A Dissertation

concerning the generation of certain new

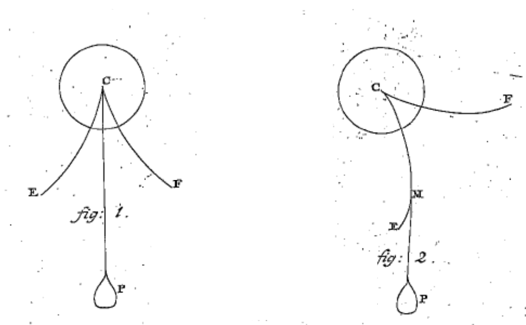
TAUTOCHRONIC CURVES.

Author

Leonard Euler.

1.

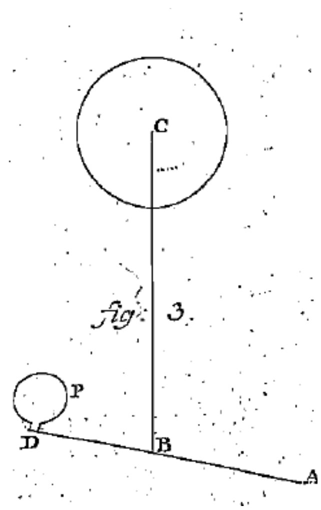
[Month of July 1727.] In a publication of the previous year, D. Sully of Paris has ran through the description of a certain navel clock ; which has a singular method of construction in order to measure time at sea, and thus to determine the position of the longitude, and which can be judged to be extremely reliable. The particular interest of this invention consists in the generation of a certain new kind of oscillation arising from the random motions of a pulley wheel inclined at some angle about an axle at some instant : these motions are effected with the help of a weight, while the pulley always tends towards a certain place in the disturbed motion. But just as these oscillations are to be isochronous, concerning which it is not yet clearly certain [- if this should in fact be the case], since the weight should hang from an accurately described [mechanical] curve by a certain line required for the motion ; which moreover is not known otherwise than by repeated trials, and Minerva [The legendary weaver of Greek mythology] has determined the coarseness of that figure. Our thoughts in the present dissertation are concerned with establishing the required tautochronous behaviour of this curve, and other ways are shown by which the equality of the oscillations can be maintained.



II. Here the manner in which Sully tried to obtain the required oscillations with the pulley wheel is more or less reproduced. From the centre of the pulley C two curved plates CE and CF are attached, between which the string CP hangs with the loading weight P, and the natural position here is that in which the string touches neither plate [as in Fig. 1]. If the weight is pushed away from this position, so that the string touches either plate at M, then from the nature of the lever, the weight P will have the strength to restore the pulley to its natural [equilibrium] position of [minimum] disturbance. And on that account, the oscillations arise, while the pulley roams about now on this side and then on the other beyond this natural place.

III. These oscillations, large or small, must have the opportunity to be restored to their natural position in an isochronous manner, and that depends on the curvature of the affixed plates, to be duly determined : and that is what D. Sully wanted to do. Moreover

this problem is greatly complex, embracing many diverse ideas, which are to be carefully distinguished from each other and explained :
Since the wheel is fixed to the plates, it must be taken into account with the plates, since there is no difference in the recovery from any position for either, thus the position of the common centre of gravity of the axle and the pulley is required. And this I assume in what follows, in order that I may simplify the calculation.
IV. Certain things about the string are to be considered closely, whether it is always vertical? or always directed against the same blow ? or truly otherwise ? Moreover, about the force applied to the string, the following points apply:
1. Whether the inertial force applies to the string, as if a weight is hung on ? [Is the string massless or not ?]; and whether or not it is almost elastic [does the string extend, and does the force acting on it determine the extension ?]. Thus indeed the forces are to be properly distinguished, for not all of the above force of inertia is expended in the movement of the pulley, but whatever is required for the motion itself. For against the force of inertia set in place, all the force should be applies to the motion of the pulley. [Euler means that only a part of the force is completely used up; some engineering texts still use the idea of the inertial force, instead of considering it as the unbalanced force that causes the acceleration, as is usual in elementary dynamics texts.]
V. Or ***2.*** uniformly, *i. e.* the force always pulls with an equal strength or with the maximum elastic tension, in order that the force on the weight during the return motion is not less great as if the same has persisted [The string is inextensible and the tension is constant]; or the motion is performed more or less in this way, but with the tension in the chord rising, as the air is either condensed or rarefied around it. All of which considerations should be carefully accounted for in the calculation, so that the curvature of the sheet can be found. The Sully machine was constructed mainly from these components [Fig. 3.] The string CB has been bound to the lever AD at B to be mobile about A, and the weight P rests on the lever at D, thus it shall be as neither the string remains vertical always, nor the weight pulls constantly, and in addition a large force of inertia is taken to be applied.



[Note: Sulley's device was presented to the French Royal Academy of Science in 1724; the diagram here does not incorporate the twin plates which were also present as above, and Euler's scheme. Essentially, the free pendulum was replaced by a lever swinging about A. More details can be found, including a diagram (plate XXXV), in the work by Defossez, *'Les Savannts du XVIIe Siecle et La Mesure du Temps'*, published in 1946 by the Swiss Clock and Watch Society. This is a fascinating book, detailing the efforts of Newton, Huygens, J. Bernoulli, and many others, in the search for a chronometer unaffected by variations in gravity with latitude, temperature, or the rough conditions at sea - which is the concern of the present work - in order that the longitude could be found; ending with the invention of the marine chronometer of John Harrison.]
VI. Moreover the simplest case to be considered is to be examined here first, and from that the curve sought can be determined; in order to show the method, which it is also

possible to conveniently apply in practise : then I will define an amount which is a part of this force, in order that the oscillations can be resolved for a given time. And finally I will disclose another case, which should not be held in low regard in nautical matters and which seems to me to be outstanding. Truly the simplest case to me is that in which the string is always vertical [in the same plane], the force acting uniformly and all the masses with inertia are set in place, connected together.

VII. I solve the problem according to this acceptable line of thought. The line CM describes either plate in some position away from the natural one [see Fig. 4], and CB is a vertical line, parallel to which will be the direction of the string MR touching the curve in M ; from the point of contact M, MT is drawn perpendicular to CB; this will also be the normal to the curve. The line CO is drawn, describing the angle BCO by which the devise has been disturbed from the natural position. With centre C and with an arbitrary radius CB the circle BO is described, the arc BO of which will measure the angle BCO, by means of which I can find the curve ; it is apparent that the force pulling along MR tries to draw the pulley [back] to the natural position. Let that force be P, and the action of this force will be as P.TM

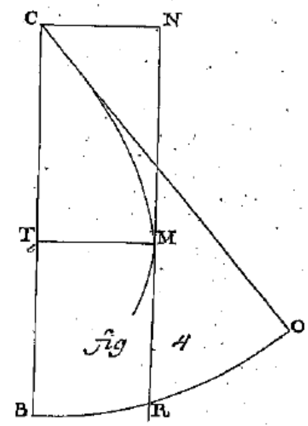


fig 4

[i. e. the moment of the force : some words describing the effects of forces are not yet in use]. TM is the perpendicular from M in the vertical CB. [original has PM.]

VIII. But since this force is applied continually to the lever formed under C [as fulcrum], I can look instead to the radius connected normally to the lever by the radius CO ending in O, which provides equal leverage. RM is produced as far as N, where it crosses the horizontal line drawn from C, the force P produces the same effect and it pulls towards NR from the radius CN at N. But from the nature of the lever, a force can be applied at O and acting along the normal to CO, and which is equivalent to the force P, as the force P is as CN or TM so it is as CO [*i. e.* F/P = (CN or TM)/CO]. Hence that equivalent force is equal to P.TM/CO or from the proportionals, on account of P and CO remaining constant, to TM itself.

