CHAPTER III

CONCERNING MULTIPLE OR COMPOSITE LENSES

DEFINITION 1

96. *A double lens arises, if two lenses may be joined together on the same axis.*

Here I assume the thickness of each lens as zero, and since the distance between the lenses is put as nothing, also the thickness of the double lens will be taken as zero.

COROLLARY 1

97. There the two lenses *PP* and *QQ* (Fig. 7) by themselves almost in contact constitute a double lens ; from which nevertheless it is observed its thickness can be ignored without harm as can the thickness of each simple lens taken separately.

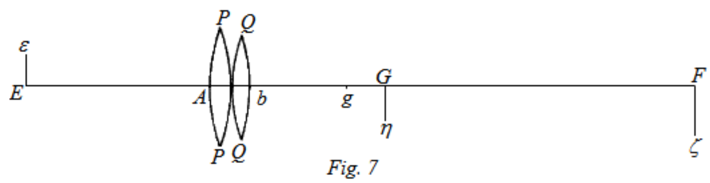

Fig. 7

For if the lenses may be touching together as a point, the phenomenon of color observed by Newton may be a concern ; then truly also we will show, how an account of the distances between the two lenses may be had.

[Newton's Rings, the colored circular white light interference fringes formed by a lens and a flat plate, or by two lenses, at the point of contact.]

COROLLARY 2

98. If the determinable distances of the foremost lens *PP* shall be *a* et α, truly of the latter lens *QQ* *b* and β, it is necessary, that there shall be $\alpha + b = 0$ or $\alpha = -b$. Then truly the position of the principal image of the object placed before the lens at the distance $AE = a$ is represented after the lens at the distance $bG = \beta$.

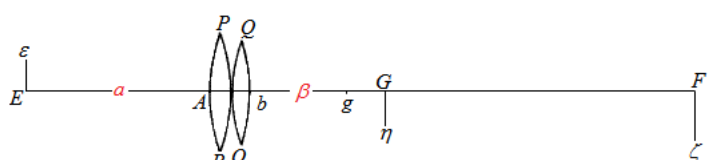

Fig. 7

COROLLARY 3

99. Therefore a and β will be as the if the determinable distances of the double lens ; and on accepting α or b at will an infinite number of suitable lenses can be shown for these distances. Then since each lens in addition may take an indefinite number λ , in addition to the infinite variety of lenses that have a place.

COROLLARY 4

100. Because the thickness is considered to be zero, the ratio of the apertures in the individual faces is the same ; clearly if the radius of the aperture of the first face shall be $= x$, in the remaining faces the aperture must be the same or at least not smaller.

SCHOLIUM

101. Clearly there is no need that the lenses be connected in contact, since perhaps the law of refraction may be disturbed: or this nevertheless may be avoided with the minimum distance interposed, that which suffices for our set up, since also the thickness of each may not be entirely zero.

PROBLEM 1

102. *To define all double lenses, for which the object* $E\varepsilon$ (Fig. 7) *at the given proposed distance before the lens AE will be represented at the given distance bG after the lens, and the diffusion of the image Gg, likewise to be defined for a given aperture of the lens.*

SOLUTION

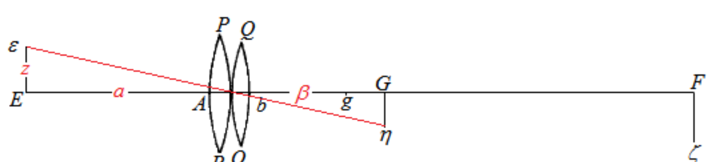

Fig. 7

The distance of the object shall be $AE = a$, and of the principal image $bG = \beta$, then truly the determinable distances of the first lens PP shall be a and α, truly of the latter lens QQ , b and β; now since the separation of the lenses is zero, there will be $\alpha + b = 0$ or $\alpha = -b$. Hence if the magnitude of the object shall be $E\varepsilon = z$, the magnitude of the principal image will be $G\eta = \frac{\beta}{a} z$, for the inverted case. Again since each lens shall be able to be formed in an infinite number of ways, the number λ shall be arbitrary for the lens *PP* and λ' for the second lens *QQ,* of which therefore the construction itself may be had :

$$\text{For lens } PP \text{ radius of face } \begin{cases} \text{anterior} = \dfrac{a\alpha}{\rho\alpha+\sigma a\pm\tau(a+\alpha)\sqrt{(\lambda-1)}} \\ \text{posterior} = \dfrac{a\alpha}{\rho\alpha+\sigma a\mp\tau(a+\alpha)\sqrt{(\lambda-1)}} \end{cases}$$

$$\text{For lens } QQ \text{ radius of face } \begin{cases} \text{anterior} = \dfrac{b\beta}{\rho\beta+\sigma b\pm\tau(b+\beta)\sqrt{(\lambda'-1)}} \\ \text{posterior} = \dfrac{b\beta}{\rho b+\sigma\beta\mp\tau(b+\beta)\sqrt{(\lambda'-1)}} \end{cases}$$

If the ratio of the refraction shall be different, for the second lens there must be written ρ', σ' and τ' in place of ρ, σ and τ .
But for the diffusion length *Gg* the radius of the aperture of the double lens shall be required to be found $= x$, and there will be

$$Gg = \mu\beta\beta xx\left\{\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)\left(\lambda\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)^2+\tfrac{\nu}{a\alpha}\right)+\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)\left(\lambda'\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)^2+\tfrac{\nu}{b\beta}\right)\right\}$$

and the inclination to the axis of the rays concurring at $g = \frac{x}{\beta}$. If the ratio of the refraction , in the part arising from the second lens there may be written μ', ν' in place of μ, ν.

COROLLARY 1

103. Therefore the two distances of this double lens required to be measured are *a* and β, truly besides the two arbitrary numbers λ and λ' together with the distance α or *b* constitute its perfect determination, from which with lenses of this kind a much greater variety occurs than with simple lenses.

COROLLARY 2

104. If from these same distances *a* and β required to be determined, by adding the arbitrary number λ^0 a simple lens may be constructed, that refers to an image with the same magnitude $G\eta = \frac{\beta}{a}z$; but for the same aperture, of which the radius $= x$, the diffusion length will be had :

$$Gg = \mu\beta\beta xx\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)\left(\lambda^0\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2+\tfrac{\nu}{a\beta}\right)$$

COROLLARY 3

105. Therefore it can happen, that a double lens may arise in one way with greater diffusion, and in another with lesser diffusion. But the simple lens is produced with the least diffusion, if $\lambda^0 = 1$; therefore then the double lenses are preferred to the simple lenses, since according to this they produce smaller diffusion.

SCHOLIUM

106. A certain simple lens can always be considered producing the same diffusion as a double lens, if we may allow all values for the number λ^0; for in whatever way the double lens were prepared, if the diffusion distance thence produced by this lens may be put equal to that arising from the simple lens, a determined value is elicited for the number λ^0 : which if it were positive or greater than one, it will be shown to be equivalent to a real simple lens, but if a number less than one or negative were produced, the simple lens would be required to be considered imaginary. But when there shall become $\lambda^0 > 1$, it is evident the same simple lens plainly can produce the same effect as the double lens, and thus it will be expedient always for the simple lens to be used rather than the double lens; but if truly there will have been produced $\lambda^0 < 1$, in which case the simple lens becomes imaginary, then the double lenses will produce an outstanding effect not hoped for from simple lenses, which thus with this added conspicuous convenience, that the diffusion length shall be smaller. Therefore we will be able to use such double lenses with maximum success , and thus these will be preferred before the simpler lenses, since the value of the number λ^0 corresponding to those had a smaller value.

PROBLEM 2

107. *With a double lens given for the two related determinable distances* $AE = a$ (Fig. 7) *and* $bG = \beta$ *, for these same distances to define a simple lens, which may produce the same diffusion of the image for the same aperture.*

SOLUTION

Therefore the whole effort reverts at this point, so that the diffusion distance *Gg* produced by a simple lens (§ 104) may be put equal to the diffusion distance arising from a double lens, of which an expression has been found in the preceding problem (§ 102), and thence the value of the number λ^0 for the construction of the simple lens may be elicited. So that which investigation may be established more conveniently, we may put :

$$\tfrac{1}{\alpha}=\tfrac{f-1}{a}+\tfrac{f}{\beta}, \text{ and there will become } \tfrac{1}{b}=\tfrac{-f+1}{a}-\tfrac{f}{\beta},$$

thus so that in place of the quantity a or b we may introduce the number f, and there will be

$$\tfrac{1}{a}+\tfrac{1}{\alpha}=f(\tfrac{1}{a}+\tfrac{1}{\beta}) \text{ and } \tfrac{1}{b}+\tfrac{1}{\beta}=(1-f)(\tfrac{1}{a}+\tfrac{1}{\beta})$$

from which the diffusion length produced in the double lens arises:

$$Gg=\mu\beta\beta xx\left\{\begin{array}{l} f\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)\left(\lambda ff\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2+\nu\left(\tfrac{f-1}{aa}+\tfrac{f}{a\beta}\right)\right) \\ +(1-f)\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)\left(\lambda'(1-f)^2\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2+\nu\left(\tfrac{1-f}{a\beta}-\tfrac{f}{\beta\beta}\right)\right)\end{array}\right\}$$

which expression is reduced to this form :

$$Gg=\mu\beta\beta xx\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)\left\{\begin{array}{l}\left(\lambda f^3+\lambda'(1-f)^2\right)\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2 \\ +\nu\left(\tfrac{f(f-1)}{aa}+\tfrac{1-2f+2ff}{a\beta}+\tfrac{f(f-1)}{\beta\beta}\right)\end{array}\right\}$$

Truly the latter term

$\tfrac{f(f-1)}{aa}+\tfrac{1-2f+2ff}{a\beta}+\tfrac{f(f-1)}{\beta\beta}$ may be changed into $f(f-1)(\tfrac{1}{a}+\tfrac{1}{\beta})^2+\tfrac{1}{a\beta}$,

and thus we will have for the double lens :

$$Gg=\mu\beta\beta xx\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)\left\{\left(\lambda f^3+\lambda'(1-f)^3-\nu f(1-f)\right)(\tfrac{1}{a}+\tfrac{1}{\beta})^2+\tfrac{\nu}{a\beta}\right\}$$

which form now may be compared more easily with the diffusion length of the simple lens and thence it may be clearly deduced :

$$\lambda^0=\lambda f^3+\lambda'(1-f)^3-\nu f(1-f).$$

But since we have introduced the number f in place of the quantities α and b, the construction of the double lens thus will itself be had :

$$\text{First lens } PP \begin{cases} \text{radius anterior face} = \dfrac{a\beta}{(\rho-\sigma(1-f))\beta+\sigma fa\pm\tau f(a+\beta)\sqrt{(\lambda-1)}} \\ \text{radius posterior face} = \dfrac{a\beta}{(\sigma-\rho(1-f))\beta+\rho fa\mp\tau f(a+\beta)\sqrt{(\lambda-1)}} \end{cases}$$

$$\text{Second lens } QQ \begin{cases} \text{radius anterior face} = \dfrac{a\beta}{(\sigma-\rho f)a+\rho(1-f)\beta\pm\tau(1-f)(a+\beta)\sqrt{(\lambda'-1)}} \\ \text{radius posterior face} = \dfrac{a\beta}{(\rho-\sigma f)a+\sigma(1-f)\beta\mp\tau(1-f)(a+\beta)\sqrt{(\lambda'-1)}}. \end{cases}$$

Then truly with the number λ^0 found, the construction of the equivalent simple lens will be :

$$\text{radius} \begin{cases} \text{of the anterior face} = \dfrac{a\beta}{\rho\beta+\sigma a\pm\tau(a+\beta)\sqrt{(\lambda^0-1)}} \\ \text{of the posterior face} = \dfrac{a\beta}{\rho\beta+\sigma a\mp\tau(a+\beta)\sqrt{(\lambda^0-1)}} \end{cases}$$

COROLLARY 1

108. Therefore whenever there were :

$$\lambda f^3+\lambda'(1-f)^3-\nu f(1-f)>1,$$

an equivalent simple lens is able to be prepared equivalent to the double lens, and therefore in these cases it will be better to use a simple lens rather than a double lens.

COROLLARY 2

109. Truly if there were

$$\lambda f^3+\lambda'(1-f)^3-\nu f(1-f)< 1,$$

on account of $\lambda^0<1$ the construction of a simple lens becomes impossible, and a double lens will be produced with less diffusion than can be obtained by any simple lens.

COROLLARY 3

110. If there were $f=0$, $\lambda^0=\lambda'$ is produced, and in the double lens no refraction will be produced in the first lens on account of the parallel faces, and the same is returned, as if only the second lens may be present. But if there may be taken $f=1$, there becomes $\lambda^0=\lambda'$, and the second lens is superfluous and therefore in each case no gain is obtained.

COROLLARY 4

111. But since the numbers λ and λ' cannot be smaller than unity and there shall be $v=0{,}232692$, from which it is apparent no value can be assumed for f between the limits 0 and 1, from which there may become $\lambda^0=0$. But if f may be taken outside these limits, certainly numbers of this kind greater than unity will be able to be assigned for λ and λ', so that there may become $\lambda^0=0$.

SCHOLIUM 1

112. Therefore on account of the number f it will be agreed to consider three cases of double lenses, just as either f is contained between the limits 0 and 1, or there were $f>1$, or $f<0$. In the first case it cannot happen, as there may become $\lambda^0=0$; but it will concern most thus to have determined a double lens, for which λ^0 may obtain a minimum value, which since it may fall less than unity, thus a more perfect lens will be required to be considered and with the greater simplicity requiring to be judged with the simple lenses being preferred. Truly for the two remaining cases $f>1$, and $f<0$ thus lenses of this kind will be required to be prepared, which produce $\lambda^0=0$, which therefore shall be required to be had for the most perfect lens. Truly here also practice is required to be respected, which since always it may be accustomed to depart from theory, it can eventuate, so that with a negligible error put together, λ^0 not only may not vanish, but thus may exceed unity, in which case certainly it would be expedite to use the simple lens.

SCHOLIUM 2

113. Since there shall be so great concern about an account of the aberration, from which in practice a lens can scarcely be had to be free, that also will be agreed to be considered carefully in simple lenses. But we have demanded that a lens can be constructed from given determinable distances, in which the number, in which the number λ may obtain a given value, as long as it shall not be less than unity; therefore here it is required to be observed, where λ was greater, thus to become more difficult to avoid an error ; for if in the construction the smallest error may be made, thus another

more diverse value of λ will arise, where λ would be greater. Truly on the other hand, since one shall be the minimum value, which λ can receive, from the minimum nature it is apparent, even if in practice it may depart notably from the described rule, yet thence scarcely a sensible difference in λ is going to be too great. From which we conclude most happily from the success of simple lenses of this kind able to be prepared, for which there shall soon become $\lambda = 1$, nor here practical errors, unless they were enormous, certainly to be a cause of concern. Then where a smaller number λ may be greater than unity, thus we will be able to be more certain of success, but not thus in the case where $\lambda = 1$; but if there shall be a need for a lens of this kind, for which the value of λ must be a number large enough, will be satisfied in practice with difficulty and perhaps it will require a large number of lenses to be prepared, before one may be obtained satisfying the goal. On account of which, in practice we may wish to consider, we need scarcely examine other lenses, unless for which the number λ either shall be unity itself or a little greater. But if other lenses may be required for some suitable design, for the labour will not be effective, since perhaps after several ineffective trials we may return finally to the initial wishes.

PROBLEM 3

114. *To define that double lens, for which, if the number f may be taken between the numbers* 0 *and* 1*, the number* λ^0 *may arrive at the minimum value.*

SOLUTION

With the same put in place, which have been established in the preceding problems, we find to be

$$\lambda^0 = \lambda f^3 + \lambda'(1-f)^3 - \nu f(1-f),$$

where, since f must be assumed to lie between the limits 0 and 1, both the terms λf^3 and $\lambda'(1-f)^3$ will be positive. Whereby so that λ^0 may obtain the smallest value of all, it is necessary each number λ and λ' be attributed the minimum value, of which it is capable.

Therefore there shall be $\lambda = 1$ and $\lambda' = 1$, and we will have

$$\lambda^0 = 1 - 3f + 3ff - \nu f + \nu ff = 1 - (3+\nu)f(1-f),$$

so that which expression may be reduced to a minimum, it will be required to make $f(1-f)$ a maximum, that which arises by making $f = \frac{1}{2}$; and hence there becomes

$$\lambda^0 = 1 - \tfrac{1}{4}(3+\nu) = \tfrac{1-\nu}{4} = 0{,}191827.$$

Whereby the construction of this double lens will be had thus:

$$\text{For the lens } PP \text{ the radius of the } \begin{cases} \text{anterior face} = \frac{2a\beta}{(2\rho-\sigma)\beta+\sigma a} \\ \text{posterior face} = \frac{2a\beta}{(2\sigma-\rho)\beta+\rho a} \end{cases}$$

$$\text{For the lens } QQ \text{ the radius of the } \begin{cases} \text{anterior face} = \frac{2a\beta}{(2\sigma-\rho)a+\sigma\beta} \\ \text{posterior face} = \frac{2a\beta}{(2\rho-\sigma)a+\sigma\beta} \end{cases}$$

and if the radius of the aperture shall be $= x$, the diffusion distance will be

$$Gg = \mu\beta\beta xx\left(\frac{1}{a}+\frac{1}{\beta}\right)\left(0{,}191827\left(\frac{1}{a}+\frac{1}{\beta}\right)^2+\frac{\nu}{\alpha\beta}\right).$$

If we may use glass of this kind, for which there is $n = 1{,}60 = \frac{8}{5}$, then on account of $\nu = \frac{4}{15}$, there will be produced $\lambda^0 = 0{,}183333$, and thus this kind of glass may at this point produce a smaller confusion.

COROLLARY 1

115. If with the same determinable distances a and β a simple lens producing the same minimum diffusion may be constructed, which happens by taking

$$\text{the radius of the anterior face } [f] = \frac{a\beta}{\rho\beta+\sigma a}, \text{ of the posterior } [g] = \frac{a\beta}{\rho a+\sigma\beta},$$

the diffusion distance will become $\mu\beta\beta xx(\frac{1}{a}+\frac{1}{\beta})((\frac{1}{a}+\frac{1}{\beta})^2+\frac{\nu}{a\beta})$.

COROLLARY 2

116. Therefore it may appear from the double lens described much smaller diffusion arises than from a simple lens, even if this now shall be constructed for the minimum diffusion. Indeed since the remaining parts shall be equal, the coefficient of the term $(\frac{1}{a}+\frac{1}{\beta})^2$ is smaller by five times in the double lens than in the single lens.

COROLLARY 3

117. If we may put $\lambda = 1$ and $\lambda' = 1$, or even $\lambda = \lambda'$, a smaller value for λ^0 is unable to be obtained, as we have found, even if we may wish to allow other values for f. From which if each lens itself may be prepared now by the minimum dimension, for the double lens the value of λ^0 is unable to become smaller than 0,191827.

SCHOLlUM.

118. Therefore double lenses of this kind are especially noteworthy, since in place of using simple ones they may produce far less diffusion, from which the use of these will be greatest in the construction of telescopes and microscopes. Not only truly are they endowed with this significant property, but also the construction of these in practice is liable to the minimum difficulties: on account of which, even if it may depart a little from the prescribed rules, the effect still may show scarcely any change. Or if indeed in the construction of each separately a small error may be committed, the values of the numbers λ and λ' hardly may sensibly exceed unity or if in the quantity f the true value may not be observed exactly $f = \frac{1}{2}$, the error scarcely is observed, since these numbers have been deduced from the nature of the minimum. But in whatever manner it may have departed from the prescribed rules, the value of λ^0 itself will be produced a little greater. Just as if errors of this kind may be committed, so that there shall be

$$\lambda = 1 + \tfrac{1}{10}, \lambda' = 1 + \tfrac{1}{10} \text{ and } f = \tfrac{1}{2} \pm \tfrac{1}{20},$$

$\lambda^0 = 0{,}191827 + 0{,}03383$ or $\lambda^0 = 0{,}2257$ will be produced, thus so that the distinction of the parts may amount only to a thirtieth part of unity. Moreover it may be understood easily, provided λ^0 may be produced less than $\frac{1}{4}$, which cannot readily be considered to be outstanding from these double lenses as significant use is to be expected. Therefore even if double lenses of this kind can be constructed, for which the number λ^0 plainly may vanish, on that account a hazardous construction of these may be seen to be preferred, as soon will appear clearer.

PROBLEM 4

119. *For the given determinable distances* $AE = a$ *and* $bG = \beta$ *to find these double lenses, in which there shall be* $\lambda^0 = 0$.

SOLUTION

Since there may be unable to become $\lambda^0 = 0$, unless the number f may be taken outside the limits 0 and 1, and likewise the numbers λ and λ' were unequal, thus so that at least one or the other lens separately may not produce minimum diffusion, we may put either to be $f > 1$ or $f < 0$.
Therefore in the first place there shall be $f = 1 + \xi$ and since there shall be $\lambda^0 = \lambda(1+\xi)^3 - \lambda'\xi^3 + \nu\xi(1+\xi)$, so that there may become $\lambda^0 = 0$, there is required to become

$$\lambda' = \lambda(1+\tfrac{1}{\xi})^3 + \tfrac{\nu(1+\xi)}{\xi^2},$$

from which λ' by necessity will be greater than one; of which the value may not be produced exceedingly large, it will be convenient to take $\lambda = 1$, thus so that there shall be

$$\lambda' = (1+\tfrac{1}{\xi})^3 + \tfrac{0{,}282692(1+\xi)}{\xi^2} \text{ et } \lambda = 1.$$

Therefore whatever value may be attributed to ξ, a double lens is had, for which there shall be $\lambda^0 = 0$, and therefore the diffusion length $= \mu\beta\beta xx(\frac{1}{a} + \frac{1}{\beta})\frac{\nu}{a\beta}$. We may consider some special cases :

$$f = \tfrac{3}{2};\ \xi = \tfrac{1}{2};\ \lambda = 1 \text{ et } \lambda' = 28{,}396152$$
$$f = 2;\ \xi = 1;\ \lambda = 1 \text{ et } \lambda' = 8{,}465384$$
$$f = 3;\ \xi = 2;\ \lambda = 1 \text{ et } \lambda' = 3{,}549519$$
$$f = 4;\ \xi = 3;\ \lambda = 1 \text{ et } \lambda' = 2{,}473789$$
$$f = 5;\ \xi = 4;\ \lambda = 1 \text{ et } \lambda' = 2{,}025841$$
$$f = 6;\ \xi = 5;\ \lambda = 1 \text{ et } \lambda' = 1{,}783846$$

etc.

For the other case there shall be $f=-\xi$, and thus $\lambda^0=-\lambda\xi^3+\lambda'(1+\xi)^3+v\xi(1+\xi)$, from which on making $\lambda^0=0$ there is produced

$$\lambda=\lambda'(1+\tfrac{1}{\xi})^3+\tfrac{v(1+\xi)}{\xi^2}.$$

Therefore it will be convenient to set $\lambda'=1$, and the following cases may be noted for λ :

$$f=-\tfrac{1}{2};\ \xi=\tfrac{1}{2};\ \lambda=28,396152 \text{ and } \lambda'=1$$
$$f=-1;\ \xi=1\ ;\ \lambda=\ \ 8,465384 \text{ and } \lambda'=1$$
$$f=-2;\ \xi=2;\ \lambda=\ \ 3,549519 \text{ and } \lambda'=1$$
$$f=-3;\ \xi=3;\ \lambda=\ \ 2,473789 \text{ and } \lambda'=1$$
$$f=-4;\ \xi=4;\ \lambda=\ \ 2,025841 \text{ and } \lambda'=1$$
$$f=-5;\ \xi=5;\ \lambda=\ \ 1,783846 \text{ and } \lambda'=1$$

etc.

and thus it is apparent double lenses can be prepared in an infinite number of ways, for which there shall be $\lambda^0=0$ and the diffusion distance

$$Gg=\mu\beta\beta xx(\tfrac{1}{a}+\tfrac{1}{\beta})\tfrac{v}{a\beta}.$$

SCHOLIUM 1

120. If lenses of this kind may be able to be prepared more accurately, there is no doubt, why not with the preceding preferred, therefore so that the diffusion from these may be diminished still more. Truly it is painful to observe, because the smallest error involved in the construction of these may destroy almost all the use. So that we may be able to decide this easily, we will examine that case, where there is

$$f=5,\ \lambda=1 \text{ and } \lambda'=2,025841$$

and hence

$$\lambda^0=\lambda f^3-\lambda'(f-1)^3+vf(f-1)=0.$$

But we may put in the construction an error to be established, so that actually there shall not be $f=5$, but $f=5\tfrac{1}{10}$, while the numbers λ et λ' may obtain their true values : but on this account an error can scarcely be avoided and there shall not become $\lambda^0=0$, but thus $\lambda^0=-\ 2,011$; and thus this double lens with the simplest length is required to be discarded; in a similar manner if f were = 5, but either λ or λ' were to err a little from the prescribed value, at once a huge difference in the value of λ^0 would emerge. Indeed an error to be less feared may be seen in that kind, in which $\lambda = 1$, $\lambda' = 28{,}396152$ and $f = \frac{3}{2}$; but in addition because the simple posterior lens may be most difficult to prepare, for which it may follow that λ' be assigned a precise value, such a lens is unsuitable for use in an instrument on account of the large curvature of the other face. Since which shall be thus, because while a small error made in the preparation of a lens of this kind, so that λ^0 thus may increase beyond one, we can scarcely hope, that such a double lens may be perfected at any time, for which λ^0 may be diminished as far as $\frac{1}{5}$. But success will scarcely be able to fail in lenses of the preceding kind, unless in practice it may greatly in error from the prescribed rules: from which it will only be possible to use these double lenses with pleasure, provided these, as we have described in the present problem, may be seen to be completely discarded.

SCHOLIUM 2

121. Therefore for the two determinable distances a and β a lens of this kind can always be prepared, from which the diffusion distance may arise

$$Gg = \mu\beta\beta xx\left(\frac{1}{a} + \frac{1}{\beta}\right)\left(\lambda^0\left(\frac{1}{a} + \frac{1}{\beta}\right)^2 + \frac{\nu}{a\beta}\right)$$

thus so that λ^0 may be able to denote some number. Indeed for this, it will be required for this equation to be satisfied :

$$\lambda^0 = \lambda f^3 + \lambda'(1-f)^3 - \nu f(1-f),$$

which can happen always, since f clearly may depend on our arbitrary choice and the numbers λ et λ' must be taken not less than one only. But with these three numbers defined λ, λ' and f thus, so that λ^0 may obtain a given value, the two simple lenses, from which the doublet is required to be composed, must be put together following the formulas given in § 107. Where indeed these are to be held in the manner which we have observed, in practice these lenses to be obtained more easily, when the numbers λ and λ' are a little greater than one, f truly may be denoted almost by $\frac{1}{2}$, since otherwise, where these numbers may depart more from these terms, there the danger will be greater, lest the effect may become fail completely. Finally since we have reduced the diffusion arising from a double lens to the same form as that arising from a single lens, by which the diffusion of a single lens is expressed, so that in a similar manner we may prevail to define the diffusion arising from more lenses .

SUPPLEMENT 1

FOR TWO LENSES

If the index of refraction for the anterior lens shall be $n:1$, truly the other ratio $n':1$ may be used for the posterior lens, here the problems will be able to be resolved treated in the following manner.

