PROBLEMATIS
Traiectoriarum Reciprocarum
Solutio.

Auctore
Leonhardo Eulero, Basil.

I.

Problema, de quo in hoc schediasmate agere constituti, est celebre illud et in Actis Lips. multum agitatum, de inveniendis curvis, quae intra datas parallelas eaedem recto et inverso situ positae et secundum parallelarum directionem hinc inde motae, mutua intersectione ubique; angulum eundem constituunt, Problema in Act. Lipf. Suppl. T. VII. a beate hic defuncto Nicolao Bernoulli propositum Ita autem cum hoc problemate res se habet, ut infinitae, tam algebraicae, quam transcendentes curvae satisfaciant. Quapropter ad plenam eius et perfectam solutionem requiritur, ut exhibeatur methodus, qua curvae satisfacientes omnes inveniri, simplicissimae autem tam algebraicae quam transcendentes re ipsa ervantur.
[Note 1 on page 1 of Euler's *Commentationes Geometricae*:
Nicolaus II Bernoulli (1695 - 1726), Johannes filius, proposuit in opere suo [Opera Omnia Johannis Bernoulli, Vol. II, p. 423 - 472]: *Exercitatio geometrica de trajectoriis orthogonalibus* sequens problema: Intra duos axes parallelos MN et FH positione datos, invenire et construere curvam ABC, eandemque DBE, sed inverso situ positam; ita ut alterutra, vel utraque mota secundum axem suum motu sibi semper parallelos, curvae ABC et DBE secant constantur se mutuo ad angelos rectos, hoc est, ut secandae et secantes sint curvae eaedem. Denique anonymous Angulus (Hendricus Pemberton 1694 - 1771) proposuit Problema, ut determinetur omnium, quae satisfaciunt in proposita quaestione, curvarum simplicissima (Joh. Bernoulli Opera t. II, p. 554). Sub finem solutionis, quam dederat Bernoulli (Opera omnia II, p. 600), scribit: Vices nunc redditurus, permettente id talionis lege, quaero ab eo, ut det aliquam Trajectoriam reciprocam algebraicam, quae sit inter omnes, quas invenire potest, ut ordine simplicitatis secunda, utque exhibeat aequationem inter coodinates naturam curvae quaesitae exprimentem.]

II. Dedi nuper, occasione quaestionis, quae Cel. Bernoullio cum Angelo quodam est nomen celante, de inveniendis traiectoriis reciprocis algebraicis simplicissimis, in Act. Lipf. A. 1727 methodum, qua ex quodlibet curvarum ordine, excepto secundo et tertio, (ex quorum posteriore quidem alia via curva satisfaciens inveniri potest), una ad minimum traiectoria reciproca exhiberi potest, una cum generali modo, omnes traiectorias reciprocas algebraicas ex curvis cuspide et circa cuspidem ramis similibus et aequalibus praeditis et algebraicis, derivandi. Animus hic est generalem huius problematis solutionem largiri, ex eaque infinitas formulas generales algebraicas easque maxime foecundas deducere. Quibus adiungam problematis cuiusdam agnati, de inveniendis traiectoriis reciprocis uno plures axes habentibus, solutionem.

III. Problema ad analysin magis accommodatum sic sonat : *Invenire curvam* CBD *circa axem* AB*, talem, ut ductis duabus rectis* MP, NQ*, ab axe utrinque aequidistantibus eique* *parallelis, summa angulorum PMB + QND, sit ubique constans, aequalis nimirum duplo anguli* DBA*, quem axis cum curva constituit.* Inversa enim CBD circa axem AB, cadet applicata QN super PM, tum moveatur, donec applicatae QN punctum N incidat in punctum M, et curva sic situ inversa sit *cbd* oportet angulum intersectionis BM*d* esse constantem. Sunt autem anguli PM*d*, et QND aequales, consequentur summa angulorum, PMB + QND, debet esse constans. Crescentibus ergo ex una parte axis AB, angulis applicatarum cum curva, ex altera parte tantundem decrescere debent.

IV. Ducta ad axem AB, normali PQ, erit AP = AQ, ducantur duae proximae respondentes applicatae, *pm, qn*. Erit P*p* = Q*q*. Ducantur ex M et N, tangentes MR, NS, ut habeantur anguli RM*m*, SN*n*, quorum ille est decrementum anguli PMB, hic incrementum anguli QND. Quocirca erit ex conditione problematis RM*m* = SN*n*. Unde natura curvae investigari debet.

V. Sumatur ubique in applicata MP producta, PF, proportionalis angulo RM*m*, assumpto elemento P*p*, abscissae AP pro constante, erit punctum F in curva quadam, cuius diameter erit axis traiectoriae BA, erit enim ubique PF = QG. Quare tota difficultas eo est reducta, ut ex curva FEG data altera CBD, in qua elementa angulorum BMP sint respondentibus applicata PF proportionales construantur; Et ut curva CBD evadat traiectoria reciproca, curva FEG debet habere diametrum, et circa eam ramos similes et aequales, cuiusmodi est FEG. Curva MBN ex ea constructa erit traiectoria reciproca, cuius axis est EB, diameter prioris curvae.

VI. Sit AP = x; PM = y ; PF = u. Erit angulus RM*m*, ut $ddy : (dx^2 + dy^2)$. Ergo $u = ddy : (dx^2 + dy^2)$, posito dx constante, ex qua aequatione datis u et x inveniri debet, y. Ponatur $dy = pdx$. erit $ddy = dp.dx$. Ergo $u = \frac{dp}{dx+ppdx}$ et $udx = \frac{dp}{1+pp}$. Ex qua aequatione, ob u et x datis, p invenitur, indeque y; est autem $\frac{dp}{1+pp}$, duplum elementum sectoris circularis, cuius radius est, 1. et tangens p; erit ergo $\frac{1}{2}\int udx$ = sectori isti circuli. Est vero $\frac{1}{2}\int udx$ = area APEF, demta vel addita constante invenitur ergo per quadratus, p, indeque rursus per quadraturas y sequenti modo.

VII. Sit data quaecunque curva IEK diametro EA praedicta ; super recta AO diametrum EA normaliter secante, accipiatur punctum quodvis O, quo centro et radio arbitrario OD describatur circulus DGH, et ex D ducatur tangens DQ. Ducta quacunque applicata PF, spatio PFID aequalis sumatur sector DOG, et producatur DG in Q. ex Q ducatur ipsi DA parallela QN, occurrens applicatae FP productae in N; Erit punctum N in curva DN tali, ut sit PN = p, si sit AP = x. Hic dimidium, quod superiore paragrapho inventum est, negligitur, cum enim

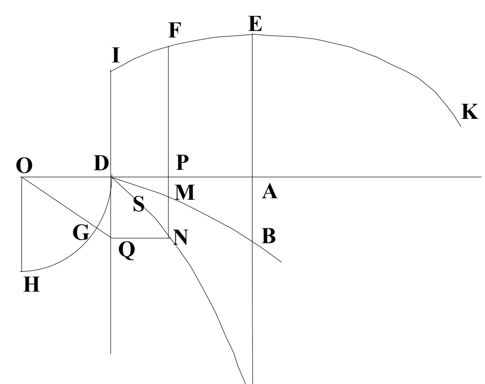

Fig. 2

$ddy:(dx^2+dy^2)$ saltem proportionetur ipsi u, etiam $\int udx$, tantum proportionalis assumi potest, sectori DOG. unde nihil interest sive dimidius sector sive totus sumatur, et dein sive $\int udx$ ab applicata DI, sive ab AF computetur.

VIII. Inventa curva DN facili negotio habetur curva DM traiectoria reciproca, cum enim sit dy = pdx accipiatur ubique PM proportionalis areae DPN, erit punctum M, in traictoria reciproca, cuius axis est AB, diameter curvae IEK assumptae. Apparet hic simul, infinitas, ex unica assumpta IEK, traiectorias reciprocas inveniri posse, prout enim transversalis DA aliter ducitur, punctaque O et D aliter assumuntur, ita aliae resultant traiectoriae reciprocae. Dein etiam pro varia ratione, quae ponitur inter PM spatium DPN, traiectoriae variae formantur. Unde patet data una traiectoria reciproca, applicatas in eadem ratione augendo vel diminuendo infinitas inveniri alias traiectorias reciprocas.

IX. Si spatium DPFI aequale accipiatur quadranti ODH, tangens DQ ipsique aequalis applicata PN evadit infinita, eritque tum PN asymptotos curvae DN; Sin spatium illud maius fuerit quadrante, applicata PN erit negativa. Traiectoriae autem DM applicata PM evadet, ubi PN est infinita, tangens curvae. Et deinde abeunte PN in negativam applicata PM decrescet, quare curva DM habebit in M punctum reversionis. Si spatium DPN, existente PN asymptoto, est infinitum, applicata PM quoque erit infinita, adeoque asymptotos etiam curvae DM.

X. Sit exempli gratia $u=\dfrac{b}{xx+aa}$ et hinc ervatur aequatio inter x et y; cum sit u ut $ddy:(dx^2+dy^2)$, erit $bdx^2+bdy^2=aaddy+xxddy$, ponatur $dy=pdx$, erit $ddy=dpdx$, quibus valoribus substitutis habetur $bdx+bppdx=aadp+xxdp$.

Ergo
$$\frac{bdx}{aa+xx}=\frac{dp}{1+pp},$$
cuius aequationis utrisque membri integratio a circuli quadratura dependet.
Huc autem aequatio ea reducitur
$$\frac{b}{a}\left(\frac{dx}{a+x\sqrt{-1}}+\frac{dx}{a-\sqrt{-1}}\right)=\frac{dp}{1+p\sqrt{-1}}+\frac{dp}{1-p\sqrt{-1}}.$$
Quae integrata abit in hanc
$$bl(a+x\sqrt{-1})-bl(a-x\sqrt{-1})=al\left(1+p\sqrt{-1}\right)-al\left(1+p\sqrt{-1}\right)+alh,$$
erit ergo
$$\left(\frac{a+x\sqrt{-1}}{a-x\sqrt{-1}}\right)^{\frac{b}{a}}=\frac{h+hp\sqrt{-1}}{1-p\sqrt{-1}}=\frac{hdx+hdy\sqrt{-1}}{dx-dy\sqrt{-1}}.$$
Sit $b=a$ et $h=\sqrt{-1}$, erit
$$\left(\frac{a+x\sqrt{-1}}{a-x\sqrt{-1}}\right)=\frac{dx\sqrt{-1}-dy}{dx-dy\sqrt{-1}}$$
quae reducta dat
$$dy=\frac{x-a}{x+a}dx.$$

Si sit $b = 2a$ manente $h = \sqrt{-1}$, erit

$$dy = \frac{2ax + xx - aa}{2ax - xx + aa} dx .$$

Et ita porro; sed huiusmodi exemplis non immiror, fusius de iis infra agetur.