IX. Now it is necessary to consider the isochronous nature of the curve in order that the curve can be determined, which is obtained, if the acceleration and the distance travelled through are always in proportion [s.h.m.]; moreover, the oscillations of the pulley can be observed by the oscillations of the pendulum CO, if they are to be isochronous, and the oscillations of the pulley are of such a kind. The point O has to traverse the arc BO, and the acceleration of this point O is as the applied force P.TM/CO *i. e.* as TM; hence, in order to obtain

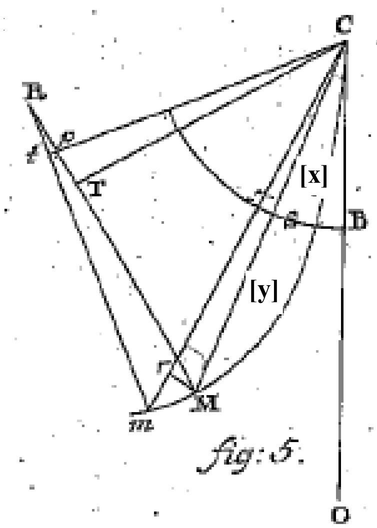


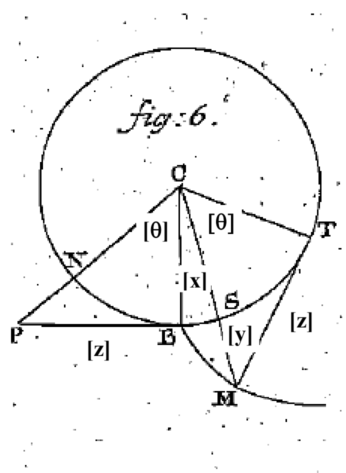
fig: 5.

isochronous motion it is necessary that the arc BO or the angle BCO is itself in proportion to TM. [This idea is expounded by J. Bernoulli in his work on vibrating strings, presented on this website, and is essentially the idea behind s.h.m.]

X. With the line TM normal to the curve [see Fig. 5, in which CO is rendered vertical for convenience; CT is the true vertical line], and the line CT is drawn perpendicular to TM from the fixed point C, and the line CO is in a fixed position relative to the curve. The problem has thus been reduced to this, in order that, for a given position of the line CO : a curve CM is to be found passing through that point C with the special property that, with MT drawn normal to the curve, and crossing that line CT is drawn perpendicular from the centre C, that the line TM shall be proportional to the angle TCO, or the differential of TM to be as the element [or differential] of the angle TCO.

XI. In order that I can obtain these elements [differentials], I take a point m very close to the point M, and from this I draw the normal mt, crossing the centre to the first osculating circle in R ; and to that line I send the perpendicular Ct cutting the first normal in p; pT is an element of the line TM; but the element of the angle TCO is the angle TCp : hence, as the element pT shall be in proportion to the element of angle TCp, it is necessary that CT is constant, which is a specific property of the curve to be found [Thus, physically, when the axle rotates a little, there is no change in the potential energy of the weight P if CT is fixed, and an element of string winds on or off the plate, as in Fig. 4; hence, mathematically, TM is the only variable quantity]. Hence it is apparent that the points T and t must fall in R, in order that the element pt of CT is equal to zero.

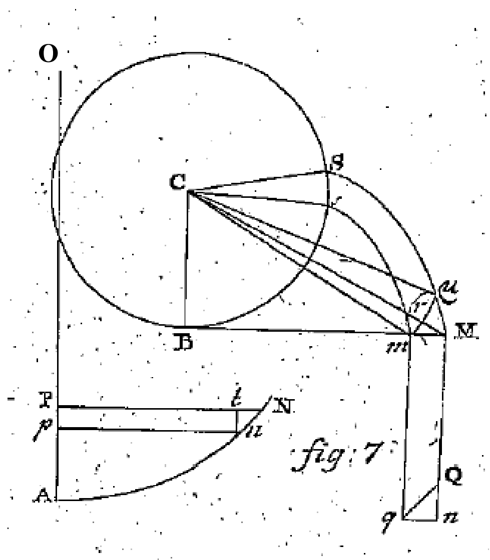
XII. In order that I can come closer to knowing this curve, I describe the circle BS with centre C, and with the interval CB = 1, which is cut by the radii CM and Cm in S and s. BS is called x and CM, y [note that these are the polar coordinates of the curve]; then Ss = dx and $mr = dy$, I draw Mr as a little arc with centre C; then on account of the similar triangles CSs and CMr, M$r = ydx$. Since CT must be constant, put CT = 1. Hence TM = $\sqrt{(yy-1)}$. Then on account of the similar triangles Mrm and MTC we have the ratio CT(1) : TM $[\sqrt{(yy-1)}] = mr(dy) : Mr(ydx)$;
from which this equation is extracted :
$dy\sqrt{(yy-1)} = ydx$; or $dx = \frac{dy}{y}\sqrt{(yy-1)}$.



XIII. To make the construction more succinct, I put $\sqrt{(yy-1)} = z$; [i. e. the tangent MT of variable length z moves around the circle and generates the involute.]
then $y = \sqrt{(zz+1)}$ and $dy = \frac{zdz}{\sqrt{(zz+1)}}$. With these values substituted I obtain this equation [for the angle] : $dx = \frac{zzdz}{zz+1} = dz - \frac{dz}{zz+1}$, which equation can be constructed with the help of the rectification circle. [The string that generates the curve will be considered to unwind from this circle in terms of polar coordinates; thus generating the involute of the circle.] With centre C and with radius CB = 1, a small circle NBST is described [Fig. 6, in which the opposite branch of the curve is used], to which the line BP is tangent at B; in which BP =

z may be taken in any case, and CP is drawn cutting the circle in N; the arc BN is given by $\int\frac{dz}{zz+1}$
[as $z = \tan\theta$; $d\theta = \sec^2\theta . d\theta$, etc.], and CP = $\sqrt{(zz+1)} = y$. Moreover, $x = z - \int\frac{dz}{zz+1}$: hence from the point B the arc BS or x = BP - BN is taken. The radius CS is produced in M, so that CM = CP = y is the point M on the curve sought. [Note that quantities in square brackets in Fig. 6 have been added by the translator for convenience to the reader. Thus, for a given y, x has been found as the difference of the arc length and the tangent length: these are the polar coordinates of the involute.]

XIV. I observe that the curve constructed in this way is generated by evolution from the circle NBST. The normal MT is drawn to the curve at M, which is a tangent to the circle at T, where (as in § 11) a perpendicular CT sent from C to this normal is equal to 1. And in addition from the same §, the normal TM is itself a radius of the osculating circle at M to the curve itself , which since by always being in contact with the circle, it is evident that this circle is the evolute of the curve to be found [i. e. the circle is traced out by the normal to this circle] : and indeed that can be more easily and conveniently described by the unwinding of a string from the perimeter of the circle.



XV. Since now it is important to resolve the problem involving the time, it is necessary also to take into account how the pulley and the force are set up, in order that the number of oscillations for a given time can be determined. Whereby, in order that the time for a complete oscillation can be found, I consider the acceleration at some instant. Let the pulley CBS be homogenous and uniform [Thus, the pulley wheel is identified as the small circle in the previous section used to generate the curves for the plates]: let its weight be Q, and its radius CB is equal to 1. The weight presents a force with the same effect everywhere and in this way the time for one oscillation will become known [The weight always acts along the tangent to the curve, which presents a line of action at a distance MT from the centre C of the pulley; hence the angular acceleration of the pulley is proportional to MT].

