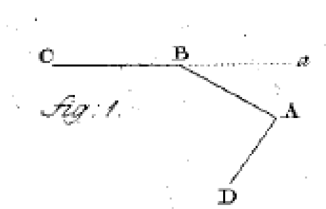
The solution is found for problems Concerning plane curves formed by lines with various kinds of elasticity, the points of which are acted on by forces of some kind.

Author
Leonard Euler.

Since that first curve that the celebrated James Bernoulli assigned for the curvature of a planar elastic curve, and afterwards by many others, the form of every elastic curve is now known, except for heavy elastic laminar curves; it is understood that their solutions are different from these previous solutions. Moreover the bending of heavy elastic laminar shapes can only be found by their departure from their natural state, yet as far as I know, this has not been determined by anyone until now. Recently the most distinguished Daniel Bernoulli and I have fallen upon this investigation, and I have approached this problem in what can be considered to be a not too inelegant manner, and the solutions that we have produced simultaneously to the problems that we pursued have agreed exceedingly well. Truly, for the first solution, it is seen that there is nothing difficult initially, and here too my solution need not be explained by the formal theory I have considered. Indeed the first investigation extends this far, that the curve formed by a heavy laminar shape, with one end fixed and having some kind of force applied to the other end, can be found. However, at the start of the work I consider a more general lamina of arbitrary elasticity, and having some hanging weights attached. And on this account, although any particular case can be worked out on its own, nevertheless it is seen that [the more difficult] problems cannot be easily worked out, without making use of this general theory: and indeed not only has it been extended to laminar curves of any elasticity, but indeed also to curves formed by perfectly flexible bodies, and the theory can account for the forces applied in any manner of displacement; so thus the curvatures of all flexible bodies can be found. It can be agreed upon besides, that the solution of the general problem can scarcely avoid the accepted thought of being more bountiful than that of any specific case. Truly, before I am able to approach the general problem itself, it is necessary to present first some matters relevant to the solution.

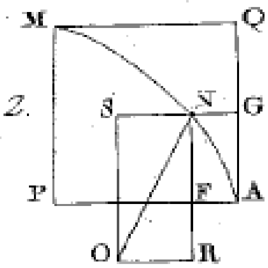


Hypothesis.

1. If two [rigid] rods *a*B and BC [on the line CB*a*] are joined at B in an elastic way and are to be rotated by a force AD into the position ABC [fig. 1,], in order that the angle AB*a* is taken up. There will be a moment of the force AD about B considered as the product of the elastic force at B and the angle AB*a* . It is known that this factor is equal to the product of the force AD by AB. [note that AD is at right-angles to AB] This hypothesis is generally assumed; however the truth of this is satisfactory only if the angle AB*a* is exceedingly small and that can probably be shown by experiment. [Euler presumably had in mind the twisting of a wire, which obeys an angular form of Hooke's law: applied torque or moment = elastic constant × angle rotated through from equilibrium, or $\tau = c\vartheta$.]

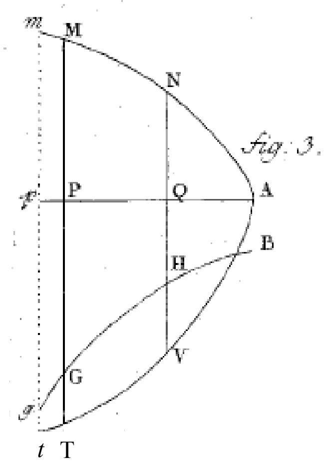
Lemma 1. If the force NO is applied to some point N of the curve ANM [fig. 2], and this is resolved into a horizontal force NS and as a vertical force NR. The size of the force NO acting on the curve AM to make it rotate about M, or the moment of this force is equal to NR.FP + NS.GQ. With MQ, AP drawn to the horizontal and AQ, PM to the vertical.

fig: 2.



Demonstration. The force NO is equivalent to the two forces NR and NS acting together. The moment truly of the force NR in M is, in order to be in agreement with the principles of statics, NR.PF ; and the moment of the force NS in M is NS. GQ. Hence since both forces are trying to turn the curve AM in the same direction, the moments of both or the moment of the force NO is equal to NR.FP + NS.GQ. Q.E.D.

Lemma 2. The curve AM, [which is a curved rod, string, chain, or plate, $f(x)$; for which Euler eventually assigns A as the origin, with a left-going x-axis], with forces to be applied at all the points N obviously parallel to the perpendiculars to AP, is to be determined from the curve BG [$g(x)$ is the force/unit length; the forces can be externally applied, or be due to the weight of the curve], thus as the force acting through the point N, for a point on the curve which is as QN [we can take QH as the size of the force, see fig. 3, acting along NQ]. Another curve AVT is constructed [$h(x)$ is the total weight or force acting vertically to right of the section considered], the lines of application QV of which shall be as the area AQHB [$h = \int g(x')dx'$]. The sum of all the moments of all the forces to the curve ANM [ANB in original text] turning about M shall be as the area APT.



Demonstration. The moment of the force QH in M is QH.PQ. A point *m* is taken near to the point M. The line of application of the force *mp* is drawn; the moment of the force QH in *m* is equal to QH.Q*p*. Hence the difference of these moments is QH.P*p*.
[Thus, for a given element of the original curve, which we can imagine to be a small section of an elastic rod, one effect of a force QH applied along the rod at Q is to apply a moment of size QH.P*p*.]
The same thing will prevail for all the individual forces, hence the difference of all the moments acting at M and *m* gives rise to the area ABGP by P*p* or PT.P*p*, [*i. e.* the area of the element P*pt*T of the curve *h*].
If now the sum of all the moments for *m* is put equal to M + *d*M, then the difference of the moments is *d*M , therefore dM = P*pt*T, and consequently M is equal to the area APT.

[We can now add the moments of all the other weights or forces acting along the rod for the points M and *m* ; the total difference of the moments is the whole moment acting on the element, and each weight gives a contribution similar to that established. The area of the curve ABGP is the total weight or force applied; note that we should really be talking about the weight or force per unit length for the loading of the rod or beam, to get the correct units. Occasionally, Euler has objects with cylindrical symmetry in mind.]

[In terms of integrals, as an alternative approach, we can write $M(x) = \int_x^X x'g(x')dx'$,

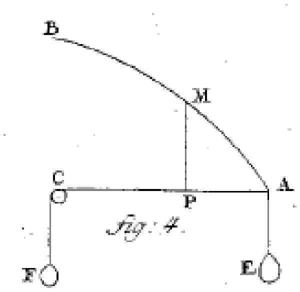
where $g(x')$ is the force at position x' due perhaps to the weight of the curve; the point A is X, and P is the point x w.r.t. some origin. The integration is performed by parts, where $\int g(x')dx' = h(x)$ to give : $M(x) = \int_x^X x'g(x')dx' = [xh(x)]_x^X - \int_x^X h(x')dx' = -\int_x^X h(x')dx'$, as required, since either $x = 0$ or $h(\mathrm{X}) = 0$ in the first term.]

Q.E.D.

Now it can be said that with the abscissa AP taken, x, and the corresponding point of application in curve AT for P ; Moreover, the sum of all the forces [acting vertically to the right] can be brought together at x and designated P [i. e. P(x)]; hence the sum of all the moments at M $= \int \mathrm{P}dx$.

The General Problem.

Let BMA be any elastic laminar plate and with displacements from forces applied at any fixed points whatever, moreover with that end in B fixed, and at A two weights E and F are have been hung on pulling on A, of which E is along the vertical and F the horizontal. It is required to determine the curve AMB that the nearby lamina has been turned into by being bent.



Solution. The horizontal line AC is taken for the axis, in which the abscissa AP is taken equal to x. Let PM = y be the perpendicular to this and AM = s, the [differential] element of which is taken as being constant. The radius of curvature at the point M is put equal to r ; and the angle that the two elements constitute vary inversely as r. The elastic strength at M is designated by the letter v; the strength producing this angle varies as $\frac{v}{r}$ (from the hypothesis above.) For this single proportional therefore ought to be the sum of all the moments delivered to M so coming from the individual forces applied to the curve AP, as from the weights E and F. Moreover the moment of the force or weight E in M = E.AP = E.x (by Lemma 1.) and the moment of the force F is equal to F.PM = F.y (cit.) Besides this point M being displaced, also the arc AM of all the individual points are displaced by the forces. For these are to be resolved acting in the vertical along AE and in the horizontal along AC, the sum of all the vertical forces is to be called P, and Q for the sum of the horizontal forces from A as far as M. The sum of the moments of the vertical forces $= \int \mathrm{P}dx$ (Lem.2.) and the sum of the moments of the horizontal forces = $\int \mathrm{Q}dy$. Thus the total strength of the moments acting at M is equal to Ex + Fy + $\int \mathrm{P}dx$ +

$\int Qdy$. Since $\frac{v}{r}$ should be proportional to that moment , the equation is obtained :

$\frac{\Delta v}{r} = Ex + Fy + \int Pdx + \int Qdy.$

If in place of P + E , P is written, and in place of Q + F , Q ; we have $\frac{\Delta v}{r} = \int Pdx + \int Qdy$, let $\frac{\Delta v}{r} = Z$; then $Z = \int Pdx + \int Qdy$ or $dZ = Pdx + Qdy.$ From which the nature of the curve AMB can be recognised. Q.E.I.

In order that the use of this equation is better understood, I will apply this equation to special cases, and some of these is now to be expanded on, in order that their similarities can be understood; this is indeed a part of analysis still in a dormant state, as many forms of curves yet unknown will see the light for the first time.

Problem. To find the general equation for curves, which perfectly flexible bodies assume under the action of any manner of forces.

