A NEW METHOD
BY MEANS OF WHICH INNUMERABLE DIFFERENTIAL EQUATIONS OF THE SECOND DEGREE CAN BE REDUCED TO DIFFERENTIAL EQUATIONS OF THE FIRST DEGREE.

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1.

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When differential equations of the second or of some higher degree present themselves for analysis, there are two ways in which they can be re-arranged for their solution. In the first place the analyst inquires whether these are ready to be integrated : if this is so, then the required integral can be found. Moreover, for an integration which is either absolutely impossible, or that seems to be even more difficult, the analyst tries to reduce these to differential equations of the first order ; obviously analysts are able to judge more easily from these whether a solution can be constructed ; and no differential equations other than first order, up to the present, can be constructed by known methods. As far as this is concerned, it is not the intention of this dissertation to consider the solution of first order equations; but rather [to establish] how differential equations of higher degree, especially how those of the second degree can be reduced to equations of the first degree; this is to be explained in the following by a certain unusal method which is of the widest applicability.

2. Now indeed mathematicians, when they have come upon differential equations of the second or higher degree, on many occasions have reduced these to differential equations of the first degree, and then constructed solutions; as one can see from the solutions of the catenary, [or problems involving] elastic shapes, projectiles in media with resistance, and of many other curves, for which first order equations have been found from second or third order equations. Indeed most of these are themselves integrable, but yet these are easier to integrate after they have been reduced to first order. Moreover the method of [constructing] these equations has thus been by comparison [of quantities], so that a distant or even more distant [value of a] variable can be ignored, and the equations are formed from so many differential ingredients of this or these differences of differentials [*i. e*. for first or second order differential equations].

3. However in a difference - of - differentials equation [which we will usually simply call a second order differential equation in this translation: the independent variable is the one with the first difference considered constant] with one or the other variable missing : it is easy to simply reduce the differential equation by substituting in place of a missing quantity a factor from a certain new variable in another differential. For by this reasoning, if a certain constant differential quantity were put in place, the difference of the differentials is simply found for the differential; and a differential equation of the first order is obtained. As for example in this equation : $\mathrm{P}dy^n = \mathrm{Q}dv^n + dv^{n-2}ddv$, where P and Q signify some functions of y, and dy is put constant. Since v itself shall not enter the equation, put $dv = zdy$, then $ddv = dzdy$. With these substituted, the equation itself becomes : $\mathrm{P}dy^n = \mathrm{Q}z^n dy^n + z^{n-2}dy^{n-1}dz$, and with this divided by dy^{n-1} : $\mathrm{P}dy = \mathrm{Q}z^n dy + z^{n-2}dz$; which is a simpler differential equation.

4. Except for equations of this kind, no one at any time hitherto as far as I know, has been able to reduce difference - of - differential equations to first order differential equations, except perhaps these that are completely ready to be integrated. With the method I present here, by which indeed not all but nevertheless innumerable second order differential equations in which other variable terms are present, can be reduced to simpler differentials. Thus truly for these to be reduced that I can change, I can transform these into other forms by a certain substitution, in which the other variable is missing. For with this done by means of the substitution, these equations set out in the preceeding § are entirely reduced to first order differentials.

5. For now I will observe a property of the exponential quantity, or a power of that dignified number, the exponent of which is a variable with the raised quantity remaining constant if it is to be differentiated, and again differentiated, then the finite variable of the exponent itself always remains unaffected ; and the differentiated parts are factors from the differentiation of the whole exponent. A quantity of this kind is c^x where c can denote the number, of which the logarithm is one; the differential of this is $c^x dx$, the second order differential is $c^x(ddx + dx^2)$, where only x in the exponent enters into the calculation. I have examined these, considering if exponentials of this kind could be substituted in place of indeterminate variables in second order equations : then the variables themselves are only present in the exponentials. Where it should be noted, as this exponential [function] can be applied in place of the indeterminate [variable] quantities by the substitution of a factor, that they can be removed by division; by this method even other indeterminate variables can be removed from the equation, as long as the differentiated quantities of these remain.

6. This operation does not indeed succeed with all equations; nevertheless I have observed that three kinds of second order differential equations are to be admitted. The first kind consists all these equations which only have two constant terms. Another consists of these equations, in which the indeterminate individual terms of equal dimension [*i. e.* having the same power] constitute a number: and truly not only the indeterminates alone, but also the differential coefficients of this and the order of each single dimension set up is to be judged. For the third kind I refer to these equations, in which the individual terms with other indeterminates maintain a number of the same dimension; they all follow the same line of thought, concerning the manner in which the the dimension they have produced is judged. Therefore, I will demonstrate here how to make the reduction for all the equations of these three kinds.

7. All the equations pertaining to the first kind are to be understood under this general formula : $ax^m dx^p = y^n dy^{p-2} ddy$, where dx is put constant [i. e. if dx is constant, then x is the independent variable running along the x-axis]. Even although in some equations neither dx nor dy can be taken as constant ; but each depends on a certain other variable, with which there is no difficulty by the known method, because the constant was a differential made into a variable and likewise for the other constant.
Truly in order to reduce this equation, I put $x = c^{\alpha v}$, and $y = c^v t$. Hence $dx = \alpha c^{\alpha v} dv$, and $dy = c^v(dt + tdv)$. And hence $ddx = \alpha c^{\alpha v}(ddv + \alpha dv^2)$ and
$ddy = = c^v(ddt + 2dtdv + tddv + tdv^2)$. But since dx is made constant then $ddx = 0$, and thus $ddv = -\alpha dv^2$. With this substituted in place of ddv, we have

$ddy = c^v(ddt + 2dtdv + (1 - \alpha)\, tdv^2)$. These values can be put in place of x and y in the proposed equation, and it is transformed into this equation:

$$ac^{\alpha v(m+p)}\alpha^p dv^p = c^{(n+p-1)v} t^n (dt + tdv)^{p-2}(ddt + 2dtdv + (1-\alpha)tdv^2)\,.$$