XVI. With a vertical line CB drawn, the curve has its beginning at some point S; and SM shall be the curve, the tangent of which MQ is vertical at the point M : and thus the radius of the osculating circle BM is horizontal ending in B. Also the direction of the force is along MQ. The curve can descent to a nearby position [by a sudden turn of the pulley through a small angle], where truly the point S becomes s, M departs from m, and MQ becomes mq; the new horizontal radius of osculation is Bm. The radii CS and Cs are drawn [s is just below S and rather indistinct on the original diagram of Fig. 7], and the lines Cm and CM; with centre C and distancr Cm, a little arc of the curve $m\mu$ is described crossing through the other former position μ, m and μ are two homologous corresponding points; hence the angles $mC\mu$ and SC*s* are equal [as everything has undergone the same small angular displacement].

XVII. The force that arrives at *q* from Q thus describes the small distance Q*q* = $m\mu$, whereby BM is drawn parallel to *qn*; Q*n* is equal Mμ because the length of string is the same and on account of CSM - C*sm* = Mμ. The force descends therefore through Q*n* by this motion; hence it generates the *work* P.Q*n* [which Euler expresses as the *vis viva* according to J. B.] , the total of which is transferred to the pulley: since the force is supposed free from inertia. Hence the *energy* in the pulley, when it is rotated through the angle SC*s* by the motion, should increase by P.Q*n*. [The words *work* and *energy* are used in italics to stop us from being too pendantic, but they represent the meaning given to the equivalent Latin words for the equations to make sense; the reader needs to remember that these concepts were only properly formalised at a later date.]

XVIII. Let the velocity of the point S be equal to that acquired by falling through a height *v* [From our viewpoint, this is an unfortunate choice of symbol for the height: we will call the associate velocity *V*; in which case $gv = V^2/2$; the other difficulty we have is not using g, the acceleration of gravity, as it was customery to compare times with that of a pendulum of some length, in which case the *g* factor cancels out; we can also get round the difficulty by setting $g = {}^1/_2$ in some units , and then using the comparison when appropriate]. The total *energy* of the pulley will equal Q*v*/2
[Note :i. e. The kinetic energy of the ring or pulley is equal to
$I\omega^2 = \frac{1}{2}mr^2.(V/r)^2 = \frac{1}{2}QV^2 = \frac{1}{2}Qv$, assuming a solid cylinder for the pulley, and not bothering with the contribution of the plates to the moment of inertia. We are neglecting the factor of half in front of the kinetic energy term, as all energies are considered as equal to a falling weight, where gravity is given the value of ${}^1/_2$. For later use, we should note that all periodic times for s.h.m. involve a square root of a force constant on an inertial term; when taken as a ratio, any inaccuracies in absolute equations disappear when a ratio is taken between the equations.];
thus the differential of this is $\frac{Qdv}{2} = \text{P.Q}n$; hence $dv = \frac{2\text{P.Q}n}{\text{Q}} = \frac{2\text{P.M}\mu}{Q}$ [thus, d*v* = 2VdV].
Moreover on account of the similar triangles $\Delta \text{M}\mu m$ and $\Delta \text{B}mC$; Mμ : M*m* = B*m* : BC [taking B*m* ~ BM]; hence
$\text{M}\mu = \frac{\text{B}m.\text{M}m}{\text{BC}}$: but $\text{M}m = \text{S}s$; thus $\text{M}\mu = \frac{\text{B}m.\text{S}s}{\text{BC}}$. Hence $dv = \frac{2\text{P.B}m.\text{S}s}{\text{Q.BC}}$, consequently the motion
$\frac{dv}{\text{S}s} = \frac{2\text{P.B}m}{\text{Q.BC}} = \frac{2\text{P.BS}}{\text{Q.BC}}$, on account of which BS is the evolute of the curve SM, and thus BM or B*m* is equal to the radius of osculation .
[The general relation between involutes and evolutes of curves is considered e. g. in Courant, *Differential and Integral Calculus*, Vol. 1, p. 310. This includes an account of the involute of the circle that arises by unwinding the tangent, which is Euler's curve. The evolute of which is the circle itself.]

XIX. The length of the synchronous pendulum can easily be found from the matter of the motion $\frac{dv}{\text{S}s}$ in this manner [the small diagram on Fig. 7] : let the isochronous pendulum be OA oscillating along the cycloid NA. And let the arc AN be equal to the arc BS and occuring at the same time. N*n* is taken equal to S*s*, and the vertical *nt* is drawn, the motion through Nn is equal to $\frac{nt}{\mathrm{N}n}$; which must be equal to the motion $[[\frac{dv}{\mathrm{S}s}=]\frac{2\mathrm{P.BS}}{\mathrm{Q.BC}}$: But from the nature of the cycloid $\frac{nt}{\mathrm{N}n}=\frac{\mathrm{AN}}{\mathrm{AO}}=\frac{\mathrm{BS}}{\mathrm{AO}}$; hence $\mathrm{AO}=\frac{\mathrm{Q.BC}}{2\mathrm{P}}$. Hence as the ratio of the weight of the equivalent force to the weight of the pulley, thus becomes the ratio of half the radius BC to the fourth [term], which latter term is the length of the isochronous pendulum.

[For the equation of the cycloid given with origin at A can be written in the form $s=4a\sin\psi$, where a is the radius of the generating circle, giving AO = 4a; the gradient $\sin\psi = dy/ds=\frac{nt}{Nn}$; and the arc length s = AN, giving $s/4a$ = AN/AO. We can also establish this result in a physical manner as follows :
The usual energy equation for a mass falling from rest through a distance h under gravity m satisfies $mgh = {}^1/_2 mV^2$; at present, $2g = 1$, and $h = v$, giving $v = V^2$ in some units. The above equation $\frac{dv}{\mathrm{S}s}=\frac{2\mathrm{P.B}m}{\mathrm{Q.BC}}=\frac{2\mathrm{P.BS}}{\mathrm{Q.BC}}$ can be written as
$\frac{dv}{arc\ Ss}=\frac{VdV}{a.\Delta\vartheta}=\frac{2\mathrm{P.B}m}{\mathrm{Q.BC}}=\frac{2\mathrm{P.arc\ BS}}{\mathrm{Q.radius\ BC}}=\frac{2mg.a\vartheta}{M.a}=\frac{m.\vartheta}{M}$, where a is the radius BC. If we set $V=a\dot{\vartheta}$, then the equation becomes $a\dot{\vartheta}.d\dot{\vartheta}=-\frac{m}{M}\vartheta.d\vartheta$, on introducing the correct signs, where the mass/weight m/mg is P as required, and M is Q. This equation can be compared with the derivative of the energy equation of a simple pendulum of lenth l for small amplitudes, which of course carries over to the corrected cycloidal pendulum for large amplitudes : $\dot{\vartheta}.d\dot{\vartheta}=-\frac{g}{l}\vartheta.d\vartheta=-\frac{\vartheta d\vartheta}{2l}$. Hence, $\omega^2=\frac{m}{Ma}=\frac{1}{2l}$, giving $l=\frac{Ma}{2m}$ as above; we have also found the period for the oscillation of the pulley as $T=2\pi\sqrt{\frac{Ma}{2mg}}$, or $2\pi\sqrt{\frac{\mathrm{Q.BC}}{2\mathrm{P}}}$ in the units used, where mg = P.]

