Leonhardi Euleri, A. L. M. METHODUS inveniendi Trajectorias reciprocas Algebraicas.

Quum temporis spatium, quod Anonymo illi Angelo ad aperiendam, quam invenire potuerit, trajectoriam algebraicam illam tertia ordinis sequentem, concesseram , elapsam sit, meam huius questionis solutionem, una cum aliis insuper ad algebraicas trajectorias reciprocas pertinentibus inventis cum publico communicare constitui.

Primum istud problema examinare atque de solutione eius meditari animum lubuit, praecipue eam, quam commendavit Celeberrimus Dominus Johannes Bernouillius, Praeceptor meus hisce in rebus atque Patronus summe colendus, selegi viam per rectificationes curvarum; & strenuam, quantum potui, navavi operam in indagatione ejusmodi curvarum algebraicarum, quae rectificari possent, ut exinde aequationes algebraicas pro trajectoriis reciprocis eruerem. Hoc in negotio occupatus, in istam incidi aequationem : $yy + {}^2/_3\, aa = a\sqrt[3]{axx}$,
quae est ad curvam, diametro et vertice in diametrum normali gaudentem, ac insuper rectificabilem. Ut itaque ex ista aequatio algebraica pro trajectoria reciproca inveniri possit, sit (Fig. 1) MBM ista curva, ABP diameter et A initium abscissarum, ut scilicet

$$AB = a\sqrt{\frac{8}{27}},$$

atque est AP = x, PM = y. Sit curva BM = s, et ex M ducatur diametro parallelo MQ, occurrens lineae AQ normali in diametrum in Q, absumaturque in ea MN = BM, erit N in trajectoria quaesita. Sit AQ = $t = y$ et QN = $z = x - s$. Est autem per aequationem

$$x = \frac{(yy + \frac{2}{3}aa)^{\frac{3}{2}}}{aa}$$

ideoque

$$dx = \frac{3ydy\sqrt{yy + \frac{2}{3}aa}}{aa};$$

ergo

$$ds^2 = dx^2 + dy^2 = \frac{9y^4dy^2 + 6aayydy^2 + a^4dy^2}{a^4}$$

ergo

$$ds = \frac{3yydy + aady}{aa}$$

consequenter

$$s = \frac{y^2}{4a} + y;$$

posito loco y, t, invenietur substituendo in ista aequatione $z = x - s$ pro x et s valores inventos

$$z = \frac{(tt + \frac{2}{3}aa)^{\frac{3}{2}}}{aa} - t - \frac{t^2}{4a}$$

Ergo

$$aaz + aat + t^2 = (tt + \tfrac{2}{3}aa)^{\frac{3}{2}}$$

et reducendo ad rationalitatem reperietur ista aequatione pro trajectoria reciproca

$$a^4zz + 2a^4tz + 2aat^3z = \tfrac{1}{3}a^4tt + \tfrac{8}{27}a^6,$$

quae dividendo per *aa* reducitur ad hanc

$$aazz + 2aatz + 2t^3z = \tfrac{1}{3}aatt + \tfrac{8}{27}a^4,$$

seu ponendo $aa = \frac{3}{2}bb$ ad hanc

$$12t^3z + 18bbtz + 9bbzz - 3bbtt - 4b^4 = 0.$$

Quae est aequatio ad quarti ordinis curvam. Ad istam ergo aequationem perventum est ope rectificationis curvae, quae solutio proinde soli fortuna esset adscribenda, nisi comperta mihi fuisset peculiaris methodus inveniendi curvas algebraicas rectificabiles, unde et aequatio illa, quam ad constructionem istius trajectoriae reciprocae selegeram, promanavit.

Postmodum vero problema istud inveniendarum trajectoriarum algebraicarum reciprocarum, ex immediata earum natura discutiens, in istam incidi aequationem, (Fig. 2) appellando

$$AQ, x; QN, y; dy = dx(x + \sqrt{xx+1})^n$$

denotante *n* numero quocunque. Quae aequatio, ut facile patet, est ad trajectoriam reciprocam, ponendo enim in expressione ipsius *dy* pro *x*, - *x*, invenietur

$$dy = -dx(-x + \sqrt{xx+1})^n$$

Quae duae aequationes respective invicem ductae dant $dy^2 = -dx^2$, id quod est de essentia trajectoriarum reciprocarum. Est autem ista aequatio

$$dy = dx\left(x + \sqrt{xx+1}\right)^n$$

generaliter integrabilis, exceptis duobus casibus, ubi *n* est vel 1 vel - 1, quibus in casibus integratio dependet a logarithmis. Antequam vero integretur ista aequatio, monendum est, sive n sit affirmativum sive negativum, aequationem ad eandem esse curvam, una quippe obtinet, cum abscissae AQ cis axem conversionis AB sumuntur, altera, cum illae trans AB accipiuntur.