FOR PROBLEM 1

If for the numbers ρ, σ and τ, which arise from n, in a similar manner the values ρ', σ' and τ' may be sought from n', they will be

$$\text{For the lens } PP \text{ the radius of the } \begin{cases} \text{anterior face} = \dfrac{a\alpha}{\rho\alpha+\sigma a\pm\tau(a+\alpha)\sqrt{(\lambda-1)}} \\ \text{posterior face} = \dfrac{a\alpha}{\rho a+\sigma\alpha\mp\tau(a+\alpha)\sqrt{(\lambda-1)}} \end{cases}$$

$$\text{For the lens } QQ \text{ the radius of the } \begin{cases} \text{anterior face} = \dfrac{b\beta}{\rho'\beta+\sigma' b\pm\tau'(b+\beta)\sqrt{(\lambda'-1)}} \\ \text{posterior face} = \dfrac{b\beta}{\rho' b+\sigma'\beta\mp\tau'(b+\beta)\sqrt{(\lambda'-1)}} \end{cases}$$

and with the values μ', ν' defined in a similar manner from the ratio $n':1$ the diffusion distance will be defined

$$\beta\beta xx\left\{\mu(\tfrac{1}{a}+\tfrac{1}{\alpha})\left(\lambda\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)^2+\tfrac{\nu}{a\alpha}\right)+\mu'(\tfrac{1}{b}+\tfrac{1}{\beta})\left(\lambda'\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)^2+\tfrac{\nu'}{b\beta}\right)\right\};$$

the rest will remain as in the problem.

FOR PROBLEM 2

Because here two kinds of glass occur, we may put the simple lens sought to be prepared from some other kind of glass, of which the ratio of refraction shall be $n^0:1$, from which the numbers μ^0 and ν^0 may be produced ; but it would be superfluous to have computed the numbers ρ^0, σ^0 and τ^0, since the radii of the faces would become imaginary, thus so that there shall be no need to express these, and because for this simple equivalent lens the number λ^0 has been introduced, the diffusion length of this lens will be

$$\beta\beta xx\cdot\mu^0(\tfrac{1}{a}+\tfrac{1}{\beta})\left(\lambda^0\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2+\tfrac{\nu^0}{a\beta}\right),$$

which may be put so that it may become equal to the diffusion length of the double lens, as has been made in the problem,

$$\tfrac{1}{\alpha}=\tfrac{f-1}{a}+\tfrac{f}{\beta}\ \text{ and }\ \tfrac{1}{b}=\tfrac{1-f}{a}-\tfrac{f}{\beta},$$

so that there may be produced

$$\tfrac{1}{a}+\tfrac{1}{\alpha}=f(\tfrac{1}{a}+\tfrac{1}{\beta})\ \text{ and }\ \tfrac{1}{b}+\tfrac{1}{\beta}=(1-f)(\tfrac{1}{a}+\tfrac{1}{\beta}),$$

from which it may arrive at this equation

$$\begin{aligned}&\mu f\left(\lambda f^2\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2+\nu\left(\tfrac{f-1}{a^2}+\tfrac{f}{a\beta}\right)\right)+\\&\mu'(1-f)\left(\lambda'(1-f)^2\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2+\nu'\left(\tfrac{1-f}{a\beta}-\tfrac{f}{\beta^2}\right)\right)\\&=\mu^0\left(\lambda^0\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2+\tfrac{\nu^0}{a\beta}\right)\end{aligned}$$

which can be represented thus :

$$\begin{aligned}&\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2\left(\mu\lambda f^3+\mu'\lambda'(1-f)^3\right)+\left(\tfrac{f-1}{a}+\tfrac{f}{\beta}\right)\left(\tfrac{\mu\nu f}{a}-\tfrac{\mu'\nu'(1-f)}{\beta}\right)\\&=\mu^0\lambda^0\left(\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2\right)+\tfrac{\nu^0\mu^0}{a\beta}.\end{aligned}$$

From which

$$\lambda^0 = \frac{\mu\lambda f^3 + \mu'\lambda'(1-f)^3}{\mu^0} + \frac{aa\beta\beta}{\mu^0(\alpha+\beta)^2}\left(\left(\frac{f-1}{a} + \frac{f}{\beta}\right)\left(\frac{\mu\nu f}{a} - \frac{\mu'\nu'(1-f)}{\beta}\right) - \frac{\nu^0\mu^0}{a\beta}\right).$$

Finally if from the ratio of the refraction n':1 the numbers ρ', σ' and τ' may be computed, the radii of the faces of the double lenses will be

For the first lens PP the radius of the
$$\begin{cases} \text{anterior face} = \dfrac{a\beta}{(\rho-\sigma(1-f))\beta+\sigma fa\pm\tau f(a+\beta)\sqrt{(\lambda-1)}} \\ \text{posterior face} = \dfrac{a\beta}{(\sigma-\rho(1-f))\beta+\rho fa\mp\tau f(a+\beta)\sqrt{(\lambda-1)}} \end{cases}$$

For the second lens QQ the radius of the
$$\begin{cases} \text{anterior face} = \dfrac{a\beta}{(\sigma'-\rho')a+\rho'(1-f)\beta\pm\tau'(1-f)(a+\beta)\sqrt{(\lambda'-1)}} \\ \text{posterior face} = \dfrac{a\beta}{(\rho'-\sigma')a+\sigma'(1-f)\beta\mp\tau'(1-f)(a+\beta)\sqrt{(\lambda'-1)}}. \end{cases}$$

FOR PROBLEM 3

Here in this problem I will deal not only with the difference in refraction n and n', but also I will have an account of the distance between the two lenses. Therefore at first I will recall each lens for the determinable distances of the double lens, a and β, on putting

$$\tfrac{1}{a} + \tfrac{1}{\alpha} = f(\tfrac{1}{a} + \tfrac{1}{\beta}) \quad \text{and} \quad \tfrac{1}{b} + \tfrac{1}{\beta} = g(\tfrac{1}{a} + \tfrac{1}{\beta})$$

so that there shall be

$$\tfrac{1}{\alpha} = \tfrac{f-1}{a} + \tfrac{f}{\beta} \quad \text{and} \quad \tfrac{1}{b} = \tfrac{g}{a} + \tfrac{g-1}{\beta}$$

and hence

$$\alpha = \frac{a\beta}{fa+(f-1)\beta} \quad \text{and} \quad b = \frac{a\beta}{(g-1)a+g\beta}$$

and thus the distance between the lenses

$$\alpha + b = \frac{a\beta(f+g-1)(a+\beta)}{(fa+(f-1)\beta)((g-1)a+g\beta)},$$

which must be $=0$, it will be required to take $g = 1-f$; but if we allow some distance between the lenses, we may establish $f+g-1=\omega$, with ω indicating some minimum fraction, either positive or negative, so that between the lenses may be produced positive and very small. Therefore since hence there shall be $g = 1+\omega - f$, the separation of the lenses will be

$$\alpha + b = \frac{a\beta(a+\beta)\omega}{(fa+(f-1)\beta)((\omega-1)a+(1+\omega-f)\beta)}.$$

Now the diffusion distance is expressed thus:

$$\beta\beta xx\cdot(\tfrac{1}{a}+\tfrac{1}{\beta})\begin{pmatrix}\lambda\mu f^3\left(\frac{1}{a}+\frac{1}{\beta}\right)^2+\lambda'\mu' g^3\left(\frac{1}{a}+\frac{1}{\beta}\right)^2\\ +\frac{\mu\nu f}{a}\left(\frac{f-1}{a}+\frac{f}{\beta}\right)+\frac{\mu'\nu' g}{\beta}\left(\frac{g}{a}+\frac{g-1}{\beta}\right)\end{pmatrix}$$

which since in this problem it must be treated thus, so that both f as well as g may be taken positive, and this minimum formula may be returned, it is clear minimum values must be attributed to the letters λ and λ' , evidently $\lambda = 1$ and $\lambda' = 1$; then for this case the minimum values may be defined conveniently for f and g, where, since $f + g - 1 = \omega$ thus must be constant on differentiation, we will have $dg = -df$;from which we obtain this equation :

$$(3\mu f^2 - 3\mu' g^2)\left(\frac{1}{a}+\frac{1}{\beta}\right)^2+\frac{\mu\nu}{a}\left(\frac{2f-1}{a}+\frac{2f}{\beta}\right)-\frac{\mu'\nu'}{\beta}\left(\frac{2g}{a}+\frac{2g-1}{\beta}\right)=0,$$

for which it will be satisfied approximately by putting $f = g = \frac{1+\omega}{2}$, with which values substituted the minimum diffusion length itself will be approximately :

$$\frac{\beta^2 x^2(1+\omega)}{8}(\tfrac{1}{a}+\tfrac{1}{\beta})\begin{pmatrix}\left(\frac{1}{a}+\frac{1}{\beta}\right)^2(1+\omega)^2(\mu+\mu')\\ +\frac{2\mu\nu}{a}\left(\frac{\omega-1}{a}+\frac{1+\omega}{\beta}\right)+\frac{2\mu'\nu'}{\beta}\left(\frac{1+\omega}{a}+\frac{\omega-1}{\beta}\right)\end{pmatrix}$$

Then truly the distance between the lenses will be

$$\alpha + b = \frac{4a\beta(a+\beta)\omega}{((1+\omega)a+(1-\omega)\beta)((\omega-1)a+(1+\omega)\beta)} = \frac{4a\beta(a+\beta)\omega}{-(1-\omega^2)(a^2+\beta^2)+2(1+\omega^2)a\beta},$$

the denominator of which shall be negative on account of the minimum *w*, it is necessary the fraction ω must be taken negative ; and thus hence the diffusion length is rendered smaller.

COROLLARY

Therefore if the distance of the object a were infinite or $a=\infty$, in the first place the separation of the lenses will be had $=\frac{-4\beta\omega}{1-\omega^2}$ and the following diffusion distance

$$\frac{x^2(1+\omega)}{8\beta}((1+\omega)^2(\mu+\mu')-2\mu'\nu'(1-\omega))$$

so that, since ω must be negative, it cannot be very much smaller, than if the separation of the lenses were zero.

FOR PROBLEM 4

Also in this problem we may not disregard the separation of the lenses ; and with the reduction made as before, we may establish the diffusion length clearly vanishing ; which cannot be allowed to happen, unless either of the letters f and g shall be negative, so that with that done, when the confusion arising from the refraction of the different rays must be returned to zero, as we will see further; this case deserves to be treated much more fully. Therefore there may be put $g=-\zeta f$, where ζ may be determined from that condition, that in the first place for the separation of the lenses there shall be :

$$\alpha+b=\frac{-a\beta(a+\beta)(f-\zeta f-1)}{(fa+(f-1)\beta)((\zeta f+1)a+\zeta f\beta)},$$

and on putting $f+g-1=\omega$ there may be produced $f-\zeta f=1+\omega$ and thus

$$f=\frac{1+\omega}{1-\zeta}\text{ and } g=\frac{-\zeta(1+\omega)}{1-\zeta}$$

and thus

$$\alpha+b=\frac{-a\beta(a+\beta)\omega(1-\zeta)^2}{((1+\omega)a+(\omega+\zeta)\beta)((1+\omega\zeta)a+\zeta(1+\omega)\beta)},$$

and if ω may be removed in the denominator, there will become

$$\alpha+b=\frac{-a\beta(a+\beta)(1-\zeta)^2\omega}{(a+\zeta\beta)^2},$$

and thus it is apparent ω must be taken negative.
Moreover on putting $g=-\zeta f$ the diffusion distance shall become :

$$\beta^2x^2(\tfrac{1}{a}+\tfrac{1}{\beta})\left(\begin{array}{l}(\lambda\mu f^3-\lambda'\mu'\zeta^3f^3)\left(\frac{1}{a}+\frac{1}{\beta}\right)^2\\ +\frac{\mu\nu f}{a}\left(\frac{f-1}{a}+\frac{f}{\beta}\right)-\frac{\mu'\nu'\zeta f}{\beta}\left(\frac{-\zeta f}{a}+\frac{(\zeta f+1)}{\beta}\right)\end{array}\right),$$

to be returned to zero ; from which there follows to become :

$$\lambda' = \frac{\lambda\mu}{\mu'\zeta^3} + \frac{\mu\nu \cdot a \cdot \beta^2}{\mu'\zeta^2 f^2 (a+\beta)^2}\left(\frac{f-1}{a} + \frac{f}{\beta}\right) + \frac{\nu' \cdot a^2 \beta}{\zeta^2 f^2 \cdot (a+\beta)^2}\left(+\frac{\zeta f}{a} + \frac{\zeta f+1}{\beta}\right)$$

since the value must be greater than one, if perhaps it may arise, that it may be produced smaller than one, then it will be agreed not for λ' but λ to be defined, where it is to be noted $f = \frac{1+\omega}{1-\zeta}$. And hence by the formulas given before the radius of each lens will be made easy to be deduced.

And if the formula given above for the diffusion length now is adapted for the case, in which the diffusion length is zero, nevertheless since here we have assumed the minimum distance, thence there is no fear of error.

DEFINITION 2

122. *A triple lens is one which consists of three simple lenses in contact on a common axis.*

Indeed here also I ignore the thickness, even if by necessity it shall be greater than with double lenses. But in the supplement it will be shown, how also an account of the distances between the lenses shall be obtained.

COROLLARY

123. Therefore it will be possible for a triplicate lens to be considered as if composed from a double lens and from a single lens, and this is a twofold manner, provided either the two anterior or the two posterior lenses are considered to be placed together.

SCHOLIUM

124. Indeed with triple lenses it must be agreed to be conceded in practice, that it does not follow to have the same convenience as with simple or double lenses. But this convenience is consistent with the smallness of the numbers λ, which provided it were greater than unity, to be outstanding always as a simple lens, since truly the circumstances in the expression of the diffusion length require a small number λ, for multiple lenses will have vanished. Therefore since we have expounded how to put together double lenses of this kind, in which the value of λ not only may be diminished as far as zero, but also thus may be diminished to negative values, the use of a triplicate lens is considered superfluous. Now truly we have observed in practice scarcely any duplicate

lenses of this kind are able to be prepared for which the value of λ shall be less than 0,191827, on account of which, if the slightest error may be committed, all the labor may be returned in vain. Therefore from these cases especially, when there is a need for a smaller value of the number λ, triple lenses are required to be called into use ; and because according to practice, by not all every equation able to be constructed, inevitable errors here too will have to be looked for especially, so that it may be apparent, until the number λ may be able to be diminished successively, and if at this stage there will be a need for a smaller value, thus quadruple lenses will be required to be considered.

PROBLEM 5

125. *With the two determinable distances given to define all triple lenses, and likewise the diffusion length arising from these.*

SOLUTION

$E\varepsilon$ (Fig. 8) shall be the object, of which the magnitude $= z$, and the distance from the lens $AE = a$; then the determinable distances of the first lens *PP* shall be a and α, of the second lens *QQ*, b and β, and of the third *RR*, c and γ. With these in place, since the lenses are assumed to be in close contact,

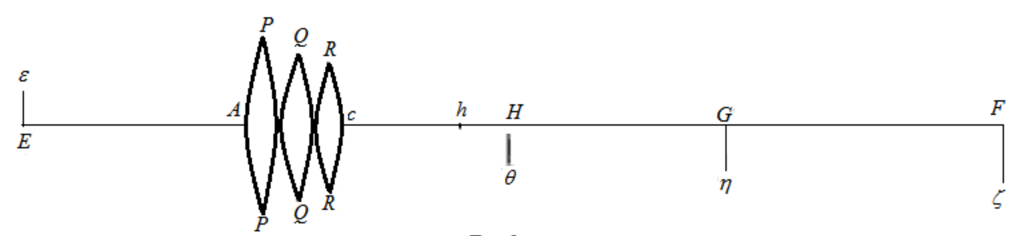

Fig. 8.

there will be $\alpha + b = 0$, $\beta + c = 0$, and both a and γ will be the determinable distances of the triple, thus so that α and β may be left to our choice. Therefore if the first lens *PP* may be present alone, the image will be represented at $F\zeta$, so that there must be $AF = \alpha$ and $F\zeta = \frac{\alpha}{a}z$; if the two first lenses *PP* and *QQ* only were present, the image is shown at $G\eta$, so that there were $AG = \beta$ and $G\eta = \frac{\beta}{a}z$; but it is returned to $H\theta$ by the third lens, so that there shall be $CH = \gamma$ and $H\theta = \frac{\gamma}{a}z$, with the inverse in place on account of $\frac{\alpha\beta}{bc} = 1$. At this point clearly the matter may be had thus, as if a simple lens may be had at A for adapted for the determinable distances a and γ.

But if we may consider the diffusion distance Hh, we must consider the shape of the individual lenses in the computation, since besides the determinable distances the arbitrary numbers $\lambda,\ \lambda',\ \lambda''$ are involved; from which the faces of the individual lenses have been defined above in § 91. But from the same place it is deduced for the aperture, of which the radius $=x$, the diffusion length on account of $\frac{\alpha}{b}=-1$ and $\frac{\beta}{c}=-1$, to become

$$Hh=\mu\gamma\gamma xx\left(\begin{array}{l}+(\frac{1}{a}+\frac{1}{\alpha})\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{v}{a\alpha}\right)+(\frac{1}{b}+\frac{1}{\beta})\left(\lambda'\left(\frac{1}{b}+\frac{1}{\beta}\right)^2+\frac{v}{b\beta}\right)\\+(\frac{1}{c}+\frac{1}{\gamma})\left(\lambda''\left(\frac{1}{c}+\frac{1}{\gamma}\right)^2+\frac{v}{c\gamma}\right)\end{array}\right)$$

which expression so that it may be reduced to the form of a single lens, we may put:

$$\tfrac{1}{a}+\tfrac{1}{\alpha}=f(\tfrac{1}{a}+\tfrac{1}{\gamma}),\ \tfrac{1}{b}+\tfrac{1}{\beta}=g(\tfrac{1}{a}+\tfrac{1}{\gamma}),\ \tfrac{1}{c}+\tfrac{1}{\gamma}=h(\tfrac{1}{a}+\tfrac{1}{\gamma}),$$

and because $\frac{1}{\alpha}+\frac{1}{b}=0$ and $\frac{1}{\beta}+\frac{1}{c}=0,$ with these equations being added we arrive at : $f+g+h=1$. Again truly there will be

$$\tfrac{1}{\alpha}=f(\tfrac{1}{a}+\tfrac{1}{\gamma})-\tfrac{1}{a},\ \tfrac{1}{b}=\tfrac{1}{a}-f(\tfrac{1}{a}+\tfrac{1}{\gamma}),$$
$$\tfrac{1}{\beta}=(f+g)(\tfrac{1}{a}+\tfrac{1}{\gamma})-\tfrac{1}{a},\ \tfrac{1}{c}=\tfrac{1}{a}-(f+g)(\tfrac{1}{a}+\tfrac{1}{\gamma}),$$

or if, because $1=f+g+h$,

$$\tfrac{1}{a}=\tfrac{f+g+h}{a},\quad \tfrac{1}{\alpha}=\tfrac{-g-h}{a}+\tfrac{f}{\gamma},$$
$$\tfrac{1}{b}=\tfrac{g+h}{a}-\tfrac{f}{\gamma},\quad \tfrac{1}{\beta}=-\tfrac{h}{a}+\tfrac{f+g}{\gamma},$$
$$\tfrac{1}{c}=\tfrac{h}{a}-\tfrac{f+g}{\gamma},\quad \tfrac{1}{\gamma}=\tfrac{f+g+h}{\gamma}.$$

From which since the diffusion length shall become

$$Hh=\mu\gamma\gamma xx(\tfrac{1}{a}+\tfrac{1}{\gamma})\left((\lambda f^3+\lambda' g^3+\lambda'' h^3)(\tfrac{1}{a}+\tfrac{1}{\gamma})^2+v(\tfrac{f}{a\alpha}+\tfrac{g}{b\beta}+\tfrac{h}{c\gamma})\right)$$

that is reduced to this form

$$Hh=\mu\gamma\gamma xx(\tfrac{1}{a}+\tfrac{1}{\gamma})\left((\lambda f^3+\lambda' g^3+\lambda'' h^3-v(1-f)(1-g)(1-h))(\tfrac{1}{a}+\tfrac{1}{\gamma})^2+\tfrac{v}{a\gamma}\right).$$

Moreover the inclination to the axis of the rays concurring at h will be $=\frac{x}{\gamma}$.

COROLLARY 1

126. Therefore this triple lens produces the same diffusion length as a simple lens constructed will produce for the same determinable distances, with an arbitrary number of this being present (indicated by the letter $\lambda^{(3)}$)

$$\lambda f^3 + \lambda' g^3 + \lambda'' h^3 - \nu(1-f)(1-g)(1-h),$$

where certainly there is $f + g + h = 1$.

COROLLARY 2

127. On which account therefore this amount not only must be returned less than one but also by the fraction 0,191827, thus it is evident, so that in practice not a great amount of aberration may be established, as far as use will be made from triple lenses.

COROLLARY 3

128. If there shall be either $f = 0,\ g = 0,$ or $h = 0$, one of the lenses will have parallel faces, and the triple lens will be equivalent to a double lens ; and if two of the letters *f, g, h* likewise may vanish, with the third going into unity, the case of a single lens will be expressed.

COROLLARY 4

129. If there shall be $f = 1$ and thus $h = -g$, the value of the number λ for the triple lens will be equivalent to the single lens, which likewise arises , if there were either $g = 1$ or $h = 1$.

COROLLARY 5

130. But with suitable numbers taken for *f, g, h* on account of $f + g + k = 1$, construction of the triple lens from the formula § 91 has shown it taken to be desired :

$$\frac{1}{\alpha} = \frac{-1+f}{a} + \frac{f}{\gamma},\quad \frac{1}{b} = \frac{1-f}{a} - \frac{f}{\gamma},\quad \frac{1}{\beta} = \frac{-h}{a} + \frac{1-h}{\gamma},\quad \frac{1}{c} = \frac{h}{a} - \frac{1-h}{\gamma}.$$

SCHOLIUM 1

131. Just as hence the expression found may be produced, it is required to note :

$$\frac{f}{a\alpha}+\frac{g}{b\beta}+\frac{h}{c\gamma}=\begin{cases}+\frac{1}{aa}(-f(1-f)-gh(1-f))\\+\frac{1}{a\gamma}(+ff+g(1-f)(1-h)+fgh+hh)\\+\frac{1}{\gamma\gamma}(-fg(1-h)-h(1-h))\end{cases}$$

But $-f(1-f)-gh(1-f)=-(1-f)(f+gh)=-(1-f)(1-g)(1-h)$ on account of $f=1-g-h$ and thus $f+gh=(1-g)(1-h)$. In a similar manner for $\frac{1}{\gamma\gamma}$ there is $-fg(1-h)-h(1-h)=-\ (1-h)(h+fg)=-\ (1-f)(1-g)(1-k)$ on account of $h=1-f-\ g$. Finally for $\frac{1}{\alpha\gamma}$, since there is

$$ff+hh=(f+h)^2-2fh=(1-g)^2-2fh=1-2g+gg-2fh,$$

with this value substituted the coefficient itself $\frac{1}{\alpha\gamma}$ will become

$$\begin{aligned}&1-2g-2fh+gg+fgh+g(1-f)(1-h)\\&=1+(g+fk)(g-2)+g(1-f)(1-h)=1-2(1-f)(1-g)(1-h)\end{aligned}$$

on account of $g+fh=(1-f)(1-h)$.

Consequently there is deduced:

$$\frac{f}{a\alpha}+\frac{g}{b\beta}+\frac{h}{c\gamma}=-(1-f)(1-g)(1-h)\left(\frac{1}{a}+\frac{1}{\gamma}\right)^2+\frac{1}{\alpha\gamma}.$$

SCHOLIUM 2

132. This problem also can be resolved more easily in the following manner with the aid of the preceding. Clearly the two lenses *PP* and *QQ* taken together may be considered as a double lens for the determinable distances a and β constructed with the arbitrary numbers λ, λ' and f, and on putting $\lambda f^3+\lambda'(1-f)^3-\nu f(1-f)=\lambda^{(2)}$, the diffusion length arising from that alone will be

$$\mu\beta\beta xx(\tfrac{1}{a}+\tfrac{1}{\beta})\left(\lambda^{(2)}(\tfrac{1}{a}+\tfrac{1}{\beta})^2+\tfrac{\nu}{a\beta}\right)$$

which lens if now put in place with the third *RR* is treated as a simple lens, thence the diffusion length is elicited and from the conjunction of these two simple lenses

$$Hh=\mu\gamma\gamma xx\Big((\tfrac{1}{a}+\tfrac{1}{\beta})\Big(\lambda^{(2)}(\tfrac{1}{a}+\tfrac{1}{\beta})^2+\tfrac{v}{a\beta}\Big)+(\tfrac{1}{c}+\tfrac{1}{\gamma})\Big(\lambda''(\tfrac{1}{c}+\tfrac{1}{\gamma})^2+\tfrac{v}{c\gamma}\Big)\Big)$$

where it is required to be observed $\beta+c=0$, and the construction of the double lens from § 107 will be desired, truly from the simple lens *RR* with the determinable distance $c=-\beta$ et γ together with the arbitrary number λ''. Now there may be put , $\frac{1}{\beta}=\frac{g-1}{a}+\frac{g}{\gamma}$, so that there shall be $\frac{1}{c}=\frac{g-1}{a}-\frac{g}{\gamma}$, and the diffusion length of this triple lens will become :

$$Hh=\mu\gamma\gamma xx(\tfrac{1}{a}+\tfrac{1}{\beta})\Big(\Big(\lambda^{(2)}g^3+\lambda''(1-g)^3-vg(1-g)\Big)(\tfrac{1}{a}+\tfrac{1}{\gamma})^2+\tfrac{v}{a\gamma}\Big).$$

Whereby if for the triple lens there may be put

$$\lambda^{(2)}g^3+\lambda''(1-g)^3-vg(1-g)=\lambda^{(3)},$$

thus so that now we may abandon our arbitrary number *g* above, we will have the diffusion distance received at this point by the customary expression :

$$Hh=\mu\gamma\gamma xx(\tfrac{1}{a}+\tfrac{1}{\gamma})\Big(\lambda^{(3)}(\tfrac{1}{a}+\tfrac{1}{\gamma})^2+\tfrac{v}{a\gamma}\Big).$$

Hence now that is understood, which is less apparent from the preceding solution : if the number $\lambda^{(2)}$ were greater than unity, in place of the two first lenses *PP* and *QQ* it would be more convenient to use a single simple lens; from which hitherto only triple lenses are agreed to result, whenever the number $\lambda^{(2)}$ is less than one. But we have seen it is not possible to hope to succeed without risk, unless $\lambda^{(2)}$ shall be equal to the fraction 0,191827 or not much greater than that; from which if in practice we may wish to deliberate, if it may not be agreed to use the smaller value $\lambda^{(2)}$, and on that account the same rational number *g* ought to be accepted between 0 and 1 ; of which the first value will be convenient in practice, if it may make the number $\lambda^{(3)}$ a minimum, since then small errors minimally disturb the setup. But if truly for $\lambda^{(2)}$ we may substitute the assigned value, we will obtain

$$\lambda^{(3)}=\lambda f^3g^3+\lambda' g^3(1-f)^3+\lambda''(1-g)^3-vg^3f(1-f)-vg(1-g).$$

But it will be appropriate here to have set out each expression for the number $\lambda^{(3)}$, by which the diffusion distance is defined arising from a triple lens, since other conclusions from that may be deduced more easily. But even if hence all the values for $\lambda^{(3)}$ can be obtained, yet only those which remain near the minimum, will be convenient to use in practice.

