L. EULER.
The Transformation of certain Differential Equations, which do not permit the separation of variables.

There are two kinds of constructions used to determine the magnitude of some unknown quantity from geometry; to the former can be referred all geometrical constructions such as those involving planes, solid shapes and curves ; the latter truly are relate to these transformations by means of which either the quadrature or rectification of curves can be brought about. We use these transformations in common geometry, in order that the roots of some kind of algebraic equation can be expressed ; and this is brought about by the intesection of curves or straight lines that we have agreed upon to use, as the equation set out requires. Truly the transformations of the latter kind, which can be called transcendental, are used in the resolution of differential equations, which cannot be transformed into algebraic equations. Moreover equations, whither algebraic or transcendental, to which two variable quantities belong, require transformations of this kind in order that, with some value to please set for one variable, the other can be determined. In order that this can be brought about for algebraic equations, it is established as a postulate that some algebraic function *Z* can be produced for a given magnitude *z*. Moreover for the above differential or transcendental equations , $\int Zdz$ is to be defined anew, in which Z denotes some function or other of z, either algebraic or transcendental, and as such can be considered with the given equation. Therefore on account of this, a proposed equation can often thus be transformed, in order that it can be equal to the other variable, or to some function of this variable, by which the whole transformation of that equation will be made evident. Moreover it is customery to call an equation of the transformed variable separated ; from which it is likewise to be understood, that a careful separation made in this manner is always required for transcendental equations. However this separation is not required in the setting out of algebraic equations. Indeed, for these with the variables mixed together, the whole business can be equally easy to complete. The truth is, for these transformations aimed at for solving differential equations, there is not a unique construction that can indeed be assigned, which can be performed or the variables to separated [in all cases]. Usually indeed all the transformations have been compared, for from these the separation of the variables themselves may follow spontaneously, even if the other has been found with the greatest difficulty. On this account, I have observed a method that is quite outstanding, that I fell upon recently while engaged in constructing the solution of a certain differential equation, for which the variables would not allow themselves to be separated from each other; and likewise I recognised that here there were more transformations to call upon than I had previously submitted to the usual observation. The first equation that occurred was of the form : $dy+\frac{y^2dx}{x}=\frac{xdx}{x^2+1}$ [This is a d. e. of the form $x(x^2+1)dy/dx+y^2(x^2+1)=x^2$, or related to the Riccati form; see Piaggio: *Differential Equations*, p. 201.]for which, not only was I unable to separate the variables from each other, but also the transformation itself shows that there is no place for a separation of this kind. If indeed a transformations should succeed, then it will be evident, that it is possible to show a comparison of the perimeters of the following dissimilar ellipses,

