Concerning the shortest line on any surface by which any two points can be joined together.

Author
Leonard Euler.

1.

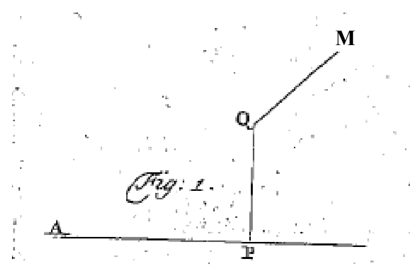
It is well-known that the shortest line or path from a give point to some other point is a straight line, which is considered as an axion by many writers. It is easily understood from this that when the surface is a plane, the shortest distance joining any two points [in the plane] is the straight line drawn from one to the other. On a spherical surface, on which it is not possible to draw straight lines, it has been established by the geometers that the shortest path between two given points is the [shorter arc of the] great circle joining them.

2. However, for any surface either convex or concave, without being a mixture of the two, what the shortest path shall be, drawn from one given point to any other, has not yet generally been determined. The most celebrated Johan Bernoulli has proposed this question to me, indicating that he himself has found the general equation, in order that the shortest line to be applied to a given surface between any two given points can be found. I too have solved this problem, and I want to set out the solution in this dissertation.

3. Mechanically this problem is easily solved with the help of a thread which is stretched between the two given points : the length it becomes will designate the shortest path on the proposed surface. Moreover it is necessary that the surface is convex, in order that this thread touches the surface everywhere, for with concave surfaces the shortest length is not represented by the arc of a curve but indeed by the chord [joining the points; though this chord does not lie on the surface]. Therefore in this case the thread ought to be applied thus, or to be so considered in this application, that it always touches the surface in a convex part.

4. Truly, anyone who wishes to examine the nature of the innermost secrets of this line, and who is accustomed to having an equation set up, cannot be satisfied with this geometrical construction. Moreover, the line sought that has been seen from a mechanical construction is hardly one that is set out [in a mathematical sense], and neither can the nature of the line be examined. On account of this, I am going to present a method by which the shortest lines [joining two points on the surface] can be determined for all surfaces, as long as they can be expressed by equations.

5. This [expression by equations] is therefore useful, as the natures of the surfaces are included by the equatons, from which the whole analytical operation can be resolved. Curved lines situated in the same plane are usually expressed by equations between two coordinates, from which the position of a point is defined, along the longitude and latitude of these coordinates. But for surfaces, three relations of the positions are to be considered, for the locus of any point on the surface must be determined along three dimensions. It is therefore appropriate for three variables to be used in the equations of surfaces; of which one variable acts along the longitude, another along the latitude, and a third along the altitude, in order to determine the position of a point on the surface.



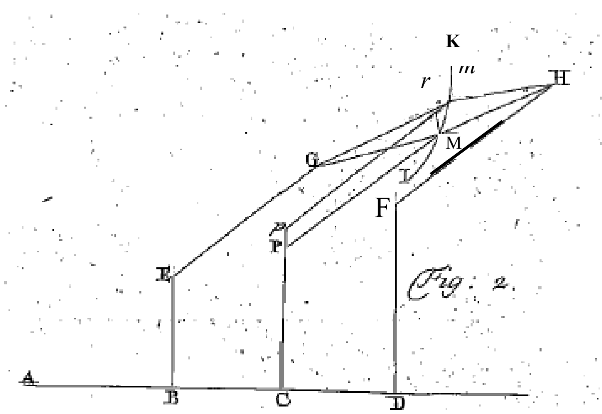
6. A plane is considered, which is congreunt with the plane of the page, and which we will call horizontal, and on this plane for argument's sake a line AP is drawn, which is to be considered as an axis. Now let M be a point of some surface situated above this plane, and from that point a perpendicular MQ is to be sent crossing the plane in Q, and from Q a perpendicular is drawn to the line or axis AP. Now it is evident that the position of the point M has been determined by the three given lines, with magnitudes AP, PQ, and QM.

7. Therefore these three lines AP, PQ and QM are our variables [Euler calls them indeterminates at this stage.], from which the equation for the surface is constructed from points with fixed values terminating in M. We will call AP, t, PQ, x, and QM, y, and for any surface, for which some quantity is sought, it is necessary to investigate the equation between these variable quantities. Hence properties of the surface will be gathered by the same method from such equations, and from which properties of these curves are derived. Thus, if the surface were a sphere, with centre A and radius equal to a, then an equation of this kind holds [between the variables] : $aa = tt + xx + yy$.

8. Again, just as a certain point on a curved line [in two dimensions] is to be determined either by the value of one or the other variable being assigned, or by some other equation being solved together with the local equation. Thus, with surfaces [in three dimensions], if certain values of the three variables are determined [in order to define a point on the surface], or some other equation is solved with the equation defining the surface, then an equation can be obtained for some line situated on that surface, which is formed by the intersection of the given surface with the other newly expressed equation. Then a point can be established fixed in the surface, either from the determination of the two variables [as one of the variables is given a fixed value], or from the solution of two new equations [from the intersection of two lines on the surface].

9. On account of the business of determining the shortest line drawn on some surface, of which the equation is known, I will investigate the equation of another line to be drawn, which when it intersects with that first line forms the shortest possible line sought. Hence,everything pertaining to knowledge about the shortest line can then be elicited from these two equations. From a horizontal projection an equation can be defined from these two lines from which it emerges that y can be eliminated. From a projection in a vertial plane passing through AP, the elimination of x is given. And from a projection in a vertical plane and perpendicular to AP, the letter t can be eliminated.

10. Now it is necessary to use the *method of maxima and minima* to solve this problem as the question itself demands. Moreover in the given surface, between all the lines having the same endpoints, that line is sought which has a minimum length. This property of being a minimum shall not only be agreed upon by the whole line sought, but also for every small part of this line [in between]; thus two contiguous elements of this line can designate the shortest path between their terminal points. From this, therefore there arises an easier method for arriving at the equation sought.



11. I now present the following lemma related to determining the shortest path between the positions of the two elements put in place. Let I and H be two fixed points [on the surface], and there is this [other] extended curve between I and K. That point M is sought such that, as that path, (with the lines GM and MH drawn) GM + MH is a minimum for all the paths which can be drawn through the other points of the curve. It is noted that the *method of the maximum and minimum* should be applied, let *m* be a point near to M itself, GM + MH = G*m* + *m*H, and from this equation the place of the point M is to be found, through which the path crossing GM + MH is a minimum.

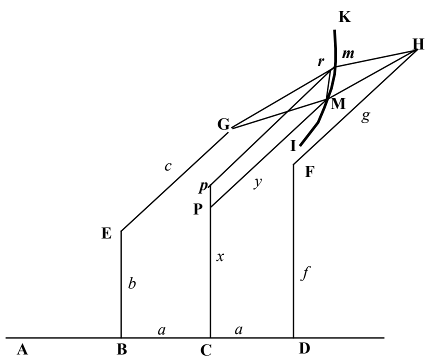


Fig. 2 (modified).