PROBLEM 6

133. *With the determinable distances* $AE = a$ (Fig. 8) *and* $cH = \gamma$ *given, to define that triple lens, which may produce the minimum diffusion length.*

SOLUTION I

This problem can be solved in two ways, just as the diffusion length thus either is expressed as in the solution of the preceding problem, or as in Scholium 2. Thus in the first way it will be required to determine the numbers f, g, h, so that this expression may return a minimum value :

$$\lambda f^2 + \lambda' g^3 + \lambda'' h^3 - v(1-f)(1-g)(1-h),$$

where it is required to be observed $f + g + h = 1$. Therefore we will have to put its differential equal to zero:

$$3\lambda ffdf + 3\lambda' ggdg + 3\lambda'' hhdh + vdf(1-g)(1-h) + vdg(1-f)(1-h) + vdh(1-f)(1-g) = 0.$$

But since there shall be $dh = -df - dg$, there will be

$$\left.\begin{array}{l} +\ 3\lambda ffdf - 3\lambda'' hhdf\ + vdf(1-g)(f-h) \\ +3\lambda' ggdg - 3\lambda'' hhdg + vdg(1-f)(g-h) \end{array}\right\} = 0;$$

since truly the two differentials *df* and *dg* will not depend on each other, both the members of this equation must themselves vanish, from which on account of $1-g = f+h$ and $1-f = g+h$ we come upon these two equations:

$$3\lambda ff - 3\lambda'' hh + vff - vhh = 0,\ 3\lambda' gg - 3\lambda'' hh + vgg - vhh = 0,$$

from which we elicit

$$f = h\sqrt{\tfrac{3\lambda''+v}{3\lambda+v}} \text{ and } g = h\sqrt{\tfrac{3\lambda''+v}{3\lambda'+v}}.$$

But since there shall be $f + g + h = 1$ or $\frac{1}{h} = 1 + \frac{f}{h} + \frac{g}{h}$, there will become

$$\frac{1}{h}=1+\sqrt{\frac{3\lambda''+v}{3\lambda+v}}+\sqrt{\frac{3\lambda''+v}{3\lambda'+v}}$$
$$\frac{1}{g}=1+\sqrt{\frac{3\lambda'+v}{3\lambda+v}}+\sqrt{\frac{3\lambda'+v}{3\lambda''+v}}$$
$$\frac{1}{f}=1+\sqrt{\frac{3\lambda+v}{3\lambda'+v}}+\sqrt{\frac{3\lambda+v}{3\lambda''+v}}.$$

Therefore whatever the numbers $\lambda,\ \lambda',\ \lambda''$ were taken in the construction of the individual lenses, hence the numbers *f, g* and *h* may be determined, from which the minimum diffusion distance may emerge. But for the construction of the lens hence the distances $\alpha,\ b,\ \beta,\ c$ are defined thus, so that there shall be

$$\frac{1}{\alpha}=\frac{-1+f}{a}+\frac{f}{\gamma},\quad \frac{1}{b}=\frac{1-f}{a}-\frac{f}{\gamma},\quad \frac{1}{\beta}=\frac{-h}{a}+\frac{1-h}{\gamma},\quad \frac{1}{c}=\frac{h}{a}-\frac{1-h}{\gamma};$$

truly there is

$$f=\frac{\sqrt{(3\lambda'+v)(3\lambda''+v)}}{\sqrt{(3\lambda+v)(3\lambda'+v)}+\sqrt{(3\lambda+v)(3\lambda''+v)}+\sqrt{(3\lambda'+v)(3\lambda''+v)}}$$
$$g=\frac{\sqrt{(3\lambda+v)(3\lambda''+v)}}{\sqrt{(3\lambda+v)(3\lambda'+v)}+\sqrt{(3\lambda+v)(3\lambda''+v)}+\sqrt{(3\lambda'+v)(3\lambda''+v)}}$$
$$h=\frac{\sqrt{(3\lambda+v)(3\lambda'+v)}}{\sqrt{(3\lambda+v)(3\lambda'+v)}+\sqrt{(3\lambda+v)(3\lambda''+v)}+\sqrt{(3\lambda'+v)(3\lambda''+v)}}.$$

But from the distances $a,\ \alpha,\ b,\ \beta,\ c,\ \gamma$ with the numbers $\lambda,\ \lambda',\ \lambda''$ the lenses themselves are constructed by the formulas shown in § 91.

COROLLARY 1

134. If for the triple lens there may be put

$$\lambda f^3+\lambda' g^3+\lambda'' h^3-v(1-f)(1-g)(1-h)=\lambda^{(3)},$$

with these values substituted for *f, g* and *h* there will be found

$$\lambda^{(3)}=\frac{1}{3\left(\frac{1}{\sqrt{(3\lambda+v)}}+\frac{1}{\sqrt{(3\lambda'+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}}\right)^2}-\frac{1}{3}v,$$

from which the diffusion distance will become

$$Hh=\mu\gamma\gamma xx\left(\frac{1}{a}+\frac{1}{\gamma}\right)\left(\lambda^{(3)}\left(\frac{1}{a}+\frac{1}{\gamma}\right)^2+\frac{v}{a\gamma}\right).$$

COROLLARY 2

135. But generally this diffusion length will become a minimum, if the minimum values of the numbers $\lambda,\ \lambda',\ \lambda''$ may be attributed, which they are able to accept. Therefore there shall be $\lambda=1,\ \lambda'=1,\ \lambda''=1$, and there shall be

$$f=\tfrac{1}{3},\quad g=\tfrac{1}{3},\quad h=\tfrac{1}{3}$$

and $\lambda^{(3)}=\frac{\nu+3}{27}-\frac{\nu}{3}=\frac{3-8\nu}{27}=0,042165$ on account of $\nu=0,232692$; which value is much smaller than for the case of the double lens.

COROLLARY 3

136. Again in this case there will be

$$\frac{1}{\alpha}=-\frac{2}{3a}+\frac{1}{3\gamma},\quad \frac{1}{b}=-\frac{1}{3a}+\frac{2}{3\gamma}\quad\text{and}\quad \frac{1}{c}=\frac{1}{3a}-\frac{2}{3\gamma},$$

from which thus the construction of the three simple lenses will be had :

For the first lens, the radius of the $\begin{cases}\text{anterior face}=\dfrac{3a\gamma}{(3\rho-2\sigma)\gamma+\sigma a}\\ \text{posterior face}=\dfrac{3a\gamma}{(3\sigma-2\rho)\gamma+\rho a}\end{cases}$

For the second lens, the radius of the $\begin{cases}\text{anterior face}=\dfrac{3a\gamma}{(2\rho-\sigma)\gamma+(2\sigma-\rho)a}\\ \text{posterior face}=\dfrac{3a\gamma}{(2\sigma-\rho)\gamma+(2\rho-\sigma)a}.\end{cases}$

For the third lens, the radius of the $\begin{cases}\text{anterior face}=\dfrac{3a\gamma}{\rho\gamma+(3\sigma-2\rho)a}\\ \text{posterior face}=\dfrac{3a\gamma}{\sigma\gamma+(3\rho-2\sigma)a}.\end{cases}$

ANOTHER SOLUTION OF THE PROBLEM

137. We will consider the two earlier lenses PP and QQ as a double lens thus constructed for the determinable distances a and β, so that on putting

$$\lambda f^3+\lambda'(1-f)^3-\nu f(1-f)=\lambda^{(2)}$$

the diffusion distance thence shall arise

$$=\mu\beta\beta xx(\tfrac{1}{a}+\tfrac{1}{\beta})\left(\lambda^{(2)}(\tfrac{1}{a}+\tfrac{1}{\beta})^2+\tfrac{v}{a\beta}\right).$$

But for the construction of the simple double lens, from which this lens is composed, it is required to be noted

$$\tfrac{1}{\alpha}=\tfrac{-1+f}{a}+\tfrac{f}{\beta} \text{ and } \tfrac{1}{b}=\tfrac{1-f}{a}-\tfrac{f}{\beta}.$$

Now with the third lens added *RR* constructed according to the arbitrary number λ'', for the determinable distances $c=-\beta$ and γ, if we may put $\tfrac{1}{\beta}=\tfrac{-1+g}{a}+\tfrac{g}{\gamma}=-\tfrac{1}{c}$ and

$$\lambda^{(2)}g^3+\lambda''(1-g)^3-vg(1-g)=\lambda^{(3)},$$

the diffusion length will be

$$Hh=\mu\gamma\gamma xx(\tfrac{1}{a}+\tfrac{1}{\gamma})\left(\lambda^{(3)}(\tfrac{1}{a}+\tfrac{1}{\gamma})^2+\tfrac{v}{a\gamma}\right)$$

which so that it may be a minimum, the value of this $\lambda^{(3)}$ must be reduced to a minimum ; therefore initially g, while $\lambda^{(2)}$ may be considered as a given number, and we will have

$$3\lambda^{(2)}gg-3\lambda''(1-g)^2-v+2vg=0,$$

from which it is elicited

$$g=\frac{-3\lambda''-v+\sqrt{(3\lambda^{(2)}+v)(3\lambda''+v)}}{3\lambda^{(2)}-3\lambda''} \text{ or } g=\frac{\sqrt{(3\lambda''+v)}}{\sqrt{(3\lambda^{(2)}+v)}+\sqrt{(3\lambda''+v)}},$$

and hence the value of $\lambda^{(3)}$ will be

$$\lambda^{(3)}=\frac{1}{3(\frac{1}{\sqrt{(3\lambda^{(2)}+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}})^2}-\tfrac{1}{3}v,$$

or

$$\frac{1}{3\lambda^{(3)}+v}=(\frac{1}{\sqrt{(3\lambda^{(2)}+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}})^2.$$

And thus there becomes :

$$\frac{1}{\sqrt{(3\lambda^{(3)}+v)}}=\frac{1}{\sqrt{(3\lambda^{(2)}+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}}.$$

In a similar manner, if f may be defined thus, so that $\lambda^{(2)}$ may become a minimum, there will be found :

$$f=\frac{\sqrt{(3\lambda'+v)}}{\sqrt{(3\lambda+v)}+\sqrt{(3\lambda'+v)}}$$

and with this value substituted

$$\frac{1}{\sqrt{(3\lambda^{(2)}+v)}}=\frac{1}{\sqrt{(3\lambda+v)}}+\frac{1}{\sqrt{(3\lambda'+v)}}.$$

Whereby if with our arbitrary numbers remaining λ, λ', λ'' the two numbers f and g thus may be defined, so that it may follow that $\lambda^{(3)}$ adopts a minimum value, there will become

$$\frac{1}{\sqrt{(3\lambda^{(3)}+v)}}=\frac{1}{\sqrt{(3\lambda+v)}}+\frac{1}{\sqrt{(3\lambda'+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}},$$

from which likewise the value for $\lambda^{(3)}$ is found, which we have arrived at before.

COROLLARY 1

138. Therefore from this solution the numbers f and g thus are defined, so that

$$\frac{1}{f\sqrt{(3\lambda+v)}}=\frac{1}{\sqrt{(3\lambda+v)}}+\frac{1}{\sqrt{(3\lambda'+v)}}$$

and

$$\frac{1}{g\sqrt{(3\lambda^{(2)}+v)}}=\frac{1}{\sqrt{(3\lambda^{(2)}+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}}$$

or if in this manner

$$(\tfrac{1}{f}-1)\frac{1}{\sqrt{(3\lambda+v)}}=\frac{1}{\sqrt{(3\lambda'+v)}} \quad\text{and}\quad (\tfrac{1}{g}-1)(\frac{1}{\sqrt{(3\lambda+v)}}+\frac{1}{\sqrt{(3\lambda'+v)}})=\frac{1}{\sqrt{(3\lambda''+v)}}.$$

COROLLARY 2

139. From the minimum values found of the numbers $\lambda^{(2)}$ and $\lambda^{(3)}$ the numbers f and g also are defined thus, so that there shall be

$$\frac{1}{f\sqrt{(3\lambda+v)}}=\frac{1}{\sqrt{(3\lambda^{(2)}+v)}} \quad\text{or}\quad f=\sqrt{(\frac{3\lambda^{(2)}+v}{3\lambda+v})}$$

and

$$\frac{1}{g\sqrt{(3\lambda^{(2)}+v)}}=\frac{1}{\sqrt{(3\lambda^{(3)}+v)}} \quad\text{seu}\quad g=\sqrt{(\frac{3\lambda^{(3)}+v}{3\lambda^{(2)}+v})}.$$

COROLLARY 3

140. But the determinable distances of the individual lenses thus expressed by a and γ will produce :

$$\frac{1}{\alpha}=-\frac{1}{b}=\frac{-1+fg}{a}+\frac{fg}{\gamma},\quad \frac{1}{\beta}=-\frac{1}{c}=\frac{-1+g}{a}+\frac{g}{\gamma},$$

where since there shall be

$$g=\sqrt{\left(\frac{3\lambda^{(3)}+\nu}{3\lambda^{(2)}+\nu}\right)},\text{ it is required to be observed } fg=\sqrt{\frac{3\lambda^{(3)}+\nu}{3\lambda+\nu}}.$$

COROLLARY 4

141. Moreover by eliminating the number $\lambda^{(2)}$, there will be

$$\frac{1}{\alpha}=-\frac{1}{b}=-\frac{1}{a}\sqrt{\frac{3\lambda^{(2)}+\nu}{3\lambda'+\nu}}-\frac{1}{a}\sqrt{\frac{3\lambda^{(3)}+\nu}{3\lambda''+\nu}}+\frac{1}{\gamma}\sqrt{\frac{3\lambda^{(3)}+\nu}{3\lambda+\nu}}$$
$$\frac{1}{\beta}=-\frac{1}{c}=-\frac{1}{a}\sqrt{\frac{3\lambda^{(3)}+\nu}{3\lambda''+\nu}}+\frac{1}{\gamma}\sqrt{\frac{3\lambda^{(3)}+\nu}{3\lambda+\nu}}+\frac{1}{\gamma}\sqrt{\frac{3\lambda^{(3)}+\nu}{3\lambda'+\nu}}$$

from which formulas, now with the minimum value $\lambda^{(3)}$ found, the individual lenses may be determined most conveniently.

COROLLARY 5

142. If besides the individual lenses were prepared thus, so that by this means the minimum confusion may be produced, which will happen, if $\lambda=\lambda'=\lambda''=1$, there will be $\frac{1}{\sqrt{(3\lambda^{(2)}+\nu)}}=\frac{2}{\sqrt{(3+\nu)}}$ and hence $3\lambda^{(2)}+\nu=\frac{3+\nu}{4}$, from which there becomes $\lambda^{(2)}=\frac{1-\nu}{4}$.
Then truly there will be had :
$\frac{1}{\sqrt{(3\lambda^{(3)}+\nu)}}=\frac{3}{\sqrt{(3+\nu)}}$ and hence $3\lambda^{(3)}+\nu=\frac{3+\nu}{9}$ and therefore $\lambda^{(3)}=\frac{3-8\nu}{27}$.
But if only $\lambda=\lambda'=\lambda''$, there will be found $\lambda^{(2)}+\nu=\frac{3\lambda-3\nu}{12}$ and $\lambda^{(3)}=\frac{3\lambda-8\nu}{27}$.

COROLLARY 6

143. But in the same case, where $\lambda=\lambda'=\lambda''$, on account of $\sqrt{(3\lambda^{(3)}+\nu)}=\frac{1}{3}\sqrt{(3+\nu)}$ the determinable distances for the simple lenses will be:

$$\frac{1}{\alpha}=\frac{-2}{3a}+\frac{1}{3\gamma},\quad \frac{1}{b}=\frac{2}{3a}-\frac{1}{3\gamma},\quad \frac{1}{\beta}=\frac{-1}{3a}+\frac{2}{3\gamma},\quad \frac{1}{c}=\frac{1}{3a}-\frac{2}{3\gamma},$$

from which these same formulas arise for their construction, which have been brought forth above (§ 136) , except that now in the denominators the terms must be added $\pm\tau(a+\gamma)\sqrt{(\gamma-1)}$.

SCHOLIUM

144. Therefore by a suitable union of the three simple lenses it can be effected, so that in the expression of the diffusion length

$$Hh=\mu\gamma\gamma xx(\tfrac{1}{a}+\tfrac{1}{\gamma})\left(\lambda^{(3)}(\tfrac{1}{a}+\tfrac{1}{\gamma})^2+\tfrac{\nu}{a\gamma}\right)$$

the number $\lambda^{(3)}$ may become $=0,042165$. Moreover with double lenses we have seen the minimum value of the number $\lambda^{(2)}$ to become $=0,191827$: and thus with triple lenses this number can be reduced almost five times smaller ; but this in turn is seen to be almost four times smaller, than what can be obtained with a single lens. I am saying here from the minimum principle sought with regard to lenses, certainly which have been adapted in practice especially, while the construction may not be disturbed by small errors to any extent. For whenever triple as well as double lenses may be able to be present, according to which the agreeing number λ not only may be equal to zero, but also may become negative, yet the construction of these is not without difficulty, as a small error may render the whole labour useless. Yet meanwhile the danger lies not only with triple but also with double lenses, according to which it is worthwhile to solve the following problem.

PROBLEM 7

145. *For the given determinable distances a and γ to define these triple lenses, for which the value of $\lambda^{(3)}$ may become zero at once.*

SOLUTION

The number $\lambda^{(2)}$ arising from the two prior lenses may be considered as given, and since there shall be the following latter solution from the preceding problem

$$\lambda^{(3)}=\lambda^{(2)}g^3+\lambda''(1-g)^3-\nu g(1-g),$$

g must be defined thus, thus so that this quantity may vanish. Truly since $\lambda^{(2)}$ conveniently may be unable to be made smaller than $\frac{1-v}{4}$, we may put $\frac{1-v}{4}=\lambda^{(2)}$, and there will be required to become:

$$0=\tfrac{1}{4}g^3+\lambda''(1-g)^3-vg(1-g)^2;$$

but since λ'' cannot become less than one, it is necessary, that g may be taken greater than one; therefore certain cases may be set out:

I. $g=\frac{5}{4}$; $0=\frac{125}{4\cdot 64}-\frac{\lambda''}{64}-\frac{5\cdot 9}{4\cdot 64}v$ and $\lambda''=\frac{125-45v}{4}$
II. $g=\frac{6}{4}$; $0=\frac{216}{4\cdot 64}-\frac{8\lambda''}{64}-\frac{6\cdot 4}{4\cdot 64}v$ and $\lambda''=\frac{126-24v}{4\cdot 8}=\frac{27-3v}{4}$
III. $g=\frac{7}{5}$; $0=\frac{343}{4\cdot 64}-\frac{27\lambda''}{64}-\frac{7\cdot 1}{4\cdot 64}v$ and $\lambda''=\frac{343-7v}{4\cdot 27}$
IV. $g=\frac{8}{4}$; $0=\frac{512}{4\cdot 64}-\frac{64\lambda''}{64}$ and $\lambda''=2$
V. $g=\frac{9}{4}$; $0=\frac{729}{4\cdot 64}-\frac{125\lambda''}{64}-\frac{9\cdot 1}{4\cdot 64}v$ and $\lambda''=\frac{729-9v}{4\cdot 125}$
VI. $g=\frac{10}{4}$; $0=\frac{1000}{4\cdot 64}-\frac{125\lambda''}{64}-\frac{10\cdot 4}{4\cdot 64}v$ and $\lambda''=\frac{1000-40v}{4\cdot 216}=\frac{125-5v}{4\cdot 27}$.

In the first and second case the value of λ'' shall become exceedingly large, as that lens may be able to be used conveniently in practice ; and if a much greater value may be attributed to g , a large unpredictable error is put into effect. For in place of g the error may be taken $g+\omega$, and since λ'' duly will have been defined from g, there becomes

$$\lambda^{(3)}=\omega(\tfrac{3}{4}gg-3\lambda''(1-g)^2-v(1-\tfrac{1}{2}g)(1-\tfrac{3}{2}g))$$

and with the value for λ'' substituted

$$\lambda^{(3)}=\tfrac{\omega}{4(1-g)}(3gg-v(4-gg)),$$

from which it is apparent, however little g may exceed one, and likewise however great g may become, the valor of $\lambda^{(3)}$ thus becomes greater on account of the error ω . But this error thus is understood to become a minimum, if there may be taken $g=1+\sqrt{\frac{3-3v}{v+3}}$; but from this value there is elicited

$$\lambda''=\tfrac{(3+v)\sqrt{3}(1-v)(3+v)+3(3+vv)}{9(1-v)}.$$

Therefore on taking $g=1{,}84384$, from which there is deduced :

$$\lambda'' = \frac{g^3 - \nu g(2-g)^2}{4(g-1)^3} = 2,60372,$$

and if these measurements are observed exactly, there becomes $\lambda^{(3)} = 0$. But if the value of g may be in error by a particular ω, so that there shall be $g = 1,84384 + \omega$, on account of this error there will be produce :

$$\lambda^{(3)} = -\ 2,981\omega \ ;$$

thus if the error may be $\omega = \pm\frac{1}{10}$, in place of $\lambda^{(3)} = 0$ there will be produced:

$\lambda^{(3)} = \mp 0,2981$, and thus the triple lens, on being placed after the double, still may be preferred by far to a simple lens.

In general therefore for any other value of $\lambda^{(2)}$ we may investigate the same conveniently: and in the first place since there shall be

$$\lambda'' = \frac{\lambda^{(2)} g^3 + \nu g(g-1)}{(g-1)^3},$$

if in place of the true value g there may be taken $g + \omega$, there becomes

$$\lambda^{(3)} = \omega(3\lambda^{(2)} g^2 - 3\lambda''(g-1)^2 - \nu + 2\nu g),$$

where if the value may be substituted for λ'', there will be

$$\lambda^{(3)} = \frac{\omega(\nu - (3\lambda^{(2)} + \nu)gg)}{g-1},$$

which so that the error may be made a minimum, there must be taken

$$g = 1 + \sqrt{\frac{3\lambda^{(2)}}{3\lambda^{(2)} + \nu}},$$

which is the maximum suitable value requiring to be taken for g, from which there is elicited

$$\lambda'' = \frac{1}{3\lambda^{(2)}}\left(\sqrt{(3\lambda^{(2)} + \nu)} + \sqrt{3\lambda^{(2)}}\right)\left(\tfrac{1}{3}(6\lambda^{(2)} + \nu)\sqrt{3\lambda^{(2)}} + (2\lambda^{(2)} + \nu)\sqrt{(3\lambda^{(2)} + \nu)}\right)$$

and with the error $g + \omega$ taken for g there will become

$$\lambda^{(3)} = -2\omega(3\lambda^{(2)} + \nu + \sqrt{3\lambda^{(2)}(3\lambda^{(2)} + \nu)})$$

or

$$\lambda^{(3)} = -2\omega(\sqrt{3\lambda^{(2)}} + \sqrt{(3\lambda^{(2)}+\nu)})\sqrt{(3\lambda^{(2)}+\nu)}.$$

Where now therefore the smaller were the value of $\lambda^{(2)}$, there the less will be the error requiring to be met, from which the previous solution derived from the value $\lambda^{(2)} = \frac{1-\nu}{4}$ is to be recommended before the rest. Then truly there will be
$f = \frac{1}{2}$, $\lambda = 1$, $\lambda' = 1$, $\lambda'' = 2,60372$ and $g = 1,84384$, from which we have for the construction of the lens:

$$\frac{1}{\alpha} = -\frac{1}{b} = \frac{-2+g}{2a} + \frac{g}{2\gamma}, \quad \frac{1}{\beta} = -\frac{1}{c} = \frac{-1+g}{a} + \frac{g}{\gamma},$$

from which the individual lenses will be constructed by the formulas § 91.

COROLLARY 1

146. Therefore the first lens *PP* must be constructed from the determinable distances a and $\frac{2a\gamma}{ga-(2-g)\gamma}$ with the number $\lambda = 1$.

Truly the following lens *QQ* from the determinable distances $\frac{-2a\gamma}{ga-(2-g)\gamma}$ and $\frac{a\gamma}{ga-(1-g)\gamma}$ with the number $\lambda = 1$.

But the third lens *RR* from the determinable distances $\frac{-a\gamma}{ga-(1-g)\gamma}$ and γ with the number $\lambda'' = 2,60372$, on taking $g = 1,84384$.

COROLLARY 2

147. Since in the formula with the diffusion length expressed , which on account of $\lambda^{(3)} = 0$ is

$$\mu\gamma\gamma xx(\tfrac{1}{a} + \tfrac{1}{\gamma})\frac{\nu}{a\gamma}$$

with the first factor $\mu\gamma\gamma xx$ removed, of which the ratio is not present in these investigations, the distances a and γ can be interchanged with each other, hence also the other third lens sought will be able to be shown to be equally satisfying.

COROLLARY 3

148. Certainly for this other triple lens the first *PP* must be constructed from the determinable distances a and $\frac{+a\gamma}{(1-g)a-g\gamma}$ with the number $\lambda = 2,60372$.

The second lens QQ from the determinable distances $\frac{-a\gamma}{(1-g)a-g\gamma}$ and $\frac{+2a\gamma}{(2-g)a-g\gamma}$ with the number $\lambda'=1$.

While the third lens from the determinable distances $\frac{-2a\gamma}{(2-g)a-g\gamma}$ and γ with the number $\lambda''=1$, with present as before $g=1,84384$.

COROLLARY 4

149. Hence therefore we have obtained the triple lenses for the distances a and γ, which produce the diffusion distance $Hh=\mu\gamma\gamma xx(\frac{1}{a}+\frac{1}{\gamma})\frac{\nu}{a\gamma}$. And these among infinitely others presenting this same outstanding effect enjoy this prerogative, that the least error connected with the construction may disturb the aim minimally.

COROLLARY 5

150. If in the construction of this lens by an error the number g may be taken a little greater than 1,84384, then for the triple lens the number produced $\lambda^{(3)}$ is neither zero nor negative. But if the number g in practice may be taken a little smaller, then the number $\lambda^{(3)}$ may become greater than zero, and thus the triplet lens will approach the nature of the doublet.

