CHAPTER VI

CONCERNING THE CONFUSION ARISING FROM THE NATURE OF DIVERSE RAYS

PROBLEM 1

287. *If rays may be transmitted through the lens PP from the given point E* (Fig. 1)*, to define the variation in the position of the image F, which arises from the diverse refrangibility of the rays.*

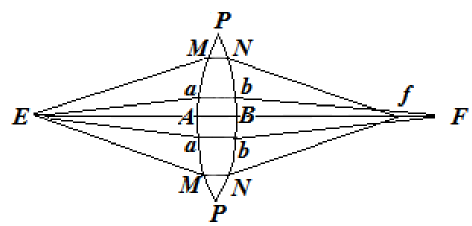

Fig. 1

SOLUTION

Let the distance of the point E before the lens be $AE = a$, moreover the radius of the anterior face of the lens shall be $= f$, and the radius of the posterior face $= g$, and the thickness of the lens $Aa = v$, which quantities are constant. Now with the ratio of the refraction from air into glass put $= n:1$, on account of the diverse nature of the rays the number n will be variable, and thus also the position F of the image expressed past the lens will be, if there may be put $aF = \alpha$, from what has been found above

$$\frac{n-1}{f} = \frac{1}{a} + \frac{2n}{k+v} \text{ and } \frac{n-1}{g} = \frac{1}{\alpha} - \frac{2n}{k-v},$$

where the quantity k also is required to be considered as a variable, because only a, f, g and v are constants. Therefore the question corresponds here, so that if the number n may be supposed to increase by its differential dn, it may be defined in terms of the differential of the distance α. Whereby both equations these equations may be differentiated :

$$\frac{dn}{f} = \frac{2dn}{k+v} - \frac{2ndk}{(k+v)^2}, \quad \frac{dn}{g} = \frac{-d\alpha}{\alpha\alpha} - \frac{2dn}{k-v} + \frac{2ndk}{(k+v)^2},$$

and thus with dk eliminated there will be obtained:

$$\frac{dn(k+v)^2}{f} + \frac{dn(k-v)^2}{g} = 2dn(k+v) - 2dn(k-v) - \frac{d\alpha(k-v)^2}{\alpha\alpha}.$$

The initial values put in place for f and g may be restored, and this equation may be found :

$$\frac{dn(k+v)^2}{a}+\frac{dn(k-v)^2}{\alpha}+4vdn+\frac{(n-1)d\alpha}{\alpha\alpha}(k-v)^2=0,$$

from which there is found :

$$\frac{d\alpha}{\alpha\alpha}=\frac{-dn}{(n-1)a}\left(\frac{k+v}{k-v}\right)^2-\frac{dn}{(n-1)\alpha}-\frac{4vdn}{(n-1)(k-v)^2},$$

or

$$d\alpha=\frac{-\alpha dn}{n-1}\left(1+\frac{\alpha}{a}\left(\frac{k+v}{k-v}\right)^2+\frac{4\alpha v}{(k-v)^2}\right).$$

Then truly, since k also shall be a variable quantity, there will become

$$dk=\frac{-(k-v)dn}{n(n-1)}\left(1+\frac{k+v}{2a}\right).$$

Therefore since we will have put $\frac{k+v}{k-v}=i$, on account of $di=\frac{2vdk}{(k+v)^2}$, there will be

$$di=\frac{-vdn}{n(n-1)}\left(\frac{1}{a}+\frac{2}{k+v}\right).$$

COROLLARY 1

288. If $n:1$ may denote the ratio of refraction of the rays of natural media, so that there shall be $n=\frac{31}{20}=1,55$, there will be for the red rays or for the smallest refraction $n=1,54$, and for the violet colored $n=1,56$, of which value the difference from the mean, since it shall be $=\frac{1}{100}$, will be able to be taken for the differential dn.

COROLLARY 2

289. Whereby if α may denote the distance of the image formed from the mean rays, for that, which is formed from the red, there will be $dn=\frac{1}{100}$ et $\frac{dn}{n-1}=\frac{-1}{55}$. Hence the distance of the red image after the lens will be

$$\alpha+\frac{\alpha}{55}\left(1+\frac{\alpha}{a}\left(\frac{k+v}{k-v}\right)^2+\frac{4\alpha v}{(k-v)^2}\right).$$

Moreover, the distance of the violet image after the lens will be

$$\alpha - \frac{\alpha}{55}\left(1 + \frac{\alpha}{a}\left(\frac{k+v}{k-v}\right)^2 + \frac{4\alpha v}{(k-v)^2}\right).$$

COROLLARY 3

290. If the thickness of the lens may vanish, so that there shall become $v = 0$, on account of the variation of the number *n,* there will be

$$d\alpha = \frac{-\alpha dn}{n-1}\left(1 + \frac{\alpha}{a}\right) = \frac{-\alpha\alpha dn}{n-1}\left(\frac{1}{a} + \frac{1}{\alpha}\right).$$

And if the distance of the focus of the lens may be put $= p$, since there shall be $\frac{1}{a} + \frac{1}{\alpha} = \frac{1}{p}$, there will be

$$d\alpha = \frac{-\alpha\alpha dn}{(n-1)p} = \frac{-20\alpha\alpha dn}{11p}.$$

SCHOLIUM

291. Therefore the value of the distance α is changed in this manner on account of the diverse nature of the rays, in which case just as from the object, the rays are emitted again to a following lens, also the distance of the object becomes the variable with respect to this lens. Whereby for this reason in place of the image from that first lens a two-fold variation of the form will arise : that which henceforth also will eventuate with many more lenses in the following. Therefore this variation, which arises for some lens in place of the image, we will determine in the following problem.

PROBLEM 2

292. *If the place of the image F* (Fig. 5) *which carries on in turn to become the object with regard to the lens QQ, which itself shall be variable on account of the diverse nature of the rays, to determine the variation arising from the same cause, on account of which the following image will be allowed at G.*

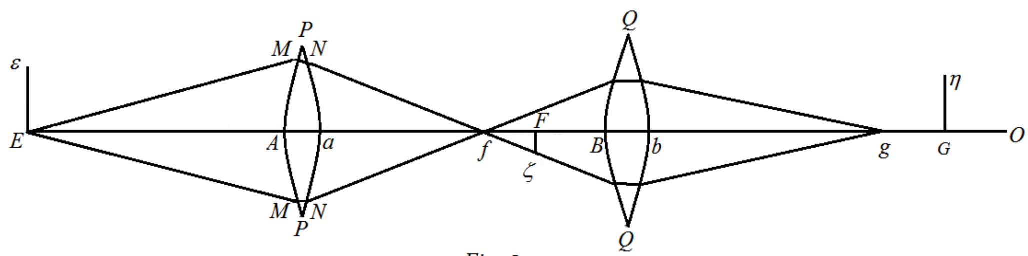

Fig. 5

SOLUTION

For rays of the mean kind, for which the number n corresponds, the distance of the object F before the lens shall be $BF = b$, and thence the distance of the image projected past the lens $bG = \beta$; but while n will be changed into $n + dn$, both these distances b and β may take their differential increments db and $d\beta$. For which requiring to be found, the radius of the anterior face of the lens QQ shall be $= f$, of the posterior $= g$ and the thickness $Bb = v$, and there will be as before :

$$\frac{n-1}{f} = \frac{1}{b} + \frac{2n}{k+v} \text{ and } \frac{n-1}{g} = \frac{1}{\beta} - \frac{2n}{k-v},$$

where k is required to be had as a variable with b and β. Therefore on differentiation there will be had in place

$$\frac{dn}{f} = \frac{-db}{bb} + \frac{2dn}{k+v} - \frac{2ndk}{(k+v)^2}, \quad \frac{dn}{g} = -\frac{d\beta}{\beta\beta} - \frac{2dn}{k-v} + \frac{2ndk}{(k-v)^2},$$

from which by eliminating dk there will become

$$\frac{dn(k+v)^2}{f} + \frac{dn(k-v)^2}{g} = \frac{-db}{bb}(k+v)^2 - \frac{d\beta}{\beta\beta}(k-v)^2 + 4vdn,$$

which multiplied by $n-1$, if the given values for f and g may be substituted, produces

$$\frac{dn(k+v)^2}{b} + 2ndn(k+v) + \frac{dn(k-v)^2}{\beta} - 2ndn(k-v)$$
$$= -\frac{(n-1)db}{bb}(k+v)^2 - \frac{(n-1)d\beta}{\beta\beta}(k-v)^2 + 4(n-1)vdn$$

or

$$\frac{dn(k+v)^2}{(n-1)b}+\frac{dn(k-v)^2}{(n-1)\beta}+\frac{4vdn}{n-1}+\frac{db}{bb}(k+v)^2+\frac{d\beta}{\beta\beta}(k-v)^2=0.$$

And hence there is deduced:

$$d\beta=\frac{-\beta\beta db}{bb}\left(\frac{k+v}{k-v}\right)^2-\frac{\beta dn}{(n-1)}\left(1+\frac{\beta}{b}\left(\frac{k+v}{k-v}\right)^2+\frac{4\beta v}{(k-v)^2}\right).$$

But for the variation of *k* there will be found:

$$\frac{dk}{(k+v)^2}=-\frac{db}{2nbb}-\frac{dn}{2n(n-1)b}-\frac{dn}{2n(n-1)(k+v)}.$$

Where if there may be put $\frac{k-v}{k+v}=i,$ on account of $di=\frac{2vdk}{(k+v)^2}$ there will be

$$di=\frac{-vdb}{nbb}-\frac{vdn}{n(n-1)b}-\frac{2vdn}{n(n-1)(k+v)},$$

and if the number i may be introduced in place of k, there will become :

$$d\beta=\frac{-\beta\beta db}{iibb}-\frac{\beta dn}{n-1}\left(1+\frac{\beta}{iib}+\frac{(1-i)^2\beta}{iiv}\right) \text{ and } di=\frac{-vdb}{nbb}-\frac{vdn}{n(n-1)}\left(\frac{1}{b}+\frac{1-i}{v}\right).$$

COROLLARY 1

293. Also this form can be represented from the differential equation found , so that there shall be:

$$\frac{id\beta}{\beta\beta}+\frac{db}{ibb}=\frac{-dn}{n-1}\left(\frac{i}{\beta}+\frac{1}{ib}+\frac{(1-i)^2}{iv}\right)$$

or if on re-instating k:

$$\frac{d\beta}{\beta\beta}\left(\frac{k-v}{k+v}\right)+\frac{db}{bb}\left(\frac{k+v}{k-v}\right)=\frac{-dn}{n-1}\left(\frac{1}{\beta}\left(\frac{k-v}{k+v}\right)+\frac{1}{b}\left(\frac{k+v}{k-v}\right)+\frac{4v}{kk-vv}\right),$$

where this analogy is required to be observed, so that in whatever way $\frac{k+v}{k-v}$ is referred to b, thus $\frac{k-v}{k+v}$ may be referred to β.

COROLLARY 2

294. If the thickness of this lens may vanish, there becomes $v=0$ and $i=1$, where the figure of the lens no further enters into the calculation, but only the distance of the focus,

from which the variation in the position of the image G thus will be prepared, so that there shall be

$$\frac{d\beta}{\beta\beta}+\frac{db}{bb}=\frac{-dn}{n-1}\left(\frac{1}{\beta}+\frac{1}{b}\right)$$

and thus

$$d\beta=\frac{-\beta\beta}{bb}db-\frac{\beta\beta dn}{n-1}\left(\frac{1}{\beta}+\frac{1}{b}\right).$$

COROLLARY 3

295. If we wish to transfer the case for the mean rays, for which the formulas have been adapted up to this stage, to the red rays, it will be required to put $dn=-\frac{1}{100}$, but if to the violet rays, then $dn=+\frac{1}{100}$.

PROBLEM 3

296. *If the rays from the object E may be transmitted through several lenses, to determine the variation in the positions of the individual images, which proceed from the different refrangibilities of the rays.*

SOLUTION

All the denominations may be retained, from which we have made use in the above chapters, and $a,\ \alpha,\ b,\ \beta,\ c,\ \gamma$ etc. shall be the determined distances for rays of the mean kind. Also these distances will be changed by the variation in the ratio of refraction, the variations of which we will indicate by differentials. But since the distances between two lenses may remain constant, these variations will be prepared thus, so that there shall be

$$d\alpha+db=0,\quad d\beta+dc=0,\quad d\gamma+dd=0\quad \text{etc.}$$

Now since the distance of the object $AE=a$ shall be invariant, from the first problem there will be, if there may be put $\frac{k-v}{k+v}=i$,

$$d\alpha=-db=\frac{-\alpha\alpha dn}{i(n-1)}\left(\frac{i}{\alpha}+\frac{1}{ia}+\frac{(1-i)^2}{iv}\right)=\frac{-\alpha\alpha dn}{n-1}\cdot\frac{k+v}{k-v}\left(\frac{k-v}{\alpha(k+v)}+\frac{k+v}{a(k-v)}+\frac{4v}{kk-vv}\right)$$

and

$$di=\frac{-vdn}{n(n-1)}\left(\frac{1}{a}+\frac{1-i}{v}\right).$$

Then for the second lens, to which the determinable distances b and β refer with the arbitrary k' and with the thickness v', from which we have composed $\frac{k'-v'}{k'+v'}=i'$, and we will have

$$d\beta=-dc=\frac{-\beta\beta db}{i'i'bb}-\frac{\beta\beta dn}{i'(n-1)}\left(\frac{i'}{\beta}+\frac{1}{i'b}+\frac{(1-i')^2}{i'v'}\right)$$

and

$$di' = \frac{-v'db}{nbb} - \frac{v'dn}{n(n-1)}\left(\frac{1}{b} + \frac{1-i'}{v'}\right).$$

In a similar manner for the third lens, to which the determinable distances c and γ refer, with the arbitrary k'' and with the thickness v'', on putting $\frac{k''-v''}{k''+v''} = i''$, we come upon

$$d\gamma = -dd = \frac{-\gamma\gamma dc}{i''i''cc} - \frac{\gamma\gamma dn}{i''(n-1)}\left(\frac{i''}{\gamma} + \frac{1}{i''c} + \frac{(1-i'')^2}{i''v''}\right)$$

and

$$di'' = \frac{-v''dc}{ncc} - \frac{v''dn}{n(n-1)}\left(\frac{1}{c} + \frac{1-i''}{v''}\right),$$

and by progressing further we will obtain the following formulas:

$$d\delta = -de = \frac{-\delta\delta dd}{i'''i'''dd} - \frac{\delta\delta dn}{i'''(n-1)}\left(\frac{i'''}{\delta} + \frac{1}{i''d} + \frac{(1-i''')^2}{i'''v'''}\right)$$

and

$$di''' = \frac{-v''dd}{ndd} - \frac{v'''dn}{n(n-1)}\left(\frac{1}{d} + \frac{1-i'''}{v'''}\right),$$

from which this easy differentiation may be extended to any number of lenses. And if here the values of the differentials *dd, de, db* defined before now may be substituted successively, all these differentials both of the determinable distances as well as of the numbers i, i', i'', i''' etc. will be expressed by the differential *dn*.

If the ratio of refraction shall be different for the individual lenses and for these the order may be expressed by the numbers n, n', n'' etc., it is evident the differentials found here are going to be expressed in the following manner:

I. $$d\alpha = -db = \frac{-\alpha\alpha dn}{i(n-1)}\left(\frac{i}{\alpha} + \frac{1}{ia} + \frac{(1-i)^2}{iv}\right)$$

$$di = \frac{-vdn}{n(n-1)}\left(\frac{1}{a} + \frac{1-i}{v}\right).$$

II. $$d\beta = -dc = \frac{-\beta\beta db}{i'i'bb} - \frac{\beta\beta dn}{i'(n-1)}\left(\frac{i'}{\beta} + \frac{1}{i'b} + \frac{(1-i')^2}{i'v'}\right)$$

$$di' = \frac{-v'db}{nbb} - \frac{v'dn}{n(n-1)}\left(\frac{1}{b} + \frac{1-i'}{v'}\right).$$

III. $$d\gamma = -dd = \frac{-\gamma\gamma dc}{i''i''cc} - \frac{\gamma\gamma dn''}{i''(n''-1)}\left(\frac{i''}{\gamma} + \frac{1}{i''c} + \frac{(1-i'')^2}{i''v''}\right)$$

$$di'' = \frac{-v''dc}{n''cc} - \frac{v''dn''}{n''(n''-1)}\left(\frac{1}{c} + \frac{1-i''}{v''}\right)$$
etc.,

from which formulas also the changes of the individual images and hence also finally of the final image will be able to be defined easily, or rather in place of the images the angles, according to which we will consider these to become apparent with the eye put at the true distance l :

For one lens $\frac{1}{i} \cdot \frac{\alpha}{a} \cdot \frac{z}{l}$

for two lenses $\frac{1}{ii'} \cdot \frac{\alpha\beta}{ab} \cdot \frac{z}{l}$

for three lenses $\frac{1}{ii'i''} \cdot \frac{\alpha\beta\gamma}{abc} \cdot \frac{z}{l}$

etc.,

in addition, clearly besides the distances a, α; b, β; c, γ etc. also the letters i, i', i'', i''' etc. are variables.

COROLLARY 1

297. Hence therefore it is allowed to define by differentiation, by how much the change in the location of the final image may arise, which constitutes the object viewed, on account of the different refrangibility of the rays.

COROLLARY 2

298. Accordingly also we will define the magnitude of each image by the above determinable distances and the numbers i, i', i'' etc., there is had $\frac{1}{ii'i''i'''} \cdot \frac{\alpha\beta\gamma\delta}{abcd} \cdot \frac{z}{l}$ for the magnitude of the image (§ 189); in a similar manner a change can be assigned, as the magnitude of the final image will be allowed to change on account of the different refrangibility of the rays.

COROLLARY 3

299. But with each change known, which the final image undergoes both with respect of position as well as of magnitude, it may be deduced without difficulty, by how much the sharpness of the image viewed will be disturbed on account of the different refrangibility of the rays.

COROLLARY 4

300. If the thickness of the lens may vanish, there will become

$$d\alpha = -db = \frac{-\alpha\alpha dn}{(n-1)}\left(\frac{1}{a}+\frac{1}{\alpha}\right)$$
$$d\beta = -dc = \frac{-\beta\beta db}{bb} - \frac{\beta\beta dn}{n-1}\left(\frac{1}{b}+\frac{1}{\beta}\right)$$
$$d\gamma = -dd = \frac{-\gamma\gamma dc}{cc} - \frac{\gamma\gamma dn}{n-1}\left(\frac{1}{c}+\frac{1}{\gamma}\right)$$
$$d\delta = -de = \frac{-\delta\delta dd}{dd} - \frac{\delta\delta dn}{n-1}\left(\frac{1}{d}+\frac{1}{\delta}\right)$$
etc.,

moreover the numbers *i, i′, i″* etc. will be changed into one and no further changes are necessary.

Therefore if the thicknesses of the lenses may vanish, for different refractions of the individual lenses the above formulas will be changed into the following:

I. $d\alpha = -db = \frac{-\alpha\alpha dn}{(n-1)}\left(\frac{1}{a}+\frac{1}{\alpha}\right)$

II. $d\beta = -dc = \frac{-\beta\beta db}{bb} - \frac{\beta\beta dn'}{n'-1}\left(\frac{1}{b}+\frac{1}{\beta}\right)$

III. $d\gamma = -dd = \frac{-\gamma\gamma dc}{cc} - \frac{\gamma\gamma dn''}{n''-1}\left(\frac{1}{c}+\frac{1}{\gamma}\right)$

etc.,

Moreover it is evident for these, when § 288 is said to indicate dn to be either $+\frac{1}{100}$ or $-\frac{1}{100}$, that only from that kind of glass, for which the mean ratio of refraction is required to be $n=\frac{31}{20}$, and for other kinds of glass the differentials *dn′, dn″, dn‴* etc. are barely able to differ from $\frac{1}{100}$. But how much this is going to differ in the future will be required to be chosen, as that may be defined from experiment rather than by some theory.

SCHOLIUM

301. Therefore on account of the difference of the refrangibility of the rays, since a two-fold change may be introduced to each image, of which one affects its magnitude, and truly the other its position, thence a two-fold confusion is brought into the image seen. For if the formulas shown in the above chapters may be restricted to the mean rays, for which there is $n=\frac{31}{20}$, on putting $dn=-\frac{1}{100}$ from the differentials of the formulas treated here, the position and magnitude of the image formed will be defined for the red rays; but on putting $dn=+\frac{1}{100}$ the position and magnitude of the violet image will be indicated. Evidently if $I\iota$ (Fig. 16) were the final image presented to the eye, which

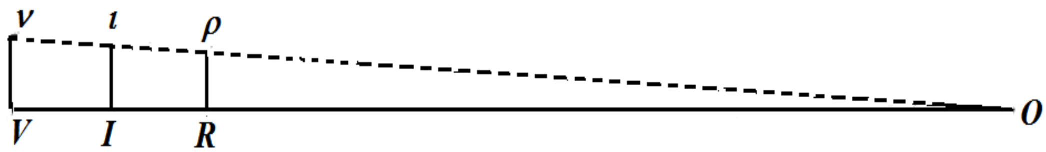

Fig. 16.

is formed from the mean rays, by the formulas just found, just as dn may be put to be either $-\frac{1}{100}$ or $+\frac{1}{100}$, both the red $R\rho$ as well as the violet image Vv may be defined ; and from the nature of the differentials it is evident since the intervals IR and IV must be equal to each other, then also the differences $I\iota - R\rho$ and $Vv - I\iota$ must be equal, thus so that a series of innumerable images situated between the extremes $R\rho$ et Vv may be provided to be discerned by the eye, so that the greater would be the difference both in the account of the position as well as the magnitude, [the further we progress from the mean]. Whereby this confusion may be removed completely, if the disposition of a lens of this kind may be able to be defined, so that both the interval RV as well as the difference between the images $R\rho$ and Vv may be reduced to zero, so that each, unless it may be performed likewise, will not be able to remove the confusion completely. Yet truly even if neither of these conditions may be able to be satisfied completely, yet a position of this kind O will be given for the eye, where the confusion may be perceived to be minimally sensible, which will be at the concurrence of the right line $v\rho$ produced to the axis: for here all the extremities ρ, ι, v will be perceived by common rays, and neither therefore will the extremities of the object appear with a colored tincture. Whereby if likewise the point O may agree as the suitable place of the eye, the other source of confusion will not be perceived, unless the origin is dragged thence, so that perhaps the extreme images $R\rho$ and Vv may differ greatly from the true distance of the eye, if indeed the mean position $I\iota$ were moved away from the true distance of the eye.

[Thus Euler indicates how the images in different colors can be made to overlap, but these images of different colors, at slightly different distances from the eye, cannot be resolved equally unless the separation is very small. Euler's geometric arguments might be more convincing if he had made use of Fermat's Principle of least action, and shown that a pulse of light emanating from the object along the axis would arrive at its conjugate point at the same time as another sent off at the same time, but following the path indicated through the lenses.]

Nor yet will the boundary of the object hence appear with iridescent colors, for which a special kind of confusion is occurring ; and thus that, which has hitherto been presented, will be able to be tolerated easily with this new kind of confusion, only if the interval RV either may be reduced to zero or perhaps be rendered small enough. Hence therefore we understand that disadvantage not to be completely unavoidable, by which objects presented by dioptric instruments often are represented together with surrounding iridescent colors, so that in no way may they be able to be separated from these ; on account of which it will be most worthwhile that we may investigate how these instruments may be able to be freed from this error. Which whole investigation may be reduced here, so that the point O may be determined, where the right line drawn through the ends of the images v, ι, ρ concurs with the axis and this point may be returned agreeing with the position of the eye now defined above, if indeed it can happen: from which it is seen the position of the eye O ought to be with this aforesaid property, so that the angle, under which the final image is discerned, may be allowed on account of the variable number n to be given no change. Then truly in addition it will be required to be seen, whether the intervals IR and IV may be able to be reduced to zero or rendered minimal.

PROBLEM 4

302. *For the proposed single lens to define the position of the eye, from which an object may be seen without a colored border.*

SOLUTION

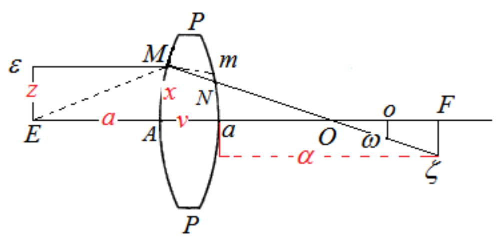

Fig. 12

The distance of the object $E\varepsilon$ (Fig. 12) before the lens shall be $EA = a$, truly the image will be represented by the rays of the mean nature at $F\zeta$, and there may be put $aF = \alpha$. Truly for the lens its thickness shall be $Aa = v$ and the arbitrary quantity $= k$, from which there may be taken $\frac{k-v}{k+v} = i$.