$$dy = dx\left(x + \sqrt{xx+1}\right)^{-n}$$

cum ista

$$dy = dx\left((-x + \sqrt{xx+1}\right)^n.$$

Ideo vero istud praemoneo, ne quis substituendo pro *n* valores negativos, se problematis solutionem ulterius promovisse existimet. Integretur iam aequatio ista

$$dy = dx(x + \sqrt{xx+1})^{-n},$$

id quod facile fieri poterit ponendo $x + \sqrt{xx+1} = t$, ut habeatur

$$x = \frac{tt-1}{2t} \text{ et } dx = \frac{dt}{2} + \frac{dt}{2tt},$$

ut adeo esset

$$2dy = t^n dt + t^{n-2}dt.$$

Quocirca erit

$$2y = \frac{1}{n+1}t^{n+1} + \frac{1}{n-1}t^{n-1} = \frac{1}{n+1}\left(x+\sqrt{xx+1}\right)^{n+1} + \frac{1}{n-1}\left(x+\sqrt{xx+1}\right)^{n-1}.$$

Quae aequatio generalis pro trajectoriis reciprocis ad casus speciales applicata suppeditat ex quolibet curvarum ordine, exceptis secundo et tertio, ad minimum unam trajectoriam reciprocam. Si enim pro n ponantur 2, reperietur haec quarti ordinis aequatio, conveniens cum iam inventa

$$12yx^3 + 18xy - 9yy + 3xx + 4 = 0.$$

Si substituatur loco n, 3, reperietur ista quinti ordinis trajectoria reciproca

$$128yx^4 + 192yxx + 48y - 64yy - 8xx - 9 = 0.$$

Sit $n = 4$, reperietur ista sexti ordinis

$$720yx^5 + 1200yx^3 + 450yx - 225yy + 15xx + 16 = 0.$$

Sit $n = 5$, reperietur aequatio septimi ordinis et in genere, quicunque numerus loco n substituatur, curva inde generata erit ordinis n + 2, si n sit numerus integer, sin vero sit fractus, n + 2 reducitur ad fractionem in minimis terminis et indicabit numerator fractionis ordinem curvae inde natae. Exempli gratia sit $n = \frac{1}{2}$, invenietur ista ordinis 5, cum sit $\frac{1}{2} + 2 = \frac{5}{2}$,

$$72yyx^3 - 81y^4 + 144x^4 - 216yyx - 96xx + 16 = 0.$$

Hinc itaque colligi potest, quot, beneficio istius aequationis generalis, ex quovis ordine trajectoriae reciprocae inveniri queant. Scilicet ex 4 unam, ex 5 duas, ex sexto unam, ex 7 tres, ex 8 duas, etc. Superest mihi praeterea alia methodus inveniendi trajectorias reciprocas algebraicas, ex qua quidem difficilius aequationes pro illis eruuntur, verum hoc ea se commendare potest, quod sit universalis et omnes, quotquot existunt, curvas satisfacientes algebraicas suppeditare queat. Est illa quidem affinis methodo per rectificationes, a Celeberrimo Domino Ionanne Bernoullio detectae et non difficulter exinde elicitur. Dependet scilicet ea ab inventione curvarum diametro et cuspide in vertice gaudentibus; quas curvas possibiles omnes cuiusvis ordinis cum invenire in promtu sit, omnes trajectoriae reciprocae algebraicae possibiles inde elicientur. Cum enim omnes trajectoriae algebraicae inveniri queant, ex curvis rectificabilibus diametro et vertice in diametrum perpendiculi praeditis. eiusmodi vero curvae omnes sint evolutae totidem curvarum algebraicarum cuspidatarum et diametro instructarum, hinc quoque trajectoriae reciprocae reperientur, hoc nempe modo. Sit (Fig. 3) CAM curva eiusmodi cuspidata. AB eius diameter, ducatur AQ eidem normalis et sumtae pro lubitu abscissae AP ducatur respondens applicata PM, ducatur radius MS circuli osculantis curvam in M, erit punctum S in evoluta AS, eritque MS = curvae AS. Sit itaque radius MS circa centrum S moveatur in SN, ita ut fiat diametro AB parallelus, erit punctum N in trajectoria reciproca, cuius axis conversionis erit linea AB. Sequenti vero modo aequatio pro ea reperietur, data aequatione ad curvam AM. Dicantur AP, x; PM, y; AQ, t, et QN, z. Sit brevitatis gratia $ds = \sqrt{dx^2 + dy^2}$, invenietur instituto calculo, posito elemento dy constante quod sit