COROLLARY 6

151. Therefore if there were the need for a lens, for which the number λ may have a negative value, it will be able for this aim to be satisfied conveniently by the triple lenses described, provided the number for g may be taken a little greater than 1,84384. Evidently if there may be taken

$$g=1,84384+\omega, \text{ there may become } \lambda^{(3)}=-\,2,981\omega$$

SUPPLEMENT II

CONCERNING TRIPLE LENSES

If the refraction may be different for the individual lenses, for the first $n:1$, for the second $n':1$ and for the third $n'':1$, and the radii of the faces of the lenses may be defined in the following manner :

	Determinable distances	Refraction and the depending letters	
I.	a and α	$n:1,\ \mu,\ \nu,\ \rho,\ \sigma,\ \tau.$	λ
II.	b and β	$n':1,\ \mu',\ \nu',\ \rho',\ \sigma',\ \tau'.$	λ'
III.	c and γ	$n'':1,\ \mu'',\ \nu'',\ \rho'',\ \sigma'',\ \tau''.$	λ''

clearly if for the first lens the radius of the anterior face may be called $=F$, of the posterior $=G$, and there will become

$$\frac{1}{F}=\frac{\rho}{a}+\frac{\sigma}{\alpha}\mp\left(\frac{1}{a}+\frac{1}{\alpha}\right)\tau\sqrt{\lambda-1} \text{ and } \frac{1}{G}=\frac{\rho}{\alpha}+\frac{\sigma}{a}\pm\tau\left(\frac{1}{\alpha}+\frac{1}{a}\right)\sqrt{\lambda-1},$$

and in a similar manner for the remaining lenses, evidently for the second

$$\frac{1}{F'}=\frac{\rho'}{a}+\frac{\sigma'}{\alpha}\mp\tau'\left(\frac{1}{a}+\frac{1}{\alpha}\right)\sqrt{\lambda'-1} \text{ et } \frac{1}{G'}=\frac{\rho'}{\alpha}+\frac{\sigma'}{a}\pm\tau'\left(\frac{1}{\alpha}+\frac{1}{a}\right)\sqrt{\lambda'-1}$$

then truly for the third

$$\frac{1}{F''}=\frac{\rho''}{a}+\frac{\sigma''}{\alpha}\mp\left(\frac{1}{a}+\frac{1}{\alpha}\right)\tau''\sqrt{\lambda''-1} \text{ et } \frac{1}{G''}=\frac{\rho''}{\alpha}+\frac{\sigma''}{a}\pm\tau''\left(\frac{1}{\alpha}+\frac{1}{a}\right)\sqrt{\lambda''-1}$$

Then because the separation of the lenses may be taken as nothing, evidently

$$\alpha+b=0 \text{ and } \beta+c=0,$$

the diffusion length will become

$$=\gamma\gamma xx\left\{\begin{array}{l}+\mu\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{\nu}{a\alpha}\right)\\+\mu'\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\lambda'\left(\frac{1}{b}+\frac{1}{\beta}\right)^2+\frac{\nu'}{b\beta}\right)\\+\mu''\left(\frac{1}{c}+\frac{1}{\gamma}\right)\left(\lambda''\left(\frac{1}{c}+\frac{1}{\gamma}\right)^2+\frac{\nu''}{c\gamma}\right)\end{array}\right\}.$$

Now there may be put

$$\tfrac{1}{a}+\tfrac{1}{\alpha}=f(\tfrac{1}{a}+\tfrac{1}{\gamma}),\ \tfrac{1}{b}+\tfrac{1}{\beta}=g(\tfrac{1}{a}+\tfrac{1}{\gamma}),\ \tfrac{1}{c}+\tfrac{1}{\gamma}=h(\tfrac{1}{a}+\tfrac{1}{\gamma}),$$

so that there may become

$$\tfrac{1}{\alpha}=\tfrac{f-1}{a}+\tfrac{f}{\gamma}=\tfrac{-1}{b}\ \text{ et }\ \tfrac{1}{c}=\tfrac{h}{a}+\tfrac{h-1}{\gamma}=\tfrac{-1}{\beta},$$

from which there becomes

$$\tfrac{1}{b}+\tfrac{1}{\beta}=\tfrac{1-f-h}{a}+\tfrac{1-f-h}{\gamma}=g(\tfrac{1}{a}+\tfrac{1}{\gamma})$$

and hence $f+g+h=1$; with which values substituted there will be for the first two lenses, the radii of the faces made:

$$\tfrac{1}{F}=\tfrac{\rho+(f-1)\sigma}{a}+\tfrac{\sigma f}{\gamma}\mp\tau f(\tfrac{1}{a}+\tfrac{1}{\gamma})\sqrt{\lambda-1}$$
$$\tfrac{1}{G}=\tfrac{\sigma+(f-1)\rho}{a}+\tfrac{\rho f}{\gamma}\pm\tau f(\tfrac{1}{a}+\tfrac{1}{\gamma})\sqrt{\lambda-1}$$
$$\tfrac{1}{F'}=\tfrac{(1-f)\rho'-h\sigma'}{a}+\tfrac{(1-h)\sigma'-f\rho'}{\gamma}\mp\tau' g(\tfrac{1}{a}+\tfrac{1}{\gamma})\sqrt{\lambda'-1}$$
$$\tfrac{1}{G'}=\tfrac{(1-f)\sigma'-h\rho'}{a}+\tfrac{(1-h)\rho'-f\sigma'}{\gamma}\mp\tau' g(\tfrac{1}{a}+\tfrac{1}{\gamma})\sqrt{\lambda'-1}$$

and for the third lens

$$\tfrac{1}{F''}=\tfrac{h\rho''}{a}+\tfrac{(h-1)\rho''+\sigma''}{\gamma}\mp(\tfrac{1}{a}+\tfrac{1}{\gamma})\tau'' h\sqrt{\lambda''-1}$$
$$\tfrac{1}{G''}=\tfrac{h\sigma''}{a}+\tfrac{(h-1)\sigma''+\rho''}{\gamma}\pm\tau'' h(\tfrac{1}{a}+\tfrac{1}{\gamma})\sqrt{\lambda''-1}.$$

Truly the diffusion distance now will be expressed thus :

$$=\gamma\gamma xx(\tfrac{1}{a}+\tfrac{1}{\gamma})\left\{\begin{array}{l}(\tfrac{1}{a}+\tfrac{1}{\gamma})^2(\mu\lambda f^3+\mu'\lambda' g^3+\mu''\lambda'' h^3)+\tfrac{\nu\mu f}{a}(\tfrac{f-1}{a}+\tfrac{f}{\gamma})\\ +\nu'\mu' g(\tfrac{1-f}{a}-\tfrac{f}{\gamma})(\tfrac{1-h}{\gamma}-\tfrac{h}{a})+\tfrac{\nu''\mu'' h}{\gamma}(\tfrac{h}{a}+\tfrac{h-1}{\gamma})\end{array}\right\}.$$

Therefore now the following are required to be observed about these triple lenses:

1. Media of differing refractions are accustomed to be used only in the end, so that not only the diffusion distance here determined may be reduced to zero, but also the confusion arising from the different refraction of the rays may be removed, certainly what cannot be obtained by rays of the same refraction. But below we will see for this condition to be implemented to require, that there shall be $\zeta f+\eta g+\theta h=0$, with the

conditions present : $\zeta=\frac{dn}{n-1}$, $\eta=\frac{dn'}{n'-1}$, and $\theta=\frac{dn''}{n''-1}$; from which forms it is now seen, if these differential dn, dn' and dn'' shall be proportional to $n-1$, $n'-1$ and $n''-1$, as Newton had established, then there is going to be produced $\zeta=\eta=\theta$ or $f+g+h=0$; at now we have seen there must be $f+g+h=1$; whereby if Newton's opinions shall be true, then lest indeed it may be able to bring the remedy arising from diverse refractions used. Therefore so far only this inconvenience will be able to be avoided, in as much as the letters ζ, η and θ are different, thus, so that both $f+g+h=1$ and $\zeta f+\eta g+\theta h=0$ may be able to be taken at the same time, from which it is understood that of the quantities *f, g* and *h* one or perhaps two must be negative, and thus with this condition added above this principle case, where all three letters *f, g* and *h* are taken positive, cannot occur here.

II. Therefore so that our diffusion distance may vanish, it is required to satisfy this equation

$$\left(\frac{1}{a}+\frac{1}{\gamma}\right)^2(\mu\lambda f^3+\mu'\lambda' g^3+\mu''\lambda'' h^3)+\frac{\nu\mu f}{a}\left(\frac{f-1}{a}+\frac{f}{\gamma}\right)$$
$$+\nu'\mu' g\left(\frac{1-f}{a}-\frac{f}{\gamma}\right)\left(\frac{1-h}{\gamma}-\frac{h}{a}\right)+\frac{\nu''\mu'' h}{\gamma}\left(\frac{h}{a}+\frac{h-1}{\gamma}\right)=0.$$

From which either λ, λ', or λ'' may be sought, as long as precautions may be taken, lest a value less than unity or exceedingly great may be produced, because the first is of an incompatible nature, but the others, because the construction of the lenses may become extremely difficult. But with this accomplished approximately by the two remaining letters λ nothing further is defined nor also about the letters *f, g* and *h,* except what is contained above by the conditions brought forth $f+g+h=1$ and $\zeta f+\eta g+\theta h=0$, from which from one of these letters given the two remaining are defined at once

$$g=\frac{(\theta-\zeta)f-\theta}{\eta-\theta} \text{ and } h=\frac{(\zeta-\eta)f+\eta}{\eta-\theta}.$$

III. But in practice triple lenses of this kind thus may be sought chiefly, so that they may be substituted in place of the objective lens in telescopes, for which there is $a=\infty$. Therefore at once we may put in place $a=\infty$ and for the radii of the faces of the individual lenses we will have

$$\frac{1}{F}=\frac{\sigma f}{\gamma}\mp\frac{\tau f}{\gamma}\sqrt{\lambda-1},\quad \frac{1}{G}=\frac{\rho f}{\gamma}\pm\frac{\tau f}{\gamma}\sqrt{\lambda-1},$$

for the second lens,

$$\frac{1}{F'}=\frac{(1-h)\sigma'-f\rho'}{\gamma}\mp\frac{\tau' g}{\gamma}\sqrt{\lambda'-1},\quad \frac{1}{G'}=\frac{(1-h)\sigma'-h\rho'}{a}+\frac{(1-h)\rho'-f\sigma'}{\gamma}\pm\frac{\tau' g}{\gamma}\sqrt{\lambda'-1},$$

for the third lens,

$$\frac{1}{F''}=\frac{(h-1)\rho''+\sigma''}{\gamma}\mp\frac{\tau''h}{\gamma}\sqrt{\lambda''-1},\ \frac{1}{G''}=\frac{(h-1)\sigma''+\rho''}{\gamma}\pm\frac{\tau''h}{\gamma}\sqrt{\lambda''-1}.$$

Moreover the diffusion distance then will be expressed thus :

$$\frac{x^2}{\gamma}(\mu\lambda f^3+\mu'\lambda' g^3+\mu''\lambda'' h^3-\nu'\mu'\cdot fg(1-h)-\nu''\mu''\cdot h(1-h)),$$

thus so that it will be required for this equation to be satisfied :

$$\mu\lambda f^3+\mu'\lambda' g^3+\mu''\lambda'' h^3-\nu'\mu'\cdot fg(1-h)-\nu''\mu''\cdot h(1-h)=0.$$

From which one of the values $\lambda,\ \lambda',\ \lambda''$, which is considered to be the most convenient to use, must be determined.

IV. If we wish to use only glass lenses [recalling that at this time telescopes used mainly concave mirrors for objectives], it will suffice to use only two kinds of glass ; therefore if we may put in place the third and first lens from the same kind of glass, so that $\mu''=\mu,\ \nu''=\nu,\ \rho''=\rho,\ \sigma''=\sigma,\ \tau''=\tau$ and $\xi=\theta$ on account of $n''=n$, but for the letters f, g and h these determinations will have to be made :

From the equation

$$\zeta f+\eta g+\zeta h=0 \text{ there becomes } f+h=\frac{-\eta}{\zeta}\cdot g,$$

with which value substituted there becomes :

$$g\frac{(\zeta-\eta)}{\zeta}=1,\ g=\frac{\zeta}{\zeta-\eta},\text{ and thus } f+h=\frac{-\eta}{\zeta-\eta}\ ;$$

from which it appears, just as the letter g were positive or negative, to become in turn the sum $f+h$ either negative or positive, and the equation requiring to be resolved now will be

$$\mu(\lambda f^3+\lambda'' h^3)+\mu'\lambda' g^3-\frac{\mu'\nu'\cdot\zeta}{\zeta-\eta}\cdot f(1-h)-\mu\nu\cdot h(1-h)=0.$$

Hence therefore it is deduced that

$$\mu'\lambda' g^3=-\mu(\lambda f^3+\lambda'' h^3)+\frac{\mu'\nu'\cdot\zeta}{\zeta-\eta}\cdot f(1-h)+\mu\nu\cdot h(1-h),$$

for which any proposed case will be able to be satisfied easily.

V. The first and third lens are then prepared from the same kind of glass, the middle may be agreed to be prepared from water or some other fluid, so that the triple lens may contain fluid between the two glass lenses, and because the fluid generally is allowed to have a smaller refraction than glass, there will be $n' < n$ and thence again $\eta < \zeta$. Therefore because for this case *g* shall be a positive or convex water lens, the glass lenses either both or at least one must be concave.

But in the first place besides the prescriptions now found it is necessary, that the radius of the anterior face for the middle lens shall be equal and opposite to the radius of the posterior face of the first lens, and in the same manner the radius of the posterior face shall be equal and opposite to the anterior face of the third lens, from which these equalities arise :

$$\tfrac{1}{F'} = \tfrac{-1}{G} \text{ or } F' + G = 0 \text{ and } \tfrac{1}{G'} = \tfrac{-1}{F''} \text{ or } F'' + G' = 0..$$

And thus from these equations there will be required to be satisfied :

$$(1-h)\sigma' - f\rho' \mp \tau' g \cdot \sqrt{\lambda' - 1} = -f\rho \mp \tau f \sqrt{\lambda - 1}$$

and

$$(1-h)\rho' - f\sigma' \pm \tau' g \sqrt{\lambda' - 1} = (1-h)\rho - \sigma \pm \tau h \sqrt{\lambda'' - 1}.$$

Therefore we have two conditions which must be satisfied, from which either the numbers λ and λ'' must be defined, or either of these together with either of the letters *f* and *h* must be defined ; which in the end it is required to observe the formulas $\sqrt{\lambda - 1}$ and $\sqrt{\lambda'' - 1}$ properly as it pleases, assumed to be either positive or negative, nor to depend on each other. But for the formula $\sqrt{\lambda' - 1}$ it is required to be noted, if these may be put positive in the first equation, in the second by necessity they must be taken negative in turn.

END OF SUPPLEMENT II.

PROBLEM 8

152. *To determine these quadruple lenses* (Fig. 9) *suitably adapted for the given determinable distances* $AE = a$ and $dI = \delta$, *which produce the minimum diffusion length Ii.*

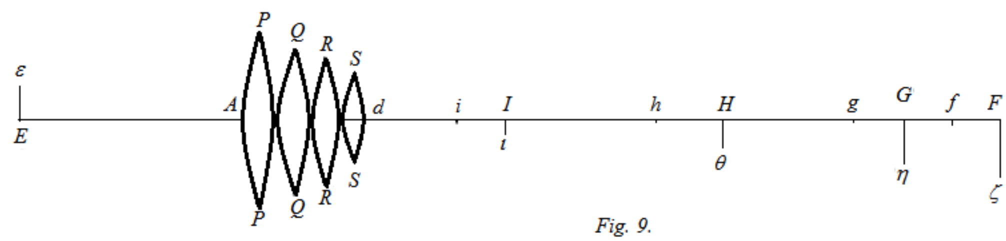

Fig. 9.

SOLUTION

The first lens *PP* may be constructed for the determinable distances $AE = a$ and $AF = \alpha$ with the number λ, with the second *QQ* for the distances $b = -\alpha$ and $AG = \beta$ with the number λ', the third *RR* for the distances $c = -\beta$ and $AH = \gamma$ with the number λ'', and the fourth *SS* for the distances $d = -\gamma$ and $dI = \delta$ with the number λ'' : where evidently we may regard the thickness of the lenses as vanishing. Now on putting the radius of the aperture *x*, the first lens alone *PP* will produce the spatial diffusion

$$Ff = \mu\alpha\alpha xx(\tfrac{1}{a}+\tfrac{1}{\alpha})\left(\lambda\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)^2+\tfrac{v}{a\alpha}\right).$$

Moreover by adding the second lens *QQ* and by putting

$$\tfrac{1}{\alpha}=\tfrac{-1+f}{a}+\tfrac{f}{\beta} \text{ and } \tfrac{1}{b}=\tfrac{1-f}{a}-\tfrac{f}{\beta},$$

if for the sake of brevity we may put

$$\lambda^{(2)}=\lambda f^3+\lambda'(1-f)^3-vf(1-f),$$

we have seen the diffusion distance to become :

$$Gg = \mu\beta\beta xx(\tfrac{1}{a}+\tfrac{1}{\beta})\left(\lambda^{(2)}\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2+\tfrac{v}{a\beta}\right).$$

In addition the third lens *RR* may be adjoined, and the number *g* may be taken thus, so that there shall be

$$\tfrac{1}{\beta}=\tfrac{-1+g}{a}+\tfrac{g}{\gamma}\ \text{ and }\ \tfrac{1}{c}=\tfrac{1-g}{a}-\tfrac{f}{\gamma},$$

and if for brevity therefore we may put

$$\lambda^{(3)}=\lambda^{(2)}g^3+\lambda''(1-g)^3-vg(1-g),$$

the diffusion length will become:

$$Hh=\mu\gamma\gamma xx(\tfrac{1}{a}+\tfrac{1}{\gamma})\left(\lambda^{(3)}\left(\tfrac{1}{a}+\tfrac{1}{\gamma}\right)^2+\tfrac{v}{a\gamma}\right).$$

Now finally the fourth lens *SS* may be added , and with the number *h* thus introduced into the calculation, so that there shall be

$$\tfrac{1}{\gamma}=\tfrac{-1+h}{a}+\tfrac{h}{\gamma}\ \text{ and }\ \tfrac{1}{d}=\tfrac{1-h}{a}-\tfrac{h}{\delta},$$

if in a similar manner we may put

$$\lambda^{(4)}=\lambda^{(3)}h^3+\lambda'''(1-h)^3-vh(1-h),$$

the diffusion length will be produced from the fourth lens :

$$Ii=\mu\delta\delta xx(\tfrac{1}{a}+\tfrac{1}{\delta})\left(\lambda^{(4)}\left(\tfrac{1}{a}+\tfrac{1}{\delta}\right)^2+\tfrac{v}{a\delta}\right),$$

which therefore must be reduced to a minimum. In the end these λ, λ', λ'' and λ''' may be considered as given numbers, and suitable values may be sought for the numbers *f, g* and *h* ; and so that the value $\lambda^{(4)}$ may emerge a minimum, it is necessary also the values $\lambda^{(3)}$ and $\lambda^{(2)}$ to become minimums. Therefore we may begin from the value $\lambda^{(2)}$, which is rendered a minimum by taking

$$f=\frac{\sqrt{(3\lambda'+v)}}{\sqrt{(3\lambda+v)}+\sqrt{(3\lambda'+v)}}\ \text{ or }\ \tfrac{1}{f}=1+\sqrt{\tfrac{3\lambda+v}{3\lambda'+v}},$$

from which there becomes

$$\frac{1}{\sqrt{(3\lambda^{(2)}+v)}}=\frac{1}{\sqrt{(3\lambda+v)}}+\frac{1}{\sqrt{(3\lambda'+v)}}.$$

Then the number $\lambda^{(3)}$ will adopt a minimum value on taking

$$g=\frac{\sqrt{(3\lambda''+v)}}{\sqrt{(3\lambda^{(2)}+v)}+\sqrt{(3\lambda''+v)}} \text{ seu } \frac{1}{g}=1+\sqrt{\frac{3\lambda^{(2)}+v}{3\lambda''+v}},$$

and hence there is deduced

$$\frac{1}{\sqrt{(3\lambda^{(3)}+v)}}=\frac{1}{\sqrt{(3\lambda^{(2)}+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}}.$$

Finally the number $\lambda^{(4)}$ and therefore the diffusion length *Ii* will be effected by taking

$$h=\frac{\sqrt{(3\lambda'''+v)}}{\sqrt{(3\lambda^{(3)}+v)}+\sqrt{(3\lambda'''+v)}} \text{ or } \frac{1}{h}=1+\sqrt{\frac{3\lambda^{(3)}+v}{3\lambda'''+v}},$$

from which there will be obtained

$$\frac{1}{\sqrt{(3\lambda^{(4)}+v)}}=\frac{1}{\sqrt{(3\lambda^{(3)}+v)}}+\frac{1}{\sqrt{(3\lambda'''+v)}}.$$

So that if we may substitute these values found above, we will arrive at

$$\frac{1}{\sqrt{(3\lambda^{(4)}+v)}}=\frac{1}{\sqrt{(3\lambda+v)}}+\frac{1}{\sqrt{(3\lambda'+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}}+\frac{1}{\sqrt{(3\lambda'''+v)}}.$$

Truly from the preceding there will be

$$\frac{1}{\sqrt{(3\lambda^{(3)}+v)}}=\frac{1}{\sqrt{(3\lambda+v)}}+\frac{1}{\sqrt{(3\lambda'+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}}$$

and

$$\frac{1}{\sqrt{(3\lambda^{(2)}+v)}}=\frac{1}{\sqrt{(3\lambda+v)}}+\frac{1}{\sqrt{(3\lambda'+v)}}.$$

Then truly again from these we obtain :

$$f=\sqrt{\frac{3\lambda^{(2)}+v}{3\lambda+v}},\ g=\sqrt{\frac{3\lambda^{(3)}+v}{3\lambda^{(2)}+v}},\ h=\sqrt{\frac{3\lambda^{(4)}+v}{3\lambda^{(3)}+v}}.$$

Therefore it remains, that we may set out the construction of the individual lenses more clearly, and we may express the determinable distances of these by the proposed *a* and δ ; therefore there will become :

$$\frac{1}{\gamma}=\frac{-1+h}{a}+\frac{h}{\delta},\ \frac{1}{\beta}=\frac{-1+gh}{a}+\frac{gh}{\delta},\ \frac{1}{\alpha}=\frac{-1+fgh}{a}+\frac{fgh}{\delta},$$
$$\frac{1}{d}=\frac{1-h}{a}-\frac{h}{\delta},\quad \frac{1}{c}=\frac{1-gh}{a}-\frac{gh}{\delta},\quad \frac{1}{b}=\frac{1-fgh}{a}-\frac{fgh}{\delta}.$$

Truly there is deduced from the above formulas :

$$h=\sqrt{\frac{3\lambda^{(4)}+v}{3\lambda^{(3)}+v}},\quad g=\sqrt{\frac{3\lambda^{(3)}+v}{3\lambda^{(2)}+v}},\quad f=\sqrt{\frac{3\lambda^{(2)}+v}{3\lambda+v}}$$

from which there becomes

$$gh=\sqrt{\frac{3\lambda^{(4)}+v}{3\lambda^{(2)}+v}}\ \text{ and }\ fgh=\sqrt{\frac{3\lambda^{(4)}+v}{3\lambda+v}};$$

and the above values thus will be able to be expressed :

$$\frac{1}{\alpha}=-\frac{1}{b}=-\frac{\sqrt{(3\lambda^{(4)}+v)}}{a}\left(\frac{1}{\sqrt{(3\lambda'+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}}+\frac{1}{\sqrt{(3\lambda'''+v)}}\right)+\frac{\sqrt{(3\lambda^{(4)}+v)}}{\delta}\cdot\frac{1}{\sqrt{(3\lambda+v)}}$$
$$\frac{1}{\beta}=-\frac{1}{c}=-\frac{\sqrt{(3\lambda^{(4)}+v)}}{a}\left(\frac{1}{\sqrt{(3\lambda''+v)}}+\frac{1}{\sqrt{(3\lambda'''+v)}}\right)+\frac{\sqrt{(3\lambda^{(4)}+v)}}{\delta}\left(\frac{1}{\sqrt{(3\lambda+v)}}+\frac{1}{\sqrt{(3\lambda'+v)}}\right)$$
$$\frac{1}{\gamma}=-\frac{1}{d}=-\frac{\sqrt{(3\lambda^{(4)}+v)}}{a}\frac{1}{\sqrt{(3\lambda'''+v)}}+\frac{\sqrt{(3\lambda^{(4)}+v)}}{\delta}\left(\frac{1}{\sqrt{(3\lambda+v)}}+\frac{1}{\sqrt{(3\lambda'+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}}\right).$$

COROLLARY 1

153. If the numbers λ, λ', λ'' and λ''' the numbers for the simple lenses may be taken equal to each other, there becomes for the minimum diffusion length :

$$\sqrt{(3\lambda^{(2)}+v)}=\tfrac{1}{2}\sqrt{(3\lambda+v)},\ \ \lambda^{(2)}=\tfrac{3\lambda-1\cdot 3v}{3\cdot 4}\ ;$$
$$\sqrt{(3\lambda^{(3)}+v)}=\tfrac{1}{3}\sqrt{(3\lambda+v)},\ \ \lambda^{(3)}=\tfrac{3\lambda-2\cdot 4v}{3\cdot 9}\ ;$$
$$\sqrt{(3\lambda^{(4)}+v)}=\tfrac{1}{4}\sqrt{(3\lambda+v)},\ \ \lambda^{(4)}=\tfrac{3\lambda-3\cdot 5v}{3\cdot 16}\ ;$$

hence $f=\frac{1}{2}$, $g=\frac{2}{3}$, $h=\frac{3}{4}$ and for the construction of the simple lenses :

$$\frac{1}{\alpha}=-\frac{3}{4a}+\frac{1}{4\delta},\ \ \frac{1}{\beta}=-\frac{2}{4a}+\frac{2}{4\delta},\ \ \frac{1}{\gamma}=-\frac{1}{4a}+\frac{3}{4\delta},$$
$$\frac{1}{b}=+\frac{3}{4a}-\frac{1}{4\delta},\quad \frac{1}{c}=+\frac{2}{4a}-\frac{2}{4\delta},\ \ \frac{1}{d}=+\frac{1}{4a}-\frac{3}{4\delta}.$$

COROLLARY 2

154. Hence from § 91 the following construction of the four simple lenses is obtained :

Radius of face

first lens PP $\begin{cases} \text{anterior} = \dfrac{4a\delta}{(4\rho-3\sigma)\delta+\sigma a\pm\tau(a+\delta)\sqrt{(\lambda-1)}} \\ \text{posterioris} = \dfrac{4a\delta}{(4\sigma-3\rho)\delta+\rho a\mp\tau(a+\delta)\sqrt{(\lambda-1)}} \end{cases}$

second lens QQ $\begin{cases} \text{anterior} = \dfrac{4a\delta}{(3\rho-2\sigma)\delta+(2\sigma-\rho)a\pm\tau(a+\delta)\sqrt{(\lambda-1)}} \\ \text{posterior} = \dfrac{4a\delta}{(3\sigma-2\rho)\delta+(2\rho-\sigma)a\mp\tau(a+\delta)\sqrt{(\lambda-1)}} \end{cases}$

third lens RR $\begin{cases} \text{anterior} = \dfrac{4a\delta}{(2\rho-\sigma)\delta+(3\sigma-2\rho)a\pm\tau(a+\delta)\sqrt{(\lambda-1)}} \\ \text{posterior} = \dfrac{4a\delta}{(2\sigma-\rho)\delta+(3\rho-2\sigma)a\mp\tau(a+\delta)\sqrt{(\lambda-1)}} \end{cases}$

fourth lens SS $\begin{cases} \text{anterior} = \dfrac{4a\delta}{\rho\delta+(4\sigma-3\rho)a\pm\tau(a+\delta)\sqrt{(\lambda-1)}} \\ \text{posterior} = \dfrac{4a\delta}{\sigma\delta+(4\rho-3\sigma)a\mp\tau(a+\delta)\sqrt{(\lambda-1)}}. \end{cases}$

COROLLARY 3

155. If in addition the numbers $\lambda,\ \lambda',\ \lambda''$ et λ''' may be put equal to unity, which is the minimum value which they can receive, there will be on account of $v = 0,232692$

$$\lambda^{(2)} = \tfrac{3-3v}{3\cdot 4} = 0,191827,\ \lambda^{(3)} = \tfrac{3-8v}{3\cdot 9} = 0,042165,\ \ \lambda^{(4)} = \tfrac{3-15v}{3\cdot 16} = -0,010216;$$

and thus for the quadruple lens the value of the number $\lambda^{(4)}$ becomes negative.

COROLLARY 4

156. Therefore a greater value will be able to be attributed to the numbers $\lambda,\ \lambda',\ \lambda''$ and λ''' , so that the value of $\lambda^{(4)}$ may be produced precisely equal to zero ; certainly this may be done by taking $\lambda = 5v = 1,163460.$ Hence there will be had $\lambda - 1 = 0,163460$ and $\tau\sqrt{(\lambda-1)} = 0,365947,$ from which the individual simple lenses will be able to be constructed in two ways by the formulas shown.