8. Now α can be determined thus, as by division the exponential terms can be taken away. In order that this can be done, it is necessary that $\alpha v(m + p) = (n + p - 1)v$, hence on collecting terms, $\alpha = \frac{n+p-1}{m+p}$. Therefore the above equation with α determined is changed into the following : $\alpha(\frac{n+p-1}{m+p})^p dv^p = t^n (dt + tdv)^{p-2}(ddt + 2dtdv + \frac{m-n+1}{m+p} tdv^2)$. Which might have been found immediately from the proposition, if I had put $x = c^{(n + p - 1)v:(m + p)}$, and $y = c^v t$. But $n + p - 1$ is the number of dimensions which constitutes y; and $m + p$ which constitutes x [from the original d.e.]. Hence in which case the particular α that is easily determined could have been substituted straight away. In the equation found, in place of v, put $dv = zdt$, then $ddv = zddt + dzdt$, but $ddv = -\alpha dv^2 = \frac{1-n-p}{m+p} z^2 dt^2$. Hence $ddt = \frac{-dzdt}{z} + \frac{1-n-p}{m+p} zdt^2$ is found. With these substituted there emerges : $a(\frac{n+p-1}{m+p})^p z^p dt^p = t^n (dt + tzdt)^{p-2}(\frac{1-n-p}{m+p} zdt^2 - \frac{dzdt}{z} + 2zdt^2 + \frac{m-n+1}{m+p} tzzdt^2)$. Which divised by dt^{p-1} gives $a(\frac{n+p-1}{m+p})^p z^p dt = t^n (1+tz)^{p-2}(\frac{1+2m-n+p}{m+p} zdt - \frac{dz}{z} + \frac{m-n+1}{m+p} tz^2 dt)$.

9. Therefore the proposed general equation $ax^m dx^p = y^n dy^{p-2} ddy$ has therefore been reduced to this first order differential equation :
$a(\frac{n+p-1}{m+p})^p z^{p+1} dt = tz^n (1+z)^{p-2}(\frac{1+2m-n+p}{m+p} z^3 dt + \frac{m-n+1}{m+p} tz^3 dt - dz)$, with the equation found multiplicatied by z. This equation can be found from that [first equation] by a single act, by $\int zdt$ being put in place of v in the first equation. Hence x and y become : $x = c^{(n+p-1)\int zdt:(m+p)}$ and $y = c^{(m+p)\int zdt} t$. If from the differential equation found, the proposed second order differential equation can be found again, we may see what kind of substitution in place of z and t ought to be used. When $x = c^{(n+p-1)\int zdt}$ then $c^{\int zdt} = x^{1:(n+p-1)}$: whereby $y = x^{(m+p):(n+p-1)} t$. Hence we have $t = yx^{-(m+p):(n+p-1)}$. Hence, since $c^{\int xdt} = x^{1:(n+p-1)}$ this gives $\int zdt = \frac{1}{n+p-1} lx$; hence $zdt = \frac{dx}{(n+p-1)x}$. But $dt = x^{-(m+p):(n+p-1)} dy - \frac{(m+p)}{n+p-1} yx^{-(m-n-2p+1):(n+p-1)} dx$. Consequently it is found that $z = dx : [(n+p-1)x^{-(m-n+1):(n+p-1)} dy - (m+p) yx^{-(m+p):(n+p-1)} dx]$. Moreover it is evident, that if z in t or t in z can also given a relation, as x and y have between each other, then it can be found. [Note that $a : b$ means a/b.]

10. We can demonstrate these which have been found in general by some particular example. Let $xdxdy = yddy$, which is reduced on division by dy, to this equation $xdx = ydy^{-1} ddy$. For this adaptation from the general equation, we have placed $a = 1$, $m = 1$, $p = 1$, $n = 1$. With these substituted in the first order differential equation that is obtained, to which the proposed equation is reduced:
$\frac{1}{2} zdt = t(1+tz)^{-1}(\frac{3}{3} z^2 dt + \frac{1}{2} tz^3 dt - dz)$, which will become
$z^2 dt + tz^3 dt = 3tz^2 dt + t^2 z^3 dt - 2tdz$. The proposed $xdxdy = yddy$ is reduced to this equation if x is made equal to $c^{\int zdt}$ and $y = c^{2\int zdt} t$. Therefore the construction of the

proposed equation depends on the construction of the differential equation found; if one can be done, then so will the other; for if this can itself be integrated, then that too will be integrated.

11. The second kind of difference - of -differentials equations [*i. e.* second order d. e.'s], that I can reduce to first order differential equations by my method, embraces these, which maintain a number of the same dimension for the individual terms, of which the variables and their differential terms are in agreement. The general equation pertaining to this is the following : $ax^m y^{-m-1}dx^p dy^{2-p} + bx^n y^{-n-1}dx^q dy^{2-q} = ddy$. With the individual terms of this equation, the variables have a single dimension: and dx is placed constant. Although this equation truly is agreed upon from only three terms taken : nevertheless any number of extra terms can be added as you please, with the same operation still remaining. Terms of the form $ex^r y^{-r-1}dx^q dy^{2-q}$ can thus be added as you please, as particular examples, to which the general equation to be reduced can be adapted, with more or less terms in agreement. However it is sufficient to take three terms, as I have said : since the method does not require more of the other kind to be reduced.

12. I can reduce the proposed equation with c^v substituted in place of x, and in place of y, $c^v t$. Therefore, since $x = c^v$ and $y = c^v t$; then $dx = c^v dv$ and $dy = c^v(dt + tdv)$: and again $ddx = c^v(ddv + dv^2)$ and $ddy = cv(ddt + 2dtdv + tdv^2 + tddv)$. Since truly dx is placed constant, then $ddx = 0$, from which therefore $ddv = -dv^2$, on account of which the equation is obtained: $ddy = c^v(ddt + 2dtdv)$. These values are put in place x, y, dx, dy and ddy in the equation, and it is transformed into the following:
$ac^v t^{-m-t}dv^p(dt + tdv)^{2-p} + bc^v t^{-n-1}dv^q(dt + tdv)^{2-q} = c^p(ddt + 2dtdv)$. Which divided with c^v removed, results in this equation:
$at^{-m-t}dv^p(dt + tdv)^{2-p} + bt^{-n-1}dv^q(dt + tdv)^{2-q} = ddt + 2dtdv$. In this since v shall be absent, I put $dv = zdt$ then $ddv = zddt + dzdt$, but $ddv = -dv^2 = -z^2dt^2$ hence $ddt = -zdt^2 - \frac{dxdt}{z}$. Hence the equation is obtained :
$at^{-m-1}z^p dt^p(dt + ztdt)^{2-p} + bt^{-n-1}z^q dt^q(dt + ztdt)^{2-q} = zdt^2 - \frac{dzdt}{z} + 2zdt^2$ or this equation, with more order: $at^{-m-r}z^p dt(1 + zt)^{2-p} + bt^{-n-1}z^q dt(dt + zt)^{2-q} = zdt - \frac{dzt}{z}$.