XI. Quum curvae genetricis IEK diameter EA sit axes trajectoriae inde genitae, manifestum est, si illa curva plures una diametros habuerit, trajectoriam inde ortam plures axes etiam habituram ; se ergo loco curvae IEK curva infinitarum diametrorum adhibetur, trajectoria infinitos etiam axes habebit. Quando autem trajectoria desireratur, quae axium datum habeat numerum, id aliter interpretandum est. Ut enim omnis curva una plures diametros habens necessario infinitas habet, ita etiam trajectoria reciproca, quae uno plures, infinitos necessario axes habebit. Sed quia trajectoria infinitorum axium infinita habere debet puncta reversionis, datus axium numerus ad unam tantum curvae portionem intra duo puncta flexus proxima comprehensam referendus est. Desideratur enim curva omni irregularitate, cuiusmodi est flexura et reflexio, destituta.

XII. Sit curva (Fig. 3) IEK*ek* infinitis praedictis diametris, EA, KL, *ea, kl*, et abscindatur area DBTI = quadranti ODH, linea TB producta asymptotos erit curvae DNV et tanget trajectoriam in C, ubi est punctum reflexionis. Portio ergo trajectoriae DMC talis erit, de qua est quaestio numeri axium dati. Haec vero portio tot habebit axes, quot diametri fuerint in spatio DBTI. Quocirca in arbitrio nostro positum erit numerum axium definire hoc modo: Proposito numero axium abscindatur spatium DBTI eundem diametrorum numerum comprehendens, tum describatur circulus ODG tantus, ut eius quadrans ODH adaequet abscissum spatium DBTI, manifestum est hoc modo generari curvam desideratam.

XI. Quum curvae genetricis IEK diameter EA sit axes trajectoriae inde genitae, manifestum est, si illa curva plures una diametros habuerit, trajectoriam inde ortam plures axes etiam habituram ; si ergo loco curvae IEK curva infinitarum diametrorum adhibetur, trajectoria infinitos etiam axes habebit. Quando autem trajectoria desideratur, quae axium datum habeat numerum, id aliter interpretandum est. Ut enim omnis curva una plures diametros habens necessario infinitas habet, ita etiam trajectoria reciproca, quae uno plures, infinitos necessario axes habebit. Sed quia trajectoria infinitorum axium infinita habere debet puncta reversionis, datus axium numerus ad unam tantum curvae portionem intra duo puncta flexus proxima comprehensam referendus est. Desideratur enim curva omni irregularitate, cuiusmodi est flexura et reflexio, destituta.

XII. Sit curva (Fig. 3) IEK*ek* infinitis praedictis diametris, EA, KL, *ea, kl*, et abscindatur area DBTI = quadranti ODH, linea TB producta asymptotos erit curvae DNV et tanget trajectoriam in C, ubi est punctum reflexionis. Portio ergo trajectoriae DMC talis erit, de qua est quaestio numeri axium dati. Haec vero portio tot habebit axes, quot diametri fuerint in spatio DBTI. Quocirca in arbitrio nostro positum erit numerum axium definire hoc modo: Proposito numero axium abscindatur spatium DBTI eundem diametrorum numerum comprehendens, tum describatur circulus ODG tantus, ut eius quadrans ODH adaequet abscissum spatium DBTI, manifestum est hoc modo generari curvam desideratam.

XIII. Si loco curvae IEKek adhibeatur linea recta ipsi DB parallela, quaelibet applicata FP erit diameter, egro et trajectoriae DMC quaevis applicata erit axis. Atque haec est illa curva, de qua Celeberrimus Bernoullius sub pantogonia nomine fusius in Actis Eruditorum 1726 egit. [*Continuatio materiae de trajectoriis reciprocis, imprimis de Pantogonia*, opera omnia Tomus II, p. 600 - 616.] Aequatio eius naturam exprimens erit : $1 = addy : (dx^2 + dy^2)$ seu $dx^2 + dy^2 = addy$, cuius haec est proprietas, ut, radius secundum axium directionem incidentibus, radii reflexi omnes sint inter se aequales. Facilius autem curva haec sic construetur, ut accipiatur
$x = \int \frac{aadp}{aa+pp}$ et $y = \int \frac{apdp}{aa+pp}$.

XIV. Methodum hanc inveniendi trajectorias reciprocas per duplicem quadraturam non eo fine attuli, ut inde trajectoriae reciprocae ervantur, id quod vix praestari posset, si simplices vel algebraicae desiderentur, sed ut inde adipiscar solutionem problematis de inveniendis trajectoriis pluribus uno axibus gaudentibus, quod Anonymus Anglus Celeberrimo Ionanni Bernoullio proposuit. At nunc ad alium pergo modum perquam foecundum in exhibendis trajectoriis simplicioribus et praecipue algebraicis. Persequar autem hic illum tantum problematis casum, quo angulus intersectionis ponitur rectus, cum facillime reliqui casus omnes ad hunc reducantur.

XV. Sit (Fig. 4) CBD trajectoria orthogonalis, cuius axis sit AB, quem ad angulos rectos secet recta PQ. Ducantur duae applicatae PM, QN, axi AB parallelae, utrinque aeque distances ab eodem, illisque proximae *pm, qn,* nec non basi PQ parallelae MR, NS. Erunt triangula MRm, Nsn similia, ob S*n*N + R*m*M = recto. Sint AP = x, PM = y, erunt
Pp = MR = dx, Rm = dy nec non
AQ = - x, Qq = NS = - dx; sit QN = z seu Sn = dz. Ex similitudine triangulorum MRm, NSn deducetur

$$MR(dx) : Rm(dy) = Sn(dz) : SN(-dx),$$

unde erit $dydz = - dx^2$.
Ex qua aequatione inveniri debet y. Etenim z ab y dependet, quia in expressione ipsius y, posito loco x, - x, habetur z.

XVI. Ponatur $dy = pdx$, est autem p functio ipsius x. Abeat ea, posito - x loco x, in q, erit $dz = - qdx$ et consquentur erit $pq = 1$. Unde patet loco p talem sumi debere ipsius x functionem, ex qua factum in eandem, sed loco x posito - x, adaequet unitatem. Totum ergo huius solutionis artificium huc redit, ut idoneae eligantur functiones ipsius x loco p substituendae. Ad hoc autem, nisi fortunae earum inventionem committere velimus, accuratior functionem requiritur cognito. Cuius ut quasi prima elementa iaciam, sequenti modo eas discernere commodum visum est.

XVII. Primo loco notandae sunt functiones, quas pares appello, quarum haic est proprietas, ut immutatae maneant, etsi loco x ponantur - x. Huiusmodi sunt omnes potentiae ipsius x , quarum exponentes sunt numeri pares, aut fractiones, quarum numeratores sunt numeri pares, denominatores vero impares. Dein, quaecunque functiones ex huiusmodi potentiis vel additione vel subtractione, vel multiplicatione vel divisione, vel denique ad potentatiam quamcunque elevatione componuntur, sunt itidem pares, ut

$$x^{\frac{4}{5}}, \quad (ax^2+bx^{\frac{2}{3}})^n.$$

XVIII. Secundo functiones impares observo, quae prorsus sui negativas producunt, si x abit in - x. Cuiusmodi sunt x ipsum, x^3, x^5, etc., omnes potentiae, quarum exponentes sunt numeri impares, vel fractiones quarum numeratores et denominatores sunt numeri impares, nec non functiones, quae harum potentiarum additione vel subtractione, etiam elevatione ad exponentis imparis dignitatem componuntur, ut

$$x^3, \quad (ax^3+bx^{\frac{5}{7}})^3.$$

XIX. Si functio impar per imparem multiplicatur, factum semper erit functio pars, ut x^3 in $x^{\frac{1}{3}}$ dat $x^{\frac{10}{3}}$. At functio par in imparem ducta semper quidem imparem producit, interdum tamen ea simul pro pari haberi potest, ut $x\sqrt{(aa+xx)}$ est functio simul par et impar, quippe eadem cum

$$\sqrt{(aaxx+x^4)}\,,$$

quae est par. Quod autem de elevatione functionis paris ad dignitatem quamvis supra dictum est, quod potentia sit quoque par, si exponens sit fractio, cuius denominator numerus par, verbi gratia $\frac{1}{2}$, restricto adhibenda est, nisi radix re ipsa extrahi queat, ut

$$(\tfrac{aa}{xx}+2a+xx)^{\frac{1}{2}}$$

non est functio par, convenit enim cum $\frac{a}{x}+x$. De huiusmodi autem functionibus iudicium facile patet.

XX. Praeterea observatu dignae sunt functiones reciprocae, quae mihi sunt functiones, posito in iis - x loco x, abeuntes in tales, quae in illas ductae producunt unitatem, ut

$$\left(\tfrac{a+x}{a-x}\right)^n,$$

quae. posito x negativo, abit in hanc

$$\left(\tfrac{a-x}{a+x}\right)^n,$$

cuius in illam factum est = 1. Huc referendae quoque sunt exponentiales $a^x, (aa+xx)^{x^3}$, etc., omnes nempe functiones pares elevatae ad functiones impares.

XXI. Hisce de functionibus praemissis manifestum est p esse functionem ipsius x reciprocam, cum sit $pq = 1$. Quemadmodum autem huiusmodi functiones reciprocae inveniendae sint, brevi ostendere conabor. Sed primo de functionibus exponentialibus nihil intermiscere constitui, cum ante omnia trajectorias reciprocas algebraicas eruere animus sit. Postmodum autem de exponentialibus quaedam subiungam.

XXII. Ut autem rem generalius absolvam, assumo tertiam variabilem t et investigabo, quomodo x et y in t determinari debeant, ut trajectoria reciproca resultet, pono itaque $dx = rdt$ et $dy = pdt$. Efficiendum ergo est, ut posito t negative, et dx in negativum abeat. Quare loco r ponantur oportet functio ipsius t par, quae sit N; erit $dx = Ndt$, et abeunde t

in negativum, erit $dx = - Ndt$. Consequenter ob $dydz = - dx^2$, posito in casu, - t loco p, q. ut ante, habebitur $pq = NN$.

