-lx) = 1$$

et dicta area circuli *A*, cuius diameter est 1, erit

$$\int dx\sqrt{-lx} = \sqrt{A},$$

unde

$$d^{\frac{1}{2}}z = \sqrt{\frac{zdz}{A}}.$$

Proposita igitur sit haec aequatio ad quampiam curvam

$$yd^{\frac{1}{2}}z = z\sqrt{dy},$$

ubi *dz* ponitur constans, et quaeratur, qualis ea sit curva. Cum sit $d^{\frac{1}{2}}z = \sqrt{\frac{zdz}{A}}$, abibit ea aequatio in hanc $y\sqrt{\frac{zdz}{A}} = z\sqrt{dy}$,

quae quadrata dat

$$\frac{yydz}{A} = zdy;$$

unde invenitur

$$\frac{1}{A}lz = c - \frac{1}{y}$$

vel

$$ylz = cAy - A,$$

quae est aequatio ad curvam quaesitam.