12.The perpendiculars GE, HF, MP, and *mp*, are sent from the points G, H, M, and *m* to the horizontal plane, [recall that we are looking down onto the plane of the diagram], *p*P is produced in C and this is joined to the normal AC situated on the horizontal plane, which is considered as the axis; and to this the perpendiculares EB and FD are drawn. We put BC and CD to be equal, such indeed as we are allowed to assume in the following. Let BC = CD = a; BE = b; EG = c ; DF = f ; FH =g. Again let CP = x and PM = y, which are the coordinates of of the point M on the curve IK. Therefore Cp = $x + dx$ and $pm = y + dy$, [giving the co-ords of *m,* a point on the curve IJK, as $(t, x + dx, y + dy)$, which is taken in a plane normal to the axis AC. Thus, the fixed point G has co-ords $(t - a, b, c)$; M is the variable point (t, x, y) ; and H is the fixed point $(t + a, f, g)$; on taking A as the origin, and AC = t; $mr = dy$ and M$r = dx$.]

13. From these, GM is found to equal $\surd[a^2 + (x - b)^2 + (y - c)^2]$: indeed $GM^2 = (PM - GE)^2 + (CP - BE)^2$. Similarly we have HM = $\surd[a^2 + (f - x)^2 + (g - y)^2]$. The whole path is therefore GM + MH = $\surd[a^2 + (x - b)^2 + (y - c)^2] + \surd[a^2 + (f - x)^2 + (g - y)^2]$, which quantity hence should have a minimum value. The variable quantites of this path are x and y, upon which the point M sought depends. Therefore this quantity expressing GM + MH is differentiated, and what arises is set equal to zero. This equation is produced : $\frac{(x-b)dx+(y-c)dy}{\sqrt{[a^2+(x-b)^2+(y-c)^2]}} = \frac{(f-x)dx+(g-y)dy}{\sqrt{[a^2+(f-x)^2+(g-y)^2]}}$. From which the locus of the point M can be determines.

14. Because the given curve IK is put in place, the equation between the coordinates x and y of this curve are given [in the intersection of the x, y plane with the known surface]: But there is a need for a differential equation, therefore we may put the relation of the elements dx and dy to be given by this equation in the form: Pdx = Qdy, or $dx : dy$

= Q : P. Now with these values of the proportions in place of *dx* and *dy*, an equation is produced [from the first diff. equation expressing the minimum value]:
$\frac{(x-b)\mathrm{Q}+(y-c)\mathrm{P}}{\sqrt{[a^2+(x-b)^2+(y-c)^2]}}=\frac{(f-x)\mathrm{Q}+(g-y)\mathrm{P}}{\sqrt{[a^2+(f-x)^2+(g-y)^2]}}$ which is devoid of differential quantities.

15. Now we will examine the lines GM and MH as two elements of the shortest line on the surface, in which the point G and H and the curve IK have been taken to be drawn. We put AC = *t*, and now the quantities are CP = *x* and PM = *y*. Let BC = CD = *a* = *dt*; DF = *f* = *x* + *dx*; FH = *g* = *y* + *dy*; BE = *b* = *x* - *dx* + *ddx*; EG = *c* = *y* - *dy* + *ddy*. [Thus, the IMK curve is give some curvature, and the following equation is symmetric.] These values are substituted for *a, b, c, f* and *g* in the equation found above, and this equation arises :

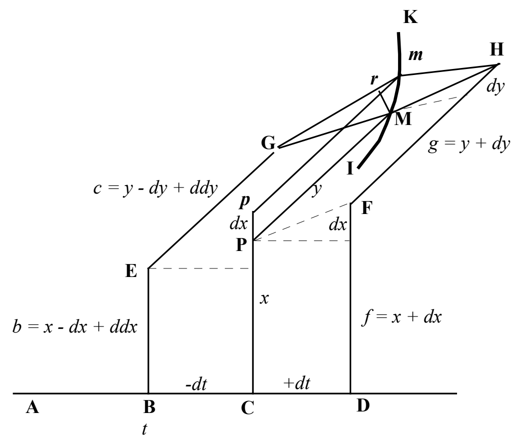


Fig. 2 (modified further).

$\frac{\mathrm{Q}(dx-ddx)+\mathrm{P}(dy-ddy)}{\sqrt{[dt^2+(dx-ddx)^2+(dy-ddy)^2]}}=\frac{\mathrm{Q}dx+\mathrm{P}dy}{\sqrt{[dt^2+dx^2+dy^2]}}$. [Note that this equation defines a turning point, as both sides have the same value for an incremental change in position; hence, normal differentiation of the r.h.s. is equivalent to subtraction of one side from the other to give zero for the derivative.]

16. The only significance of this equation is that the differential of this quantity $\frac{\mathrm{Q}dx+\mathrm{P}dy}{\sqrt{[dt^2+dx^2+dy^2]}}$ can be made equal to zero, with P, Q, and *dt* made constant (indeed *dt* from the construction is made constant) ; hence we have this equaton from this differentiation :

$$(\mathrm{Q}ddx+\mathrm{P}ddy)\sqrt{(dt^2+dx^2+dy^2)}=(\mathrm{Q}dx+\mathrm{P}dy)(dxddx+dyddy):\sqrt{(dt^2+dx^2+dy^2)}$$

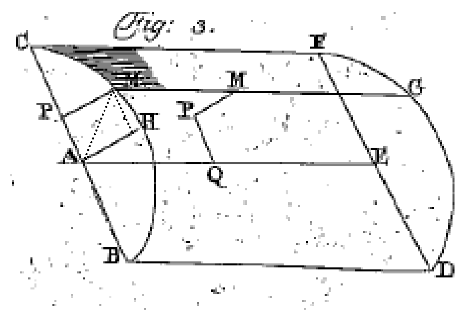
Which can be changed into the following form by reducing the order of the terms
$\frac{\mathrm{Q}ddx+\mathrm{P}ddy}{\mathrm{Q}dx+\mathrm{P}dy}=\frac{dxddx+dyddy}{dt^2+dx^2+dy^2}$. [This is the general equation found for the shortest path for a surface specified by P and Q.]

17. We can now also introduce into the caculuation the nature of the surface ; which is most conveniently expressed by an equation between the three coordinates *t, x* and *y* . Since indeed we make use of the differential so much, here this shall be written as P*dx* = Q*dy* + R*dt*. From this the equation for the curve IK ought to be elicited, which is naturally situated in the proposed surface. But the equation for this is found if *t* is put equal to a constant value AC in the equation of the surface, *i. e.* if *dt* = 0. Thus the equation for the curve IK, which is P*dx* = Q*dy* emerges, the same that was assumed above. But we can also assume the other equation P*dx* = Q*dy* + R*dt*, as from this other things may result, and there will not be the need for new substitutions.

18. From these two equations in the the two preceeding paragraphs : with the one given for the proposed surface, and from the other the nature of the minimum can be deduced, and the shortest line can be found that is to be drawn on the surface. Moreover, these two equations ought to be joined together, and from them a new

equation formed,which involves as many as two variables. And this new equation will determine the same projection of the shortest line in the other plane, from which the two remaining coordinates should be recognised. Thus the shortest line sought is to be elicited from these two equations Pdx = Qdy + Rdt, and $\frac{\mathrm{Q}ddx+\mathrm{P}ddy}{\mathrm{Q}dx+\mathrm{P}dy}=\frac{dxddx+dyddy}{dt^2+dx^2+dy^2}$.