13. This differential equation of the first order can be obtained from the proposed equation in a single act, if x and y are replaced at once by $x = c^{\int zdt}$ and $y = c^{\int zdt}t$; to become thus : $dx = c^{\int zdt}zdt$ and $dy = c^{\int zdt}(dt + tzdt)$; and $ddx = c^{\int zdt}(zdtt + dzdt + zzdt^2) = 0$, whereby $ddt = -zdt^2 - dzdt : z$. With this called upon for use we will have $ddy = c^{\int zdt}(zdt^2 - dzdt : z)$.
This example is to be considered: $y^{\alpha+1}ddy = x^\alpha dx^2$, it can be changed into $ddy = x^\alpha y^{-\alpha-1}dx^2$. With this brought together with the general equation, there becomes : $a = 1$, $b = 0$, $m = a$, and $p = 2$. If this equation is therefore reduced as with the general formula, this equation is found : $t^{-\alpha-1}z^2dt = zdt - dz : z$. Or this: $t^{-\alpha-1}z^3dt = z^2dt - dz$. [These sorts of constructions] may be permitted, if the construction and a second order differential equation can to be made. It is to be observed that nearly always one comes

upon differential equations of such a kind that certainly they can only be resolved with difficulty, or which absolutely defy resolution.

14. I insert another example, $xdxdy - ydx^2 = y^2ddy$, which adopts this form of the general equation: $xy^{-2}dxdy - y^{-1}dx^2 = ddy$. Here the general equation is reduced, and a = 1, m = 1, p = 1, b = -1, n = 0, q = 2. Therefore the proposed example answers to the following differential equation : $t^{-2}zdt(1+zt)^{-t-1}z^2dt = zdt - dz : z$. This is multiplied by t^2z, and $z^2dt + z^3tdt - z^3tdt = z^2t^2dt - t^2dz$ or $z^2dt = zt^2dt^2 - t^2dz$ is obtained, which on separation gives $dz : z^2 = dt(t^2 - 1) : tt$ and on integration - 1: $z = t + 1 : t - a$ or $atz - t = t^2z + z$. Truly put $z = dv : dt$. Hence $atdv - tdt = t^2dv + dv$, or $dv = tdt(at - tt - 1)$. Since indeed $c^v = x$ then $v = lx$ and $t = y : x$ hence $dv = dx : x$ and $dt = (xdy - ydx) : xx$ consequently $ydy + xdx = aydx$. This equation can be integrated again, where truly I only note the case that if a = 0 then it will pass over into the equation of a circle.

15. I now accept the case where there shall be more terms, as in the general equation. $yydx^3 + xxdy^3 + yxdxdy^2 - yxdx^2dy + yx^2dxddy - y^2dxddy = 0$. This exemple can be reduced in the above manner. With dx placed constant, there remain the same substitutions of course, $x = c^v$, $y = c^vt$; $dx = c^vdv$; $dy = c^v(dt + tdv)$ and $ddy = c^v(ddt + 2dtdv)$. Truly $ddv = -dv^2$. With these put in place and with the equation put in order, it is found that $dt^3 + 2tdt^2dv - ttdv^2 + tdtdv^2 + tdvddt - ttdvddt = 0$. Since here v is to be missing, put $dv = zdt$, then as before the equation will be : $ddt = -zdt^2 - dxdt:z$. After that this equaton of the reduced order is found : $dt + 2tzdt - tdx + ttdz = 0$. Which, since z has only a single dimension, can be separated by the method set out by the celebrated Johan Bernoulli in the *Actis Lips*. But without any substitution, this equation and these of a similar kind, can be integrated straight away or reduced to an integrable form, by the following method.

16. Our equation can be reduced to this : $dz + \frac{2zdt}{t-1} + \frac{dt}{tt-t} = 0$, in order that dz is affected by the coefficient that it takes, by which z is affected, truly $\frac{2dt}{t-1}$ with the integral of this is expressed by $2\int\frac{dt}{t-1}$. Now the proposed equation is multiplied by $c^{2\int\frac{dt}{t-1}}$ and we have: $c^{2\int\frac{dt}{t-1}}dz + \frac{2c^{2\int\frac{dt}{t-1}}zdt}{1-t} + \frac{c^{2\int\frac{dt}{t-1}}dt}{tt-t} = 0$. Moreover the equation of the integration has been done, indeed the integral of the first two terms is $c^{2\int\frac{dt}{t-1}}z$. Therefore $c^{2\int\frac{dt}{t-1}}z + \int\frac{c^{2\int\frac{dt}{t-1}}dt}{tt-t} = a$. But $\int\frac{dt}{t-1} = l(t-1)$ hence $c^{2\int\frac{dt}{t-1}} = (t-1)^2$. Thus $(t-1)^2z + \int\frac{(t-1)dt}{t} = a$, and hence $(t-1)^2z + t - lt = a$. By this method all the differential equations for which either of the variables never has more than one dimensions, can be integrated or they can even be arranged to be reduced. Here, regarding the industrious method I have used, from which more can be understood of how great the uses of the exponential function shall be in solving equations.

17. The equation arrived at is this $(t-1)^2z + t - lt = a$. This can be reduced further, in order that an equation can at last be obtained between x and y : since $dv = zdt$ then

$z = dv : dt$; on account of which the equation will be changed from
$(t-1)^2 dv + tdt - dtlt = adt$ to this $dv = \frac{adt-tdt+dtlt}{(t-1)^2}$. Which again allows integration; the integration truly has this form $v = \frac{-a+t-tlt}{(t-1)}$, indeed with a constant added has this form, $v = \frac{b-a+t--bt-tlt}{(t-1)}$. Since truly $x = c^v$; then $v = lx$. And since $y = c^v t$ then $y = tx$ and therefore $t = y : x$. The following equation is obtained with these substituted :
$lx = \frac{bx-ax+y-by-yly+ylx}{y-x}$. From which this equation arises : $(b - a)x + (1 - b)y = yly - xlx$. For the sake of brevity, put $b - a = f$, and $1 - b = g$; then the equation becomes :
$fx + gy = yly - xlx$. Which is the integral for the proposed equation in §15. If we put $f = 0$, and $g = 0$, then $yly = xlx$. From which, with the numbers to be taken [from the logs], the equation $y^y = x^x$ is found.