Hence putting $E\varepsilon = z$ there will be $F\zeta = \frac{1}{i} \cdot \frac{\alpha}{a} z$ (§ 86); whereby if for the position of the eye there may be put the distance $aO = O$, which is fixed, there will be $OF = \alpha - O$ and the tangent of the angle $FO\zeta = \frac{1}{i} \cdot \frac{\alpha}{a} \cdot \frac{z}{\alpha - O}$, which formula must undergo no change on account of the different refrangibilities of the rays. Thence moreover only the quantities α and i may be changed, while the rest remain constant. Whereby the logarithmic differential of this formula put equal to zero will produce this equation

$$-\frac{di}{i} + \frac{d\alpha}{\alpha} - \frac{d\alpha}{\alpha - O} = 0 \text{ or } -\frac{di}{i} - \frac{Od\alpha}{\alpha(\alpha - O)} = 0,$$

where if the above values found may be substituted, there becomes

$$\frac{vdn}{in(n-1)}\left(\frac{1}{a}+\frac{1-i}{v}\right)+\frac{O\alpha dn}{i(n-1)(\alpha-O)}\left(\frac{1}{\alpha}+\frac{1}{ia}+\frac{(1-i)^2}{iv}\right)=0,$$

which equation divided by $\frac{dn}{i(n-1)}$ provides :

$$\frac{v}{na}+\frac{1-i}{n}+\frac{O\alpha}{(\alpha-O)}\left(\frac{1}{\alpha}+\frac{1}{ia}+\frac{(1-i)^2}{iv}\right)=0,$$

from which the position of the eye will be able to be defined ; which if it must agree with that found above (§ 238), where we found
$O=\frac{-i\alpha v}{n\alpha-iv}$, there will be $\alpha-O=\frac{n\alpha\alpha}{n\alpha-iv}$ and $\frac{O\alpha}{\alpha-O}=\frac{-iv}{n}$, and hence our equation multiplied by n will become $\frac{v}{a}+1-i-\frac{iiv}{\alpha}-\frac{v}{a}-(1-i)^2=0$ or $i-ii-\frac{iiv}{\alpha}=0$, and thus $i=\frac{\alpha}{\alpha+v}=\frac{k-v}{k+v}$. On account of which thus it will be appropriate to define the arbitrary quantity k, so that there shall be $k=2\alpha+v$; and since $i=\frac{\alpha}{\alpha+v}$, for the position of the eye we will have $O=\frac{-\alpha v}{n\alpha+(n-1)v}=\frac{-20\alpha v}{31\alpha+11v}$ on account of $n=\frac{31}{20}$.

So that hence if again we may desire the variation in the position of this image, it will be required to define the differential $d\alpha$, which becomes :

$$d\alpha=\frac{-\alpha\alpha dn}{i(n-1)}\left(\frac{1}{\alpha}+\frac{1}{ia}+\frac{(1-i)^2}{iv}\right)$$

and with the value $\frac{\alpha}{\alpha+v}$ put in place for i:

$$d\alpha=\frac{-(\alpha+v)dn}{n-1}\left(1+\frac{\alpha+v}{a}\right),$$

which value if it may be reduced to nothing, may remove completely the confusion arising from the diverse refrangibility of all the different rays.

COROLLARY 1

303. Therefore for the construction of the lens an arbitrary quantity k thus must be taken, so that there shall be $k=2\alpha+v$; and then the eye may be put in that place, where it may perceive the whole apparent field of view, likewise no confusion may be experienced from the nature of the diverse rays.

COROLLARY 2

304. But likewise in order that the eye may view the image at the true distance, it shall be required that $\alpha-O=-l$, and thus $l=\frac{-31\alpha(\alpha+v)}{31\alpha+11v}$. From which it is deduced :

$$\alpha = \tfrac{-1}{2}l - \tfrac{1}{2}v - \sqrt{\left(\tfrac{1}{4}ll + \tfrac{9}{62}vl + \tfrac{1}{4}vv\right)},$$

and hence

$$O = \tfrac{1}{2}l - \tfrac{1}{2}v - \sqrt{\left(\tfrac{1}{4}ll + \tfrac{9}{62}vl + \tfrac{1}{4}vv\right)},$$

and

$$k = -l - 2\sqrt{\left(\tfrac{1}{4}ll + \tfrac{9}{62}vl + \tfrac{1}{4}vv\right)}.$$

COROLLARY 3

305. Therefore in addition it can be effected, so that also $d\alpha$ may vanish, which happens, if $a + \alpha + v = 0$, that is

$$a = \tfrac{1}{2}l - \tfrac{1}{2}v + \sqrt{\left(\tfrac{1}{4}ll + \tfrac{9}{62}vl + \tfrac{1}{4}vv\right)}.$$

Truly since in this case on account of $\alpha = -a - v$ the image may fall in the object itself, thus so that the rays may be considered to be allowed no refraction, and the view through the lens will be comparable to the naked eye.

COROLLARY 4

306. Plainly if the thickness of the lens v may vanish, both on account of the field of view as well as the different refrangibility of the rays, there shall become $O = 0$; therefore in this case the eye applied immediately to the eye will perceive no confusion on account of the nature of the diverse rays. Therefore while there were $\alpha = -l$, the vision will be distinct.

SCHOLIUM

307. Here evidently we have extricated the essential parts thoroughly from the confusion now determined above, which arises from the aperture of the lens, and thus we may consider the aperture of the first lens as vanishing. Therefore here we consider only that kind of confusion, which originates from the different refrangibility of the rays ; which generally we have observed to be removed, if the angle at O may be considered to be invariable ; for then the boundaries of the object may be seen to be pretty well defined nor surrounded by iridescent. Yet meanwhile at this point some other kind of confusion may be able to arise, so that, if the mean image may be held at the true distance from the eye, one of the extreme images may be exceedingly close, the other exceedingly far away; truly if the intervals of these shall not be exceedingly large, this confusion will be barely perceived. Thus we find here, what experiments demonstrate well enough, if we may view an object by a single lens, that appears without a colored outline, provided the eye may be applied next to it; so that if when it may be seen to happen otherwise, the

cause without doubt will have to be attributed to the aperture of the lens, for which condition here the alleged reasons are refractory.

PROBLEM 5

308. *If a dioptric instrument may be constructed from two lenses, to define the position of the eye, from which the object may be seen without a colored fringe.*

SOLUTION

[Recall that in all these diagrams, the positions and heights of the intermediate and final images have been found algebraically using thick or thin lens formulae from determinable lengths and known radii of curvature, and with an undetermined variable k which can be given a specific value for a known ray, and the ray tracing techniques have not been developed yet in full. Evidently if the lenses can be made very thin, then the lines connecting the object and the images come to resemble rays, such as when $M'N'$ tends to zero.]

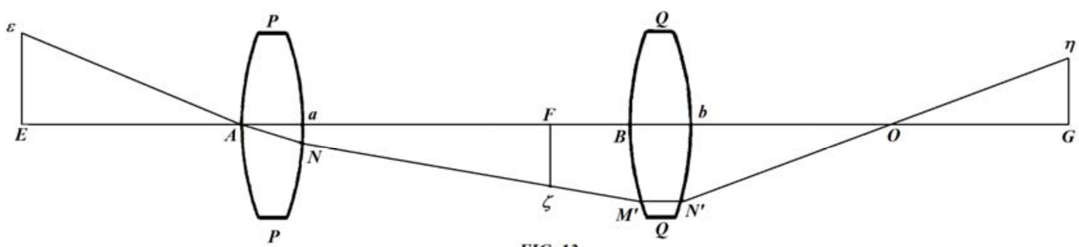

FIG. 13.

With the distance of the object $AE = a$ (Fig. 13) the nature of the remaining determinable distances for the mean rays shall be $aF = \alpha$, $BF = b$ and $bG = \beta$, truly the thicknesses of the lenses $AA = v$, $Bb = v'$ and the arbitrary distances k and k', and there may be put $\frac{k-v}{k+v} = i$ and $\frac{k'-v}{k'+v} = i'$. With these in place if the magnitude of the object $E\varepsilon$ may be called $= z$, the image will be $G\eta = \frac{1}{ii'} \cdot \frac{\alpha\beta}{ab} \cdot z$, from which, if the distance of the eye may be put $bO = O$, on account of $GO = \beta - O$ the tangent of the angle $GO\eta$ will be $\frac{1}{ii'} \cdot \frac{\alpha\beta}{ab} \cdot \frac{z}{\beta - O}$, of which the logarithmic differential equated to zero provides :

$$-\frac{di}{i} - \frac{di'}{i'} + \frac{d\alpha}{\alpha} - \frac{db}{b} + \frac{d\beta}{\beta} - \frac{d\beta}{\beta - O} = 0,$$

which with the above values found substituted (§ 296) will become :

$$\frac{vdn}{in(n-1)}\left(\frac{1}{a}+\frac{1-i}{v}\right)+\frac{v'db}{n'i'bb}+\frac{v'dn'}{i'n'(n'-1)}\left(\frac{1}{b}+\frac{1-i'}{v'}\right)-\frac{db}{\alpha}-\frac{db}{b}$$
$$+\frac{O}{\beta-O}\left(\frac{\beta db}{i'i'bb}+\frac{\beta dn'}{i'(n'-1)}\left(\frac{i'}{\beta}+\frac{1}{i'b}+\frac{(1-i')^2}{i'v'}\right)\right)=0.$$

Truly the field condition required $O=\frac{\beta\mathfrak{b}}{\mathfrak{b}+\frac{1}{ii'}\cdot\frac{\alpha\beta}{ab}\cdot z}$, with

$$\mathfrak{b}=\left(\frac{i'}{i}\cdot\frac{\alpha+b}{a}-\frac{i'bv}{n a\alpha}-\frac{1}{i}\cdot\frac{\alpha v'}{n'ab}\right)z$$

(§ 245), from which there becomes

$$\frac{O}{\beta-O}=\frac{\beta\mathfrak{b}}{\frac{1}{ii'}\cdot\frac{\alpha\beta\beta}{ab}\cdot z}=\frac{ii'ab}{\alpha\beta}\left(\frac{i'}{i}\cdot\frac{\alpha+b}{a}-\frac{i'bv}{n a\alpha}-\frac{1}{i}\cdot\frac{\alpha v'}{n'ab}\right).$$

Now truly there is $db=\frac{\alpha\alpha dn}{i(n-1)}\left(\frac{i}{\alpha}+\frac{1}{ia}+\frac{(1-i)^2}{iv}\right)$, which previous value we may substitute, and we will transform our equation into this form [putting $n'=n$]

$$\frac{dn}{n-1}\left(\frac{v}{ina}+\frac{1-i}{in}+\frac{v'}{i'nb}+\frac{1-i'}{i'n}+\frac{\beta O}{i'(\beta-O)}\left(\frac{i'}{\beta}+\frac{1}{i'b}+\frac{(1-i')^2}{i'v'}\right)\right)+db\left(\frac{v'}{n'i'bb}-\frac{1}{\alpha}-\frac{1}{b}+\frac{\beta O}{i'i'bb(\beta-O)}\right)=0,$$

where the latter term will change into $-\frac{iv}{n\alpha\alpha}db$; then truly there will be

$$0=\frac{dn}{n-1}\begin{Bmatrix}1+\frac{b}{\alpha}+\frac{i'i'b}{\beta}\left(1+\frac{b}{\alpha}\right)+\frac{(1-i')^2b(\alpha+b)}{\alpha v'}+\frac{2-i-i'}{n}\\-\frac{iv}{n\alpha}-\frac{i'v'}{n\beta}-\frac{ibv}{n\alpha\alpha}-\frac{ii'i'bbv}{n\alpha\alpha\beta}-\frac{i(1-i')^2bbv}{n\alpha\alpha v'}\end{Bmatrix}$$

By distinguishing n' from n there will be

$$\frac{dn}{n-1}\left(\frac{iv}{\alpha n}-\frac{(1-i)}{n}\right)=\frac{dn'}{n'-1}\begin{Bmatrix}1+\frac{b}{\alpha}+\frac{i'i'b}{\beta}\left(1+\frac{b}{\alpha}\right)-\frac{ii'i'bbv}{n\alpha\alpha\beta}-\frac{i'v'}{n'\beta}+\frac{1-i'}{n'}\\-\frac{ibv}{n\alpha\alpha}+\frac{b(1-i')^2}{v'}\left(1+\frac{b}{\alpha}\right)-\frac{i(1-i')^2bbv}{n\alpha\alpha v'}\end{Bmatrix},$$

the complicated nature of this equation shall prevent anything further convenient to be concluded
.

COROLLARY 1

309. If both lenses may lack thickness, so that there shall be
$v=0$, $v'=0$ and $i=i'=1$, the first differential equation is [with $n'=n$]

$$\frac{d\alpha}{\alpha}-\frac{db}{b}-\frac{Od\beta}{\beta(\beta-O)}=0\,;$$

then truly:

$$d\alpha=-db=\frac{-\alpha\alpha dn}{n-1}\left(\frac{1}{\alpha}+\frac{1}{a}\right)$$

and

$$d\beta=\frac{-\beta\beta db}{bb}-\frac{\beta\beta dn}{n-1}\left(\frac{1}{\beta}+\frac{1}{b}\right)=\frac{-dn}{n-1}\left(\frac{\alpha\alpha\beta\beta}{bb}\left(\frac{1}{\alpha}+\frac{1}{a}\right)+\beta\beta\left(\frac{1}{\beta}+\frac{1}{b}\right)\right),$$

with which values substituted, and with the division made by $\frac{dn}{n-1}$, there becomes

$$-\alpha\alpha\left(\frac{1}{\alpha}+\frac{1}{a}\right)\left(\frac{1}{\alpha}+\frac{1}{b}\right)+\frac{O\beta}{\beta-O}\left(\frac{\alpha\alpha}{bb}\left(\frac{1}{\alpha}+\frac{1}{a}\right)+\frac{1}{\beta}+\frac{1}{b}\right)=0.$$

From which. if the eye may be applied next to the latter lens, so that there shall be $O=0$, there will become :

$$\left(\frac{1}{\alpha}+\frac{1}{a}\right)\left(\frac{1}{\alpha}+\frac{1}{b}\right)=0.$$

COROLLARY 2

810. Truly by the same hypothesis, so that the position of the eye may agree with that, which the field of view requires, there must be $O=\frac{b\beta(\alpha+b)}{b(\alpha+b)+\alpha\beta}$ from which there becomes $\beta-O=\frac{\alpha\beta\beta}{b(\alpha+b)+\alpha\beta}$, and thus $\frac{O}{\beta-O}=\frac{b(\alpha+b)}{\alpha\beta}=\frac{bb}{\beta}\left(\frac{1}{\alpha}+\frac{1}{b}\right)$, with which value substituted our equation will be

$$0=\left(\frac{1}{\alpha}+\frac{1}{b}\right)\left(-\alpha\alpha\left(\frac{1}{\alpha}+\frac{1}{a}\right)+bb\left(\frac{\alpha\alpha}{bb}\left(\frac{1}{\alpha}+\frac{1}{a}\right)+\frac{1}{\beta}+\frac{1}{b}\right)\right),$$

which is reduced to this form $0=bb\left(\frac{1}{\alpha}+\frac{1}{b}\right)\left(\frac{1}{\beta}+\frac{1}{b}\right)$. On distinguishing n' from n the terms depending on dn cancel each other, and here arises $0=\frac{dn'}{n'-1}bb\left(\frac{1}{\alpha}+\frac{1}{b}\right)\left(\frac{1}{\beta}+\frac{1}{b}\right)$, which is shown thus :

Since the first differential equation provides :

$$\frac{d\alpha}{\alpha}-\frac{d\beta}{\beta}-\frac{Od\beta}{\beta(\beta-O)}=0 \text{ or } d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)-\frac{Od\beta}{\beta(\beta-O)}=0,$$

since from the condition of the apparent field there shall be $\frac{O}{\beta-O}=\frac{bb}{\beta}\left(\frac{1}{\alpha}+\frac{1}{b}\right)$, the equation will become

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)-\frac{bb}{\beta^2}\left(\frac{1}{\alpha}+\frac{1}{b}\right)d\beta=0$$

and thus in place of $d\beta$, on substituting its value :

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)-\frac{bb}{\beta^2}\left(\frac{1}{\alpha}+\frac{1}{b}\right)\left(\frac{\beta\beta d\alpha}{bb}-\frac{\beta^2 dn'}{n'-1}\left(\frac{1}{\beta}+\frac{1}{b}\right)\right)=0,$$

where the terms which contain $d\alpha$ evidently disappear, and the whole question is led to this equation

$$\frac{dn'}{n'-1}bb\left(\frac{1}{\alpha}+\frac{1}{b}\right)\left(\frac{1}{\beta}+\frac{1}{b}\right)=0.$$

COROLLARY 3

311. So that if it may be possible to satisfy this condition, the object will appear without colored fringes; besides truly the confusion may be completely removed, if it may be allowed to render $d\beta=0$, since by this equation there becomes

$$\frac{\alpha\alpha}{bb}\left(\frac{1}{\alpha}+\frac{1}{a}\right)+\frac{1}{\beta}+\frac{1}{b}=0 \text{ or } \alpha\alpha\left(\frac{1}{\alpha}+\frac{1}{a}\right)+bb\left(\frac{1}{\beta}+\frac{1}{b}\right)=0$$

or by distinguishing n' from n,

$$\frac{dn}{n-1}\cdot\frac{\alpha\alpha}{bb}\left(\frac{1}{\alpha}+\frac{1}{a}\right)+\frac{dn'}{n'-1}\cdot\left(\frac{1}{\beta}+\frac{1}{b}\right)=0.$$

COROLLARY 4

312. But the first equation cannot be satisfied, unless there were either $\alpha+b=0$ or $\left(\frac{1}{\beta}+\frac{1}{b}\right)=0$. In both these cases both lenses will be joined together, so that they will make a single lens; truly here the distance of the second lens will be infinite, which again may return again to the case of a single lens ; for there will become $O=-\alpha-b$, on account of $\beta=-b$, and the eye must be applied next to the first lens.

SCHOLIUM

313. If we may wish to extend these investigations to several lenses, without neglecting the thickness of these plainly we may well fall into inextricable formulas, from which hardly anything may be able to be concluded. Truly since in nearly all dioptric instruments, especially those which are constructed from several lenses, with these the thickness is accustomed to be present so thin, that without notable error it may be taken as zero, we will be able easily to pass over so tedious an investigation. According to which it may be agreed, so that here it may not be acted on with geometrical rigor, but we

may be able to be content with this, provided that we will understand this confusion well enough: from which it will suffice in the consideration of several lenses the thickness of these evidently may be ignored.

SUPPLEMENT IV

If the ratio of the refraction in the individual lenses shall be different, the solution may be resolved in the following manner, with the thickness of the lenses ignored :

I. The first differential equation will be had at once, as in the problem, thus so that there shall be

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)+d\beta\left(\frac{1}{\beta}+\frac{1}{c}\right)-\frac{Od\gamma}{\gamma(\gamma-O)}=0,$$

and since also, as before, there is

$$\frac{O}{\gamma-O}=\frac{bbcc}{\alpha\beta^2\gamma}\left(1+\frac{\alpha}{b}\right)+\frac{cc}{\beta\gamma}\left(1+\frac{\beta}{c}\right),$$

our equation will be

$$\left(\frac{1}{\alpha}+\frac{1}{b}\right)\left(d\alpha-\frac{bbcc}{\beta^2\gamma^2}d\gamma\right)+\left(\frac{1}{\beta}+\frac{1}{c}\right)\left(d\beta-\frac{cc}{\gamma\gamma}d\gamma\right)=0.$$

II. But now the ratio of the different refractions is required to be had ; from which we have found to be in the superior additions :

$$d\alpha=-\frac{\alpha\alpha dn}{n-1}\left(\frac{1}{a}+\frac{1}{\alpha}\right);\quad d\beta=\frac{\beta\beta d\alpha}{bb}-\frac{\alpha\alpha dn'}{n'-1}\left(\frac{1}{b}+\frac{1}{\beta}\right);\quad d\gamma=\frac{\gamma\gamma d\beta}{cc}-\frac{\gamma\gamma dn''}{n''-1}\left(\frac{1}{c}+\frac{1}{\gamma}\right).$$

And hence therefore there will become

$$d\alpha-\frac{bbcc}{\beta^2\gamma^2}d\gamma=d\alpha-\frac{bbd\beta}{\beta^2}+\frac{bbccdn''}{\beta^2(n''-1)}\left(\frac{1}{c}+\frac{1}{\gamma}\right)=\frac{bbdn'}{n'-1}\left(\frac{1}{b}+\frac{1}{\beta}\right)+\frac{bbccdn''}{\beta^2(n''-1)}\left(\frac{1}{c}+\frac{1}{\gamma}\right).$$

from which

$$d\beta-\frac{cc}{\gamma\gamma}d\gamma=\frac{ccdn''}{n''-1}\left(\frac{1}{c}+\frac{1}{\gamma}\right).$$

III. Therefore from these our differential equation will be changed into this form :

$$\left(\frac{1}{\alpha}+\frac{1}{b}\right)\left(\frac{bbdn'}{n'-1}\left(\frac{1}{b}+\frac{1}{\beta}\right)+\frac{bbc^2dn''}{\beta^2(n''-1)}\left(\frac{1}{c}+\frac{1}{\gamma}\right)\right)+\left(\frac{1}{\beta}+\frac{1}{c}\right)\frac{ccdn''}{n''-1}\left(\frac{1}{c}+\frac{1}{\gamma}\right)=0,$$

or

$$\frac{dn'}{n'-1}bb\left(\frac{1}{\alpha}+\frac{1}{b}\right)\left(\frac{1}{\beta}+\frac{1}{b}\right)+\frac{dn''}{n''-1}cc\left(\frac{1}{c}+\frac{1}{\gamma}\right)\left(\frac{bb}{\beta\beta}\left(\frac{1}{\alpha}+\frac{1}{b}\right)+\frac{1}{\beta}+\frac{1}{c}\right)=0.$$

IV. But for that singular case, where the eye must be applied next to the final lens, on account of $O=0$ this simpler equation will be had :

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)+d\beta\left(\frac{1}{\beta}+\frac{1}{c}\right)=0\,,$$

or with the value $d\beta$ substituted:

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)+\frac{\beta\beta d\alpha}{bb}\left(\frac{1}{\beta}+\frac{1}{c}\right)-\frac{bbdn'}{n'-1}\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\frac{1}{\beta}+\frac{1}{c}\right)=0,$$
$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}+\frac{\beta\beta}{bb}\left(\frac{1}{\beta}+\frac{1}{c}\right)\right)-\frac{b^2dn'}{n'-1}\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\frac{1}{\beta}+\frac{1}{c}\right)=0$$

or finally

$$+\frac{dn}{n-1}\alpha\alpha\left(\frac{1}{\alpha}+\frac{1}{a}\right)\left(\frac{1}{\alpha}+\frac{1}{b}+\frac{\beta^2}{b^2}\left(\frac{1}{\beta}+\frac{1}{c}\right)\right)+\frac{dn'}{n'-1}\beta\beta\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\frac{1}{\beta}+\frac{1}{c}\right)=0.$$

V. In this way only the colored margin is removed; but in order that the whole confusion may be removed, which happens, if $d\gamma=0$, in addition for this equation to be satisfied

$$0=\frac{dn}{n-1}\frac{\alpha\alpha\beta\beta\gamma\gamma}{bbcc}\left(\frac{1}{a}+\frac{1}{\alpha}\right)+\frac{dn'}{n'-1}\frac{\beta\beta\gamma\gamma}{cc}\left(\frac{1}{b}+\frac{1}{\beta}\right)+\frac{dn''}{n''-1}\gamma\gamma\left(\frac{1}{c}+\frac{1}{\gamma}\right).$$

PROBLEM 6

314. *If an optical instrument may depend on three lenses, of which the thickness may vanish, to define that disposition, so that with the eye in that place which the field demands, the constitution of the object may be seen without a colored fringe.*

SOLUTION

Therefore with the distance of the object put before the lens $AE=a$ (Fig. 14) of which the magnitude $E\varepsilon=z$, the distances of the images formed from the mean rays may be called as follows :

$$aF=\alpha,\ BF=b,\ BG=\beta,\ \ CG=c\ \text{ and }\ cH=\gamma,$$

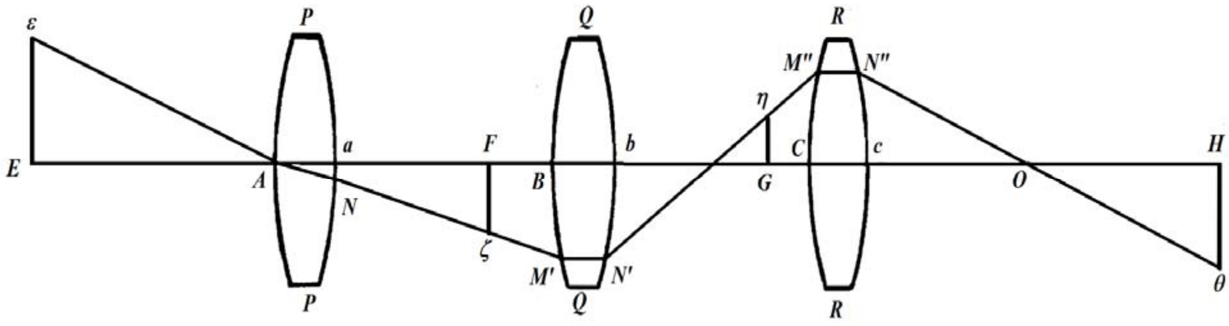

Fig. 14.