XX. Thus I have used the force without inertia, in that the force is not required in the generation of the velocity. It is hence readily apparent, if the force shall thus be small, in order that the weight provides nothing sufficient to the weight of the pulley to give a detectable ratio, the *energy change* arising from that can be rejected; hence a weight for P is put in place, as Sully wishes, but much smaller than Q. But nevertheless, in order that the number of oscillations of the pulley can be determined for a given time found for BC, the ratio hence becomes: as Q to 2P thus as AO to BC : let Q be several hundred times greater than P, and let the length of the pendulum AO be truly equal to 3166 scruples [~ 1 m] of Rhenish feet for a one second swing from one side to the other [See, e.g. Euler's *de Sono* in this series; the period defined in the modern way it thus 2 seconds], BC will be 63 scruples [~ 2 cm], which is a large enough quantity for the radius BC. And for this ratio of lengths a satisfactory weight is attached, as Sully wished. In order that the weight maintains a sense to the vertical and not to be oscillating [the motion of the pendulum bob or mass P is hence a vertical oscillation only], the author's precautions [below] should be born in mind; in particular the string should be

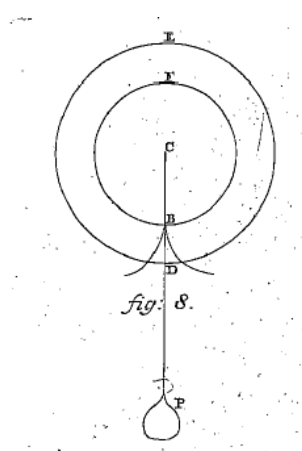


fig: 8.

long enough, thus in order that the radius BC can be small, and the direction of the string will always be kept almost vertical.

XXI. In this manner we may correct this difficulty [Fig. 8]. The pulley CD can be made much larger than the generating circle BF of the assigned plates of the curve, in this manner a larger pulley should have more impression on the motion, since the weight hung on on account of the small nature of the circle BF is scarcely to be moved, in order that the motion rightly to be generated by that with respect to the motion of the pulley can be rejected, particularly if the above weight has a small ratio to the weight of the pulley. Moreover in that case I say that if the radius of the pulley CD is equal to *a,* then the length of the isochronous pendulum will be equal to $\frac{Q.a^4.BC}{2P}$. Therefore with this corrected ratio, Sully's clock should become more useful than ever.
[The moment of *I* has increased by a factor a^2 , while the mass has increased to a^2.Q for the same thickness; while the torque has remained the same; hence the momen of inertia is multiplies by the factor a^4. The clock would then have a very long period, which would contribute to its stability.]

XXII. From what has been said it is apparent that there is a particular difficulty with the direction of the string moving around, as it is not always staying vertical. Here indeed the inconvenience is taken up by construction of the following curve. Before the string is drawn, to which the force is applied, through some fixed opening, that can be made in this manner, in order that the string always points towards a given fixed point. Hence, I have fallen upon the following property that has enabled me to produce the curve with the tautochronous property in this case. Let C be the centre of the pulley, BM the curve required, A that fixed point or opening, through which the string always passes [see Fig. 9, in which lengths in square brackets have again been added for convenience]. With the said small part AC = *a*, and for some given radius CM = *y*.

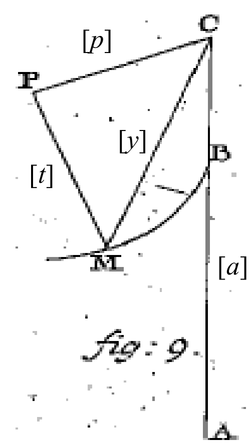


fig: 9.

Again, let PM be the normal to the curve, and CP put at right angles to the normal MP with CP = *p* and PM = *t*, and assigning the constant *b* as I please; I obtain this characteristic trial equation of the curve :

$$b\sqrt{(aa-tt)}-bp=p\sqrt{(aa-tt)} .$$

From this equation, as it is algebraic, it will be constructed according to the known rules with the help of the circle for rectification, in the same way as the curve found there in §13 had been constructed.

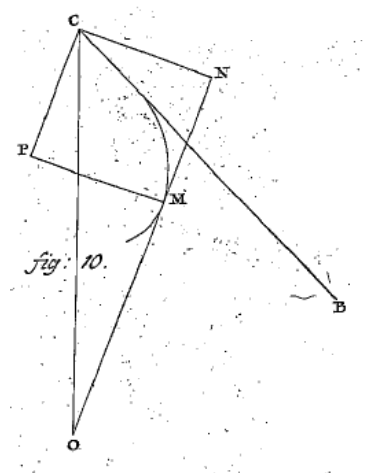
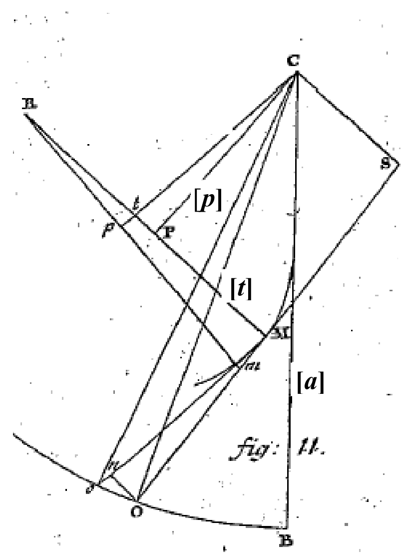


fig: 10.

XXIII. Moreover, the equation for the given curve, $b\sqrt{(aa-tt)}-bp=p\sqrt{(aa-tt)}$ is found in this way : [See Fig. 10] Let C be the centre of the pulley, O the fixed point to which the string is always drawn, or O shall be the opening through which the thread has been drawn, to which the force without inertia shall be attached below the hole, in order that the string can always pass through this opening O : Let CM be the place of the curve which is to be found, that touches the the line OM, in the direction of the string, in this situation a perpendicular CN is sent from the centre of the circle C of the pulley to the curve produced in OM; this quantity CN expresses the size of the force, which is attached below the the opening, which is applied to the movement of the pulley, since that force is put constant.

XXIV. Moreover, from the isochronous principle, the force must be applied to make the pulley move in the manner to be described, so that it can be returned to the natural resting position: this way in which the measurement is to be described is from the angle by which CM is displaced from the resting position, and the line CB is drawn from the resting position which it meets in CO; BCO will be that angle, which expresses the described way, therefore it is necessary that the line CN, which expresses how the force contributed to the pulley movement in the position CM, is proportional to the angle BCO; or from the point M, MP is drawn normal to the curve at the point M, and from C the perpendicular CP is sent to this line so that MP = CN and indeed the length MP should be in proportion to angle BCO.



XXV. Now let CB be in the resting position, [Fig. 11, where *p, t*, and *a* are added in square brackets as an aid] and this distance is made equal to the distance C from the centre of the pulley to the hole, and CM is the curve to be found. M is some point on the curve, and the tangent line MO is drawn to the curve; with centre C and radius CB the arc of a circle is described cutting the tangent at M in O : here will be the position of the hole O corresponding to the point M of the curve. The line CO is drawn; the angle BCO is the same as the angle BCO in fig. X. From M the perpendicular to the curve MP is erected, which crosses the perpendicular sent from C at P : thus it is necessary that the line MP is in proportion to the angle BCO. [Thus, the variable component of the torque is proportional to the (-ve of the) angle of displacement, as required for s.h.m.]

XXVI. In order that I can obtain this element [i. e. differential] I take a point *m* near to the point M, and the corresponding lines *mo* and *mp* are drawn, tangent to the curve in *m*, and *mp* is perpendicular to the curve in *m*, which is cut in *p* by the perpendicular C*p* ; this perpendicular C*p* first cuts the first perpendicular in *t*, and P*t* is the increment of the normalPM. The points C and *o* are joined by the line C*o*, OC*o* is the increment of the angle BCO: it is hence required for the determination of the sought curve CM, that P*t* is proportional to the element of the angle OC*o*. But P*t* is [the arc] corresponding to the angle PC*t* drawn for the radius PC [thus, PC.ang.PC*t* = P*t*] ; therefore the angle OC*o* will be to the arc P*t*, or to PC*t*. CP in a given ratio, which shall be as 1 to *b* [which is the constant of proportionality], as a consequence OC*o* : PC*t* = CP : *b*. [There is an exta *t* in the original which is obviously a mis-print.]