18. Here I relate the method by which a third kind of equation can be reduced, these are embraced, in the individual terms of which each variable maintains the same number of dimensions. Here there are two cases to be distinguished, as either with the differential of that variable itself of the same dimensions put constant everywhere, or otherwise. In the first case, consider the following general equation :
$\mathrm{P}x^m dy^{m+2} + \mathrm{Q}x^{m-b} dx^b dy^{m+2-b} = dx^m ddy$. In which x hs the dimensions m in each term, amd dx is put constant. Moreover, P and Q signify some functions of y itself. This reduction can be effected with the help of a single substitution ; for x can be made equal to c^v and $dx = c^v dv$ and $ddx = c^v(ddv + dv^2)$; hence $ddv = -dv^2$. From these changes made $\mathrm{P}dy^{m+2} + \mathrm{Q}x^{m-b} dv^b dy^{m+2-b} = dv^m ddy$ is obtained. Obviously, after it has been divided by c^{mv}.

19. Since in the equation found, v is not kept, for the equation is reduced by substituting zdy in place of dv. Then it becomes : $ddv = zddy + dydz = -dv^2 = -z^2 dy^2$. Hence $ddy = -zdy^2 - dydz.z$ is found. Hence these values found are substituted into the equation, in place of dv and ddy, and the following equation is obtained :
$\mathrm{P}dy^{m+2} + \mathrm{Q}z^b dy^{m+2-b} = -dz^{m+1} dy^{m+2} - z^{m-1} dy^{m+1} dz$. Which results in this equation, on division by dy^{m+1} : $\mathrm{P}dy + \mathrm{Q}z^b dy = -dz^{m+1} dy - z^{m-1} dz$. Which is of first order, as was proposed. It is possible to reach this immediately, if x is put equal to $c^{\int zdy}$. From which we have :
$dx = c^{\int zdy} zdy$ and $ddx = c^{\int zdy}(zdy^2 + dzdy + zddy) = 0$ and hence $ddy = -zdy^2 - dxdy : z$.
The values from these are to be substituted in place of x, dx, ddy , and they immediately present the equation found .

20. The other case pertaining to an equation of the third kind considers the following general equation : $\mathrm{P}x^m dy^{m+1} + \mathrm{Q}x^{m-b} dx^b dy^{m-b+1} = dx^{m-1} ddx$. In which equation dy is put constant, P and Q designate some functions of y itself. And in order that all the individual tems of x are observed to have the dimension m. As before, z is put equal to c^v; then $dx = c^v dv$, and
$ddx = c^v(ddv + dv^2)$. With these substituted in the equation, this equation results on division by the factor c^{mv}, $\mathrm{P}dy^{m+1} + \mathrm{Q}dv^b dy^{m-b+1} = dv^{m+1} + dv^{m-1} ddv$. This equatio, in order to be further reduced, with v removed, dv is put equal to zdy since dy is constant,

and $ddv = dzdy$. This on account of the this, is finaly changed into :
$\mathrm{P}dy^{m+1} + \mathrm{Q}z^{b}dy^{m+1} = z^{m+1}dy^{m+1} + z^{m-1}dy^{m}dz$. Moreover this, if it is divided by dy^{m}, gives the equation: $\mathrm{P}dy + \mathrm{Q}z^{b}dy = z^{m+1}dy + z^{m-1}dz$. Therefore construction of the proposed equation follows from that found.

21. From these I can observe, it is understood, the manner in which differential equations of the second order pertaining to these three kinds ought to be handled to reduce them to the first order. I concede that it is indeed very rare to come upon such equations, in which neither variable is missing; However I think that nobody is going to argue over this discovery, on account of its usefulness. It may well be the case, as some new field of study is opened up, suggesting problems the resolution of which leads to such equations. I myself remember when certain physical problems came down to resolving this equation $y^2ddy = xdxdy$, for which then at the time neither by me nor by others with whom I communicated, could any way of resolving the problem be found. Now truly, since it belongs both to the first and the second kind, a reduction can be easily accomplished, as can be seen from §10.

22. Truly with this besides, concerning the constant to be assumed, I have brought a word of caution : For equations related to the second kind there is nothing between whatever differential shall be taken as constant. It can be either the differential of one or the other variable, or another differential from both variables with differentials freely composed, but which is homogeneous, as the nature of the problem requires. That indeed has been the general position from the example; but at the same time it is to be understood how, if the constant differential is to be any given quantity, then the equations are required to be tractable. The constant differential is still to be accounted for in the two remaining first and the third kinds. Indeed there it is necessary, as the constant can be put in place from either differentiable variable. If this is not done by the method explained, then the reduction will not succeed. Here truly the constant ought to be unchanged for these cases, and an equation can be transformed into another, in which either differentiable variable can remain the same.

23. The method expounded in this dissertation is concerned with the reduction of second order differential equations to simpler differential equations by the substitution of suitable exponential quantities for the variables. This truly is of wider application, as we have shown here. Endless differential equations of the third order are able to be reduced to others which are of second order, with the aid of this method. And in general, differential equations of order n are reduced to others of order as much as n - 1. Truly equation of this order of the differentials, which are reduced by this method, also are to be made up from three kinds, the same as those shown here. From these therefore it is to be considered, how much use substitutions of this kind might have in handling first order differential equations. But this is not the work to explain many things about these.

NOVA METHODUS
INNUMERABILES AEQUATIONES DIFFERENTIALES SECUNDI GRADUS REDUCENDI AD AEQUATIONES DIFFERERNTIALES PRIMI GRADUS.

Auctore
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1.

M. Sept. 1728.

Quando ad aequationes differentiales secundi vel altioris cuiuspiam gradus perveniunt analystae, in iis resoluendis duplici modo versantur. Primo inquirunt, an in promtu sit eas integrare : id si fuerit, obtinuerunt, quod desiderant. Cum autem integratio vel prorsus impossibilis, vel saltem difficultior videtur, conantur eas ad differentiales primi gradus reducere; quippe de quibus facilius iudicari potest, an construi queant ; nullaeque aequationes differentiales, nisi primi gradus, adhuc cognitis methodis construi possint. Quod ad illud attinet, de eo hac dissertatione explicare non est propositum; quomodo autem aequationes differentiales altiorum graduum praesertim vero secundi ad differentiales primi gradus sint reducendae, methodum quandam adhuc inusitatam, et quae latissime patet in sequentibus sum expositurus.