SCHOLIUM 1

157. If we may compare between these two cases, in which there is either $\lambda^{(4)} = -0,010216$, or $\lambda^{(4)} = 0$ we see, even if the difference may scarcely exceed a hundredth part of unity, yet in the construction of the simple lenses the difference can be taken large enough, since the denominators of the formulas (§ 154) either to be increased or decreased by the amount $0,365947(a+\delta)$; which difference perhaps is greater than the not exceedingly coarse errors which may be committed by the artisan. From which in turn we gather, even if in the construction of these quadruple lenses small errors may be committed by the artisan, thence scarcely a perceptible effect to be measured in the diffusion length or in the value of $\lambda^{(4)}$, as on account of this reason these lenses are seen to be especially well adapted in practice. Clearly if there were a need for a lens, for which the value of λ were to become zero, much more use would be required to be made from these quadruple lenses described in corollary 4 than from the triple lenses defined above. Also even if these quadruple lenses will be allowed to be used, in which there is $\lambda^{(4)} = -0,010216$, since this number is scarcely less than zero; and if it may depart from the prescribe measurements in the construction, this value may be led rather towards zero, as in this case the errors taken together may be serving to attain the goal. But below we will show several cases to be given, but we will see several cases to be given, in which it will be convenient for lenses of this kind to be used, for which the value of λ not only shall be equal to zero, but also may be depressed a little below zero; then we will name quadruple lenses of this kind to be in use with the greatest success. But if lenses of this kind may suffice, for which the value of λ shall be 0,042165 or may exceed that by a little amount, triple lenses will be recommended, of which we have given the construction above in (§ 136) ; just as it may consist of a doublet, if there is no need for a value of λ less than 0,191827. Moreover the whole perfection of dioptric instruments rests on this maxim, that lenses may be had, for which the value of λ shall be as small as possible, since these shall be the most suitable for removing the confusion completely ; for which the use of the quadruple lenses described here will be the greatest.

SCHOLIUM 2

158. If those people, who may have considered most carefully which we have presented here concerning quadruple lenses, it will be readily apparent, how quintuple lenses and those of greater multiples may be adapted for use. Evidently there will be a need for a construction of this kind, which may recede a little from the natural minimum, because in this way errors produced in practice disturb the proposed effect minimally. But since scarcely any use may arise, so that there shall be a need for lenses, for which the value of the number λ may be diminished below zero, it would be superfluous to pursue the construction of quintuple or lenses of greater multiples further. Yet meanwhile it will help, if the number λ for lenses of this kind may be indicated conveniently by the

symbols $\lambda^{(5)}$, $\lambda^{(6)}$ etc., we may present the values which they take from the natural minima . We may take all the numbers λ, λ', λ'', λ''', λ'''', λ''''' etc. with all the simple corresponding lenses equal to unity, and of these, which agree with the multiple lenses from the natural minimas, thus will be had:

For the lens:

solitary	$\lambda^{(1)} = \frac{3-0v}{3\cdot 1} = 1,000000$
duplicate	$\lambda^{(2)} = \frac{3-3v}{3\cdot 4} = 0,191827$
triplicate	$\lambda^{(3)} = \frac{3-8v}{3\cdot 9} = 0,421653$
quadruplicate	$\lambda^{(4)} = \frac{3-15v}{3\cdot 16} = -0,010216$
quintuplicate	$\lambda^{(5)} = \frac{3-24v}{3\cdot 25} = -0,034461$
sextuplicate	$\lambda^{(6)} = \frac{3-35v}{3\cdot 36} = -0,047632$
septuplicate	$\lambda^{(7)} = \frac{3-48v}{3\cdot 49} = -0,055573$
octuplicate	$\lambda^{(8)} = \frac{3-63v}{3\cdot 64} = -0,060727$
noncuplicate	$\lambda^{(9)} = \frac{3-80v}{3\cdot 81} = -0,064261$
decuplicate	$\lambda^{(10)} = \frac{3-99v}{3\cdot 100} = -0,066788$

and if lenses of this kind may be multiplied indefinitely, the value of the corresponding number $\lambda^{(\infty)}$ will be $-\frac{v}{3} = -0,077564,$ thus so that at no time shall the number be able to be depressed further : from which it is apparent scarcely any case can exist ever, in which there shall be a need perhaps for a lens of the fifth order. Moreover also in general, whatever values besides unity may be attributed to the numbers λ, λ', λ'', λ''' etc., from which the above numbers for the above numbers $\lambda^{(5)}$, $\lambda^{(6)}$ etc. will be derived easily ; indeed also the determinable distances of the individual simple lenses may be defined without difficulty from the same place, since the law of the progression shall be evident enough. Truly through this whole chapter it is required to be understood everywhere I have considered the thickness of the lenses to be as vanishing, but in the following chapter I am going to consider lenses again in general without ignoring the thickness.

CAPUT III

DE LENTIBUS COMPOSITIS SEU MULTIPLICATIS

DEFINITIO 1

96. *Lens duplicata oritur, si duae lentes super communi axe sibi immediate iungantur.*

Crassitiem hic utriusque lentis tanquam nullam assumo, et quia distantia inter lentes nulla ponitur, crassities etiam lentis duplicatae pro nulla haberi poterit.

COROLLARIUM 1

97. Binae ergo lentes *PP* et *QQ* (Fig. 7) sibi ad contactum fere coniunctae lentem duplicatam constituunt; de qua tamen notandum est eius crassitiem minus tuto negligi posse quam utriusque lentis simplicis seorsim

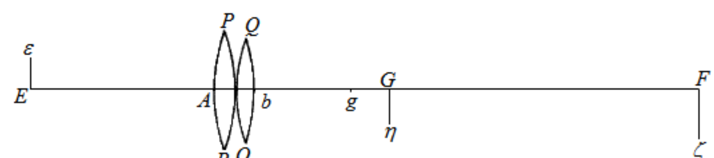

Fig. 7

sumtae. Si enim lentes immediate se in puncto contingerent, phaenomena colorum a NEUTONO observata essent metuenda; tum vero etiam ostendemus, quomodo ratio distantiae inter binas lentes haberi possit.

COROLLARIUM 2

98. Si lentis anterioris *PP* distantiae determinatrices sint a et α, lentis posterioris vero *QQ* b et β, necesse est, ut sit $\alpha + b = 0$ seu $\alpha = -b$. Tum vero obiecti ante lentem ad distantiam $AE = a$ positi imago principalis repraesentabitur post lentem ad distantiam $bG = \beta$.

COROLLARIUM 3

99. Erunt ergo a et β quasi distantiae determinatrices lentis duplicatae; ac sumendo α vel b ad libitum infinita paria lentium pro his distantiis exhiberi possunt. Cum deinde

utraque lens praeterea numerum indefinitum λ recipiat, insuper infinita varietas locum habet.

COROLLARIUM 4

100. Quia crassities pro nihilo reputatur, aperturae in singulis faciebus eadem est ratio; scilicet si in prima facie semidiameter aperturae sit $= x$, in reliquis quoque faciebus apertura eadem vel saltem non minor esse debet.

SCHOLION

101. Non opus est, ut lentes plane ad contactum coniungantur, quoniam forte refractionis lex turbari posset: hoc autem vel minima interposita distantia evitabitur, id quod ad institutum nostrum sufficit, cum etiam crassities utriusque non omnino sit nulla.

PROBLEMA 1

102. *Omnes lentes duplicatas describere, quibus obiectum* $E\varepsilon$ (Fig. 7) *in data ante lentem distantia AE propositum post lentem in data distantia bG repraesentetur, simulque diffusionem imaginis Gg pro data lentis apertura definire.*

SOLUTIO

Sit distantia obiecti $AE = a$, imaginis principalis $bG = \beta$, tum vero lentis primae *PP* distantiae determinatrices a et α, lentis posterioris vero *QQ* b et β; iam quia distantia lentium est nulla, erit $\alpha + b = 0$ seu $\alpha = -b$. Hinc si obiecti magnitudo sit $E\varepsilon = z$, erit imaginis principalis magnitudo $G\eta = \frac{\beta}{a} z$ pro situ inverso. Cum porro utraque lens infinitis modis formari possit, sit λ numerus arbitrarius pro prima *PP* et λ' pro secunda *QQ*, quarum ergo constructio ita se habebit:

$$\text{Pro lente } PP \text{ radius faciei} \begin{cases} \text{anterioris} = \dfrac{a\alpha}{\rho\alpha + \sigma a \pm \tau(a+\alpha)\sqrt{(\lambda-1)}} \\ \text{posterioris} = \dfrac{a\alpha}{\rho\alpha + \sigma a \mp \tau(a+\alpha)\sqrt{(\lambda-1)}} \end{cases}$$

$$\text{Pro lente } QQ \text{ radius faciei} \begin{cases} \text{anterioris} = \dfrac{b\beta}{\rho\beta + \sigma b \pm \tau(b+\beta)\sqrt{(\lambda'-1)}} \\ \text{posterioris} = \dfrac{b\beta}{\rho b + \sigma\beta \mp \tau(b+\beta)\sqrt{(\lambda'-1)}} \end{cases}$$

Si ratio refractionis sit diversa, pro secunda lente scribi debet ρ', σ' et τ' loco ρ, σ et τ.

Pro spatio autem diffusionis Gg inveniendo sit semidiameter aperturae lentis duplicatae $= x$, eritque

$$Gg = \mu\beta\beta xx\left\{\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{\nu}{a\alpha}\right)+\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\lambda'\left(\frac{1}{b}+\frac{1}{\beta}\right)^2+\frac{\nu}{b\beta}\right)\right\}$$

et radiorum in g concurrentium inclinatio ad axem $=\frac{x}{\beta}$. Si ratio refractionis discrepet, in parte ex lente secunda orta scribatur μ', ν' loco μ, ν.

COROLLARIUM 1

103. Huius ergo lentis duplicatae distantiae determinatrices sunt a et β, praeterea vero duo numeri arbitrarii λ et λ' una cum distantia α vel b eius perfectam determinationem constituunt, unde in huiusmodi lentibus multo maior varietas locum habet quam in lentibus simplicibus.

COROLLARIUM 2

104. Si ex eisdem distantiis determinatricibus a et β adiungendo numero arbitrario λ^0 lens simplex construatur, ea imaginem eadem magnitudine $G\eta = \frac{\beta}{a}z$ referet; sed pro eadem apertura, cuius semidiameter $= x$, habebitur spatium diffusionis

$$Gg = \mu\beta\beta xx\left(\frac{1}{a}+\frac{1}{\beta}\right)\left(\lambda^0\left(\frac{1}{a}+\frac{1}{\beta}\right)^2+\frac{\nu}{a\beta}\right)$$

COROLLARIUM 3

105. Fieri ergo poterit, ut lens duplicata modo maiorem, modo minorem diffusionem gignat. Lens autem simplex minimam parit diffusionem, si $\lambda^0 = 1$; ergo tum lentes duplicatae simplicibus erunt praeferendae, cum adhuc minorem diffusionem producent.

SCHOLION

106. Concipi quidam semper poterit lens simplex eandem diffusionem gignens ac lens duplicata, si pro numero λ^0 omnes valores admittamus; quomodocunque enim lens duplicata fuerit comparata, si spatium diffusionis inde productum huic ex lente simplici nato aequale statuatur, determinatus valor pro numero λ^0 elicitur: qui si fuerit positivus et unitate maior, realis lens simplex aequivalens exhiberi poterit, sin autem prodeat unitate minor vel adeo negativus, lens simplex inter imaginaria erit referenda. Quando autem fit $\lambda^0 > 1$, evidens est lentem simplicem eundem plane effectum esse edituram

ac duplicatam, ideoque semper expediet lente simplici potius uti quam duplicata; quodsi vero prodierit $\lambda^0 < 1$, quo casu lens simplex fit imaginaria, tum lentes duplicatae effectum praestabunt a simplicibus non expectandum, qui adeo cum insigni hoc commodo, quod spatium diffusionis futurum sit minus, erit coniunctus. Talibus ergo lentibus duplicatis maximo cum successu uti poterimus, eoque magis eae simplicibus erunt anteponendae, quo minor fuerit valor numeri λ^0 iis respondens.

PROBLEMA 2

107. *Data lente duplicata ad binas distantias determinatrices* $AE = a$ (Fig. 7) *et* $bG = \beta$ *relata, pro iisdem distantiis definire lentem simplicem, quae pro eadem apertura eandem diffusionem imaginis producat.*

SOLUTIO

Totum ergo negotium huc redit, ut spatium diffusionis Gg a lente simplici productum (§ 104) aequale ponatur spatio diffusionis a lente duplicata orto, cuius expressio in problemate praecedente (§ 102) est inventa, indeque valor numeri λ^0 pro constructione lentis simplicis eliciatur. Quae investigatio quo commodius institui possit, ponamus:

$$\tfrac{1}{\alpha} = \tfrac{f-1}{a} + \tfrac{f}{\beta}, \text{ eritque } \tfrac{1}{b} = \tfrac{-f+1}{a} - \tfrac{f}{\beta},$$

ita ut loco quantitatis a vel b numerum f introducamus, eritque

$$\tfrac{1}{a} + \tfrac{1}{\alpha} = f(\tfrac{1}{a} + \tfrac{1}{\beta}) \text{ et } \tfrac{1}{b} + \tfrac{1}{\beta} = (1-f)(\tfrac{1}{a} + \tfrac{1}{\beta})$$

unde spatium diffusionis lente duplicata ortum prodit:

$$Gg = \mu\beta\beta xx \left\{ \begin{aligned} & f\left(\tfrac{1}{a} + \tfrac{1}{\beta}\right)\left(\lambda ff\left(\tfrac{1}{a} + \tfrac{1}{\beta}\right)^2 + \nu\left(\tfrac{f-1}{aa} + \tfrac{f}{a\beta}\right)\right) \\ & +(1-f)\left(\tfrac{1}{a} + \tfrac{1}{\beta}\right)\left(\lambda'(1-f)^2\left(\tfrac{1}{a} + \tfrac{1}{\beta}\right)^2 + \nu\left(\tfrac{1-f}{a\beta} - \tfrac{f}{\beta\beta}\right)\right) \end{aligned} \right\}$$

quae expressio reducitur ad hanc formam:

$$Gg=\mu\beta\beta xx\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)\left\{\begin{array}{l}\left(\lambda f^3+\lambda'(1-f)^2\right)\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2\\ +\nu\left(\frac{f(f-1)}{aa}+\frac{1-2f+2ff}{a\beta}+\frac{f(f-1)}{\beta\beta}\right)\end{array}\right\}$$

Verum postremum membrum

$\frac{f(f-1)}{aa}+\frac{1-2f+2ff}{a\beta}+\frac{f(f-1)}{\beta\beta}$ mutatur in $f(f-1)(\frac{1}{a}+\frac{1}{\beta})^2+\frac{1}{a\beta}$,

sicque habebimus pro lente duplicata:

$$Gg=\mu\beta\beta xx\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)\left\{\left(\lambda f^3+\lambda'(1-f)^3-\nu f(1-f)\right)(\tfrac{1}{a}+\tfrac{1}{\beta})^2+\tfrac{\nu}{a\beta}\right\}$$

quae forma iam facillime cum spatio diffusionis lentis simplicis comparatur indeque manifesto colligitur:

$$\lambda^0=\lambda f^3+\lambda'(1-f)^3-\nu f(1-f).$$

Cum autem loco quantitatum *a* et *b* numerum *f* introduxerimus, constructio lentis duplicatae ita se habebit:

Pro lente radius faciei

prima *PP* $\begin{cases}\text{anterioris}=\dfrac{a\beta}{(\rho-\sigma(1-f))\beta+\sigma fa\pm\tau f(a+\beta)\sqrt{(\lambda-1)}}\\ \text{posterioris}=\dfrac{a\beta}{(\sigma-\rho(1-f))\beta+\rho fa\mp\tau f(a+\beta)\sqrt{(\lambda-1)}}\end{cases}$

secunda *QQ* $\begin{cases}\text{anterioris}=\dfrac{a\beta}{(\sigma-\rho f)a+\rho(1-f)\beta\pm\tau(1-f)(a+\beta)\sqrt{(\lambda'-1)}}\\ \text{posterioris}=\dfrac{a\beta}{(\rho-\sigma f)a+\sigma(1-f)\beta\mp\tau(1-f)(a+\beta)\sqrt{(\lambda'-1)}}\end{cases}$

Tum vero invento numero λ^0 constructio lentis simplicis aequivalentis erit

radius faciei $\begin{cases}\text{anterioris}=\dfrac{a\beta}{\rho\beta+\sigma a\pm\tau(a+\beta)\sqrt{(\lambda^0-1)}}\\ \text{posterioris}=\dfrac{a\beta}{\rho\beta+\sigma a\mp\tau(a+\beta)\sqrt{(\lambda^0-1)}}\end{cases}$

COROLLARIUM 1

108. Quoties ergo fuerit

$$\lambda f^3 + \lambda'(1-f)^3 - \nu f(1-f) > 1,$$

semper lens simplex parari potest duplicatae aequivalens, iisque ergo casibus praestabit lente simplici uti potius quam lente duplicata.

COROLLARIUM 2

109. Verum si fuerit

$$\lambda f^3 + \lambda'(1-f)^3 - \nu f(1-f) < 1,$$

ob $\lambda^0 < 1$ constructio lentis simplicis fit impossibilis, ac lens duplicata minorem pariet diffusionem, quam per ullam lentem simplicem obtineri potest.

COROLLARIUM 3

110. Si esset $f = 0$, prodiret $\lambda^0 = \lambda'$, et in lente duplicata anterior nullam refractionem produceret ob facies parallelas, resque eodem rediret, ac si posterior sola adesset. Sin autem sumatur $f = 1$, fit $\lambda^0 = \lambda'$, et lens posterior superfluas utroque ergo casu nullum lucrum impetratur.

COROLLARIUM 4

111. Cum autem numeri λ et λ' unitate nequeant esse minores sitque $\nu = 0{,}232692$, patet pro *f* nullum valorem inter limites 0 et 1 assumi posse, unde fiat $\lambda^0 = 0$. At si *f* extra hos limites capiatur, utique pro λ et λ' eiusmodi numeri unitate maiores assignari poterunt, ut fiat $\lambda^0 = 0$.

SCHOLION 1

112. Ratione ergo numeri *f* tres casus lentium duplicatarum considerari conveniet, prout vel *f* intra limites 0 et 1 continetur, vel fuerit $f > 1$, vel $f < 0$. Primo casu evenire non potest, ut fiat $\lambda^0 = 0$; sed plurimum intererit eam determinasse lentem duplicatam, pro qua λ^0 minimum obtineat valorem, qui quo magis infra unitatem cadat, eo perfectior lens erit censenda maioreque iure simplicibus anteferenda. Binis reliquis vero casibus $f > 1$, et $f < 0$ eiusmodi adeo lentes duplicatae parari poterunt, quae praebeant $\lambda^0 = 0$, quae ergo pro perfectissimis essent habendae. Verum hic quoque ad praxin est

respiciendum, quae cum semper a praeceptis theoriae aberrare soleat, evenire potest, ut levi errore commisso numerus λ^0 non solum non evanescat, sed adeo unitatem excedat, quo casu utique expediret lente uti simplici.

SCHOLION 2

113. Cum tanti sit momenti rationem aberrationis, a qua praxis vix liberari potest, habere, eam etiam in lentibus simplicibus perpendi conveniet. Postulavimus autem pro datis distantiis determinatricibus lentem construi posse, in qua numerus λ datum obtineat valorem, dum ne sit unitate minor; hic igitur observari oportet, quo maior fuerit λ, eo difficilius fore errorem evitare; si enim in constructione levissimus error committatur, alius eo magis diversus valor pro λ orietur, quo maior fuerit λ. Verum e contrario, cum unitas sit minimus valor, quem λ recipere potest, ex natura minimi liquet, etiamsi in praxi a praescripta regula notabiliter recedatur, tamen inde vix sensibile discrimen in valorem λ esse redundaturum. Ex quo concludimus felicissimo cum successu eiusmodi lentes simplices parari posse, pro quibus futurum sit $\lambda = 1$, neque hic errores praxeos, nisi fuerint enormes, admodum esse pertimescendos. Deinde quo minus numerus λ unitatem superare debeat, eo certiores esse poterimus de successu, sed non eo gradu, quo casu $\lambda = 1$; at si opus sit eiusmodi lente, pro qua valor ipsius λ debeat esse numerus satis magnus, difficillime per praxin satisfiet ac fortasse ingentem lentium numerum parare oportebit, antequam una obtineatur scopo satisfaciens. Quamobrem, si praxi consulere velimus, vix alias lentes exigere debemus, nisi pro quibus numerus λ vel sit unitas ipsa vel parumper maior. Sin autem ad insigne aliquod commodum aliae lentes requirantur, labori non erit parcendum, cum fortasse non nisi post plurimos conatus irritos voti tandem compotes reddi queamus.

PROBLEMA 3

114. *Definire eam lentem duplicatam, pro qua, si numerus f intra limites* 0 *et* 1 *accipiatur, numerus* λ^0 *minimum adipiscatur valorem.*

SOLUTIO

Positis iisdem, quae in praecedentibus problematibus sunt constituta, invenimus esse

$$\lambda^0 = \lambda f^3 + \lambda'(1-f)^3 - \nu f(1-f),$$

ubi, cum f intra limites 0 et 1 assumi debeat, ambo termini λf^3 et $\lambda'(1-f)^3$ erunt positivi. Quare ut λ^0 omnium minimum valorem nanciscatur, necesse est utrique numero λ et λ' minimum valorem, cuius est capax, tribui.

Sit ergo $\lambda = 1$ et $\lambda' = 1$, ac habebimus

$$\lambda^0 = 1-3f+3ff-vf+vff = 1-(3+v)f(1-f),$$

quae expressio ut minima reddatur, oportet fieri $f(1-f)$ maximum, id quod fit sumendo $f=\frac{1}{2}$; hincque oritur

$$\lambda^0 = 1-\tfrac{1}{4}(3+v) = \tfrac{1-v}{4} = 0{,}191827.$$

Quare constructio huius lentis duplicatae ita se habebit:

$$\text{Pro lente prima } PP \text{ radius faciei } \begin{cases} \text{anterioris} = \frac{2a\beta}{(2\rho-\sigma)\beta+\sigma a} \\ \text{posterioris} = \frac{2a\beta}{(2\sigma-\rho)\beta+\rho a} \end{cases}$$

$$\text{Pro lente prima } QQ \text{ radius faciei } \begin{cases} \text{anterioris} = \frac{2a\beta}{(2\sigma-\rho)a+\sigma\beta} \\ \text{posterioris} = \frac{2a\beta}{(2\rho-\sigma)a+\sigma\beta} \end{cases}$$

et si aperturae semidiameter sit $=x$, erit spatium diffusionis

$$Gg = \mu\beta\beta xx\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)\left(0{,}191827\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2+\tfrac{v}{\alpha\beta}\right).$$

Si eiusmodi vitro utamur, pro quo est $n=1{,}60=\frac{8}{5}$, tum ob $v=\frac{4}{15}$ prodiret $\lambda^0 = 0{,}183333$, ideoque haec vitri species adhuc minorem confusionem pareret.

COROLLARIUM 1

115. Si pro iisdem distantiis determinatricibus a et β lens simplex minimam diffusionem pariens construatur, quod fit sumendo

$$\text{radium faciei anterioris} = \tfrac{a\beta}{\rho\beta+\sigma a},\ \text{posterioris} = \tfrac{a\beta}{\rho a+\sigma\beta},$$

spatium diffusionis foret $\mu\beta\beta xx(\frac{1}{a}+\frac{1}{\beta})((\frac{1}{a}+\frac{1}{\beta})^2+\frac{v}{a\beta})$.

COROLLARIUM 2

116. Apparat ergo a lente duplicata descripta multo minorem oriri diffusionem quam a lente simplici, etiamsi haec iam ad minimam diffusionem sit instructa. Cum enim ceterae partes sint pares, coefficiens membri $(\frac{1}{a}+\frac{1}{\beta})^2$ plus quam quintuplo minor est in duplicata quam in simplici.

COROLLARIUM 3

117. Si ponamus $\lambda=1$ et $\lambda'=1$, vel saltem $\lambda=\lambda'$, minor valor pro λ^0 obtineri nequit, quam invenimus, etiamsi pro *f* alios valores admittere velimus. Unde si utraque lens per se iam minimam diminsionem pariat, pro lente duplicata valor ipsius λ^0 minor quam 0,191827 fieri nequit.

SCHOLlON

118. Huiusmodi ergo lentes duplicatae maxime sunt notatu dignae, cum loco simplicium adhibitae multo minorem diffusionem pariant, ex quo in constructione Telescopiorum et Microscopiorum earum amplissimus erit usus. Neque vero hac insigni proprietate sunt praeditae, sed etiam earum constructio in praxi minimis difficultatibus est obnoxia: propterea quod, etsi a praescriptis regulis parumper aberretur, effectus tamen inde vix ullam mutationem patiatur. Sive enim in constructione utriusque seorsim levis error committatur, valores numerorum λ et λ' unitatem haud sensibiliter excedent sive in quantitate *f* valor iustus $f=\frac{1}{2}$ non exacte observetur, error vix sentietur, quoniam hi numeri ex natura minimi sunt eruti. Quomodocunque autem a regulis praescriptis aberretur, valor ipsius λ^0 inde paulisper maior prodibit. Veluti si eiusmodi errores committantur ·, ut sit

$$\lambda=1+\frac{1}{10}, \lambda'=1+\frac{1}{10} \text{ et } f=\frac{1}{2}\pm\frac{1}{20},$$

prodibit $\lambda^0=0,191827+0,03383$ seu $\lambda^0=0,2257$, ita ut discrimen partem tantum tricesimam unitatis conficiat. Facile autem intelligitur, dummodo λ^0 prodeat minus quam $\frac{1}{4}$, quod nunquam non facile praestari posse videtur, ab his lentibus duplicatis insignem utilitatem expectandam esse. Etsi ergo eiusmodi lentes duplicatae confici possunt, pro quibus numerus λ^0 plane evanescat, ob harum lubricam constructionem illae istis anteferendae videntur, ut mox clarius patebit.

PROBLEMA 4

119. *Pro datis distantiis determinatricibus $AE = a$ et $bG = \beta$ eas lentes duplicatas invenire, in quibus sit $\lambda^0 = 0$.*

SOLUTIO

Cum fieri nequeat $\lambda^0 = 0$, nisi numerus f extra limites 0 et 1 accipiatur, simulque numeri λ et λ' fuerint inaequales, ita ut alterutra saltem lens non debeat seorsim minimam diffusionem parare, ponamus esse vel $f > 1$ vel $f < 0$.
Sit ergo primo $f = 1 + \xi$ et cum sit $\lambda^0 = \lambda(1+\xi)^3 - \lambda'\xi^3 + \nu\xi(1+\xi)$, ut fiat $\lambda^0 = 0$, oportet esse

$$\lambda' = \lambda(1+\tfrac{1}{\xi})^3 + \tfrac{\nu(1+\xi)}{\xi^2},$$

unde λ' necessario unitatem superabit; cuius valor ne prodeat nimis magnus, sumi conveniet $\lambda = 1$, ita ut sit

$$\lambda' = (1+\tfrac{1}{\xi})^3 + \tfrac{0,282692(1+\xi)}{\xi^2} \text{ et } \lambda = 1.$$

Quicunque ergo valor ipsi ξ tribuatur, lens duplicata habetur, pro qua sit $\lambda^0 = 0$, ac propterea spatium diffusionis $= \mu\beta\beta xx(\frac{1}{a} + \frac{1}{\beta})\frac{\nu}{a\beta}$. Videamus nonnullus casus speciales:

$$f = \tfrac{3}{2};\ \xi = \tfrac{1}{2};\ \lambda = 1 \text{ et } \lambda' = 28,396152$$
$$f = 2;\ \xi = 1;\ \lambda = 1 \text{ et } \lambda' = 8,465384$$
$$f = 3;\ \xi = 2;\ \lambda = 1 \text{ et } \lambda' = 3,549519$$
$$f = 4;\ \xi = 3;\ \lambda = 1 \text{ et } \lambda' = 2,473789$$
$$f = 5;\ \xi = 4;\ \lambda = 1 \text{ et } \lambda' = 2,025841$$
$$f = 6;\ \xi = 5;\ \lambda = 1 \text{ et } \lambda' = 1,783846$$
etc.