19. We will adopt this general solution to three particular kinds of forms of surfaces : these which are cylindrical, conical and these made round or turned [about an axis; surfaces of revolution]. I do not refer so much to the common kind of cylinder having a circular base, but all bodies, the sections of which to a perpendicular axis are between each other equal and similar. BHCFGD is a cylinder of this kind, of which the axis is the line AE. In this if the abscissa is placed on the axis AQ = t, and any perpendicular to this in the plane of the horizontal BCFD, with QP taken equal to x and the vertical PM pertaining to [a point on] the surface is equal to y. It is required that for any constant made equal to t, the same equation is always produced between x and y.



20. Therefore the equation for surfaces of this kind will be $Pdx = Qdy$, in which P and Q do not involve the letter t. If indeed either the third term were present Rdt, or P and Q were dependent on t, equations for the various sections to the perpendicular axis would be produced, that would be contrary to the kind of the cylindrical bodies. Therefore the same equation Pdx = Qdy will express the nature of the base BHC. Indeed put AP = x and PM = y in this base, then the equation for this base is now also Pdx = Qdy. Hence for common cylinders, for which BHC is a circle, if A is the centre of this, will be $xdx = -ydy$.

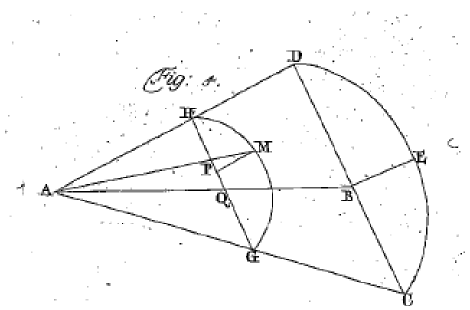
21. In order that the shortest line can be determined on the surface of the cylinder, in place of the general equation $Pdx = Qdy + Rdt$, we must use here $Pdx = Qdy$, or P : Q = $dy : dx$. Therefore by substitution from these proportions in place of P and Q in the equation $\frac{Qddx+Pddy}{Qdx+Pdy}=\frac{dxddx+dyddy}{dt^2+dx^2+dy^2}$, this equation is produced: $\frac{dxddx+dyddy}{dx^2+dy^2}=\frac{dxddx+dyddy}{dt^2+dx^2+dy^2}$, which on integration [by logs, and taking the square root] gives $\sqrt{(dx^2+dy^2)}=m\sqrt{(dt^2+dx^2+dy^2)}$, which is equivalent to $dx^2 + dy^2 = mndt^2$ [where $n = m/(1 - m^2)$ on re-arranging]. Consequently $nt=\int\sqrt{(dx^2+dy^2)}+C$; or t is always proportional to the arc in the corresponding transverse section cut by the shortest line, to be increased or decreased by some constant .

22. We can put this equation $dx^2 + dy^2 + dt^2 = n^2dt^2$ in place of the equation $dx^2 + dy^2 = n^2dt^2$, for any number can be put in place of n, this will become $nt=\int\sqrt{(dx^2+dy^2+dt^2)}$. From which the length of the shortest line is to be understood everywhere, as corresponds to the point t on the axis. Moreover, from the above equaton $nt=\int\sqrt{(dx^2+dy^2)}+\text{const.}$, it is concluded, if $n = 0$, that all the arcs for the transverse sections with the shortest line are all to be equal for all abscissa, and therefore the shortest line in the surface is a straight line drawn parallel to the axis.

Again let $nt = \int\sqrt{(dx^2 + dy^2)} + \text{const.} = n\int\sqrt{(dx^2 + dy^2 + dt^2)}$; in which if $n = 1$, then the perimeter of the transvere section itself will be the shortest line.

23. Let the transverse section of the cylinder be a circle with the axis passing through the centre A [Fig. 3 again], then $xdx =- ydy$ or $xx + yy = aa$; the equation $nt + b = \int\sqrt{(dx^2 + dy^2 + dt^2)}$ is joined with this , or with differentials as $n^2dt^2 = dx^2 + dy^2$. From these equations a new equation is derived, from which y is missing, which will determine the projection of the shortest line in the horizontal plane. Moreover this gives :
$ndt = \frac{adx}{\sqrt{(aa-xx)}}$ or $nt = \int\frac{adx}{\sqrt{(aa-xx)}}$. Which gives this construction : take AQ = t and the arc HM = nt is marked off on the base, AP is the sine of this arc [shown dotted by extra lines in Fig. 3], which is equal to the applied line QP and the point P in projection, which therefore is the line of the sine.

24. Here conical bodies are regarded by me as solids bounded by straight lines drawn from the individual points of any curve to a fixed point taken beyond the plane of the curve. These are changed into ordinary cones if the curves of these solids are conic sections. ACFDA [Fig. 4] is a body of this kind, and with ACD the horizontal plane. We place the base CBDE to be perpendicular to the axis AB. Now it is manifest that all the sections perpendicular to the axis are similar to each other, and proportional to the square of the distance from the vertex A. As before, AQ is called t; OP, x, and PM, y.



25. Since all the transverse sections are similar, the equation between $t, x,$ and y ought to be such that, as with two of these coordinates increased or diminished, the third should also be increased or diminished in the same ratio. Or if nt, nx, and ny are put in place of t, x, and y in the equation, then the equation persists unchanged. Truly this is a property of homogeneous equations, in which t, x, and y everywhere establish a number of the same dimension. Indeed, with the mentioned substitution n made, all these terms will have the same power, and on accout of that by division these can be removed, and the original equation will be produced.

26. Conoidal bodies hence have this property that the equation made between t, x and y shall be homogenous, *i. e.* so that from all the terms of this, a number is formed from the variables t, x and y of the same dimensions. Therefore if from this equation it is asked what t shall be, it is found to be equal to a homogeneous function composed from x and y and of the one dimension [*i. e.* of first degree]. On account of which $\frac{t}{x}$ is equal to a homogeneous function composed from x and y and of zero dimension.

27. This function is called F; and $\frac{t}{x} = \text{F}$. For truly the differential of this function F will have the form Mdx + Ndy. In which the letters M and N will have this relation to each other, that Mx + Ny = 0. [as it has zero dimension] Now Fy = qx is put in the function, this is changed since it is homogeneous and of zero dimension, into another [L = F/q or dF = Ldq, where $q = y/x$], in which as the letter q will occur, neither x nor y are

found bigger in that equation. Therefore the differential of this will have the form Ldq. Truly,
$\mathrm{L}dq = \frac{\mathrm{L}dy}{x} - \frac{\mathrm{L}ydx}{xx} = \mathrm{M}dx + \mathrm{N}dy$. Hence we have $\mathrm{M} = \frac{-\mathrm{L}y}{xx}$ and $\mathrm{N} = \frac{\mathrm{L}}{x}$. From which it is apparent that Mx + Ny = 0. Hence $\mathrm{N} = \frac{-\mathrm{M}x}{y}$ or $\mathrm{M} = \frac{-\mathrm{N}y}{x}$.