2. Iam quidem saepenumero Mathematici, quando aequationes differentiales secundi vel altiorum graduum occurrerunt, eas ad differentiales primi gradus reduxerunt, atque deinde constrexerunt; quemadmodum videre licet in constructionibus catenariae, elasticae, projectoriae in medio quocunque resistenti pluriumque aliarum curvarum, quarum aequationes primo differentiales secundi vel tertii gradus sunt inventae. Pleraeque quidem earum reipsa integrabiles sunt, sed tamen eas facilius erat integrare, postquam ad differentiales primi gradus fuerant reductae. Earum autem aequationum ratio ita est comparata, ut vel ultra vel saltem alterutra indeterminata ipsa desit, earum eiusve differentialibus et differentio - differentialibus aequationes tantum ingredientibus.

3. Si autem in aequatione differentio - differentiali alterutra indeterminata caret: facile est eam ad simpliciter differentialem reducere substituendo loco differentialis quantitatis deficientis factum ex nova quadam indeterminata in alterum differentiale. Hac enim ratione, si constans quoddam differentiale fuerit positum, differentio-differentiali aequale invenitur simpliciter differentiale; quo substituto aequatio habetur differentialis primi gradus. Ut in hac aequatione $\mathrm{P}dy^{n}=\mathrm{Q}dv^{n}+dv^{n-2}ddv$, ubi P et Q significant functiones quascunque ipsius *y*, atque *dy* constans ponatur. Quia ipsa *v* non ingreditur aequationem, fiat $dv=xdy,$ erit $ddv=dzdy$. His substitutis ista oritur aequatio
$\mathrm{P}dy^{n}=\mathrm{Q}z^{\mathrm{n}}dy^{n}+z^{n-2}dy^{n-1}dz$, divisaque hac per ista $\mathrm{P}dy=\mathrm{Q}z^{n}dy+z^{n-2}dz$; quae est simpliciter differentialis.

4. Alias aequationes differentio differentiales, nisi huiusmodi, nemo adhyc quantum scio, ad differentiales primi gradus unquam reduxit, nisi forre in promtu fuerit eas prorsus integrare. Hic autem methodum exponam, qua non quidem omnes, sed tamen innumerabiles aequationes differentio-differentiales utut ab utraque indeterminata affectae ad simpliciter differentiales reduci poterunt. Ita vero in iis reducendis versor, ut eas certa quadam substitutione in alias transformem, in quibus alterutra indeterminata de

est. Quo facto ope substitutionis § praeced. expositae eae aequationes penitus ad differentiales primi gradus reducentur.

5. Cum observassem eam esse quantitatum exponentialium, seu potius earum dignitatum, quarum exponens est variabilis manente quantitate elevata constante, proprietatem, ut si differentientur, denuoque differentientur, semper variabilis finita ipsa nonnisi exponentem afficiat; atque differentialia sint facta ex ipso integrali in differentialia exponentis. Quantitas huiusmodi est c^x ubi c denotet numerum , cuius logarithmus est unitas; erit eius differentiale $c^x dx$, differentio - differentiale $c^x(ddx + dx^2)$, ubi x nonnisi in exponentem ingreditur. Haec considerans perspexi, si in aequatione differentio - differentiali loco indeterminatarum huiusmodi exponentialia substituantur : tum ipsas variabiles tantummodo in exponentibus superfuturas esse. Quo cognito oportet, ut ea exponentialia loco indeterminatarum substituenda ita accommodentur, ut facta substitutione, ea divisione tolli queant; hoc modo alterutra saltem indeterminata ex aequatione eliminabitur, eiusque duntaxat differentialia supererunt.

6. Haec quidem operatio non in omnibus aequationibus succedit; verumtamen eam tria aequationum differentialium 2[di] gradus genera admittere observavi. Primum genus est omnium earum aequationum , quae nonnisi duobus constant terminis. Alterutrum eas comprehendit aequationes, in quarum singulis terminis indeterminatae aequalem dimensionum numerum constituunt : neque vero indeterminata ipsa solum, sed etiam eius differentialia cuiusque gradus dimensionem unam constituere existimanda sunt. Ad tertium genus eas refero aequationes, in quarum singulis terminis alterutra indeterminata eundem obtinet dimensionum numerum; quorsum eadem pertinent, quae modo de aestimatione dimensionum allata sunt. Omnes igitur aequationes ad haec tria genera pertinentes hic reducere docebo.

7. Omnes aequationes ad primum genus pertinentes sub hac generali formula comprehenduntur : $ax^m dx^p = y^n dy^{p-2} ddy$, ubi dx constans ponitur. Etsi enim in aequatione quapiam neque dx neque dy constans accipiatur; sed aliud quoddam differentiale inde pendens, id nihil difficultatis habet, cum cognita sit methodus, quod constans erat differentiale, variabile faciendi et vice eius aliud quoddam constans. Ad hanc vero aequationem reducendam pono $x = c^{\alpha v}$, et $y = c^v t$. Erit $dx = \alpha c^{\alpha v} dv$, et $dy = c^v(dt + tdv)$. Atque hinc $ddx = \alpha c^{\alpha v}(ddv + \alpha dv^2)$ et $ddy = = c^v(ddt + 2dtdv + tddv + tdv^2)$. Sed cum dx ponitur constans erit $ddx = 0$, adeoque $ddv = -\alpha dv^2$. Hoc substituto loco ddv, habevitur $ddy = c^v(ddt + 2dtdv + (1 - \alpha)\, tdv^2)$. Surrogentur hi valores loco x et y in aequatione proposita, transformabitur ea in hanc

$$\alpha c^{\alpha v(m+p)} \alpha^p dv^p = c^{(n+p-1)v} t^n (dt + tdv)^{p-2} (ddt + 2dtdv + (1-\alpha) tdv^2) .$$

8. Iam α determinari debet ita, ut exponentialia divisione tolli possint. Hoc ut fiat, oportet sit $\alpha v(m + p) = (n + p - 1)v$, inde colligitur $\alpha = \frac{n+p-1}{m+p}$. Superior igitur aequatio determinato α abibit in sequentem

$$a(\tfrac{n+p-1}{m+p})^p dv^p = t^n (dt + tdv)^{p-2} (ddt + 2dtdv + \tfrac{m-n+1}{m+p} tdv^2) .$$

Quae protinus ex proposita eruta fuisset, si posuissem $x = c^{(n + p - 1)v:(m + p)}$, et $y = c^v t$. Est autem $n + p - 1$ numerus dimension, quas y constituit; et $m + p$ quas x. Facile ergo in quovis casu particulari α determinatur statimque debita substitutio habebitur. In aequatione inventa, cum abset v, poaatur $dv = zdt$, erit $ddv = zddt + dxdt$, sed $ddv = -\alpha dv^2 = \frac{1-n-p}{m+p} z^2 dt^2$. Hinc invenitur

$ddt = \frac{-dzdt}{z} + \frac{1-n-p}{m+p} zdt^2$. His substitutis emergit
$a(\frac{n+p-1}{m+p})^p z^p dt^p = t^n (dt + tzdt)^{p-2} (\frac{1-n-p}{m+p} zdt^2 - \frac{dzdt}{z} + 2zdt^2 + \frac{m-n+1}{m+p} tzzdt^2)$. Quae divisa per dt^{p-1} dabit $a(\frac{n+p-1}{m+p})^p z^p dt = t^n (1+tz)^{p-2} (\frac{1+2m-n+p}{m+p} zdt - \frac{dz}{z} + \frac{m-n+1}{m+p} tz^2 dt)$.