and the final image will be $H\theta = \frac{\alpha\beta\gamma}{abc}z$, and with the position of the eye past the final lens the distance $cO = O$ and there will be $OH = \gamma - O$ and the tangent of the angle $HO\theta = \frac{\alpha\beta\gamma}{abc}\cdot\frac{z}{\gamma - O}$, which must be invariable. Therefore on putting its logarithmic differential $= 0$ we will have this equation:

$$\frac{d\alpha}{\alpha} - \frac{db}{b} + \frac{d\beta}{\beta} - \frac{dc}{c} + \frac{d\gamma}{\gamma} - \frac{d\gamma}{\gamma - O} = 0.$$

But we have elicited from the above formulas (§ 300)

$$d\alpha = -db = \frac{-dn}{n-1}\alpha\alpha\left(\frac{1}{a} + \frac{1}{\alpha}\right),\ \ d\beta = -dc = -\frac{\beta\beta dn}{n-1}\left(\frac{\alpha\alpha}{bb}\left(\frac{1}{a} + \frac{1}{\alpha}\right) + \frac{1}{b} + \frac{1}{\beta}\right),$$
$$d\gamma = -\frac{\gamma\gamma dn}{n-1}\left(\frac{\alpha\alpha\beta\beta}{bbcc}\left(\frac{1}{a} + \frac{1}{\alpha}\right) + \frac{\beta\beta}{cc}\left(\frac{1}{b} + \frac{1}{\beta}\right) + \frac{1}{c} + \frac{1}{\gamma}\right),$$

from which our equation will be

$$d\alpha\left(\frac{1}{\alpha} + \frac{1}{b}\right) + d\beta\left(\frac{1}{\beta} + \frac{1}{c}\right) - \frac{Od\gamma}{\gamma(\gamma - O)} = 0.$$

But on account of the apparent field, we have found above (§ 256) :

$$\mathfrak{B} = \mathfrak{b} = b\left(1 + \frac{\alpha}{b}\right)\frac{z}{a};\ \mathfrak{C} = \mathfrak{c} = \frac{bc}{\beta}\left(1 + \frac{\alpha}{b}\right)\frac{z}{a} + \frac{\alpha c}{b}\left(1 + \frac{\beta}{c}\right)\frac{z}{a}$$

and hence

$O = \frac{\gamma\mathfrak{c}}{\mathfrak{c} + H\theta}$ and $\frac{O}{\gamma - O} = \frac{\mathfrak{c}}{H\theta}$, from which there becomes, on account of $H\theta = \frac{\alpha\beta\gamma}{bc}\cdot\frac{z}{a}$,

$$\frac{O}{\gamma - O} = \frac{bbcc}{\alpha\beta\beta\gamma}\left(1 + \frac{\alpha}{b}\right) + \frac{cc}{\beta\gamma}\left(1 + \frac{\beta}{c}\right) \text{ or } \frac{O}{\gamma(\gamma - O)} = \frac{bbcc}{\alpha\beta\gamma\gamma}\left(\frac{1}{\alpha} + \frac{1}{b}\right) + \frac{cc}{\gamma\gamma}\left(\frac{1}{\beta} + \frac{1}{c}\right).$$

Now with these values substituted we will have :

$$\left(\tfrac{1}{\alpha}+\tfrac{1}{b}\right)\left(d\alpha-\tfrac{bbcc}{\beta\beta\gamma\gamma}d\gamma\right)+\left(\tfrac{1}{\beta}+\tfrac{1}{c}\right)\left(d\beta-\tfrac{cc}{\gamma\gamma}d\gamma\right)=0.$$

But there is

$$d\alpha-\tfrac{bbcc}{\beta\beta\gamma\gamma}d\gamma=\tfrac{dn}{n-1}\left(bb\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)+\tfrac{bbcc}{\beta\beta}\left(\tfrac{1}{c}+\tfrac{1}{\gamma}\right)\right)$$

$$d\beta-\tfrac{cc}{\gamma\gamma}d\gamma=\tfrac{dn}{n-1}cc\left(\tfrac{1}{c}+\tfrac{1}{\gamma}\right).$$

Whereby with the division made by $\frac{dn}{n-1}$ we will obtain this equation :

$$bb\left(\tfrac{1}{\alpha}+\tfrac{1}{b}\right)\left(\tfrac{1}{b}+\tfrac{1}{\beta}+\tfrac{cc}{\beta\beta}\left(\tfrac{1}{c}+\tfrac{1}{\gamma}\right)\right)+cc\left(\tfrac{1}{\beta}+\tfrac{1}{c}\right)\left(\tfrac{1}{c}+\tfrac{1}{\gamma}\right)=0.$$

Truly so that if the eye may be applied next to the lens or there shall be $O=0$, the prescribed condition demands this equation :

$$\alpha\alpha\left(\tfrac{1}{\alpha}+\tfrac{1}{b}\right)\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)+\beta\beta\left(\tfrac{1}{\beta}+\tfrac{1}{c}\right)\left(\tfrac{\alpha\alpha}{bb}\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)+\tfrac{1}{b}+\tfrac{1}{\beta}\right)=0.$$

Truly the confusion arising from the different refrangibility of rays will be removed completely, if in addition there were $d\gamma=0$ or

$$\tfrac{\alpha\alpha\beta\beta}{bbcc}\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)+\tfrac{\beta\beta}{cc}\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)+\tfrac{1}{c}+\tfrac{1}{\gamma}=0.$$

COROLLARY 1

815. If the ratios of the apertures of the lenses may be introduced into the calculation and that for the second lens may be put $=\pi$, for the third $=\pi'$, there will be

$$bN'=\tfrac{\pi b\beta}{b+\beta}\ \text{ and }\ CM''=cN''=\tfrac{\pi' c\gamma}{c+\gamma}.$$

Then truly on putting $\frac{z}{a}=\Phi$ there will be $G\eta=\frac{\alpha\beta}{ab}\cdot a\Phi$ and $H\theta=\frac{\alpha\beta\gamma}{abc}\cdot a\Phi$. Hence there will become

$$\tfrac{O}{\gamma-O}=\tfrac{cN''}{H\theta}=\tfrac{\pi' abcc\gamma}{\alpha\beta(c+\gamma)a\Phi}\ \text{ and }\ \tfrac{O}{\gamma(\gamma-O)}=\tfrac{\pi' abcc}{\alpha\beta\gamma(c+\gamma)}\cdot\tfrac{1}{a\Phi}.$$

COROLLARY 2

316. So that again if as above there may be put $\alpha = Aa,\ \beta = Bb,\ \gamma = Cc$, there will be

$$\frac{O}{\gamma(\gamma - O)} = \frac{\pi'}{ABC(C+1)a\Phi};$$

then truly $bN' = \frac{\pi Bb}{B+1}$ and $CM'' = \frac{\pi' Cc}{C+1}$, and also $G\eta = ABa\Phi$; since now there shall be $bN' + G\eta : bG = CM'' - G\eta : CG,$ there will be $\frac{bN'+G\eta}{bG} = \frac{CM''-G\eta}{CG}$, and thus $\frac{1}{bG} + \frac{1}{CG} = \frac{CM''}{CG\cdot G\eta} - \frac{bN'}{bG\cdot G\eta}$, that is $\frac{1}{\beta} + \frac{1}{c} = \frac{\pi' C}{AB(C+1)a\Phi} - \frac{\pi}{AB(B+1)a\Phi}$;

Indeed in a like manner there is $\frac{1}{\alpha} + \frac{1}{b} = \frac{\pi B}{A(B+1)a\Phi}$.

COROLLARY 3

317. With the same substitutions there becomes

$$\frac{1}{a} + \frac{1}{\alpha} = \frac{A+1}{Aa},\quad \frac{1}{b} + \frac{1}{\beta} = \frac{B+1}{Bb},\quad \frac{1}{c} + \frac{1}{\gamma} = \frac{C+1}{Cc} \text{ etc.}$$

and hence :

$$\alpha\alpha\left(\frac{1}{a} + \frac{1}{\alpha}\right) = A(A+1)a$$
$$\beta\beta\left(\frac{1}{b} + \frac{1}{\beta}\right) = B(B+1)b$$
$$\gamma\gamma\left(\frac{1}{c} + \frac{1}{\gamma}\right) = C(C+1)c$$

and thus so on.

COROLLARY 4

318. Therefore from these new denominations introduced the differentials from § 300 will be expressed thus:

$$d\alpha = -db = -\frac{dn}{n-1}\cdot A(A+1)a$$
$$d\beta = -dc = -BBdb - \frac{dn}{n-1}\cdot B(B+1)b$$
$$d\gamma = -dd = -CCdc - \frac{dn}{n-1}\cdot C(C+1)c,$$

which more convenient formulas will be introduced into the calculation.

Just as here the new forms acquired are going to be used in the following, and thus will be allowed to be used henceforth for the different cases of refraction in the following refractions :

$$\frac{O}{\gamma(\gamma-O)}=\frac{\pi'}{ABC(C+1)a\Phi}$$
$$\frac{1}{\alpha}+\frac{1}{b}=\frac{\pi B}{A(B+1)a\Phi}$$
$$\frac{1}{\beta}+\frac{1}{c}=\frac{\pi' C}{AB(C+1)a\Phi}-\frac{\pi}{AB(B+1)a\Phi}$$
$$\frac{1}{\gamma}+\frac{1}{d}=\frac{\pi'' D}{ABC(D+1)a\Phi}-\frac{\pi'}{ABC(B+1)a\Phi}$$
etc.;

then truly the differential formulas will be

$$d\alpha=-db=-\frac{dn}{n-1}\cdot A\left(A+1\right)a$$
$$d\beta=-dc=B^2d\alpha-\frac{dn'}{n'-1}\cdot B\left(B+1\right)b$$
$$d\gamma=-dd=C^2d\beta-\frac{dn''}{n''-1}\cdot C\left(C+1\right)c$$
$$d\delta=-de=D^2d\gamma-\frac{dn'''}{n'''-1}\cdot D\left(D+1\right)d$$
etc.

And from these formulas, so that the colored margin may vanish, for this equation to become satisfied

$$0=\frac{dn'}{n'-1}\cdot\frac{\pi b}{Aa\Phi}+\frac{dn''}{n''-1}\cdot\frac{\pi' c}{ABa\Phi}.$$

But in order that this confusion may be completely removed, must become

$$d\gamma=-\frac{dn}{n-1}\cdot A(A+1)B^2C^2a-\frac{dn'}{n'-1}\cdot B(B+1)C^2b-\frac{dn''}{n''-1}\cdot C(C+1)c=0$$

or

$$\frac{dn}{n-1}\cdot\frac{(A+1)a}{A}-\frac{dn'}{n'-1}\cdot\frac{(B+1)b}{AAB}+\frac{dn''}{n''-1}\cdot\frac{(C+1)c}{A^2B^2C}=0.$$

PROBLEM 7

319. *If a dioptric instrument depends on four lenses or which the thickness may be ignored, to define that arrangement, so that the eye may be in that place, may observe the object put in place which the apparent field of view demands, without confusion arising from the diverse nature of the rays.*

SOLUTION

With the distance of the object before the instrument put $AE=a$ (Fig. 15) and its magnitude $E\varepsilon=z$, the distances of the images formed from the nature of the mean rays may be called as follows:

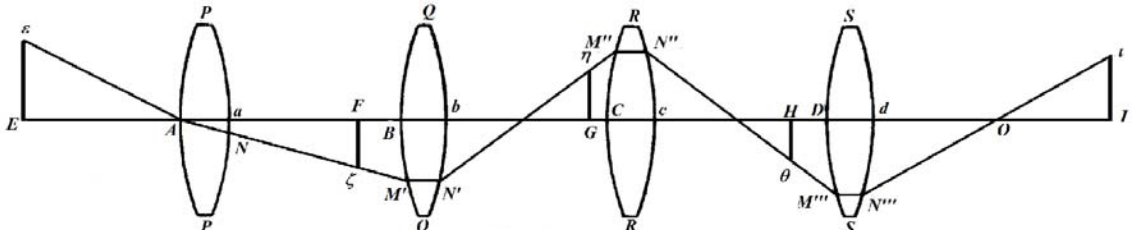

Fig. 15.

$aF = \alpha,\ BF = b,\ bG = \beta,\ CG = c,\ cH = \gamma,\ DH = d,\ \ dI = \delta.$ Then truly we may put in addition :

$$\alpha = Aa,\ \beta = Bb,\ \gamma = Cc,\ \delta = Dd,$$

and the ratios of the apertures for the individual lenses may be introduced after the first, which shall be π for the second *QQ,* π' for the third *RR* and π'' for the fourth *SS,* on assuming $\frac{z}{a} = \Phi$ for the field. With these in place the images will be

$F'\zeta = Aa\Phi,\ G\eta = ABa\Phi,\ \ H\theta = ABCa\Phi$ and $I\iota = ABCDa\Phi = \frac{\alpha\beta\gamma\delta}{abcd}z.$ Now with the eye put after the instrument at the distance $dO = O,$ so that there shall be $OI = \delta - O,$ the tangent of the angle $IO\iota$ will be $\frac{\alpha\beta\gamma\delta}{abcd}\cdot\frac{z}{\delta - O}$, which, since on account of the difference of the refrangibility of the rays must remain unchanged, the differences will give this equation

$$\frac{d\alpha}{\alpha} - \frac{db}{b} + \frac{d\beta}{\beta} - \frac{dc}{c} + \frac{d\gamma}{\gamma} - \frac{dd}{d} - \frac{Od\delta}{\delta(\delta - O)} = 0,$$

which on account of $db = -\,d\alpha,\ dc = -d\beta$ and $dd = -d\gamma$ will be changed into this

$$d\alpha\left(\frac{1}{\alpha} + \frac{1}{b}\right) + d\beta\left(\frac{1}{\beta} + \frac{1}{c}\right) + d\gamma\left(\frac{1}{\gamma} + \frac{1}{d}\right) - \frac{Od\delta}{\delta(\delta - O)} = 0.$$

Truly, in the manner we have found before, to become (§ 318)

$$d\alpha = \frac{-dn}{n-1}\cdot A(A+1)a,\quad d\beta = BBd\alpha - \frac{dn}{n-1}\cdot B(B+1)b,$$
$$d\gamma = CCd\beta - \frac{dn}{n-1}\cdot C(C+1)c,\quad d\delta = DDd\gamma - \frac{dn}{n-1}\cdot D(D+1)d,$$
etc.

where the values assigned for b, c, d of § 266 must be substituted. Now truly I put as we have noticed to be (§ 316)

$$\frac{1}{\alpha}+\frac{1}{b}=\frac{1}{ABa\Phi}\frac{\pi B}{B+1},$$
$$\frac{1}{\beta}+\frac{1}{c}=\frac{1}{ABa\Phi}\left(\frac{\pi' C}{C+1}-\frac{\pi}{B+1}\right),$$
$$\frac{1}{\gamma}+\frac{1}{d}=\frac{1}{ABCa\Phi}\left(\frac{\pi'' D}{D+1}-\frac{\pi'}{C+1}\right)$$

and

$$\frac{O}{\delta(\delta-O)}=\frac{\pi''}{ABCD(D+1)a\Phi}.$$

So that if now the former values may be substituted in the latter, we will have

$$d\alpha=-\frac{dn}{n-1}\cdot A(A+1)a$$
$$d\beta=-\frac{dn}{n-1}\cdot B\big(AB(A+1)a+(B+1)b\big)$$
$$d\gamma=-\frac{dn}{n-1}\cdot C\big(AB^2C(A+1)a+BC(B+1)b+(C+1)c\big)$$
$$d\delta=-\frac{dn}{n-1}\cdot D\big(AB^2C^2D(A+1)a+BC^2D(B+1)b+CD(C+1)c+(D+1)d\big).$$

Therefore with these values substituted into the differential equation and division made by $\frac{dn}{(n-1)a\Phi}$, our equation will be set out by the following individual members

$$-\frac{\pi B}{(B+1)}\cdot(A+1)a$$
$$-\frac{1}{A}\left(\frac{\pi' C}{C+1}-\frac{\pi}{B+1}\right)\big(AB(A+1)a+(B+1)b\big)$$
$$-\frac{1}{AB}\left(\frac{\pi'' D}{D+1}-\frac{\pi'}{C+1}\right)\big(AB^2C(A+1)a+BC(B+1)b+(C+1)c\big)$$
$$+\frac{1}{ABC}\cdot\frac{\pi'' D}{D+1}\big(AB^2C^2D(A+1)a+BC^2D(B+1)b+CD(C+1)c+(D+1)d\big)=0.$$

But the negative terms only of the third member collected together provide

$$-\frac{BCD\pi''(A+1)}{D+1}a-\frac{CD(B+1)\pi''}{A(D+1)}b-\frac{D(C+1)\pi''}{AB(D+1)}c+\frac{\pi}{A}b+\frac{\pi'}{AB}c,$$

to which if the fourth be added, provide

$$\frac{\pi}{A}b+\frac{\pi'}{AB}c+\frac{\pi''}{ABC}d=0.$$

Now here two cases are required to be considered, the one, where the point O falls after the final lens, the other truly, where on account of the negative distance O the eye is applied next to the final lens. For the former case, where the distance $dO=O$ appears positive, this equation itself will be had, if indeed in the equation just found the values for b, c, d found in § 266 may be substituted :

$$\frac{(B+1)\pi}{B\pi-(B+1)\Phi}+\frac{(C+1)\pi'}{C\pi'-(C+1)(\pi-\Phi)}+\frac{(D+1)\pi''}{D\pi''-(D+1)(\pi'-\pi+\Phi)}=0.$$

For the latter case, where the distance O is assumed vanishing, with the multiplication made by $\frac{D+1}{a\Phi}$:

$$\frac{BCD(A+1)\pi''}{\Phi}+\frac{CD(B+1)^2\pi''}{B\pi-(B+1)\Phi}+\frac{D(C+1)^2\pi''}{C\pi'-(C+1)(\pi-\Phi)}$$
$$=\frac{(B+1)(D+1)\pi}{B\pi-(B+1)\Phi}+\frac{(C+1)(D+1)\pi'}{C\pi'-(C+1)(\pi-\Phi)}.$$

In this manner it may be effected, so that an object may appear without a colored fringe ; but with all the confusion removed, if besides there were

$$AB^2C^2D(A+1)+\frac{ABC^2D(B+1)^2\Phi}{B\pi-(B+1)\Phi}+\frac{ABCD(C+1)^2\Phi}{C\pi'-(C+1)(\pi-\Phi)}$$
$$+\frac{ABC(D+1)^2\Phi}{D\pi''-(D+1)(\pi'-\pi+\Phi)}=0.$$

or on dividing by ABC :

$$BCD(A+1)+\frac{CD(B+1)^2\Phi}{B\pi-(B+1)\Phi}+\frac{D(C+1)^2\Phi}{C\pi'-(C+1)(\pi-\Phi)}+\frac{(D+1)^2\Phi}{D\pi''-(D+1)(\pi'-\pi+\Phi)}=0.$$

or

$$BCD(A+1)+\frac{CD(B+1)b}{Aa}+\frac{D(C+1)c}{ABa}+\frac{(D+1)d}{ABCa}=0.$$

COROLLARY 1

320. If it can be able to satisfy each condition, so that neither the position nor the magnitude of any image may be allowed to change, the position of the eye no further enters into the calculation, but the image, in whatever manner it may be determined, in short will be free from all confusion.

COROLLARY 2

321. Therefore so that we may follow this highest order of perfection, it will be required to satisfy these two equations:

$$\frac{(B+1)\pi}{B\pi-(B+1)\Phi}+\frac{(C+1)\pi'}{C\pi'-(C+1)(\pi-\Phi)}+\frac{(D+1)\pi''}{D\pi''-(D+1)(\pi'-\pi+\Phi)}=0$$

and

$$A+1+\frac{(B+1)^2\Phi}{B(B\pi-(B+1)\Phi)}+\frac{(C+1)^2\Phi}{BC(C\pi'-(C+1)(\pi-\Phi))}+\frac{(D+1)^2\Phi}{BCD(D\pi''-(D+1)(\pi'-\pi+\Phi))}=0.$$

COROLLARY 3

322. If again, as we have made before, we may put

$$\frac{A}{A+1}=\mathfrak{A},\ \frac{B}{B+1}=\mathfrak{B},\ \frac{C}{C+1}=\mathfrak{C},\ \text{and}\ \frac{D}{D+1}=\mathfrak{D},$$

we may express the equations by the following simpler :

$$\frac{\pi}{\mathfrak{B}\pi-\Phi}+\frac{\pi'}{\mathfrak{C}\pi'-\pi+\Phi}+\frac{\pi''}{\mathfrak{D}\pi''-\pi'+\pi-\Phi}=0$$

and

$$\frac{1}{\mathfrak{A}}+\frac{\Phi}{A\mathfrak{B}(\mathfrak{B}\pi-\Phi)}+\frac{\Phi}{AB\mathfrak{C}(\mathfrak{C}\pi'-\pi+\Phi)}+\frac{\pi''}{ABC\mathfrak{D}(\mathfrak{D}\pi''-\pi'+\pi-\Phi)}=0;$$

here, if a ratio may be had of different refractions, the first members may be multiplied by $\frac{dn}{n-1}$, the second by $\frac{dn'}{n'-1}$, the third by $\frac{dn''}{n''-1}$ etc.

COROLLARY 4

323. But if it may not be allowed to implement both these equations, it is required so that perhaps the first may be satisfied, from which the apparent magnitude may be freed from the confusion ; for in this manner the object will appear without a colored fringe, where indeed it is agreed with respect of the two cases, just as the distance of the eye dO will be produced positive or negative.

SCHOLIUM

324. Therefore with the ratios of the apertures requiring to be introduced, with which above now we have made the most convenient use in defining the apparent field, also these same equations arising from the diverse refrangibilities of the rays have been removed simply enough, as they may be able to be treated without difficulty, if indeed the thickness of the lenses may be ignored. Therefore in this way the general problem may be agreed to be carried out, whatever were the number of lenses.

SUPPLEMENT V

If the ratio of the refraction may differ in the individual lenses, in the first place the same differential equation appears [§319]

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)+d\beta\left(\frac{1}{\beta}+\frac{1}{c}\right)+d\gamma\left(\frac{1}{\gamma}+\frac{1}{d}\right)-\frac{Od\delta}{\delta(\delta-O)}=0,$$

therefore in the case where $O=0$ the last term must be removed.

I. But if O may have a positive value, there will be as before

$$\frac{O}{\delta(\delta-O)}=\frac{\pi''}{ABCD(D+1)a\Phi}$$

and also the values $\frac{1}{\alpha}+\frac{1}{b},\ \frac{1}{\beta}+\frac{1}{c},\ \frac{1}{\gamma}+\frac{1}{d}$ remain the same as before.

