The Solution of an Astronomical Problem: From three given Altitudes of a Fixed Star to find the Elevation of the Pole Star and the Declination of the Star.

LEMMA

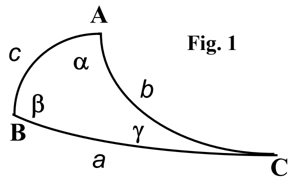


Fig. 1

In any spherical triangle *ABC* (Fig. 1), the cosine of the angle *A* is given by:
cos A = $\frac{\cos BC - \cos AB \cos AC}{\sin AB \sin AC}$, with the radius or the whole sine placed as 1.
[This is a standard result for spherical triangles defined in the 'Euler ' sense, so that all the angles are less than π; see any standard work on spherical triangles, e. g. *The VNR Concise Encyclopedia of Mathematics* (1975), p.262 onwards, which has some nice 3D effect diagrams, though it is rather dated; perhaps there is a newer edition. We have added $\alpha, \beta,$ and , and γ as the angles; as well as *a, b*, and *c* for the sides in Fig. 1 for convenience.
For reference, we give the standard cosine rules for sides and angles :
For sides: $\cos a = \cos b \cos c + \sin b \sin c \cos \alpha$; and likewise by permutation for the other sides.
For angles: $\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a$; and likewise for the other angles by permutation.
and the sine rule : $\sin b \sin c \sin \alpha = \sin c \sin a \sin \beta = \sin a \sin b \sin \gamma$.]

This is proven from these theorems that the most distinguished Professor MATER has set out in his Trigonometry.

COROLLARY

From these it follows that $\cos BC = \cos AB \cos AC + \cos A \sin AB \sin AC$.

THEOREM

In any spherical triangle ABC:

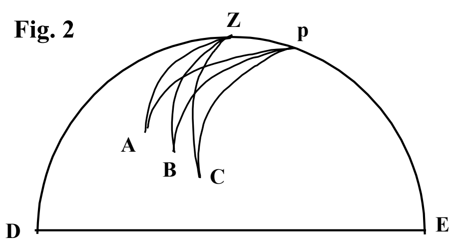
$$\cos BC = \frac{\cos(AB+AC)+\cos(AB-AC)}{2} + \frac{\cos A \cos(AB-AC) - \cos A \cos(AB+AC)}{2}.$$

With the whole sine taken as 1.
The product of two cosines is equal to half the sum of the cosines and the difference of the cosines of the arcs or angles. And the product of two sines is equal to half the cosine of the difference, with half the cosine of the sum of the arcs or angles taken away. As can be easily gathered or become apparent from these cited [For angles, the usual rules of plane trigonometry apply]. Hence :
$\cos BC \cos AC = \frac{\cos(AB+AC)+\cos(AB-AC)}{2}$, and
$\sin BC \sin AC = \frac{\cos(AB-AC)-\cos(AB+AC)}{2}$.

With these substituted into the corollary to the lemma, this gives the equation :
$\cos BC = \frac{\cos(AB+AC)+\cos(AB-AC)}{2} + \frac{\cos A\cos(AB-AC)-\cos A\cos(AB+AC)}{2}.$
Q.E.D.

PROBLEM

Fig. 2



For a given fixed star successively observed in three places (Fig. 2) *ABC*, with the altitudes or the complements of these *ZA, ZB, ZC*, and with the elapsed times between the observations given, or from the angles to the pole *P, APB* and *BPC*, to find the elevation of the pole star or the complement PZ of this, and the declination of the star, or the complements of this, at either AP, BP or CP.
[In modern terms roughly, with apologies to astronomers, the celestial coordinates of a star are given at some instant by two angles; the first is its right ascension or longitude in the celestial sphere, measured from the Vernal Equinox, represented here by the angles DPA, DPB, and DPC and usually expressed as a time, as the celestial sphere appears to rotate from east to west once every 24 hours; where Z is the local vertical or zenith and P is the position of the pole star; and the second angle is the latitude or the declination, measured by the angles or arcs AP, BP, and CP from the North Celestial Pole or pole star.]