Solution. We will have perfectly flexible bodies [such as formed by the links of a chain, or by a string] when the elastic strength vanishes everywhere, then indeed even the smallest force will produce an angle between two elements. Moreover, the size of the elastic force [torque really] is expressed by the letter *v*, therefore v is set equal to zero, to give the resulting equation :

$0 = E.x + F.y + \int Pdx + \int Qdy$, which satisfies the question. Q.E.I.

Moreover, as P and Q , which obviously can be eliminated from the depending summation, and in place of these dP and dQ are produced, which denote the forces themselves applied to the points M; the equation is differentiated and we have:

$E.dx + F.dy + P.dx + Q.dy = 0.$ Hence $Q + \frac{Pdx}{dy} + F + \frac{Edx}{dy} = 0$. From which,

$dQ + \frac{Pdyddx - Pdxddy}{dy^2} + \frac{dPdx}{dy} + \frac{Edyddx - Edxddy}{dy^2} = 0$. But since is agreed to put *ds* constant then: $dyddy = -dxddx$, and $dyddx - dxddy = \frac{ds^2 ddx}{dy}$.

[These results follow by differentiation of $ds^2 = dx^2 + dy^2$ = constant;

$0 = dx.ddx + dy.ddy$, etc.

Note that *x* and *y* are treated as separate variables; the closest we can get to this is to regard both as functions of some parameter, such as the arc length from some fixed point on the curve, or as the time taken to travel a certain length along the curve at a constant speed $x(t)$, $y(t)$ in the same direction; this was the legacy of Newton, who was still alive at this time, though the notation is that of Leibnitz, which Euler obviously thought more convenient. Thus, we can consider *dx*, *ddx, dy*, and *ddy* as equivalent to Newton's $\dot{x}, \ddot{x}, \dot{y}, \ddot{y}$, etc., when treated in this manner. Parametric equations are often handled with the dot notation, which we present here along with the one used by Euler. Hence,

$ds^2 = dx^2 + dy^2$, for constant ds^2 gives $\dot{x}^2 + \dot{y}^2 = \dot{s}^2$, from which

$\dot{x}\ddot{x} + \dot{y}\ddot{y} = 0 = dxddx + dyddy$ (*) , and

$dyddx - dxddy = \dot{y}\ddot{x} - \dot{x}\ddot{y} = \dot{y}\ddot{x} + \dot{x}\dot{x}\ddot{x} / \dot{y} = (\dot{y}^2 + \dot{x}^2)\ddot{x} / \dot{y} = \dot{s}^2 \ddot{x} / \dot{y} = ds^2 ddx / dy$ (**);]

Whereby :

$dQ + \frac{Pds^2 ddx}{dy^3} + \frac{Eds^2 ddx}{dy^3} + \frac{dPdx}{dy} = 0$. And from this: $\frac{dQdy^3}{ds^2 ddx} + P + E + \frac{dPdy^2 dx}{ds^2 ddx} = 0$.

Hence [on differentiation again to eliminate E, with ds^2 constant]:

$$\frac{dy^3 ddQ}{ds^2 ddx} + \frac{3dQdy^2 ddxddy - dQdy^3 d^3 x}{ds^2 ddx^2} + dP + \frac{dy^2 dxddP}{ds^2 ddx} + \frac{2dPdydxddyd\,dx + dPdy^2 ddx^2 - dPdy^2 dxd^3 x}{ds^2 ddx^2} = 0.$$

From this equation we find, using $dx.ddx = -dy.ddy$:

$$dy^3 dd\mathrm{Q}ddx - 3d\mathrm{Q}dydxddx^2 - d\mathrm{Q}dy^3 d^3 x + dy^2 dxdd\mathrm{P}ddx - 3d\mathrm{P}dx^2.ddx^2$$
$$+ d\mathrm{P}dy^2 ddx^2 - d\mathrm{P}dy^2 dxd^3 x = 0.$$

This equation will be made much simpler, if the radius of the osculating circle *r* is introduced, which is :

$$r = \frac{dsdy}{ddx} = \frac{-dsdx}{ddy}.$$

[This follows from the standard formula for *r,* and from the above relations : note initially that $\frac{dy}{dx} = \dot{y}/\dot{x}$; hence $\frac{d(dy/dx)}{dx} = \frac{d(\dot{y}/\dot{x})}{dt}(dt/dx) = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^3} = \frac{ddydx - dyddx}{dx^3}$.

Hence, $\frac{1}{r} = \frac{\frac{d^2y}{dx^2}}{(1+(dy/dx)^2)^{3/2}} = \frac{\frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{x}^3}}{\dot{s}^3/\dot{x}^3} = \frac{\ddot{y}\dot{x} - \dot{y}\ddot{x}}{\dot{s}^3}$

or, $\frac{1}{r} = \frac{ddy.dx - ddx.dy}{ds^3} = \frac{ds^2.ddx/dy}{ds^3} = \frac{ddx}{ds.dy} = \frac{-ddy}{ds.dx}$.

$r = \frac{ds.dy}{ddx} = \frac{-ds.dx}{ddy}$, and from ** $dddx = \frac{ds.(r.ddy - drdy)}{r^2}$; $dddy = -ds.\frac{(ddx.r - dx.dr)}{r^2}$.

In terms of the dot notation, $\ddot{x} = \dot{s}\dot{y}/r$; $\ddot{y} = -\dot{s}\dot{x}/r$; $\dddot{x} = \frac{-\dot{s}}{r^2}(\dot{s}\dot{x} + \dot{y}\dot{r})$;

$\dddot{y} = \frac{-\dot{s}}{r^2}(\dot{s}\dot{y} + \dot{x}\dot{r})$;]

From this we find [this is the fundamental result used throughout this paper.] :

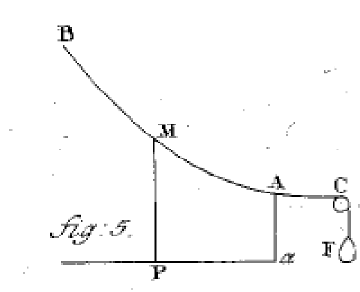
$$0 = rdxdd\,\mathrm{P} + 2d\mathrm{P}dsdy + d\mathrm{P}drdx + rdydd\,\mathrm{Q} - 2d\mathrm{Q}dsdx + d\mathrm{Q}drdy$$

[+ extra terms $dPds^3/dy + dPdx^2ds^2/dy$ + those arising from $2dx.dy.ddx.ddy.dP$.

It is observed that these terms involve higher orders of *ds* than the first power, and the last term involves a product of second order derivatives. For the sake of completeness, the entire equation is quoted here in the dot notation, before the stage of substituting for the third derivatives, which can of course be done, when further simplification can then be made:

$$\dot{\mathrm{P}} + \frac{3\dot{y}^2\ddot{y}}{\dot{s}^2\ddot{x}}\dot{\mathrm{Q}} + \frac{\dot{y}^3}{\dot{s}^2\ddot{x}}\ddot{\mathrm{Q}} - \frac{\dot{y}^3\dddot{x}}{\dot{s}^2\ddot{x}^2}\dot{\mathrm{Q}} + \frac{\dot{y}^2}{\dot{s}^2}\dot{\mathrm{P}} + \frac{\dot{x}\dot{y}^2}{\dot{s}^2\ddot{x}}\ddot{\mathrm{P}} - \frac{\dot{x}\dot{y}^2\dddot{x}}{\dot{s}^2\ddot{x}^2}\dot{\mathrm{P}} + \frac{2\dot{x}\dot{y}\ddot{y}}{\dot{s}^2\ddot{x}}\dot{\mathrm{P}} = 0.\ ]$$

This equation gives all the possible curves which perfectly flexible bodies are able to form by being displaced in some manner, as understood below.



Problem. To find the curve that a perfectly flexible string forms, to which vertical forces are applied to all points.

Solution. The horizontal forces vanish in this case, leading to Q = 0. Whereby

$0 = \mathrm{E}x + \mathrm{F}y + \int \mathrm{P}dx$; and hence $0 = \mathrm{E}dx +$

Fdy + Pdx = 0; and thus $dP + \frac{F(dxddy - dyddx)}{dx^2} = 0;$
or $dPdx^2dy = Fds^2ddx.$ Truly, $\frac{dsdy}{ddx} = r,$ hence
[from **], $rdPdx^2 = Fds^3.$ And from these equations the curve can be recognised in special examples. Q.E.I.

This equation gives catenaries of all kinds.

Let dP = *ads,* as the forces shall be everywhere equal, which should produce the common catenary for chains of uniform thickness. Moreover we have the equation $adx^2dy = Fdsddx;$ which integrated gives $ay = \mathrm{C} - \frac{\mathrm{F}ds}{dx}$ [recall that *ds* is a constant quantity] or with the constant C ignored, which does not change the nature of the curve, and with - F put in place of F : $aydx = \mathrm{F}ds$ is obtained, and hence [on squaring, re-arranging and taking the square root again], $dx = \frac{Fdy}{\sqrt{(aayy - FF)}}$, which is the equation for catenary curves [Fig. 5], to be applied in this way : on account of assuming F negative, the weight indicated by that letter ought to act in the opposite direction to that in which we put F to pull in the first place [*i. e.* to the right in Fig. 5, as opposed to the left in Fig. 4]. From A the vertical is sent Aa = F/a, and the horizontal aP is the axis of the curve AMB, which is convex towards aP and the tangent at A is parallel to the axis aP, hence AB is the common catenary.

Let $d\mathrm{P} = adx$, then $adx^3dy = \mathrm{F}ds^2ddx$ hence $ay = -\frac{\mathrm{F}ds^2}{2dx^2}$, or with F taken negative, the equation will be, [on substituting $ds^2 = dx^2 + dy^2$] :
$2aydx^2 - \mathrm{F}dx^2 = \mathrm{F}dy^2$. Hence $dx = \frac{dy\sqrt{\mathrm{F}}}{\sqrt{(2ay - F)}}$, which is in agreement with the equation for the parabola of Appolonius [*i. e.* a curve of the form $x^2 = 2a(y - y_0)$ where $\mathrm{F} = 4a^2$].