XXVII. The perpendiculars MP and *mp* are concurrent in R, the centre of the osculating circle is at M. The angles PC*t* and MR*m* are equal on account of the similar triangles PC*t* and *p*R*t*, but angle MR*m* = angle O*mo*, which is formed from the nearby tangents OM and *om*; hence PC*t* = O*mo*, hence it is required that OCo: O*mo* = CP: *b* : the tangent OM is produced to S, hence it cuts the perpendicular CS sent from C, the triangles O*no* (formed from the perpendicular O*n* sent from O to *mo*) and OSC are similar ; hence O*o* : O*n* = CO : OS. But the ratio of the angles OC*o* : O*mo* = $\frac{\mathrm{O}o}{\mathrm{OC}}:\frac{\mathrm{O}n}{\mathrm{OM}}=[\frac{\mathrm{O}o}{\mathrm{O}n}\times\frac{\mathrm{OM}}{\mathrm{OC}}=\frac{OC}{OS}\times\frac{OM}{OC}]=\frac{\mathrm{CO}}{o\mathrm{C}}:\frac{\mathrm{OS}}{\mathrm{OM}}$ (with the proportionals OC and OS put in place of O*o* to O*n*) = OM:OS. Hence OC*o*:OM*o* = OM:OS.

XXVIII. Moreover, since it is required that OC*o* : OM*o* = CP : *b*, this ratio shall be found CP : *b* = OM : OS, which is free from all angles, and hence this equation is obtained : CP.OS = *b*.OM. But on account of ΔCOS to rectangle OS =$\sqrt{(CO^2 - CS^2)}$ =(since OC = CB and CS = PM) =$\sqrt{(CB^2- PM^2)}$. Hence OM = OS - SM = OS - CP =$\sqrt{(CB^2- PM^2)}$ - CP; hence this equation is obtained CP$\sqrt{(CB^2- PM^2)}$ = $b\sqrt{(CB^2- PM^2)}$ - *b*.CP. Hence the desired curve is determined [algebraically; compare this with the initial analytic method].

XXIX. With the application of symbols, and CB is called the distance of the centre of the pulley from the hole *a*, CP, *p*, and MP, *t*; this equation is had for the curve sought : $p\sqrt{(aa-tt)}=b\sqrt{(aa-tt)}-bp$, which is the same as that shown in § XXII. Hence $p=\frac{b\sqrt{(aa-tt)}}{b+\sqrt{(aa-tt)}}$. From which equation the curve is able to be constructed and to be applied for use. If the hole is placed an infinite distance from the centre of the pulley, the direction of the string will always be parallel to the same direction, and the first case is had for which CP is always constant. For indeed with *a* tends to infinity, $\sqrt{(aa - tt)}$ will result in *a* and $p=\frac{ab}{a+b}$ is obtained, which is equal to *b* on account of *a* in the evanescent denominator.

XXX. If it should seem more favourable to change the weight in place of that given, as was seen to be true in the first case too, then the same warnings should be heeded. For a given weight hung on small enough to make the machine work can be increased as the radius of the pulley is diminished, and this weight can be increased as long as the error from the imperceptible force of inertia of the falling weight can be avoided, which may lead to irregularities in the working of the machine, that may go unnoticed. Moreover, as the construction of the curves required in this machine shall be very labourous, indeed contrary to that of the first case, in which the direction of the string maintains the same stroke, with an easier construction, I think the first method to be preferred to that of the second.

Dissertatio

de nouo quadam

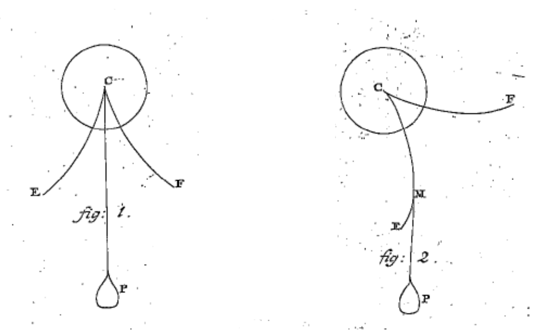
CURVARUM TAUTOCHRONARUM GENERE

Auctore

Leonh. Eulero.

1.

M. Jul. 1727. Edidit ante annum et quod excurrit D. Sully Parisiis descriptionem navi cuisdam horologii ; quod peculiari modo fabricatum ad dimetienda mari tempora, et inde determinandam locorum longitudinem, perquam idoneum iudicat. Praecipuum eius inventum, consistit in nouo quodam oscillationum genere a vacillatione trochleae circa axem petito. Idque efficit ope ponderis, trochleam semper versus certum situum sollicitantis. At quomodo istae oscillationes isochronae efficiendae sint, de eo nondum plane certus est, cum id pendeat ab accurata descriptione curvae cuiusdam lineae ad id requisitae; quam autem aliter non nisi crebra tentatione cognovit, eiusque figuram crassa Minerva determinavit. De hac curva ad tautochronismum desiderata in praesenti dissertatione agere animus est, aliosque exhibere modos, quibus aequalitas oscillationum conservari poterit.



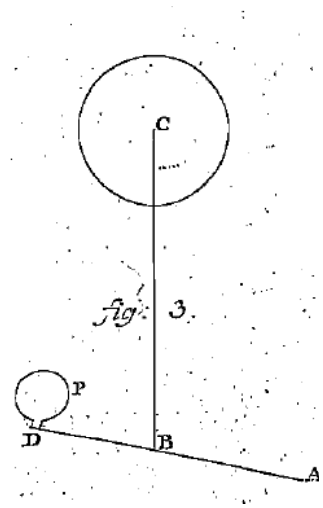
II. Huc fere autem reducitur modus, quo Sully in trochlea oscillationes obtinere conatur. In centro trochleae C applicat duas laminas incurvatas CE, CF inter quas dependet filum CP, pondere P oneratum, et hic situs, quo filum neutram laminam tangit, est naturalis. Ex quo si pellatur, ut filum in M alterutram laminam tangat, ex natura vectis pondis pondus P vim habebit trochleam in situm naturalem sollicitandi. Et ea propter oscillationes orientur, dum trochlea nunc cis nunc ultra situm hunc naturalem extravagabitur.

III. Hae oscillationes, sine minus sine magis sint amplae, ut isochronae reddantur, id pendet a curvatura laminarum affixarum, ut haec rite determinetur: et id ipsum est, quod D. Sully desiderat. Est autem hoc problema valde intricatum, plurimaque deversa complectens, quae diligenter sunt evoluenda et distinguenda:

Quod rotam attinet, ea cum laminis ita debet esse comparata, ut indifferens sit ad quemuis situm recipiendum, unde centrum commune gravitatis in axe trochleae positum sit oportet. Atque id in posterum assumam, ut nimiam culculi prolixitatem evitem.