Pro altero casu sit $f = -\xi$, ideoque $\lambda^0 = -\lambda\xi^3 + \lambda'(1+\xi)^3 + \nu\xi(1+\xi)$, unde facto $\lambda^0 = 0$ prodit

$$\lambda = \lambda'(1+\tfrac{1}{\xi})^3 + \tfrac{\nu(1+\xi)}{\xi^2}.$$

Statui ergo conveniet $\lambda' = 1$, ac pro λ notentur casus sequentes:

$$f = -\tfrac{1}{2};\ \xi = \tfrac{1}{2};\ \lambda = 28{,}396152 \text{ et } \lambda' = 1$$
$$f = -1;\ \xi = 1;\ \lambda = 8{,}465384 \text{ et } \lambda' = 1$$
$$f = -2;\ \xi = 2;\ \lambda = 3{,}549519 \text{ et } \lambda' = 1$$
$$f = -3;\ \xi = 3;\ \lambda = 2{,}473789 \text{ et } \lambda' = 1$$
$$f = -4;\ \xi = 4;\ \lambda = 2{,}025841 \text{ et } \lambda' = 1$$
$$f = -5;\ \xi = 5;\ \lambda = 1{,}783846 \text{ et } \lambda' = 1$$
etc.

sicque patet infinitis modis huiusmodi lentes duplicatas parari posse, pro quibus sit $\lambda^0 = 0$ et spatium diffusionis

$$Gg = \mu\beta\beta xx(\tfrac{1}{a} + \tfrac{1}{\beta})\tfrac{\nu}{a\beta}.$$

SCHOLION 1

120. Si huiusmodi lentes accuratissime parari possent, nullum est dubium, quin praecedentibus essent anteferendae, propterea quod diffusio iis adhuc magis diminuitur. Verum dolendum est, quod minimus error in earum constructione commissus omnem fere usum destruat. Quo hoc facilius diiudicare queamus, examinemus eum casum, quo est

$$f = 5,\ \lambda = 1 \text{ et } \lambda' = 2{,}025841$$

hincque

$$\lambda^0 = \lambda f^3 - \lambda'(f-1)^3 + \nu f(f-1) = 0.$$

Ponamus autem in constructione errorem esse commissum, ut revera non sit $f = 5$, sed $f = 5\tfrac{1}{10}$, dum numeri λ et λ' suos iustos valores obtineant: ob hunc autem vix vitandum errorem non fit $\lambda^0 = 0$, sed adeo $\lambda^0 = -2{,}011$; sicque haec lens duplicata simplicibus longe est postponenda; simili modo si *f* esset = 5, sed vel λ vel λ' tantillum a praescripto valore aberraret, enorme statim discrimen in valorem ipsius λ^0 redundaret. Minus quidem error metuendus videtur in ea specie, qua $\lambda = 1$, $\lambda' = 28{,}396152$ et $f = \tfrac{3}{2}$; sed praeterquam quod lens simplex posterior difficillime parari queat, pro qua λ' praecise valorem assignatum consequatur, talis lens ad usum dioptricum plane est inepta ob ingentem alterius faciei curvaturam. Quae cum ita sint, quia tam exiguus error in paratione huiusmodi lentium commissus facit, ut λ^0 adeo supra unitatem excrescat, vix sperare poterimus, ut unquam talis lens duplicata perficiatur, pro qua λ^0 usque ad $\frac{1}{5}$ diminuatur. In lentibus autem praecedentis generis successus vix fallere poterit, nisi in praxi enormiter a praescripta regula aberretur: ex quo his solis lentibus duplicatis cum fructu uti licebit, dum contra eae, quas in praesente problemate descripsimus, penitus profligandae videntur.

SCHOLION 2

121. Pro datis ergo binis distantiis determinatricibus a et β semper eiusmodi lens duplicata parari potest, ex qua nascatur spatium diffusionis

$$Gg = \mu\beta\beta xx\left(\frac{1}{a}+\frac{1}{\beta}\right)\left(\lambda^0\left(\frac{1}{a}+\frac{1}{\beta}\right)^2+\frac{\nu}{a\beta}\right)$$

ita ut λ^0 numerum quemcunque denotare possit. Ad hoc enim satisfieri oportet huic aequationi:

$$\lambda^0 = \lambda f^3 + \lambda'(1-f)^3 - \nu f(1-f),$$

id quod semper fieri potest, cum f plane ab arbitrio nostro pendeat et numeri λ et λ' tantum non unitate minores accipi debeant. Definitis autem his tribus numeris λ, λ' et f ita, ut λ^0 datum valorem obtineat, binae lentes simplices, ex quibus duplicata est componenda, secundum formulas § 107 datas confici debent. Ubi quidem tenenda sunt ea, quae modo observavimus, in praxi eas lentes facillime obtineri, quando numeri λ et λ' unitatem parum superant, f vero propemodum $\frac{1}{2}$ denotat, cum contra, quo magis hi numeri ab istis terminis recedant, eo maius sit periculum, ne effectus enormiter fallat. Ceterum cum diffusionem a lente duplicata oriundam ad eandem formam reduxerimus, qua diffusio lentis simplicis exprimitur, inde id commodi consequimur, ut simili modo diffusionem a lentibus magis multiplicatis ortam definire valeamus.

SUPPLEMENTUM 1

DE LENTIBUS DUPLICATIS

Si pro lente anteriori ratio refractionis sit $n : 1$, pro lente posteriori vero alia ratio $n' : 1$ locum habeat, problemata hic tractata sequenti modo resolvi poterunt.

PRO PROBLEMATE 1

Si pro numeris ρ, σ et τ, qui ex n oriuntur, quaerantur simili modo ex n' valores ρ', σ' et τ', erunt

$$\text{Pro lente } PP \text{ radius faciei } \begin{cases} \text{anterioris} = \dfrac{a\alpha}{\rho\alpha+\sigma a\pm\tau(a+\alpha)\sqrt{(\lambda-1)}} \\ \text{posterioris} = \dfrac{a\alpha}{\rho a+\sigma\alpha\mp\tau(a+\alpha)\sqrt{(\lambda-1)}} \end{cases}$$

$$\text{Pro lente } QQ \text{ radius faciei } \begin{cases} \text{anterioris} = \dfrac{b\beta}{\rho'\beta+\sigma' b\pm\tau'(b+\beta)\sqrt{(\lambda'-1)}} \\ \text{posterioris} = \dfrac{b\beta}{\rho' b+\sigma'\beta\mp\tau'(b+\beta)\sqrt{(\lambda'-1)}} \end{cases}$$

et definitis simili modo valoribus μ', v' ex ratione $n':1$ reperietur spatium diffusionis

$$\beta\beta xx\left\{\mu(\tfrac{1}{a}+\tfrac{1}{\alpha})\left(\lambda\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)^2+\tfrac{v}{a\alpha}\right)+\mu'(\tfrac{1}{b}+\tfrac{1}{\beta})\left(\lambda'\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)^2+\tfrac{v'}{b\beta}\right)\right\};$$

reliqua manent ut in problemate.

PRO PROBLEMATE 2

Quoniam hic duae vitri species occurrunt, ponamus lentem simplicem quaesitam ex alio quocunque vitri genere parari, cuius ratio refractionis sit $n^0:1$, unde prodeant numeri μ^0 et v^0 ; superfluum autem foret numeros ρ^0, σ^0 et τ^0 computari, quoniam radii facierum fiunt imaginarii, ita ut eos exprimere non sit opus, et quoniam pro hac lente simplici aequivalente numerus λ^0 est introductus, erit huius lentis spatium diffusionis

$$\beta\beta xx\cdot\mu^0(\tfrac{1}{a}+\tfrac{1}{\beta})\left(\lambda^0\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2+\tfrac{v^0}{a\beta}\right),$$

quod ut cum spatio diffusionis lentis duplicatae aequale fiat, ponatur, uti in problemate est factum,

$$\tfrac{1}{\alpha}=\tfrac{f-1}{a}+\tfrac{f}{\beta} \text{ et } \tfrac{1}{b}=\tfrac{1-f}{a}-\tfrac{f}{\beta},$$

ut prodeat

$$\tfrac{1}{a}+\tfrac{1}{\alpha}=f(\tfrac{1}{a}+\tfrac{1}{\beta}) \text{ et } \tfrac{1}{b}+\tfrac{1}{\beta}=(1-f)(\tfrac{1}{a}+\tfrac{1}{\beta}),$$

unde pervenietur ad hanc aequationem

$$\mu f\left(\lambda f^2\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2+\nu\left(\tfrac{f-1}{a^2}+\tfrac{f}{a\beta}\right)\right)+$$
$$\mu'(1-f)\left(\lambda'(1-f)^2\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2+\nu'\left(\tfrac{1-f}{a\beta}-\tfrac{f}{\beta^2}\right)\right)$$
$$=\mu^0\left(\lambda^0\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2+\tfrac{\nu^0}{a\beta}\right)$$

quae ita repraesentari potest

$$\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2\left(\mu\lambda f^3+\mu'\lambda'(1-f)^3\right)+\left(\tfrac{f-1}{a}+\tfrac{f}{\beta}\right)\left(\tfrac{\mu\nu f}{a}-\tfrac{\mu'\nu'(1-f)}{\beta}\right)$$
$$=\mu^0\lambda^0\left(\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2\right)+\tfrac{\nu^0\mu^0}{a\beta}.$$

Unde

$$\lambda^0=\frac{\mu\lambda f^3+\mu'\lambda'(1-f)^3}{\mu^0}+\frac{aa\beta\beta}{\mu^0(\alpha+\beta)^2}\left(\left(\frac{f-1}{a}+\frac{f}{\beta}\right)\left(\frac{\mu\nu f}{a}-\frac{\mu'\nu'(1-f)}{\beta}\right)-\frac{\nu^0\mu^0}{a\beta}\right).$$

Denique si ex ratione refractionis n':1 computentur numeri ρ', σ' et τ', radii facierum lentis duplicatae erunt

Pro lente *PP* radius faciei
$$\begin{cases}\text{anterioris}=\dfrac{a\beta}{(\rho-\sigma(1-f))\beta+\sigma fa\pm\tau f(a+\beta)\sqrt{(\lambda-1)}}\\[2ex]\text{posterioris}=\dfrac{a\beta}{(\sigma-\rho(1-f))\beta+\rho fa\mp\tau f(a+\beta)\sqrt{(\lambda-1)}}\end{cases}$$

Pro lente *QQ* radius faciei
$$\begin{cases}\text{anterioris}=\dfrac{a\beta}{(\sigma'-\rho')a+\rho'(1-f)\beta\pm\tau'(1-f)(a+\beta)\sqrt{(\lambda'-1)}}\\[2ex]\text{posterioris}=\dfrac{a\beta}{(\rho'-\sigma')a+\sigma'(1-f)\beta\mp\tau'(1-f)(a+\beta)\sqrt{(\lambda'-1)}}.\end{cases}$$

AD PROBLEMA 3

Hoc problema non solum pro diversa refractione n et n' hic generalius pertractabo, sed etiam rationem distantiae inter binas lentes habebo. Primum igitur utramque lentem ad distantias determinatrices lentis duplicatae, a et β, revocabo, ponendo

$$\tfrac{1}{a}+\tfrac{1}{\alpha}=f\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)\quad\text{et}\quad\tfrac{1}{b}+\tfrac{1}{\beta}=g\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)$$

ut sit

$$\frac{1}{\alpha}=\frac{f-1}{a}+\frac{f}{\beta} \quad \text{et} \quad \frac{1}{b}=\frac{g}{a}+\frac{g-1}{\beta}$$

hincque

$$\alpha=\frac{a\beta}{fa+(f-1)\beta} \quad \text{et} \quad b=\frac{a\beta}{(g-1)a+g\beta}$$

ideoque distantia lentium

$$\alpha+b=\frac{a\beta(f+g-1)(a+\beta)}{(fa+(f-1)\beta)((g-1)a+g\beta)},$$

quae si deberet esse $=0$, capi oporteret $g=1-f$; sed si distantiam aliquam inter lentes admittamus, statuamus $f+g-1=\omega$, denotante ω fractionem quandam minimam, sive positivam sive negativam, ut distantia lentium prodeat positiva et valde parva. Cum hinc igitur sit $g=1+\omega-f$, erit lentium distantia

$$\alpha+b=\frac{a\beta(a+\beta)\omega}{(fa+(f-1)\beta)((\omega-1)a+(1+\omega-f)\beta)}.$$

Spatium autem diffusionis nunc ita exprimetur:

$$\beta\beta xx\cdot\left(\frac{1}{a}+\frac{1}{\beta}\right)\left(\begin{array}{l}\lambda\mu f^{3}\left(\frac{1}{a}+\frac{1}{\beta}\right)^{2}+\lambda'\mu'g^{3}\left(\frac{1}{a}+\frac{1}{\beta}\right)^{2}\\ +\frac{\mu\nu f}{a}\left(\frac{f-1}{a}+\frac{f}{\beta}\right)+\frac{\mu'\nu'g}{\beta}\left(\frac{g}{a}+\frac{g-1}{\beta}\right)\end{array}\right)$$

quae cum in hoc problemate ita tractari debeat, ut tam *f* quam *g* positive sumantur et haec formula minima reddatur, evidens est litteris λ et λ' minimos valores tribui debere, scilicet $\lambda=1$ et $\lambda'=1$; deinde pro hoc casu minimi *f* et *g* convenienter definiantur, ubi, quia $f+g-1=\omega$ ideoque constans in differentiatione, habebimus $dg=-df$; unde obtinebimus hanc aequationem

$$(3\mu f^{2}-3\mu'g^{2})\left(\frac{1}{a}+\frac{1}{\beta}\right)^{2}+\frac{\mu\nu}{a}\left(\frac{2f-1}{a}+\frac{2f}{\beta}\right)-\frac{\mu'\nu'}{\beta}\left(\frac{2g}{a}+\frac{2g-1}{\beta}\right)=0,$$

cui proxime satisfit ponendo $f=g=\frac{1+\omega}{2}$, quibus valoribus substitutis spatium diffusionis ipsum minimum erit proxima

$$\frac{\beta^{2}x^{2}(1+\omega)}{8}\left(\frac{1}{a}+\frac{1}{\beta}\right)\left(\begin{array}{l}\left(\frac{1}{a}+\frac{1}{\beta}\right)^{2}(1+\omega)^{2}(\mu+\mu')\\ +\frac{2\mu\nu}{a}\left(\frac{\omega-1}{a}+\frac{1+\omega}{\beta}\right)+\frac{2\mu'\nu'}{\beta}\left(\frac{1+\omega}{a}+\frac{\omega-1}{\beta}\right)\end{array}\right)$$

Tum vero distantia lentium erit

$$\alpha + b = \frac{4a\beta(a+\beta)\omega}{((1+\omega)a+(1-\omega)\beta)((\omega-1)a+(1+\omega)\beta)} = \frac{4a\beta(a+\beta)\omega}{-(1-\omega^2)(a^2+\beta^2)+2(1+\omega^2)a\beta},$$

cuius denominator cum sit negativus ob *w* minimum, necesse est fractionem ω sumi debere negativam; hincque adeo spatium diffusionis minus reddetur.

COROLLARIUM

Si ergo distantia obiecti a fuerit infinita seu $a = \infty$, habebitur primo distantia lentium $= \frac{-4\beta\omega}{1-\omega^2}$ et secundo spatium diffusionis

$$\frac{x^2(1+\omega)}{8\beta}((1+\omega)^2(\mu+\mu') - 2\mu'\nu'(1-\omega))$$

quod, cum ω debeat esse negativum, non mediocriter minus erit, quam si distantia lentium esset nulla.

AD PROBLEMA 4

In hoc problemate etiam distantiam lentium non negligamus; factaque reductione, ut ante, statuamus spatium diffusionis plane evanescens; id quod fieri nequit, nisi altera litterarum *f* et *g* sit negativa, quod cum etiam fiat, quando confusio a diversa radiorum refrangibilitate oriunda ad nihilum redigi debet, ut infra videbimus; hic casus multo magis evolutionem meretur. Ponatur igitur $g = -\zeta f$, ubi ζ ex illa conditione determinatur, ut primo pro distantia lentium sit

$$\alpha + b = \frac{-a\beta(a+\beta)(f-\zeta f-1)}{(fa+(f-1)\beta)((\zeta f+1)a+\zeta f\beta)},$$

et posito $f + g - 1 = \omega$ prodeat $f - \zeta f = 1 + \omega$ ideoque

$$f = \frac{1+\omega}{1-\zeta} \text{ et } g = \frac{-\zeta(1+\omega)}{1-\zeta}$$

sicque

$$\alpha + b = \frac{-a\beta(a+\beta)\omega(1-\zeta)^2}{((1+\omega)a+(\omega+\zeta)\beta)((1+\omega\zeta)a+\zeta(1+\omega)\beta)},$$

ac si in denominatore ω reiiciatur, erit

$$\alpha + b = \frac{-a\beta(a+\beta)(1-\zeta)^2\omega}{(a+\zeta\beta)^2}$$

sicque patet ω negative capi debere.

Posito autem $g=-\zeta f$ fit spatium diffusionis

$$\beta^2 x^2 (\tfrac{1}{a}+\tfrac{1}{\beta})\left(\begin{array}{l}(\lambda\mu f^3-\lambda'\mu'\zeta^3 f^3)\left(\frac{1}{a}+\frac{1}{\beta}\right)^2\\ +\frac{\mu\nu f}{a}\left(\frac{f-1}{a}+\frac{f}{\beta}\right)-\frac{\mu'\nu'\zeta f}{\beta}\left(\frac{-\zeta f}{a}+\frac{(\zeta f+1)}{\beta}\right)\end{array}\right)$$

ad nihilum redigendum; unde sequitur fore

$$\lambda'=\frac{\lambda\mu}{\mu'\zeta^3}+\frac{\mu\nu\cdot a\cdot\beta^2}{\mu'\zeta^2 f^2(a+\beta)^2}(\frac{f-1}{a}+\frac{f}{\beta})+\frac{\nu'\cdot a^2\beta}{\zeta^2 f^2\cdot(a+\beta)^2}(+\frac{\zeta f}{a}+\frac{\zeta f+1}{\beta})$$

valor cum debeat esse maior unitate, si forte eveniat, ut minor prodeat, tunc non λ' sed λ definiri conveniet, ubi notandum est esse $f=\frac{1+\omega}{1-\zeta}$. Hincque per formulas ante datas facile eruuntur radii facierum utriusque lentis.

Etsi formula superius data pro spatio diffusionis iam ad casum, quo distantia lentium est nulla, est adcommodata, tamen, quia hic distantiam minimam assumimus, nullus inde error est metuendus.

DEFINITIO 2

122. *Lens triplicata est, quae consistit ex tribus lentibus simplicibus sibi immediate iunctis ad communem axem.*

Hic quidem etiam crassitiem negligo, etiamsi necessario maior sit quam in lentibus duplicatis. In supplemento autem ostendetur, quomodo etiam distantiarum inter lentes ratio sit habenda.

COROLLARIUM

123. Potest ergo lens triplicata considerari quasi composita ex lente duplicata et lente simplici, hocque duplici modo, prout vel binae anteriores vel binae posteriores lentem duplicatam constituere concipiuntur.

SCHOLION

124. Lentibus triplicatis tum demum usum in praxi concedi conveniet, cum eadem commoda per lentes simplices vel duplicatas consequi non licet. Haec autem commoda in parvitate numeri λ consistunt, qui quamdiu unitate fuerit maior, semper lente simplici uti praestat, cum vero circumstantiae in expressione diffusionis minorem numerum λ requirunt, ad lentes multiplicatas erit confugiendum. Quoniam igitur eiusmodi lentes

duplicatas conficere docuimus, in quibus valor ipsius λ non solum ad nihilum usque, sed adeo ad negativa diminui queat, usus lentium triplicatarum superfluus videtur. Verum iam observavimus in praxi aegre eiusmodi duplicatas lentes parari posse, pro quibus valor ipsius λ minor sit quam 0,191827, propterea quod, si levissimus error committatur, omnis labor irritus reddatur. His igitur casibus imprimis, quando minori valore numeri λ opus est, lentes triplicatae in usum erunt vocandae; et quia respectu ad praxin habito non omnes aequo successu construere licet, errores inevitabiles hic quoque imprimis spectari oportet, ut pateat, quousque numerus λ cum successu diminui queat, ac si adhuc minore valore opus fuerit, lentes adeo quadruplicatae erunt inducendae.

PROBLEMA 5

125. *Datis binis distantiis determinatricibus omnes lentes triplicatas definire, simulque spatium diffusionis ab iis ortum.*

SOLUTIO

Sit $E\varepsilon$ (Fig.8) obiectum, cuius magnitudo $= z$, et distantia a lente $AE = a$; tum primae lentis *PP* distantiae determinatrices sint a et α, secundae lentis *QQ*, b et β, ac tertiae *RR*, c et γ. His positis, quia lentes immediate iunctae

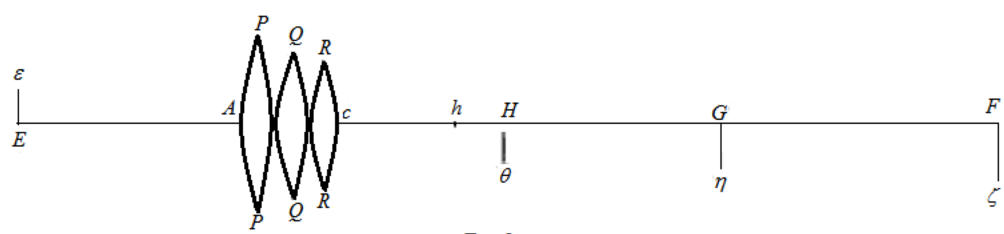

Fig. 8.

sumuntur, erit $\alpha + b = 0, \beta + c = 0$, et a et γ distantiae determinatrices lentis triplicatae, ita ut α et β arbitrio nostro relinquantur. Si igitur lens prima sola *PP* adesset, imago repraesentaretur in $F\zeta$, ut esset $AF = \alpha$ et $F\zeta = \frac{\alpha}{a}z$; si binae priores *PP* et *QQ* solae adessent, imago exhiberetur in $G\eta$, ut esset $AG = \beta$ et $G\eta = \frac{\beta}{a}z$; at per lentem triplicatam referetur in $H\theta$, ut sit $CH = \gamma$ et $H\theta = \frac{\gamma}{a}z$, pro situ inverso ob $\frac{\alpha\beta}{bc} = 1$. Hactenus scilicet res perinde se habet, ac si in *A* haberetur lens simplex ad distantias determinatrices a et γ accommodata.

At si ad spatium diffusionis *Hh* respiciamus, figuram singularum lentium in computum ducere debemus, quatenus praeter distantias determinatrices numeri arbitrarii λ, λ', λ'' involvuntur; ex quibus facies singularum lentium supra § 91 sunt definitae. Indidem autem colligitur pro apertura, cuius semidiameter $= x$, spatium diffusionis ob $\frac{\alpha}{b} = -1$ et $\frac{\beta}{c} = -1$ fore

$$Hh = \mu\gamma\gamma xx \left(\begin{array}{l} +(\frac{1}{a}+\frac{1}{\alpha})\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2 + \frac{\nu}{a\alpha}\right) + (\frac{1}{b}+\frac{1}{\beta})\left(\lambda'\left(\frac{1}{b}+\frac{1}{\beta}\right)^2 + \frac{\nu}{b\beta}\right) \\ +(\frac{1}{c}+\frac{1}{\gamma})\left(\lambda''\left(\frac{1}{c}+\frac{1}{\gamma}\right)^2 + \frac{\nu}{c\gamma}\right) \end{array} \right)$$

quae expressio ut ad formam uni lenti respondentem reducatur, statuamus:

$$\frac{1}{a}+\frac{1}{\alpha} = f(\frac{1}{a}+\frac{1}{\gamma}),\ \frac{1}{b}+\frac{1}{\beta} = g(\frac{1}{a}+\frac{1}{\gamma}),\ \frac{1}{c}+\frac{1}{\gamma} = h(\frac{1}{a}+\frac{1}{\gamma}),$$

et quia $\frac{1}{\alpha}+\frac{1}{b} = 0$ et $\frac{1}{\beta}+\frac{1}{c} = 0$, his aequationibus addendis adipiscimur: $f+g+h=1$.
Porro vero erit

$$\frac{1}{\alpha} = f(\frac{1}{a}+\frac{1}{\gamma}) - \frac{1}{a},\ \frac{1}{b} = \frac{1}{a} - f(\frac{1}{a}+\frac{1}{\gamma}),$$
$$\frac{1}{\beta} = (f+g)(\frac{1}{a}+\frac{1}{\gamma}) - \frac{1}{a},\ \frac{1}{c} = \frac{1}{a} - (f+g)(\frac{1}{a}+\frac{1}{\gamma}),$$

sive ob $1 = f+g+h$

$$\frac{1}{a} = \frac{f+g+h}{a},\quad \frac{1}{\alpha} = \frac{-g-h}{a} + \frac{f}{\gamma},$$
$$\frac{1}{b} = \frac{g+h}{a} - \frac{f}{\gamma},\quad \frac{1}{\beta} = -\frac{h}{a} + \frac{f+g}{\gamma},$$
$$\frac{1}{c} = \frac{h}{a} - \frac{f+g}{\gamma},\quad \frac{1}{\gamma} = \frac{f+g+h}{\gamma}.$$

Unde cum spatium diffusionis fiat

$$Hh = \mu\gamma\gamma xx(\frac{1}{a}+\frac{1}{\gamma})\Big((\lambda f^3 + \lambda' g^3 + \lambda'' h^3)(\frac{1}{a}+\frac{1}{\gamma})^2 + \nu(\frac{f}{a\alpha} + \frac{g}{b\beta} + \frac{h}{c\gamma})\Big)$$

reducetur id ad hanc formam

$$Hh = \mu\gamma\gamma xx(\frac{1}{a}+\frac{1}{\gamma})\Big((\lambda f^3 + \lambda' g^3 + \lambda'' h^3 - \nu(1-f)(1-g)(1-h))(\frac{1}{a}+\frac{1}{\gamma})^2 + \frac{\nu}{a\gamma}\Big).$$

Radiorum autem in h concurrentium inclinatio ad axem erit $= \frac{x}{\gamma}$.

COROLLARIUM 1

126. Haec igitur lens triplicata idem producit spatium diffusionis, quod produceret lens simplex ad easdem distantias determinatrices instructa, numero eius arbitrario (per litteram $\lambda^{(3)}$ indicato) existente

$$\lambda f^3 + \lambda' g^3 + \lambda'' h^3 - v(1-f)(1-g)(1-h),$$

ubi quidam est $f+g+h=1$.

COROLLARIUM 2

127. Quatenus ergo haec quantitas reddi potest minor non solum unitate, sed etiam fractione 0,191827, ita scilicet, ut praxis non enormi aberrationi sit exposita, eatenus lentibus triplicatis usus erit concedendus.

COROLLARIUM 3

128. Si sit vel $f=0,$ vel $g=0,$ vel $h=0$, una lentium habebit facies parallelas, et lens triplicata aequivalebit duplicatae; ac si duae litterarum *f, g, h* simul evanescant, tertia in unitatem abeunte, casus habebitur lentis simplicis.

COROLLARIUM 4

129. Si sit $f=1$ ideoque $h=-g$, valor numeri λ pro lente triplicata simplici aequivalebit, quod idem evenit, si fuerit vel $g=1$ vel $h=1$.