Problem. To find the curve that a flexible string forms, to which normal forces are applied at each and every point.

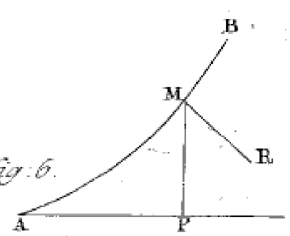


Fig. 6.

Solution. The curve shall be AMB [Fig. 6], and as above AP = x, PM = y; and AM = s. The normal force dN at M acting along the normal MR is resolved to the vertical and horizontal, the vertical force is $d\mathrm{P} = \frac{d\mathrm{N}dx}{ds}$
and the horizontal force $dQ = \frac{d\mathrm{N}dy}{ds}$ thus $\mathrm{P} = \int\frac{dNdx}{ds}$ and $\mathrm{Q} = \int\frac{dNdy}{ds}$; and
$dd\mathrm{P} = \frac{d\mathrm{N}ddx + dxddN}{ds}$;
$ddQ = \frac{d\mathrm{N}ddy + dyddN}{ds};$ these values are substituted into the general equation for perfectly flexible bodies; but in place of ddP, $\frac{dNdy}{r} + \frac{dxddN}{ds}$, and in place of ddQ, $-\frac{dNdx}{r} + \frac{dyddN}{ds}$, as the radius of osculation is applied in the computation [recall that $-\frac{dsdx}{ddy} = \frac{dsdy}{ddx} = r.$];
it is then found that $d\mathrm{N}dr + rdd\mathrm{N} = 0$, thus $rd\mathrm{N} = \mathrm{C}ds$. Therefore the radius of osculation varies inversely as the normal force. Q.E.I.

[The general equation is:

$$0 = rdxdd\,\mathrm{P} + 2d\mathrm{P}dsdy + d\mathrm{P}drdx + rdydd\,\mathrm{Q} - 2d\mathrm{Q}dsdx + d\mathrm{Q}drdy;$$

Hence; $0 = \frac{dNdy.rdx}{ds} + \frac{dx.ddN.rdx}{ds} + \frac{2dNdx.dsdy}{ds} + \frac{dNdx.drdx}{ds}$

$- \frac{dNdx.rdy}{ds} + \frac{dy.ddN.rdy}{ds} - \frac{2dNdy.dsdx}{ds} + \frac{dNdy.drdy}{ds}$; giving :

$(ddN.rdx^2 + dN.drdx^2 + ddN.rdy^2 + dN.drdy^2)/ds = ddN.rds + dN.dr.ds$;

or $(ddN.r + dN.dr).ds = 0$, which integrates to give $r.dN = Cds$ as required.]

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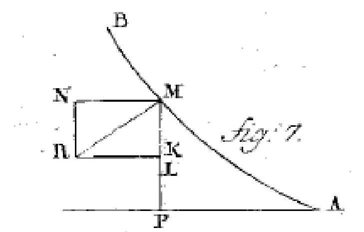
Since this property is in agreement with these recently found, I shall not delay deriving from it the curves for awnings and sails, and which follow from that property.

The solutions to these two last problems just done have been obtained from geometry. When the directions of the forces are of course either parallel amongst themselves or normal to the shape of the curve. Moreover, the curves for any kind of flexible bodies, for which any kind of forces have been applied, ought to be found, which no-one has shown before, except the celebrated Jacob Hermann, the solution of this problem is set out in his *Phoronomia*.



Problem. To find the form of the curve AMB that a perfectly flexible string assumes when two forces, one vertical and the other normal, are applied to each point M.

Solution. The normal MR is resolved in the sides MN and MK, of which MK is vertical and MN horizontal. Let $d\mathrm{P} = \mathrm{ML} + \mathrm{MK}$ and $d\mathrm{Q} = \mathrm{MN}$. It can be said that $\mathrm{ML} = dL$ (the vertical force) and $\mathrm{MR} = d\mathrm{N}$. Then $\mathrm{MK} = \frac{d\mathrm{N}dx}{ds}$ and $\mathrm{MN} = \frac{d\mathrm{N}dy}{ds}$; hence $d\mathrm{P} = d\mathrm{L} + \frac{d\mathrm{N}dx}{ds}$ and $d\mathrm{Q} = \frac{d\mathrm{N}dy}{ds}$. And for the other

$$dd\mathrm{P} = dd\mathrm{L} + \frac{dNddx + dxddN}{ds} = dd\mathrm{L} + \frac{d\mathrm{N}dy}{r} + \frac{dxdd\mathrm{N}}{ds}.$$

Also $dd\mathrm{Q} = \frac{d\mathrm{N}ddy + dydd\mathrm{N}}{ds} = -\frac{d\mathrm{N}dx}{r} + \frac{dydd\mathrm{N}}{ds}$. From which with the variables substituted [in the general equation

$0 = rdxdd\,\mathrm{P} + 2d\mathrm{P}dsdy + d\mathrm{P}drdx + rdydd\,\mathrm{Q} - 2d\mathrm{Q}dsdx + d\mathrm{Q}drdy$, to give :

$$rdx\left(ddL + \frac{dNdy}{r} + \frac{dxddN}{ds}\right) + 2dsdy\left(dL + \frac{dNdx}{ds}\right) + drdx\left(dL + \frac{dNdx}{ds}\right)$$

$$+ rdy\left(-\frac{dNdx}{r} + \frac{dyddN}{ds}\right) - 2dsdx.\frac{dNdy}{ds} + drdy.\frac{dNdy}{ds};$$

to give as required :

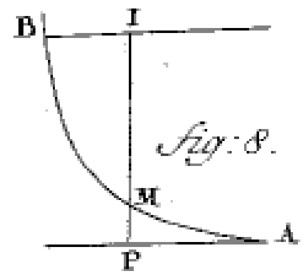
$rdx.ddL + rdsddN + 2dsdy.dL + drdx.dL + dr.ds.dN$]

$d\mathrm{N}dsdr + rdsdd\mathrm{N} + rdxdd\mathrm{L} + d\mathrm{L}dxdr + 2d\mathrm{L}dsdy = 0$ is found,

i.e. $d(rdN)ds + d(rdL)dx + 2dLdsdy = 0$;

or $rdsd\mathrm{N} + rdxd\mathrm{L} + \int d\mathrm{L}dsdy = Cds^2$, where $2d\mathrm{L} \to d\mathrm{L}$.

From which the nature of the curve will be apparent. Q.E.I.
The vertical force *d*L is put proportional to the element of the curve *ds*. Since the string shall have the same weight and density everywhere, then *d*L = *ads* and we have



$rdsdN + ardsdx + ayds^2 = Cds^2$,

which divided by *ds*, gives $rdN + ardx + ayds = Cds$.

[There are several obvious misprints in this last derivation, which must agree with the previous case; hence the integration constant C*ds*.]

AMB shall be a heavy linen sail filled with liquid as far as BI, the normal force *d*N shall be as MI, let PI = *b*; then MI = *b - y*. Therefore put *d*N = *bds - yds ;* this gives $brds - yrds + ardx + ayds = Cds$. But

$r = \frac{dsdy}{ddx}$; hence $bdsdy - ydsdy + adxdy + ayddx = Cddx$, which

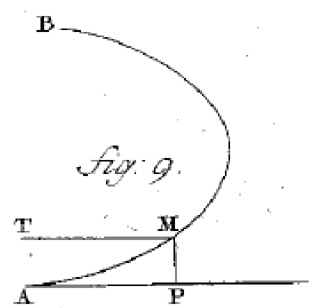
integrated gives $byds - \frac{1}{2}yyds + aydx = eds + Cdx$ or

$(yy\text{-}by + e)ds = aydx - Cdx$. [The term *adxdy* has been ignored]

With the constants changed in order that numbers can be avoided : this equation becomes the following, [on squaring, collecting terms, and taking the square root again]:

$$dx = \frac{(yy - by + e)dy}{\sqrt{(ay-c)^2 - (yy-by+e)^2}}$$

in order that this equation becomes valid for the axis AP; it is necessary in order that for vanishing *y*, that *dy:dx* becomes equal to 0:1; and *cc* = *ee*, or *c* = ± *e*. If *a* becomes 0, the equation is obtained for the noted canvas curve [as the pressure of the liquid acting on the element is far greater than the weight of the canvas element].