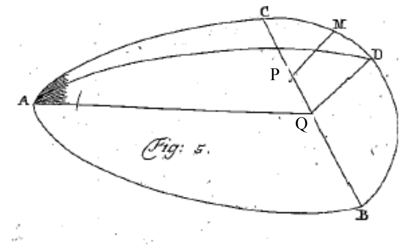
28. Since $\frac{t}{x} = \mathrm{F}$; then $\frac{xdt\text{-}tdx}{xx} = d\mathrm{F} = \mathrm{M}dx + \mathrm{N}dy = \frac{\text{-}\mathrm{N}ydx}{x} + Ndy$. From this the following equation is produced : $tdx - \mathrm{N}xydx = -\mathrm{N}xxdy + xdt$ which compared with the general equation Pdx = Qdy + Rdt will give
P = t - Nxy; Q = -Nxx; and R = x. Truly from these two equations
$tdx - Nxydx = Ndddy + xdt$ and $Mx + Ny = 0$, it is found that
$\mathrm{N} = \frac{xdt\text{-}tdx}{xxdy\text{-}xydx}$ and $\mathrm{M} = \frac{yxdt - tydx}{xxydx - x^3dy}$.

Therefore $\mathrm{P} = \frac{txdy - xydt}{xdy\text{-}ydx}$ and $\mathrm{Q} = \frac{yxdt - tydx}{xydx - ydy}$. With these factors substituted in the general equation $\frac{\mathrm{Q}ddx + \mathrm{P}ddy}{\mathrm{Q}dx + \mathrm{P}dy} = \frac{dxddx + dyddy}{dt^2 + dx^2 + dy^2}$, this equation is found :
$\frac{yxdtddx - tydxddx - txdyddy + xydtddy}{yxdtdx - tydx^2 + txdy^2 - xydtdy} = \frac{dxddx + dyddy}{dt^2 + dx^2 + dy^2}$.

29. For the reduction of this equation I put $tt + xx + yy = zz$, and $dt^2 + dx^2 + dy^2 = ds^2$; then $xdx + ydy = zdz - tdt$, and $dx^2 + dy^2 = ds^2 - dt^2$. Again $dxddx + dyddy = dsdds$, and $yddy + xddx = zddz + dz^2 - ds^2$. With the help of these values, this equation is arrived at :
$\frac{xdtddx + dtdx^2 - dtds^2 - tdsdds}{zdzdt - tds^2} = \frac{dds}{ds}$. From this $\frac{zdsddz + dz^2dt - zdxdds}{ds^2} = ds$. Which integrated gives $\frac{zdx}{ds} = s$, and this integrated again gives this equation: $ss = zz + \mathrm{C} = tt + xx + yy + \mathrm{C}$. Therefore the length of the shortest line $s = \sqrt{(tt + xx + yy + \mathrm{C})}$. From this property the shortest line sought in any particular case can be found. For the right cone, in which all the transverse sections are the circles $yy + xx = nntt$. Therefore the shortest length is given by $s = \sqrt{[(nn+1)tt + \mathrm{C}]}$.

30. These things that we have related up to the present are concerned with the shortest line that can be drawn on the surfaces of cylinders, can be found by another easier method from a property of these bodies, that can change their surfaces by evolving into planes. Therefore the line which is the shortest in these planes, will also be the shortest on the surfaces for the cones and cylinders. Whereby the shortest line for surfaces according to this method ought to have this property, that with the evolved surfaces changed into planes, the shortest line is changed into a straight line.

31. Truly this method cannot be extended to be applied to other surfaces which cannot be transformed into planes. For such truly the method is equally explained in this dissertation, and for these it is valid. Therefore we will use this method for surfaces of revolution of round or turned bodies, which are generated by the rotation of some figure around a fixed axis; as the sphere is generated by the rotation of a semicircle about a diameter; the right cone by the rotation of a right-angled triangle about another side ; and a right cylinder by the rotation of a parallelogram about a side.



32. Some kind of rounded body ABMC will have been generated by the rotation of the curve AQC about the axis AQ. In that BMC is some transverse section, which is a circle, of which the centre is Q. As before, BQ is called t; QP, x and PM, y. [Note: P and Q have been interchanged in Fig. 5 from the original diagram.] The equation between these coordinates ought to have this property, as with a constant position t or $dt = 0$, it can be changed to the equation of a circle $xx + yy =$ Const. or $xdy = -ydx$. On account of this, the equation for the solid round shape is $xx + yy =$ T, where T denotes some function of t itself and of the constant. Therefore by differentiation this gives $xdx = -ydy + Rdt$, in which RQ depends only on t and on constants.

33. With the general equation from §16, $\frac{Qddx+Pddy}{Qdx+Pdy} = \frac{-dxddx+dyddy}{dt^2+dx^2+dy^2}$ substituted, this equation is formed: $\frac{xddy-yddx}{xdy-ydx} = \frac{dxddx+dyddy}{dt^2+dx^2+dy^2}$, of which the integral is the equation :

$l(xdy - ydx) = l\sqrt{(dt^2 + dx^2 + dy^2)} + la$, or $xdy - ydx = a\sqrt{(dt^2 + dx^2 + dy^2)}$. If this is joined with the natural equation for the surface expressed by, $xdx = -ydy + Rdt$, then the shortest line will be determined.

34. The letter a [the constant in the above logarithmic integration] is arbitrary or depends in the location of some point which the shortest line must pass through. If a is put equal to 0; then $xdy = ydx$ and $y = nx$. Hence the periphery of the curve is known around the axis of rotation in which in some position the shortest line between its terminals is represented. Hence for these it is the case, that if the shortest line is to be drawn beween two points then they are in the same plane as the axis. And from these it is apparent that on a sphere the shortest line is always the great circle : since the sphere is generated by the rotation of the circle about a diameter, which is everywhere equal and similar to itself.

35. To bring about a more manageable equation, I put $xx + yy = zz$, and $dx^2 + dy^2 = ds^2$; then $xdx + ydy = zdz$. From these it is apparent that $zzds^2 - zzdz^2 = (xdy - ydx)^2$. Whereby with

$xdy - ydx = a\sqrt{(dt^2 + dx^2 + dy^2)}$, then $z^2ds^2 - zzdx^2 = aadt^2 + aads^2$

and $ds = z\sqrt{\left(\frac{dz^2+dt^2}{zz-aa}\right)}$.

Although two variables z and t are seen to occur here, nevertheless in any case t can be determined in terms of z from the equation for the surface, and the length of the shortest line will even become known by quadrature.

36. These are three particular kinds of bodies, on the surfaces of which the shortest lines are to be delineated, and here the method [for doing this] has been set out. These have this property before other cases [that can be considered], that the general equation adapted for these can be reduced to a differential equation of the first order. From these truly other similar cases admitting integration can be brought forward. As for bodies with a cylindrical shape the equation is $Pdx = Qdy$. In which P and Q have been said to depend on x and y. But it is evident that the reduction is equally likely to succeed, if P and Q also depent on t , in which case the equation is no larger for the bodies of

cylinders. In a similar way in the equation for rounded bodies [generated by rotationg a curve about an axis of symmetry] $xdx = ydy + Rdt$, so far as R depends on t. Therefore if R also is understood in terms of x and y, the equation will be for a new kind of surface to be generated, and nothing will be allowed to be reduced.
The celebrated Johan. Bernoulli proposed this question to me, after I had written my solution for him, as without doubt besides these three expositions I could investigate other kinds of surfaces, which may also lead to integrable equations. Therefore the soluton of this question, which flows so easily from the preceeding, I wish to add here. [In the following paper.]