But truly on account of the difference of the refraction we will have :

$$d\alpha=\frac{-dn}{n-1}\cdot A(A+1)a$$
$$d\beta=BBd\alpha-\frac{dn'}{n'-1}\cdot B(B+1)b$$
$$d\gamma=CCd\beta-\frac{dn''}{n''-1}\cdot C(C+1)c$$
$$d\delta=DDd\gamma-\frac{dn'''}{n'''-1}\cdot D(D+1)d.$$

Hence therefore our equation may be formed successively thus :

$$\frac{-O}{\delta(\delta-O)}=\frac{-\pi''d\delta}{ABCD(D+1)a\Phi}=\frac{-\pi''Dd\gamma}{ABC(D+1)a\Phi}+\frac{dn'''}{n'''-1}\cdot\frac{\pi''\cdot d}{ABCa\Phi}.$$

There may be added:

$$d\gamma\left(\frac{1}{\gamma}+\frac{1}{d}\right)=\frac{\pi''Dd\gamma}{ABC(D+1)a\Phi}-\frac{\pi'd\gamma}{ABC(C+1)a\Phi},$$

and there will be produced

$$\frac{-\pi''d\gamma}{ABC(C+1)a\Phi}+\frac{dn'''}{n'''-1}\cdot\frac{\pi''\cdot d}{ABCa\Phi},$$

and with the value substituted for $d\gamma$

$$\frac{-\pi'Cd\beta}{AB(C+1)a\Phi}+\frac{dn''}{n''-1}\cdot\frac{\pi'\cdot c}{ABa\Phi}+\frac{dn'''}{n'''-1}\cdot\frac{\pi''\cdot d}{ABCa\Phi}.$$

Now there may be added :

$$d\beta\left(\tfrac{1}{\beta}+\tfrac{1}{c}\right)=\frac{\pi' Cd\beta}{AB(C+1)a\Phi}-\frac{\pi d\beta}{AB(B+1)a\Phi},$$

and there is produced

$$\frac{-\pi d\beta}{AB(B+1)a\Phi}+\frac{dn''}{n''-1}\cdot\frac{\pi'\cdot c}{AB\cdot a\Phi}+\frac{dn'''}{n'''-1}\cdot\frac{\pi''\cdot d}{ABCa\Phi},$$

and with the value substituted for $d\beta$:

$$\frac{-\pi d\alpha}{A(B+1)a\Phi}+\frac{dn'}{n'-1}\cdot\frac{\pi\cdot b}{Aa\Phi}+\frac{dn''}{n''-1}\cdot\frac{\pi'\cdot c}{AB\cdot a\Phi}+\frac{dn'''}{n'''-1}\cdot\frac{\pi''\cdot d}{ABCa\Phi}.$$

And finally there may be added:

$$d\alpha\left(\tfrac{1}{\alpha}+\tfrac{1}{b}\right)=\frac{\pi Bd\alpha}{A(B+1)a\Phi},$$

and the equation sought, from which the colored margin vanishes, will be:

$$\frac{dn'}{n'-1}\cdot\frac{\pi\cdot b}{Aa\Phi}+\frac{dn''}{n''-1}\cdot\frac{\pi'\cdot c}{AB\cdot a\Phi}+\frac{dn'''}{n'''-1}\cdot\frac{\pi''\cdot d}{ABCa\Phi}=0.$$

II. But if O may have a negative value, then there must be taken $O=0$, and for the same goal the equation will be formed thus :
Since there will be :

$$d\gamma\left(\tfrac{1}{\gamma}+\tfrac{1}{d}\right)=\frac{\pi'' Dd\gamma}{ABC(D+1)a\Phi}-\frac{\pi' d\gamma}{ABC(C+1)a\Phi},$$

with the value $d\gamma$ substituted there becomes :

$$CCd\beta\left(\frac{\pi'' D}{ABC(D+1)a\Phi}-\frac{\pi'}{ABC(C+1)a\Phi}\right)$$
$$-\frac{dn'}{n'-1}\cdot C(C+1)c\left(\frac{\pi'' D}{ABC(D+1)a\Phi}-\frac{\pi'}{ABC(C+1)a\Phi}\right).$$

There may be added:

$$d\beta\left(\tfrac{1}{\beta}+\tfrac{1}{c}\right)=\frac{\pi' Cd\beta}{AB(C+1)a\Phi}-\frac{\pi d\beta}{AB(B+1)a\Phi},$$

and there will become :

$$d\beta\left(\frac{CD\pi''}{AB(D+1)a\Phi}-\frac{\pi}{AB(B+1)a\Phi}\right)-\frac{dn''}{n''-1}\cdot(C+1)c\left(\frac{\pi'' D}{AB(D+1)a\Phi}-\frac{\pi'}{AB(C+1)a\Phi}\right)$$

and with the value of $d\beta$ substituted:

$$d\alpha\left(\frac{BCD\pi''}{A(D+1)a\Phi}-\frac{B\pi}{A(B+1)a\Phi}\right)-\frac{dn'}{n'-1}\cdot\left(\frac{(B+1)CD\pi''}{A(D+1)a\Phi}-\frac{\pi}{Aa\Phi}\right)$$
$$-\frac{dn''}{n''-1}\cdot(C+1)c\left(\frac{\pi''D}{AB(D+1)a\Phi}-\frac{\pi'}{AB(C+1)a\Phi}\right).$$

There may be added :

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)=\frac{\pi Bd\alpha}{A(B+1)a\Phi},$$

and there will become:

$$-\frac{dn}{n-1}\cdot\frac{(A+1)BCDa\cdot\pi''}{(D+1)a\Phi}-\frac{dn'}{n'-1}\cdot\left(\frac{(B+1)CD\pi''-(D+1)b\pi}{A(D+1)a\Phi}\right)$$
$$-\frac{dn''}{n''-1}\cdot\left(\frac{(C+1)Dc\pi''-(D+1)c\pi'}{AB(D+1)a\Phi}\right).$$

Therefore for the case $O=0$ the equation, by which the colored margin is destroyed, will become:

$$0=\frac{adn}{n-1}\cdot(A+1)BCD\pi''+\frac{bdn'}{n'-1}\cdot\left(\frac{(B+1)CD\pi''-(D+1)\pi}{A}\right)$$
$$+\frac{cdn''}{n''-1}\cdot\left(\frac{(C+1)D\pi''-(D+1)\pi'}{AB}\right).$$

III. But so that besides all the confusion may be removed, in addition it will be required to return $d\delta=0$; from which this equation arises :

$$0=\begin{cases}\frac{adn}{n-1}\cdot(A+1)B^2C^2D^2+\frac{bdn'}{n'-1}\cdot(B+1)C^2D^2\\+\frac{cdn''}{n''-1}\cdot(C+1)D^2+\frac{ddn'''}{n'''-1}(D+1),\end{cases}$$

which divided by $A^2B^2C^2D^2$ gives

$$0=\frac{adn}{n-1}\cdot\frac{A+1}{A}+\frac{bdn'}{n'-1}\cdot\frac{B+1}{A^2B}+\frac{cdn''}{n''-1}\cdot\frac{C+1}{A^2B^2C}+\frac{ddn'''}{n'''-1}\frac{D+1}{A^2B^2C^2D}.$$

Concerning this equation it is to be observes especially, if all the lenses shall be established with the same amount of refraction, it is not possible for that to be satisfied at all ; from which here the equation pertains properly to the case, were different refractions are used.

PROBLEM 8

325. *If a dioptric instrument may be constructed from some number of lenses the thickness of which may be ignored, to determine that disposition, so that the eye put into that position, which the field of view demands, may experience no confusion.*

SOLUTION

The distance of the object before the first lens shall be $AE = a$ and its magnitude $E\varepsilon = z$, which indeed may be able to be observed, and there may be put $\frac{z}{a} = \Phi$. Accordingly the distances of the images of the nature formed by the mean rays shall be as above :

$$AE = a,\ BF = b,\ CG = c,\ DH = d,\ EI = e$$
$$aF = \alpha,\ bG = \beta,\ cH = \gamma,\ dI = \delta,\ eK = \varepsilon$$
$$\text{etc.},$$

and for the sake of brevity we may call

$$\alpha = Aa,\ \beta = Bb,\ \gamma = Cc,\ \ \delta = Dd,\ \delta = Ee \quad \text{ete.},$$

then truly also

$$\tfrac{A}{A+1} = \mathfrak{A},\ \tfrac{B}{B+1} = \mathfrak{B},\ \tfrac{C}{C+1} = \mathfrak{C},\ \tfrac{D}{D+1} = \mathfrak{D},\ \tfrac{E}{E+1} = \mathfrak{E}\ \text{ etc.}$$

Now the aperture of the first lens *PP* may be considered as vanishing, the ratio of the aperture for the remaining lenses

$$QQ = \pi,\ RR = \pi',\ SS = \pi'',\ TT = \pi''' \ \text{ etc.}$$

With these in place we have seen (§ 270) besides to become $\alpha = Aa$

$$b = \tfrac{Aa\Phi}{\mathfrak{B}\pi - \Phi} \qquad \beta = \tfrac{ABa\Phi}{\mathfrak{B}\pi - \Phi}$$
$$c = \tfrac{ABa\Phi}{\mathfrak{C}\pi' - \pi + \Phi} \qquad \gamma = \tfrac{ABCa\Phi}{\mathfrak{C}\pi' - \pi + \Phi}$$
$$d = \tfrac{ABCa\Phi}{\mathfrak{D}\pi'' - \pi' + \pi - \Phi} \qquad \delta = \tfrac{ABCDa\Phi}{\mathfrak{D}\pi'' - \pi' + \pi - \Phi}$$
$$e = \tfrac{ABCDa\Phi}{\mathfrak{E}\pi''' - \pi'' + \pi' - \pi + \Phi} \qquad \varepsilon = \tfrac{ABCDEa\Phi}{\mathfrak{E}\pi''' - \pi'' + \pi' - \pi + \Phi}$$
$$\text{etc.} \qquad \text{etc.}$$

Then the magnitude itself of the individual images thus will be had :

$$F\zeta = Aa\Phi,\ G\eta = ABa\Phi,\ H\theta = ABCa\Phi,\ I\iota = ABCDa\Phi \quad \text{etc.}$$

and the radii of the apertures :

Second lens $QQ = \frac{A\mathfrak{B}a\Phi}{\mathfrak{B}\pi-\Phi}\pi$

third lens $RR = \frac{AB\mathfrak{C}a\Phi}{\mathfrak{C}\pi'-\pi+\Phi}\pi'$

fourth lens $SS = \frac{ABC\mathfrak{D}a\Phi}{\mathfrak{D}\pi''-\pi'+\pi-\Phi}\pi''$

fifth lens $TT = \frac{ABCD\mathfrak{E}a\Phi}{\mathfrak{E}\pi'''-\pi''+\pi'-\pi+\Phi}\pi'''$

etc.

Now with the change of the refraction law $n:1$ by infinitely small parts of $\alpha,\ b,\ \beta,\ c,\ \gamma\ \ d,\ \delta$ etc. such changes are received :

$$d\alpha = \frac{-dn}{n-1}\cdot Aa\left(A+1\right)$$
$$d\beta = \frac{-dn}{n-1}\cdot ABa\left(\left(A+1\right)B + \frac{(B+1)\Phi}{\mathfrak{B}\pi-\Phi}\right)$$
$$d\gamma = \frac{-dn}{n-1}\cdot ABCa\left(\left(A+1\right)BC + \frac{(B+1)C\Phi}{\mathfrak{B}\pi-\Phi} + \frac{(C+1)\Phi}{\mathfrak{C}\pi'-\pi+\Phi}\right)$$
$$d\delta = \frac{-dn}{n-1}\cdot ABCDa\left(\left(A+1\right)BCD + \frac{(B+1)CD\Phi}{\mathfrak{B}\pi-\Phi} + \frac{(C+1)D\Phi}{\mathfrak{C}\pi'-\pi+\Phi} + \frac{(D+1)\Phi}{\mathfrak{D}\pi''-\pi'+\pi-\Phi}\right).$$

Truly again there is had:

$$\frac{1}{\alpha}+\frac{1}{b} = \frac{1}{Aa\Phi}\mathfrak{B}\pi,$$
$$\frac{1}{\beta}+\frac{1}{c} = \frac{1}{ABa\Phi}\left(\mathfrak{C}\pi' - \frac{\mathfrak{B}\pi}{B}\right),$$
$$\frac{1}{\gamma}+\frac{1}{d} = \frac{1}{ABCa\Phi}\left(\mathfrak{D}\pi'' - \frac{\mathfrak{C}\pi'}{C}\right),$$
$$\frac{1}{\delta}+\frac{1}{e} = \frac{1}{ABCDa\Phi}\left(\mathfrak{E}\pi''' - \frac{\mathfrak{C}\pi''}{D}\right)$$

etc.

With these in place for any number of lenses , we may set out the number of satisfying formulas sought separately with the distance of the eye after the final lens put $=O$:

I. For a single lens

This differential equation is had $\frac{-Od\alpha}{\alpha(\alpha-O)} = 0$, for which case both the magnitude of the image as well as its position shall remain fixed, if there were $d\alpha = 0$, that is $A(A+1) = 0$, from which there must be either $A = 0$ or $A = -1$, of which the former does not permit vision, but the latter lens removes it. Then truly on that account the field must be $O = 0$.

II. For two lenses

This differential equation is had, by which the colored fringe is removed :

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{a}\right)-\frac{Od\beta}{\beta(\beta-O)}=0;$$

but on account of the apparent field of view there is $\frac{O}{\beta(\beta-O)}=\frac{1}{BB}\left(\frac{1}{\alpha}+\frac{1}{b}\right)$, thus so that we may have:

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)-\frac{d\beta}{BB}\left(\frac{1}{\alpha}+\frac{1}{b}\right)=0.$$

Truly there is :

$$d\beta=BBd\alpha-\frac{dn}{n-1}\cdot ABa\cdot\frac{(B+1)\Phi}{\mathfrak{B}\pi-\Phi}.$$

Whereby if O may have a positive value, there will be on account of $\mathfrak{B}(\mathrm{B}+1)=B$

$$\frac{\pi}{\mathfrak{B}\pi-\Phi}=0.$$

But if the value O may become negative, in which case there is taken $O=0$, there will be :

$$\frac{(A+1)\mathfrak{B}\pi}{\Phi}=0.$$

Moreover all the confusion will be removed completely, if in addition there were :

$$\frac{1}{\mathfrak{A}}+\frac{\Phi}{A\mathfrak{B}(\mathfrak{B}\pi-\Phi)}=0.$$

III. For three lenses

If we may pursue the calculation in the same manner, an object will be seen without a colored fringe:

1. If the distance O may be produced positive from the apparent field of view, it will be required for this equation to be satisfied:

$$\frac{\pi}{\mathfrak{B}\pi-\Phi}+\frac{\pi'}{\mathfrak{C}\pi'-\pi+\Phi}=0.$$

2. If on account of the distance O arising negative there may be taken $O=0$, for this equation requiring to be satisfied there will be :

$$\frac{\pi'}{\mathfrak{A}\Phi}+\frac{\pi'}{A\mathfrak{B}(\mathfrak{B}\pi-\Phi)}=\frac{\pi}{ABC(\mathfrak{B}\pi-\Phi)}.$$

But all the confusion will be removed completely, if in addition there were :

$$\frac{1}{\mathfrak{A}}+\frac{\Phi}{A\mathfrak{B}(\mathfrak{B}\pi-\Phi)}+\frac{\Phi}{AB\mathfrak{C}(\mathfrak{C}\pi'-\pi+\Phi)}=0.$$

IV. For four lenses

So that the object may be seen without a colored fringe :

1. If the distance O may be produced positive from the apparent field of view, it will be required for this equation to be satisfied:

$$\frac{\pi}{\mathfrak{B}\pi-\Phi}+\frac{\pi'}{\mathfrak{C}\pi'-\pi+\Phi}+\frac{\pi''}{\mathfrak{D}\pi''-\pi'+\pi-\Phi}=0.$$

2. But if there may be taken $O=0$, for this :

$$\frac{\pi''}{\mathfrak{A}\Phi}+\frac{\pi''}{A\mathfrak{B}(\mathfrak{B}\pi-\Phi)}+\frac{\pi''}{AB\mathfrak{C}(\mathfrak{C}\pi'-\pi+\Phi)}=\frac{\pi}{ABC\mathfrak{D}(\mathfrak{B}\pi-\Phi)}+\frac{\pi'}{ABC\mathfrak{D}(\mathfrak{C}\pi'-\pi+\Phi)}.$$

Truly with all the confusion removed completely, if there were besides :

$$\frac{1}{\mathfrak{A}}+\frac{\Phi}{A\mathfrak{B}(\mathfrak{B}\pi-\Phi)}+\frac{\Phi}{AB\mathfrak{C}(\mathfrak{C}\pi'-\pi+\Phi)}+\frac{\Phi}{ABC\mathfrak{D}(\mathfrak{D}\pi''-\pi'+\pi-\Phi)}=0.$$

V. For five lenses

So that the object may be seen only without a colored fringe :

1. If the distance O may be produced positive from the apparent field, it will be sufficient for the equation to be satisfied:

$$\frac{\pi}{\mathfrak{B}\pi-\Phi}+\frac{\pi'}{\mathfrak{C}\pi'-\pi+\Phi}+\frac{\pi''}{\mathfrak{D}\pi''-\pi'+\pi-\Phi}+\frac{\pi'''}{\mathfrak{E}\pi'''-\pi''+\pi'-\pi+\Phi}=0.$$

2. But if there may be taken $O=0$, to this :

$$\frac{\pi'''}{\mathfrak{A}\Phi}+\frac{\pi'''}{A\mathfrak{B}(\mathfrak{B}\pi-\Phi)}+\frac{\pi'''}{AB\mathfrak{C}(\mathfrak{C}\pi'-\pi+\Phi)}+\frac{\pi'''}{ABC\mathfrak{D}(\mathfrak{D}\pi''-\pi'+\pi-\Phi)}$$
$$=\frac{1}{ABCD\mathfrak{E}}\left(\frac{\pi}{\mathfrak{B}\pi-\Phi}+\frac{\pi'}{\mathfrak{C}\pi'-\pi+\Phi}+\frac{\pi''}{\mathfrak{D}\pi''-\pi'+\pi-\Phi}\right).$$

Moreover all the confusion will be removed completely, if in addition for this equation to be satisfied :

$$\frac{1}{\mathfrak{A}}+\frac{\Phi}{A\mathfrak{B}(\mathfrak{B}\pi-\Phi)}+\frac{\Phi}{AB\mathfrak{C}(\mathfrak{C}\pi'-\pi+\Phi)}+\frac{\Phi}{ABC\mathfrak{D}(\mathfrak{D}\pi''-\pi'+\pi-\Phi)}$$
$$+\frac{\Phi}{ABCE\mathfrak{E}(\mathfrak{E}\pi'''-\pi''+\pi'-\pi+\Phi)}=0.$$

And hence the progression to a greater number of terms is evident.

COROLLARY 1

326. Therefore in the case of a single lens although indeed it may be freed from the colored fringe truly it cannot be freed completely from the confusion. But in the case of two lenses indeed not only may the colored fringe can be removed, but if the eye may be held in that place, which the condition of the apparent field postulates, it may be freed from the confusion.

COROLLARY 2

327. Just as truly if more than two lenses may be had, a sufficient number of quantities may be present, of which not only for the colored fringe to be removed, but also perhaps all the confusion may be seen to be taken away completely, especially if the number of lenses may exceed three.

SCHOLIUM

328. Therefore just as the serious or greatest inconvenience arising from the diverse nature of the rays as has indeed been observed Newton, that it may be judged that dioptric instruments cannot be freed from these in any way; that indeed at least, which concerns the colored fringe, which Newton was considering especially, now happily certainly can be removed in a satisfactory manner, thus so that at any rate on this account there shall be no need to depart to reflecting telescopes. But with this fault removed if in addition we may block off the other source of confusion, evidently lenses bearing no confusion will be required to be used, so that there is no doubt, why dioptric instruments may not be able to reach the highest degree of perfection. Which therefore at this point we have proposed small changes concerning the uses of these instruments, that it may be agreed to gather together, so that in the following chapter the general precepts for the general construction of all dioptric instruments is seen to be treated .

SUPPLEMENT VI

From these , which have been added before, we will be able also to show the solution of the problem for the case, where the individual lenses have been endowed with a particular refraction, where indeed only the latter equations demand a certain change for avoiding the confusion; yet meanwhile also we may represent the former formulas more carefully, by which the location of the eye may be determined, which the apparent field of view requires.

I. *The distance of the eye after the final lens* for any number of lenses will itself be had, as follows:

Number of lenses	O is the distance of the eye past the final lens
I	0
II	$\frac{A\mathfrak{B}a\pi\Phi}{(\pi-\Phi)(\mathfrak{B}\pi-\Phi)}$ or $\frac{\mathfrak{B}b\pi}{(\pi-\Phi)}$
III	$\frac{ABa\mathfrak{C}\pi'\Phi}{(\pi'-\pi+\Phi)(\mathfrak{C}\pi'-\pi+\Phi)}$ or $\frac{\mathfrak{C}c\pi'}{(\pi'-\pi+\Phi)}$
IV	$\frac{ABC\mathfrak{D}a\pi''\Phi}{(\pi''-\pi'+\pi-\Phi)(\mathfrak{D}\pi''-\pi'+\pi-\Phi)}$ or $\frac{\mathfrak{D}d\pi''}{(\pi''-\pi'+\pi-\Phi)}$
V	$\frac{ABCD\mathfrak{E}a\pi'''\Phi}{(\pi'''-\pi''+\pi'-\pi+\Phi)(\mathfrak{E}\pi'''-\pi''+\pi'-\pi+\Phi)}$ or $\frac{\mathfrak{E}e\pi'''}{(\pi'''-\pi''+\pi'-\pi+\Phi)}$

II. If the value of O may be positive, for the colored fringe being removed the following equations are required to be fulfilled:

Number of lenses	
I	$O=0$
II	$0=\frac{bdn'}{n'-1}\cdot\frac{\pi}{Aa\Phi}$
III	$0=\frac{bdn'}{n'-1}\cdot\frac{\pi}{Aa\Phi}+\frac{cdn''}{n''-1}\cdot\frac{\pi'}{ABa\Phi}$
IV	$0=\frac{bdn'}{n'-1}\cdot\frac{\pi}{Aa\Phi}+\frac{cdn''}{n''-1}\cdot\frac{\pi'}{ABa\Phi}+\frac{ddn'''}{n'''-1}\cdot\frac{\pi''}{ABCa\Phi}$

III. If the value of O may be produced negative, in which case there must be taken $O=0$, for the colored fringe being removed the following equations are required to be fulfilled:

Number of lenses	
I	$O=0$
II	$0=\frac{adn}{n-1}(A+1)B\pi$
III	$0=\frac{adn}{n-1}(A+1)BC\pi'+\frac{bdn'}{n'-1}\frac{(B+1)C\pi'-(C+1)\pi}{A}$
IV	$0=\frac{adn}{n-1}(A+1)BCD\pi''+\frac{bdn'}{n'-1}\frac{(B+1)CD\pi''-(D+1)\pi}{A}$ $+\frac{Cdn''}{n''-1}\frac{(C+1)D\pi''-(D+1)\pi'}{AB}$
	etc.

IV. But so that in addition all the confusion of this kind may be removed , the following equations are required to be fulfilled :

Number of lenses	
I	$0=\frac{adn}{n-1}\cdot\frac{A+1}{A}$
II	$0=\frac{adn}{n-1}\cdot\frac{A+1}{A}+\frac{bdn'}{n'-1}\cdot\frac{B+1}{A^2B}$
III	$0=\frac{adn}{n-1}\cdot\frac{A+1}{A}+\frac{bdn'}{n'-1}\cdot\frac{B+1}{A^2B}+\frac{cdn''}{n''-1}\cdot\frac{C+1}{A^2B^2C}$
IV	$0=\frac{adn}{n-1}\cdot\frac{A+1}{A}+\frac{bdn'}{n'-1}\cdot\frac{B+1}{A^2B}+\frac{cdn''}{n''-1}\cdot\frac{C+1}{A^2B^2C}+\frac{ddn'''}{n'''-1}\cdot\frac{D+1}{A^2B^2C^2D}$
	etc.

The order of which formulas hence may be seen more clearly than in the problem.

CAPUT VI

DE CONFUSIONE A DIVERSA RADIORUM INDOLE ORIUNDA

PROBLEMA 1

287. *Si a puncto dato E* (Fig. 1) *radii per lentem PP transmittantur, definire variationem in loco imaginis F, quae a diversa radiorum refrangibilitate oritur.*

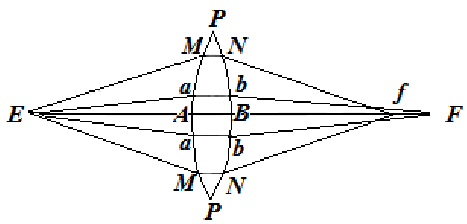

Fig. 1

SOLUTIO

Sit distantia puncti E ante lentem $AE = a$, facierum autem lentis radius anterioris $= f$, posterioris $= g$ et crassities $Aa = v$, quae quantitates sunt constantes. Posita nunc refractionis ratione ex aere in vitrum $= n:1$, ob diversam radiorum naturam numerus n erit variabilis, ideoque etiam locus imaginis F post lentem expressae, cuius distantia si ponatur $aF = \alpha$, erit ex supra inventis

$$\frac{n-1}{f} = \frac{1}{a} + \frac{2n}{k+v} \text{ et } \frac{n-1}{g} = \frac{1}{\alpha} - \frac{2n}{k-v}$$

ubi quantitas k etiam pro variabili est habenda, quia tantum a, f, g et v sunt constantes. Quaestio ergo huc redit, ut, si numerus n differentiali suo dn crescere sumatur, definiatur differentiale distantiae α. Quare differentientur ambae aequationes illae:

$$\frac{dn}{f} = \frac{2dn}{k+v} - \frac{2ndk}{(k+v)^2}, \quad \frac{dn}{g} = \frac{-d\alpha}{\alpha\alpha} - \frac{2dn}{k-v} + \frac{2ndk}{(k+v)^2},$$

indeque eliminato dk habebitur

$$\frac{dn(k+v)^2}{f} + \frac{dn(k-v)^2}{g} = 2dn(k+v) - 2dn(k-v) - \frac{d\alpha(k-v)^2}{\alpha\alpha}.$$

Restituantur pro f et g valores initio positi, ac pervenietur ad hanc aequationem:

$$\frac{dn(k+v)^2}{a}+\frac{dn(k-v)^2}{\alpha}+4vdn+\frac{(n-1)d\alpha}{\alpha\alpha}(k-v)^2=0,$$

unde reperitur

$$\frac{d\alpha}{\alpha\alpha}=\frac{-dn}{(n-1)a}\left(\frac{k+v}{k-v}\right)^2-\frac{dn}{(n-1)\alpha}-\frac{4vdn}{(n-1)(k-v)^2},$$

seu

$$d\alpha=\frac{-\alpha dn}{n-1}\left(1+\frac{\alpha}{a}\left(\frac{k+v}{k-v}\right)^2+\frac{4\alpha v}{(k-v)^2}\right).$$

Tum vero, cum etiam k sit quantitas variabilis, erit

$$dk=\frac{-(k-v)dn}{n(n-1)}\left(1+\frac{k+v}{2a}\right).$$

Cum ergo posuerimus $\frac{k+v}{k-v}=i$, ob $di=\frac{2vdk}{(k+v)^2}$ erit

$$di=\frac{-vdn}{n(n-1)}\left(\frac{1}{a}+\frac{2}{k+v}\right).$$

COROLLARIUM 1

288. Si $n:1$ denotet rationem refractionis radiorum mediae naturae, ut sit $n=\frac{31}{20}=1,55$, erit pro radiis rubris seu minime refractis $n=1,54$, et violaceis $n=1,56$, quorum valorum discrimen a medio, cum sit $=\frac{1}{100}$, pro differentiali *dn* haberi poterit.