The curvature of a uniform heavy sail AMB is sought [Fig.9]. The wind may intrude following TM parallel to the axis AP. The strength of this will be, since the wind acts normally to the curve, as the square of the sine of the angle AMT [from Hermann's *Phoronomia*, Ch.XXI.], *i. e.* this is put as $\frac{dy^2}{as^2}$ and

hence $dN = \frac{dy^2}{ds}$, giving $\frac{rdy^2}{ds} + ardx + ayds = Cdx$. Moreover since

$r = \frac{dsdy}{ddx}$ this will be $dy^3 + adsdydx + aydsddx = Cdsddx$. But $ddx = \frac{-dyddy}{dx}$

hence $dy^2dx + adsdx^2 - aydsddy + Cdsddy = 0$; put $ds = pdy$; this gives

$ddy = \frac{dpdy}{p}$ and $dx = dy\sqrt{pp-1}$. From this $dy\sqrt{pp-1} + ap(pp-1)dy = cdp - aydp$,

or $\frac{dy}{c-ay} = \frac{dp}{\sqrt{(pp-1)} + ap(pp-1)}$. Let $\sqrt{pp-1} = p - q$; then $\frac{dy}{c-ay} = \frac{-4qdq}{4qq + a - aq^4}$;

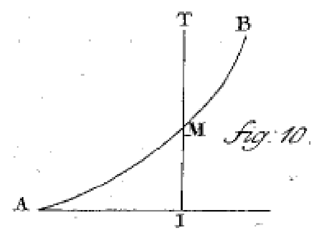
$4qq + a - aq^4$ is resolved in the two factors $1 + \beta qq$ and $a + \delta qq$; where $\beta = \frac{2}{a} + \sqrt{(\frac{4}{aa}+1)}$ and $\delta = 2 - \sqrt{(4+aa)}$. And this becomes $\frac{dy}{c-ay} = \frac{mqdq}{1+\beta qq} + \frac{nqdq}{a+\delta qq}$, where $m = \frac{-4-2\sqrt{(4+aa)}}{a\sqrt{(4+aa)}}$ and $n = \frac{4-2\sqrt{(4+aa)}}{\sqrt{(4+aa)}}$. Therefore

$$-\frac{1}{a}l(c-ay) = \frac{m}{2\beta}l(1+\beta qq) + \frac{n}{2\delta}l(a+\delta qq) = \frac{-l(1+\beta qq)+l(a+\delta qq)}{\sqrt{(4+aa)}} =$$

$$\frac{1}{\sqrt{(4+aa)}}l\frac{a2+aqq(2-\sqrt{(4+aa)})}{a+qq(2+\sqrt{(4+aa)})};$$ but $q = p - \sqrt{pp-1} = \frac{ds-dx}{dy}$;

hence

$$\left(\frac{c-ay}{D}\right)^{\sqrt{(4+aa)}:a} = \frac{ady^2+(ds-dx)^2(2+\sqrt{(4+aa)})}{ady^2+(ds-dx)^2(2-\sqrt{(4+aa)})}$$ is obtained for the heavy sail. [Note the use of l for the natural log function.]



If the wind is applied downwards along the direction TM, the force of this on the sail will be as the square of the sine of the angle AMT : *i. e.* as $\frac{dx^2}{ds^2}$. Therefore put $dN = \frac{dx^2}{ds}$ and this gives $\frac{dx^2}{ds} + ardx + ayds = Cds$. Since moreover $r = \frac{dsdy}{adx}$, it will be $dx^2dy + adsdsdy + aydsddx = Cdsddx$, hence

$$\frac{dy}{Cds-ayds} = \frac{ddx}{dx^2+adsdx} = \frac{1}{ads}\cdot\frac{ddx}{dx} - \frac{1}{ads}\cdot\frac{ddx}{dx+ads}$$ which integrated becomes

$$= \frac{-1}{ads}l(cds-ayds)\frac{-1}{ads}l\frac{dx+ads}{dx} + \frac{1}{ads}lbds.$$ Hence

$\frac{c-ay}{b} = \frac{dx+ads}{dx}$ or $cdx - aydx = bdx + abds$. Let $c-b=e$; then $dx^2(e-ay)^2 = aabbds^2$, consequently $dx = \frac{abdy}{\sqrt{(s-ay)^2-aabb}}$ as AP is the axis, it is necessary that $dy : dx = 0 : 1$; if $y = 0$, therefore $e = \pm ab$. Thus $dx = \frac{bdy}{\sqrt{(yy\pm 2by)}}$. Which is the equation for a catenary and the same variable quantity a is kept, but not from the weight of a string that may be hanging. Hence from this example it is apparent that this can happen, for the force of the wind, as gravity, can produce the same catenary separately.

The normal force is constant too, for

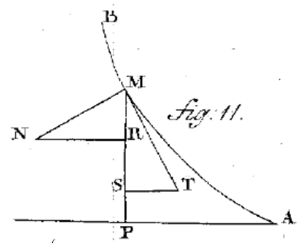
$dN = bds$ erit $brds + ardx + ayds = cdx$, since truly $r = \frac{dsdy}{ddx}$, this becomes

$bdsdy + adxdy + ayddx = cddx$, which integrated gives:

$\frac{eds}{c-ay} = bds + adx$. Let $e = bc + ac$; then $cds = cdx - byds - aydx$,

and $dx = \frac{(c+by)dy}{\sqrt{(c-ay)^2-(a+ay)^2}}$.

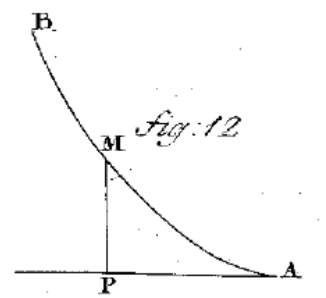
Problem. If two forces are applied to any point whatever M on the curve AMB, [Fig. 11] of which one is applied along the normal as MN, and the other along the tangent MT; the equation for the curve is found, that a perfectly flexible forms.



Solution. Both forces are resolved along the vertical and horizontal sides [of rectangles], truly MN into MR and RN, and MT into MS and TS; Hence dP = MR + MS and dQ =NR - TS. But MN = dN and MT = dT. Hence $dP = \frac{dNdx + dTdy}{ds}$ and $dQ = \frac{dNdy - dTdx}{ds}$; from which $ddP = \frac{dNddx + dxddN + dTddy + dyddT}{ds} = \frac{dNdy}{r} + \frac{dTdx}{r} + \frac{dxddN}{ds} + \frac{dyddT}{ds}$. And $ddQ = \frac{-dNdx}{r} - \frac{dRdy}{r} - \frac{dyddN}{ds} - \frac{dxddT}{ds}$. From which by substituting these values the equation is obtainded : $dNdr + rddNd + Tds = 0$, or after integration rdN +Tds = C$ds;$ on account of $r = \frac{dsdy}{ddx}$, it follows that this becomes $dNdy + Tddx = Cddx$. Q.E.I.

This equation has this use that, not only is it easy to be applied to a very simple case, but also it can easily be applied to all cases, but in the general case it is possible to resolve the forces along the tangent and the normal. Besides this advantage I think that there should be caution too on going into a computation, for in place of the vertical and horizontal components, the most general equation is returned in the most succinct way by the normal and tangential components, for which moreover this equation arises :
$$dNdrds + rdsddN + dTds^2 = rd^3Z + drddZ + \frac{ds^2dz}{r} \text{ ubi } Z = \frac{\Delta v}{r}.$$

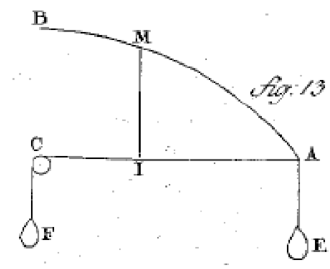
Problem. To find the general equation for curves, that strings of any elastic nature should have, having no forces applied to the individual points.



Solution. In this case both $dP = 0$ and $dQ = 0$. Hence
$$rd^3z + drddz + \frac{ds^2dz}{r} = 0,$$
suitably arranged from the general equation ; in which
$dN = 0$ and $dT = 0$:
$$\text{hence } rrddzd^3z + rdrddz^2 + ds^2dzddz = 0,$$
which integrated gives $rrddz^2 + ds^2dz^2 = ads^4$ or $rddz = ds\sqrt{(ads^2 - dz^2)}$. But $r = \frac{dsdy}{ddx}$, hence $\frac{ddz}{\sqrt{(ads^2 - dz^2)}} = \frac{ddx}{dy} = \frac{ddx}{\sqrt{ds^2 - dx^2}}$. This with the help of logarithmic integration gives :
$$\frac{1}{\sqrt{-1}} l \frac{dz + \sqrt{(dz^2 - ads^2)}}{ds\sqrt{-a}} = \frac{1}{\sqrt{-1}} l \frac{dx + \sqrt{(dx - ds)^2}}{ds\sqrt{-b}}, \text{ or } Ex + Fy = \frac{\Delta v}{r}.$$

The same equation is found from the first equation found, where $\int Pdx$ et $\int$Qdy vanish
and thus there remains $\mathrm{E}x+\mathrm{F}y=\frac{Av}{r}$. Q.E.I.

Problem. To find the curvature with elasticity but without gravity, where there is the same elastic force everywhere [Fig. 13].



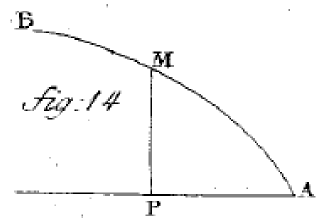
Solution. Since the elastic force is the same everywhere, put
$v=a$; and hence $\mathrm{E}x+\mathrm{F}y=\frac{Aa}{r}$. It is possible to put Ex + Fy = Gt, where t can denote the abscissa in place of the other assumed, and hence $\mathrm{G}t=\frac{Aa}{r}$, which equation now allows the elastic curve to be recognised, that truly in this way can be integrated. Let
$\mathrm{E}xdsdy+\mathrm{F}ydsdy=\mathrm{A}addx$ or $xdy+\frac{\mathrm{F}ydy}{\mathrm{E}}=\frac{\mathrm{A}addx}{\mathrm{E}ds}$.
Also let $\mathrm{E}xdsdy+\mathrm{F}ydsdy=-\mathrm{A}addy$ or $ydx+\frac{\mathrm{E}xdx}{\mathrm{F}}=-\frac{\mathrm{A}addy}{\mathrm{F}ds}$. Consequently on addition and
integration we have $yx+\frac{Exx}{2F}+\frac{Fyy}{2E}=\frac{Aadx}{Ed}+\frac{Aady}{Fds}+C$. Hence 2FE$dxds$ + EE$xxds$ + FF$yyds$ =
2AFadx -2AEady +2EFCds = ds(Ex + Fy)^2. Which also gives the elastic curve. Q.E.I.

Problem. To find the curve formed by a filament AMB of uniform weight and with the same elasticity everywhere.