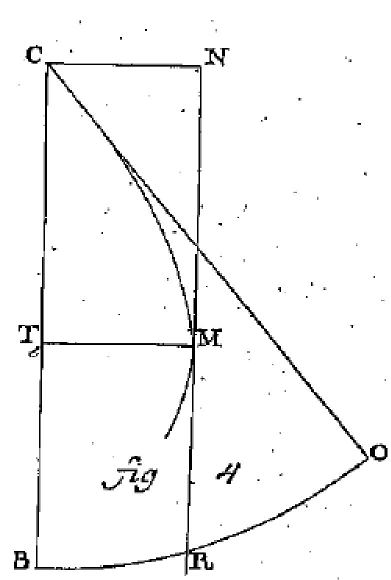
IV. Circa filum consideranda sunt, an sit semper verticale? an semper versus eandem plagam dirigatur? an vero secus ? Circa potentiam autem filo applicatam sequentia. 1. An ipsa habeat vim inertiae? ut si pondus appendatur ; an vero non, ut elastra fere. Haec

probe sunt distinguenda, potentia enim vi inertiae praedita non omnem vim ad trochleam movendam impendit, sed quidquam ad sui ipsius motum requiritur. Cum econtra potentia inertia destituta omnem vim ad motum trochleae impendere queat.
V. 2. An uniformiter, i. e. semper aequali vi trahat, ut pondus, vel elastrum maxime tensum, cuius vis in remissionibus non nimis magnis quasi eadem persistit; an autem modo magis modo minus agat, ut elater, chorda tensa, aer condensatus, vel rarefactus. Quae



considerationes omnes diligenter in computum duci debent, vt laminarum curvatura inveniatur. Machina Sulliana maxime ex hisce est composita. Filum CD vecti AD circa A mobili, in B est alligatum , et vecti in D pondus P incumbit, unde sit, ut nec filum semper verticale maneat, neque pondus uniformiter trahat, et insuper vis inertiae non exigua adsit.
VI. Casum autem simplicissimum hic primo examini subiicere animus est, et pro eo curvam quaesitam determinare, nec non modum monstrare, quo in praxi commode applicari possit : dein quantitatem cuius vis partis definiam, ut oscillationes absolvantur dato tempore. Et tandem alium evoluam casum, qui non contemnendum in re nautica usum mihi praestare videtur. Simplicissimus vero mihi est casus, quo filum perpetuo verticale persisistit, potentia uniformiter agens et omni inertia distituta applicatur.

VII. Problema hoc sensu acceptum sic soluo. Designet linea CM laminam alterutram in quovis situ non naturali, sitque CB linea verticalis, cui parallela erit fili directio MR curvam in M tangens; ex puncto contactus M ducatur in CB perpendicularis MT; erit haec etiam normalis in curvam. Ducatur recta CO, designans angulum BCO, quo machina ex situ naturali est deturbata. Centro C radio arbitrario CB describatur circulus BO, eius arcus BO metietur angulum BCO, quibus factis hoc modo curvam detego; patet potentiam secundum MR trahentem trochleam in situm naturalem perducere conari. Sit ea potentia P, erit illa vis ut P. TM. Est PM perpendiculum ex M in verticalem CB.

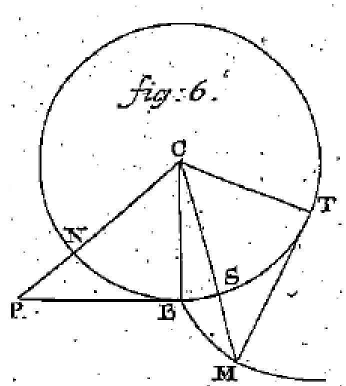


VIII. Cum autem haec vis continuo aliter respectu hypomochlii C applicetur, ei quaero aequipollentem radio CO in O normaliter applicandam. Producatur RM in N usque, ubi occurrat horizontali ex C ductae, potentia P eundem edit effectum ac si radio CN in N applicata versus NR trahet. At ex natura vectis est potentia in O applicanda et secundum normalem ad CO agens, aequipollensque potentiae P, ad potentiam P ut CN sive TM ad CO. Erit ergo ea = P.TM/CO seu proportionalis, ob P et CO constantes,ipsi TM.

IX. Nunc ad curvam determinandam isochronisum considerare oportet, qui obtinetur, si acceleratio spatio percurrendo semper proportionatur, possunt autem oscillationes trochleae tanquam oscillationes penduli CO spectari, quae si sint isochronae, et trochleae oscillationes tales erunt. Percurrnedus vero est puncto O arcus BO, et huius puncti O acceleratio est ut vis applicata P.TM/CO i. e. ut TM; ad obtinendum ergo isochronismum oportet, ut arcus BO vel angulus BCO proportionetur ipsi TM.

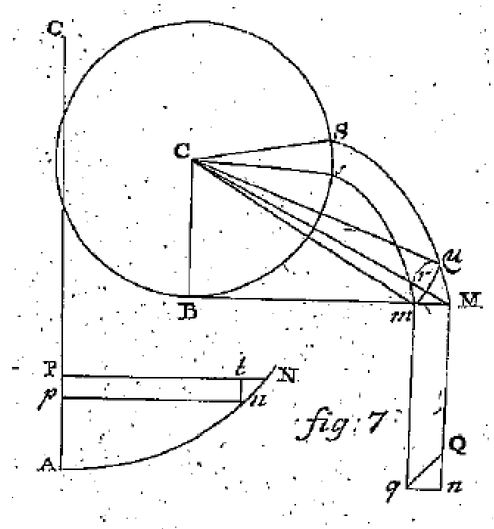
X. Quum linea TM sit in curvam normalis, in eamque CT, ex puncto fixo C, perpendicularis, atque linea CO ad curvam habeat ubique eundem situm; Problema huc reductum est, ut, data recta CO positione, in eaque puncto C, inveniatur curva CM huius proprietatis, ut, ducta normali MT, in eamque ex centro C perpendiculo CT, sit linea TM proportionalis angulo TCO, seu differentiale ipsius TM elemento anguli TCO.

XI. Ut obtineam haec elementa, puncto M accipio proximum m, et ex eo duco normalem mt, priori occurens in R centro circuli osculatoris; in eamque demitto perpendicularum Ct priorem normalem in p fecans; erit pT elementum lineae TM; at elementum anguli TCO est angulus TCp : ut ergo pT elemento ang. TCp proportionalis sit, oportet, ut sit CT constans, quae est proprietas specifica curvae inveniendae. Patet hinc puncta T et t in R cadere debere, ut pt elementum ipsius CT sit = 0.



XII. Ut ad huius curvae cognitionem propius accedam, centro C intervallo CB = 1 describo circulum BS, qui secatur a radiis CM Cm in S et s. Vocetur BS, x et CM, y ; erit Ss = dx et mr = dy, duco arculo Mr centro C; unde ob triangula CSs, CMr similia, obtinetur Mr = ydx. Cum CT constans esse debeat, ponatur CT = 1. Erit TM = $\sqrt{(y - 1)}$. Dein ob similia ΔΔMrm, MTC habetur CT(1) : TM [$\sqrt{(y - 1)}$] = $mr(dy)$: M$r(ydx)$; unde elicitur haec aequatio $dy\sqrt{(yy - 1)} = ydx$; seu $dx = \frac{dy}{y}\sqrt{(yy-1)}$.

XIII. Ad construendam succinctius hanc aequationem, pono $\sqrt{(yy - 1)} = z$; erit y =$\sqrt{(yy + 1)}$ et $dy = \frac{zdz}{\sqrt{(zz+1)}}$. His valoribus substitutis obtineo hanc aequationem $dx = \frac{zzdz}{zz+1} = dz - \frac{dz}{zz+1}$, quae aequatio ergo ope rectificationis circuli construi potest. Centro C radio CR = 1 describatur circulus NBST, quem in B tangat recta BP; in qua accipiatur utcunque BP = z, ducaturque CP secans circulum in N; erit arcus BN $\int \frac{dz}{zz+1}$, et CP = $\sqrt{(zz+ 1)} = y$. Est autem $x = z - \int \frac{dz}{zz+1}$: sumatur ergo a puncto B



arcus BS = BP - BN; erit BS = x. Radius CS in M producatur, ut sit CM = CP = y erit punctum M in curva quaesita.