COROLLARIUM 2

289. Quare si α denotet distantiam imaginis a radiis mediis formatae, pro ea, quae a rubris formatur, erit $dn=\frac{1}{100}$ et $\frac{dn}{n-1}=\frac{-1}{55}$. Hinc distantia imaginis rubrae post lentem erit

$$\alpha+\frac{\alpha}{55}\left(1+\frac{\alpha}{a}\left(\frac{k+v}{k-v}\right)^2+\frac{4\alpha v}{(k-v)^2}\right).$$

Distantia autem imaginis violaceae post lentem erit

$$\alpha-\frac{\alpha}{55}\left(1+\frac{\alpha}{a}\left(\frac{k+v}{k-v}\right)^2+\frac{4\alpha v}{(k-v)^2}\right).$$

COROLLARIUM 3

290. Si crassities lentis evanescat, ut sit $v=0$, ob variabilitatem numeri n erit

$$d\alpha=\frac{-\alpha dn}{n-1}\left(1+\frac{\alpha}{a}\right)=\frac{-\alpha\alpha dn}{n-1}\left(\frac{1}{a}+\frac{1}{\alpha}\right).$$

Ac si distantia focalis lentis ponatur $=p$, cum sit $\frac{1}{a}+\frac{1}{\alpha}=\frac{1}{p}$, erit

$$d\alpha=\frac{-\alpha\alpha dn}{(n-1)p}=\frac{-20\alpha\alpha dn}{11p}.$$

SCHOLION

291. Hoc ergo modo ob diversam radiorum naturam valor distantiae α immutatur, unde, si ab ea tanquam ab obiecto radii porro ad lentem secundam emittantur, fiet etiam respectu huius lentis distantia obiecti variabilis. Quaro ob causam in loco imaginis ab ea formatae duplex variatio orietur: id quod deinceps etiam in lentibus sequentibus multo magis eveniet. Hanc igitur variationem, quae pro quavis lente in loco imaginis nascitur, in sequente problemate determinemus.

PROBLEMA 2

292. *Si locus imaginis F* (Fig. 5), *quae respectu lentis QQ vicem obiecti gerit, ob diversam radiorum naturam ipse sit variabilis, determinare variationem, quam ob eandem causam imago sequens in G patietur.*

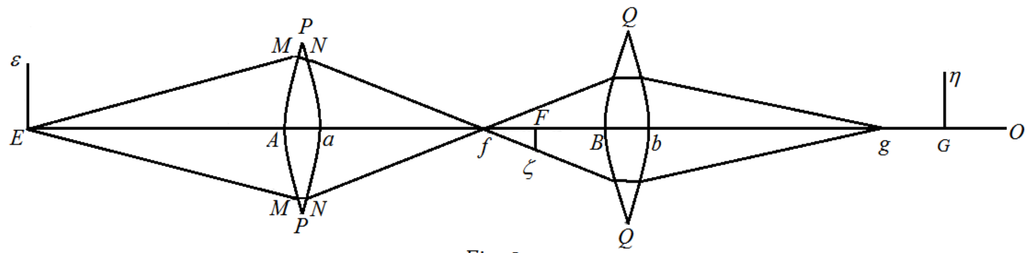

Fig. 5

SOLUTIO

Sit pro radiis mediae naturae, quibus respondet numerus n, distantia obiecti F ante lentem $BF=b$ imaginisque inde proiectae distantia post lentem $bG=\beta$; dum autem n abit in $n+dn$, hae distantiae ambae b et β capiant sua incrementa differentialia db et $d\beta$. Ad quae invenienda sit lentis QQ radius faciei anterioris $=f$, posterioris $=g$ et crassities $Bb=v$, eritque ut ante:

$$\frac{n-1}{f}=\frac{1}{b}+\frac{2n}{k+v} \text{ et } \frac{n-1}{g}=\frac{1}{\beta}-\frac{2n}{k-v},$$

ubi k cum b et β pro variabili est habenda. differentiatione ergo instituta habebitur

$$\frac{dn}{f}=\frac{-db}{bb}+\frac{2dn}{k+v}-\frac{2ndk}{(k+v)^2}, \quad \frac{dn}{g}=-\frac{d\beta}{\beta\beta}-\frac{2dn}{k-v}+\frac{2ndk}{(k-v)^2},$$

unde eliminato dk fit

$$\frac{dn(k+v)^2}{f}+\frac{dn(k-v)^2}{g}=\frac{-db}{bb}(k+v)^2-\frac{d\beta}{\beta\beta}(k-v)^2+4vdn,$$

quae multiplicata per $n-1$, si pro f et g valores dati substituantur, prodit

$$\frac{dn(k+v)^2}{b}+2ndn(k+v)+\frac{dn(k-v)^2}{\beta}-2ndn(k-v)$$
$$=-\frac{(n-1)db}{bb}(k+v)^2-\frac{(n-1)d\beta}{\beta\beta}(k-v)^2+4(n-1)vdn$$

seu

$$\frac{dn(k+v)^2}{(n-1)b}+\frac{dn(k-v)^2}{(n-1)\beta}+\frac{4vdn}{n-1}+\frac{db}{bb}(k+v)^2+\frac{d\beta}{\beta\beta}(k-v)^2=0.$$

Atque hinc elicitur:

$$d\beta=\frac{-\beta\beta db}{bb}\left(\frac{k+v}{k-v}\right)^2-\frac{\beta dn}{(n-1)}\left(1+\frac{\beta}{b}\left(\frac{k+v}{k-v}\right)^2+\frac{4\beta v}{(k-v)^2}\right).$$

Pro variabilitate autem ipsius k reperietur

$$\frac{dk}{(k+v)^2}=-\frac{db}{2nbb}-\frac{dn}{2n(n-1)b}-\frac{dn}{2n(n-1)(k+v)}.$$

Quare si ponatur $\frac{k-v}{k+v}=i$, ob $di=\frac{2vdk}{(k+v)^2}$ erit

$$di=\frac{-vdb}{nbb}-\frac{vdn}{n(n-1)b}-\frac{2vdn}{n(n-1)(k+v)},$$

ac si loco k numerus i introducatur, erit

$$d\beta=\frac{-\beta\beta db}{iibb}-\frac{\beta dn}{n-1}\left(1+\frac{\beta}{iib}+\frac{(1-i)^2\beta}{iiv}\right) \text{ et } di=\frac{-vdb}{nbb}-\frac{vdn}{n(n-1)}\left(\frac{1}{b}+\frac{1-i}{v}\right).$$

COROLLARIUM 1

293. Inventa aequatio differentialis etiam hac forma repraesentari potest, ut sit

$$\frac{id\beta}{\beta\beta}+\frac{db}{ibb}=\frac{-dn}{n-1}\left(\frac{i}{\beta}+\frac{1}{ib}+\frac{(1-i)^2}{iv}\right)$$

sive restituendo k

$$\frac{d\beta}{\beta\beta}\left(\frac{k-v}{k+v}\right)+\frac{db}{bb}\left(\frac{k+v}{k-v}\right)=\frac{-dn}{n-1}\left(\frac{1}{\beta}\left(\frac{k-v}{k+v}\right)+\frac{1}{b}\left(\frac{k+v}{k-v}\right)+\frac{4v}{kk-vv}\right),$$

ubi haec observanda est analogia, ut quemadmodum ad b refertur $\frac{k+v}{k-v}$, ita ad β referatur $\frac{k-v}{k+v}$.

COROLLARIUM 2

294. Si lentis huius crassities evanescat, fit $v=0$ et $i=1$, ubi figura lentis non amplius in computum ingreditur, sed sola distantia focalis, unde variatio in loco imaginis G ita erit comparata, ut sit

$$\frac{d\beta}{\beta\beta}+\frac{db}{bb}=\frac{-dn}{n-1}\left(\frac{1}{\beta}+\frac{1}{b}\right)$$

ideoque

$$d\beta=\frac{-\beta\beta}{bb}db-\frac{\beta\beta dn}{n-1}\left(\frac{1}{\beta}+\frac{1}{b}\right).$$

COROLLARIUM 3

295. Si casum a radiis mediae naturae, ad quos formulae hactenus traditae sunt accommodatae, ad radios rubros transferre velimus, poni oportet $dn=-\frac{1}{100}$, sin autem ad radios violaceos, $dn=+\frac{1}{100}$.

PROBLEMA 3

296. *Si radii ab obiecto E per lentes quotcunque transmittantur, determinare variationem in locis singularum imaginum, quae a diversa radiorum refrangibilitate proficiscitur.*

SOLUTIO

Retineantur omnes denominationes, quibus in superioribus capitibus sumus usi, ac sint $a,\ \alpha,\ b,\ \beta,\ c,\ \gamma$ etc. distantiae determinatrices lentium pro radiis mediae naturae. Variata ergo ratione refractionis etiam hae distantiae variabuntur, quarum variationes differentialibus indicemus. Cum autem distantiae inter binas lentes maneant constantes, illae variationes ita erunt comparatae, ut sit

$$d\alpha+db=0,\ \ d\beta+dc=0,\ \ d\gamma+dd=0\quad \text{etc.}$$

Cum iam distantia obiecti $AE = a$ sit invariabilis, erit ex problemate primo, si ponatur $\frac{k-v}{k+v} = i$,

$$d\alpha = -db = \frac{-\alpha\alpha dn}{i(n-1)}\left(\frac{i}{\alpha} + \frac{1}{ia} + \frac{(1-i)^2}{iv}\right) = \frac{-\alpha\alpha dn}{n-1}\cdot\frac{k+v}{k-v}\left(\frac{k-v}{\alpha(k+v)} + \frac{k+v}{a(k-v)} + \frac{4v}{kk-vv}\right)$$

et

$$di = \frac{-vdn}{n(n-1)}\left(\frac{1}{a} + \frac{1-i}{v}\right).$$

Deinde pro secunda lente, ad quam referuntur distantiae determinatrices b et β cum arbitraria k' et crassitie v', unde fecimus $\frac{k'-v'}{k'+v'} = i'$ habebimus

$$d\beta = -dc = \frac{-\beta\beta db}{i'i'bb} - \frac{\beta\beta dn}{i'(n-1)}\left(\frac{i'}{\beta} + \frac{1}{i'b} + \frac{(1-i')^2}{i'v'}\right)$$

et

$$di' = \frac{-v'db}{nbb} - \frac{v'dn}{n(n-1)}\left(\frac{1}{b} + \frac{1-i'}{v'}\right).$$

Simili modo pro tertia lente, ad quam referuntur distantiae determinatrices c et γ cum arbitraria k'' et crassitie v'', posito $\frac{k''-v''}{k''+v''} = i''$ adipiscemur

$$d\gamma = -dd = \frac{-\gamma\gamma dc}{i''i''cc} - \frac{\gamma\gamma dn}{i''(n-1)}\left(\frac{i''}{\gamma} + \frac{1}{i''c} + \frac{(1-i'')^2}{i''v''}\right)$$

et

$$di'' = \frac{-v''dc}{ncc} - \frac{v''dn}{n(n-1)}\left(\frac{1}{c} + \frac{1-i''}{v''}\right),$$

atque ulterius progrediendo obtinebimus sequentes formulas:

$$d\delta = -de = \frac{-\delta\delta dd}{i'''i'''dd} - \frac{\delta\delta dn}{i'''(n-1)}\left(\frac{i'''}{\delta} + \frac{1}{i''d} + \frac{(1-i''')^2}{i'''v'''}\right)$$

et

$$di''' = \frac{-v''dd}{ndd} - \frac{v'''dn}{n(n-1)}\left(\frac{1}{d} + \frac{1-i'''}{v'''}\right),$$

unde haec differentialia facile ad quotcunque lentes extenduntur. Atque si hic successive valores differentialium *dd, de, db* iam ante definiti substituantur, omnia haec differentialia tam distantiarum determinatricium quam numerorum i, i', i'', i''' etc. per differentiale *dn* exprimentur.

Si ratio refractionis pro singulis lentibus sit diversa pro iisque ordine exprimatur numeris n, n', n'' etc., perspicuum est differentialia hic inventa sequenti modo expressum iri:

I. $$d\alpha = -db = \frac{-\alpha\alpha dn}{i(n-1)}\left(\frac{i}{\alpha}+\frac{1}{ia}+\frac{(1-i)^2}{iv}\right) = \frac{-\alpha\alpha dn}{n-1}\cdot\frac{k+v}{k-v}\left(\frac{k-v}{\alpha(k+v)}+\frac{k+v}{a(k+v)}+\frac{4v}{kk-vv}\right)$$

$$di = \frac{-vdn}{n(n-1)}\left(\frac{1}{a}+\frac{1-i}{v}\right).$$

II. $$d\beta = -dc = \frac{-\beta\beta db}{i'i'bb} - \frac{\beta\beta dn}{i'(n-1)}\left(\frac{i'}{\beta}+\frac{1}{i'b}+\frac{(1-i')^2}{i'v'}\right)$$

$$di' = \frac{-v'db}{nbb} - \frac{v'dn}{n(n-1)}\left(\frac{1}{b}+\frac{1-i'}{v'}\right).$$

III. $$d\gamma = -dd = \frac{-\gamma\gamma dc}{i''i''cc} - \frac{\gamma\gamma dn''}{i''(n''-1)}\left(\frac{i''}{\gamma}+\frac{1}{i''c}+\frac{(1-i'')^2}{i''v''}\right)$$

$$di'' = \frac{-v''dc}{n''cc} - \frac{v''dn''}{n''(n''-1)}\left(\frac{1}{c}+\frac{1-i''}{v''}\right)$$
etc.,

ex quibus formulis etiam mutationes singularum imaginum ac proinde etiam tandem ultimae imaginis facile definiri poterunt, seu potius loco imaginum angulos, sub quibus eae oculo ad iustam distantiam *l* positae sint adpariturae, considereremus:

Pro una lente $\frac{1}{i}\cdot\frac{\alpha}{a}\cdot\frac{z}{l}$

pro duabus lentibus $\frac{1}{ii'}\cdot\frac{\alpha\beta}{ab}\cdot\frac{z}{l}$

pro tribus lentibus $\frac{1}{ii'i''}\cdot\frac{\alpha\beta\gamma}{abc}\cdot\frac{z}{l}$

etc.,

quatenus scilicet praeter distantias $a,\ \alpha;\ b,\ \beta;\ c,\ \gamma$ etc. etiam litterae $i,\ i',\ i'',\ i'''$ etc. sunt variabiles.

COROLLARIUM 1

297. Hinc igitur per differentiationem definira licet, quanta mutatio in loco ultimae imaginis, quae obiectum visionis constituit, ob diversam radiorum refrangibilitatem oriri debeat.

COROLLARIUM 2

298. Deinde cum etiam magnitudinem cuiusque imaginis supra per distantias determinatrices et numeros $i,\ i',\ i''$ etc. definiverimus, pro magnitudine imaginis habetur $\frac{1}{ii'i''i'''}\cdot\frac{\alpha\beta\gamma\delta}{abcd}\cdot\frac{z}{l}$ (§ 189); simili modo mutatio assignari potest, quam magnitudo ultimae imaginis ob diversam refrangibilitatem radiorum patietur.

COROLLARIUM 3

299. Cognita autem utraque mutatione, quam ultima imago tam respectu loci quam magnitudinis subit, non difficulter colligetur, quanta confusione ipsa visio ob diversam radiorum refrangibilitatem perturbetur.

COROLLARIUM 4

300. Si crassities lentium evanescat, fiet

$$d\alpha = -db = \frac{-\alpha\alpha dn}{(n-1)}\left(\frac{1}{a}+\frac{1}{\alpha}\right)$$
$$d\beta = -dc = \frac{-\beta\beta db}{bb} - \frac{\beta\beta dn}{n-1}\left(\frac{1}{b}+\frac{1}{\beta}\right)$$
$$d\gamma = -dd = \frac{-\gamma\gamma dc}{cc} - \frac{\gamma\gamma dn}{n-1}\left(\frac{1}{c}+\frac{1}{\gamma}\right)$$
$$d\delta = -de = \frac{-\delta\delta dd}{dd} - \frac{\delta\delta dn}{n-1}\left(\frac{1}{d}+\frac{1}{\delta}\right)$$
etc.,

numeri autem i, i', i'' etc. abeunt in unitatem nullique mutationi amplius sunt obnoxii.
Si ergo crassities lentium evanescat, pro diversa refractione singularum lentium formulae superiores abibunt in sequentes:

$$\text{I.}\quad d\alpha = -db = \frac{-\alpha\alpha dn}{(n-1)}\left(\frac{1}{a}+\frac{1}{\alpha}\right)$$
$$\text{II.}\quad d\beta = -dc = \frac{-\beta\beta db}{bb} - \frac{\beta\beta dn'}{n'-1}\left(\frac{1}{b}+\frac{1}{\beta}\right)$$
$$\text{III.}\quad d\gamma = -dd = \frac{-\gamma\gamma dc}{cc} - \frac{\gamma\gamma dn''}{n''-1}\left(\frac{1}{c}+\frac{1}{\gamma}\right)$$
etc.,

Ceterum per se manifestum est, quando § 288 dn vel $+\frac{1}{100}$ vel $-\frac{1}{100}$ significare dicitur, id tantum de illa vitri specie, pro qua est refractio radiorum mediorum $n=\frac{31}{20}$, esse intelligendum, et pro aliis vitri speciebus differentialia dn', dn'', dn''' etc. haud mediocriter ab $\frac{1}{100}$ discrepare posse. Quanta autem futura sit haec diversitas, optandum esset, ut ea potius experimentis quam ex theoria quapiam definiretur.

SCHOLION

301. Cum igitur ob diversam radiorum refrangibilitatem cuique imagini duplex alteratio inducatur, quarum altera eius magnitudinem, altera vero eius locum afficit, duplex inde confusio in visionem infertur. Si enim formulae in superioribus capitibus exhibitae ad

radios mediae naturae restringantur, pro quibus est $n=\frac{31}{20}$, posito $dn=-\frac{1}{100}$ ex formulis hic traditis differentialibus locus et magnitudo imaginis a radiis rubris formatae definietur; posito autem $dn=+\frac{1}{100}$ locus et magnitudo imaginis violaceae declarabitur. Scilicet si

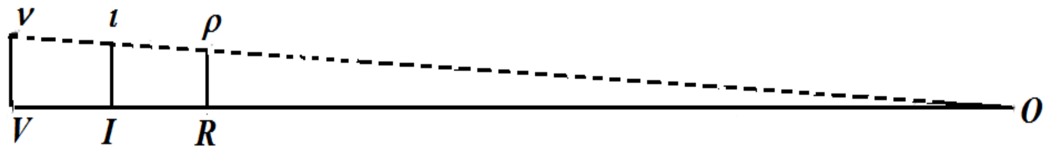

Fig. 16.

$I\iota$ (Fig. 16) fuerit imago ultima visioni obiecta, quae a radiis mediae naturae formatur, per formulas modo inventas, prout vel dn vel $-\frac{1}{100}$ vel $+\frac{1}{100}$ ponatur, definietur tam imago rubra $R\rho$ quam violacea $V\nu$; atque ex natura differentialium manifestum est cum intervalla IR et IV inter se aequalia esse debere, tum etiam differentias $I\iota-R\rho$ et $V\nu-I\iota$, ita ut oculo series iunumerabilium imaginum inter extremas $R\rho$ et $V\nu$ sitarum simul cernenda offeratur, unde eo maior confusio oriatur necesse est, quo maior fuerit differentia tam ratione loci quam magnitudinis. Quare haec confusio penitus tolleretur, si eiusmodi lentium dispositio definiri posset, ut tam intervallum RV quam differentia inter imagines $R\rho$ et $V\nu$ ad nihilum redigeretur, quod utrumque nisi simul praestari queat, confusionem perfecte tollere non licet. Verumtamen etiamsi neutri harum conditionum satisfieri possit, tamen dabitur pro oculo eiusmodi locus O, ubi confusio minime sensibilis percipiatur, qui erit in concursu rectae $\nu\rho$ productae cum axe: ibi enim omnes extremitates ρ, ι, ν communibus radiis cernentur, neque propterea extremitas obiecti colore tincta apparebit. Quare si simul punctum O conveniat cum loco oculi idoneo, alia confusio non percipietur, nisi quae inde originem trahit, quod forte imagines extremae $R\rho$ et $V\nu$ nimis a distantia iusta discrepent, siquidem media $I\iota$ ad distantiam iustam ab oculo fuerit remota. Neque tamen hinc ora obiecti coloribus iridis cincta apparebit, cui confusionis speciei maxime est occurrendum; ideoque ea, quae adhuc adfuerit, confusio facile tolerari poterit, quae vero etiam omnino tolleretur, si modo intervallum RV vel in nihilum redigi vel saltem satis parvum reddi posset. Hinc ergo intelligimus vitium illud, quo obiecta coloribus iridis circumdata saepe repraesentantur, non tam necessario cum instrumentis dioptricis esse coniunctum, ut nullo pacto ab iis separari queat; quamobrem eo magis operae erit pretium, ut investigemus, quomodo haec instrumenta ab isto vitio liberari possint. Quae tota investigatio huc redit, ut determinetur punctum O, ubi recta per terminas imaginum ν, ι, ρ ducta cum axe concurrit hocque punctum cum loco oculi iam supra definito conveniens reddatur, si quidem fieri potest: unde perspicitur locum oculi O hac proprietate praeditum esse oportere, ut angulus, sub quo ultima imago cernitur, ob variabilitatem numero n tributam nullam mutationem patiatur. Tum vero in super videndum erit, num intervalla IR et IV vel ad nihilum reduci vel minima reddi queant.

PROBLEMA 4

302. *Proposita unica lente definire locum oculi, unde obiectum sine margine colorato cernatur.*

SOLUTIO

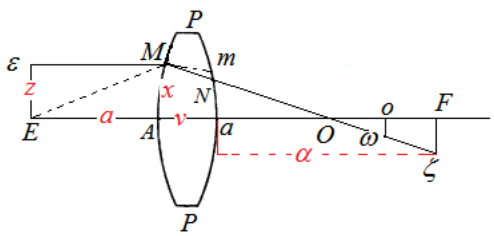

Fig. 12

Sit obiecti $E\varepsilon$ (Fig. 12) ante lentem distantia $EA = a$, imago vero per radios mediae naturae in $F\zeta$ repraesentetur, ponaturque $aF = \alpha$. Pro lente vero sit eius crassities $Aa = v$ et quantitas arbitraria $= k$, unde capiatur $\frac{k-v}{k+v} = i$.