DE LINEA BREVISSIMA IN SUPERFICIE QUACUNQUE DUO QUAELIBET PUNCTA IUNGENTE.

Auctore
Leonh. Eulero.

1.

Cuisque notum est, et a multis tanquam axioma ponitur, lineam seu viam brevissimam a dato puncto ad aliud quodcunque esse lineam rectam. Ex hoc facile intelligitur, in superficie plana lineam brevissimam duo quaelibet puncta iungentem esse rectam, quae ab altero ad alterum ducitur. In superficie sphaerica, in qua recta duci non potest, statuitur a Geometris viam brevissimam esse circulum maximum, que data duo puncta coniungit.

2. Quae autem in superficie quacunque sive convexa, sive concava, sine ex his mixta sit via brevissima, quae ex dato puncto ad aliud quodcunque ducitur, nondum est generaliter determinatum. Proposuit mihi hanc quaestionem Cel. Ioh. Bernoulli, significans se universalem invenisse aequationem, quae ad lineam brevissimam determinandam cuique superficiei accomodari possit. Solvi ego etiam hoc problema, solutionemque hac dissertatione exponere volui.

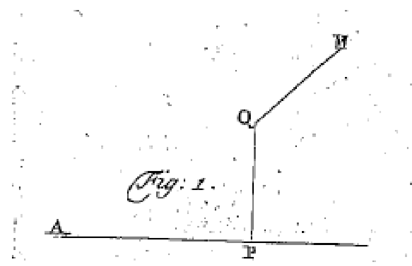
3. Mechanice hoc problema facillime solvitur ope file, quod per data duo puncto ductum tenditur, quantum fieri potest, hoc enim filum in superficie proposita designabit viam brevissimam. Necesse est autem, ut hoc filum ubique superficiem tangat, quamadmodum si superficies convexa sit, in superficiebus quidem concavis non arcum curvae sed chordam repraesentabit. Hoc igitur in casu filum ita applicata debet, vel applicatum concipi, ut semper superficiem in parte convexa tangat.

4. Hac vero constructione geometra contentus esse non potest, qui naturam huius lineae intimam perspicere desiderat, eamque , ut fieri solet, aequatione exponere. In mechanica autem constructione linea quaesita tantum aspectui exponitur, neque ex hoc natura eius potest perspici. Propterea hic methodum sum tradituris qua pro omnibus superficiebus, dummodo aequationibus exprimi possunt, linea brevissima determinari potest.

5. Ad hoc igitur opus est, ut superficierum naturae aequationibus includantur; quo tota operatio analytice possit absolvi. Solent lineae curvae in eadem plano sitae exprimi aequationibus inter duas coordinatas, ex quibus cuiusque puncti situs secundum longitudinem et latitudinem definitur. In superficiebus autem tres considerandae sunt

positionum relationes, cum puncti cuiuslibet in superficiae locus secundum tres dimensiones debeat esse determinatus. Tribus igitur in aequationibus superficierum uti convenit variabilibus; quarum una locum puncti secundum longitudinem, altera secundum latitudinem et tertia altidudinem determinat.

6. Concipiatur planum, quod in figura congruis cum plano chartae, et quod horizontale appellabimus, in eoque recta pro lubitu ducta AP, quae tanquam axis erit consideranda. Sit nunc M punctum cuiuspiam superficiei extra hoc planum situm, demittatur ex eo in planum horizontale perpendiculum MQ plano in Q occureus, et ex Q in lineam seu axem AP ducatur perpendicularis QP. Perspicuum nunc est datis tribus lineis AP, PQ, et QM quantitate, situm puncti M fore determinatum.

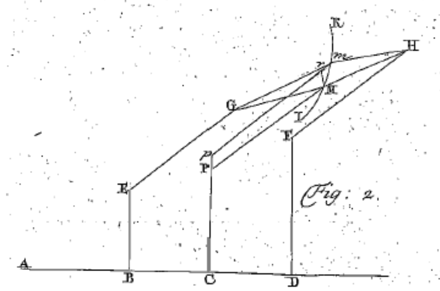


7. Hae igitur tres lineae AP, PQ, et QM nobis erunt indeterminatae, ex quibus cum constantibus aequatio pro superficie punctis M terminata conficitur. Vocabimus AP, t, PQ, x, et QM, y, atque pro qualibet superficie, de qua quidquam quaeritur, oportet aequationem inter has indeterminatas investigare. Simili deinceps modo ex huiusmodi aequationibus proprietates erventur, quo ex aequationibus curvarum earum proprietates derivantur. Uti si superficies fuerit sphaerica, cuius centrum in A et radius = a, erit aequatio eius naturam continens $aa = tt + xx + yy$.

8. Quamadmodum porro in linea curva certum punctum definitur vel determinato valore alterutri indeterminatae assignando, vel alia quadam aequatione cum aequatione locali coniungenda. Sic in superficiebus, si quaedam trium indeterminataram determinatur, vel alia aequatio cum aequatione superficiem definiente coniungatur; habibitur aequatio pro linea quadam in ea superficie sita, quae formatur intersectione datae superficiei et alius nova aequatione experessa. Punctum denique fixum in superficie constituetur, vel duabus indeterminatis determinandis, vel duabus novis aequationibus adiungendis.

9. Quamobrem ad lineam brevissimaem in superficie quacunque, cuius cognita est aequatio, ducendam aliam aequationem investigabo, quae cum illa iuncta definet in superficie ea lineam brevissimam quaesitam. Ex his deinde huabus aequationibus omnia, quae ad situm lineae brevissimae cognoscendum pertinent elici poterunt. Proiectio scilicet in plano horizontale definietur aequatione, quae ex illis duabus prodit exterminata y. Proiectio in plano verticali horizontale in AP secanate habitur exterminanda x. Et proiectio in plano verticali et perpendiculari ad AP habetur eliminanda littera t.

10. Ad solvendum nunc hoc problema uti oportet *methodo maximorum et minimorum* prout ipsa quaesito postulat. Quareitur autem in superficie data inter omnes lineas eosdem terminos habentes ea, quae est minima. Proprietas haec minimi non solum in integram lineam quaesitam competet, sed etiam in in singulas eius particulas; ita ut duo elementa eius contigua designent intra suos



terminos viam brevissimam. Ex hoc igitur facilior nascitur modu ad aequationem perveniendi.

11. Ad determinandam nunc positionem duorum elementorum viam intra suos terminos brevissimam constituentium sequens praemitto lemma. Sint duo puncto fixa I et H et curva inter ea extensa IK. Quaerendum est is ea punctum M tale, ut via(ductis rectis GM et MH) GM + MH sit omnium , quae per alia puncta curvae IK duci possunt, minima. Notuem est ex *methodo maximorum et minimorum* porri oportere, sumto m puncto proximo ipsi M, GM + MH = Gm + mH, ex hacque aequatione inveniri locum puncti M, per quod transiens via GM + MH est minima.