XIV. Curvam hoc modo constructam ex ipsius circuli NBST evolutione generari observo. Ducatur enim ad curvam in M normalis MT, tanget ea circulum in T, cum ex. § 11 perpendiculum CT ex C in eam normalem demissum sit = 1. Et insuper ex eodem § normalis TM est ipse curvae in M radius osculi, qui cum circulum continuo tangat, liquet, circulum esse evolutam huius curvae inventae : adeoque ea facilius et commodius evolutione fili circulo circumducti describetur.

XV Quod iam attinet ad tempus absolutum, id quoque supputandum est, ut liqueat, quo modo trochlea et potentiae sint instituendae, ut oscillationes dato tempore absoluantur. Quare ad tempus totius oscillationis inveniendum considerabo accelerationem quamvis momentaneam. Sit trochlea CBS homogenea et aequabilis ubiuis : sit eius pondus = Q; et radius eius CB = 1. Praestet potentia eundem ubique effectum ac pondus P hoc modo innotescet tempus unius oscillationis.

XVI. Ducta verticali CB consistat curvae initium in loco quovis S; sitque curva SM, in cuius puncto M tangens MQ sit verticalis : adeoque radius osculi BM erit horizontalis in B terminatus. Erit itaque MQ directio potentiae. Descendat curva in situm proximum, nempe punctum S in s, abibit M in m et MQ in mq; erit denuo Bm radius osculi horizontalis. Ducantur radii CS, Cs, et rectae Cm, CM; centro C , intervallo Cm, describatur arculus mμ curvae in altero priori situ in μ occurrens erunt puncta m, μ duo puncta homologia et respondentia; ergo ang. $m\mathrm{C}\mu = \mathrm{SC}s$.

XVII. Pervenit porro potentia ex Q in q descripsit adeo spatiolum Qq = $m\mu$, quare ducta qn parallela BM; erit Qn = Mμ propter eandem fili longitudinem et ob CSM - Csm = Mμ. Descendit igitur potentia hoc momento per Qn; unde generari debet vis viva P. Qn, quae tota in trochleam transferetur : quia potentia inertiae expers supponitur. Vis ergo viva in trochleae, dum motu angulari SCs gyratur, augeri debet vi P.Qn.

XVIII. Sit velocitas puncti S. aequalis acquisitae ex altitudine v. Erit vis viva totius trochleae = Qv/2; unde eius differentiale $\frac{\mathrm{Q}dv}{2} = \mathrm{P.Q}n$; Ergo $dv = \frac{2\mathrm{P.Q}n}{\mathrm{Q}} = \frac{2\mathrm{P.M}\mu}{2}$.Est autem ob ΔΔ similia Mμm, et BmC; Mμ: Mm = Bm : BC; ergo $\mathrm{M}\mu = \frac{\mathrm{B}m.\mathrm{M}m}{\mathrm{BC}}$: at $\mathrm{M}m = \mathrm{S}s$; unde $\mathrm{M}\mu = \frac{\mathrm{B}m.gs}{\mathrm{BC}}$.Ergo $dv = \frac{2\mathrm{P.B}m.gs}{\mathrm{Q.BC}}$, consequenter momentum $\frac{dv}{\mathrm{S}s} = \frac{2\mathrm{P.B}m}{\mathrm{Q.BC}} = \frac{2\mathrm{P.BS}}{\mathrm{Q.BC}}$, ob BS evolutam curvae SM, adeoque aequalem radio osculi BM seu Bm.

XIX. Invento momento $\frac{dv}{\mathrm{S}s}$ facili negotio reperietur longitudo penduli isochroni hoc modo: sit pendulum isochronum OA oscillans in cycloide NA. Sitque arcus AN = arcui BS et contemporaneus. Sumatur Nn = Ss, ducaturque verticalis nt, erit momentum per Nn = $\frac{nt}{\mathrm{N}n}$; id quod aequari debet momento $\frac{2\mathrm{P.BS}}{\mathrm{Q.BC}}$: Sed ex natura cyloidis est $\frac{nt}{\mathrm{N}n} = \frac{\mathrm{AN}}{\mathrm{AO}} = \frac{\mathrm{BS}}{\mathrm{AO}}$; ergo $\mathrm{AO} = \frac{\mathrm{Q.BC}}{2\mathrm{P}}$. Fiat ergo vt pondus potentiae aequivalens, ad

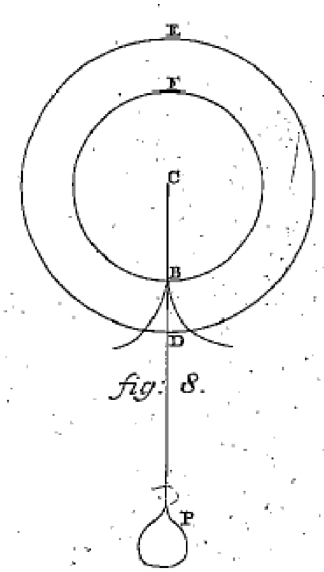


fig. 8.

pondus trochleae, ita dimidius radius BC ad quartam, quae erit longitudo penduli isochroni.

XX. Potentiam ideo adhibui inertia destitutam, ne ad velocitatem in ea generandam vis requiratur. Hinc igitur facile patet, si potentia ita sit exigua, ut pondus ei suffectum nullum ad trochkeae pondus habeat rationem sendibilem, vim in eo generandam reiici posse; adeoque loco P poterit, ut Sully vult, pondus substitui, modo valde exiguum respectu Q. Ut autem nihilominus oscillationes trocheae dato tempore absolvantur BC, inde determinari debet, fiat enim, ut Q as 2P ita AO ad BC : sit Q centies maius quam P, sitque AO longitudo penduli oscillantis singulis minutis secundis, nempe = 3166 scrup. ped. Rhen. erit BC = 63 scrup. quae est quantitas satis magna pro radio BC. Atque hoc sensu pondus satisfaciet appensum, ut Sully desiderat. Id ut ad sensum verticale perseveret, neque oscilletur, cautelae ab Autore adhibitae locum obtinebunt; praecipue vero filum satis longum esse debet, unde ob radium BC exiguum, directio fili semper fere verticalis obtinebitur.

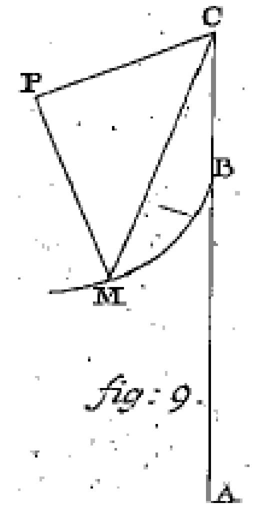


fig: 9.

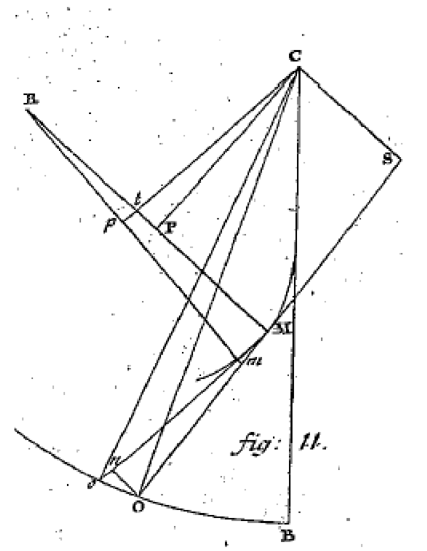
XXI. Quin et hoc modo commode isti difficulatiti medebimur. Construatur trochlea ED multo maior, quam circulus generator BF curvae laminis tributae, hoc modo trocheae ingens erit imprimendus motus, cum tamen pondus appensum ob circuli BF parvitatem vix moveatur, ut motus in eo generandus merito respectu motus trochleae reiici queat, praecipue si insuper pondus P ad trochleae pondus exiguam habuerit rationem. Illo autem casu dicto radio trochleae CD = a erit longitudo penduli isochroni = $\frac{Q.a^4.ac}{2P}$. Hac ergo ratione horologium Sully emendatum , multo maiorem praestare poterit utilitatem.