without indeed conceding the quadrature of the circle. Truly I transform the above equation in the following manner. An infinite number of ellipses are established on the same conjugate axis, which therefore can only be distinguished from each other by the transverse axis. A new curve can be constructed from these, in which, if the abscissae are taken from the transverse axis of the ellipses, the applied lines [or *y*-coordinates] will be equal to the peripheries of the same ellipses. With this accomplished, the constant conjugate axis is taken as 1, the abscissa [x-coordinate] of this new curve, or the transverse axis of this ellipse is made equal to *r*, & the applied line, or the perimeter of the same ellipse is equal to *q*.
[Note: For an ellipse written in the standard form
$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1$, this becomes $\frac{x'^2}{r^2}+\frac{y'^2}{1^2}=1$ or $x'^2=r^2(1-y'^2)$ for the family of ellipses; and the perimeter of such an ellipse *q* is a function of *r,* where
$q(r)=\int ds=\int dx'\sqrt{(1+(dy'/dx')^2)}=\int dx'\left[1+\frac{x'^2}{r^4y'^2}\right]^{\frac{1}{2}}=\int dx'\left[1+\frac{z^2}{r^4}\right]^{\frac{1}{2}}$, where $z=x'/y'$.
Now, $\frac{dq}{dr}=\int dx'\frac{1}{2}\left[1+\frac{z^2}{r^4}\right]^{-\frac{1}{2}}.-\frac{4z^2}{r^5}=-\frac{2}{r^5}\int dx'.Z^{-\frac{1}{2}}z^2$, where $Z=1+\frac{z^2}{r^4}$.
Similarly, $\frac{d^2q}{dr^2}=\frac{10}{r^6}\int dx'.Z^{-\frac{1}{2}}z^2-\frac{4}{r^{10}}\int dx'.Z^{-\frac{3}{2}}z^4$]
Now, these equations : $x=\sqrt{(r^2-1)}$ and $y=\frac{(r^2-1)dq}{qrdr}$ can be taken, which transformation is in short legitimate, since *q* is obtained from the given *r* by the rectification of a known curve.
[Thus, $x^2+1=r^2$, and $xdx=rdr$; thus the r.h.s. of the above d.e. becomes :
$\frac{xdx}{x^2+1}=\frac{rdr}{r^2}=\frac{dr}{r}$. Hence, $dy+\frac{y^2dx}{x}=\frac{xdx}{x^2+1}=\frac{dr}{r}$ giving: $\frac{dy}{dr}+\frac{y^2r}{r^2-1}=\frac{1}{r}$ now, assume a solution to be: $y=\frac{(r^2-1)dq}{qrdr}$ then $\frac{dy}{dr}=\frac{2}{q}\frac{dq}{dr}-\frac{r^2-1}{qr^2}\frac{dq}{dr}+\frac{r^2-1}{qr}\frac{d^2q}{dr^2}-\frac{r^2-1}{q^2r}(\frac{dq}{dr})^2$; for which $\frac{y^2r}{r^2-1}=\frac{r^2-1}{q^2r}(\frac{dq}{dr})^2$;
hence
$\frac{dy}{dr}+\frac{y^2r}{r^2-1}=\frac{1}{qr}(\frac{r^2+1}{r}\frac{dq}{dr}+(r^2-1)\frac{d^2q}{dr^2})$;
Thus, the transformed equation becomes $\frac{r^2+1}{r}\frac{dq}{dr}+(r^2-1)\frac{d^2q}{dr^2}=q$. Euler does not present this result, or go on to solve the resulting d. e.]
In a similar way I soon deduced the construction of that famous equation $ax^ndx=dy+y^2dx$, which the celebrated Count *Riccati* first brought to light, and set forth for examination by all the Geometers. Following this, certain lines of thought have become evident, moreover all of which are not related to this other equation, except as particular cases, or they have shown values that can be put in place of *n*, from which the equation is permitted to be separated and integrated. Truly no one, as far as I know, has indeed assigned a single case for which a solution can be found beyond those shown. Therefore, I can pass over to the universal transformation, for whatever *n* may signify, for which equations without the aid of my method hardly any transformations can be found : I can resolve this equation by the following account. That differential quantity

$n(n+4)dz(1-z^2)^{\frac{-n-4}{2n+4}} + 2dz(1-z^2)^{\frac{-n-4}{2n+4}}(c^{\frac{2z\sqrt{f}}{n+2}} + c^{\frac{-2z\sqrt{f}}{n+2}})$, in which z is a variable, f is a constant, and c the number for which the hyperbolic logarithm is 1, thus may be integrated, as with the factor $z = 0$, the whole can vanish. Because indeed the integral, even if the thing cannot be shown, is nevertheless constructed by quadrature, and thus can be considered as known. Hence in this integral z is put equal to 1, and a quantity is obtained, which will be a function of a certain f. Again the function $y = \frac{dP}{Pdx}$ can be written, which is in truth the value taken for y itself in the proposed equation : $ax^n dx = dy + y^2 dx$. Moreover, it is to be noted, that there is no solution in place when n lies between these terms 0 and -2. But there is an easy remedy for this inconvenience, thus, in order that nevertheless a transformation will be had for everything. Sine indeed, the use agrees with these , which *Daniel Bernoulli* has published concerning these equatons, for that equation, if it is separable in the case $n = m$, can also be separated in the case $n = \frac{-m}{m+1}$ or $n = -m-4$; it is evident that in all the cases between the limits 0 and -2 the contents of the equation can be reduced to the other cases taken within the limits -2 and -4, and on account of this are no longer to be excluded. Moreover I observe that the differential formula $n(n+4)dz(1-z^2)^{\frac{-n-4}{2n+4}} + 2dz(1-z^2)^{\frac{-n-4}{2n+4}}(c^{\frac{2z\sqrt{f}}{n+2}} + c^{\frac{-wz\sqrt{f}}{n+2}})$, as often as $\frac{-n-4}{2n+4}$ is either 0 or a positive whole number, so the whole thing itself can be integrated. For truly this happens whenever $n = \frac{-4K}{2K-1}$, with K denoting some positive whole number. Hence from these the equation, if the exponent x itself is $\frac{-n}{n+1}$, can be reduced to $ax^n dx = dy + y^2 dx$, that formula too is integrable if $n = \frac{-4K}{2K+1}$. And thus those cases emerges, now to be established from the other, for which the variables in a proposed equaton can be separated from each other.