XXIII. Ponatur $p = (P+Q)^n$ denotante P functione pare et Q impare ipsius t, erit $q = (P-Q)^n$ adeoque

$$(PP-QQ)^n = N^2;$$

ergo

$$PP = N^{\frac{2}{N}} + QQ \text{ et } P = \sqrt{N^{\frac{2}{N}} + QQ};$$

erit ergo

$$p = (Q + \sqrt{(N^{\frac{2}{N}} + QQ)}\,)^n.$$

Nihil contradictorii hic latet in aequatione $P = \sqrt{N^{\frac{2}{N}} + QQ}$, etenim P, quae functio par esse debet, talis etiam in aequatione exhibetur. At si Q erueretur, inveniretur $Q = \sqrt{N^{\frac{2}{N}} - PP}$, id quod contradictionem involvit; nam Q, quae functionem imparem denotat, aequatur hic functioni pari. Ut fractiones in exponentibus evitaem, scribo loco N, N^n et erit

$$dx = N^n dt \text{ et } dy = dt(Q + \sqrt{(N^2 + Q^2)}\,)^n;$$

potest hic loco Q scribi NQ (§19) et dein loco N^n, ut ante, N; habebitur

$$dx = Ndt \text{ et } dy = Ndt(Q + \sqrt{(1+QQ)}\,)^n$$

XXIV. Ut novae formulae resultent, tollo irrationalitatem, ponendo

$$\sqrt{(N^2 + Q^2)} = N + RQ,$$

erit

$$Q = \frac{2NR}{1-RR},$$

unde R functio impar sit ipsius t necesse est, ob Q imparem, erit ergo

$$Q + \sqrt{(N^2 + Q^2)} = \frac{N(1+R)}{1-R} = (\text{scripto loco R}, \frac{\text{Q}}{\text{P}}) \frac{N.(P+Q)}{P-Q}.$$

Denotabunt semper P pares et Q impares functiones ipsius t. Erit ergo

$$dx = N^n dt \text{ et } dy = dt\left(\frac{NP+NQ}{P-Q}\right)^n;$$

altera formula eodem modo tractata dat

$$dx = Ndt \text{ et } dy = Ndt\left(\frac{P+Q}{P-Q}\right)^n.$$

Huiusmodi formulae generales infinitae possunt inveniri, alias aequationes loco $p = (P+Q)^n$ assumendo : cuiusmodi est haec formula

$$dx = Ndt \text{ et } dy = Ndt(P+Q)^{mn}.\left(S + \sqrt{\left(SS + (PP-QQ)^{-\frac{m}{k}}\right)}\right)^{nk},$$

denotante S functione impare; sed duabus formulis inventis tanquam simplicissimis et foecundissimis in productione trajectoriarum algebraicarum contentus ero, quarum altera irrationalitate est affecta, altera vero rationalis.

XXV. Accipio formulam priorem, casus, quibus dy integrabile redditur, evoluturus. Sit primo N = 1, erit $dx = dt$, unde

$$dy = dx(Q + \sqrt{(1+QQ)}\,)^n .$$

Pono porro Q = x, erit

$$dy = dx(x + \sqrt{(1+xx)}\,)^n ,$$

cuius integrale observo generaliter haberi posse; ponatur $x + \sqrt{(1+xx)} = u$, erit

$$x = \frac{uu-1}{2u}, \text{ consequenter } dy = \frac{u^n du}{2} + \frac{u^{n-2} du}{2}$$

et hinc

$$2y = \frac{u^{n+1}}{n+1} + \frac{u^{n-1}}{n-1} = \frac{\left(x+\sqrt{(1+xx)}\right)^{n+1}}{n+1} + \frac{\left(x+\sqrt{(1+xx)}\right)^{n-1}}{n-1};$$

habetur ergo hic aequatio algebraica generalis infinitas curvas suppeditans, numeros rationales loco n substituendo.

XXVI. Antequam autem ad derivationem aequationum determinatarum ex generali pergam, quaedam ex aequatione differentiali deducenda sunt, quae ex integrata difficilius eruerentur. Primo palam est, si sit $n = 0$, trajectoriam tum esse lineam rectam, cum axe angulum semirectum constituentem, propter $dy = dx$. Dein, si fuerit $n = 1$, erit

$$dy = xdx + dx\sqrt{(1+xx)} ;$$

unde patet hanc aequationem esse ad trajectoriam reciprocam, quae methodo BERNOULLIANA ope rectificationis parabolae construitur, quo unico casu non absolute est integrabilis.

XXVII. Tertio, etsi loco n ponatur - n, aequationem nihilominus ad eandem fore curvam, abscissis saltem ex axis altera parte sumptis seu existentibus negativis. Conveniunt enim duae hae expressiones

$$(-x + \sqrt{(1+xx)}\,)^n \text{ et } (x + \sqrt{(1+xx)}\,)^{-n} ,$$

ut cuivis examinanti facile patebit. Nihil ergo in posterum lucraturus essem loco n valores negativos substituendo. Quare substitutione numerorum affirmativorum tantum utar, cum ii soli sufficiant ad universalem aequationem exhauriendam.

XXVIII. Excussi iam sunt casus, ubi $n = 0$ et $n = 1$, progredior ulterius, sed integralem aequationem in usum vocando, et pono $n = 2$. Erit

$$3y = 2x^3 + 3x + (2 + 2xx)\sqrt{(1+xx)} ,$$

quea ad rationalitatem reducta huc redit

$$12yx^3 + 18xy - 9yy + 3xx + 4 = 0 .$$

Et haec aequatio quatuor dimensionum sine dubio simplicissima est, post illam paraboloidem tertii ordinis: satisfacit adeo quaestioni, quam Celeberrimus

BERNOULLIUS Anonymo Angelo proposuit, (et ego repeti in Actis Eruditorum 1727) de invenienda trajectoria algebraica, eam tertii ordinis, in simplicitatis ordine proxime excipiente.

XXIX. Si ponatur n = 3, prodibit aequatio pro linea 5 ordinis haec

$$128yx^4+192xy^2+48y-64yy-8xx-9=0.$$

Sit $n=4$, resultabit aequatio 6 ordinis, et hinc legitima inductione inferri potest aequationem generalem ad rationalitatem reductam esse semper ordinis $n+2$. Id quod etiam in valoribus fractis loco n subrogatis obtinet. Si sit $n=\frac{1}{2}$, aequatio erit ordinis $\frac{5}{2}$. Quae autem, cum adhuc sit irrationalis, reducta erit ordinis quinti, et generaliter si fuerit $n=\frac{p}{q}$ aequatio reducta ascendet ad $p+2q$ ordinem.

XXX. Patet ergo aequationem generalem loco n alios atque alios valores substituendo, ex quolibet curvarum ordine, si excipias secundum et tertium, unam ad minimum trajectoriam reciprocam exhibere. Et dato ordine curvarum, quot ex ope huius aequationis inveniri possint trajectoriae, facile determinare erit, nempe dispiciendum est, quoties $p+2q$ numerum dati ordinis producere queat, sed loco p et q numeri saltem affirmativi et integri substitui possunt et eiusmodi insuper, ut $p:q$ ad minores terminos reduci nequeat. Sed de hac formula generali fusius in Actis Lipsiensibus 1727 actum est a me, ideoque hic ad alias me converto.

XXXI. Adhaereo adhuc aequationi paragraphi 25 ad hanc reductae

$$dy=dx\left(Q+\sqrt{(QQ+1)}\right)^n.$$

Circa quam observavi nullis eam substitutionibus potentiarum rationalium, quales sunt x^3, x^5, etc. nec non x^{-1}, x^{-2}, etc., loco Q factis generaliter integrabilem reddit, quanquam utique passim reperiantur casus particulares integrabiles, quos autem persequi institutum minime permittit. At substituendo loco Q potentias ipsius x irrationalibus, sed ubi numerator exponentis est unitas, semper formulam integrabilem reddit observavi.

XXXII. Sit itaque $Q=x^{\frac{1}{3}}$, erit

$$dy=dx\left(x^{\frac{1}{3}}+\sqrt{\left(x^{\frac{2}{3}}+1\right)}\right)^n.$$

Quae ut integretur, pono

$$x^{\frac{1}{3}}+\sqrt{\left(x^{\frac{2}{3}}+1\right)}=t,$$

erit

$$x^{\frac{1}{3}}=\tfrac{tt-1}{2t},$$

unde

$$x=\tfrac{t^3}{8}-\tfrac{3t}{8}+\tfrac{3}{8t}-\tfrac{3}{8t^2},$$

adeoque

$$dx = \frac{3tt}{8} - \frac{3}{8} - \frac{3}{8tt} + \frac{3}{8t^4},$$

ergo

$$dy = \frac{3t^{n+2}dt}{8} - \frac{3t^{n}dt}{8} - \frac{3t^{n-2}dt}{8} + \frac{3t^{n-4}dt}{8}.$$

Consequenter

$$\frac{8y}{3} = \frac{t^{n+3}}{n+3} - \frac{t^{n+1}}{n+1} - \frac{t^{n-1}}{n-1} + \frac{t^{n-3}}{n-3}$$

$$= \frac{\left(x^{\frac{1}{3}} + \sqrt{(x^{\frac{2}{3}}+1)}\right)^{n+3}}{n+3} - \frac{\left(x^{\frac{1}{3}} + \sqrt{(x^{\frac{2}{3}}+1)}\right)^{n+1}}{n+1} - \frac{\left(x^{\frac{1}{3}} + \sqrt{(x^{\frac{2}{3}}+1)}\right)^{n-1}}{n-1} + \frac{\left(x^{\frac{1}{3}} + \sqrt{(x^{\frac{2}{3}}+1)}\right)^{n-3}}{n-3}.$$

XXXIII. Sunt autem quidam casus, quibus integratio a logarithmis dependet, nempe si fuerit $n = 1$ vel 3. Ceterae substitutiones omnes loco n factae suppeditant curvas algebraicas, idque ut superior secundum legem. Quod de superiori formula enunciatum est valores ipsius n negativos superfluos esse, idem etiam de hac, nec non de generalissima tendendum est. Quaemadmodem et semper obtinet, si fiat $n = 0$, tum trajectoriam degenere in lineam rectam.

XXXIV. Casus huius aequationis simplicimissus sine dubio erit, quo $n = 2$. In eoque posito brevitatis ergo loco $x^{\frac{1}{3}}, t$, reperietur

$$5y = 6t^5 + 5t^3 + (6t^4 + 2tt - 4)\sqrt{(1+tt)}.$$

Consequenter ad rationalitatem reducendo pervenietur ad hanc aequationem

$$60t^5y + 50t^3y - 25y^2 - 60t^4 - 45t^6 + 16 = 0.$$

Atque haec tandem, substituto $x^{\frac{1}{3}}$ loco t, abibit in aequationem 8 ordinis. Si ponatur $n = 4$, aequationem ad 10 dimensiones assurecturam, facile praevidere potui. Et aequationem generalem ad ordinem linearum $n + 6$ esse referendam. Ut adeo et haec formula ex quolibet curvarum ordine ad minimum unam, si excipiantur 2, 3, 4, 5, 6, 7, et 9, trajectoriam exhibeat.