Hinc posito $E\varepsilon = z$ erit $F\zeta = \frac{1}{i}\cdot\frac{\alpha}{a}z$ (§ 86); quare si pro loco oculi statuatur distantia $aO = O$, quae est fixa, erit $OF = \alpha - O$ et anguli $FO\zeta$ tangens $= \frac{1}{i}\cdot\frac{\alpha}{a}\cdot\frac{z}{\alpha-O}$, quae formula ob diversam radiorum refrangibilitatem nullam mutationem subira debet. Inde autem quantitates α et i tantum variantur, dum reliquae manent constantes. Quare istius formulae differentiale logarithmicum nihilo aequale positum praebet hanc aequationem

$$-\frac{di}{i} + \frac{d\alpha}{\alpha} - \frac{d\alpha}{\alpha-O} = 0 \quad \text{seu} \quad -\frac{di}{i} - \frac{Od\alpha}{\alpha(\alpha-O)} = 0,$$

ubi si valores supra inventi substituantur, prodit

$$\frac{vdn}{in(n-1)}\left(\frac{1}{a} + \frac{1-i}{v}\right) + \frac{O\alpha dn}{i(n-1)(\alpha-O)}\left(\frac{1}{\alpha} + \frac{1}{ia} + \frac{(1-i)^2}{iv}\right) = 0,$$

quae aequa per $\frac{dn}{i(n-1)}$ divisa praebet :

$$\frac{v}{na} + \frac{1-i}{n} + \frac{O\alpha}{(\alpha-O)}\left(\frac{1}{\alpha} + \frac{1}{ia} + \frac{(1-i)^2}{iv}\right) = 0,$$

unde locus oculi definiri poterit; qui si debeat convenire cum supra invento (§ 238), ubi invenimus $O = \frac{-i\alpha v}{n\alpha - iv}$, erit $\alpha - O = \frac{n\alpha\alpha}{n\alpha - iv}$ et $\frac{O\alpha}{\alpha - O} = \frac{-iv}{n}$, hincque nostra aequatio per n multiplicata abit in $\frac{v}{a} + 1 - i - \frac{iiv}{\alpha} - \frac{v}{a} - (1-i)^2 = 0$ seu $i - ii - \frac{iiv}{\alpha} = 0$, ideoque $i = \frac{\alpha}{\alpha+v} = \frac{k-v}{k+v}$. Quamobrem quantitatem arbitrariam k ita definiri conveniet, ut sit

$k = 2\alpha + v$; et cum sit $i = \frac{\alpha}{\alpha+v}$, pro loco oculi habebimus

$O = \frac{-\alpha v}{n\alpha+(n-1)v} = \frac{-20\alpha v}{31\alpha+11v}$ ob $n = \frac{31}{20}$.

Quod si porro hinc variationem in loco imaginis desideremus, definiri oportet differentiale $d\alpha$, quod fiet:

$$d\alpha = \frac{-\alpha\alpha dn}{i(n-1)}\left(\frac{1}{\alpha} + \frac{1}{ia} + \frac{(1-i)^2}{iv}\right)$$

et pro i posito valore $\frac{\alpha}{\alpha+v}$

$$d\alpha = \frac{-(\alpha+v)dn}{n-1}\left(1 + \frac{\alpha+v}{a}\right),$$

qui valor si ad nihilum redigi posset, confusio omnis a diversa radiorum refrangibilitate oriunda perfecte tolleretur.

COROLLARIUM 1

303. Pro lentis ergo constructione quantitas arbitraria k ita accipi debet, ut sit $k = 2\alpha + v$; atque tum oculus in eo loco constitutus, ubi totum campum apparentem percipiat, simul nullam confusionem a diversa radiorum indole sentiet.

COROLLARIUM 2

304. Ut autem oculus simul imaginem in distantia iusta aspiciat, oportet sit $\alpha - O = -l$, ideoque $l = \frac{-31\alpha(\alpha+v)}{31\alpha+11v}$. Unde colligitur:

$$\alpha = \frac{-1}{2}l - \frac{1}{2}v - \sqrt{\left(\frac{1}{4}ll + \frac{9}{62}vl + \frac{1}{4}vv\right)},$$

hincque

$$O = \frac{1}{2}l - \frac{1}{2}v - \sqrt{\left(\frac{1}{4}ll + \frac{9}{62}vl + \frac{1}{4}vv\right)},$$

et

$$k = -l - 2\sqrt{\left(\frac{1}{4}ll + \frac{9}{62}vl + \frac{1}{4}vv\right)}.$$

COROLLARIUM 3

305. Potest vero insuper effici, ut etiam $d\alpha$ evanescat, quod evenit, si $a + \alpha + v = 0$, hoc est

$$a=\frac{1}{2}l-\frac{1}{2}v+\sqrt{\left(\frac{1}{4}ll+\frac{9}{62}vl+\frac{1}{4}vv\right)}.$$

Verum cum hoc casu ob $\alpha=-a-v$ imago in ipsum obiectum cadat, ita ut radii nullam refractionem pati sint censendi, visio per lentem perinde erit comparata atque nudis oculis.

COROLLARIUM 4

306. Si crassities lentis v plane evanescat, tam ob campum apparentem quam diversam radiorum refrangibilitatem fit $O=0$; hoc ergo casu oculus lenti immediate applicatus nullam confusionem ob diversam radiorum naturam percipiet. Dum ergo fuerit $\alpha=-l$, visio erit distincta

SCHOLION

307. Hic scilicet penitus mentem abstrahimus a confusione iam supra determinata, quae a lentium apertura oritur, ideoque aperturam primae lentis ut evanescentem spectamus. Eam igitur hic tantum confusionis speciem contemplamur, quae a diversa radiorum refrangibilitate originem ducit; quam plerumque tolli observavimus, si angulus ad O invariabilis reddatur; tum enim ora obiecti satis bene terminata conspicietur neque coloribus iridis cincta. Interim tamen adhuc aliqua confusio sentiri poterit inde oriunda, quod, si imago media iustam ab oculo distantiam teneat, imaginum extremarum altera sit nimis propinqua, altera nimis remota; verum si earum intervallum non sit admodum magnum, confusio haec parum erit sensibilis. Ita hic invenimus, quod experientia satis comprobatur, si obiecta per unicam lentem spectemus, ea margine colorato destituta apparere, dummodo oculus immediate applicetur; quod si quando secus evenire videatur, causa aperturae lentis sine dubio erit tribuenda, cui conditioni rationes hic allegatae refragantur.

PROBLEMA 5

308. *Si instrumentum dioptricum duabus instructum sit lentibus, definire locum oculi, unde obiectum sine margine colorato videatur.*

SOLUTIO

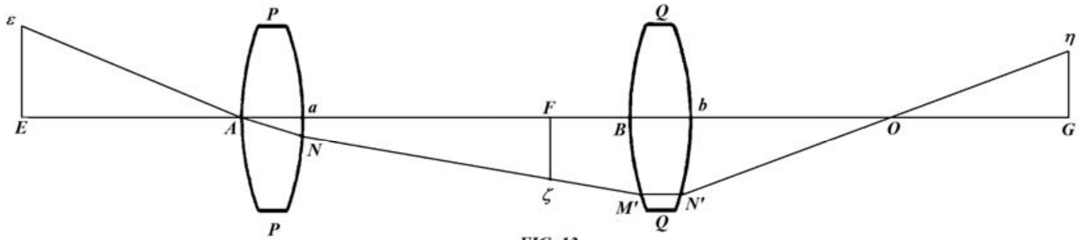

FIG. 13.

Posita obiecti distantia $AE = a$ (Fig. 13) sint pro radiis mediae naturae reliquae distantiae determinatrices $aF = \alpha$, $BF = b$ et $bG = \beta$, crassities vero lentium $AA = v$, $Bb = v'$ et distantiae arbitrariae k et k', ponaturque $\frac{k-v}{k+v} = i$ et $\frac{k'-v}{k'+v} = i'$. His positis si magnitude obiecti $E\varepsilon$ vocetur $= z$, erit imago $G\eta = \frac{1}{ii'} \cdot \frac{\alpha\beta}{ab} \cdot z$, unde, si oculi distantia ponatur $bO = O$, ob $GO = \beta - O$ erit anguli $GO\eta$ tangens $\frac{1}{ii'} \cdot \frac{\alpha\beta}{ab} \cdot \frac{z}{\beta - O}$, cuius differentiale logarithmicum nihilo aequatum praebet:

$$-\frac{di}{i} - \frac{di'}{i'} + \frac{d\alpha}{\alpha} - \frac{db}{b} + \frac{d\beta}{\beta} - \frac{d\beta}{\beta - O} = 0,$$

quae valoribus supra (§ 296) inventis substitutis abit in:

$$\frac{vdn}{in(n-1)}\left(\frac{1}{a} + \frac{1-i}{v}\right) + \frac{v'db}{n'i'bb} + \frac{v'dn'}{i'n'(n'-1)}\left(\frac{1}{b} + \frac{1-i'}{v'}\right) - \frac{db}{\alpha} - \frac{db}{b}$$
$$+ \frac{O}{\beta - O}\left(\frac{\beta db}{i'i'bb} + \frac{\beta dn'}{i'(n'-1)}\left(\frac{i'}{\beta} + \frac{1}{i'b} + \frac{(1-i')^2}{i'v'}\right)\right) = 0.$$

Verum conditio campi exigebat $O = \frac{\beta\mathfrak{b}}{\mathfrak{b} + \frac{1}{ii'} \cdot \frac{\alpha\beta}{ab} \cdot z}$, existente

$$\mathfrak{b} = \left(\frac{i'}{i} \cdot \frac{\alpha + b}{a} - \frac{i'bv}{n\alpha} - \frac{1}{i} \cdot \frac{\alpha v'}{n'ab}\right) z$$

(§ 245), unde fit

$$\frac{O}{\beta - O} = \frac{\beta\mathfrak{b}}{\frac{1}{ii'} \cdot \frac{\alpha\beta\beta}{ab} \cdot z} = \frac{ii'ab}{\alpha\beta}\left(\frac{i'}{i} \cdot \frac{\alpha + b}{a} - \frac{i'bv}{n\alpha} - \frac{1}{i} \cdot \frac{\alpha v'}{n'ab}\right).$$

Nunc vero est $db = \frac{\alpha\alpha dn}{i(n-1)}\left(\frac{i}{\alpha} + \frac{1}{ia} + \frac{(1-i)^2}{iv}\right)$, quem valorem antequam substituamus, transformemus aequationem nostram in hanc formam [posito $n' = n$]

$$\frac{dn}{n-1}\left(\frac{v}{ina} + \frac{1-i}{in} + \frac{v'}{i'nb} + \frac{1-i'}{i'n} + \frac{\beta O}{i'(\beta - O)}\left(\frac{i'}{\beta} + \frac{1}{i'b} + \frac{(1-i')^2}{i'v'}\right)\right) + db\left(\frac{v'}{n'i'bb} - \frac{1}{\alpha} - \frac{1}{b} + \frac{\beta O}{i'i'bb(\beta - O)}\right) = 0,$$

ubi posterius membrum abit in $-\frac{iv}{n\alpha\alpha} db$; tum vero erit

$$0 = \frac{dn}{n-1}\begin{cases} 1 + \frac{b}{\alpha} + \frac{i'i'b}{\beta}\left(1 + \frac{b}{\alpha}\right) + \frac{(1-i')^2 b(\alpha + b)}{\alpha v'} + \frac{2 - i - i'}{n} \\ -\frac{iv}{n\alpha} - \frac{i'v'}{n\beta} - \frac{ibv}{n\alpha\alpha} - \frac{ii'i'bbv}{n\alpha\alpha\beta} - \frac{i(1-i')^2 bbv}{n\alpha\alpha v'} \end{cases}$$

Distinguendo n' ab n erit

$$\frac{dn}{n-1}\left(\frac{iv}{\alpha n}-\frac{(1-i)}{n}\right)=\frac{dn'}{n'-1}\left\{\begin{array}{l}1+\frac{b}{\alpha}+\frac{i'i'b}{\beta}\left(1+\frac{b}{\alpha}\right)-\frac{ii'i'bbv}{n\alpha\alpha\beta}-\frac{i'v'}{n'\beta}+\frac{1-i'}{n'}\\ -\frac{ibv}{n\alpha\alpha}+\frac{b(1-i')^2}{v'}\left(1+\frac{b}{\alpha}\right)-\frac{i(1-i')^2bbv}{n\alpha\alpha v'}\end{array}\right\}$$

cuius aequationis complicatio obstat, quominus quicquam commode inde concludi possit.

COROLLARIUM 1

309. Si ambae lentes crassitie careant, ut sit $v=0$, $v'=0$ et $i=i'=1$, aequatio differentialis prima est [existante $n'=n$]

$$\frac{d\alpha}{\alpha}-\frac{db}{b}-\frac{Od\beta}{\beta(\beta-O)}=0$$

tum vero:

$$d\alpha=-db=\frac{-\alpha\alpha dn}{n-1}\left(\frac{1}{\alpha}+\frac{1}{a}\right)$$

et

$$d\beta=\frac{-\beta\beta db}{bb}-\frac{\beta\beta dn}{n-1}\left(\frac{1}{\beta}+\frac{1}{b}\right)=\frac{-dn}{n-1}\left(\frac{\alpha\alpha\beta\beta}{bb}\left(\frac{1}{\alpha}+\frac{1}{a}\right)+\beta\beta\left(\frac{1}{\beta}+\frac{1}{b}\right)\right),$$

quibus valoribus substitutis et per $\frac{dn}{n-1}$ divisione facta fit

$$-\alpha\alpha\left(\frac{1}{\alpha}+\frac{1}{a}\right)\left(\frac{1}{\alpha}+\frac{1}{b}\right)+\frac{O\beta}{\beta-O}\left(\frac{\alpha\alpha}{bb}\left(\frac{1}{\alpha}+\frac{1}{a}\right)+\frac{1}{\beta}+\frac{1}{b}\right)=0.$$

Unde si oculus lenti posteriori immediate applicaretur, ut esset $O=0$, deberet esse

$$\left(\frac{1}{\alpha}+\frac{1}{a}\right)\left(\frac{1}{\alpha}+\frac{1}{b}\right)=0.$$

COROLLARIUM 2

810. Verum in eadem hypothesi, ut locus oculi congruat cum eo, quem visio campi exigit, debet esse $O=\frac{b\beta(\alpha+b)}{b(\alpha+b)+\alpha\beta}$ unde fit $\beta-O=\frac{\alpha\beta\beta}{b(\alpha+b)+\alpha\beta}$, ideoque $\frac{O}{\beta-O}=\frac{b(\alpha+b)}{\alpha\beta}=\frac{bb}{\beta}\left(\frac{1}{\alpha}+\frac{1}{b}\right)$, quo valore substituto nostra aequatio erit

$$0=\left(\frac{1}{\alpha}+\frac{1}{b}\right)\left(-\alpha\alpha\left(\frac{1}{\alpha}+\frac{1}{a}\right)+bb\left(\frac{\alpha\alpha}{bb}\left(\frac{1}{\alpha}+\frac{1}{a}\right)+\frac{1}{\beta}+\frac{1}{b}\right)\right),$$

quae reducitur ad hanc formam $0=bb\left(\frac{1}{\alpha}+\frac{1}{b}\right)\left(\frac{1}{\beta}+\frac{1}{b}\right)$. Distinguendo n' ab n membra a dn pendentia se destruunt, et oritur $0=\frac{dn'}{n'-1}bb\left(\frac{1}{\alpha}+\frac{1}{b}\right)\left(\frac{1}{\beta}+\frac{1}{b}\right)$, quod ita ostenditur:

Cum aequatio prima differentialis praebeat:

$$\frac{d\alpha}{\alpha}-\frac{d\beta}{\beta}-\frac{Od\beta}{\beta(\beta-O)}=0 \text{ sive } d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)-\frac{Od\beta}{\beta(\beta-O)}=0,$$

cum igitur ex conditione campi apparentis sit $\frac{O}{\beta-O}=\frac{bb}{\beta}\left(\frac{1}{\alpha}+\frac{1}{b}\right)$, erit

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)-\frac{bb}{\beta^2}\left(\frac{1}{\alpha}+\frac{1}{b}\right)d\beta=0$$

ideoque loco $d\beta$ suum valorem substituendo

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)-\frac{bb}{\beta^2}\left(\frac{1}{\alpha}+\frac{1}{b}\right)\left(\frac{\beta\beta d\alpha}{bb}-\frac{\beta^2 dn'}{n'-1}\left(\frac{1}{\beta}+\frac{1}{b}\right)\right)=0,$$

ubi membra, quae $d\alpha$ continent, manifesto se destruunt, et tota quaestio ad hanc aequationem perducitur

$$\frac{dn'}{n'-1}bb\left(\frac{1}{\alpha}+\frac{1}{b}\right)\left(\frac{1}{\beta}+\frac{1}{b}\right)=0.$$

COROLLARIUM 3

311. Quod si ergo huic conditioni satisfieri possit, obiectum sine margine colorato apparebit; praeterea vero confusio penitus tolleretur, si reddi liceret $d\beta=0$, quod fit per hanc aequationem

$$\frac{\alpha\alpha}{bb}\left(\frac{1}{\alpha}+\frac{1}{a}\right)+\frac{1}{\beta}+\frac{1}{b}=0 \text{ sive } \alpha\alpha\left(\frac{1}{\alpha}+\frac{1}{a}\right)+bb\left(\frac{1}{\beta}+\frac{1}{b}\right)=0,$$

vel distinguendo n' ab n

$$\frac{dn}{n-1}\cdot\frac{\alpha\alpha}{bb}\left(\frac{1}{\alpha}+\frac{1}{a}\right)+\frac{dn'}{n'-1}\cdot\left(\frac{1}{\beta}+\frac{1}{b}\right)=0.$$

COROLLARIUM 4

312. Priori autem aequationi satisfieri nequit, nisi fuerit vel $\alpha+b=0$ vel $\left(\frac{1}{\beta}+\frac{1}{b}\right)=0$. Illo casu ambae lentes coniungerentur, ut unicam constituerent; hoc vero posterioris

distantia focalis fieret infinita, qui casus iterum ad casum unicae lentis rediret; foret enim $O=-\alpha-b,$ ob $\beta=-b,$ et oculus priori lenti immediate applicari deberet.

SCHOLION

313. Si simili modo has investigationes ad plures lentes extendere vellemus, non neglecta earum crassitie in formulas plane in extricabiles delaberemur, unde vix quicquam concludi posset. Verum quia in omnibus fere instrumentis dioptricis, praecipue quae pluribus lentibus constant, iis tam exigua crassities tribui solet, ut sine notabili errore pro nihilo haberi possit, tam taediosae indagationi facile supersedere poterimus. Ad quod accedit, quod hic non de summo rigore geometrico agatur, sed contenti esse queamus, dummodo hanc confusionem satis prope cognoverimus: ex quo sufficiet in consideratione plurium lentium earum crassitiem prorsus neglexisse.

SUPPLEMENTUM IV

Si ratio refractionis in singulis lentibus sit diversa, solutio sequenti modo absolvetur [neglecta lentium crassitie]:

I. Prima aequatio differentialis prorsus se habebit, ut in problemate, ita ut sit

$$d\alpha\left(\tfrac{1}{\alpha}+\tfrac{1}{b}\right)+d\beta\left(\tfrac{1}{\beta}+\tfrac{1}{c}\right)-\tfrac{Od\gamma}{\gamma(\gamma-O)}=0,$$

et quia etiam, ut ante, est

$$\tfrac{O}{\gamma-O}=\tfrac{bbcc}{\alpha\beta^2\gamma}\left(1+\tfrac{\alpha}{b}\right)+\tfrac{cc}{\beta\gamma}\left(1+\tfrac{\beta}{c}\right),$$

erit nostra aequatio

$$\left(\tfrac{1}{\alpha}+\tfrac{1}{b}\right)\left(d\alpha-\tfrac{bbcc}{\beta^2\gamma^2}d\gamma\right)+\left(\tfrac{1}{\beta}+\tfrac{1}{c}\right)\left(d\beta-\tfrac{cc}{\gamma\gamma}d\gamma\right)=0.$$

II. Nunc autem ratio diversae refractionis est habenda; unde in superioribus additamentis invenimus esse

$$d\alpha=-\tfrac{\alpha\alpha dn}{n-1}\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right);\quad d\beta=\tfrac{\beta\beta d\alpha}{bb}-\tfrac{\alpha\alpha dn'}{n'-1}\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right);\quad d\gamma=\tfrac{\gamma\gamma d\beta}{cc}-\tfrac{\gamma\gamma dn''}{n''-1}\left(\tfrac{1}{c}+\tfrac{1}{\gamma}\right).$$

Hincque ergo fiet

$$d\alpha-\tfrac{bbcc}{\beta^2\gamma^2}d\gamma=d\alpha-\tfrac{bbd\beta}{\beta^2}+\tfrac{bbccdn''}{\beta^2(n''-1)}\left(\tfrac{1}{c}+\tfrac{1}{\gamma}\right)=\tfrac{bbdn'}{n'-1}\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)+\tfrac{bbccdn''}{\beta^2(n''-1)}\left(\tfrac{1}{c}+\tfrac{1}{\gamma}\right).$$

deinde

$$d\beta-\frac{cc}{\gamma\gamma}d\gamma=\frac{ccdn''}{n''-1}\left(\frac{1}{c}+\frac{1}{\gamma}\right).$$

III. Ex his ergo nostra aequatio differentialis abibit in hanc formam:

$$\left(\frac{1}{\alpha}+\frac{1}{b}\right)\left(\frac{bbdn'}{n'-1}\left(\frac{1}{b}+\frac{1}{\beta}\right)+\frac{bbc^2dn''}{\beta^2(n''-1)}\left(\frac{1}{c}+\frac{1}{\gamma}\right)\right)+\left(\frac{1}{\beta}+\frac{1}{c}\right)\frac{ccdn''}{n''-1}\left(\frac{1}{c}+\frac{1}{\gamma}\right)=0,$$

sive

$$\frac{dn'}{n'-1}bb\left(\frac{1}{\alpha}+\frac{1}{b}\right)\left(\frac{1}{\beta}+\frac{1}{b}\right)+\frac{dn''}{n''-1}cc\left(\frac{1}{c}+\frac{1}{\gamma}\right)\left(\frac{bb}{\beta\beta}\left(\frac{1}{\alpha}+\frac{1}{b}\right)+\frac{1}{\beta}+\frac{1}{c}\right)=0.$$

IV. Pro casu autem illo singulari, quo oculus lenti ultimae immediate debet applicari, ob $O=0$ habebitur simpliciter haec aequatio

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)+d\beta\left(\frac{1}{\beta}+\frac{1}{c}\right)=0$$

sive substituto valore $d\beta$

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)+\frac{\beta\beta d\alpha}{bb}\left(\frac{1}{\beta}+\frac{1}{c}\right)-\frac{bbdn'}{n'-1}\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\frac{1}{\beta}+\frac{1}{c}\right)=0,$$

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}+\frac{\beta\beta}{bb}\left(\frac{1}{\beta}+\frac{1}{c}\right)\right)-\frac{b^2dn'}{n'-1}\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\frac{1}{\beta}+\frac{1}{c}\right)=0$$

seu tandem

$$+\frac{dn}{n-1}\alpha\alpha\left(\frac{1}{\alpha}+\frac{1}{a}\right)\left(\frac{1}{\alpha}+\frac{1}{b}+\frac{\beta^2}{b^2}\left(\frac{1}{\beta}+\frac{1}{c}\right)\right)+\frac{dn'}{n'-1}\beta\beta\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\frac{1}{\beta}+\frac{1}{c}\right)=0.$$

V. Hoc modo tantum margo coloratus tollitur; ut autem tota confusio tollatur, quod fit, si $d\gamma=0$, insuper satisfieri debet huic aequationi

$$0=\frac{dn}{n-1}\frac{\alpha\alpha\beta\beta\gamma\gamma}{bbcc}\left(\frac{1}{a}+\frac{1}{\alpha}\right)+\frac{dn'}{n'-1}\frac{\beta\beta\gamma\gamma}{cc}\left(\frac{1}{b}+\frac{1}{\beta}\right)+\frac{dn''}{n''-1}\gamma\gamma\left(\frac{1}{c}+\frac{1}{\gamma}\right).$$

PROBLEMA 6

314. *Si instrumentum dioptricum tribus constet lentibus, quarum crassities evanescat, eam definire dispositionem, ut oculus in eo loco, quem campus postulat, constitutus obiectum sine margine colorato conspiciat.*

SOLUTIO

Posita ergo distantia obiecti ante lentem obiectivam $AE=a$ (Fig. 14) eiusque magnitudine $E\varepsilon=z$, vocentur distantiae imaginum a radiis mediae naturae formatarum ut hactenus

$$aF=\alpha,\ BF=b,\ BG=\beta,\ \ CG=c\ \text{ et }\ cH=\gamma,$$

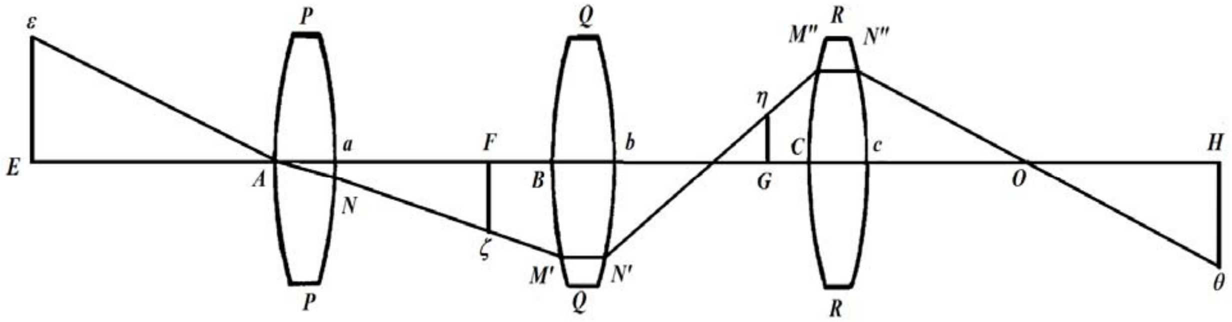

Fig. 14.