9. Reducta ergo est aequatio generalis proposita $ax^m dx^p = y^n dy^{p-2} ddy$ ad hanc differentialem primi gradus
$a(\frac{n+p-1}{m+p})^p z^{p+1} dt = tz^n (1+z)^{p-2} (\frac{1+2m-n+p}{m+p} z^3 dt + \frac{m-n+1}{m+p} tz^3 dt - dz)$, multiplicata aequatione inventa per *z*. Haec aequatio unico actu ex ea inveniri potest, posito in prima substitutione loco *v* hoc $\int zdt$. Fieri ergo debet $x = c^{(n+p-1)\int xdt:(m+p)}$ et $y = c^{(m+p)\int zdt}$. Si ex aequatione differentiali inventa iterum proposita differentialis secundi gradus inveniri debeat, videamus quales loco *z* et *t* substitutiones adhiberi debeant. Cum sit $x = c^{(n+p-1)\int xdt:(m+p)}$ erit $t^{\int xdt} = x^{1:(n+p-1)}$: quare $y = x^{(m+p):(n+p-1)} t$. Unde habetur $t = yx^{-(m+p):(n+p-1)}$. Deinde quia $c^{\int xdt} = x^{1:(n+p-1)}$ erit $\int zdt = \frac{1}{n+p-1} lx$; ergo $zdt = \frac{dx}{(n+p-1)x}$. Sed est $dt = x^{-(m+p):(n+p-1)} dy - \frac{(m+p)}{n+p-1} yx^{-(m-n-2p+1):(n+p-1)} dx$. Consequenter invenietur $z = dx:[(n+p-1)x^{-(m-n+1):(n+p-1)} dy - (m+p) yx^{-(m+p):(n+p-1)} dx]$. Perspicuum autem est, si *z* in t vel *t* in *z* detur etiam relationem, quam *x* et *y* inter se habeant, inveniri posse.

10. Illustremus haec, quae generaliter inventa sunt exemplo quodam particulari. Sit *xdxdy* = *yddy*, quae reducitur dividendo per *dy*, ad hanc $xdx = ydy^{-1} ddy$. Huic generali accommodata, habebitur *a* = 1, *m* = 1, *p* = 1, *n* = 1. Substitutis his in aequatione differentiali primi gradus, habebitur ea, ad quam proposita reducitur,
$\frac{1}{2} zdt = t(1+tz)^{-1} (\frac{3}{3} z^2 dt + \frac{1}{2} tz^3 dt - dz)$, quae abit in $z^2 dt + tz^3 dt = 3tz^2 dt + t^2 z^3 dt - 2tdz)$.
Ad hanc aequationis proposita *xdxdy* = *yddy* si fiat $x = c^{\int zdt}$ et $y = c^{2\int zdt} t.$ Constructio ergo aequationis propositae pendet a constructione aequationis differentialis inventae; haec si construi poterit, et ea construetur; si fuerit reipsa integrabilis, ea quoque integrari poterit.

11. Secundum genus aequationum differentio-differentialium, quas mea methodo ad differentiales primi gradus reducere possum, eas complectitur, quae in singulis terminis eundem dimensionum, quas indeterminatae earumque differentialia constituunt, numerum tenent. Aequatio generlais huc pertinens est sequens
$ax^m y^{-m-1} dx^p dy^{2-p} + bx^n y^{-n-1} dx^q dy^{2-q} = ddy$. In huius singulis terminis est unica dimensio indeterminatarum: ponturque dx constans. Etsi vero aequatio haec assumta [assumpta] tribus tantum constat terminis: tamen quodcunque libuerit insuper adiici possunt, operatio enim eadem manet. Possent adhuc addi $ex^r y^{-r-1} dx^q dy^{2-q}$ et huiusmodi libuerit; prout exempla particularia, ad quae reducenda generalis accomodari debet, pluribus paucioribusve constant terminis. Tres vero terminos, ut dixi, assumsisse sufficit : cum plures alium reducendi modum non requirant.

12. Aequationem propositam reduco substituendis loco *z*, c^v et loco *y*, $c^v t$. Cum igitur sit $x = c^v$ et $y = c^v t$; erit $dx = c^v dv$ et $dy = c^v (dt + tdv)$: porroque
$ddx = c^v (ddv + dv^2)$ et $ddy = cv(ddt + 2dtdv + tdv^2 + tddv)$. Quia vero dx ponitur constans,

erit $ddx = 0$, hinc igitur $ddv = -dv^2$, hanc ob rem habebitur $ddy = c^v(ddt + 2dtdv)$. Ponatur hi valores in acquatione loco *x, y, dx, dy* et *ddy*, transformabitur ea in sequentem : $ac^v t^{-m-t} dv^p (dt + tdv)^{2-p} + bc^v t^{-n-1} dv^q (dt + tdv)^{2-q} = c^p(ddt + 2dtdv)$. Quae divisa per c^v abibit in hanc $at^{-m-t} dv^p (dt + tdv)^{2-p} + bt^{-n-1} dv^q (dt + tdv)^{2-q} = ddt + 2dtdv$. In hac cum desit *v* pono $dv = zdt$ erit $ddv = zddt + dzdt$, sed $ddv = -dv^2 = -z^2dt^2$ ergo $ddt = -zdt^2 - \frac{dxdt}{z}$. Hinc ista obtinitur aequatio, $at^{-m-1} z^p dt^p (dt + ztdt)^{2-p} + bt^{-n-1} z^q dt^q (dt + ztdt)^{2-q} = zdt^2 - \frac{dzdt}{z} + 2zdt^2$ seu haec ordinatior $at^{-m-r} z^p dt(1 + zt)^{2-p} + bt^{-n-1} z^q dt(dt + zt)^{2-q} = zdt - \frac{dzt}{z}$.