12. Demissis ex punctis G, H, M, et m ad planum horizontale perpendiculis GE, HE, MP, et mp, producatur pP in C horizontalii sita, quae tanquam axis consideretur; ad hancque ducantur perpendiculares EB et FD. Ponamus BC et CD esse aequales, tales enim in sequentibus assumerer licebit. Sint BC = CD = a; BE = b; EG = c ; DF = f ; FH =g. Sit porro CP = x et PM = y, quae sunt coordinatae curvae IK. Erit igitur Cp = $x + dx$ et $pm = y + dy$.

13. Ex his invenietur GM = $\sqrt{[a^2 + (x - b)^2 + (y - c)^2]}$: est enim $GM^2 = (PM - GE)^2 + (CP - BE)^2$. Similiter habebitur HM = $\sqrt{[a^2 + (f - x)^2 + (g - y)^2]}$. Tota igitur via GM + MH erit = $\sqrt{[a^2 + (x - b)^2 + (y - c)^2]} + \sqrt{[a^2 + (f - x)^2 + (g - y)^2]}$, quae ergo quantitas debet naturam minimi habere. Variabiles eius quantitates sunt x et y a quibus punctum M quaesitum pendet. Differentietur igitur isat quantitas exprimens GM + MH, et, quod provenit, ponatur = 0. Orieturque haec aequatio,
$\frac{(x-b)dx+(y-c)dy}{\sqrt{[a^2+(x-b)^2+(y-c)^2]}} = \frac{(f-x)dx+(g-y)dy}{\sqrt{[a^2+(f-x)^2+(g-y)^2]}}$. Ex qua locus puncti M determinabitur.

14. Quia curva IK ponitur data, dabitur aequatio inter eius coordinatus x et y : Opus autem est tantum aequatione differentiali, propterea ponamus relationem elementorum dx et dy dari hac aequatione Pdx = Qdy, seu $dx : dy$ = Q : P. Positis nunc his valoribus proportionalibus loco dx et dy, prodibit aequatio
$\frac{(x-b)\mathrm{Q}+(y-c)\mathrm{P}}{\sqrt{[a^2+(x-b)^2+(y-c)^2]}} = \frac{(f-x)\mathrm{Q}+(g-y)\mathrm{P}}{\sqrt{[a^2+(f-x)^2+(g-y)^2]}}$ quae vacua est differentialibus quantitatibus.

15. Consideremus iam lineas GM et MH tanquam duo elementa lineae brevissimae in superficie, in qua sumta sunt puncta G et H et curva IK, ducendae. Ponamus AC = t, suntque iam factae CP = x et PM = y. Erit BC = CD = $a = dt$; DF = $f = x + ds$; FH = $g = y + dy$; BE = $b = x - dx + ddx$; EG = $c = y - dy + ddy$. Substituantur hi valores pro a, b, c, f et g in aequatione supra inventa, orietur aequatio haec
$\frac{\mathrm{Q}(dx-ddx)+\mathrm{P}(dy-ddy)}{\sqrt{[dt^2+(dx-ddx)^2+(dy-ddy)^2]}} = \frac{\mathrm{Q}dx+\mathrm{P}dy}{\sqrt{[dt^2+dx^2+dy^2]}}$.

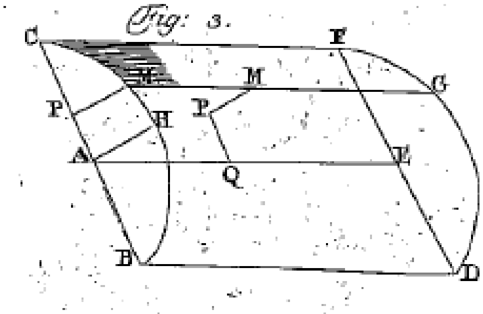
16. Aequatio haec allud non significat, nisi quod differentiale huius quantitatis $\frac{\mathrm{Qdx+Pdy}}{\sqrt{[dt^2+dx^2+dy^2]}}$ aequale sit faciendum nihilo, positis P, Q, et dt constantibus (dt quidem re ipsa constans ponitur) habebitur ergo ex hac differentiatione aequatio haec
$(Qddx + Pddy)\sqrt{(dt^2 + dx^2 + dy^2)} = (Qdx + Pdy)(dxddx + dyddy) : \sqrt{(dt^2 + dx^2 + dy^2)}$.
Quae in ordinem reducta abit in hanc $\frac{Qddx+Pddy}{Qdx+Pdy} = \frac{dxddx+dyddy}{dt^2+dx^2+dy^2}$.

17. Introducamus nunc etiam in culculum nataram superficiei; quae aequatione inter tres coordinatas t, x et y commodissime exprimitur. Quia vero hic tantum utimur differentiali, sit ea $Pdx = Qdy + Rdt$. Ex hac elici debet aequatio pro curva IK, quippe

quae superficie proposita est sita. Aequatio autem pro ea invenitur si in aequatione superficiei t ponitur aequalis constanti AC, i. e. Si sit $dt = 0$. Proveniet itaquae pro curva IK aequatio $Pdx = Qdy$, eadem, quae supra erat assumpta. Propterea autem aequationem $Pdx = Qdy + Rdt$ assumsimus, ut ex ea altera resultaret, atque ne novis substitutionibus opus esset.

18. Ex his ergo duabus aequationibus in duabus praecedentibus paragr: datis, quarum altera est pro superficie proposita, altera ex natura minimi deducta, inveniri poterit linea brevissima in superficie ducenda. Ad hoc coniungi debent duae illae aequationes ex iisque nova confici, quae tantum duas indeterminatas involuit. Aaecque nova aequatione determinabit proiectionem quampiam lineae brevissimae in aliquo plano, quod ex coordinatis binis remanentibus cognoscetur. Linea itaque brevissima quaesita elicienda est ex his duabus aequationibus $Pdx = Qdy + Rdt$, et $\frac{Qddx+Pddy}{Qdx+Pdy} = \frac{dxddx+dyddy}{dt^2+dx^2+dy^2}$.

Fig: 3.



19. Generalem hanc solutionem ad tria praecipua formarum superficierum genera accommodabimus : quae sunt cylindrica, conica et rotunda seu tornata. Ad genus cylindricum non tantum refero cylindros communes bases circulares habentes, sed omnia corpora, quorum sectiones ad axem perpendiculares sunt inter se aequales et similes. Huiusmodi cylinder sit BHCFGD, cuius axis est linea AE. In hoc si ponatur abscissa in axe AQ = t, huicque perpendicularis quacunque in plano horizontali BCFD sumto QP = x et verticalis PM as superficiem pertingens = y. Oportet ut cuicunque constanti aequali facta t semper eadem prodeat aequatio inter x et y.

20. Aequatio igitur pro huiusmodi super ficiebus erit $Pdx = Qdy$, in qua P et Q non involuunt litteram t. Si enim vel adesset tertius terminus Rdt vel P et Q a t penderent aequationes pro variis sectionibus ad axem perpendicularibus variae prodirent, quod esset contra naturam corporum cylindricorum. Eadem igitur aequatio $Pdx = Qdy$ exprimet naturam basis BHC. Facta enim in basi hac AP = x et PM = y, aequatio pro hac basi erit etiam $Pdx = Qdy$. Pro cylindris igitur communibus, in quibus BHC est circulus, si A fuerit eius centrum erit $xdx = -ydy$.