L. EULERI. CONSTRUCTIO AEQUATIONUM

quarundam differentialium, que indeterminarum separationem non admittunt.

Constructiones, quibus Geometrae ad determinandas quasvis magnitudines utuntur, duplicis sunt generis; ad quorum alterum referri possunt omnes constructiones Geometicae, tam planae, quam solidae & lineares; ad alterum vero pertinent eae constructiones, quae vel quadraturis curvarum, vel rectificationibus, perficiuntur. Illas adhibemus in Geometria communi radices aequationum algebraicarum quarumcunque exprimendas; id quod efficitur, uti constat, intersectione linearum vel rectarum, vel curvarum, prout aequatio oblata postulat. Posterioris vero generis constructiones, quas transscendentes appellare licet inserviunt ad aequationes differentiales resolvendas, quae in algebraicas transmutari nequeunt. Aequationes autem, sive algebraicae, sive transscendentes, in quibus duae insunt quantitates indeterminatae, huiusmodi requirunt constructiones, ut, altera indeterminatarum pro lubitu assumta, altera determinetur; in quo efficiendo pro aequationibus algebraicis, tanquam postulatum, praemittitur, ut data magnitudine z, ijus quaecunque functio algebraica Z exhiberi. Pro differentialibus autem vel transcendentibus aequationibus insuper $\int Zdz$ in qua Z significat functionem quamcunque ipsius z, sive algebraicam, sive transcendentem, denuo definiri, atque adeo tanquam data considerari possit. Hanc ob rem igitur, quoties aequatio proposita ita potest transformavi, ut altera indeterminata, vel eius quaedam functio, aequalis esse debeat certae cuidam functioni alterius indeterminatae, toties illius aequationis constructio erit in promptu. Vocari autem solet ista aequationum transmutatio indeterminatarum separatio; ex quo simul intellegitur, quare semper ad aequationes transsendentes construendas hujusmodi separaratio tam solicite requiratur. In algebraicis quidem aequationibus haec separatio non est necessaria ad constructionem adornandam. Quomodocunque enim in iis indeterminatae sint permixtae, totum negotium aeque facile perficitur. Verum, quod ad differentiales aequationes attinet, ne unica quidem poterit assignari, quae construi, neque tamen separari, queat. Usitatae enim constructiones omnes ita sunt comparatae, ut ex iis ipsis separatio indeterminatarum, etiamsi alias fuerit inventu difficillima, sponte sequatur. Hanc ob rem non parum me praestitisse arbitror, cum nuper in constructiones aequationum quarundam differentialium, quae indeterminatas a se invicem separari non patiuntur, incidissem, simulque cognovissem, hac constructiones plus postulare, quam ante concedi folere observaveram. Prima aequatio, quae occurrebat, erat huius formae $dy+\frac{y^2dx}{x}=\frac{xdx}{x^2+1}$ in qua non solum indeterminatas a se invicem separare ego non possum, sed ipsa etiam constructio demonstrabit, huiusmodi separationem locum habere non posse. Si enim succederet, perspicuum erit, comparationem perimetrorum ellipsium dissimilium ex ea esse secuturam, quae tamen, ne circuli quidem concessa quadratura, exhiberi potest. Istam vero aequationem sequenti modo construo. Fiant super eodem axe conjugato infinitae ellipses, quae ergo solo axe transverso a se invicem discrepant. Ex his conficiatur nova curva, in qua, si abscissae aequales capiantur axibus ellipsium transversis, applicatae sint aequales peripheriis earundem ellipsium. Hoc facto, vocetur axis ille conjugatus constans *i*, abscissa huius novae curvae, seu axis transversus ellipsis cuius ponatur = *r*, & applicata, seu perimeter eiusdem ellipsis = *q*. Accipiatur hunc $x=\sqrt{(r^2-i)}$, eritque $y=\frac{(r^2-i)dq}{qrdr}$, quae constructio, quia *q* ex data *r* per rectificationem curvae cognitae habetur, prorsus est legitima. Simili modo deductus sum mox ad constructionem celebris huius aequatonis $ax^n dx=dy+y^2dx$, quam primum Cl. Comes *Riccati* in lucem produxit, cunctisque Geometris examinandam tradidit. Deinceps quidem variae comparverunt meditationes, quae autem omnes nihil aliud continent, nisi ut casus particulares, seu valores loco n substituendos, exhibuerint, quibus ista aequatio separationem & integrationem quoque admittet. Nemo vero, quantum scio ne unicum quidem assignavit casum, quo constructio perfici possit, praeter illos exhibitos. Ut taceam igitur universalem, quicquid *n* significet, constructionem, quae, nisi meae methodi beneficio, vix a quoquam poterit inveniri : sequenti ratione ego istam aequationem resolvo. Quantitas ista differentialis : $n(n+4)dz(1-z^2)^{\frac{-n-4}{2n+4}}+2dz(1-z^2)^{\frac{-n-4}{2n+4}}$