$$AQ, t = x + \tfrac{ds^2}{ddx} \quad \text{et} \quad QN, z = y + \tfrac{ds^2.(ds-dx)}{4yddx} \ .$$

unde data relatione y ad x, reperiri poterit aequatio inter t et x, seu coordinatae trajectoriae reciprocae.

Sunt autem aequationes ad curvas eiusmodi cuspidatas ex quovis ordine sequentes.

I. Ex ordine tertio $(a+b).xx = y^3 + cy^2$.

II Ex ordine quarto $ax^4 + (b+cy+eyy).xx = y^4 + fy^3 + gy^2$.

Sunt scilicet in aequationibus generalissimis ad cuiusvis ordinis curvas omittendi sequentes termini. Primo illi termini, in quibus continentur pontentiae ipsius *x* imparium exponentium. Secundo terminus, in quo merae constantes et tertio terminus, in quo *y* unius tantum est dimensionis. Cum autem ex istis aequationibus difficulter ad aequationes pro trajectoriis reciprocis perveniatur, etiamsi eae admodum faciles atque simplices reperiantur, operae pretium esset, si quis ad ulteriorem rei analyticae promotionem, perpendat, quomodo minori negotio aequationes pro trajectoriis reciprocis, determinatis *t* et *z* in *x* et *y*, dataque insuper relatione inter *x* et *z*, ervi possent, quomodo exterminata alterutra x vel y, altera quoque exterminanda sit atque aequatio, quam nonnisi t et z una cum constantibus ingrediantur, inde elici possit.

A METHOD FOR FINDING ALGEBRAIC RECIPROCAL TRAJECTORIES.

Leonhard Euler.

I concede that it is now some time since the discovery by that anonymous Englishman [who was in fact Henry Pemberton, (1694-1771), the editor of the 3rd edition of Newton's *Principia*], of how to solve the problem of the following third order algebraic trajectory. My solution to this question is presented here, together with additional related algebraic reciprocal trajectories that I have found.

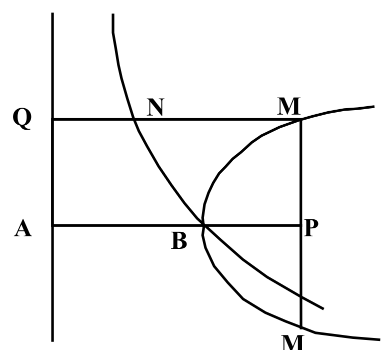

Fig. 1

Initially, I intend to examine a particular problem of this kind, and I am pleased to consider especially the solution pointed out by that most renowned of masters, Johan Bernoulli, who not only was my teacher, greatly fostering my inquiries into such matters, but also looked after me as a patron. I have found a way to solve these problems for the rectification of curves, and I have done this work in the investigating of rectifiable algebraic curves with much zeal and vigour, in order that I might hence find the algebraic equations for the reciprocal trajectories. To become occupied with this business, I start with this equation:

$$yy + {}^2/_3\, aa = a\sqrt[3]{axx},$$

which is the equation of a curve with the vertex B on the diameter along a normal to the curve, and which is rectifiable as above. Thus in order that the reciprocal trajectory can be found for this algebraic equation for the [original] trajectory, as in Fig. 1, let MBM be that curve, with ABP diameter and A the initial point of the abscissa [i. e. the origin], as desired: $AB = a\sqrt{\frac{8}{27}}$, and AP = *x*, PM = *y*. Let the length of the curve be BM = *s*, and from M a parallel diameter MQ is drawn, crossing the line AQ normal to the diameter in Q, and it is assumed that in this figure, MN = BM, and N is a point on the trajectory sought. Let AQ = $t = y$ and QN = $z = x - s$. But from the equation of the curve :

$$x = \frac{(yy + \frac{2}{3}aa)^{\frac{3}{2}}}{aa}$$

and thus on differentiation by x:

$$dx = \frac{3ydy\sqrt{yy + \frac{2}{3}aa}}{aa};$$

hence:

$$ds^2 = dx^2 + dy^2 = \frac{9y^4dy^2 + 6aayydy^2 + a^4dy^2}{a^4};$$

and hence:

$$ds = \frac{3yydy + aady}{aa},$$

and consequently,

$$s = \frac{y^3}{aa} + y;$$

with y put in place, t can be found by substituting into that equation $z = x - s$ for x and s the values found :

$$z = \frac{(tt + \frac{2}{3}aa)^{\frac{3}{2}}}{aa} - t - \frac{t^3}{aa}.$$

Hence,

$$aaz + aat + t^3 = (tt + \tfrac{2}{3}aa)^{\frac{3}{2}},$$

and on reduction [of the powers to whole numbers]:

$$a^4zz + 2a^4tz + 2aat^3z = \tfrac{1}{3}a^4tt + \tfrac{8}{27}a^6,$$

can be found for the equation of the reciprocal trajectory, which on division by aa is reduced to this:

$$aazz + 2aatz + 2t^3z = \tfrac{1}{3}aatt + \tfrac{8}{27}a^4,$$

or by putting $aa = \frac{3}{2}bb$ to this:

$$12t^3z + 18bbtz + 9bbzz - 3bbtt - 4b^4 = 0.$$

Which is an equation of the fourth order for this curve, [where z is the abscissa and t the ordinate]. Hence, this equation has been arrived at by means of the rectification of a curve, and that solution might only be ascribed to good fortune, except that I have found a method for producing individual rectifiable curves, and the equation which I selected resulted in the construction of that reciprocal trajectory.

Presently indeed, setting aside that problem of finding algebraic reciprocal trajectories according to their nature, I have come upon this equation, see Fig. 2, to be called :

$$AQ, x; \quad QN, y; \quad dy = dx\left(x + \sqrt{xx+1}\right)^n,$$

with n denoting some number. This equation can readily be shown to be that for a reciprocal trajectory. For putting - x for x in the expression for dy,

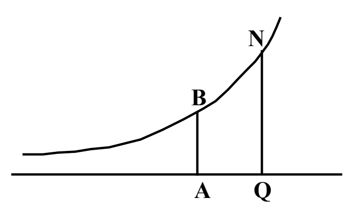

Fig. 2

$$dy = -dx\left(-x + \sqrt{xx+1})\right)^n \text{ is found.}$$

Which two equations respectively multiplied together give : $dy^2 = -dx^2$, which is of the essence of reciprocal trajectories. Moreover that equation

$$dy = dx\left(x + \sqrt{xx+1}\right)^n$$

is generally capable of being integrated directly, except for the cases when n is 1 or - 1. For these cases the integration depends upon the use of logarithms. However, before the equation is integrated, care should be taken regarding the positive and negative values of n in the equation used to describe [the branches of] the same curve, obviously the one given with the abscissa AQ on this side of the axis of rotation AB are taken, the others are taken with that across AB .

$$dy = dx\left(x + \sqrt{xx+1}\right)^{-n}$$

with that

$$dy = dx\left((-x + \sqrt{xx+1}\right)^n .$$

Truly I give this forewarning, that you cannot substitute negative values for n, if it is to be considered that the solution of the problem has moved further forwards. The equation can now be integrated:

$$dy = dx\left(x + \sqrt{xx+1}\right)^n ,$$

which it is easy to perform be putting $x + \sqrt{xx+1} = t$, giving

$$x = \frac{tt-1}{2t} \text{ and } dx = \frac{dt}{2} + \frac{dt}{2tt} .$$

in order that

$$2dy = t^n dt + t^{n-2} dt .$$

Wherefore on integrating,

$$2y = \frac{1}{n+1} t^{n+1} + \frac{1}{n-1} t^{n-1} = \frac{1}{n+1}\left(x + \sqrt{xx+1}\right)^{n+1} + \frac{1}{n-1}\left(x + \sqrt{xx+1}\right)^{n-1} .$$

This is the general equation that can be used to generate the reciprocal trajectory of curves from any order for any case that we may wish to consider, with the second and third orders taken as the smallest cases with a single reciprocal trajectory. If n is put equal to 2, this fourth order equation is found, agreeing with that found:

$$12yx^3 + 18xy - 9yy + 3xx + 4 = 0.$$

If 3 is substituted in place of n, that fifth order reciprocal trajectory is found :

$$128yx^4 + 192yxx + 48y - 64yy - 8xx - 9 = 0.$$

Let $n = 4$, that sixth order curve is found :