XXXV. Haec eadem formula, ut et reliquae, quae ex substitutionibus paragrapho 31 determinatis deducuntur, alia via ex altera aequationis generalis forma derivantur. Et quam ideo paucis hic complectar, quod insuper ex ea plures formulae algebraicae generales, aliunde altioris indaginis, fluant. Aequatio generalis haec est

$$dx = Ndt, dy = Ndt\left(Q + \sqrt{(QQ+1)}\right)^n.$$

In quia si fiat Q = x et successive N = 1 vel xx vel x^4, etc. nec non vel $a + bxx, axx + bx^4$ et eiusmodi compositae functiones pares ipsius x subrogentur, aequatio generalis semper erit integrabilis et algebraicarum aequationum summopere foecunda.

XXXVI. Exposito modo, quo ad aequationes algebraicas generales pervenitur, examinandi sunt alii casus, quibus quidem aequatio generalis non integrabilis redditur, nihilo tamen minus infinitis modis facilibus determinatu algebraicas exhibere potest aequationes. Assumpto hanc formulam

$$dx = N^n dt, \text{ et } dy = dt\left(Q + \sqrt{(QQ+NN)}\right)^n,$$

fiat N = tt et Q = 1, erit

$$dx = t^{2n}dt, \text{ et } x = \tfrac{t^{2n+1}}{2n+1} \text{ et } dy = dt\left(t + t\sqrt{(1+tt)}\right)^n.$$

Unde patet hanc aequationem semper esse integrabilem, si fuerit *n* numerus integer impar; id quod facile videre est, si reipsa ad dignitatem elevetur.

XXXVII. Sunt autem insuper alii casus, quibus formula nostra integrabilis redditur, quos sic invenio: ponatur

$$t + t\sqrt{(1+tt)} = utt\,,$$

consequenter

$$dy = -\tfrac{2^{2n+1}u^{2n}du(uu+1)}{(uu-1)^{2n+2}}.$$

Unde patet, si fuerit n numerus negative par, fore aequationem integrabilem, id quod patebit, si $(uu - 1)^{-2n-2}$ ipso facto elevetur. Plures casus elicientur si fiat

$$t + t\sqrt{(1+tt)} = ut\,;$$

et obtinebitur

$$dy = du(u^{\frac{3n+1}{2}} - u^{\frac{3n-1}{2}})(u-2)^{\frac{n-1}{2}}\,,$$

quae erit integrabilis primo, si sit n quilibet numerus impar, dein, si $\tfrac{3n+1}{2}$ fuerit numerus integer. Fiat ergo $\tfrac{3n+1}{2} = m$; erit $n = \tfrac{2m-1}{3}$ adeoque loco *n* poni potest fractio, cuius denominator = 3 et numerator numerus impar.

XXXVIII. Unicum exemplum attulisse sufficiat; sit n = 1, erit

$$dx = ttdt, \text{ et } t = \sqrt[3]{3x}\ ;$$

deinde

$$dy = tdt + tdt\sqrt{(1+tt)}$$

ergo

$$y = \tfrac{tt}{2} + \tfrac{(1+tt)\sqrt{(1+tt)}}{3}$$

unde elicitur haec aequatio ordinis sexti

$$(12yy - 12xx)^3 = 3456y^5 + 12528x^2y^3 - 432yx^4 - 2304y^4 - 288x^2y^2 + 81x^4 + 512y^3.$$

Possunt itaque infinitae aequationes algebraicae etiam ex hac aequatione

$$dy = dt(t + t\sqrt{(1+tt)}\ )^n$$

erui, et simili modo ex aliis formis, loco N vel Q ; alios valores substituendo, casus, quibus hoc contingit, non superiori absimili modo detegentur.

XXXIX. Quae de priori duarum generalium formularum irrationali hucusque tradita sunt, usum eius et foecunditatem satis superque commonstrant. Progredior nunc ad alteram formulam rationalem, quae est

$$dx = N^n dt \text{ et } dy = dt\left(\frac{N.(P+Q)}{P-Q}\right)^n$$

seu, quod eodem redit,

$$dy = dt\left(\frac{N.(1+Q)}{1-Q}\right)^n .$$

Non immoror hic derivandis hinc curvis transcendentabilus, nempe logarithmicae semirectangulae, si $Q = t$, $N = 1$ et $n = 1$, aut cycloidi, si $n = \frac{1}{2}$, quippe quae ab aliis iam fusius pertractatae sunt; propositum mihi est, ut in priori, quas ea sub se comprehendit curvas algebraicas, persequi et regulas, quibus algebraicae inveniri queant, eruere.

XL. Ne autem fractio in causa sit, cur difficilius casus algebraici dignoscantur, eam tollo loco N ponendo $N(1-QQ)$. Debet enim N esse functio par ipsius t. Habitur

$$dx = dt(N - NQQ)^n \text{, et } dy = dt\left(N(1+Q)^2\right)^n$$

seu

$$dx = N^n dt(1-QQ)^n \text{, et } dy = N^n dt(1+Q)^{2n} .$$

Ut hinc aequatio algebraica derivari queat, oportet, ut et dx et dy integrabile fiat. Ponatur $N = 1$; erit

$$dx = dt(1-QQ)^n \text{, et } dy = dt(1+Q)^{2n} ;$$

sit $Q = t$, erit

$$dx = dt(1-tt)^n \text{, et } dy = dt(1+t)^{2n} ,$$

unde

$$y = \tfrac{(1+t)^{2n+1}}{2n+1} ;$$

ergo

$$\sqrt[2n+1]{(2n+1)y} - 1 = t.$$

Ut igitur dx integrari queat, patet loco n substitui debere numerum integrum affirmativum.

XLI. Cum n sit numerus integer affirmativus, constituet $(1-tt)^n$, si in seriem convertur, progressionem numeri terminorum finiti hanc

$$1 - \frac{n}{1}tt + \frac{n(n-1)}{1.2}t^4 - \frac{n(n-1)(n-2)}{1.2.3}t^6 + etc.$$

Unde obtenibitur

$$x = t - \frac{n}{1}\frac{t^3}{3} + \frac{n(n-1)}{1.2}\frac{t^5}{5} - \frac{n(n-1)(n-2)}{1.2.3}\frac{t^7}{7} + etc.,$$

in qua si loco t substituatur valor invenus

$$\sqrt[2n+1]{(2n+1)y} - 1$$

habebitur aequatio inter y et x adeoque pro curva quaesita.

XLII. Sit $n = 1$, erit

$$x = t - \tfrac{t^3}{3} = \sqrt[3]{3y} - 1 - \tfrac{1}{3}(\sqrt[3]{3y} - 1)^3 = \sqrt[3]{9}y^2 - y - \tfrac{2}{3} \text{, ergo } (x + y + \tfrac{2}{3})^3 = 9yy .$$

Quae aequatio evadit tertii ordinis et exprimit parabolam cubicalem semirectangulam, quae pro simplicissima omnium trajectoriarum reciprocarum algebraicarum habetur. De qua Celeberrimus JOHANNES BERNOULLIUS in Actis Eruditorum 1725 [July, pp.318-325] peculiari schediasmate egit. Sunt autem reliquae substitutiones loco n factae minus felices in exhibendis curvis simplicibus, posito enim $n = 2$, aequatio iam ultra trigesimum gradum assurgit.

XLII. Possunt loco Q aliae functiones ipsius t substituti, ut t^3, t^5, aut $t^{\frac{1}{3}}$ etc. quae omnes formulam infinitis modis integrabilem reddent, semper nimirum, quando n fuerit numerus affirmativus integer. Simili modo res se habet, si alii loco N valores subrogentur. Sit nimirum $N = tt$, erit

$$dx = t^{2n}dt(1-QQ)^n\text{, et } dy = t^{2n}dt(1+Q)^{2n}.$$

Ponatur $Q = t$, erit

$$t^{2n}dt(1-tt)^n\text{, et } dy = t^{2n}dt(1+t)^{2n}.$$

Unde patet et x et y haberi posse, modo sit $2n$ numerus integer. Si enim fuerit numerus par, facile patet omnia esse in simplices terminos resolubilia, si $2n$ fuerit numerus impar, erit $(1-tt)^n$ irrationale, sed licet $2n$ sit numerus impar, nihilomnus $t^{2n}dt(1-tt)^n$ erit integrabile.

XLIV. Sit $2n = 1$, erit

$$dx = tdt\sqrt{(1-tt)}\text{, et } dy = tdt + ttdt.$$

Quare

$$x = -\tfrac{1}{3}(1-tt)^{\frac{3}{2}} \text{ et } y = \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3.$$

Erit igitur

$$t = \sqrt{(1-\sqrt[3]{9xx})};$$

ponatur $y + \frac{1}{6} = u$, habebitur reductione peracta haec aequatio sexti gradus

$$(12(u^2+x^2)-8u)^3 + 3^5.4^3x^2u^3 - 81x^4 + 3^4.4(12(u^2+x^2)-8u)x^2u = 0$$

[corrected from original]
Alii loco n numeri substituti alias exhibebunt curvas algebraicas, deinde innumerabiles aliae loco Q et loco N substitutiones fieri possunt, quae semper, si n est numerus affirmativus integer, algebraicas efficient aequationes. Et haec praecipua sunt, quae de Algebraicis curvis afferri possunt.

XLV. Hisce coronidis loco subiungo alias formulas generales, quae resultant, si loco (vide paragraphum 22) p et q functiones exponentiales subrogentur. Habentur autem in exponentialibus et functiones pares et reciprocae, ut P^R est functio pares, si sit et P et R functiones pares, at si fuerit R functio impar, erit ea functio reciproca, priori in casu abeunte x in - x manet P^R , in posteriori mutatur in P^{-R}.

XLVI. Quibus ergo in locis, antea functiones pares substituere opus fuerat, poterunt huiusmodi exponentiales adhiberi, et loco functionum imparium similes exponentiales ductae in functionem imparem quandam, ut P^RQ, existentibus P et R paribus functionibus et Q impari.

XLVII. Cum functiones reciprocae ita sint comparatae, ut factum earum in se ipsas, sed loco x posito $-x$, aequetur unitati, patet, quicquid sit p, semper ei insuper eiusmodi functionem reciprocam multiplicatione adjuncti posse, nempe, ubi fuerat $dy = pdt$, potest etiam sumi

$$dy = T^V pdt,$$

denotante T functione pari et V impari. Nihilominus enim factum ex dy in se, sed abeunte t in $-t$, idem erit ac ante.

XLVIII. Formulis ergo generalibus paragraphis 23 et 24 inventis adiungi poterit functio reciproca, immutato earum usu. Et habebitur

$$dx = Ndt \text{ et } dy = T^V Ndt\left(Q + \sqrt{(1+QQ)}\right)^n,$$

deinde loco formulae rationalis habebitur

$$dx = Ndt \text{ et } dy = T^V Ndt\left(\frac{P+Q}{P-Q}\right)^n.$$

Atque his formulis in amplissimos curvarum exponentialium, quae problemati trajectoriarum reciprocarum satisfaciunt, campos deducimur.