$(c^{\frac{2z\sqrt{f}}{n+2}} + c^{\frac{-wz\sqrt{f}}{n+2}})$, in qua *z* est variabilis, *f* constans, & *c* numerus, cuius logarithmus hyperbolicus est *i*, ita integretur, ut facto $z = 0$, tota evanescat. Quod quidem integrale, etiamsi re ipsa exhiberi nequeat, tamen per quadraturas construi, ideoque tanquam cognitum considerari poterit. In hoc deinceps integrali ponatur $z = 1$, & habebitur quantitas, quae erit functio quaedam ipsius *f*. Scribatur porro in hac functione $y = \frac{dP}{Pdx}$, qui est versu ipsius y valor in aequatione proposita $ax^n dx = dy + y^2 dx$. Notandum est autem, hanc solutionem locum non habere, quoties n fuerit numerus intra hos terminos 0 & -2 contentus. At huic incommodo facile remedium adhibetur, ita, ut ista constructio nihilominus pro universali sit habenda. Cum enim, uti constat ex iis, quae *Cl. Daniel Bernoulli* hac de aequations in publicum edidit, ista aequatio, si sit separabilis in casu $n = m$, separari quoque possit in casu $n = \frac{-m}{m+1}$ vel $n = -m-4$; perspicuum est, casus omnes intra limites 0 & -2 contentos reduci posse ad alios, qui intra limites -2 & -4 comprehenduntur, & hanc ob rem non amplius excluduntur. Observo autem, formulam illam differentialem $n(n+4)dz(1-z^2)^{\frac{-n-4}{2n+4}} + 2dz(1-z^2)^{\frac{-n-4}{2n+4}}(c^{\frac{2z\sqrt{f}}{n+2}} + c^{\frac{-wz\sqrt{f}}{n+2}})$, quoties $\frac{-n-4}{2n+4}$ sit vel 0 vel numerus integer affirmativus, toties re ipsa posse integrari. Hoc vero accidit quoties fuerit $n = \frac{-4K}{2K-i}$, denotante *K* numerum quemcunqur affirmativum integrum. Quis deinde aequatio, si exponens ipsius *x* est $\frac{-n}{n+i}$, ad hanc $ax^n dx = dy + y^2 dx$ potest reduci, erit illa formula quoque integrabilis, si fuerit $n = \frac{-4K}{2K+i}$. Atque sic prodeunt illi ipsi casus, iam ab alius eruti, quibus indeterminatae in aequatione proposita a se invicem possunt separari.