13. Aequatio haec differentialis primi gradus unico actu ex proposita elici potuisset, si statim positum esset $x = c^{\int zdt}$ et $y = c^{\int zdt} t$; unde foret $dx = c^{\int zdt} zdt$ et d$y = c^{\int zdt}(dt + tzdt)$; atque $ddx = c^{\int zdt}(zdtt + dzdt + zzdt^2) = 0$, quare $ddt = -zdt^2 - dzdt : z$. Hoc in usum vocoto habebitur $ddy = c^{\int zdt}(zdt^2 - dzdt : z)$. Propositum sit hoc exemplum $y^{\alpha+1} ddy = x^\alpha dx^2$, mutetur id in $ddy = x^\alpha y^{-\alpha-1} dx^2$. Collato hoc cum generali aequatione fiet $a = 1$, $b = 0$. $m = a$, $p = 2$. Si ergo hoc exemplum, ut generalis formulae, reductur, haec invenietur aequatio $t^{-\alpha-1} z^2 dt = zdt - dz : z$. Sive haec $t^{-\alpha-1} z^3 dt = z^2 dt - dz$. Quae si constructionem admitteret, et differentialis secundi gradus ex ea construi posset. Notandum est, semper fere ad eiusmodi aequationes differentiales perveniri, quae admodum difficulter vel prorsus non construi queant.

14. Assumo aliud exemplum, $xdxdy - ydx^2 = y^2 ddy$, quod ad modum generalis aequationis hanc induit formam $xy^{-2} dxdy - y^{-1} dx^2 = ddy$. Reducatur huc generalis aequatio, et erit $a = 1$, $m = 1$, $p = 1$, $b = -1$, $n = 0$, $q = 2$. Respondet ergo exemplo proposito sequens aequatio differentialis $t^{-2} zdt(1 + zt)^{-t-1} z^2 dt = zdt - dz : z$. Multiplicatur haec per $t^2 z$, habebitur $z^2 dt + z^3 tdt - z^3 tdt = z^2 t^2 dt - t^2 dz$ sive $z^2 dt = zt^2 dt^2 - t^2 dz$, quae separata dat $dz : z^2 = dt(t^2 - 1) : tt$ et integrata hanc - 1: $z = t + 1 : t - a$ sive $atz - t = t^2 z + z$. Est vero $z = dv : dt$. Itaque $atdv - tdt = t^2 dv + dv$, seu $dv = tdt(at - tt - 1)$. Quia vero $c^v = x$ erit $v = lx$ et $t = y : x$ ergo $dv = dx : x$ et $dt = (xdy - ydx) : xx$ consequenter $ydy + xdx = aydx$. Haec aequatio iterum integrari potest, cum vero tantum noto casum, quod si $a = 0$ ea transeat in aequationem circuli.

15. Accipio nunc casum, quo plures, quam in generali aequatione, sint termini $yydx^3 + xxdy^3 + yxdxdy^2 - yxdx^2 dy + yx^2 dxddy - y^2 dxddy = 0$. Hoc exemplum modo supra reducere licebit. Cum *dx* ponatur constans, maneant eaedem substitutiones scilicet, $x = c^v$, $y = c^v t$; $dx = c^v dv$; $dy = c^v(dt + tdv)$ et $ddy = c^v(ddt + 2dtdv)$. Est vero $ddv = -dv^2$. His substitutis atque aequatione proveniente ordinata, invenitur $dt^3 + 2tdt^2 dv - ttdv^2 + tdtdv^2 + tdvddt - ttdvddt = 0$. Hic cum desit *v*, ponatur $dv = zdt$, erit ut ante $ddt = -zdt^2 - dxdt{:}z$. Exinde reperitur haec aequatio in ordinem reducta, $dt + 2tzdt - tdx + ttdz = 0$. Quae, cum *z* unicam tantum habeat dimensionem separari potest methodo a Cel. Ioh. Bernoulli in Actis Lips. tradita. Sed sine ulla substitutione eam eique similes quascunque statim integrare seu ad integralem formam solum reducere possum, sequenti modo.

16. Reducatur aequatio nostra ad hanc $dz+\frac{2zdt}{t-1}+\frac{dt}{tt-t}=0$, ut *dz* nullo affectum sit coefficiente tum sumatur id, quo z est affectum, nempe $\frac{2dt}{t-1}$ cuius integrale exprimatur per $2\int\frac{dt}{t-1}$. Iam aequatio proposita multiplicetur per $c^{2\int\frac{dt}{t-1}}$ et habebitur $c^{2\int\frac{dt}{t-1}}dz+\frac{2c^{2\int\frac{dt}{t-1}}zdt}{1-t}+\frac{c^{2\int\frac{dt}{t-1}}dt}{tt-t}=0$. Nunc autem aequatio integrabilis est facta, duorum enim priorum terminorum integrale est $c^{2\int\frac{dt}{t-1}}z$. Est igitur $c^{2\int\frac{dt}{t-1}}z+\int\frac{c^{2\int\frac{dt}{t-1}}dt}{tt-t}=a$. Sed cum sit $\int\frac{dt}{t-1}=l(t-1)$ erit $c^{2\int\frac{dt}{t-1}}=(t-1)^2$. Ergo $(t-1)^2z+\int\frac{(t-1)dt}{t}=a$, hincque $(t-1)^2z+t-lt=a$. Hoc modo omnes aequationes differentiales in quibus alterutra variabilis una plures dimensiones nusquam habetm integrari seu saltem construibiles redduntur. Hac de industria methodo sum usus, quo magis intelligatur quanti sint usus exponentialia in tractandis aequationibus.

17. Aequatio ad quam est perventum haec est $(t-1)^2z+t-lt=a$. Haec ulterius reducatur, ut tandem aequatio inter *x* et *y* rursus obtineatur : quoniam erat $dv = zdt$ erit $z = dv : dt$; quamobrem aequatio in $(t-1)^2dv+tdt-dtlt=adt$ haec vero in $dv=\frac{adt-tdt+dtlt}{(t-1)^2}$. Quae denuo integrationem admitit; integrata vero hanc habet formam $v=\frac{-a+t-tlt}{(t-1)}$ constante vero addita hanc $v=\frac{b-a+t--bt-tlt}{(t-1)}$. Quia vero est $x = c^v$; erit $v = lx$. Et cum sit $y = c^vt$ erit $y = tx$ et ideo $t = y : x$. His substitutis habebitur sequens aequatio $lx=\frac{bx-ax+y-by-yly+ylx}{y-x}$. Unde oritur haec $(b - a)x + (1 - b)y = yly - xlx$. Ponatur brevitatis causa $b - a = f$, et $1 - b = g$; erit $fx + gy = yly - xlx$. Quae est integralis aequatio propositae § 15. Si fiat $f = 0$, et $g = 0$, erit $yly = xlx$. Ex qua sumendis numeris reperitur haec $y^y = x^x$.