eritque imago ultima $H\theta=\frac{\alpha\beta\gamma}{abc}z$, et posita oculi post lentem ultimam distantia $cO=O$ erit $OH=\gamma-O$ et anguli HOθ tangens $=\frac{\alpha\beta\gamma}{abc}\cdot\frac{z}{\gamma-O}$, quae debet esse invariabilis. Posito ergo eius differentiali logarithmico $=0$ habebimus hanc aequationem:

$$\frac{d\alpha}{\alpha}-\frac{db}{b}+\frac{d\beta}{\beta}-\frac{dc}{c}+\frac{d\gamma}{\gamma}-\frac{d\gamma}{\gamma-O}=0.$$

At ex formulis supra (§ 300) erutis habemus

$$d\alpha=-db=\frac{-dn}{n-1}\alpha\alpha\left(\frac{1}{a}+\frac{1}{\alpha}\right),\ \ d\beta=-dc=-\frac{\beta\beta dn}{n-1}\left(\frac{\alpha\alpha}{bb}\left(\frac{1}{a}+\frac{1}{\alpha}\right)+\frac{1}{b}+\frac{1}{\beta}\right),$$
$$d\gamma=-\frac{\gamma\gamma dn}{n-1}\left(\frac{\alpha\alpha\beta\beta}{bbcc}\left(\frac{1}{a}+\frac{1}{\alpha}\right)+\frac{\beta\beta}{cc}\left(\frac{1}{b}+\frac{1}{\beta}\right)+\frac{1}{c}+\frac{1}{\gamma}\right),$$

unde nostra aequatio erit

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)+d\beta\left(\frac{1}{\beta}+\frac{1}{c}\right)-\frac{Od\gamma}{\gamma(\gamma-O)}=0.$$

Sed ob campum apparentem supra (§ 256) invenimus:

$$\mathfrak{B}=\mathfrak{b}=b\left(1+\frac{\alpha}{b}\right)\frac{z}{a};\ \mathfrak{C}=\mathfrak{c}\ =\frac{bc}{\beta}\left(1+\frac{\alpha}{b}\right)\frac{z}{a}+\frac{\alpha c}{b}\left(1+\frac{\beta}{c}\right)\frac{z}{a}$$

hincque

$$O=\frac{\gamma\mathfrak{c}}{\mathfrak{c}+H\theta}\ \text{ et }\ \frac{O}{\gamma-O}=\frac{\mathfrak{c}}{H\theta},\ \text{unde fit ob }\ H\theta=\frac{\alpha\beta\gamma}{bc}\cdot\frac{z}{a}$$

$$\frac{O}{\gamma-O}=\frac{bbcc}{\alpha\beta\beta\gamma}\left(1+\frac{\alpha}{b}\right)+\frac{cc}{\beta\gamma}\left(1+\frac{\beta}{c}\right)\ \text{ seu }\ \frac{O}{\gamma(\gamma-O)}=\frac{bbcc}{\alpha\beta\gamma\gamma}\left(\frac{1}{\alpha}+\frac{1}{b}\right)+\frac{cc}{\gamma\gamma}\left(\frac{1}{\beta}+\frac{1}{c}\right).$$

Valoribus iam his substitutis habebimus:

$$\left(\frac{1}{\alpha}+\frac{1}{b}\right)\left(d\alpha-\frac{bbcc}{\beta\beta\gamma\gamma}d\gamma\right)+\left(\frac{1}{\beta}+\frac{1}{c}\right)\left(d\beta-\frac{cc}{\gamma\gamma}d\gamma\right)=0.$$

At est

$$d\alpha-\frac{bbcc}{\beta\beta\gamma\gamma}d\gamma=\frac{dn}{n-1}\left(bb\left(\frac{1}{b}+\frac{1}{\beta}\right)+\frac{bbcc}{\beta\beta}\left(\frac{1}{c}+\frac{1}{\gamma}\right)\right)$$

$$d\beta-\frac{cc}{\gamma\gamma}d\gamma=\frac{dn}{n-1}cc\left(\frac{1}{c}+\frac{1}{\gamma}\right).$$

Quare facta divisione per $\frac{dn}{n-1}$ nanciscemur hanc aequationem:

$$bb\left(\frac{1}{\alpha}+\frac{1}{b}\right)\left(\frac{1}{b}+\frac{1}{\beta}+\frac{cc}{\beta\beta}\left(\frac{1}{c}+\frac{1}{\gamma}\right)\right)+cc\left(\frac{1}{\beta}+\frac{1}{c}\right)\left(\frac{1}{c}+\frac{1}{\gamma}\right)=0.$$

Quodsi vero oculus lenti postremae immediate applicetur seu sit $O=0$, conditio praescripta hanc postulat aequationem:

$$\alpha\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)\left(\frac{1}{a}+\frac{1}{\alpha}\right)+\beta\beta\left(\frac{1}{\beta}+\frac{1}{c}\right)\left(\frac{\alpha\alpha}{bb}\left(\frac{1}{a}+\frac{1}{\alpha}\right)+\frac{1}{b}+\frac{1}{\beta}\right)=0.$$

Confusio vero a diversa radiorum refrangibilitate oriunda perfecte tolletur, si praeterea fuerit $d\gamma=0$ seu

$$\frac{\alpha\alpha\beta\beta}{bbcc}\left(\frac{1}{a}+\frac{1}{\alpha}\right)+\frac{\beta\beta}{cc}\left(\frac{1}{b}+\frac{1}{\beta}\right)+\frac{1}{c}+\frac{1}{\gamma}=0.$$

COROLLARIUM 1

815. Si rationes aperturarum lentium in computum ducantur eaque pro lente secunda ponatur $=\pi$, pro tertia $=\pi'$, erit

$$bN'=\frac{\pi b\beta}{b+\beta} \quad \text{et} \quad CM''=cN''=\frac{\pi' c\gamma}{c+\gamma}.$$

Tum vero posito $\frac{z}{a}=\Phi$ erit $G\eta=\frac{\alpha\beta}{ab}\cdot a\Phi$ et $H\theta=\frac{\alpha\beta\gamma}{abc}\cdot a\Phi$. Hinc fiet

$$\frac{O}{\gamma-O}=\frac{cN''}{H\theta}=\frac{\pi' abcc\gamma}{\alpha\beta(c+\gamma)a\Phi} \quad \text{et} \quad \frac{O}{\gamma(\gamma-O)}=\frac{\pi' abcc}{\alpha\beta\gamma(c+\gamma)}\cdot\frac{1}{a\Phi}.$$

COROLLARIUM 2

316. Quodsi porro ut supra ponatur $\alpha = Aa,\ \beta = Bb,\ \gamma = Cc$, erit

$$\frac{O}{\gamma(\gamma - O)} = \frac{\pi'}{ABC(C+1)a\Phi};$$

tum vero $bN' = \frac{\pi Bb}{B+1}$ et $CM'' = \frac{\pi' Cc}{C+1}$ ac $G\eta = ABa\Phi$; cum iam sit
$bN' + G\eta : bG = CM'' - G\eta : CG,$ erit $\frac{bN' + G\eta}{bG} = \frac{CM'' - G\eta}{CG}$, ideoque
$\frac{1}{bG} + \frac{1}{CG} = \frac{CM''}{CG \cdot G\eta} - \frac{bN'}{bG \cdot G\eta}$, hoc est $\frac{1}{\beta} + \frac{1}{c} = \frac{\pi' C}{AB(C+1)a\Phi} - \frac{\pi}{AB(B+1)a\Phi}$;

Simili vero modo est $\frac{1}{\alpha} + \frac{1}{b} = \frac{\pi B}{A(B+1)a\Phi}$.

COROLLARIUM 3

317. per easdem substitutiones fit

$$\frac{1}{a} + \frac{1}{\alpha} = \frac{A+1}{Aa},\quad \frac{1}{b} + \frac{1}{\beta} = \frac{B+1}{Bb},\quad \frac{1}{c} + \frac{1}{\gamma} = \frac{C+1}{Cc}\quad \text{etc.}$$

hincque:

$$\alpha\alpha\left(\frac{1}{a} + \frac{1}{\alpha}\right) = A(A+1)a$$
$$\beta\beta\left(\frac{1}{b} + \frac{1}{\beta}\right) = B(B+1)b$$
$$\gamma\gamma\left(\frac{1}{c} + \frac{1}{\gamma}\right) = C(C+1)c$$

sicque porro.

COROLLARIUM 4

318. His ergo novis denominationibus introductis differentialia ex § 300 ita exprimentur

$$d\alpha = -db = -\frac{dn}{n-1} \cdot A(A+1)a$$
$$d\beta = -dc = -BBdb - \frac{dn}{n-1} \cdot B(B+1)b$$
$$d\gamma = -dd = -CCdc - \frac{dn}{n-1} \cdot C(C+1)c,$$

quae formulae commodius in calculum introducentur.

Quemadmodum hic novae formae adhibentur in sequentibus usurpandae, ita et pro casu diversae refractionis sequentibus formulis in posterum uti licebit:

$$\frac{O}{\gamma(\gamma-O)}=\frac{\pi'}{ABC(C+1)a\Phi}$$
$$\frac{1}{\alpha}+\frac{1}{b}=\frac{\pi B}{A(B+1)a\Phi}$$
$$\frac{1}{\beta}+\frac{1}{c}=\frac{\pi' C}{AB(C+1)a\Phi}-\frac{\pi}{AB(B+1)a\Phi}$$
$$\frac{1}{\gamma}+\frac{1}{d}=\frac{\pi'' D}{ABC(D+1)a\Phi}-\frac{\pi'}{ABC(B+1)a\Phi}$$
etc.;

tum vero formulae differentiales erunt

$$d\alpha=-db=-\tfrac{dn}{n-1}\cdot A\left(A+1\right)a$$
$$d\beta=-dc=B^2 d\alpha-\tfrac{dn'}{n'-1}\cdot B\left(B+1\right)b$$
$$d\gamma=-dd=C^2 d\beta-\tfrac{dn''}{n''-1}\cdot C\left(C+1\right)c$$
$$d\delta=-de=D^2 d\gamma-\tfrac{dn'''}{n'''-1}\cdot D\left(D+1\right)d$$
etc.

Atque ex his formulis, ut margo coloratus evanescat, satisfieri debet huic aequationi

$$0=\tfrac{dn'}{n'-1}\cdot\tfrac{\pi b}{Aa\Phi}+\tfrac{dn''}{n''-1}\cdot\tfrac{\pi' c}{ABa\Phi}.$$

Ut autem haec confusio penitus tollatur, fieri debet

$$d\gamma=-\tfrac{dn}{n-1}\cdot A(A+1)B^2C^2a-\tfrac{dn'}{n'-1}\cdot B(B+1)C^2b-\tfrac{dn''}{n''-1}\cdot C(C+1)c=0$$
sive
$$\tfrac{dn}{n-1}\cdot\tfrac{(A+1)a}{A}-\tfrac{dn'}{n'-1}\cdot\tfrac{(B+1)b}{AAB}+\tfrac{dn''}{n''-1}\cdot\tfrac{(C+1)c}{A^2B^2C}=0.$$

PROBLEMA 7

319. *Si instrumentum dioptricum quatuor constet lentibus, quarum crassities negligi queat, eam definire dispositionem, ut oculus in eo loco, quem campus apparens postulat, constitutus obiectum sine confusione a diversa radiorum indole oriunda conspiciat.*

SOLUTIO

Posita distantia obiecti ante instrumentum $AE=a$ (Fig.15) eiusque magnitudine $E\varepsilon=z$, vocentur distantiae imaginum a radiis mediae naturae formatarum ut hactenus

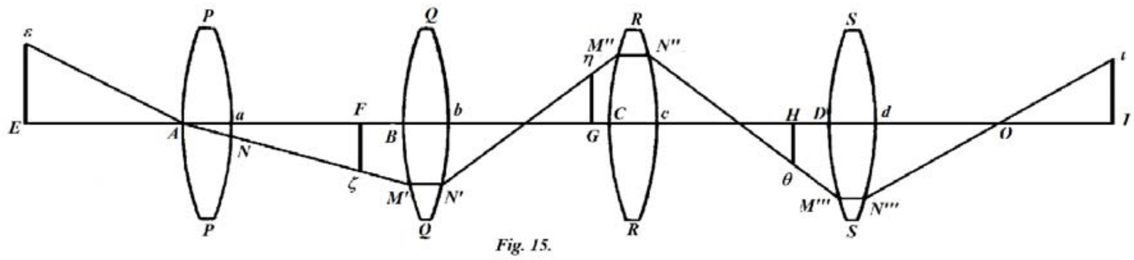

Fig. 15.

$aF=\alpha,\ BF=b,\ bG=\beta,\ CG=c,\ cH=\gamma,\ DH=d,\ \ dI=\delta.$ Tum vero ponamus praeterea

$$\alpha=Aa,\ \beta=Bb,\ \gamma=Cc,\ \delta=Dd,$$

atque introducantur rationes aperturarum pro singulis lentibus post primam, quae sint π pro secunda *QQ*, π' pro tertia *RR* et π'' pro quarta *SS*, sumto pro campo $\frac{z}{a}=\Phi$. His positis erunt imagines $F'\zeta=Aa\Phi,\ G\eta=ABa\Phi,\quad H\theta=ABCa\Phi$ et $I\iota=ABCDa\Phi=\frac{\alpha\beta\gamma\delta}{abcd}z.$ Iam posita oculi distantia post instrumentum $dO=O,$ ut sit $OI=\delta-O,$ erit anguli $IO\iota$, tangens $\frac{\alpha\beta\gamma\delta}{abcd}\cdot\frac{z}{\delta-O}$, quae, cum ob diversam radiorum refrangibilitatem immutata manere debeat, differentiata dabit hanc aequationem

$$\frac{d\alpha}{\alpha}-\frac{db}{b}+\frac{d\beta}{\beta}-\frac{dc}{c}+\frac{d\gamma}{\gamma}-\frac{dd}{d}-\frac{Od\delta}{\delta(\delta-O)}=0,$$

quae ob $db=-\,d\alpha,\ dc=-d\beta$ et $dd=-d\gamma$ abit in hanc

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)+d\beta\left(\frac{1}{\beta}+\frac{1}{c}\right)+d\gamma\left(\frac{1}{\gamma}+\frac{1}{d}\right)-\frac{Od\delta}{\delta(\delta-O)}=0.$$

Verum modo ante notavimus fore (§ 318)

$$d\alpha=\frac{-dn}{n-1}\cdot A(A+1)a,\quad d\beta=BBd\alpha-\frac{dn}{n-1}\cdot B(B+1)b,$$
$$d\gamma=CCd\beta-\frac{dn}{n-1}\cdot C(C+1)c,\quad d\delta=DDd\gamma-\frac{dn}{n-1}\cdot D(D+1)d,$$

etc.

ubi pro *b, c, d* valores § 266 assignati substitui debent. Pono vero iam animadvertimus esse (§ 316)

$$\frac{1}{\alpha}+\frac{1}{b}=\frac{1}{ABa\Phi}\frac{\pi B}{B+1},$$
$$\frac{1}{\beta}+\frac{1}{c}=\frac{1}{ABa\Phi}\left(\frac{\pi'C}{C+1}-\frac{\pi}{B+1}\right),$$
$$\frac{1}{\gamma}+\frac{1}{d}=\frac{1}{ABCa\Phi}\left(\frac{\pi''D}{D+1}-\frac{\pi'}{C+1}\right)$$

atque

$$\frac{O}{\delta(\delta-O)}=\frac{\pi''}{ABCD(D+1)a\Phi}.$$

Quodsi iam priores valores in posterioribus successive substituantur, habebimus

$$d\alpha=-\frac{dn}{n-1}\cdot A\left(A+1\right)a$$
$$d\beta=-\frac{dn}{n-1}\cdot B\left(AB\left(A+1\right)a+\left(B+1\right)b\right)$$
$$d\gamma=-\frac{dn}{n-1}\cdot C\left(AB^2C\left(A+1\right)a+BC\left(B+1\right)b+\left(C+1\right)c\right)$$
$$d\delta=-\frac{dn}{n-1}\cdot D\left(AB^2C^2D\left(A+1\right)a+BC^2D\left(B+1\right)b+CD\left(C+1\right)c+\left(D+1\right)d\right).$$

His igitur valoribus in aequatione differentiali substitutis et divisione per $\frac{dn}{(n-1)a\Phi}$ facta aequatio nostra secundum singula membra distributa erit

$$-\frac{\pi B}{(B+1)}\cdot(A+1)a$$
$$-\frac{1}{A}\left(\frac{\pi'C}{C+1}-\frac{\pi}{B+1}\right)\left(AB\left(A+1\right)a+\left(B+1\right)b\right)$$
$$-\frac{1}{AB}\left(\frac{\pi''D}{D+1}-\frac{\pi'}{C+1}\right)\left(AB^2C\left(A+1\right)a+BC\left(B+1\right)b+\left(C+1\right)c\right)$$
$$+\frac{1}{ABC}\cdot\frac{\pi''D}{D+1}\left(AB^2C^2D\left(A+1\right)a+BC^2D\left(B+1\right)b+CD\left(C+1\right)c+\left(D+1\right)d\right)=0.$$

Terna autem priora membra negativa sola collecta praebent

$$-\frac{BCD\pi''(A+1)}{D+1}a-\frac{CD(B+1)\pi''}{A(D+1)}b-\frac{D(C+1)\pi''}{AB(D+1)}c+\frac{\pi}{A}b+\frac{\pi'}{AB}c,$$

quibus si quartum addatur, prodit

$$\frac{\pi}{A}b+\frac{\pi'}{AB}c+\frac{\pi''}{ABC}d=0.$$

Iam duo hic casus considerari oportet, alterum, quo punctum O post lentem ultimam cadit, alterum vero, quo ob distantiam O negativam oculus lenti ultimae immediate applicatur. Pro priori casu, quo distantia $dO=O$ prodit positiva, habetur ista aequatio, si quidem in aequatione modo inventa pro *b, c, d* valores § 266 inventi substituantur:

$$\frac{(B+1)\pi}{B\pi-(B+1)\Phi}+\frac{(C+1)\pi'}{C\pi'-(C+1)(\pi-\Phi)}+\frac{(D+1)\pi''}{D\pi''-(D+1)(\pi'-\pi+\Phi)}=0.$$

Pro casu posteriori, quo distantia O evanescens assumitur, facta multiplicatione per $\frac{D+1}{a\Phi}$:

$$\frac{BCD(A+1)\pi''}{\Phi}+\frac{CD(B+1)^2\pi''}{B\pi-(B+1)\Phi}+\frac{D(C+1)^2\pi''}{C\pi'-(C+1)(\pi-\Phi)}$$
$$=\frac{(B+1)(D+1)\pi}{B\pi-(B+1)\Phi}+\frac{(C+1)(D+1)\pi'}{C\pi'-(C+1)(\pi-\Phi)}.$$

Hoc modo efficitur, ut obiectum sine margine colorato appareat; at omnis confusio tolletur, si praeterea fuerit

$$AB^2C^2D(A+1)+\frac{ABC^2D(B+1)^2\Phi}{B\pi-(B+1)\Phi}+\frac{ABCD(C+1)^2\Phi}{C\pi'-(C+1)(\pi-\Phi)}$$
$$+\frac{ABC(D+1)^2\Phi}{D\pi''-(D+1)(\pi'-\pi+\Phi)}=0.$$

seu per *ABC* dividendo:

$$BCD(A+1)+\frac{CD(B+1)^2\Phi}{B\pi-(B+1)\Phi}+\frac{D(C+1)^2\Phi}{C\pi'-(C+1)(\pi-\Phi)}+\frac{(D+1)^2\Phi}{D\pi''-(D+1)(\pi'-\pi+\Phi)}=0.$$

seu

$$BCD(A+1)+\frac{CD(B+1)b}{Aa}+\frac{D(C+1)c}{ABa}+\frac{(D+1)d}{ABCa}=0.$$

COROLLARIUM 1

320. Si utrique conditioni satisfieri potest, ut nec locus nec magnitudo imaginis ullam mutationem patiatur, locus oculi non amplius in computum ingreditur, sed imago, undecunque cernatur, ab omni confusione prorsus libera erit.

COROLLARIUM 2

321. Ut ergo hunc summum perfectionis gradum consequamur, binis his aequationibus satisfieri oportet:

$$\frac{(B+1)\pi}{B\pi-(B+1)\Phi}+\frac{(C+1)\pi'}{C\pi'-(C+1)(\pi-\Phi)}+\frac{(D+1)\pi''}{D\pi''-(D+1)(\pi'-\pi+\Phi)}=0.$$

et

$$A+1+\frac{(B+1)^2\Phi}{B(B\pi-(B+1)\Phi)}+\frac{(C+1)^2\Phi}{BC(C\pi'-(C+1)(\pi-\Phi))}+\frac{(D+1)^2\Phi}{BCD(D\pi''-(D+1)(\pi'-\pi+\Phi))}=0.$$

COROLLARIUM 3

322. Si porro, ut supra fecimus, ponamus

$$\frac{A}{A+1}=\mathfrak{A},\ \frac{B}{B+1}=\mathfrak{B},\ \frac{C}{C+1}=\mathfrak{C},\ \text{et}\ \frac{D}{D+1}=\mathfrak{D},$$

istae aequationes sequenti modo simplicius exprimentur:

$$\frac{\pi}{\mathfrak{B}\pi-\Phi}+\frac{\pi'}{\mathfrak{C}\pi'-\pi+\Phi}+\frac{\pi''}{\mathfrak{D}\pi''-\pi'+\pi-\Phi}=0$$

et

$$\frac{1}{\mathfrak{A}}+\frac{\Phi}{A\mathfrak{B}(\mathfrak{B}\pi-\Phi)}+\frac{\Phi}{AB\mathfrak{C}(\mathfrak{C}\pi'-\pi+\Phi)}+\frac{\pi''}{ABC\mathfrak{D}(\mathfrak{D}\pi''-\pi'+\pi-\Phi)}=0;$$

hic [si diversae refractionis ratio habeatur] prima membra multiplicari debent per $\frac{dn}{n-1}$, secunda per $\frac{dn'}{n'-1}$, tertia per $\frac{dn''}{n''-1}$ etc.

COROLLARIUM 4

323. Sin autem non liceat has ambas aequationes adimplere, curandum est, ut saltem priori, qua magnitudo apparens a confusione liberatur, satisfiat; hoc enim modo obiectum sine margine colorato apparebit, ubi quidem ad duos casus respici convenit, prout distantia oculi dO prodierit positiva vel negativa

SCHOLION

324. Introducendis ergo rationibus aperturarum, quibus supra iam commode sumus usi ad campum apparentem definiendum, aequationes etiam istae confusionem a diversa radiorum refrangibilitate oriundam tollentes satis fiunt simplices, ut sine molestia tractari queant, si quidem crassities lentium negligatur. Hoc ergo modo problema generale, quicunque fuerit lentium numerus, expediri conveniet.

SUPPLEMENTUM V

Si ratio refractionis in singulis lentibus discrepet, prodit primo quidem eadem aequatio differentialis

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)+d\beta\left(\frac{1}{\beta}+\frac{1}{c}\right)+d\gamma\left(\frac{1}{\gamma}+\frac{1}{d}\right)-\frac{Od\delta}{\delta(\delta-O)}=0,$$

in qua ergo casu $O=0$ ultimum membrum abiici debet.

1. Sin autem O habeat valorem positivum, erit ut ante

$$\frac{O}{\delta(\delta-O)}=\frac{\pi''}{ABCD(D+1)a\Phi}$$

atque etiam valores $\frac{1}{\alpha}+\frac{1}{b}$, $\frac{1}{\beta}+\frac{1}{c}$, $\frac{1}{\gamma}+\frac{1}{d}$ manent iidem ut ante.