COROLLARIUM 5

130. Sumtis autem pro *f, g, h* numeris idoneis ob $f+g+k=1$, constructio lentis triplicatae ex formulis § 91 exhibitis est petenda sumendo:

$$\frac{1}{\alpha}=\frac{-1+f}{a}+\frac{f}{\gamma},\quad \frac{1}{b}=\frac{1-f}{a}-\frac{f}{\gamma},\quad \frac{1}{\beta}=\frac{-h}{a}+\frac{1-h}{\gamma},\quad \frac{1}{c}=\frac{h}{a}-\frac{1-h}{\gamma}.$$

SCHOLION 1

131. Quemadmodum hinc expressio inventa prodeat, notandum est fore

$$\frac{f}{a\alpha}+\frac{g}{b\beta}+\frac{h}{c\gamma}=\begin{cases}+\frac{1}{aa}(-f(1-f)-gh(1-f))\\+\frac{1}{a\gamma}(+ff+g(1-f)(1-h)+fgh+hh)\\+\frac{1}{\gamma\gamma}(-fg(1-h)-h(1-h))\end{cases}$$

Sed $-f(1-f)-gh(1-f)=-(1-f)(f+gh)=-(1-f)(1-g)(1-h)$ ob $f=1-g-h$ ideoque $f+gk=(1-g)(1-h)$. Simili modo pro $\frac{1}{\gamma\gamma}$ est $-fg(1-h)-h(1-h)=-\ (1-h)(h+fg)=-\ (1-f)(1-g)(1-k)$ ob $h=1-f-\ g$. Denique pro $\frac{1}{\alpha\gamma}$, quia est

$$ff+hh=(f+h)^2-2fh=(1-g)^2-2fh=1-2g+gg-2fh,$$

hoc valore substituto coefficiens ipsius $\frac{1}{\alpha\gamma}$ erit

$$\begin{aligned}&1-2g-2fh+gg+fgh+g(1-f)(1-h)\\&=1+(g+fh)(g-2)+g(1-f)(1-h)=1-2(1-f)(1-g)(1-h)\end{aligned}$$

ob $g+fh=(1-f)(1-h)$.

Consequenter colligitur

$$\frac{f}{a\alpha}+\frac{g}{b\beta}+\frac{h}{c\gamma}=-(1-f)(1-g)(1-h)\left(\frac{1}{a}+\frac{1}{\gamma}\right)^2+\frac{1}{\alpha\gamma}.$$

SCHOLION 2

132. Hoc problema etiam ope praecedentium facilius sequenti modo resolvi potest. Considerentur scilicet binae lentes *PP* et *QQ* iunctim sumtae tanquam lens duplicata ad distantias determinatrices a et β per numeros arbitrarios $\lambda,\ \lambda'$ et f instructa, ac posito $\lambda f^3+\lambda'(1-f)^3-\nu f(1-f)=\lambda^{(2)}$ spatium diffusionis ex ea sola ortum erit

$$\mu\beta\beta xx(\tfrac{1}{a}+\tfrac{1}{\beta})\left(\lambda^{(2)}(\tfrac{1}{a}+\tfrac{1}{\beta})^2+\tfrac{\nu}{a\beta}\right)$$

quae lens si iam in compositione cum tertia *RR* tanquam simplex tractetur, exinde elicitur spatium diffusioriis perinde atque ex coniunctione duarum simplicium

$$Hh=\mu\gamma\gamma xx\left((\tfrac{1}{a}+\tfrac{1}{\beta})\left(\lambda^{(2)}(\tfrac{1}{a}+\tfrac{1}{\beta})^2+\tfrac{\nu}{a\beta}\right)+(\tfrac{1}{c}+\tfrac{1}{\gamma})\left(\lambda''(\tfrac{1}{c}+\tfrac{1}{\gamma})^2+\tfrac{\nu}{c\gamma}\right)\right)$$

ubi notandum est esse $\beta + c = 0$, et constructio lentis duplicatae ex § 107 erit petenda, lentis vero simplicis *RR* ex distantiis determinatricibus $c = -\beta$ et γ una cum numero arbitrario λ''. Ponatur iam, $\frac{1}{\beta} = \frac{g-1}{a} + \frac{g}{\gamma}$, ut sit $\frac{1}{c} = \frac{g-1}{a} - \frac{g}{\gamma}$, eritque spatium diffusionis huius lentis triplicatae

$$Hh = \mu\gamma\gamma xx(\tfrac{1}{a} + \tfrac{1}{\beta})\Big(\big(\lambda^{(2)}g^3 + \lambda''(1-g)^3 - vg(1-g)\big)(\tfrac{1}{a} + \tfrac{1}{\gamma})^2 + \tfrac{v}{a\gamma}\Big).$$

Quare si pro lente triplicata ponatur

$$\lambda^{(2)}g^3 + \lambda''(1-g)^3 - vg(1-g) = \lambda^{(3)},$$

ita ut iam numerus *g* insuper arbitrio nostro relinquatur, habebitur spatium diffusionis more hactenus recepto expressum

$$Hh = \mu\gamma\gamma xx(\tfrac{1}{a} + \tfrac{1}{\gamma})\Big(\lambda^{(3)}(\tfrac{1}{a} + \tfrac{1}{\gamma})^2 + \tfrac{v}{a\gamma}\Big).$$

Hinc iam id intelligitur, quod ex praecedente solutione minus patet: si numerus $\lambda^{(2)}$ fuerit unitate maior, loco binarum priorum lentium *PP* et *QQ* commodius unicam simplicem adhiberi; ex quo eatenus tantum lentes triplicatae resultare censendae sunt, quatenus numerus $\lambda^{(2)}$ unitate est minor. Vidimus autem successum tuto sperari non posse, nisi $\lambda^{(2)}$ aequalis sit fractioni 0,191827 vel ea non multo maior; unde si praxi consulere velimus, ipsi $\lambda^{(2)}$ minorem valorem tribui non convenit, atque ob eandem rationem numerus *g* intra terminos 0 et 1 accipi debebit; cuius valor imprimis ad praxin erit accommodatus, si reddat numerum $\lambda^{(3)}$ minimum, quia tum leves errores negotium minime turbant. Quodsi vero pro $\lambda^{(2)}$ valorem assignatum substituamus, obtinebimus

$$\lambda^{(3)} = \lambda f^3 g^3 + \lambda' g^3(1-f)^3 + \lambda''(1-g)^3 - vg^3 f(1-f) - vg(1-g).$$

Conducet autem utramque expressionem pro numero $\lambda^{(3)}$, quo spatium diffusionis a lente triplicata ortum definitur, hic exposuisse, cum aliae conclusiones ex altera facilius deducantur. Etsi autem hinc omnes valores pro $\lambda^{(3)}$ obtineri possunt, tamen eos tantum, qui prope minimum subsistunt, ad praxin adhiberi conveniet.

PROBLEMA 6

133. *Datis distantiis determinatricibus* $AE = a$ (Fig. 8) *et* $cH = \gamma$ *, definire eam lentem triplicatam, quae minimum spatium diffusionis producat.*

SOLUTIO I

Duplici modo hoc problema solvi potest, prout spatium diffusionis vel ita exprimitur uti in solutione problematis praecedentis, vel in scholio 2. Priori modo numeros *f, g, h* ita determinari oportet, ut minima reddatur haec expressio:

$$\lambda f^2 + \lambda' g^3 + \lambda'' h^3 - v(1-f)(1-g)(1-h),$$

ubi notandum est esse $f+g+h=1$. Eius ergo differentiali nihilo aequali posito habebimus:

$$3\lambda ffdf + 3\lambda' ggdg + 3\lambda'' hhdh + vdf(1-g)(1-h) + vdg(1-f)(1-h) + vdh(1-f)(1-g) = 0.$$

Cum autem sit $dh = -df - dg$, erit

$$\left.\begin{array}{l} +\ 3\lambda ffdf - 3\lambda'' hhdf \ + vdf(1-g)(f-h) \\ + 3\lambda' ggdg - 3\lambda'' hhdg + vdg(1-f)(g-h) \end{array}\right\} = 0;$$

quia vero bina differentialia *df* et *dg* a se invicem non pendent, ambo membra huius aequationis seorsim evanescere debent, unde ob $1-g = f+h$ et $1-f = g+h$ has duas nanciscimur aequationes:

$$3\lambda ff - 3\lambda'' hh + vff - vhh = 0,\ 3\lambda' gg - 3\lambda'' hh + vgg - vhh = 0,$$

ex quibus elicimus

$$f = h\sqrt{\tfrac{3\lambda''+v}{3\lambda+v}} \ \text{ et } \ g = h\sqrt{\tfrac{3\lambda''+v}{3\lambda'+v}}.$$

Cum autem sit $f+g+h=1$ seu $\frac{1}{h} = 1 + \frac{f}{h} + \frac{g}{h}$, erit

$$\frac{1}{h}=1+\sqrt{\frac{3\lambda''+v}{3\lambda+v}}+\sqrt{\frac{3\lambda''+v}{3\lambda'+v}}$$
$$\frac{1}{g}=1+\sqrt{\frac{3\lambda'+v}{3\lambda+v}}+\sqrt{\frac{3\lambda'+v}{3\lambda''+v}}$$
$$\frac{1}{f}=1+\sqrt{\frac{3\lambda+v}{3\lambda'+v}}+\sqrt{\frac{3\lambda+v}{3\lambda''+v}}.$$

Quicunque ergo numeri $\lambda,\ \lambda',\ \lambda''$ in constructione singularum lentium fuerint usurpati, hinc numeri *f, g* et *h* determinantur, ex quibus spatium diffusionis minimum resultet. Pro lentium autem constructione hinc distantiae $\alpha,\ b,\ \beta,\ c$ ita definiuntur, ut sit

$$\frac{1}{\alpha}=\frac{-1+f}{a}+\frac{f}{\gamma},\quad \frac{1}{b}=\frac{1-f}{a}-\frac{f}{\gamma},\quad \frac{1}{\beta}=\frac{-h}{a}+\frac{1-h}{\gamma},\quad \frac{1}{c}=\frac{h}{a}-\frac{1-h}{\gamma};$$

est vero

$$f=\frac{\sqrt{(3\lambda'+v)(3\lambda''+v)}}{\sqrt{(3\lambda+v)(3\lambda'+v)}+\sqrt{(3\lambda+v)(3\lambda''+v)}+\sqrt{(3\lambda'+v)(3\lambda''+v)}}$$
$$g=\frac{\sqrt{(3\lambda+v)(3\lambda''+v)}}{\sqrt{(3\lambda+v)(3\lambda'+v)}+\sqrt{(3\lambda+v)(3\lambda''+v)}+\sqrt{(3\lambda'+v)(3\lambda''+v)}}$$
$$h=\frac{\sqrt{(3\lambda+v)(3\lambda'+v)}}{\sqrt{(3\lambda+v)(3\lambda'+v)}+\sqrt{(3\lambda+v)(3\lambda''+v)}+\sqrt{(3\lambda'+v)(3\lambda''+v)}}$$

Ex distantiis autem $a,\ \alpha,\ b,\ \beta,\ c,\ \gamma$ cum numeris $\lambda,\ \lambda',\ \lambda''$ lentes ipsae per formulas § 91 exhibitas construuntur.

COROLLARIUM 1

134. Si pro hac lente triplicata ponatur

$$\lambda f^3+\lambda' g^3+\lambda'' h^3-v\big(1-f)(1-g\big)(1-h)=\lambda^{(3)},$$

substituendis his valoribus pro *f, g* et *h* reperietur

$$\lambda^{(3)}=\frac{1}{3\left(\frac{1}{\sqrt{(3\lambda+v)}}+\frac{1}{\sqrt{(3\lambda'+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}}\right)^2}-\frac{1}{3}v,$$

unde spatium diffusionis fit

$$Hh=\mu\gamma\gamma xx(\tfrac{1}{a}+\tfrac{1}{\gamma})\Big(\lambda^{(3)}(\tfrac{1}{a}+\tfrac{1}{\gamma})^2+\tfrac{v}{a\gamma}\Big).$$

COROLLARIUM 2

135. Hoc autem spatium diffusionis omnium fiet minimum, si numeris λ, λ', λ'' minimi valores, quos accipere possunt, tribuantur. Sit ergo $\lambda = 1$, $\lambda' = 1$, $\lambda'' = 1$, eritque

$$f = \tfrac{1}{3}, \quad g = \tfrac{1}{3}, \quad h = \tfrac{1}{3}$$

et $\lambda^{(3)} = \frac{\nu+3}{27} - \frac{\nu}{3} = \frac{3-8\nu}{27} = 0{,}042165$ ob $\nu = 0{,}232692$; qui ergo valor multo est minor quam casu lentium duplicatarum.

COROLLARIUM 3

136. Hoc porro casu erit

$$\frac{1}{\alpha} = -\frac{2}{3a} + \frac{1}{3\gamma}, \quad \frac{1}{b} = -\frac{1}{3a} + \frac{2}{3\gamma} \quad \text{et} \quad \frac{1}{c} = \frac{1}{3a} - \frac{2}{3\gamma}$$

unde constructio lentium ternarum simplicium ita se habebit:

$$\text{Pro lente prima radius faciei} \begin{cases} \text{anterioris} = \dfrac{3a\gamma}{(3\rho-2\sigma)\gamma+\sigma a} \\ \text{posterioris} = \dfrac{3a\gamma}{(3\sigma-2\rho)\gamma+\rho a} \end{cases}$$

$$\text{Pro lente secunda radius faciei} \begin{cases} \text{anterioris} = \dfrac{3a\gamma}{(2\rho-\sigma)\gamma+(2\sigma-\rho)a} \\ \text{posterioris} = \dfrac{3a\gamma}{(2\sigma-\rho)\gamma+(2\rho-\sigma)a}. \end{cases}$$

$$\text{Pro lente tertia radius faciei} \begin{cases} \text{anterioris} = \dfrac{3a\gamma}{\rho\gamma+(3\sigma-2\rho)a} \\ \text{posterioris} = \dfrac{3a\gamma}{\sigma\gamma+(3\rho-2\sigma)a}. \end{cases}$$

SOLUTIO ALTERA PROBLEMATIS

137. Consideremus binas lentes priores *PP* et *QQ* ut lentem duplicatam ad distantias determinatrices a et β ita instructam, ut posito

$$\lambda f^3 + \lambda'(1-f)^3 - \nu f(1-f) = \lambda^{(2)}$$

spatium diffusionis inde oriundum sit

$$= \mu\beta\beta xx(\tfrac{1}{a} + \tfrac{1}{\beta})\left(\lambda^{(2)}(\tfrac{1}{a} + \tfrac{1}{\beta})^2 + \tfrac{\nu}{a\beta}\right).$$

Sed pro constructione binarum lentium simplicium, ex quibus haec lens est composita, recordandum est esse

$$\frac{1}{\alpha}=\frac{-1+f}{a}+\frac{f}{\beta}\ \text{ et }\ \frac{1}{b}=\frac{1-f}{a}-\frac{f}{\beta}.$$

Adiuncta iam tertia lente *RR* ad distantias determinatrices $c=-\beta$ et γ per numerum arbitrarium λ'' instructa, si ponamus $\frac{1}{\beta}=\frac{-1+g}{a}+\frac{g}{\gamma}=-\frac{1}{c}$ et

$$\lambda^{(2)}g^3+\lambda''(1-g)^3-vg(1-g)=\lambda^{(3)},$$

erit spatium diffusionis

$$Hh=\mu\gamma\gamma xx(\tfrac{1}{a}+\tfrac{1}{\gamma})\left(\lambda^{(3)}(\tfrac{1}{a}+\tfrac{1}{\gamma})^2+\tfrac{v}{a\gamma}\right)$$

quod ut fiat minimum, valor ipsius $\lambda^{(3)}$ minimus reddi debet; quaeratur ergo primo *g*, dum $\lambda^{(2)}$ ut numerus datus spectatur, et habebimus

$$3\lambda^{(2)}gg-3\lambda''(1-g)^2-v+2vg=0,$$

unde elicitur

$$g=\frac{-3\lambda''-v+\sqrt{(3\lambda^{(2)}+v)(3\lambda''+v)}}{3\lambda^{(2)}-3\lambda''}\ \text{ sive }\ g=\frac{\sqrt{(3\lambda''+v)}}{\sqrt{(3\lambda^{(2)}+v)}+\sqrt{(3\lambda''+v)}},$$

et hinc valor ipsius $\lambda^{(3)}$ erit

$$\lambda^{(3)}=\frac{1}{3(\frac{1}{\sqrt{(3\lambda^{(2)}+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}})^2}-\frac{1}{3}v,$$

vel

$$\frac{1}{3\lambda^{(3)}+v}=(\frac{1}{\sqrt{(3\lambda^{(2)}+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}})^2.$$

Sicque fit:

$$\frac{1}{\sqrt{(3\lambda^{(3)}+v)}}=\frac{1}{\sqrt{(3\lambda+v)}}+\frac{1}{\sqrt{(3\lambda'+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}},$$

Simili modo si *f* ita definiatur, ut $\lambda^{(2)}$ fiat minimum, reperietur:

$$f=\frac{\sqrt{(3\lambda'+v)}}{\sqrt{(3\lambda+v)}+\sqrt{(3\lambda'+v)}}$$

hocque valore substituto

$$\frac{1}{\sqrt{(3\lambda^{(2)}+\nu)}}=\frac{1}{\sqrt{(3\lambda+\nu)}}+\frac{1}{\sqrt{(3\lambda'+\nu)}}.$$

Quare si numeris λ, λ', λ'' arbitrio nostro relictis bini numeri f et g ita definiantur, ut $\lambda^{(3)}$ consequatur valorem minimum, erit

$$\frac{1}{\sqrt{(3\lambda^{(3)}+\nu)}}=\frac{1}{\sqrt{(3\lambda+\nu)}}+\frac{1}{\sqrt{(3\lambda'+\nu)}}+\frac{1}{\sqrt{(3\lambda''+\nu)}},$$

unde idem valor pro $\lambda^{(3)}$ reperitur, quem ante invenimus.

COROLLARIUM 1

138. Ex hac ergo solutione numeri f et g ita definiuntur, ut sit

$$\frac{1}{f\sqrt{(3\lambda+\nu)}}=\frac{1}{\sqrt{(3\lambda+\nu)}}+\frac{1}{\sqrt{(3\lambda'+\nu)}}$$

et

$$\frac{1}{g\sqrt{(3\lambda^{(2)}+\nu)}}=\frac{1}{\sqrt{(3\lambda^{(2)}+\nu)}}+\frac{1}{\sqrt{(3\lambda''+\nu)}}$$

sive hoc modo

$$\left(\frac{1}{f}-1\right)\frac{1}{\sqrt{(3\lambda+\nu)}}=\frac{1}{\sqrt{(3\lambda'+\nu)}} \quad \text{et} \quad \left(\frac{1}{g}-1\right)\left(\frac{1}{\sqrt{(3\lambda+\nu)}}+\frac{1}{\sqrt{(3\lambda'+\nu)}}\right)=\frac{1}{\sqrt{(3\lambda''+\nu)}}.$$

COROLLARIUM 2

139. Ex inventis minimis valoribus numerorum $\lambda^{(2)}$ et $\lambda^{(3)}$ numeri f et g etiam ita definiuntur, ut sit

$$\frac{1}{f\sqrt{(3\lambda+\nu)}}=\frac{1}{\sqrt{(3\lambda^{(2)}+\nu)}} \quad \text{seu} \quad f=\sqrt{\left(\frac{3\lambda^{(2)}+\nu}{3\lambda+\nu}\right)}$$

et

$$\frac{1}{g\sqrt{(3\lambda^{(2)}+\nu)}}=\frac{1}{\sqrt{(3\lambda^{(3)}+\nu)}} \quad \text{seu} \quad g=\sqrt{\left(\frac{3\lambda^{(3)}+\nu}{3\lambda^{(2)}+\nu}\right)}.$$

COROLLARIUM 3

140. Distantiae autem determinatrices singularum lentium ita per a et γ prodibunt expressae :

$$\frac{1}{\alpha}=-\frac{1}{b}=\frac{-1+fg}{a}+\frac{fg}{\gamma}, \quad \frac{1}{\beta}=-\frac{1}{c}=\frac{-1+g}{a}+\frac{g}{\gamma},$$

ubi cum sit

$g=\sqrt{(\frac{3\lambda^{(3)}+v}{3\lambda^{(2)}+v})}$, notandum est esse $fg=\sqrt{\frac{3\lambda^{(3)}+v}{3\lambda+v}}$.

COROLLARIUM 4

141. Eliminando autem numero $\lambda^{(2)}$ erit

$$\frac{1}{\alpha}=-\frac{1}{b}=-\frac{1}{a}\sqrt{\frac{3\lambda^{(2)}+v}{3\lambda'+v}}-\frac{1}{a}\sqrt{\frac{3\lambda^{(3)}+v}{3\lambda''+v}}+\frac{1}{\gamma}\sqrt{\frac{3\lambda^{(3)}+v}{3\lambda+v}}$$
$$\frac{1}{\beta}=-\frac{1}{c}=-\frac{1}{a}\sqrt{\frac{3\lambda^{(3)}+v}{3\lambda''+v}}+\frac{1}{\gamma}\sqrt{\frac{3\lambda^{(3)}+v}{3\lambda+v}}+\frac{1}{\gamma}\sqrt{\frac{3\lambda^{(3)}+v}{3\lambda'+v}}$$

ex quibus formulis invento iam valore minimo $\lambda^{(3)}$ singulae lentes commodissime determinantur.

COROLLARIUM 5

142. Si praeterea singulae lentes ita fuerint comparatae, ut per se minimam confusionem pariant, quod fit, si $\lambda=\lambda'=\lambda''=1$, erit
$\frac{1}{\sqrt{(3\lambda^{(2)}+v)}}=\frac{2}{\sqrt{(3+v)}}$ hincque $3\lambda^{(2)}+v=\frac{3+v}{4}$, unde fit $\lambda^{(2)}=\frac{1-v}{4}$.
Deinde vero habebitur:
$\frac{1}{\sqrt{(3\lambda^{(3)}+v)}}=\frac{3}{\sqrt{(3+v)}}$ hincque $3\lambda^{(3)}+v=\frac{3+v}{9}$ ac propterea $\lambda^{(3)}=\frac{3-8v}{27}$.
Sin autem tantum $\lambda=\lambda'=\lambda''$, reperietur $\lambda^{(2)}+v=\frac{3\lambda-3v}{12}$ et $\lambda^{(3)}=\frac{3\lambda-8v}{27}$.

COROLLARIUM 6

143. Eodem autem casu, quo $\lambda=\lambda'=\lambda''$, ob $\sqrt{(3\lambda^{(3)}+v)}=\frac{1}{3}\sqrt{(3+v)}$ distantiae determinatrices pro lentibus simplicibus erunt:

$$\frac{1}{\alpha}=\frac{-2}{3a}+\frac{1}{3\gamma},\quad \frac{1}{b}=\frac{2}{3a}-\frac{1}{3\gamma},\quad \frac{1}{\beta}=\frac{-1}{3a}+\frac{2}{3\gamma},\quad \frac{1}{c}=\frac{1}{3a}-\frac{2}{3\gamma},$$

unde eaedem formulae pro earum constructione nascuntur, quae supra (§ 136) sunt allatae, nisi quod iam in denominatoribus membra $\pm\tau(a+\gamma)\sqrt{(\gamma-1)}$ adiungi debeant.

SCHOLION

144. Ternarum ergo lentium simplicium idonea coniunctione effici potest, ut in expressione spatii diffusionis

$$Hh = \mu\gamma\gamma xx(\tfrac{1}{a}+\tfrac{1}{\gamma})\left(\lambda^{(3)}(\tfrac{1}{a}+\tfrac{1}{\gamma})^2+\tfrac{\nu}{a\gamma}\right)$$

numerus $\lambda^{(3)}$ fiat $=0,042165$. In lentibus autem duplicatis vidimus minimum valorem numeri $\lambda^{(2)}$ esse $=0,191827$: sicque in triplicatis hic numerus fere quinquies minor reddi potest; at is fere vicies quater est minor, quam per lentes simplices obtineri potest. Loquor hic autem de lentibus ex principio minimi petitis, quippe quae ad praxin maxime sunt accommodatae, dum constructio levibus erroribus non admodum turbatur. Quanquam enim lentes triplicatae perinde ac duplicatae parari possent, pro quibus numerus conveniens λ non solum nihilo aequalis, sed etiam negativus resultaret, tamen earum constructio tam est lubrica, ut minimus error totum laborem irritum reddat. Interim tamen periculum in triplicatis non tantum est quam duplicatis, unde sequens problema solvisse opera erit pretium.

PROBLEMA 7

145. *Pro datis distantiis determinitricibus a et γ eas definire lentes triplicatas, pro quibus valor ipsius $\lambda^{(3)}$ prorsus in nihilum abeat.*

SOLUTIO

Consideretur numerus $\lambda^{(2)}$ ex duabus prioribus lentibus natus ut datus, et cum sit secundum solutionem posteriorem praecedentis problematis

$$\lambda^{(3)} = \lambda^{(2)}g^3 + \lambda''(1-g)^3 - \nu g(1-g),$$

definiri debet g ita, ut ista quantitas evanescat. Verum cum $\lambda^{(2)}$ commode nequeat minor effici quam $\frac{1-\nu}{4}$, statuamus $\frac{1-\nu}{4} = \lambda^{(2)}$, fierique oportet:

$$0 = \tfrac{1}{4}g^3 + \lambda''(1-g)^3 - \nu g(1-g)^2;$$

sed quia λ'' unitate minor esse nequit, necesse est, ut g capiatur unitate maior; evolvantur ergo quidam casus

I. $g = \frac{5}{4}$; $0 = \frac{125}{4\cdot 64} - \frac{\lambda''}{64} - \frac{5\cdot 9}{4\cdot 64}v$ et $\lambda'' = \frac{125-45v}{4}$

II. $g = \frac{6}{4}$; $0 = \frac{216}{4\cdot 64} - \frac{8\lambda''}{64} - \frac{6\cdot 4}{4\cdot 64}v$ et $\lambda'' = \frac{126-24v}{4\cdot 8} = \frac{27-3v}{4}$

III. $g = \frac{7}{5}$; $0 = \frac{343}{4\cdot 64} - \frac{27\lambda''}{64} - \frac{7\cdot 1}{4\cdot 64}v$ et $\lambda'' = \frac{343-7v}{4\cdot 27}$

IV. $g = \frac{8}{4}$; $0 = \frac{512}{4\cdot 64} - \frac{64\lambda''}{64}$ et $\lambda'' = 2$

V. $g = \frac{9}{4}$; $0 = \frac{729}{4\cdot 64} - \frac{125\lambda''}{64} - \frac{9\cdot 1}{4\cdot 64}v$ et $\lambda'' = \frac{729-9v}{4\cdot 125}$

VI. $g = \frac{10}{4}$; $0 = \frac{1000}{4\cdot 64} - \frac{125\lambda''}{64} - \frac{10\cdot 4}{4\cdot 64}v$ et $\lambda'' = \frac{1000-40v}{4\cdot 216} = \frac{125-5v}{4\cdot 27}$.