XXII. Ex dictis patet praecipuam difficultatem circa directionem fili, quod non semper verticale persistat, versari. Hoc vero incommodum sequentis curvae constructione tollitur. Ducatur filum ante, quam ipsi potentia applicetur, per foramen quoddam fixam, hoc modo fiet, ut filum perpetuo versus datum punctum directum sit. Quaesivi igitur pro hoc casu curvam tautochronismum producentem, et incidi in sequentem proprietatem. Sit C centrum trochleae, BM curva quaesita A punctum illud fixum seu foramen, per quod filum semper transit. Dictis uncia AC = a, et quovis radio CM = y. Sit porro PM normalis in curvam, et CP normalis in MP positis CP = p et PM = t, designanteque b constantem pro lubitu accipiendam; hanc obtinui aequationem naturam curvae experimentem

$$b\sqrt{(aa-tt)}-bp=p\sqrt{(aa-tt)}\ .$$

Ex hac aequatione, cum sit algebraica, per notas regulas curva desiderata ope circuli rectificatione construetur, simili modo, quo in § 13 curva ibi inventa erat constructa.

XXIII. Obtinetur autem data aequatio $b\sqrt{(aa-tt)}-bp=p\sqrt{(aa-tt)}$ hoc modo : Sit C centrum trocheae, O punctum fixum ad quod filum semper tendit, seu in O sit foramen per quod filum est ductum, cui infra foramen potentia inertia carens sit applicata, ut filum perpetuo per hoc foramen O transeat : Sit CM situs quius curvae inveniendae, quam tangat recta OM, directio fili, in hoc curvae situ ex centro trochleae C dimittatur in OM productam, perpendiculum; CN exprimit haec CN quantitatem vis, quam potentia filo infra foramen applicata, ad trochleam movendam impendit, cum potentia ea ponatur constans.



XXIV. Ex isochronismi principio autem vis ad trochleam movendam applicata debet esse, ut via describenda, donec in situm naturalem revertatur, haec via describenda mensuranda est ex angulo, quo situs hic CM a naturali distat, ducatur linea CB quae ex naturali situ pervenit in CO; erit angulus BCO ille, qui exprimit viam describendam, oportet ergo ut sit linea CN, quae exprimit vim ad movendam trochleam in situ CM, proportionalis angulo BCO, seu ex puncto M ducantur MP normalis in curvam in punctum M, et ex C in eam demittatur perpendicularis CP, erit MP = CN adeoque debet esse MP, ut angulos BCO.

XXV. Sit iam CB in situ naturali, fiatque ea aequalis distantiae foraminis a centro trochleae C, sitque CM curva invenienda, accipiatur punctum quodvis M in in curva, in eoque tangatur curva in linea MO, centro C, radio CB describatur arcus circuli secans tangentem in M in O, erit hoc punctum O foraminis situs respondens puncto curvae M. Ducatur linea CO; erit angulus BCO idem cum angulo BCO in fig. X ex M erigatur perpendicularis in curvam MP, cui in P occurrat perpendiculum CP ex C in eam demissum, oportet hanc MP proportionalem esse angulo BCO.

XXVI. Ut obtineam haec elementa assumo puncto M proximum *m*, et ducantur lineae respondentes *mo, mp*, illa tangens in *m*, et haec *mp* perpendicularis in *m*, quae in *p* secetur a perpendiculari C*p* in ipsam; secabit haec C*p* priorem perpendicularem in *t*, eritque Pt incrementum normalis PM. Iungantur puncta C et *o*, recta C*o*, erit angulus OC*o*, incrementum anguli BCO: ad determinationem curvae CM quaesitae igitur requiritur, ut sit P*t* proportionale elemento angulari OC*o*. Est autem P*t* ut angulus PC*t* doctus in radium PC, erit ergo ang. OCo ad PC*t*, CP in data ratione, quae sit 1 ad *b* ut per consequens sit OC*o* : PCt = CP : *b*.

XXVII. Concurrant perpendiculares MP, *mp* in R, centro circuli oscularoris in M erit ang. PCt = MR*m* ob ΔΔ PCt et pR*t* similia, sed angulus MR*m* = ang. O*mo*, qui formatur a tangentibus proximis OM, *om*; ergo PCt = O*mo*, oportet ergo OCo: O*mo* = CP: *b* : producatur tangens OM in S, donec occurrat perpendiculo CS in se demisso, erunt demisso ex O in *mo* perpendiculo O*n*, triangula O*no*, OSC simila; ergo O*o* : O*n* = CO : OS. Sunt autem anguli OCo : Omo = $\frac{Oo}{OC}:\frac{On}{OM}=\frac{CO}{oC}:\frac{OS}{OM}$ (substitutis loco O*o* et O*n* proportionalibus OC et OS) = OM:OS. Est ergo OC*o*:OM*o* = OM:OS.

XXVIII. Cum autem requiratur, ut sit OCo : Oo = CP : b, obtinebitur haec analogia CP : b = OM : OS, quae tota ab angulis libera est, et inde habitur haec aequatio CP.OS = *b*.OM. Est autem ob ΔCOS ad rectang. OS $=\sqrt{(CO^2 - CS^2)}$ =(ob OC = CB et CS = PM) $=\sqrt{(CB^2 - PM^2)}$. Dein est OM = OS - SM = OS - CP $=\sqrt{(CB^2 - PM^2)} - CP$; unde haec aequatio obtinetur $CP\sqrt{(CB^2 - PM^2)} = b\sqrt{(CB^2 - PM^2)} - b.CP$. Unde curva desiderata determiari debet.

XXIX. Applicentur symbola, et vocetur CB distantia centri trochleae a foramine a, CP, p, et MP, t; habitur pro curva quaesita haec aequatio $p\sqrt{(aa-tt)} = b\sqrt{(aa-tt)} - bp$, quae eadem est cum ea quam § XXII. exhibui. Erit ergo $p = \frac{b\sqrt{(aa-tt)}}{b+\sqrt{(aa-tt)}}$. Ex qua aequatione curva construi poterit atque ad usum applicari. Si foramen ponatur infinite distans a centro trochleae, erit fili directio sibi semper parallela, adeoque habetur casus prior quo inventa erat CP semper constans. Posito enim a infinito, abibit $\sqrt{(aa - tt)}$ in a et obtinebitur $p = \frac{ab}{a+b}$ = ob b respecu ipsius a in denominatore evanescens.

XXX. Si etiam in hac machina loco elateris pondus applicare commodius visum fuerit, id ut in priori casu quoque praestari poterit, iisdem observandis monitis, ut pondus satis exiguum appendatur radius trochleae diminuatur, aucto eius pondere, idque tantum, quoad error a vi inertia ponderis oriundus insensibilis evadat, machinaeque irregularitatem, quae animadverti nequeat, inducat. Cum autem curvae ad istam machinam requisitae constructio valde sit operosa, contra vero curva priori casui, quo directio fili ad eandem perpetuo plagam tendit, constructu sit facillima, priorem modum huic posteriori fere praeferendum existimo.