SOLUTION

In the first place, the sine of the altitude or the cos *AZ*, is called *a* ; likewise cos *BZ* is *b,* and cos *CZ, c*; and the sin *APB* is *P*, the cosine of which is *p*; and sin *APC* is *Q*, the cosine of which is *q*. Moreover, the sin *ZPA* = *Z* and the cosine of this is *z*. Then on account of addition, cos *ZPB* = *r* and cos *ZPC* = *s*. Again, put cos(*ZP* + *AP*) = *x* and cos (*ZP* - *AP*) = *y*. We will now have in the spherical triangle *ZPA* :
$\cos AZ$ or $a = \frac{x+y+zy-zx}{2} = \frac{(1-z)x+(1+z)y}{2}$. [I]
And similarly in the triangle *ZPB,* $b = \frac{x+y+ry-rx}{2} = \frac{(1-r)x+(1+r)y}{2}$. [II]
And similarly in the triangle Z*PC*, $c = \frac{(1-s)x+(1+s)y}{2}$. [III]
In which from the three equations and from the three unknowns, it is necessary to determine *x, y*, and *z*.
Equations I and II give :

$$y = \frac{a(1-r)-b(1-z)}{z-r}.$$

And truly the second and third give :

$$y = \frac{b(1-s)-c(1-r)}{r-s}.$$

Thus on collation this equation is found:

$$a(1-r)(r-s) - b(1-z)(r-s) = b(1-s)(z-r) - c(1-r)(z-r).$$

Which changes to this : $a(1-r)(r-s)+c(1-r)(z-r)=b(1-r)(z-s)$, and on division by 1 - *r* gives $a(r-s)+c(z-r)=b(z-s)$. But from the construction of the sine, it follows that
r = *pz* - *P.Z* and *s* = *qz* - *Q.Z*. [These follow from the simple addition of cosines rule]
Hence we have :
$az(p-q)-aZ(P-Q)+cz(1-p)+cZP=b.z(1-q)+bZ.Q$.
From which this equation can be formed :

$$\frac{Z}{z}=\frac{a(p-q)-b(1-q)+c(1-p)}{aP-aQ+bQ-cP}=\frac{a(p-q)-b(1-q)+c(1-p)}{P(a-c)-Q(a-b)}.$$

Moreover, $\frac{Z}{z}$ is the tangent of the angle *ZPA*; this is called *T*, and also 1 - *p* = π and 1 - *q* = χ, where π and χ, denote the versed sines of the angles APB and APC.
Therefore, this equation arises : $T=\frac{a(\chi-\pi)-b\chi+c\pi}{P(a-c)-Q(a-b)}=\frac{\chi(a-b)-\pi(a-c)}{P(a-c)-Q(a-b)}$.
From which the angle *ZPA* can be determined, and the rest from that. Moreover $y=\frac{a(1-r)-b(1-z)}{z-r}$ and $x=\frac{b(1+z)-a(1+r)}{z-r}$ as is apparent from the preceding. Truly with the angle *ZPA* given, *ZPB* is given and hence *r*. Moreover,
$\frac{y+z}{2}=a-\frac{z(a-b)}{z-r}$ and $\frac{y-x}{2}=\frac{a-b}{z-r}$.
Hence *y* and *x* are easily found, the sum and difference of the cosines sought.
Q.E.D.

Here I put an example, which before I had computed from the altitude of the pole star assumed to be 54^0, 43', in order that I might investigate whether or not the same numbers might arise from this method. The first altitude is 71^0, 15', the second 68^0, 34', and the third 63^0, 54'. The time between the observations I and II or the angle *APB* is 7^0, 52'. The time between I and III or the angle APC is 20^0, 36'. Hence *a* = 9469502, *b* = 9308279, and *c* = 8979213. Hence *a* - *c* = 490289, *a* - *b* = 161223, again *P* = 1368683 and π = 94107, Q = 3518416, χ = 639404. Hence,
$\chi(a-b)-\pi(a-c)=5692700$ and $P(a-c)-Q(a-b)=10380060$.
Thus T is found: *T* = 5484423 = tang 28^0, 45'.
Hence the angle *ZPA* = 28^0, 44', and *ZPB* = 36^0, 37'. and thus we have :
cos *ZPA* = *z* = 8767267 and cos *ZPB* = *r* = 8026440.
Hence *z* - *r* = 0740727. Since indeed *a* - *b* = 162223, then
$\frac{a-b}{z-r}=2176264=\frac{y-x}{2}$. Hence $\frac{z(a-b)}{z-r}=1907988$. From this, with *a* = 9469502 taken away there remains : $\frac{y+x}{2}=7501514$.
Thus we find *y* = 9737778 and *x* = 5385250.
Hence the sum of the arcs is *AP* + *ZP* = 57^0, 25', and the difference of the arcs *AP* - *ZP* or *ZP* - *AP* = 13^0, 9'.
From these two values 35^0, 17' and 22^0, 8' are found for *ZP* and *AP* . And for the elevation of the pole star and the declination of the star, consequently these two which are the complements of these 54^0, 43' and 67^0, 52'. But from these it is not determined which shall be the declination and which the elevation of the pole star. Nevertheless it is surely the case that another elevation of the pole star gives rise to another declination of the star.