IL. Exemplum nobis sit hypothesis, qua $n = 0$, $N = 1$, $T = a$ et $V = t = x$; erit

$$dy = a^t dt = a^x dx,$$

quae est aequatio ad logarithicam ordinarium, quae satisfaciet applicatam subtangenti aequalem pro axe conversionis assumendo. Pluribus exemplis, quippe curvas ignotas exhibentibus, haec persequi minime consultum duco.

L. Hisce tandem, quae hactenus attuli, quaestioni ex asse me satisfecisse non dubito, quaecunque enim ad enodationem huiusmodi quaestionum iure requiri possunt, abunde hic exhibuisse mihi videor. Dedi enim primo generalissimas aequationes applicatu faciles: secundo methodum dedi infinitas aequationes universales inveniendi, ex quibus simplicissimas reipsa deduxi.

Tandem, quae ex transcendentalibus curvis cognitae sunt, etiam ex aequationibus generalibus facile derivantur. Hisce omnibus praemisi solutionem duorum problematum agnatorum, de Pantogonia infinitorum axium trajectoria et de trajectoriis datum axium numerum habentibus, quippe quae ex consideratione naturae trajectoriarum reciprocarum sponte fluunt.

THE SOLUTION OF THE PROBLEM OF RECIPROCAL TRAJECTORIES.

By Leonhard Euler of Basle.

§1. The problem, which is the basis of the explanatory paper presented here for consideration, is that celebrated question presented in the Acta Eruditorum [1 a], and which was the cause of much agitation. The problem is concerned with finding the curves which lie between given parallel lines, with the conditon that an inverted version of the curve can be placed at any point on the original curve - the motion of a point along the trajectory following the direction of the parallel lines to and fro, with the curves always mutually intersecting each other at the same angle[1 b]. The problem was proposed originally by the late Nicolas Bernoulli in Act. Leipzig, Suppl. Book VII [1 c]. The problem can be satisfied everywhere not only by algebraic but also by transcendental curves. On account of this, a full and complete solution is required, which the method presented here shows, by means of which all the satisfactory curves can be found, including the simplest algebraic ones as well as transcendental curves.
[1 a] *Acta E.* Leipzig (1720), p. 223; [1 b] i. e. for points on the curve with equal positive and negative abscissae; [1 c] see note in E001.
§2. Recently I have given a method (Act. Leipzig. A. 1727), following on the question which the renowned Johan Bernoulli and the anonymous Englishman had considered, regarding the simplest algebraic trajectories, by means of which (for a curve of any order, except the second and third - for which indeed it is possible to find later other satisfactory curves by other means) it is possible to establish the single reciprocal trajectory of the smallest order; together with a general method, by means of which all algebraic reciprocal trajectories for curves with cusps can be found, provided the similar branches around the same cusps are both equal and algebraic. Here we have in mind to give a more general solution to this problem, and to deduce from it endless general algebraic formulas of the greatest diversity, to which I add the solution of certain related problems, concerning the finding of reciprocal trajectories having more than one axis.

§3. The problem to be analysed can be more conveniently expressed thus: *To find a curve* CBD *about the axis* AB *of such a kind that with the two straight lines* MP *and* NQ *drawn, equidistant from and parallel to the axis* AB *on both sides, the sum of the angles PMB + QND is constant everywhere, obviously equal to twice the angle* DBA *that the axis makes with the curve.* Indeed, as regards the [laterally] inverted curve *cbd* to the curve CBD about the axis AB: the line QN applied to the curve [3 a] is moved horizontally across to PM, and then vertically until N coincides with M, and it is necessary that the angle of intersection BM*d* is constant for the inverse curve *cbd* thus put in place[3 b]. Moreover the angles PM*d*, and QND are equal, and it follows that the sum of the angles PMB + QND must be constant. Hence, with the angles associated with one part of the curve increasing, the angles associated with the other part of the curve are required to be decreasing.

[3 a] containing the angle DNQ, and we would now call QN the y-coordinate of the point N;
[3 b] The inverse curve *cbd* is hence a laterally inverted and translated copy of the original curve, with the point N translated to M on the original curve, in order that the sum of the angles at the points M and N is equal to the constant single angle BM*d*.

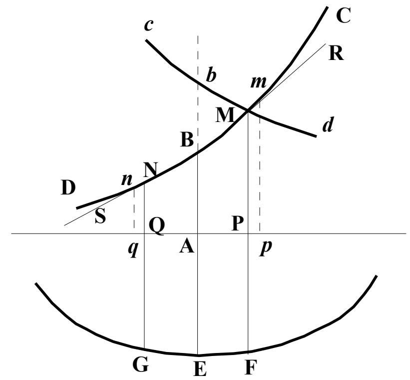

Fig. 1

§4. PQ is drawn normal to the axis AB, and AP = AQ; two nearby corresponding lines *pm* and *qn* are drawn for which P*p* = Q*q*. The tangents MR and NS are drawn from M and N, in order that they have the angles RM*m* and SN*n*, of which the former is a decrement of the angle PMB and the latter is an increment of the angle QND; whereby from the condition of the problem, the angles RM*m* and SN*n* are equal, and from which the nature of the curve CBD is to be investigated.

§5. For any point M taken on the curve, the applied coordinate line MP is produced by a length PF, which is taken to be proportional to the size of the angle RM*m*, for the assumed element P*p* of the abscissa AP, which is considered constant, then the point F will lie on a certain curve, the diameter of which is the axis of the trajectory BA, and indeed PF = QG everywhere. Whereby the whole difficult has been reduced to this: as from the curve FEG the other CBD is given, in which the elements of the angle BMP can be arranged in correspondence with the proportional coordinate lines PF. Thus as the curve CBD results in the reciprocal trajectory, the curve FEG should have a diameter, and have similar and equal branches about the diameter, and the curve FEG is constructed in this manner. The curve MBN, constructed from this curve, is the reciprocal trajectory, the axis of which is EB, the diameter of the other curve.

§ 6. Let AP = x, PM = y, and PF = u. The elemental angle RM*m* is in the ratio $d^2y : (dx^2 + dy^2)$ [6 a]. Hence, on setting $u = \frac{d\varphi}{dx}$:
$u = ddy : (dx^2 + dy^2)$, from which, with dx made constant the given equation for u, x and y can be found. For on putting $dy = pdx$, and $ddy = dp.dx$, then $u = \frac{dp}{dx + ppdx}$ and $udx = \frac{dp}{1+pp}$.
From which p can be found, as u and x have been given, and hence y can be found; but $\frac{dp}{1+pp}$ is twice the element of the sector of the circle of radius 1 and tangent p; hence the area of this sector of the circle is $\frac{1}{2}\int udx$ [6 b].
But truly $\int udx$ = area APEF, p is hence found by quadrature with the constant term either taken or added, and hence again in the same way y follows by quadrature.

[6 a] For, on setting $dy = pdx$, $ddy = dp.dx$, with dx held constant and $\frac{dy}{dx} = \tan\varphi$; then

$$\frac{dp}{dx} = \frac{d(\tan\varphi)}{dx} = \sec^2\varphi.\frac{d\varphi}{dx} = \frac{dx^2+dy^2}{dx^2}.\frac{d\varphi}{dx};$$

hence $\frac{d\varphi}{dx} = \frac{dp}{dx}.\frac{dx^2}{dx^2+dy^2} = \frac{dp.dx}{dx^2+dy^2} = \frac{ddy}{dx^2+dy^2}$; and $d\varphi = \frac{dp}{\sec^2\varphi} = \frac{dp}{1+p^2}$.

[6 b] For the area of a sector is $\frac{1}{2}\int r^2 d\varphi = \frac{1}{2}\int d\varphi = \frac{1}{2}\int udx$, and also $d\varphi = \frac{dp}{\sec^2\varphi} = \frac{dp}{1+p^2}$ from above.

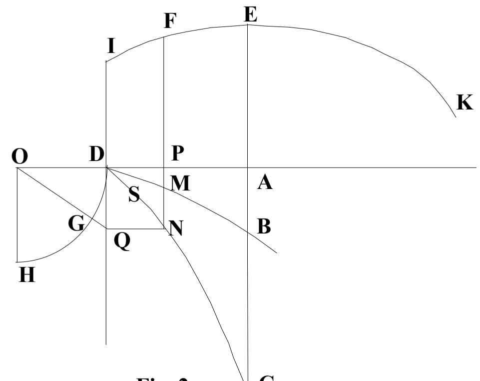

Fig. 2

§ 7. Let IEK be some given curve with the diameter EA as above [note the vertical inversion, which does not affect the argument]; some point O is taken on the line AO, cutting the diameter EA normally. From the point O as centre and with an arbitrary radius OD, the circle DGH is drawn, and from D the tangent DQ is drawn. Some line applied to the curve PF is drawn, with the area PFID taken equal to the sector DOG, and OG is produced to Q. From Q, QN is drawn parallel to DA, meeting the line FP produced in N. There is a point N in the curve DN, such that PN = p, if AP = x. Here the factor of the half, which has been found in the above paragraph, is neglected, since ddy: $(dx^2 + dy^2)$ is at least proportional to u, and so also $\int udx$ can be assumed to be proportional to the sector DOG. Hence nothing between taking the area of the sector or the sum, and hence $\int udx$ can be found either from the coordinate DI or from AF.

§ 8. With the easy task of finding the curve DN, the curve for the reciprocal trajectory DM can be obtained. Indeed since $dy = pdx,$ and PM is taken everywhere to be proportional to the area DPN, then the point M lies on the reciprocal trajectory of which AB is the axis, taken to be the diameter of the curve IEK. It is likewise apparent, from the one assumed curve IEK, that an infinite number of reciprocal trajectories can be found, as indeed the transversal DA can be drawn otherwise, and the points O and D taken otherwise, thus resulting in other reciprocal trajectories. Also of course different reciprocal trajectories are formed by different ratios, with different areas DPN corresponding to PM. From which it is apparent that given one reciprocal trajectory, with the coordinate lines [such as PM] increased or diminished in the same ratio, an infinite number of other reciprocal trajectories can be found.

§ 9. If the area DPFI is taken equal to the quadrant ODH, and the tangent DQ itself equal to the line PN tends to infinity then the line PN is an asymptote to the curve DN. But if the area DPFI is greater than the area of the quadrant, then the line PN is negative. Moreover the y coordinate PM to the trajectory DM does not intersect the curve, PN is infinite everywhere, a tangent to the curve. Hence with PN moving off in the negative direction, the line PM decreases, whereby the curve DM has a point in the reversed curve.

If the area DPN present with the asymptote PN is infinite, the coordinate PM is also infinite, and also is an asymptote to the curve DM.