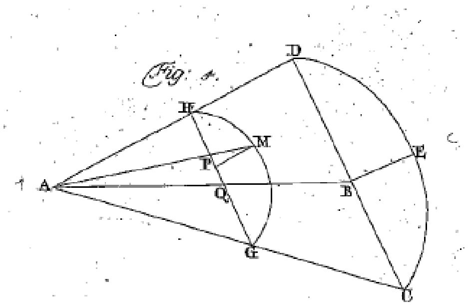
21. Ad lineam nunc brevissimam in superficiebus cylindricis determinandam loco aequationis generalis $Pdx = Qdy + Rdt$ hac uti debemus $Pdx = Qdy$, seu P:Q dy:dx. His igitur proportionalibus loco p et Q in aequatione $\frac{Qddx+Pddy}{Qdx+Pdy} = \frac{dxddx+dyddy}{dt^2+dx^2+dy^2}$, substitutis prodibit haec $\frac{dxddx+dyddy}{dx^2+dy^2} = \frac{dxddx+dyddy}{dt^2+dx^2+dy^2}$, quae integrata dat

$\sqrt{(dx^2+dy^2)} = m\sqrt{(dt^2+dx^2+dy^2)}$ cui aequivalet haec $dx^2 + dy^2 = mndt^2$.

Consequenter est $nt = \int\sqrt{(dx^2+dy^2)} + C$ seu t semper est proportionalis arcui in respondente sectione transversa a linea brevissima abscissi, constante quipiam aucti vel minuti.

22. Sumamus loco aequationis $dx^2 + dy^2 = n^2dt^2$ hanc $dx^2 + dy^2 + dt^2 = n^2dt^2$, licet enim pro n numerum quemlibet substituere, erit $nt = \int \sqrt{(dx^2 + dy^2 + dt^2)}$. Ex quo intelligitur longitudinem lineae brevissimae esse ubique, ut respondentem abscissam in axe t. Ex superiore aequatione autem $nt = \int \sqrt{(dx^2 + dy^2)} + \text{const.}$ concluditur, si n= 0 fore arcus in sectionibus transversis a linea brevissima abscissos omnes aequales, et propterea lineam brevissima esse rectam in superficie ductam ex axi parallelam. Porro est etiam $nt = \int \sqrt{(dx^2 + dy^2)} + \text{const.} = n\int \sqrt{(dx^2 + dy^2 + dt^2)}$; in qua si $n = 1$, erit linea brevissima ipse perimeter sectionis transversae.

23. Sit, transversa cylindri sectio circulus axisque per centrum transeat, erit $xdx = -ydy$ seu $xx + yy = aa$; cum hac coniungatur aequatio $nt + b = \int \sqrt{(dx^2 + dy^2 + dt^2)}$ seu differentialis tantum $n^2dt^2 = dx^2 + dy^2$. Ex his aequationibus derivetur nova y carens, quae proiectionem lineae brevissimae in plano horizontali determinabit. Prodibit autem $ndt = \frac{adx}{\sqrt{(aa-xx)}}$ seu $nt = \int \frac{adx}{\sqrt{(aa-xx)}}$. Quae dat hanc constructionem : sumta AQ = t abscindatur in basi arcus HM = nt, eiusque sinus AP erit aequalis applicatae QP et punctum P in proiectione, quae igitur est linea sinuum.



24. Corpora conoidica hic mihi denotant solida lineis rectis ex curvae cuiuslibet singulis punctis ad punctum fixum extra planum curvae assumtum ductis terminata. Haec in conos ordinarios abeunt si curvae illae fuerint sectiones conicae. Huiusmodi copus conoidicum sit ACFDA, eius axis AQB, et planum horizontale ACD. Ponamus basem CBDE esse perpendicularem ad axem AB. Manifestum nunc est, omnes sectiones axi perpendiculares fore singulas similes, et proportionales quadratis distantiarum a vertice A. Vocentur ut ante AQ, t; OP, x, et PM, y.

25. Quia omnes sectiones transversae sunt similes, aequatio inter t, x, et y talis esse debet, ut auctis vel minutis duabus harum coordinatarum tertia eadem ratione augeatur vel minuatur. Sive si in aequatione ponatur loco t, x, et y hae nt, nx, et ny, ut aequatio immutata persistat. Haec vero est proprietas aequationum homogenearum, in quibus t, x, et y ubique eundem dimensionum numerum constituunt. In his enim facta substitutione memorata in omnibus terminus n eandem habebit potestatem, et propterea ea divisione tolli poterit, et aequatio prior prodibit.

26. Hanc ergo corpora conoidica habent proprietatem, ut aequatio eorum inter t, x et y facta sit homogenea, i. e. ut in singulis eius terminis idem sit dimensionum numerus ab indeterminatis t, x et y formatus. Si igitur ex hac aequatione quaeratur quid sit t, reperietur aequalis functioni ex x et y composiae homogeneae et unius dimensionis. Quamobrem $\frac{t}{x}$ aequabitur functioni ex x et y compositae etiam homogeneae et nullius dimensionis.

27. Vocetur haec functio F; erit $\frac{t}{w} = \text{F}$. Differentiale vero huius functionis F habebit hanc formam Mdx + Ndy. In qua litterae M et N hanc habebunt inter se relationem, ut sit Mx + Ny = 0. Nam ponatur in functione Fy = qx, mutabitur ea, quia est homogenea et nullius dimensionis, in aliam, in qua tantum littera q occurret, neque x neque y amplius in ea reperiatur. Propterea eius differentiale habebit hanc formam Ldq. Est vero
$\text{L}dq = \frac{\text{L}dy}{x} - \frac{\text{L}ydx}{xx} = \text{M}dx + \text{N}dy.$ Erit igitur $\text{M} = \frac{-\text{L}y}{xx}$ and $\text{N} = \frac{\text{L}}{x}$. Ex quo apparet fore M$x$ + Ny = 0. Habetur ergo $\text{N} = \frac{-Mx}{y}$ vel $M = \frac{-Ny}{x}$.

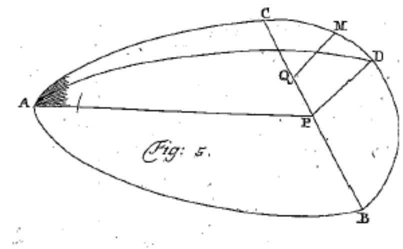
28. Quia est $\frac{1}{x} = \text{F}$; erit $\frac{xdt-tdx}{xx} = d\text{F} = \text{M}dx + \text{N}dy = \frac{-\text{N}ydx}{x} + Ndy.$ Ex hac prodibit ista aequatio $tdx - \text{N}xydx = -\text{N}xxdy + xdt$ quae comparata cum generali Pdx = Qdy + Rdt dabit P = t - Nxy; Q = -Nxx; et R = x. Ex aequationibus vero duabus
$tdx - Nxydx = Ndddy + xdt$ et $Mx + Ny = 0,$ invenitur
$\text{N} = \frac{xdt\text{-}tdx}{xxdy\text{-}xydx}$ et $\text{M} = \frac{yxdt-tydx}{xxydx-x^3dy}$.
Erit igitur
$\text{P} = \frac{txdy-xydt}{xdy\text{-}ydx}$ et $\text{Q} = \frac{yxdt-tydx}{xydx-ydy}$. Facis his substitutionibus in aequatione generali
$\frac{\text{Q}ddx+\text{P}ddy}{\text{Q}dx+\text{P}dy} = \frac{dxddx+dyddy}{dt^2+dx^2+dy^2}$ invenietur haec aequatio
$\frac{yxdtddx-tydxddx-txdyddy+xydtddy}{yxdtdx-tydx^2+txdy^2-xydtdy} = \frac{dxddx+dyddy}{dt^2+dx^2+dy^2}$.