Hence indeed the time for the star to execute its motion is known : for how distant it stands is known from the angle *ZPA,* from the time of the first observation, since *PZ* is the arc of the meridian [South or midnight]. For this angle is *ZPA* = 28^0, 45', which on reduction to time gives 1 hour and 55 minutes, and to or from this, the time for the motion of the first observation by either added or subtracted as the circumstances require, and hence the final time can be found; if the sun itself it to take part in these observations, then the time of noon itself can be found.

SOLUTIO PROBLEMATIS ASTRONOMICI EX DATIS TRIBUS STELLAE FIXAE ALTITUDINIBUS ET TEMPORUM DIFFERENTIIS INVENIRE ELEVATIONEM POLI ET DECLINATIONEM STELLAE.

LEMMA

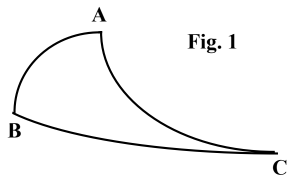


Fig. 1

In triangulo sphaerico (Fig. 1) quocunque ABC est cos anguli A $= \frac{\cos BC - \cos AB \cos AC}{\sin AB \sin AC}$, posito radio vel sinu toto 1.

Liquet hoc ex iis, quae Clar. Professor MATER in suis Trigonometris tradit.

COROLLARIUM

Ex his fluit esse $\cos BC = \cos AB \cos AC + \cos A \sin AB \sin AC$.

THEOREMA

In omni triangulo sphaerico ABC, est

$$\cos BC = \frac{\cos(AB+AC)+\cos(AB-AC)}{2} + \frac{\cos A \cos(AB-AC) - \cos A \cos(AB+AC)}{2}.$$

Posito sinu toto 1.

Factorum duorum cosinuum aequatur semissi cosinus summae cum semissi cosinus differentiae arcuum vel angulorum. Atque factum duorum sinuum aequae est semissi cosinus differentiae, dempta semissi cosinus summae arcuum vel angulorum. Ut ex iisdem citatis vel apparebit, vel facile colligetur. Erit igitur

$$\cos BC \cos AC = \frac{\cos(AB+AC)+\cos(AB-AC)}{2}$$

et

$$\sin BC \sin AC = \frac{\cos(AB-AC)-\cos(AB+AC)}{2}.$$

His ad aequationem in lemmatis corollario accommodatis prodibit

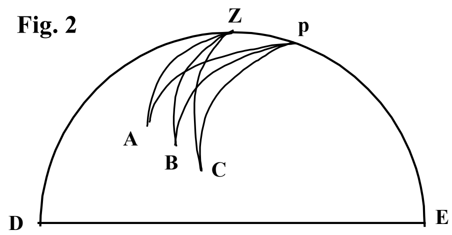
$$\cos BC = \frac{\cos(AB+AC)+\cos(AB-AC)}{2} + \frac{\cos A \cos(AB-AC) - \cos A \cos(AB+AC)}{2}.$$ Q.E.D.

PROBLEMA

Datis (Fig. 2) stellae fixae in tribus locis ABC successive observatae altitudinibus sive earum complementis ZA, ZB, ZC, temporibusque inter observationes praeterlapsis, vel angulis ad polum P, APB, BPC, invenire elevationem poli seu eius complementum PZ, et declinationem stellae seu eius complementum AP vel BP vel CP.

SOLUTIO

Fig. 2



Dicantur sinus altitudinis primae vel cos *AZ*, *a*; cos *BZ, b* et cos *CZ, c*. Atque sin *APB, P*, eiusque cosinus, *p*; sin *APC, Q*, eiusque cosinus, *q*. Sit autem sin *ZPA* = *Z* eiusque cosinus = *z*. Tum compendii causa sit cos *ZPB* = *r* et cos *ZPC* = *s*. Ponatur porro cos(*ZP* + *AP*) = *x* et cos (*ZP* - *AP*) = *y*. Habebitur in triangulo sphaerico *ZPA*,

cos *AZ* or $a = \frac{x+y+zy-zx}{2} = \frac{(1-z)x+(1+z)y}{2}$.

Deinde in triangulo *ZPB* est $b = \frac{x+y+ry-rx}{2} = \frac{(1-r)x+(1+r)y}{2}$.

Et similiter in triangulo *ZPC* erit $c = \frac{(1-s)x+(1+s)y}{2}$.