§ 10. For the sake of an example, let $u = \frac{b}{xx + aa}$ and the equation between x and y is to be established; since u shall be in the ratio ddy: $(dx^2 + dy^2)$, $bdx^2 + bdy^2 = aaddy + xxddy$, and setting $dy = pdx$, then $ddy = dpdx$, from which with the values substituted the equation $bdx + bppdx = aadp + xxdp$ is obtained.

Hence: $$\frac{bdx}{aa + xx} = \frac{dp}{1 + pp},$$

for this equation, and the integration of the other curve depends on the quadrature of the circle.

But here the equation is reduced to this :

$$\frac{b}{a}\left(\frac{dx}{a + x\sqrt{-1}} + \frac{dx}{a - \sqrt{-1}}\right) = \frac{dp}{1 + p\sqrt{-1}} + \frac{dp}{1 - p\sqrt{-1}}.$$

Which on integration results in:

$$b\ln(a + x\sqrt{-1}) - b\ln(a - x\sqrt{-1}) = a\ln\left(1 + p\sqrt{-1}\right) - a\ln\left(1 + p\sqrt{-1}\right) + a\ln h,$$

hence

$$\left(\frac{a + x\sqrt{-1}}{a - x\sqrt{-1}}\right)^{\frac{b}{a}} = \frac{h + hp\sqrt{-1}}{1 - p\sqrt{-1}} = \frac{hdx + hdy\sqrt{-1}}{dx - dy\sqrt{-1}}.$$

Let $b = a$ and $h = \sqrt{-1}$, then the above reduces to :

$$\left(\frac{a + x\sqrt{-1}}{a - x\sqrt{-1}}\right) = \frac{dx\sqrt{-1} - dy}{dx - dy\sqrt{-1}};$$

which on reduction gives:

$$dy = \frac{x - a}{x + a}dx.$$

If $b = 2a$ with $h = \sqrt{-1}$ remaining, then the equation becomes :

$$dy = \frac{2ax + xx - aa}{2ax - xx + aa}dx.$$

And so on; but I do not wish to delay matters with examples of this kind, as more are considered below.

§ 11. When the diameter EA of the original [*i. e.* mother] curve IEK is the axis of the trajectory thus produced, it is clear that if the original curve has more than one diameter, then the arising trajectory thus also has more than one axis. Therefore if in place of the curve IEK already considered a curve with an infinite number of diameters is used, then the trajectory also

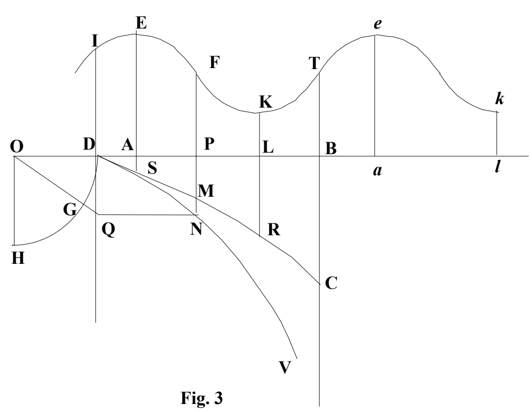

Fig. 3

has an infinite number of axes. But when a trajectory is required which has a fixed number of axes then this has to be explained in some other way. As indeed the whole curve with more than one axis [*i. e.* diameter] is by necessity unbounded, and thus also the reciprocal trajectory which has more than one axis, by necessity are unbounded. Also since a trajectory with unbounded axes must have turning points at infinity, then for a given number of axis, for the part of such a curve between two nearby points of inflection, the curve is understood to have returned. Indeed a [generating] curve is required free from all irregularities except for turning points and points of inflection.

§ 12. Let IEK*ek* be a curve (Fig. 3) with unbounded diameters EA, KL, *ea, kl*, as considered above, and the area DBTI can be selected equal to the area of the quadrant ODH, the line TB produced is a tangent to the curve DNV and touches the trajectory in C, where the point of inflection is present. Hence DMC is part of such a trajectory, concerning which the question is the number of given axes. This portion truly has as many axis, as truly there are axes in the area DBTI. Whereupon by our choice of position the number of axes is defined in this way : For the proposed number of axes which are part of the space DBTI is understood to be the same as the number of diameters, then the circle ODG is divided in such a way that the quadrant ODH of this is equal to the abscissa of the space DBTI, in this way it is shown how the desired curve can be generated.

§ 13. If in place of the curve IEKek a straight line parallel to DB is taken, any vertical line to the curve FP can be the diameter, and hence any other line can be the axis to the trajectory DMC. This is the curve that the renowned Bernoulli started to publish in the pages of the Acta Eruditorum in1726 under the name pantogonia. [*Continuatio materiae de trajectoriis reciprocis, imprimis de Pantogonia*, opera omnia Tomus II, p. 600 - 616.] An equation of this kind can be expressed as :
$1 = addy : (dx^2 + dy^2)$ or $dx^2 + dy^2 = addy$, this has the property that a ray following the line of the axis for incidence, all the reflected rays are equal to each other. But this curve is thus more easily constructed, as $x = \int \frac{aadp}{aa + pp}$ and $y = \int \frac{apdp}{aa + pp}$ may be taken.

§ 14. This method by means of which reciprocal trajectories are to be found by double integration is not finished by this example, as this is a method for the production of reciprocal trajectories that can hardly be surpassed, if simple or algebraic trajectories are desired. But in order that I may further arrive at the solution of the problem concerning the finding trajectories with more than one axis present, that the anonymous English mathematician proposed to the renowned Jonan Bernoulli, then I must move on now to consider that other extremely fertile method for showing simpler and especially algebraic trajectories. But here I will consider the case of the problem in which the angle of intersection is made right, as all the other cases can be reduced to this case.

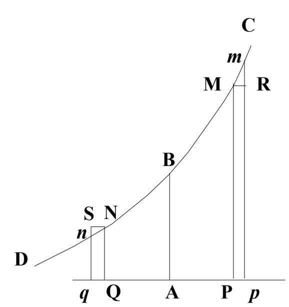

Fig. 4

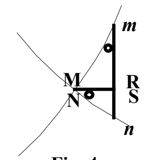

Fig. 4a

§ 15. Let (Fig. 4) CBD be an orthogonal trajectory, the axis of which is AB, cutting the line PQ at right angles. Two vertical lines PM and QN are drawn attached to the curve, parallel to the axis AB at equal distances on either side; and for these the nearby lines *pm* and *qn* are drawn, also the bases MR and NS are parallel to PQ. The triangles MR*m* and NS*n* are similar, on account of S*n*N + R*m*M = one right angle. Let AP = *x*, PM = *y*, P*p* = MR = *dx*, R*m* = *dy* , and also AQ = - *x*, Qq = NS = - *dx*; let QN = *z* or S*n* = *dz*. From the similar triangles MR*m* and NS*n* it may be deduced that

$$MR(dx):mR(dy)=Sn(dz):SN(-dx),$$

and thus $dydz = - dx^2$.
From which equation *y* must be found; and indeed *z* follows from *y* , since in the expression for *y*, with - *x* in the place of *x, z*.

§ 16. Place $dy = pdx$, since *p* is a function of x [at M]. This can be changed [to the other point N] with - *x* put in place of *x*, to give $dz = - qdx$ in terms of the *q*, and consequently $pq = 1$. It is thus apparent that in place of *p* such a function of *x* should be taken, that the product of the same by a factor with - *x* in place of *x* equals one. Therefore all of the skill of finding the solution returns to this, in order that suitable functions of *x* can be chosen with *x* to be substituted for *p*. But to this end, unless we wish to commit the discovery of these to chance, the function is required to be known more accurately. It is seen that these can be determined conveniently by the following method, of which I now present the first elements, as it were.

§ 17. In the first place functions are to be noted that I call even, of which this is the property: that they remain unchanged if - *x* is put in place of *x*. All the powers of *x* with even exponents are of this kind, and of fractions with even numerators and odd denominators. Then, any such functions of powers of this kind composed either by addition or subtraction, or by multiplication or division, or hence any such construction raised to sum are likewise even powers, such as

$$x^{\frac{4}{5}}, \quad (ax^2+bx^{\frac{2}{3}})^n.$$

§ 18. In the second place, I take heed of odd functions, which in short produce the negative of these, if *x* is changed to - *x* . All the powers of *x* with odd exponents, such as *x* itself, x^3, x^5, etc. are of this kind; or fractions of which the numerators and denominators are odd; also functions composed of these powers either added or subtracted, and which are also dignified to be raised to odd exponents, such as

$$x^3, \quad (ax^3+bx^{\frac{5}{7}})^3.$$

§19. If an odd function is multiplied by an odd function, the product is always an even function, such as x^3 by $x^{\frac{1}{3}}$ gives $x^{\frac{10}{3}}$. While an even function multiplied by an odd function indeed always produces an odd function, nevertheless sometimes such a function can have an even part at the same time, as $x\sqrt{(aa+xx)}$ is an odd function and it is obviously the same as

$$\sqrt{(aaxx+x^4)}\,,$$

which is even. But since [in this last example] we are concerned with raising an even function to some power, as we have discussed above, the power which shall be suitable too, (if the exponent is a fraction with an even denominator such as $\frac{1}{2}$), has to be restricted, for if the root of the expression itself can be extracted, as with

$$(\tfrac{aa}{xx}+2a+xx)^{\frac{1}{2}}$$

then this cannot be an even function, since it must agree indeed with $\frac{a}{x}+x$. Moreover, judgement can readily be extended to functions of this kind .

§20. In addition reciprocal functions are worth looking at, which according to me are functions with -*x* in place of *x*, which change in such a way that the product of these gives rise to unity, as :

$$\left(\frac{a+x}{a-x}\right)^n,$$

which, with the negative of *x* put in place, changes to:

$$\left(\frac{a-x}{a+x}\right)^n,$$

of which the product is equal to 1. Also, we should refer here to exponential functions such as $a^x, (aa+xx)^{x^3}$, etc., all of which are even functions raised to odd powers [which are themselves] odd function.

§21. From these functions so set out, it is clear that *p* is a reciprocal function of *x* when *pq = 1*. Moreover, as reciprocal functions of this kind are required to be found, I will try to show this shortly. But initially, nothing concerning exponential functions is to be put in place, since all the algebraic reciprocal trajectories that come to mind are put in place first, and later I can join on exponential functions.

§22. Since I am solving a more general problem, I assume a third variable *t* and I will investigate how *x* and *y* should be determined in terms of *t*, in order that reciprocal trajectories may result, thus I put $dx = rdt$ and $dy = pdt$. This has the effect that if *t* is made negative, then *dx* becomes negative. Whereby in place of *r* it is necessary to put an even function of *t*, which shall be *N*; then $dx = Ndt$, and with changing to negative *t*, $dx = - Ndt$. Consequently on account of $dydz = - dx^2$, for the case - *t* with *p* and *q* put in place as before, $pq = NN$ is given.