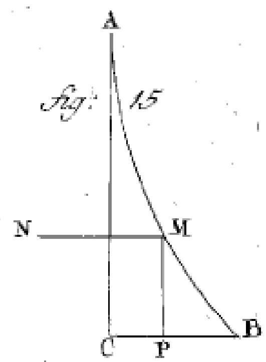
Solution. When the string or wire has the same weight[/unit length] everywhere, dP is constant, equal to ads : hence P = a and Q = 0. Besides, since the elasticity is everywhere the same, put v = b; and the equation becomes
$\frac{Ab}{r}=\mathrm{E}x+\mathrm{F}y+\int asdx$. If as is increased by the constant E then the equation is not changed,
and it becomes $\frac{Ab}{r}=\mathrm{F}y+\int asdx$ or $-\frac{A\mathrm{bdr}}{\mathrm{rr}}=\mathrm{F}dy+asdx$: and with excess constants discarded :
$asdx+cdy+\frac{bdr}{rr}=0$. But $r=\frac{dsdy}{ddx}$; hence $\frac{1}{r}=\frac{ddx}{dsdy}$: from which
$\frac{dr}{rr}=\frac{ddxddy-dyd^3x}{dsds^2}=\frac{-dxddx^2-dy^2d^3x}{dsdy^3}$. Hence $asdsdxdy^3+cdsdy^4=bdxddx^2+bdy^2d^3x$. Put dx $=pds$,
then $dy=ds\sqrt{(1-pp)}$ and $ddx=dpds$ et $d^3x=dsddp$. From which by substitution we obtain
$aps(1-pp)^{3:2}ds^2+c(1-pp)^2ds^2=bpdp^2+bddp(1-pp)$. Truly there is no agreement that
it is possible to reduce this equation to construct a solution.

There is this elegant problem remaining, what truly is the curve produced by perfectly elastic filaments in the plane: and indeed nothing stand out more than a rope,

which shall be perfectly elastic more than any other [example] and without gravity, when bend in a fluid of the same specific gravity.
This same problem has been published in the Act. Eruditorum Lips. A 1724, in order that the curvature of both elastic and inelastic ropes can be found, truly I know no more about the solution of this problem than that given at about the same time as by myself, by the most distinguished Daniel Bernoulli.

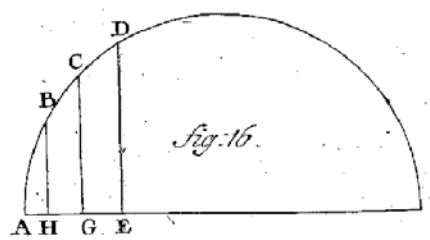


Problem. To find the curve BMA [Fig. 14] that a planar elastic curve fixed at B forms under its own weight.
Solution. This follows on from the preceding, only when the applied weights F and E have disappeared, where we have this equation $\frac{Ab}{r} = \int asdx$, or by getting rid of superfluous constants, $\frac{Ab}{r} = \int sdx$,
which as above is reduced as follows:
$sp^2(1-pp)^{\frac{3}{2}} ds^2 = Apdp^2 + A(1-pp)ddp.$ But neither this adaptation is
is able to effect a construction of the curve.



Problem. To find the curve AB [Fig.15] of a filament fixed at B, but free to be moved by the wind NM.

Solution. Let the force of gravity acting on M = *ads*, and the force of the
wind $= \frac{-bdy^2}{ds}$.
This equation is found
$brdy^2 = ardsdx + ayds^2$. The radius of osculation $r = \frac{dsdy}{ddx}$.
Whereby $bdsdy^3 = ards^2dxdy + ayds^2ddx.$ seu $bdy^3 = adsdxdy + aydsddx.$ Put $dx = pdy$ then
$ddx = pddy + dpdy = (\text{ob } dxddx = -dyddy)\,\frac{-pdxddx}{dy} + dpdy$, from which $ddx = \frac{dpdy}{1+pp}$ is found. With these
factors substituted, the equation becomes :
$bdy\sqrt{(1+pp)} = apdy(1+pp) + aydp$ et $\frac{dy}{y} = \frac{adp}{b\sqrt{(1+pp)} - ap(1+pp)}$. In which the indeterminate quantities are separated from each other, and on account of which the curve sought can be constructed.



[There is an unused diagram, Fig. 16, at the end of the pdf file used in this translation; either this was not used in the original publication, or a page is missing from the copy of the original paper. IB.]

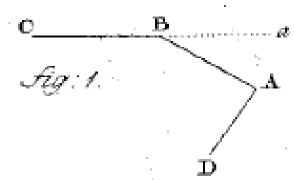
SOLUTIO PROBLEMATIS
DE
Invenienda curva, quam format lamina utcunque elastica in singulis punctis a potentiis quibuscunque sollicitata.

Auctore
Leonhardo Eulero.

Curva, quam Cel. Jacobus Bernoulli primus et postea plures alii laminae elasticae incurvatae assignarunt, quaeque nomine curvae elasticae nota est, nonnisi laminae elasticae gravitatis experti competere ex solutionibus eorum intelligitur. Curvatura autem laminae elasticae gravis, tametsi haec sola in rerum natura locum obtinere queat, tamen a nemine adhuc, quantum scio, determinata est. Incidimus nuper Clar. D. Bernoulli, et ego in hanc quaestionem, eamque non inelegantem existimantes aggressi, atque solutiones eodem tempore et egregie inter se congruentes consequuti sumus. Cum vero solutio prima fronte non nihil difficilis inventa visa sit solutionem meam hic quoque exponere non abs re fore arbitratus sum. Quaestio quidem primum tantum ad hoc extendebatur, ut inveniatur curva, quam lamina elastica gravis uno termino firmata, altero potentiam quamvis applicatam habens format. Nunc vero hanc rem generalius complectar laminam in singulis punctis utcunque elasticam, et praeter pondus appensum qualescunque applicatas habentem positurus; Idque eapropter, tum, quod, etsi quis unum vel alterum casum particularem elicuerit, problema tamen hoc modo perceptum exinde non adeo facile solvatur; tum propter summam eius universalitatem : etenim non solum ad curvaturas laminarum quomdocunque elasticarum extenditur. Verum etiam ad curvas corporum perfecte flexibilium et a potentiis quomodocunque sollicitatorum accommodari potenst; ita ut ex eo omnium corporum flexibilium curvaturae inveniri possint. Accedit praeterea , quod solutio problematis hoc sensu accepti vix prolixior evadat, quam in quovis casu speciali. Ante vero quam ipsum problema aggredi possum, necesse est nonnulla praemittere solutioni inservientia.

Hypothesis.

1. Si duae virgae A*b*, BC in B elatere iunctae a potentia AD in situm ABC torqueantur, ut ang. AB*a* comprehendat. Erit momentum potentiae AD in B ut vis elastica in B et angulus Aba coniunctium. Hoc scilicet factum aequipollet facto ex potentia AD in AB. Assumitur vulgo haec hypothesis; eius tamen veritas si angulus Aba est vehementer parvus satis probabiliter potest physice demonstrari.



Lemma 1. Si curvae ANM in puncto quecunque N applicata sit potentia NO, eaque resolvatur in horizontalem NS et verticalem NR. Erit vis pot. NO ad curvam AM circa M rotandam seu eius momentum aequale NR.FP + NS.GQ. Ductis MQ, AP hirizontalibus et AQ, PM verticalibus.

Demonstratio. Potentia NO aequivalet duabus NR at NS simul agentibus. Momentum vero potentiae NR in M est, ut ex principiis staticis constat, NR.PF; et momentum potentiae NS in M est NS. GQ. Ergo cum ambae potentiae curvam AM versus eandem plagam convertere conentur, erit momentum ambarum seu momentum potentiae NO = NR.FP + NS.GQ. Q.E.D.

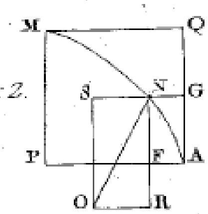
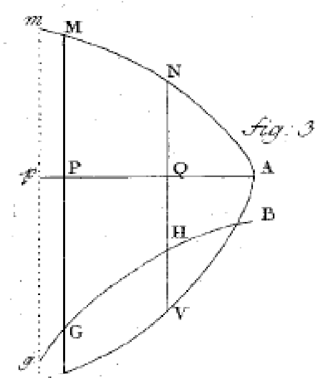


fig: 2.

Lemma 2. Si curvae AM in singulis punctis N potentiae parallelae perpendiculares nimirum un AP, applicatae, fuerint determinatae a curva BG, ita ut in punctum N agat potentia. quae est ut QN. Construaturque alia curva AVT, cuius applicatae QV sint ut areae AQHB. Erit summa momentorum omnium potentiarum ad curvam ANB circa M flectendam ut area APT.

Demonstratio. Momentum potentiae QH in M est QH.PQ. Puncto M assumatur proximum *m*. ducaturque applicata *mp*; erit momentum potentiae QH in m aequale QH.Q*p*. Ergo differnetia horum momentorum est QH.P*p*. Idem cum de singulis potentiis valeat, erit differentia omnium momentorum in M et in *m* agentium aequalis areae ABGP in P*p* seu PT.P*p*, i. e. elemento P*pt*T. Si iam ponatur summa omnium momentorum in m = M + *d* M, quarum differentia est *d* M , erit igitur d M = P*pt*T, consequenter M erit = areae - APT. Q.E.D.

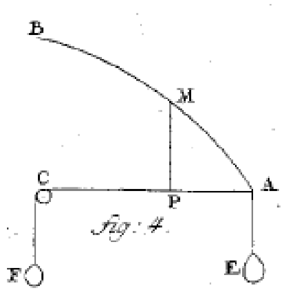


Dicantur iam abscissae in AP assumtae, x, et repondentes applicatae in curva AT, P; Designat autem P summam omnium potentiarum in x contentarum; ergo summa omnium momentorum in M $= \int \mathrm{P}dx$.

Problema Generale.