$$720yx^5 + 1200yx^3 + 450yx - 225yy + 15xx + 16 = 0.$$

Let $n = 5$, an equation of the seventh order is found, and with curves of this kind, whatever number is substituted in place of n, the curve thus generated is of order $n + 2$, if n is a whole number, but if n is a fraction, $n + 2$ is reduced to a fraction of the smallest terms and the numerator of the fraction indicates the order of the curve thus produced. For example, let $n = \frac{1}{2}$, then a curve of order 5 is found since $\frac{1}{2} + 2 = \frac{5}{2}$,

$$72yyx^3 - 81y^4 + 144x^4 - 216yyx - 96xx + 16 = 0.$$

Hence the number of these reciprocal trajectories for a given order can thus be collected together, with the help of this general equation, from which any order of the reciprocal trajectories can be found. Of course from 4 there is one, from 5 there are 2, from 6 one, from 7 three, from 8 two, etc. It remains for me [to discuss] in addition the other method of finding algebraic reciprocal trajectories, from which indeed more difficult equations are brought to light for these curves, this truth one can work out for oneself. The method is general, and all such curves, however many arise, are able to satisfy the required algebraic conditions. That method is indeed related to the method for rectification found by the renowned Johan Bernoulli and it is not thus difficult to elicit. It depends of course on finding the diameter and the cusp of the curves meeting together at the vertex; when it is evident which orders of all the possible curves are to be found, all the possible algebraic reciprocal trajectories can then be elicited. Since indeed all the algebraic trajectories can be found, from the diameter and vertex of the rectifiable curve, in the diameter of the perpendicular provided. Truly all the curves of this kind are evolutes with the equivalent number of the cusps of the algebraic curves constructed on a diameter; hence likewise the reciprocal trajectories truly can be found by this method.

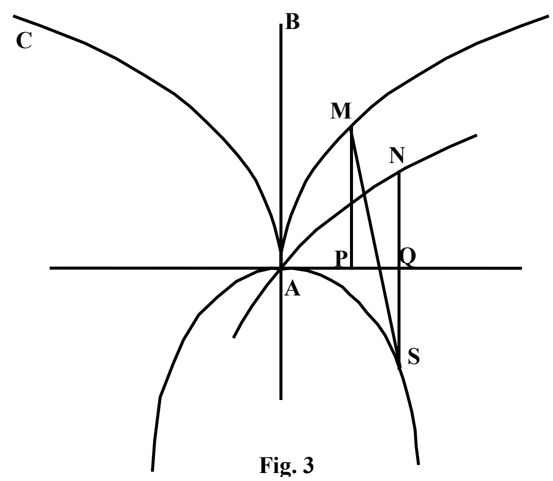

Fig. 3

In Fig. 3, CAM is a curve with a given cusp. AB is the diameter of this curve, AQ is the normal of the same, which is taken for the abscissa, and from AP the corresponding perpendicular PM is drawn in agreement, the radius MS of the osculating circle touching the curve in M is drawn, it is the point S on the evolute AS, and MS is equal to the length of the curve AS. Thus the radius MS is moved around the centre S to SN, hence as it is made parallel to the diameter AB, it is a point N in the reciprocal trajectory, of which the axis of conversion is the line AB. Truly by following in this way the equation for that is found, with the given equation for the curve AM. It may be said that AP is x; PM is y; AQ is t, and QN is z. For the sake of briefness, let $ds=\sqrt{dx^2+dy^2}$, with the calculation set up, with the position of the element y constant, which shall be :

$$AQ, t = x+\frac{ds^2}{ddx} \quad \text{and} \quad QN, z = y+\frac{ds^2.(ds-dx)}{4yddx} .$$

from which from the given relation between y and x, it is possible to find the equation between t and x, or the coordinates of the reciprocal trajectory.

Moreover there are equations for curves with the same kinds of cusps for any of the following orders.

I. From the third order $(a+b).xx = y^3+cy^2$.

II. From the fourth order $ax^4+(b+cy+eyy).xx = y^4+fy^3+gy^2$.

Obviously in the most general equations to curves of any order the following terms should be omitted. In the first case those terms, in which the power of x itself is to some odd power. In the second case terminus, in which there are isolated constants, and in the third case terms in which y if of one dimension. For moreover, from these equations it is difficult to find equations for reciprocal trajectories, even if they can be found very easily and simply, the price of the work may be, if these are advanced to the final analytical stage, it is to be carefully considered, how with less bother equations for reciprocal trajectories can be found by determining t and z in terms of x and y, from the above relation between x and z, how by eliminating either x or y, the other term also is eliminated, and the equation, since not unless t together with z are advancing in step, is it possible for the equation to be found.