§23. Placing $p=(P+Q)^n$ with P denoting an even function and Q an odd function of *t*, then

$$q=(P-Q)^n \text{ and thus}$$
$$(PP-QQ)^n = N^2;$$

hence

$$PP=N^{\frac{2}{N}}+QQ \text{ and } P=\sqrt{N^{\frac{2}{N}}+QQ};$$

and hence

$$p = (Q + \sqrt{(N^{\frac{2}{N}} + QQ)}\ )^n .$$

Nothing of an objectionable kind is hidden here in the equation : $P = \sqrt{N^{\frac{2}{N}} + QQ}$, and indeed P ought to be an even function, such as the equation shows also. But if Q is to be elicited, it can be shown that $Q = \sqrt{N^{\frac{2}{N}} - PP}$, and since that involves a contradiction ; for Q, which denotes an odd function, is here equal to an even function. Thus in order that I may avoid fractions in the exponents, I write N^n in place of *N*, and

$$dx = N^n dt \text{ et } dy = dt(Q + \sqrt{(N^2 + Q^2)}\ )^n ;$$

here it is possible to write NQ in place of Q (§19) and then in place of N^n , as before *N*; giving $dx = Ndt$ et $dy = Ndt(Q + \sqrt{(1 + Q^2)}\ )^n$.

§ 24. As new formulae arise, I can remove irrationality, by placing

$$\sqrt{(N^2 + Q^2)} = N + RQ ,$$

and hence

$$Q = \frac{2NR}{1 - RR} ,$$

thus it is necessary that R is an odd function of *t*, on account of Q being odd, hence

$$Q + \sqrt{(N^2 + Q^2)} = \frac{N(1 + R)}{1 - R} = (\text{I write } \frac{\text{Q}}{\text{P}} \text{ in place of R,}) \ \frac{N.(P + Q)}{P - Q} .$$

P is always to be denotated an even function of *t*, and Q likewise always an odd function of t. Hence [in §23] :

$$dx = N^n dt \text{ and } dy = dt\left(\frac{NP + NQ}{P - Q}\right)^n ;$$

the other formula gives on being handled in the same way :

$$dx = Ndt \text{ et } dy = Ndt\left(\frac{P + Q}{P - Q}\right)^n .$$

Endless general formulae of this kind can now be found, by taking other equations in place of $p = (P + Q)^n$: the following formula is of this kind

$$dx = Ndt \text{ and } dy = Ndt(P + Q)^{mn} . \left(S + \sqrt{\left(SS + (PP - QQ)^{-\frac{m}{k}}\right)}\right)^{nk} ,$$

with S specifying an odd function ; but I shall be content with the two formulas found as they are the are the simplest and the most fruitful in producing algebraic trajectories, of which one is associated with irrational and the other with rational funcions.

§ 25. I consider developing the case of the first formula, for which *dy* is integrable.
Let N = 1 first, and *dx* = *dt*, from which

$$dy = dx(Q + \sqrt{(1 + QQ)}\ )^n .$$

Again, I put Q = *x*, then

$$dy = dx(x + \sqrt{(1 + xx)}\ )^n ,$$

and I note that the integration can generally be performed, for putting $x+\sqrt{(1+xx)}=u$, then

$$x=\frac{uu-1}{2u}\text{, consequently } dy=\frac{u^n du}{2}+\frac{u^{n-2}du}{2}$$

and hence

$$2y=\frac{u^{n+1}}{n+1}+\frac{u^{n-1}}{n-1}=\frac{\left(x+\sqrt{(1+xx)}\right)^{n+1}}{n+1}+\frac{\left(x+\sqrt{(1+xx)}\right)^{n-1}}{n-1}$$

is obtained, thus this general algebraic equation can furnish any number of curves, by substituting rational numbers in place of *n*.

§ 26. However, before I go on to the derivation of equations determined from general principles, certain formulae can be deduced from the equation of the differentials, which can be extracted by more difficult integration. Clearly in the first place, if $n = 0$, then the trajectory is a straight line, making an angle of half a right angle with the axis, on account of $dy = dx$. Then, if $n = 1$,

$$dy=xdx+dx\sqrt{(1+xx)}\ ;$$

from which it is apparent that this equation is a reciprocal trajectory, which can be integrated with the help of the Bernoulli method for the rectification of the parabola, but except for this single case is not completely integrable.

§ 27. In the third case, if - *n* is put in place of *n*, then the equation nevertheless will be for the same curve, with the abscissus taken from the other part of the curve or arising from the negative axis. For these two expressions can be brought together:

$$(-x+\sqrt{(1+xx)}\ )^n \text{ and } (x+\sqrt{(1+xx)}\ )^{-n},$$

in order that they can easily be examined: there is nothing to be gained by substituting negative values in place of *n*. Whereupon I will only consider the substitution of such positive numbers, since it is sufficient to use these alone for the deduction of the general equation.

§ 28. Now that the cases where $n = 0$ and $n = 1$ have been discarded, I progress further and call upon the integrated equation in use, and set $n = 2$. It will be

$$3y=2x^3+3x+\left(2+2xx\right)\sqrt{(1+xx)}\ ,$$

which is here reduced to rationality

$$12yx^3+18xy-9yy+3xx+4=0\,.$$

And this equation of the fourth dimension is without doubt the most simple, after the parabola of the third order: which thus answered the question that the anonymous Englishman proposed to the most celebrated JOHAN BERNOULLI, (and which I have repeated in the Acta Eruditorum 1727) concerning the discovery of that third order algebraic trajectory, thus removing the next order of simplicity.

§ 29. If *n* is put equal to 3, then this equation for the line of order 5 is produced

$$128yx^4+192xy^2+48y-64yy-8xx-9=0\,.$$

If $n = 4$, an equation of order 6 results. and thus by the law of induction it can be inferred that the general equation can always be reduced to a rational equation of order $n + 2$. It can also be used to obtain equations with fractional values of n. If $n = \frac{1}{2}$, then the equation is of order $\frac{5}{2}$. But which, as the order may be irrational, will be reduced to order 5, and generally if $n = \frac{p}{q}$ the reduced equation will go as high as order $p + 2q$.

§ 30. It is hence apparent that by substituting one value after another in place of n in the general equation, that any order of curves, if the second and third orders are excepted, exhibit the one trajectory of minimum order. Also, from the given order of the curves, so many trajectories can be found with the help of this equation, and it will be easy to determine, surely by inspection, how many times a number of the given order $p + 2q$ can be produced; but in place of p and q even positive whole numbers can be substituted and in the same way as as above, as the ratio $p{:}q$ cannot be reduced to small terms. But I have set out this general formula in the Liepzig Acta Eruditorum of 1727, and so here I can turn to consider another matter.

§ 31. Until now I have adhered to the equation of §25, reduced to this

$$dy = dx\left(Q + \sqrt{(QQ+1)}\right)^n.$$

Concerning which I have observed that generally an integrable function is not returned by the substitution of rational powers, such as x^3, x^5, etc. , and also x^{-1}, x^{-2}, etc., in place of the factor Q, though certainly here and there integrable cases may be found, but to pursue these is considered of little importance. But with substituting irrational values of x in place of Q , and moreover where the numerator of the exponent is one, an integrable formula is always observed to be returned.

§ 32. Thus, let $Q = x^{\frac{1}{3}}$, and

$$dy = dx\left(x^{\frac{1}{3}} + \sqrt{\left(x^{\frac{2}{3}}+1\right)}\right)^n.$$

In order to integrate, I put

$$x^{\frac{1}{3}} + \sqrt{\left(x^{\frac{2}{3}}+1\right)} = t,$$

and it becomes

$$x^{\frac{1}{3}} = \frac{tt-1}{2t},$$

then

$$x = \frac{t^3}{8} - \frac{3t}{8} + \frac{3}{8t} - \frac{3}{8t^2},$$

hence

$$dx = \frac{3tt}{8} - \frac{3}{8} - \frac{3}{8tt} + \frac{3}{8t^4},$$

therefore

$$dy = \tfrac{3t^{n+2}dt}{8} - \tfrac{3t^{n}dt}{8} - \tfrac{3t^{n-2}dt}{8} + \tfrac{3t^{n-4}dt}{8}.$$

Consequently

$$\frac{8y}{3} = \frac{t^{n+3}}{n+3} - \frac{t^{n+1}}{n+1} - \frac{t^{n-1}}{n-1} + \frac{t^{n-3}}{n-3}$$

$$= \frac{\left(x^{\frac{1}{3}} + \sqrt{(x^{\frac{2}{3}}+1)}\right)^{n+3}}{n+3} - \frac{\left(x^{\frac{1}{3}} + \sqrt{(x^{\frac{2}{3}}+1)}\right)^{n+1}}{n+1} - \frac{\left(x^{\frac{1}{3}} + \sqrt{(x^{\frac{2}{3}}+1)}\right)^{n-1}}{n-1} + \frac{\left(x^{\frac{1}{3}} + \sqrt{(x^{\frac{2}{3}}+1)}\right)^{n-3}}{n-3}.$$

§ 33. But there are certain cased upon which the integration depends on logarithms, surely if n = 1 or 3. All the other substitutions made in place of n give rise to algebraic curves, and according to the above rule. Concerning which with the former formula it has been shown that negative values of n are superfluous, and the same also applies here, and this also applies for the most general case; and as always it is found that if n = 0 then the trajectory degenerates into a straight line.

§ 34. The simplest case of this equation without doubt will be that for which n = 2. Hence with t put in place of $x^{\frac{1}{3}}$ for short, it is found that

$$5y = 6t^5 + 5t^3 + (6t^4 + 2tt - 4)\sqrt{(1+tt)}.$$

Consequently on reducing to rationality, this equation is obained :

$$60t^5y + 50t^3y - 25y^2 - 60t^4 - 45t^6 + 16 = 0.$$

Finally, upon substituting $x^{\frac{1}{3}}$ in place of t , the above is changed into an equation of the 8th order. If n = 4, then I can easily see that an equation of 10 dimensions arises, and a general equation of the lines is returned of order n + 6 . Thus the trajectory for any order of curves can be shown from this equation, as long as we do not include 2, 3, 4, 5, 6, 7, and 9.

§ 35. This same formula, as with the rest, which can be deduced by substitution in the equations found in §31, can also be derived in another way from another form of the general equation. Since from above several general algebraic formulae are obtained, thus I will consider here only a few examples by a different way of investigating. This general equation is

$$dx = Ndt, dy = Ndt\left(Q + \sqrt{(QQ+1)}\right)^n.$$

In which if Q is set equal to x and successively N = 1 or xx or x^4, etc. , or also $a + bxx, axx + bx^4$, and in the same way composite functions of x itself can be nominated, the general equation is always integrable and with a great abundance of algebraic equations.