29. Ad hanc aequationem reducendam pono $tt + xx + yy = zz$, et $dt^2 + dx^2 + dy^2 = ds^2$; erit $xdx + ydy = zdz - tdt$, et $dx^2 + dy^2 = ds^2 - dt^2$. Porro
$dxddx + dyddy = dsdds,$ et $yddy + xddx = zddz + dz^2 - ds^2$. Ope horum valorum pervenitur ad hanc aequationem
$\frac{xdtddx+dtdx^2-dtds^2-tdsdds}{zdzdt-tds^2} = \frac{dds}{ds}$. Ex hac erit $\frac{zdsddz+dz^2dt-zdxdds}{ds^2} = ds$. Quae integrata dat $\frac{zdx}{ds} = s$, haecque iterum integrata hance $ss = zz$ + C = $tt + xx + yy$ + C. Erit igitur longitudo lineae brevissimae $s = \sqrt{(tt + xx + yy + \text{C})}$. Ex hac proprietate in quolibet casu particulari determinabitur linea brevissima quaesita. Pro cono recto, in quo omnes sectiones transversae sunt circuli est $yy + xx = nntt$. Erit ergo $s = \sqrt{[(nn+1)tt + \text{C}]}$.

30. Haec quae hactenus delinea brevissima ducenda in superficiebus cylindricis tradidimus, alia methodo facilius inveniuntur ex ea horum corporum propriate, quod eorum superficies evulutione in planas transmutentur. Quae igitur linea in his planis est brevissima, erit etiam in ipsis superficiebus cylindricis et conicis brevissima. Quare linea brevissima in huius modi superficiebus hanc habere debet proprietatem, ut superficiebus evolutis et in planas transmutatis linea brevissima transmutetur in rectam.

31. Haec vero methodus latius non patet, neque ad alias superficies, quae non possunt evolutione in planas mutari, potest accommodari. Pro talibus vero methodus hac dissertatione exposita aeque as pro illis valet. Utamur igitur hac methodo in superficiebus corporum rotundorum seu tornatorum, quae generantur circumrotatione cuiusque figurae circa axem immobilem; quemadmodum sphaera generatur conversione semicirculi circa dismetrum; conus rectus trianguli conversione circa alterutrum latus; cylinder rectus conversione parallelogrammi circa latus.

32. Sit huiusmodi corpus rotundum ABMC generatum conversione curvae AQC circa axem AQ. In eo sit BMC sectio transversa, quae erit circulus, cuius centrum in Q. Vocentur ut ante BQ, t; QP, x et PM, y. Aequatio inter has coordinatas hanc debebit habere proprietatem, ut posito t constante seu $dt = 0$, ea abeat in aequationem circuli $xx + yy =$ Const. seu $xdy = -ydx$. Quamobrem aequatio aequatio pro solidis rotundis est $xx + yy =$ T, ubi T denotat functionem quamcunque ipsius t et constantium. Haec igitur differentiata dat $xdx = -ydy + Rdt$, in qua RQ solis t et constantibus pendet.



33. Hac aequatione in generali §16, $\frac{Qddx+Pddy}{Qdx+Pdy} = \frac{-dxddx+dyddy}{dt^2+dx^2+dy^2}$ substitutis orietur aequatio ista $\frac{xddy-yddx}{xdy-ydx} = \frac{dxddx+dyddy}{dt^2+dx^2+dy^2}$. Cuius integralis aequatio est $l(xdy-ydx) = l\sqrt{(dt^2+dx^2+dy^2)} + la$, vel $xdy-ydx = a\sqrt{(dt^2+dx^2+dy^2)}$. Haec si coniungatur cum aequatione naturam superficiei exprimente $xdx = -ydy + Rdt$ determinabit lineam brevissimam.

34. Litera a est arbitraria seu pendet a loco punctorum per quae linea brevissima transire debet. Si ponatur $a = 0$; erit $xdy = ydx$ atque $y = nx$. Unde cognoscitur peripheriam curvae circa axem rotatae in quolibet situ repraesentare lineam brevissimam inter suos terminos. His ergo casus valet, si duo puncta inter quae linea brevissima duci debet, sunt cum axe in eodem plano. Ex hisce apparet in spheara lineam brevissimam semper esse circulum maximum: quia sphaera conversione circuli circa diametrum generatur, et sibi ubique est aequalis et similis.

35. Ad aequatonem tractabiliorem efficiendam pono $xx + yy = zz$, et $dx^2 + dy^2 = ds^2$; erit $xdx + ydy = zdz$. Ex his apparet fore $zzds^2 - zzdz^2 = (xdy-ydx)^2$. Quare cum sit
$xdy-ydx = a\sqrt{(dt^2+dx^2+dy^2)}$, erit $z^2ds^2 - zzdx^2 = aadt^2 + aads^2$
atque $ds = z\sqrt{\left(\frac{dz^2+dt^2}{zz-aa}\right)}$.
Etsi hic duae variabiles z et t occurere videntur, tamen in quolibet casu ex aequatione pro superficie determinabitur t in z, et longitudo lineae brevissimae saltem per quadraturas cognoscetur.

36. Haec sunt tria praecipua corporum genera in quorum superficiebus lineas brevissimas delineandi methodus hic fusius est tradita. Habent hi cusus hanc prae aliis proprietatem, ut generalis aequatio ad hos accommodata reduci possit ad differentialem primi gradus. Ex his vero alii se produnt casus similiter integrationem admittentes. Ut pro corporibus cylindricis aequatio est $Pdx = Qdy$. In qua P et Q ab x et y pendere dicta sunt. Perspicuum autem est reductionem aeque succedere, si P et Q etiam a t penderent, quo in casu aequatio non est amplius pro corporibus cylindricis. Simili modo in aequatione pro corporibus rotundis $xdx = ydy +$ Rdt, R tantum a t pendet. Si igitur R etiam x et y in se comprehendat, aequatio erit pro novo superficierum genere, et nihilominus reductionem admittit.

Proposuit mihi hanc quaestionem Cel. Ioh. Bernoulli, postquam ipsi hanc meam solutionem scripsissem, ut nimirum praeter tria exposita superficierum genera alia investigarem, quae etiam ad aequationes integrabiles perducant. Solutonem igitur huius quaestionis, quia tam facile ex antecedentibus fluit, hic adiungere volui.