18. Tertium genus aequationum quarum hic reducendarum methodum trado, eas complectitur, in quarum singulus terminis alterutra indeterminata eundem tenet dimensionum numerum. Hic duo distinguendi sunt cusau, prout vel ipsius illius variabilis ubique eundem dimensionum habentis differentiale constans ponitur vel secus. Ad primum casum spectat sequens aequatio universalis $\mathrm{P}x^mdy^{m+2}+\mathrm{Q}x^{m-b}dx^bdy^{m+2-b}=dx^mddy$. In qua *x* in singulis terminus m habet dimensiones, et dx ponitur constans. Significant autem P et Q functiones quascunque ipsius y. Ad hanc reducendam unica substitutione opus est ; nempe fiat $x = c^v$ erit $dx = c^vdv$ et $ddx = c^v(ddv + dv^2)$; ergo $ddv = -dv^2$. His subrogatis habetur $\mathrm{P}dy^{m+2}+\mathrm{Q}x^{m-b}dv^bdy^{m+2-b}=dv^mddy$. Postquam nimirum divisa est per c^{mv}.

19. Cum in aequatione inventa v non deprehendatur reducetur substituendo loco *dv*, *zdy*. Erit $ddv = zddy + dydz = -dv^2 = -z^2dy^2$. Hinc invenietur $ddy = -zdy^2 - dydz.z$. substituantur ergo in aequatione inventa loco dv et ddy hi volores reperti et habebitur haec aequatio $\mathrm{P}dy^{m+2}+\mathrm{Q}z^bdy^{m+2-b}=-dz^{m+1}dy^{m+2}-z^{m-1}dy^{m+1}dz$. Quae divisa per dy^{m+1} abit in hanc $\mathrm{P}dy+\mathrm{Q}z^bdy=-dz^{m+1}dy-z^{m-1}dz$. Quae est primi gradus, ut erat propositum. Ad hanc statim perveniri potuisset, si positum esset $x=c^{\int zdy}$. Unde foret

$dx = c^{\int zdy} zdy$ et $ddx = c^{\int zdy}(zdy^2 + dzdy + zddy) = 0$ et hinc $ddy = -zdy^2 - dxdy : z$. His valores loco *x, dx, ddy* substituti statim inventam aequationem praebent.

20. Alter casus aequationum aequationum ad genus tertium pertinentium respicit sequentem generalem generalem aequationem. $\mathrm{P}x^m dy^{m+1} + \mathrm{Q}x^{m-b}dx^b dy^{m-b+1} = dx^{m-1}ddx$. In qua aequatione dy ponitur constans, P et Q designant functiones ipsius y quascunque. Et ut perspicuum est *x* in singulis terminis *m* tenet dimensiones. Ponatur, ut ante, $z = c^v$; erit $dx = c^v dv$, et $ddx = c^v(ddv + dv^2)$. Hisce in aequatione substitutis, resultat haec aequatio divisione facta per c^{mv}, $\mathrm{P}dy^{m+1} + \mathrm{Q}dv^b dy^{m-b+1} = dv^{m+1} + dv^{m-1}ddv$. Haec aequatio, vt ulterius reducatur, cum *v* desit, ponatur $dv = zdy$ ob *dy* constans $ddv = dzdy$. Hance ob rem aequatio ultima transmutabitur in

$\mathrm{P}dy^{m+1} + \mathrm{Q}z^b dy^{m+1} = z^{m+1}dy^{m+1} + z^{m-1}dy^m dz$. Haec autem, si dividatur per dy^m, dabit istam $\mathrm{P}dy + \mathrm{Q}z^b dy = z^{m+1}dy + z^{m-1}dz$. Pendet ergo constructio propositae aequationis a constructione huius inventae.

21. Ex hisce, arbitror, intelligitur, quomodo aequationes differentiales secundi gradus ad unum truim expositorum genus pertinentes tractari oporteat. Facile quidem concedo raro admodum ad tales aequationes perveniri, in quibus non alterutra indeterminata desit; Tamen a nemine hoc nomine utilitatem huius inventi impugnatum iri puto. Fieri potest, ut nouus aliquis campus aperiatur problemata suggerens quorum resolutio ad aequatones tales deducat. Memini me aliquando physicum problema quoddam resoluentem ad hanc pervenisse aequationem $y^2 ddy = xdxdy$. Qua tum temporis neque a me neque ab aliis, cum quibus communicaveram, ulla modo reduci potuerat. Nunc vero, cum et ad primum et ad secundum genus pertineat, reductio facile successit ut ex § 10 videre licet.

22. Hoc vero praeterea de assumenda constante monendum duxi : In aequationibus ad secundum genus relatis nihil interest, quodcunque differentiale constans sit assumtum. Potest id esse vel differentiale alterutrius variabilis, vel aliud differentiale ex utriusque variabilis differentialibus ut libet compositum, modo id sit, ut natura rei exigit, homogeneum. Illud quidem in generali exemplo locum obtinuit; sed ex illa operatione simul intelligitur, quemodo, si differentiale constans sit qualecunque, aequationes tractari oporteat. Aliter res se habet in duobus reliquis generibus primo et tertio; ibi enim necesse est, ut alterutrius variabilis differentiale constans sit positum. Id si non fuerit methodo exposita reductio non succedit. Hic vero in casibus constans debet immutari, et aequatio in aliam transformari, in qua alterutrius variabilis differentiale sit constans.

23. Methodus in hac dissertatione exposita aequationes differentiales secundi gradus ad simpliciter differentiales reducendi consistit in idonea substitutione quantitatum exponentialium pro indeterminatis. Ea vero adhuc latius patet, quam hic est expositum. Possunt eius beneficio infinitae aequationes differentiales tertii ordinis ad alias, quae sint tantum secundi ordinis reduci. Et generaliter aequationes differentiales ordinis *n* as alias reducentur, quae ordinis tantum *n - 1*. Aequationum vero cuiusque ordinis differentialum, quae hac methodo reducuntur, quoque sunt tria genera constituenda, eademque, quae hic sunt exposita. Ex his igitur etiam intelligitur, quantum huiusmodi substitutiones in aequationibus differentialibus primi gradus tractandis usum habere possint. Sed de his non opus est plura exponere.