§ 36. I show the method by which general algebraic equations can be found, other cases are to be examined, from which a certain non integrable equation can be deduced, yet with nothing less than endless easy ways for the determination of algebraic equations it is able to show. I assume this formula

$$dx = N^n dt, \text{ and } dy = dt\left(Q + \sqrt{(QQ + NN)}\right)^n,$$

N is made equal to tt and Q = 1, then

$$dx = t^{2n} dt,\ \ \mathrm{x} = \tfrac{t^{2n+1}}{2n+1} \text{ and } dy = dt\left(t + t\sqrt{(1+tt)}\right)^n.$$

Thus it is apparent that this equation is always integrable, if n is an odd whole number; that is easy to see , if the equation itself is integrated.

§ 37. But there are different cases above, by which our formula of integration is returned, which I show thus : putting

$$t + t\sqrt{(1+tt)} = utt,$$

consequently

$$dy = -\frac{2^{2n+1}u^{3n}du(uu+1)}{(uu-1)^{2n+2}}.$$

Thus it is apparent that if *n* is an even negative number then the equation is integrable, as that will be apparent, if $(uu - 1)^{-2n-2}$ is itself raised to a product [Note: in which case the integral becomes a sum of powers of u]. More cases are forthcoming if

$$t + t\sqrt{(1+tt)} = ut;$$

and

$$dy = du(u^{\frac{3n+1}{2}} - u^{\frac{3n-1}{2}})(u-2)^{\frac{n-1}{2}},$$

is obtained which is integrable in the first place, if n is any odd number, and then if $\frac{3n+1}{2}$ is an integer. Hence set $\frac{3n+1}{2} = m$; and $n = \frac{2m-1}{3}$ and thus some fraction can be put in place of *n*, the denominator of which is 3 and the numerator can be some odd number.

§ 38. It should suffice to give a single example; let *n* = 1, then

$$dx = ttdt, \text{ and } t = \sqrt[3]{3x};$$

hence

$$dy = tdt + tdt\sqrt{(1+tt)};$$

therefore

$$y = \tfrac{tt}{2} + \tfrac{(1+tt)\sqrt{(1+tt)}}{3},$$

and thus an equation of order six is established on rationalising :

$$(12yy - 12xx)^3 = 3456y^5 + 12528x^2y^3 - 432yx^4 - 2304y^4 - 288x^2y^2 + 81x^4 + 512y^3.$$

Thus an infinite number of algebraic equations can be generated from this equation :

$$dy = dt(t + t\sqrt{(1+tt)})^n,$$

and in a like manner for other forms in place of N or Q; differenent values are to be substituted, and cases which this touches on are found to be the same as the above for this method.

§ 39. Concerning the former of the two general irrational formulae that have hitherto been propounded, its use and richness have aready been pointed out above. Now I proceed to another rational formula, which is :

$$dx = N^n dt \text{ and } dy = dt\left(\frac{N.(P+Q)}{P-Q}\right)^n$$

or, which gives the same,

$$dy = dt\left(\frac{N.(1+Q)}{1-Q}\right)^n.$$

I will not delay here for the derivations of the transcendental curves from this formula, surely the semirectangular logarithmic curve if $Q = t$ and $N = 1$, and $n = 1$; or the cycloid, if $n = \frac{1}{2}$; obviously curves which have been considered and enlarged upon by others. My intention is, as before, to determine which algebraic curves can be accomplished from the following considerations, and the rules that arise by which algebraic curves can be found.

§ 40. But to prevent a fraction being the cause of a more difficult case of an algebraic equation to be found, I acknowledge this by putting $N(1-QQ)$ in place of N. Indeed, N should be an even function of t. We thus have

$$dx = dt(N - NQQ)^n, \text{ and } dy = dt\left(N(1+Q)^2\right)^n$$

or

$$dx = N^n dt(1-QQ)^n, \text{ and } dy = N^n dt(1+Q)^{2n}.$$

In order that an algebraic equation can be derived from this equation, it is necessary that both dx and dy are made integrable. Put $N = 1$; then

$$dx = dt(1-QQ)^n, \text{ and } dy = dt(1+Q)^{2n};$$

if $Q = t$, then

$$dx = dt(1-tt)^n, \text{ and } dy = dt(1+t)^{2n},$$

therefore

$$y = \frac{(1+t)^{2n+1}}{2n+1};$$

and hence

$$\sqrt[2n+1]{(2n+1)y} - 1 = t.$$

Therefore, in order that dx can become integrable, it is necessary that a whole positive number should be substituted in place of n.

§ 41. Since n is a whole positive number, set up $(1-tt)^n$, if this is changed into a series, then this progression of a finite number of terms results:

$$1 - \frac{n}{1}tt + \frac{n(n-1)}{1.2}t^4 - \frac{n(n-1)(n-2)}{1.2.3}t^6 + etc.$$

Thus the value of x can be found:

$$x = t - \frac{n}{1}\frac{t^3}{3} + \frac{n(n-1)}{1.2}\frac{t^5}{5} - \frac{n(n-1)(n-2)}{1.2.3}\frac{t^7}{7} + etc.,$$

in which if in place of t the value found is substituted:

$$\sqrt[2n+1]{(2n+1)y}-1$$

then an equation is obtained between x and y and thus for the curve sought.

§ 42. Let $n = 1$, then the equation becomes :

$x = t - \frac{t^3}{3} = \sqrt[3]{3y} - 1 - \frac{1}{3}(\sqrt[3]{3y} - 1)^3 = \sqrt[3]{9}y^2 - y - \frac{2}{3}$, hence $(x + y + \frac{2}{3})^3 = 9yy$.

For which an equation of the third order arose and the semirectangular cubical parabola is expressed, which is the equation for the simplest of all the algebraic reciprocal trajectories. Concerning which the most renowned Johanen Bernoulli gave in an extraordinary paper in the Actis Eruditorum 1725 [July, pp.318-325]. But the remaining substitutions in place of n are less lucky in showing simple curves, indeed for n placed equal to 2, an equation with more than 30 terms arises.

§ 43. Other functions of t can be substituted in place of Q , such as t^3, t^5, or $t^{\frac{1}{3}}$ etc., which all return a formula integrable an indefinite number of times, always without doubt, when n is a positive whole number. Moreover, the same sort of thing happens if other values are proposed in place of N. For if $N = tt$, then the equations are :

$$dx = t^{2n}dt(1 - QQ)^n, \text{and } dy = t^{2n}dt(1 + Q)^{2n}.$$

Placing $Q = t$, then

$$dx = t^{2n}dt(1 - tt)^n, \text{and } dy = t^{2n}dt(1 + t)^{2n}.$$

Thus it is apparent that both x and y can have the whole number $2n$ only. If the number is even, it is easy to show that all can be resolved in simple terms; but if $2n$ is an odd number, then $(1 - tt)^n$ is irrational, but if $2n$ is allowed to be an odd number, then the function $t^{2n}dt(1 - tt)^n$ nevertheless is integrable.

§ 44. Let $2n = 1$, then

$$dx = tdt\sqrt{(1 - tt)}, \text{and } dy = tdt + ttdt.$$

Whereby

$$x = -\tfrac{1}{3}(1 - tt)^{\frac{3}{2}} \text{ and } y = \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3.$$

Therefore

$$t = \sqrt{(1 - \sqrt[3]{9}xx)};$$

and putting $y + \frac{1}{6} = u$, and by reduction carried through to the end, this sixth power equation is obtained :

$$(12(u^2 + x^2) - 8u)^3 + 3^5.4^3x^2u^3 - 81x^4 + 3^4.4(12(u^2 + x^2) - 8u)x^2u = 0$$

[corrected from original]

Other numbers substituted in place of n show other algebraic curves, thereupon innumerable other numbers can be substituted in place of Q and in place of N, which always, if n is itself a positive whole number, bring about algebraic equations. And these are the particulars which can be reported about algebraic curves.

§ 45. In the place of these curves, I can introduce other general formulas that arise, if exponential functions are selected in place of the functions p and q (see §22). But with exponential functions there are even and reciprocal functions, as P^R is an even function if both P and R are even functions; but if R is an odd function, then it will be the reciprocal funcition with that, as in the former case by changing x to $-x$, P^R remains, while in the latter it is changed to P^{-R}.

§ 46. Therefore for such situations , before it is worthwhile to substitute even functions in place of [other] even functions, exponential functions of this kind can be applied, and in the place of an odd function like exponentials are multiplied by a certain odd function, such as P^RQ, for existing even functions P and R, and for odd Q.

§ 47. When reciprocal functions thus are to be composed, as the product of any of these by itself, but with $-x$ put in place of x, is equal to one, then it is apparent for any fuction p, that it can always be multipled by the above kind of reciprocal function in such a way, that where $dy = pdt$, it is also to take

$$dy = T^V pdt\,,$$

with T denoting an even function and with V odd. Nevertheless the product of dy by itself, but with $-t$ replacing t, will indeed give the same as before.

§ 48. Thus for the general formulas found in §23 and §24, it is possible to adjoin a reciprocal funcion, with the use of these unchanged. We now have [the augmented form]:

$$dx = Ndt \text{ and } dy = T^V Ndt\left(Q + \sqrt{(1+QQ)}\right)^n,$$

then in place of the rational formula we have :

$$dx = Ndt \text{ and } dy = T^V Ndt\left(\frac{P+Q}{P-Q}\right)^n.$$

And from these formulas involving the most general of exponential curves, which satisfy the problems of reciprocal trajectories, we can deduce anything possible.

§ 49. We take as our example the hypothesis, where $n = 0$, $N = 1$, $T = a$ and $V = t = x$; then

$$dy = a^t dt = a^x dx\,,$$

which is the equation for the ordinary logarithm [i. e. $y = e^x$] . This equaton satisfies the pre-requisites of reciprocal trajectories, as the co-tangent of the y coordinate of a point with its assumed laterally inverted form are equal [i. e. x and $-x$, giving $e^x.\ e^{-x} = 1$ and the product of the slopes at the same point is - 1.]. For many examples, obviously unknown curves are to be shown, and these I have decided not to pursue.

§ 50. Finally regarding these things I have reported on up to this stage, in an investigation every part of which has without doubt afforded me the greatest satisfaction, [we may note that] there are indeed several possible ways of solving such questions, and rightly so, and I consider that this aspect has been made abundantly clear by me. For I have given at first the most general algebraic equations with easy applications; and then in the following method I presented the general transcendal equations that were to be found, from which I deduced the simplest things.

Finally, what was known [before] about transcendantal curves can also be easily derived from the general equations. From these I have presented the solutions of two related problems, the one concerning the Pantogonal trajectories [of Bernoulli] with an infinite number of axes, and the other for trajectories with a given number of axes, which usually follow immediately from a consideration of the nature of the reciprocal trajectories.