In quibus tribus aequationibus tres incognitas *x, y*, et *z* determinari oportet. Aequationibus I et II dabunt

$$y = \frac{a(1-r)-b(1-z)}{z-r}.$$

Secunda vero et tertia dant

$$y = \frac{b(1-s)-c(1-r)}{r-s}.$$

Unde colligitur haec aequatio

$$a(1-r)(r-s)-b(1-z)(r-s) = b(1-s)(z-r)-c(1-r)(z-r).$$

Quae abit in hanc, $a(1-r)(r-s)+c(1-r)(z-r) = b(1-r)(z-s)$, atque divisa per 1 - *r* dat $a(r-s)+c(z-r) = b(z-s)$. Sed ex constructione sinuum sequitur esse *r* = *pz* - *PZ* et *s* = *qz* - *QZ*. Unde habebitur

$$az(p-q)-aZ(P-Q)+cz(1-p)+cZP = bz(1-q)+bZQ.$$

Ex qua conficitur haec

$$\frac{Z}{z} = \frac{a(p-q)-b(1-q)+c(1-p)}{aP-aQ+bQ-cP} = \frac{a(p-q)-b(1-q)+c(1-p)}{P(a-c)-Q(a-b)}.$$

Est autem $\frac{Z}{z}$ tangens anguli ZPA; dicatur ea *T*, sitque etiam 1 - *p* = π et 1 - *q* = χ, denotabunt π and χ, sinus versos angulorum APB, APC. Eruitur igitur haec aequatio

$$T = \frac{a(\chi-\pi)-b\chi+c\pi}{P(a-c)-Q(a-b)} = \frac{\chi(a-b)-\pi(a-c)}{P(a-c)-Q(a-b)}.$$

Ex qua determinatur angulus *ZPA*, ex eoque reliqua. Est autem $y = \frac{a(1-r)-b(1-z)}{z-r}$ et $x = \frac{b(1+z)-a(1+r)}{z-r}$ ut ex praecedentibus apparet. Dato vero angulo *ZPA*, dabitur et *ZPB* et proinde *r*. Erit autem

$$\frac{y+z}{2} = a - \frac{z(a-b)}{z-r} \text{ et } \frac{y-x}{2} = \frac{a-b}{z-r}.$$

Hinc facile inveniuntur *y* et *x*, cosinus summae et differentiae arcuum quaesitorum. Q.E.D.

Exemplum hic appono, quod antea ex altitudine poli 54^0, 43' assumta computaveram, ut investigarem iidemne hac methodo ervantur numeri. Est altitudo prima 71^0, 15', secunda 68^0, 34', et tertia 63^0, 54'. Tempus inter I et II observationem seu angulus APB est 7^0, 52'. Tempus inter I et III seu angulus APC est 20^0, 36'. Erit ergo *a* = 9469502, *b* = 9308279, *c* = 8979213. Ergo *a* - *c* = 490289, *a* - *b* = 161223, porro *P* = 1368683 et π = 94107, Q = 3518416, χ = 639404. Erit

$\chi(a-b)-\pi(a-c)=5692700$ et $P(a-c)-Q(a-b)=10380060$.
Unde invenitur T = 5484423 = tang 28^0, 45'.
Est ergo angulus ZPA = 28^0, 44', ZPB = 36^0, 37'. Habetur itaque
cos ZPA = z = 8767267 et cos ZPB = r = 8026440.
Ergo z - r = 0740727. Cum vero sit a - b = 162223, erit
$\frac{a-b}{z-r}=2176264=\frac{y-x}{2}$. Deinde est $\frac{z(a-b)}{z-r}=1907988$. Hoc ab a = 9469502 ablato restat
$\frac{y+x}{2}=7501514$.
Hinc invenitur y = 9737778 et x = 5385250.
Est ergo summa arcuum AP + ZP = 57^0, 25', et differentia arcuum
AP - ZP vel ZP - AP = 13^0, 9'.
Ex his pro ZP et AP inveniuntur hi duo valores 35^0, 17' et 22^0, 8'. Et pro elevatione poli et declinatione stellae consequenter hi duo qui sunt illorum complementa 54^0, 43' atque 67^0, 52'. Quis autem horum sit pro declinatione aut elevatione poli, ex problemate non determintur. Id tamen certum est alterum elevationem poli, alterum declinationem stellae praebere.

Verum etiam hinc stellae tempus culminationis cognoscitur: distat enim a tempore primae observationis angulo ZPA, quia PZ est arcus meridiani. Inventus vero est ang. ZPA = 28^0, 45', qui ad tempus reductas dat 1 hor. 55', hocque tempore vel addendo vel subtrahendo a momento observationis primae, prout circumstantiae requirunt, invenitur tempus culminationis; si ipse sol in observationibus hisce adhibeatur, invenietur verum meridiei tempus.