Sit lamina BMA utcunque elastica et in singulis punctis a potentiis quibusuis follicitata, fixa autem ea sit in B, atque in A duo pondera appensa habens E et F, quorum E secundum verticalem AE, et alterum F secundum horizontalem AC trahit. Oportet determinare curvam AMB iuxta quam lamina hoc modo sollicitata flectitur.

Solutio. Sumatur horizontalis AC pro axe, in quo capiantur abscissae AP = x et AM = s, cuius elementa constantias accipiantur. Ponatur radius curvedinis in M = r; erit angulus, quem duo elementa in M constituunt, reciproce ut r. Designetur vis elastica in M litera v; erit vis hunc angulum producens ut $\frac{v}{r}$ (hyp.) Huic ergo proportionalis esse debet summa omnium momenturum in M agentium tam ortorum a singulis potentiis curvae AP applicatis, quam a ponderibus E et F. Est autem momentum potentiae seu ponderis E in M = E.AP = E.x (Lemma 1.) et ponderis F momentum est F.PM = Fy (cit.) Praeter haec sollicitatur punctum M etiam a potentiis singulorum punctorum arcus AM.



Iis in verticales secundum AE et horizontales secundum AC agentes resolutis, vocetur summa omnium verticalium P, et horizontalium ab A in M usque, Q. Erit summa momentorum potentiarum verticalium = $\int Pdx$ (Lem.2.) et summa momentorum potentiarum horizontalium = $\int Qdy$. Erit itaque tota vis in M agens = Ex + Fy + $\int Pdx$ + $\int Qdy$. Cui cum proportionalis esse debeat $\frac{v}{r}$, habebitur haec aequatio $\frac{Av}{r} = Ex + Fy + \int Pdx + \int Qdy.$ Si loco P + E scribatur tantum P, et Q loco Q + F; habitur $\frac{Av}{r} = \int Pdx + \int Qdy$, sit $\frac{Av}{r} = Z$; erit $Z = \int Pdx + \int Qdy$ seu $dZ = Pdx + Qdy.$ Ex qua natura curvae AMB cognoscitur. Q.E.I.

Ut usus huius aequationis melius percipiatur, ad casus particulares eam accommodabo, eosque partim iam tractatos, ut congruentia eorum perspici queat, partim vero ad nondum agitatos, ut plurimas a natura formatas curvas adhuc ignotas in lucem producam.

Problema. Invenire aequationem generalem pro curvis, quas corpora perfecte flexibilia a potentiis quomodocunque sollicitata formant.

Solutio. Obtinebimus corpora perfecte flexibilia, quando vis elastica ubique evanescit, tum enim vel minima vis duo elementa ad quemvis angulum inclinare valebit; Exprimitur autem quantitas vis elasticae litera v ponatur igitur v = 0 et resultat aequatio $0 = Ex + Fy + \int Pdx + \int Qdy$, quae ergo satisfaciet quaesito. Q.E.I.

Ut autem P et Q, quippe quae a summatione pendent, eliminentur, et loco eorum dP et dQ producantur, quae denotant potentias ipsas in punctis M applicatas, differentietur aequatio et habebitur $E.dx + F.dy + P.dx + Q.dy = 0.$ Ergo $Q + \frac{Pdx}{dy} + F + \frac{Edx}{dy} = 0$. Unde $dQ + \frac{Pdyddx - Pdxddy}{dy^2} + \frac{dPdx}{dy} + \frac{Edyddx - Edxddy}{dy^2} = 0$. Sed cum ponatur ds constans erit $dyddy = -dxddx$, et $dyddx - dxddy = \frac{ds^2 ddx}{dy}$. Quare

$$dQ + \frac{Pds^2ddx}{dy^3} + \frac{Eds^2ddx}{dy^3} + \frac{dPdx}{dy} = 0.$$

Et ex hac $\frac{dQdy^3}{ds^2ddx} + P + E + \frac{dPdy^2dx}{ds^2ddx} = 0$. Unde porre

$$\frac{dy^3ddQ}{ds^2ddx} + \frac{3dQdy^2ddxddy - dQdy^3dx^3}{ds^2ddx^2} + dP + \frac{dy^2dxddP}{ds^2ddx} + \frac{2dPdydxddyd\,dx + dPdy^2ddx^2 - dPdy^2dxd^3x}{ds^2ddx^2} = 0.$$

Ex hac prodibit haec :

$$dy^3ddQddx - 3dQdydxddx^2 - dQdy^3d^3x + dy^2dxddPddx - 3dPdx^2ddx^2$$
$$+ dPds^2ddx^2 - dPdy^2dxd^3x = 0.$$

Haec aequatio fiet multum simpicior, si introducatur radius osculi r, que est $= \frac{dsdy}{ddx} = \frac{-dsdx}{ddy}$ tunc enim prodibit

$$0 = rdxddP + 2dPdsdy + dPdrdx + rdyddQ - 2dQdsdx + dQdrdy.$$

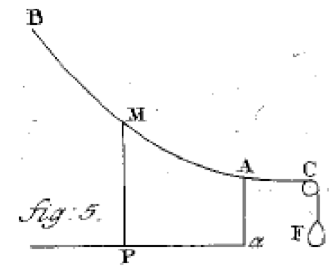
Quae aequatio omnes possilibes curvas, quas corpora perfecte flexibilia quomodocunque sollicitata formare possunt sub se comprehendit.

Probl. Invenire curvam, quam format filum perfecte flexile, cui in singulis punctis potentiae verticales sunt applicatae.

Solut. Evanescunt igitur hoc in casu potentiae horizontales, unde Q = 0. Quare $0 = Ex + Fy + \int Pdx$; et hinc $0 = Edx + Fdy + Pdx = 0$; porroque $dP + \frac{Fdxddy - dyddx}{dx^2} = 0;$

seu $dPdx^2dy = Fds^2ddx$. Est vero $\frac{dsdy}{ddx} = r$, unde $rdPdx^2 = Fds^3$. Et ex hisce aequationibus curva in exemplis specialibus cognoscetur. Q.E.I. Dat haec aequatio omnis generis catenarias.

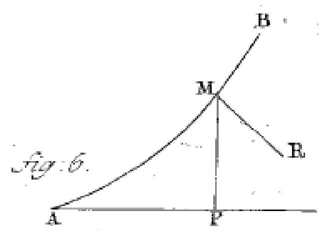
Sit dP = *ads* seu potentiae sint ubique aequales, prodire debet catenaria communis pro catenis aequaliter crassis. Habebitur autem $adx^2dy = Fdsddx$; quae integrata dat $ay = \mathrm{C} - \frac{\mathrm{F}ds}{dx}$ seu neglecta constante C, quae naturam curvae non immutat, et loco F posito -F obtinetur $aydx = \mathrm{F}ds$, et hinc $dx = \frac{Fdy}{\sqrt{(aayy-FF)}}$ quae est aequatio pro cantenaria [Fig. 5], hoc modo applicanda : ob F negative assumptum, pondus ea littera indicatum trahere debet in plagam contrariam ei , versus quam F trahere initio ponebamus. Ex A demittatur verticalis A*a* = F:*a*, et horizontalis *a*P erit axis curvae AMB, quae erit convexa versus *a*P et tangentem in A habebit parallelam axi *a*P, erit igitur AB catenaria vulgaris.



Sit $d\mathrm{P} = adx$, erit $adx^3dy = \mathrm{F}ds^2ddx$ ergo $ay = -\frac{\mathrm{F}ds^2}{2dx^2}$, seu sumpto F negatio erit $2aydx^2 - \mathrm{F}dx^2 = \mathrm{F}dy^2$. Unde $dx = \frac{dy\sqrt{\mathrm{F}}}{\sqrt{(2ay-F)}}$ quae est aequatio pro parabola appoloniana, ut constat.

Problema. Invenire curvam, quam format filium flexile, cui in singulis punctis potentiae normales sunt applicatae.

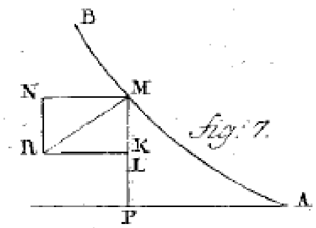
Solutio. Sit curva AMB, et ut supra AP = *x*, PM = *y*; et AM = *s*. Sit potentia in M normalis MR = *d*N resolvatur ea in verticalem et horizontalem , erit verticalis $d\mathrm{P} = \frac{d\mathrm{N}dx}{ds}$ et horizontalis $dQ = \frac{d\mathrm{N}dy}{ds}$ unde $\mathrm{P} = \int \frac{dNdx}{ds}$ et $\mathrm{Q} = \int \frac{dNdy}{ds}$; atque $dd\mathrm{P} = \frac{d\mathrm{N}ddx+dxddN}{ds}$ et $ddQ = \frac{d\mathrm{N}ddy+dyddN}{ds}$; quibus valoribus substitutis in aequatione generali pro corporibus perfecte flexibilibus; sed loco *dd*P, $\frac{dNdx}{r} + \frac{dxddN}{ds}$, $\frac{dNdx}{r} + \frac{dxddN}{ds}$, ut radius osculi in computum ducatur; invenietur $d\mathrm{N}dr + rdd\mathrm{N} = 0$ unde $rd\mathrm{N} = \mathrm{C}ds$. Est igitur potentia normalis reciproce ut radius osculi. Q.E.I.



Convenit haec proprietas cum iam inventis, quare non immorabor derivandas ex ea curvius lintheariis, velariis, et quae ex hac proprietate consequuntur.

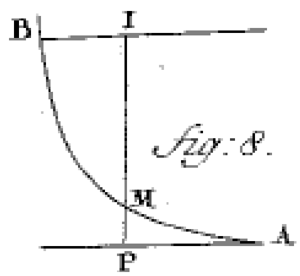
Haec duo postremo problemata iam dudum a Geometris solutiones nacta sunt. Quando scilicet potentiarum directiones vel inter se parallelae vel in curvam formatam normales sunt. Quomodo autem curvae corporum flexibilium, quibus potentiae qualescunque sunt applicatae, inveniri debeant, nemo adhuc mostravit, praeter Celeberrimum Iac. Hermannum, cuius in Phoronomia extat huius problematis solutio.