At vero ob diversitatem refractionis habebimus

$$\begin{aligned} d\alpha &= \tfrac{-dn}{n-1}\cdot A(A+1)a \\ d\beta &= BBd\alpha - \tfrac{dn'}{n'-1}\cdot B(B+1)b \\ d\gamma &= CCd\beta - \tfrac{dn''}{n''-1}\cdot C(C+1)c \\ d\delta &= DDd\gamma - \tfrac{dn'''}{n'''-1}\cdot D(D+1)d. \end{aligned}$$

Hinc ergo aequatio nostra successive ita formetur:

$$\frac{-O}{\delta(\delta-O)} = \frac{-\pi'' d\delta}{ABCD(D+1)a\Phi} = \frac{-\pi'' Dd\gamma}{ABC(D+1)a\Phi} + \frac{dn'''}{n'''-1}\cdot\frac{\pi''\cdot d}{ABCa\Phi}.$$

Addatur

$$d\gamma\left(\frac{1}{\gamma}+\frac{1}{d}\right) = \frac{\pi'' Dd\gamma}{ABC(D+1)a\Phi} - \frac{\pi' d\gamma}{ABC(C+1)a\Phi},$$

et prodibit

$$\frac{-\pi'' d\gamma}{ABC(C+1)a\Phi} + \frac{dn'''}{n'''-1}\cdot\frac{\pi''\cdot d}{ABCa\Phi}$$

et pro $d\gamma$ substituto valore

$$\frac{-\pi' Cd\beta}{AB(C+1)a\Phi} + \frac{dn''}{n''-1}\cdot\frac{\pi'\cdot c}{ABa\Phi} + \frac{dn'''}{n'''-1}\cdot\frac{\pi''\cdot d}{ABCa\Phi}.$$

Iam addatur

$$d\beta\left(\frac{1}{\beta}+\frac{1}{c}\right) = \frac{\pi' Cd\beta}{AB(C+1)a\Phi} - \frac{\pi d\beta}{AB(B+1)a\Phi},$$

proditque

$$\frac{-\pi d\beta}{AB(B+1)a\Phi} + \frac{dn''}{n''-1}\cdot\frac{\pi'\cdot c}{AB\cdot a\Phi} + \frac{dn'''}{n'''-1}\cdot\frac{\pi''\cdot d}{ABCa\Phi}$$

et substituto valore ipsius $d\beta$

$$\frac{-\pi d\alpha}{A(B+1)a\Phi} + \frac{dn'}{n'-1}\cdot\frac{\pi\cdot b}{Aa\Phi} + \frac{dn''}{n''-1}\cdot\frac{\pi'\cdot c}{AB\cdot a\Phi} + \frac{dn'''}{n'''-1}\cdot\frac{\pi''\cdot d}{ABCa\Phi}.$$

Denique addatur

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right) = \frac{\pi Bd\alpha}{A(B+1)a\Phi},$$

ac aequatio quaesita, qua margo coloratus evanescit, erit:

$$\frac{dn'}{n'-1}\cdot\frac{\pi\cdot b}{Aa\Phi}+\frac{dn''}{n''-1}\cdot\frac{\pi'\cdot c}{AB\cdot a\Phi}+\frac{dn'''}{n'''-1}\cdot\frac{\pi''\cdot d}{ABCa\Phi}=0.$$

II. Sin autem O habeat valorem negativum, tunc sumi debet $O=0$, et pro eodem scopo aequatio ita formabitur:
Cum sit

$$d\gamma\left(\frac{1}{\gamma}+\frac{1}{d}\right)=\frac{\pi''Dd\gamma}{ABC(D+1)a\Phi}-\frac{\pi'd\gamma}{ABC(C+1)a\Phi},$$

substituto valore $d\gamma$ fiet

$$CCd\beta\left(\frac{\pi''D}{ABC(D+1)a\Phi}-\frac{\pi'}{ABC(C+1)a\Phi}\right)$$
$$-\frac{dn'}{n'-1}\cdot C(C+1)c\left(\frac{\pi''D}{ABC(D+1)a\Phi}-\frac{\pi'}{ABC(C+1)a\Phi}\right).$$

Addatur

$$d\beta\left(\frac{1}{\beta}+\frac{1}{c}\right)=\frac{\pi'Cd\beta}{AB(C+1)a\Phi}-\frac{\pi d\beta}{AB(B+1)a\Phi},$$

eritque

$$d\beta\left(\frac{CD\pi''}{AB(D+1)a\Phi}-\frac{\pi}{AB(B+1)a\Phi}\right)-\frac{dn''}{n''-1}\cdot(C+1)c\left(\frac{\pi''D}{AB(D+1)a\Phi}-\frac{\pi'}{AB(C+1)a\Phi}\right)$$

et substituto valore ipsius $d\beta$

$$d\alpha\left(\frac{BCD\pi''}{A(D+1)a\Phi}-\frac{B\pi}{A(B+1)a\Phi}\right)-\frac{dn'}{n'-1}\cdot\left(\frac{(B+1)CD\pi''}{A(D+1)a\Phi}-\frac{\pi}{Aa\Phi}\right)$$
$$-\frac{dn''}{n''-1}\cdot(C+1)c\left(\frac{\pi''D}{AB(D+1)a\Phi}-\frac{\pi'}{AB(C+1)a\Phi}\right).$$

Addatur

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)=\frac{\pi Bd\alpha}{A(B+1)a\Phi},$$

eritque

$$-\frac{dn}{n-1}\cdot\frac{(A+1)BCDa\cdot\pi''}{(D+1)a\Phi}-\frac{dn'}{n'-1}\cdot\left(\frac{(B+1)CD\pi''-(D+1)b\pi}{A(D+1)a\Phi}\right)$$
$$-\frac{dn''}{n''-1}\cdot\left(\frac{(C+1)Dc\pi''-(D+1)c\pi'}{AB(D+1)a\Phi}\right).$$

Unde pro casu $O=0$ aequatio, qua margo coloratus destruitur, erit

$$0=\frac{adn}{n-1}\cdot(A+1)BCD\pi''+\frac{bdn'}{n'-1}\cdot\left(\frac{(B+1)CD\pi''-(D+1)\pi}{A}\right)$$
$$+\frac{cdn''}{n''-1}\cdot\left(\frac{(C+1)D\pi''-(D+1)\pi'}{AB}\right).$$

III. Ut autem praeterea omnis confusio tollatur, insuper reddi oportet $d\delta=0$; unde haec aequatio nascitur

$$0=\begin{cases}\frac{adn}{n-1}\cdot(A+1)B^2C^2D^2+\frac{bdn'}{n'-1}\cdot(B+1)C^2D^2\\+\frac{cdn''}{n''-1}\cdot(C+1)D^2+\frac{ddn'''}{n'''-1}(D+1),\end{cases}$$

quae per $A^2B^2C^2D^2$ divisa dat

$$0=\frac{adn}{n-1}\cdot\frac{A+1}{A}+\frac{bdn'}{n'-1}\cdot\frac{B+1}{A^2B}+\frac{cdn''}{n''-1}\cdot\frac{C+1}{A^2B^2C}+\frac{ddn'''}{n'''-1}\frac{D+1}{A^2B^2C^2D}.$$

Circa hanc autem aequationem inprimis notandum est, si omnes lentes per facultate refringente sint praeditae, ei satisfieri haud posse; ex quo hic aequatio proprie pertinet ad casum, quo diversae refractiones locum habent.

PROBLEMA 8

325. *Si instrumentum dioptricum ex quotcunque lentibus sit compositum quarum crassitiem negligere liceat, eam determinare dispositionem, ut oculus in eo loco, quem campus apparens postulat, positus nullam confusionem sentiat.*

SOLUTIO

Sit obiecti ante lentem primam distantia $AE=a$ eiusque magnitudo $E\varepsilon=z$, quae quidem conspici queat, et statuatur $\frac{z}{a}=\Phi$. Deinde sint distantiae imaginum a radiis mediae naturae formatarum ut supra:

$$AE=a,\ BF=b,\ CG=c,\ DH=d,\ EI=e$$
$$aF=\alpha,\ bG=\beta,\ cH=\gamma,\ dI=\delta,\ eK=\varepsilon$$
etc.,

ac vocemus brevitatis gratia

$$\alpha = Aa,\ \beta = Bb,\ \gamma = Cc,\ \ \delta = Dd,\ \delta = Ee \quad \text{ete.},$$

tum vero etiam

$$\tfrac{A}{A+1} = \mathfrak{A},\ \tfrac{B}{B+1} = \mathfrak{B},\ \tfrac{C}{C+1} = \mathfrak{C},\ \tfrac{D}{D+1} = \mathfrak{D},\ \tfrac{E}{E+1} = \mathfrak{E} \ \text{ etc.}$$

Iam apertura primae lentis PP ut evanescente considerata sit ratio aperturae pro reliquis lentibus

$$QQ = \pi,\ RR = \pi',\ SS = \pi'',\ TT = \pi''' \ \text{ etc.}$$

His positis supra vidimus (§ 270) fore praeter $\alpha = Aa$

$$\begin{array}{ll} b = \frac{Aa\Phi}{\mathfrak{B}\pi - \Phi} & \beta = \frac{ABa\Phi}{\mathfrak{B}\pi - \Phi} \\ c = \frac{ABa\Phi}{\mathfrak{C}\pi' - \pi + \Phi} & \gamma = \frac{ABCa\Phi}{\mathfrak{C}\pi' - \pi + \Phi} \\ d = \frac{ABCa\Phi}{\mathfrak{D}\pi'' - \pi' + \pi - \Phi} & \delta = \frac{ABCDa\Phi}{\mathfrak{D}\pi'' - \pi' + \pi - \Phi} \\ e = \frac{ABCDa\Phi}{\mathfrak{E}\pi''' - \pi'' + \pi' - \pi + \Phi} & \varepsilon = \frac{ABCDEa\Phi}{\mathfrak{E}\pi''' - \pi'' + \pi' - \pi + \Phi} \\ \text{etc.} & \text{etc.} \end{array}$$

Deinde singularum imaginum magnitudo ita se habebit:

$$F\zeta = Aa\Phi,\ G\eta = ABa\Phi,\ H\theta = ABCa\Phi,\ I\iota = ABCDa\Phi \quad \text{etc.}$$

et ipsi aperturarum semidiametri:

$$\begin{array}{lll} \text{Lentis secundae} & QQ & = \frac{A\mathfrak{B}a\Phi}{\mathfrak{B}\pi - \Phi}\pi \\ \text{lentis tertiae} & RR & = \frac{AB\mathfrak{C}a\Phi}{\mathfrak{C}\pi' - \pi + \Phi}\pi' \\ \text{lentis quartae} & SS & = \frac{ABC\mathfrak{D}a\Phi}{\mathfrak{D}\pi'' - \pi' + \pi - \Phi}\pi'' \\ \text{lentis quintae} & TT & = \frac{ABCD\mathfrak{E}a\Phi}{\mathfrak{E}\pi''' - \pi'' + \pi' - \pi + \Phi}\pi''' \end{array}$$

etc.

Mutata iam refractionis lege $n:1$ infinite parum distantiae $\alpha,\ b,\ \beta,\ c,\ \gamma\ \ d,\ \delta$ etc. tales mutationes recipiunt:

$$d\alpha = \frac{-dn}{n-1}\cdot Aa\left(A+1\right)$$
$$d\beta = \frac{-dn}{n-1}\cdot ABa\left(\left(A+1\right)B + \frac{(B+1)\Phi}{\mathfrak{B}\pi-\Phi}\right)$$
$$d\gamma = \frac{-dn}{n-1}\cdot ABCa\left(\left(A+1\right)BC + \frac{(B+1)C\Phi}{\mathfrak{B}\pi-\Phi} + \frac{(C+1)\Phi}{\mathfrak{C}\pi'-\pi+\Phi}\right)$$
$$d\delta = \frac{-dn}{n-1}\cdot ABCDa\left(\left(A+1\right)BCD + \frac{(B+1)CD\Phi}{\mathfrak{B}\pi-\Phi} + \frac{(C+1)D\Phi}{\mathfrak{C}\pi'-\pi+\Phi} + \frac{(D+1)\Phi}{\mathfrak{D}\pi''-\pi'+\pi-\Phi}\right).$$

Porro vero habetur:

$$\frac{1}{\alpha}+\frac{1}{b}=\frac{1}{Aa\Phi}\mathfrak{B}\pi,$$
$$\frac{1}{\beta}+\frac{1}{c}=\frac{1}{ABa\Phi}\left(\mathfrak{C}\pi'-\frac{\mathfrak{B}\pi}{B}\right),$$
$$\frac{1}{\gamma}+\frac{1}{d}=\frac{1}{ABCa\Phi}\left(\mathfrak{D}\pi''-\frac{\mathfrak{C}\pi'}{C}\right),$$
$$\frac{1}{\delta}+\frac{1}{e}=\frac{1}{ABCDa\Phi}\left(\mathfrak{E}\pi'''-\frac{\mathfrak{C}\pi''}{D}\right)$$
etc.

His positis pro quolibet lentium numero seorsim formulas quaesito satisfacientes expediamus posita distantia oculi post lentem ultimam $=O$:

I. Pro unica lente

Habetur haec aequatio differentialis $\frac{-Od\alpha}{\alpha(\alpha-O)}=0$, pro quo casu tam magnitudo imaginis quam eius locus manesit invariatus, si fuerit $d\alpha=0$, hoc est $A(A+1)=0$, unde deberet esse vel $A=0$ vel $A=-1$, quorum prius visio non admittit, posterius autem lentem tollit. Tum vero ob campum esse debet $O=0$.

II. Pro duabus lentibus

Habetur haec aequatio differentialis, qua margo coloratus tollitur:

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{a}\right)-\frac{Od\beta}{\beta(\beta-O)}=0;$$

at ob campum apparentem est $\frac{O}{\beta(\beta-O)}=\frac{1}{BB}\left(\frac{1}{\alpha}+\frac{1}{b}\right)$, ita ut habeamus:

$$d\alpha\left(\frac{1}{\alpha}+\frac{1}{b}\right)-\frac{d\beta}{BB}\left(\frac{1}{\alpha}+\frac{1}{b}\right)=0.$$

Verum est

$$d\beta = BBd\alpha - \tfrac{dn}{n-1} \cdot ABa \cdot \tfrac{(B+1)\Phi}{\mathfrak{B}\pi - \Phi}.$$

Quare si O habeat valorem positivum, erit ob $\mathfrak{B}(\mathrm{B}+1) = B$

$$\tfrac{\pi}{\mathfrak{B}\pi - \Phi} = 0.$$

Sin autem valor O prodeat negativus, quo casu capitur $O = 0$, erit

$$\tfrac{(A+1)\mathfrak{B}\pi}{\Phi} = 0.$$

Omnis autem confusio penitus tolletur, si insuper fuerit

$$\tfrac{1}{\mathfrak{A}} + \tfrac{\Phi}{A\mathfrak{B}(\mathfrak{B}\pi - \Phi)} = 0.$$

III. Pro tribus lentibus

Si calculum eodem modo prosequamur, obiectum sine margine colorato conspicietur:

1. Si ex campo apparente distantia O prodeat positiva, hanc aequationem adimplendo:

$$\tfrac{\pi}{\mathfrak{B}\pi - \Phi} + \tfrac{\pi'}{\mathfrak{C}\pi' - \pi + \Phi} = 0.$$

2. Si ob distantiam O prodeuntem negativam capiatur $O = 0$, huic aequationi erit satisfaciendum:

$$\tfrac{\pi'}{\mathfrak{A}\Phi} + \tfrac{\pi'}{A\mathfrak{B}(\mathfrak{B}\pi - \Phi)} = \tfrac{\pi}{AB\mathfrak{C}(\mathfrak{B}\pi - \Phi)}.$$

Omnis autem confusio penitus tolletur, si fuerit insuper

$$\tfrac{1}{\mathfrak{A}} + \tfrac{\Phi}{A\mathfrak{B}(\mathfrak{B}\pi - \Phi)} + \tfrac{\Phi}{AB\mathfrak{C}(\mathfrak{C}\pi' - \pi + \Phi)} = 0.$$

IV. Pro quatuor lentibus

Ut obiectum sine margine colorato spectetur:

1. Si ex campo apparente distantia O prodeat positiva, huic aequationi erit satisfaciendum:

$$\tfrac{\pi}{\mathfrak{B}\pi - \Phi} + \tfrac{\pi'}{\mathfrak{C}\pi' - \pi + \Phi} + \tfrac{\pi''}{\mathfrak{D}\pi'' - \pi' + \pi - \Phi} = 0.$$

2. Sin autem capiatur $O = 0$, huic:

$$\frac{\pi''}{\mathfrak{A}\Phi} + \frac{\pi''}{A\mathfrak{B}(\mathfrak{B}\pi - \Phi)} + \frac{\pi''}{ABC(\mathfrak{C}\pi' - \pi + \Phi)} = \frac{\pi}{ABC\mathfrak{D}(\mathfrak{B}\pi - \Phi)} + \frac{\pi'}{ABC\mathfrak{D}(\mathfrak{C}\pi' - \pi + \Phi)}.$$

Omnis vero confusio penitus tolletur, si fuerit praeterea

$$\frac{1}{\mathfrak{A}} + \frac{\Phi}{A\mathfrak{B}(\mathfrak{B}\pi - \Phi)} + \frac{\Phi}{ABC(\mathfrak{C}\pi' - \pi + \Phi)} + \frac{\Phi}{ABC\mathfrak{D}(\mathfrak{D}\pi'' - \pi' + \pi - \Phi)} = 0.$$

V. Pro quinque lentibus

Ut obiectum tantum sine margine colorato conspiciatur:

1. Si ex campo apparente distantia O prodeat positiva, huic aequationi erit satisfaciendum:

$$\frac{\pi}{\mathfrak{B}\pi - \Phi} + \frac{\pi'}{\mathfrak{C}\pi' - \pi + \Phi} + \frac{\pi''}{\mathfrak{D}\pi'' - \pi' + \pi - \Phi} + \frac{\pi'''}{\mathfrak{E}\pi''' - \pi'' + \pi' - \pi + \Phi} = 0.$$

2. Sin autem capiatur $O = 0$, huic:

$$\frac{\pi'''}{\mathfrak{A}\Phi} + \frac{\pi'''}{A\mathfrak{B}(\mathfrak{B}\pi - \Phi)} + \frac{\pi'''}{ABC(\mathfrak{C}\pi' - \pi + \Phi)} + \frac{\pi'''}{ABC\mathfrak{D}(\mathfrak{D}\pi'' - \pi' + \pi - \Phi)}$$
$$= \frac{1}{ABCD\mathfrak{E}}\left(\frac{\pi}{\mathfrak{B}\pi - \Phi} + \frac{\pi'}{\mathfrak{C}\pi' - \pi + \Phi} + \frac{\pi''}{\mathfrak{D}\pi'' - \pi' + \pi - \Phi}\right).$$

Omnis autem confusio penitus tolletur, si praeterea satisfiat huic aequationi:

$$\frac{1}{\mathfrak{A}} + \frac{\Phi}{A\mathfrak{B}(\mathfrak{B}\pi - \Phi)} + \frac{\Phi}{ABC(\mathfrak{C}\pi' - \pi + \Phi)} + \frac{\Phi}{ABC\mathfrak{D}(\mathfrak{D}\pi'' - \pi' + \pi - \Phi)}$$
$$+ \frac{\Phi}{ABCE\mathfrak{E}(\mathfrak{E}\pi''' - \pi'' + \pi' - \pi + \Phi)} = 0.$$

Atque hinc manifesta est progressio ad maiorem lentium numerum.

COROLLARIUM 1

326. In casu ergo unicae lentis licet quidem obiectum a margine colorato liberare neque vero confusionem penitus tollere. Casu autem duarum lentium ne margo quidem coloratus tolli potest, siquidem oculus in eo loco, quem campi apparentis conditio postulat, teneatur.

COROLLARIUM 2

327. Quodsi vero plures duabus habeantur lentes, sufficiens quantitatum numerus adest, quarum determinatione non solum margo coloratus deleri, sed etiam forte omnis confusio penitus auferri posse videtur, praecipue si lentium numerus ternarium superet.

SCHOLION

328. Quod ergo incommodum a diversa radiorum natura oriundum adeo grave vel summo NEUTONO est visum, ut instrumenta dioptrica nullo modo ab eo liberari posse sit arbitratus; id quidem saltem, quod ad marginem coloratum attinet, ad quem NEUTONUS inprimis spectabat, iam satis feliciter tolli posse certum est, ita ut saltem ob hanc causam non opus sit ad Telescopia catoptrica confugere. Hoc autem vitio sublato si praeterea alterum confusionis fontem obstruamus, lentes scilicet nullam confusionem parientes adhibendo, nullum est dubium, quin instrumenta dioptrica ad summum perfectionis gradum evehi queant. Quae igitur hactenus particulatim circa singulas horum instrumentorum affectiones proposuimus, ea colligi conveniet, unde in capite sequente praecepta generalia pro omnium instrumentorum dioptricorom constructione tradere est visum.

SUPPLEMENTUM Vl

Ex iis, quae ante sunt adiecta, poterimus etiam problematis solutionem pro casu exhibere, quo singulae lentes peculiari refractione sunt praeditae, ubi quidem tantum postremae aequationes pro confusione vitanda mutationem quandam postulant; interim tamen etiam priores formulas, quibus locus oculi, quem campus apparens requirit, determinatur, distinctius repraesentemus.

I. *Distantia oculi post ultimam lentem* pro quovis lentium numero se habebit, ut sequitur:

Numerus lentium	O id est distantia oculi post lentem ultimam
I	0
II	$\frac{A\mathfrak{B}a\pi\Phi}{(\pi-\Phi)(\mathfrak{B}\pi-\Phi)}$ seu $\frac{\mathfrak{B}b\pi}{(\pi-\Phi)}$
III	$\frac{ABa\mathfrak{C}\pi'\Phi}{(\pi'-\pi+\Phi)(\mathfrak{C}\pi'-\pi+\Phi)}$ seu $\frac{\mathfrak{C}c\pi'}{(\pi'-\pi+\Phi)}$
IV	$\frac{ABC\mathfrak{D}a\pi''\Phi}{(\pi''-\pi'+\pi-\Phi)(\mathfrak{D}\pi''-\pi'+\pi-\Phi)}$ seu $\frac{\mathfrak{D}d\pi''}{(\pi''-\pi'+\pi-\Phi)}$
V	$\frac{ABCD\mathfrak{E}a\pi'''\Phi}{(\pi'''-\pi''+\pi'-\pi+\Phi)(\mathfrak{E}\pi'''-\pi''+\pi'-\pi+\Phi)}$ seu $\frac{\mathfrak{E}e\pi'''}{(\pi'''-\pi''+\pi'-\pi+\Phi)}$

II. Si valor ipsius O sit positivus, ad marginem coloratum tollendum sequentes aequationes sunt adimplendae:

Numerus lentium	
I	$O=0$
II	$0=\frac{bdn'}{n'-1}\cdot\frac{\pi}{Aa\Phi}$
III	$0=\frac{bdn'}{n'-1}\cdot\frac{\pi}{Aa\Phi}+\frac{cdn''}{n''-1}\cdot\frac{\pi'}{ABa\Phi}$

IV $\quad 0=\frac{bdn'}{n'-1}\cdot\frac{\pi}{Aa\Phi}+\frac{cdn''}{n''-1}\cdot\frac{\pi'}{ABa\Phi}+\frac{ddn'''}{n'''-1}\cdot\frac{\pi''}{ABCa\Phi}$

III. Si valor ipsius O prodeat negativus, quo casu capi debet $O=0$, ad marginem coloratum tollendum sequentes aequationes sunt adimplendae:

Numerus lentium	
I	$O=0$
II	$0=\frac{adn}{n-1}(A+1)B\pi$
III	$0=\frac{adn}{n-1}(A+1)BC\pi'+\frac{bdn'}{n'-1}\frac{(B+1)C\pi'-(C+1)\pi}{A}$
IV	$0=\frac{adn}{n-1}(A+1)BCD\pi''+\frac{bdn'}{n'-1}\frac{(B+1)CD\pi''-(D+1)\pi}{A}$ $+\frac{Cdn''}{n''-1}\frac{(C+1)D\pi''-(D+1)\pi'}{AB}$
	etc.

IV. Ut autem insuper omnis confusio huius generis tollatur, sequentes aequationes sunt adimplendae:

Numerus lentium	
I	$0=\frac{adn}{n-1}\cdot\frac{A+1}{A}$
II	$0=\frac{adn}{n-1}\cdot\frac{A+1}{A}+\frac{bdn'}{n'-1}\cdot\frac{B+1}{A^2B}$
III	$0=\frac{adn}{n-1}\cdot\frac{A+1}{A}+\frac{bdn'}{n'-1}\cdot\frac{B+1}{A^2B}+\frac{cdn''}{n''-1}\cdot\frac{C+1}{A^2B^2C}$
IV	$0=\frac{adn}{n-1}\cdot\frac{A+1}{A}+\frac{bdn'}{n'-1}\cdot\frac{B+1}{A^2B}+\frac{cdn''}{n''-1}\cdot\frac{C+1}{A^2B^2C}+\frac{ddn'''}{n'''-1}\cdot\frac{D+1}{A^2B^2C^2D}$
	etc.

Quarum formularum ordo hinc distinctius perspicitur quam in problemate.