Problema. Invenire curvam, quam format filum AMB perfecte flexile, cui in singulis punctis M applicatae sunt duae potentiae verticales et normales.



Solutio. Resolvatur normalis MR, in laterales MN et MK, quarum MK sit verticalis et MN horizontalis. Erit $dP = ML + MK$ et $dQ = MN$. Dicatur $ML = dL$ et $MR = dN$. Erit $MK = \frac{dNdx}{ds}$ et $MN = \frac{dNdy}{ds}$; unde $dP = dL + \frac{dNdx}{ds}$ et $dQ = \frac{dNdx}{ds}$. Et ulterius $ddP = ddL + \frac{dNddx + dxddN}{ds} = ddL + \frac{dNdy}{r} + \frac{dxddN}{ds}$. Atque $ddQ = \frac{dNddy + dyddN}{ds} = \frac{dNdx}{r} + \frac{dyddN}{ds}$. Quibus varoribus substitutis obtinebitur $dNdsdr + rdsddN + rdxddL + dLdxdr + 2dLdsdy = 0$, seu $rdsdN + rdxdL + \int dLdsdy = 0$. Unde natura curvae patebit. Q.E.I.

Ponatur potentia verticalis dL, proportionalis elemento curvae ds. Ut habeatur filum ubique aequaliter crassum et grave. Sit igitur $dL = ads$ habebitur

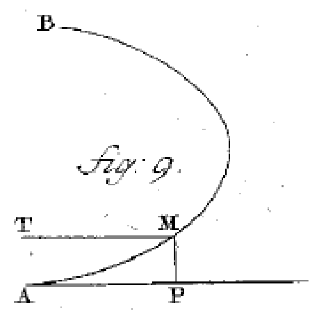


$rdsdN + ardsdx + ayds^2C = ds^2$,

quae divisa per ds, dat $rdN + ardx + ayds = Cds$.

Sit AMB lintearia gravis usque in BI liquore repleta, erit vis normalis dN ut MI, sit $PI = b$; erit $MI = b - y$. Ponatur igitur $dN = b - y$; erit $brds - yrds + ardx + ayds = Cds$. Est autem $r = \frac{dsdy}{ddx}$; unde $bsdy - ydsdy + adxdy + ayddx = Cddx$, quae integrata dat $byds - \frac{1}{2}yyds + aydx = eds + Cdx$ seu $(yy\text{-}by + e)ds = aydx - Cdx$.

Mutatis constantibus, ut numeri evitentur : haec aequatio transibit in sequentem $dx = \frac{(yy - by + e)dy}{\sqrt{(ay-c)^2 - (yy-by+c)^2}}$ ut haec aequatio pro axe AP valeat; oportet ut evanescente y fiat $dy:dx = 0:1$; eritque $cc = ee$, et $c = \pm e$. Si a fiat = 0, habebitur aequatio pro nota lintei curva.



Quaeratur curvatura velariae gravis uniformis AMB. Irruat ventus secundum TM parallelam axi AP. Erit eius vis, qua in curvam normaliter agit ut quadratam sinus anguli AMT id est ut $\frac{dy^2}{as^2}$ ponatur itaque $dN = \frac{dy^2}{ds}$, habebitur $\frac{rdy^2}{ds} + ardx + ayds = Cdx$. Cum autem sit $r = \frac{dsdy}{ddx}$ erit $dy^3 + adsdydx + aydsddx = Cdsddx$. Est autem $ddx = \frac{-dyddy}{dx}$ ergo $dy^2dx + adsdx^2 - aydsddy + Cdsddy = 0$; ponatur $ds = pdy$; erit

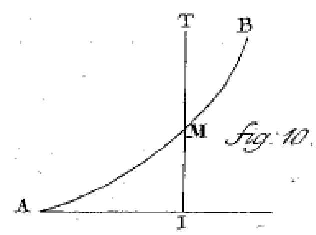
$ddy = \frac{dpdy}{p}$ et $dx = dy\sqrt{pp-1}$. Unde erit $dy\sqrt{pp-1} + ap(pp-1)dy = cdp - aydp$, seu $\frac{dy}{c-ay} = \frac{dp}{\sqrt{(pp-1)}+ap(pp-1)}$. Sit $\sqrt{pp-1} = p - q$; erit $\frac{dy}{c-ay} = \frac{-4qdq}{4qq+a-aq^4}$; resolvatur $4qq + a - aq^4$ in duos factores $1 + \beta qq$ et $a + \delta qq$; ubi est $\beta = \frac{2}{a} + \sqrt{(\frac{4}{aa}+1)}$ et $\delta = 2 - \sqrt{(4+aa)}$. Et hinc erit $\frac{dy}{c-ay} = \frac{mqdq}{1+\beta qq} + \frac{nqdq}{a+\delta qq}$, ubi

$m = \frac{-4-2\sqrt{(4+aa)}}{a\sqrt{(4+aa)}}$ et $n = \frac{4-2\sqrt{(4+aa)}}{\sqrt{(4+aa)}}$. Erit igitur

$$-\tfrac{1}{a}l(c-ay) = \frac{m}{2\beta}l(1+\beta qq) + \frac{n}{2\delta}l(a+\delta qq) = \frac{-l(1+\beta qq)+l(a+\delta qq)}{\sqrt{(4+aa)}} =$$

$\frac{1}{\sqrt{(4+aa)}}l\frac{a2+aqq(2-\sqrt{(4+aa)})}{a+qq(2+\sqrt{(4+aa)})}$; est autem $q = p - \sqrt{pp-1} = \frac{ds-dx}{dy}$; unde habebitur

$(\frac{c-ay}{D})^{\sqrt{(4+aa)}:a}\ \frac{ady^2+(ds-dx)^2(2+\sqrt{(4+aa)})}{ady^2+(ds-dx)^2(2-\sqrt{(4+aa)})}$ pro velaria gravi.

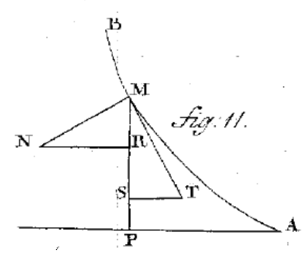


Si ventus incidat deorsum iuxta TM, erit eius vis in velum ut quadratum sinus anguli AMT : id est ut $\frac{dx^2}{ds^2}$. Ponatur igitur $DN = \frac{dx^2}{ds}$ habebitur $ardx + ayds = Cds$. Quia autem $r = \frac{dsdy}{adx}$, erit $dx^2dy + adsdsdy + aydsddx = Cdsddx$, unde

$\frac{dy}{Cds-ayds} = \frac{ddx}{dx^2+adsdx} = \frac{1}{ads}.\frac{ddx}{dx} - \frac{1}{ads}.\frac{ddx}{dx+ads}$ quae integrata abit in hanc

$= \frac{-1}{ads}l(cds - ayds)\frac{-1}{ads}l\frac{dx+ads}{dx} + \frac{1}{ads}lbds.$ Ergo

$\frac{c-ay}{b} = \frac{dx+ads}{dx}$ seu $cdx - aydx = bdx + abds$. Sit $c - b = e$; erit $dx^2(e-ay)^2 = aabbds^2$, consequenter $dx = \frac{abdy}{\sqrt{(s-ay)^2-aabb}}$ ut AP sit axis, oportet sit $dy : dx = 0 : 1$. si $y = 0$ erit ergo $e = \pm ab$. Ideoque $dx = \frac{bdy}{\sqrt{(yy\pm 2by)}}$. Quae est aequatio pro catenaria eademque manet quomodocunque a varietur, ut ergo non a pondre fili pendeat. Hoc autem ita accidere oportere ex eo patet, quod tam vis venti, quam gravitatis seorsum eandem catenariam producant.



Sit vis normalis quoque constans, nempe $dN = bds$ erit $brds + ardx + ayds = cdx$, quia vero $r = \frac{dsdy}{ddx}$, erit

$bdsdy + adxdy + ayddx = cddx$, quae integra dat

$\frac{eds}{c-ay} = bds + adx$. Sit $e = bc + ac$; erit $cds = cdx - byds - aydx$,

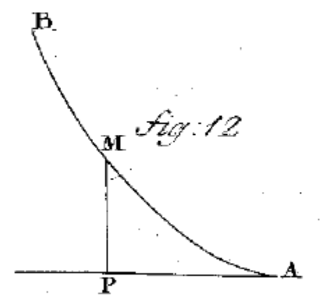
atque $dx = \frac{(c+by)dy}{\sqrt{(c-ay)^2-(a+ay)^2}}$.

Problema. Si curva AMB in quovis puncto M duae potentiae applicatae fuerint , [Fig. 11] quarum altera normalis in curvam ut MN, altera tangentialis MT; invenire aequationem pro curva, quam format filum perfecte flexible.

Solutio. Resolvantur ambae potentiae in laterales, quarum una verticalis altera horizontalis, nempe MN in MR et RN, et MT in MS et TS; Erit dP = MR + MS et dQ =NR - TS. Sit autem MN = dN et MT = dT. Erit $d\mathrm{P} = \frac{d\mathrm{N}dx+d\mathrm{T}dy}{ds}$ et $d\mathrm{Q} = \frac{d\mathrm{N}dy-d\mathrm{T}dx}{ds}$; unde
$dd\mathrm{P} = \frac{d\mathrm{N}ddx+dxdd\mathrm{N}+d\mathrm{T}ddy+dydd\mathrm{T}}{ds} = \frac{d\mathrm{N}dy}{r} + \frac{d\mathrm{T}dx}{r} + \frac{dxdd\mathrm{N}}{ds} + \frac{dydd\mathrm{T}}{ds}$. Et
$dd\mathrm{Q} = \frac{-d\mathrm{N}dx}{r} - \frac{dRdy}{r} - \frac{dydd\mathrm{N}}{ds} - \frac{dxdd\mathrm{T}}{ds}$. Quibus valoribus substitutis obtinebitur sequens aequatio $d\mathrm{N}dr + rdd\mathrm{N}d + \mathrm{T}ds = 0$ seu post integrationem rdN +Tds = C$ds;$ unde ob $r = \frac{dsdy}{ddx}$, erit $d\mathrm{N}dy + \mathrm{T}ddx = \mathrm{C}ddx$. Q.E.I.

Haec aequatio hunc habet usum, ut, cum admodum simplex sit, facile ad omnes casus applicari possit, sed nihilominus generalis est, etenim omnis potentia in normalem et tangentialem resolvi potest. Praeterea istud adhuc monendum esse puto, aequationem generalissamam quoque hoc modo succinctiorem reddi, loco verticalium et horizontalium normales et tangentiales in computum ducendo, haec autem oritur
$d\mathrm{N}drds + rdsdd\mathrm{N} + d\mathrm{T}ds^2 = rd^3\mathrm{Z} + drdd\mathrm{Z} + \frac{ds^2dz}{r}$ ubi $\mathrm{Z} = \frac{\Delta v}{r}$.

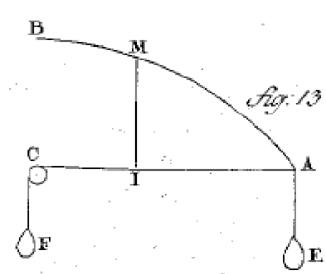
Problema. Invenire aequationem generalem pro curvis, quas fila utcunque elastica in singulis punctis nullas potentias applicatas habentia, formare debent.



Solutio. Erit ergo hoc in casu et
$d\mathrm{P} = 0$, et $d\mathrm{Q} = 0$. Ergo
$rd^3z + drddz + \frac{ds^2dz}{r} = 0,$
ex aequatione generali modo concinnata; erit enim in ea
$d\mathrm{N} = 0$ et $\mathrm{dT} = 0$:
unde $rrddzd^3z + rdrddz^2 + ds^2dzddz = 0,$
quae integrata dat $rrddz^2 + ds^2dz^2 = ads^4$ seu $rddz = ds\sqrt{(ads^2 - dz^2)}$. Est autem $r = \frac{dsdy}{ddx}$, unde $\frac{\mathrm{ddz}}{\sqrt{(\mathrm{ads}^2\text{-}\mathrm{dz}^2)}} = \frac{ddx}{dy} = \frac{ddx}{\sqrt{ds^2=dx^2}}$. Hac ope logarithmorum integrata obtinebitur

$$\frac{1}{\sqrt{-1}} l \frac{dz+\sqrt{(dz^2-ads^2)}}{ds\sqrt{-a}} = \frac{1}{\sqrt{-1}} l \frac{dx+\sqrt{(dx-ds)^2}}{ds\sqrt{-b}}, \text{ seu } \mathrm{E}x + \mathrm{F}y = \frac{\Delta v}{r}.$$

Eadem aequatio invenitur ex primo inventa aequatioe, ubi $\int Pdx$ et $\int \mathrm{Q}dy$ evanescunt adeoque restat
$\mathrm{E}x + \mathrm{F}y = \frac{\Delta v}{r}$. Q.E.I.



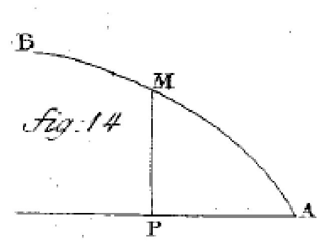
Problema. Invenire curvaturam elateris gravitatis expertis, et ubiuis eiusdem vis elasticae.

Solutio. Cum vis elastica ubique sit eadem ponatur
$v = a$; eritque $\mathrm{E}x + \mathrm{F}y = \frac{\Delta a}{r}$. Poni potest loco E$x$

+ Fy = Gt, ubi t denotet abscissam alio loco assumtam, eritque $\mathrm{G}t = \frac{Aa}{r}$, quae aequatio praebet curvam elasticam iam cognitam, hoc vero modo integratibitur. Erit
$\mathrm{E}xdsdy + \mathrm{F}ydsdy = \mathrm{A}addx$ seu $xdy + \frac{\mathrm{F}ydy}{\mathrm{E}} = \frac{\mathrm{A}addx}{\mathrm{E}ds}$.

Etiam autem est $\mathrm{E}xdsdy + \mathrm{F}ydsdy = -\mathrm{A}addy$ seu $ydx + \frac{\mathrm{E}xdx}{\mathrm{F}} = -\frac{\mathrm{A}addy}{\mathrm{F}ds}$. Consequenter addendo et
integrando habebitur $yx + \frac{Exx}{2F} + \frac{Fyy}{2E} = \frac{Aadx}{Ed} + \frac{Aady}{Fds} + C$. Ergo 2FE$dxds$ + EE$xxds$ + FF$yyds$ =
2AFadx -2AEady +2EFCds = ds(Ex + Fy)^2. Quae dabit etiam elasticam. Q.E.I.

Problema. Invenire curvam quam format filum, AMB aequabiliter grave, et aequabiliter
ubique elasticum.



Solutio. Cum filum sit ubique aequaliter grave, erit dP constans, nempe = ads : unde P = a et Q = 0. Praeterea quia elasticitas ubius eadem , ponatur v = b; erit
$\frac{Ab}{r} = \mathrm{E}x + \mathrm{F}y + \int asdx$. Augeatur as constanti E et aequatio
non immutabitur, eritque $\frac{Ab}{r} = \mathrm{F}y + \int asdx$ seu - $\frac{A\mathrm{bdr}}{\mathrm{rr}} = \mathrm{F}dy + asdx$: atque abiiciendis superfluis
constantibus $asdx + cdy + \frac{bdr}{rr} = 0$. Est autem $r = \frac{dsdy}{ddx}$; ergo $\frac{1}{r} = \frac{ddx}{dsdy}$: ex quo
$\frac{dr}{rr} = \frac{ddxddy - dyd^3x}{dsds^2} = \frac{-dxddx^2 - dy^2d^3x}{dsdy^3}$. Ergo $asdsdxdy^3 + cdsdy^4 = bdxddx^2 + bdy^2d^3x$. Ponatur $dx = pds$,
erit $dy = ds\sqrt{(1-pp)}$ et $ddx = dpds$ et $d^3x = dsddp$. Quibus substitutis habebitur
$aps(1-pp)^{3:2}ds^2 + c(1-pp)^2ds^2 = bpdp^2 + bddp(1-pp)$. Hanc vero aequationem nullo pacto
eo reducere potui, ut construi possit.

Caeteum elegans est hoc problema, quod veram curvuam tam filorum perfecte flexibilium
quam laminarum elasticarum exhibeat : etenim nullus extat funis, qui perfecte sit flexilis, neque ulla lamina elastica, quae non sit gravis, nisi forte in fluido aequalis gravitatis specificae flectatur.
Idem hoc problema in Act. Efuditorum Lips. A 1724 propositum est, ut curvatura funis elastici seu
non perfecte flexilis inveniatur, nec vero quantum scio solutionem hucusque ullus dedit, praeter Clar.
Dan. Bernoulli, qui eadem propemodum tempore, quo ego, solutionem nactus est.

Problema. Invenire curvam, quam format lamina elastica BMA in B fixa proprioque pondere
incurvata.

Solutio. Fluit ex praecedents, ubi duntaxat F et E pondera applicata evanescere debent, quare haec habebitur aequatio $\frac{Ab}{r} = \int asdx$ seu neglectio superfluis constantibus $\frac{Ab}{r} = \int sdx$

quae ut supra ad sequentem reducitur $sp^2(1-pp)^{\frac{3}{2}}ds^2 = Apdp^2 + A(1-pp)ddp$. At neque haec ad construendum accommoda effici potest.

Problema. Invenire curvam fili AB in fixi in B liberi, agatati a vento NM.

Solutio. Sit vis gravitatis in M = *ads*, et vis venti $= \frac{-bdy^2}{ds}$. Inveniatur haec aequatio $brdy^2 = ardsdx + ayds^2$. Est r radius osculi $= \frac{dsdy}{ddx}$. Quare

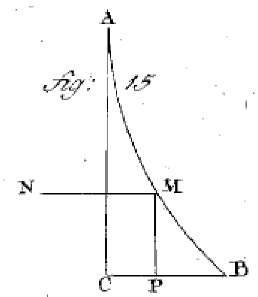


fig: 15

$bdsdy^3 = ards^2dxdy + ayds^2ddx.$ seu $bdy^3 = adsdxdy + aydsddx.$ Ponatur $dx = pdy$ erit $ddx = pddy + dpdy =$ (ob $dxddx = -dyddy$) $\frac{-pdxddx}{dy} + dpdy$, ex qua invenitur $ddx = \frac{dpdy}{1+pp}$. His factis substitutionibus orietur aequatio $bdy\sqrt{(1+pp)} = apdy(1+pp) + aydp$ et $\frac{dy}{y} = \frac{adp}{b\sqrt{(1+pp)} - ap(1+pp)}$. In qua indeterminatae sunt a se invicem separatae, et propterea curva quaesita constui potest.

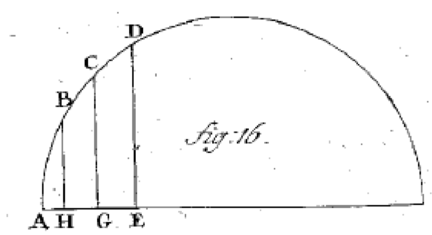


fig:16.