In primo et secundo casu fit valor ipsius λ'' nimis magnus, quam ut ista lens commode in praxin recipi queat; ac si ipsi g multo maior tribuatur valor, levis error ingentem effectum producit. Ponamus enim loco g per errorem sumi $g+\omega$, et cum λ'' ex g rite fuerit definitum, fiet

$$\lambda^{(3)} = \omega(\tfrac{3}{4}gg - 3\lambda''(1-g)^2 - v(1-\tfrac{1}{2}g)(1-\tfrac{3}{2}g))$$

et pro λ'' substituto valore

$$\lambda^{(3)} = \frac{\omega}{4(1-g)}(3gg - v(4-gg)),$$

unde patet, quo minus g unitatem excedat, ac simul quo maius fuerit g, valorem ipsius $\lambda^{(3)}$ ob errorem ω eo fieri maiorem. Intelligitur autem hunc errorem fieri minimum, si capiatur $g = 1 + \sqrt{\frac{3-3v}{v+3}}$; ex hoc autem valore elicitur

$$\lambda'' = \frac{(3+v)\sqrt{3}(1-v)(3+v)+3(3+vv)}{9(1-v)}.$$

Capi ergo debebit $g = 1{,}84384$, unde colligitur:

$$\lambda'' = \frac{g^3 - vg(2-g)^2}{4(g-1)^3} = 2{,}60372,$$

et si hae mensurae exacte observentur, fiet $\lambda^{(3)} = 0$. At si in valore ipsius g particula ω aberretur, ut sit $g = 1{,}84384 + \omega$, prodibit ob hunc errorem:

$$\lambda^{(3)} = -\,2{,}981\omega\ ;$$

ita si esset error $\omega = \pm\frac{1}{10}$, loco $\lambda^{(3)} = 0$ prodiret: $\lambda^{(3)} = \mp 0{,}2981$, ideoque lens triplicata, postponenda duplicatae, longe tamen praeferenda foret simplici.

In genere igitur pro quovis valore alio ipsius $\lambda^{(2)}$ idem commodum investigemus: ac primo cum sit

$$\lambda'' = \frac{\lambda^{(2)}g^3 + vg(g-1)}{(g-1)^3},$$

si loco iusti valoris g capiatur $g+\omega$, fiet

$$\lambda^{(3)} = \omega(3\lambda^{(2)}g^2 - 3\lambda''(g-1)^2 - v + 2vg),$$

ubi si pro λ'' valor substituatur, erit

$$\lambda^{(3)} = \frac{\omega(v-(3\lambda^{(2)}+v)gg)}{g-1},$$

qui error ut minimus reddatur, capi debet

$$g = 1 + \sqrt{\frac{3\lambda^{(2)}}{3\lambda^{(2)}+v}},$$

qui est valor maxime idoneus pro g sumendus, ex quo elicitur

$$\lambda'' = \frac{1}{3\lambda^{(2)}}\left(\sqrt{(3\lambda^{(2)}+v)} + \sqrt{3\lambda^{(2)}}\right)\left(\tfrac{1}{3}(6\lambda^{(2)}+v)\sqrt{3\lambda^{(2)}} + (2\lambda^{(2)}+v)\sqrt{(3\lambda^{(2)}+v)}\right)$$

et sumto per errorem $g+\omega$ pro g erit

$$\lambda^{(3)} = -2\omega(3\lambda^{(2)} + v + \sqrt{3\lambda^{(2)}(3\lambda^{(2)}+v)})$$

seu

$$\lambda^{(3)} = -2\omega(\sqrt{3\lambda^{(2)}} + \sqrt{(3\lambda^{(2)}+v)})\sqrt{(3\lambda^{(2)}+v)}.$$

Quo minor igitur iam fuerit valor ipsius $\lambda^{(2)}$, eo minus erit error metuendus, unde solutio ante ex valore $\lambda^{(2)} = \frac{1-v}{4}$ eruta prae ceteris est commendanda. Tum autem erit $f = \frac{1}{2}$, $\lambda = 1$, $\lambda' = 1$, $\lambda'' = 2{,}60372$ et $g = 1{,}84384$, unde pro lentium constructione habemus:

$$\frac{1}{\alpha} = -\frac{1}{b} = \frac{-2+g}{2a} + \frac{g}{2\gamma}, \quad \frac{1}{\beta} = -\frac{1}{c} = \frac{-1+g}{a} + \frac{g}{\gamma},$$

unde singulae lentes per formulas § 91 construentur.

COROLLARIUM 1

146. Lens ergo prima PP construi debet ex distantiis determinatricibus a et $\frac{2a\gamma}{ga-(2-g)\gamma}$ cum numero $\lambda = 1$.

Lens vero secunda QQ ex distantiis determinatricibus $\frac{-2a\gamma}{ga-(2-g)\gamma}$ et $\frac{a\gamma}{ga-(1-g)\gamma}$ cum numero $\lambda = 1$.

At lens tertia *RR* ex distantiis determinatricibus $\frac{-a\gamma}{ga-(1-g)\gamma}$ et γ cum numero $\lambda'' = 2,60372$, existante $g = 1,84384$.

COROLLARIUM 2

147. Quoniam in formula spatium diffusionis exprimente, quae ob $\lambda^{(3)} = 0$ est

$$\mu\gamma\gamma xx(\tfrac{1}{a} + \tfrac{1}{\gamma})\tfrac{\nu}{a\gamma}$$

reiecto factore primo $\mu\gamma\gamma xx$, cuius ratio in his investigationibus non est habita, distantiae a et γ inter se permutari possunt, hinc etiam alia lens triplicata quaesito aeque satisfaciens exhiberi poterit.

COROLLARIUM 3

148. Nempe pro hac altera lente triplicata lens prima *PP* construi debet ex distantiis determinatricibus a et $\frac{+a\gamma}{(1-g)a-g\gamma}$ cum numero $\lambda = 2,60372$.

Lens secunda *QQ* ex distantiis determinatricibus $\frac{-a\gamma}{(1-g)a-g\gamma}$ et $\frac{+2a\gamma}{(2-g)a-g\gamma}$ cum numero $\lambda' = 1$.

At lens tertia ex distantiis determinatricibus $\frac{-2a\gamma}{(2-g)a-g\gamma}$ et γ cum numero $\lambda'' = 1$, existante ut ante $g = 1,84384$.

COROLLARIUM 4

149. Hinc ergo duas nacti sumus lentes triplicatas pro distantiis a et γ, quae producunt spatium diffusionis $Hh = \mu\gamma\gamma xx(\tfrac{1}{a} + \tfrac{1}{\gamma})\tfrac{\nu}{a\gamma}$. Atque hae inter infinitas alias eundem effectum praestantes hac gaudent praerogativa, ut levis error in constructione commissus scopum minime perturbet.

COROLLARIUM 5

150. Si in constructione harum lentium per errorem numerus g parumper maior accipiatur quam 1,84384, tum pro lente triplicata numerus $\lambda^{(3)}$ prodit nihilo minor seu negativus. Sin autem numerus g in praxi aliquantillum minor sumatur, numerus $\lambda^{(3)}$ fit nihilo maior, sicque lens triplicata ad naturam duplicatarum accedet.

COROLLARIUM 6

151. Si ergo opus fuerit lente, pro qua numerus λ valorem habeat negativum, huic scopo satisfieri commode poterit per lentes descriptas triplicatas, dummodo pro g numerus aliquanto maior quam 1,84384 assumatur. Scilicet si sumatur

$$g = 1{,}84384 + \omega,\ \text{fiet}\ \lambda^{(3)} = -\,2{,}981\omega$$

SUPPLEMENTUM II

DE LENTIBUS TRIPLICATIS

Si pro singulis lentibus refractio sit diversa, pro prima $n:1$, pro secunda $n':1$ pro tertia $n'':1$, radiique facierum lentium sequenti modo definiantur:

	Distantiae determinatrices	Refractio et litterae inde pendentes	
I.	a et α	$n:1,\ \mu,\ \nu,\ \rho,\ \sigma,\ \tau.$	λ
II.	b et β	$n':1,\ \mu',\ \nu',\ \rho',\ \sigma',\ \tau'.$	λ'
III.	c et γ	$n'':1,\ \mu'',\ \nu'',\ \rho'',\ \sigma'',\ \tau''.$	λ''

scilicet si pro lente prima vocetur radius faciei anterioris $= F$, posterioris $= G$, erit

$$\tfrac{1}{F} = \tfrac{\rho}{a} + \tfrac{\sigma}{\alpha} \mp (\tfrac{1}{a} + \tfrac{1}{\alpha})\tau\sqrt{\lambda - 1}\ \text{ et }\ \tfrac{1}{G} = \tfrac{\rho}{\alpha} + \tfrac{\sigma}{a} \pm \tau(\tfrac{1}{\alpha} + \tfrac{1}{a})\sqrt{\lambda - 1}\,,$$

similique modo pro reliquis lentibus, nempe pro secunda

$$\tfrac{1}{F'} = \tfrac{\rho'}{a} + \tfrac{\sigma'}{\alpha} \mp \tau'(\tfrac{1}{a} + \tfrac{1}{\alpha})\sqrt{\lambda' - 1}\ \text{ et }\ \tfrac{1}{G'} = \tfrac{\rho'}{\alpha} + \tfrac{\sigma'}{a} \pm \tau'(\tfrac{1}{\alpha} + \tfrac{1}{a})\sqrt{\lambda' - 1}$$

tum vero pro tertia

$$\tfrac{1}{F''} = \tfrac{\rho''}{a} + \tfrac{\sigma''}{\alpha} \mp (\tfrac{1}{a} + \tfrac{1}{\alpha})\tau''\sqrt{\lambda'' - 1}\ \text{ et }\ \tfrac{1}{G''} = \tfrac{\rho''}{\alpha} + \tfrac{\sigma''}{a} \pm \tau''(\tfrac{1}{\alpha} + \tfrac{1}{a})\sqrt{\lambda'' - 1}$$

Deinde quia distantiae lentium pro nihilo habentur, scilicet

$$\alpha + b = 0\ \text{ et }\ \beta + c = 0\,,$$

erit spatium diffusionis

$$=\gamma\gamma xx\left\{\begin{aligned}&+\mu(\tfrac{1}{a}+\tfrac{1}{\alpha})(\lambda(\tfrac{1}{a}+\tfrac{1}{\alpha})^2+\tfrac{\nu}{a\alpha})\\&+\mu'(\tfrac{1}{b}+\tfrac{1}{\beta})(\lambda'(\tfrac{1}{b}+\tfrac{1}{\beta})^2+\tfrac{\nu'}{b\beta})\\&+\mu''(\tfrac{1}{c}+\tfrac{1}{\gamma})(\lambda''(\tfrac{1}{c}+\tfrac{1}{\gamma})^2+\tfrac{\nu''}{c\gamma})\end{aligned}\right\}.$$

Statuatur nunc

$$\tfrac{1}{a}+\tfrac{1}{\alpha}=f(\tfrac{1}{a}+\tfrac{1}{\gamma}),\ \tfrac{1}{b}+\tfrac{1}{\beta}=g(\tfrac{1}{a}+\tfrac{1}{\gamma}),\ \tfrac{1}{c}+\tfrac{1}{\gamma}=h(\tfrac{1}{a}+\tfrac{1}{\gamma}),$$

ut fiat

$$\tfrac{1}{\alpha}=\tfrac{f-1}{a}+\tfrac{f}{\gamma}=\tfrac{-1}{b}\ \text{ et }\ \tfrac{1}{c}=\tfrac{h}{a}+\tfrac{h-1}{\gamma}=\tfrac{-1}{\beta},$$

unde fit

$$\tfrac{1}{b}+\tfrac{1}{\beta}=\tfrac{1-f-h}{a}+\tfrac{1-f-h}{\gamma}=g(\tfrac{1}{a}+\tfrac{1}{\gamma})$$

hincque $f+g+h=1$; quibus valoribus substitutis erit primo pro radiis facierum

$$\begin{aligned}\tfrac{1}{F}&=\tfrac{\rho+(f-1)\sigma}{a}+\tfrac{\sigma f}{\gamma}\mp\tau f(\tfrac{1}{a}+\tfrac{1}{\gamma})\sqrt{\lambda-1}\\ \tfrac{1}{G}&=\tfrac{\sigma+(f-1)\rho}{a}+\tfrac{\rho f}{\gamma}\pm\tau f(\tfrac{1}{a}+\tfrac{1}{\gamma})\sqrt{\lambda-1}\\ \tfrac{1}{F'}&=\tfrac{(1-f)\rho'-h\sigma'}{a}+\tfrac{(1-h)\sigma'-f\rho'}{\gamma}\mp\tau' g(\tfrac{1}{a}+\tfrac{1}{\gamma})\sqrt{\lambda'-1}\\ \tfrac{1}{G'}&=\tfrac{(1-f)\sigma'-h\rho'}{a}+\tfrac{(1-h)\rho'-f\sigma'}{\gamma}\mp\tau' g(\tfrac{1}{a}+\tfrac{1}{\gamma})\sqrt{\lambda'-1}\end{aligned}$$

et pro lente tertia

$$\begin{aligned}\tfrac{1}{F''}&=\tfrac{h\rho''}{a}+\tfrac{(h-1)\rho''+\sigma''}{\gamma}\mp(\tfrac{1}{a}+\tfrac{1}{\gamma})\tau'' h\sqrt{\lambda''-1}\\ \tfrac{1}{G''}&=\tfrac{h\sigma''}{a}+\tfrac{(h-1)\sigma''+\rho''}{\gamma}\pm\tau'' h(\tfrac{1}{a}+\tfrac{1}{\gamma})\sqrt{\lambda''-1}.\end{aligned}$$

Spatium vero diffusionis iam ita exprimetur

$$=\gamma\gamma xx(\tfrac{1}{a}+\tfrac{1}{\gamma})\left\{\begin{aligned}&(\tfrac{1}{a}+\tfrac{1}{\gamma})^2(\mu\lambda f^3+\mu'\lambda' g^3+\mu''\lambda'' h^3)+\tfrac{\nu\mu f}{a}(\tfrac{f-1}{a}+\tfrac{f}{\gamma})\\&+\nu'\mu' g(\tfrac{1-f}{a}-\tfrac{f}{\gamma})(\tfrac{1-h}{\gamma}-\tfrac{h}{a})+\tfrac{\nu''\mu'' h}{\gamma}(\tfrac{h}{a}+\tfrac{h-1}{\gamma})\end{aligned}\right\}.$$

Nunc igitur circa has lentes triplicatas sequentia sunt observanda:

1. Diversa media refringentia eum tantum in finem adhiberi solent, ut non solum spatium diffusionis hic determinatum ad nihilum redigatur, sed etiam confusio a diversa radiorum refrangibilitate oriunda tollatur, quippe quod per lentes eiusdem refractionis obtineri nequit. Infra autem videbimus ad hanc conditionem implendam requiri, ut sit

$\zeta f + \eta g + \theta h = 0$, existante $\zeta = \frac{dn}{n-1}$, $\eta = \frac{dn'}{n'-1}$, et $\theta = \frac{dn''}{n''-1}$; ex quibus formis iam perspicitur, si haec differentialia dn, dn' et dn'' essent ipsis $n-1$, $n'-1$ et $n''-1$ proportionalia, uti NEUTONUS statuerat, tum proditura esse $\zeta = \eta = \theta$ sive $f + g + h = 0$; at iam vidimus esse debere $f + g + h = 1$; quare si NEUTONI sententia esset vera, tum ne quidam diversis refractionibus adhibendis diffusioni a diversa refrangibilitate oriundae remedium adferri posset. Eatenus igitur tantum hoc incommodum vitari poterit, quatenus litterae ζ, η et θ sunt diversae, ita, ut simul esse possit et $f + g + h = 1$ et $\zeta f + \eta g + \theta h = 0$, ex quo perspicuum est quantitatum *f, g* et *h* unam vel adeo duas esse debere negativas, sicque hac conditione superaddita casus ille principalis, quo omnes tres litterae *f, g* et *h* positivae sunt assumtae, hic locum invenire nequit.

II. Ut ergo nostrum spatium diffusionis evanescat, satisfaciendum est huic aequationi

$$(\tfrac{1}{a} + \tfrac{1}{\gamma})^2(\mu\lambda f^3 + \mu'\lambda' g^3 + \mu''\lambda'' h^3) + \tfrac{\nu\mu f}{a}(\tfrac{f-1}{a} + \tfrac{f}{\gamma})$$
$$+\nu'\mu' g(\tfrac{1-f}{a} - \tfrac{f}{\gamma})(\tfrac{1-h}{\gamma} - \tfrac{h}{a}) + \tfrac{\nu''\mu'' h}{\gamma}(\tfrac{h}{a} + \tfrac{h-1}{\gamma}) = 0.$$

Unde vel λ vel λ' vel λ'' quaeri potest, dummodo caveatur, ne valor vel unitate minor vel nimis magnus prodeat, quia prius naturae repugnat, alterum autem, quia constructio lentium fieret nimis lubrica. Hoc autem praestito circa binas reliquas litteras λ nihil amplius definitur neque etiam circa litteras *f, g* et *h,* praeterquam quod supra allatis conditionibus $f + g + h = 1$ et $\zeta f + \eta g + \theta h = 0$ continetur, unde ex data una harum litterarum duae reliquae sponte definiuntur, sequenti scilicet modo:

$$g = \tfrac{(\theta-\zeta)f-\theta}{\eta-\theta} \text{ et } h = \tfrac{(\zeta-\eta)f+\eta}{\eta-\theta}.$$

III. In praxi autem huiusmodi lentes triplicatae ideo potissimum quaeruntur, ut loco lentis obiectivae in Telescopiis substitui queant, pro quibus est $a = \infty$. Statuamus ergo statim $a = \infty$ et pro radiis facierum singularum lentium habebimus

$$\tfrac{1}{F} = \tfrac{\sigma f}{\gamma} \mp \tfrac{\tau f}{\gamma}\sqrt{\lambda - 1}, \quad \tfrac{1}{G} = \tfrac{\rho f}{\gamma} \pm \tfrac{\tau f}{\gamma}\sqrt{\lambda - 1}$$

pro secunda lente

$$\tfrac{1}{F'} = \tfrac{(1-h)\sigma' - f\rho'}{\gamma} \mp \tfrac{\tau' g}{\gamma}\sqrt{\lambda' - 1}, \; \tfrac{1}{G'} = \tfrac{(1-h)\sigma' - h\rho'}{a} + \tfrac{(1-h)\rho' - f\sigma'}{\gamma} \pm \tfrac{\tau' g}{\gamma}\sqrt{\lambda' - 1}$$

pro tertia lente

$$\frac{1}{F''}=\frac{(h-1)\rho''+\sigma''}{\gamma}\mp\frac{\tau''h}{\gamma}\sqrt{\lambda''-1},\ \ \frac{1}{G''}=\frac{(h-1)\sigma''+\rho''}{\gamma}\pm\frac{\tau''h}{\gamma}\sqrt{\lambda''-1}.$$

Spatium autem diffusionis tum ita exprimetur

$$\frac{x^2}{\gamma}(\mu\lambda f^3+\mu'\lambda' g^3+\mu''\lambda'' h^3-\nu'\mu'\cdot fg(1-h)-\nu''\mu''\cdot h(1-h)),$$

ita ut satisfieri oporteat huic aequationi

$$\mu\lambda f^3+\mu'\lambda' g^3+\mu''\lambda'' h^3-\nu'\mu'\cdot fg(1-h)-\nu''\mu''\cdot h(1-h)=0.$$

Unde unus valorum $\lambda,\ \lambda',\ \lambda''$, qui ad usum commodissimus videtur, determinari debet.

IV. Si tantum lentibus vitreis uti velimus, sufficiet duas tantum vitri species adhiberi; si igitur statuamus lentem tertiam et primam ex eadem vitri specie parari, ut $\mu''=\mu,\ \nu''=\nu,\ \rho''=\rho,\ \sigma''=\sigma,\ \tau''=\tau$ et $\xi=\theta$ ob $n''=n$, pro litteris autem *f, g* et *h* hae determinationes habebuntur:

Ex aequatione

$$\zeta f+\eta g+\zeta h=0\ \text{ fit }\ f+h=\frac{-\eta}{\zeta}\cdot g\,,$$

quo valore substituto fiet

$$g\frac{(\zeta-\eta)}{\zeta}=1,\ g=\frac{\zeta}{\zeta-\eta},\ f+h=\frac{-\eta}{\zeta-\eta}\ ;$$

unde patet, prouti littera *g* fuerit vel positiva vel negativa, fore vicissim summam $f+h$ vel negativam vel positivam, et aequatio resolvenda iam erit

$$\mu(\lambda f^3+\lambda'' h^3)+\mu'\lambda' g^3-\frac{\mu'\nu'\cdot\zeta}{\zeta-\eta}\cdot f(1-h)-\mu\nu\cdot h(1-h)=0.$$

Hinc igitur elicitur

$$\mu'\lambda' g^3=-\mu(\lambda f^3+\lambda'' h^3)+\frac{\mu'\nu'\cdot\zeta}{\zeta-\eta}\cdot f(1-h)+\mu\nu\cdot h(1-h),$$

cui facile erit pro casu quovis proposito satisfacere.

V. Dum lens prima et tertia ex eadem vitri specie parantur, media constet ex aqua vel alla materia fluida, ut lens triplicata intra duas lentes vitreas contineat fluidum, et quia fluidum plerumque minorem refractionem patitur quam vitrum, erit $n'<n$ indeque porro

$\eta < \zeta$. Quia igitur pro hoc casu fit *g* positivum seu lens aquea convexa, lentes vitreae vel ambae vel una saltem debent esse concavae.

Inprimis autem praeter determinationes iam inventas necesse est, ut radius faciei anterioris pro lente media aequalis et contrarius sit radio faciei posterioris lentis primae, eodemque modo radius faciei posterioris aequalis et contrarius radio faciei anterioris lentis tertiae, unde hae aequalitates nascentur

$$\frac{1}{F'} = \frac{-1}{G} \text{ seu } F' + G = 0 \text{ et } \frac{1}{G'} = \frac{-1}{F''} \text{ seu } F'' + G' = 0..$$

ldeoque satisfieri oportet istis aequationibus:

$$(1-h)\sigma' - f\rho' \mp \tau' g \cdot \sqrt{\lambda' - 1} = -f\rho \mp \tau f \sqrt{\lambda - 1}$$

et

$$(1-h)\rho' - f\sigma' \pm \tau' g \sqrt{\lambda' - 1} = (1-h)\rho - \sigma \pm \tau h \sqrt{\lambda'' - 1}.$$

En igitur duas conditiones, quibus satisfieri oportet, unde vel numeri λ et λ'' vel alter eorum cum alterutra litterarum *f* et *h* definiri debent; quem in finem probe observandum est formulas $\sqrt{\lambda - 1}$ et $\sqrt{\lambda'' - 1}$ pro lubitu sive positivas sive negativas assumi posse neque a se invicem pendere. Pro formula autem $\sqrt{\lambda' - 1}$ notandum est, si ea in priore aequatione positive ponatur, in posteriore necessario negative sumi debere et vicissim.

PROBLEMA 8

152. *Determinare eas lentes quadruplicatas* (Fig. 9) *ad datas distantias determinatrices* $AE = a$ *et* $dI = \delta$ *accommodatas, quae minimum spatium diffusionis Ii producant.*

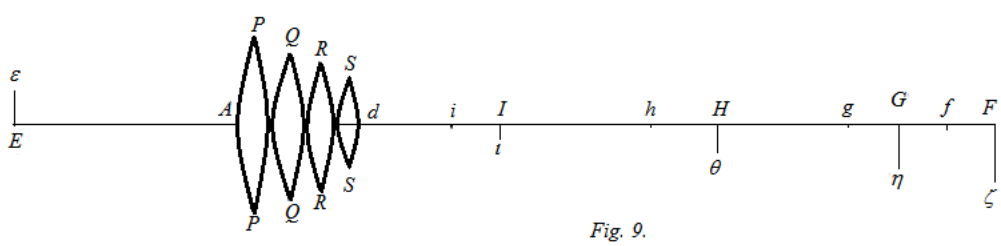

Fig. 9.

SOLUTIO

Prima lens *PP* ad distantias determinatrices $AE = a$ et $AF = \alpha$ cum numero λ construatur, secunda *QQ* ad distantias $b = -\alpha$ et $AG = \beta$ cum numero λ', tertia *RR* ad distantias $c = -\beta$ et $AH = \gamma$ cum numero λ'', et quarta *SS* ad distantias $d = -\gamma$ et $dI = \delta$ cum numero λ'' construatur: ubi scilicet crassitiem lentium ut evanescentem spectamus. Posito iam semidiametro aperturae lentis *x,* sola prima lens *PP* produceret spatium diffusionis

$$Ff = \mu\alpha\alpha xx(\tfrac{1}{a}+\tfrac{1}{\alpha})\left(\lambda\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)^2+\tfrac{\nu}{a\alpha}\right).$$

Adiuncta autem secunda lente *QQ* positoque

$$\tfrac{1}{\alpha}=\tfrac{-1+f}{a}+\tfrac{f}{\beta} \text{ et } \tfrac{1}{b}=\tfrac{1-f}{a}-\tfrac{f}{\beta},$$

si brevitatis gratia statuamus

$$\lambda^{(2)}=\lambda f^3+\lambda'(1-f)^3-\nu f(1-f),$$

vidimus fore spatium diffusionis:

$$Gg = \mu\beta\beta xx(\tfrac{1}{a}+\tfrac{1}{\beta})\left(\lambda^{(2)}\left(\tfrac{1}{a}+\tfrac{1}{\beta}\right)^2+\tfrac{\nu}{a\beta}\right).$$

Adiungatur insuper tertia lens *RR,* numerusque *g* ita sumatur, ut sit

$$\tfrac{1}{\beta}=\tfrac{-1+g}{a}+\tfrac{g}{\gamma} \text{ et } \tfrac{1}{c}=\tfrac{1-g}{a}-\tfrac{f}{\gamma},$$

ac si brevitatis ergo ponamus

$$\lambda^{(3)}=\lambda^{(2)}g^3+\lambda''(1-g)^3-\nu g(1-g),$$

erit spatium diffusionis:

$$Hh = \mu\gamma\gamma xx(\tfrac{1}{a}+\tfrac{1}{\gamma})\left(\lambda^{(3)}\left(\tfrac{1}{a}+\tfrac{1}{\gamma}\right)^2+\tfrac{\nu}{a\gamma}\right).$$

Nunc denique adiungatur lens quarta *SS,* et numero *h* ita in calculum introducto, ut sit

$$\tfrac{1}{\gamma}=\tfrac{-1+h}{a}+\tfrac{h}{\gamma} \text{ et } \tfrac{1}{d}=\tfrac{1-h}{a}-\tfrac{h}{\delta},$$

si simili modo ponamus

$$\lambda^{(4)}=\lambda^{(3)}h^3+\lambda'''(1-h)^3-\nu h(1-h),$$

erit spatium diffusionis a lente quadruplicata productum:

$$Ii=\mu\delta\delta xx(\tfrac{1}{a}+\tfrac{1}{\delta})\left(\lambda^{(4)}\left(\tfrac{1}{a}+\tfrac{1}{\delta}\right)^2+\tfrac{v}{a\delta}\right),$$

quod igitur minimum reddi debet. Hunc in finem considerentur numeri λ, λ', λ'' et λ''' ut dati, et quaerantur idonei valores pro numeris *f, g* et *h* ; atque ut valor $\lambda^{(4)}$ minimus evadat, necesse est quoque valores $\lambda^{(3)}$ et $\lambda^{(2)}$ minimos fieri. Incipiamus ergo a valore $\lambda^{(2)}$, qui minimus redditur sumendo

$$f=\frac{\sqrt{(3\lambda'+v)}}{\sqrt{(3\lambda+v)}+\sqrt{(3\lambda'+v)}} \text{ seu } \frac{1}{f}=1+\sqrt{\frac{3\lambda+v}{3\lambda'+v}},$$

unde fit

$$\frac{1}{\sqrt{(3\lambda^{(2)}+v)}}=\frac{1}{\sqrt{(3\lambda+v)}}+\frac{1}{\sqrt{(3\lambda'+v)}}.$$

Deinde numerus $\lambda^{(3)}$ minimum induet valorem capiendo

$$g=\frac{\sqrt{(3\lambda''+v)}}{\sqrt{(3\lambda^{(2)}+v)}+\sqrt{(3\lambda''+v)}} \text{ seu } \frac{1}{g}=1+\sqrt{\frac{3\lambda^{(2)}+v}{3\lambda''+v}},$$

hincque colligitur

$$\frac{1}{\sqrt{(3\lambda^{(3)}+v)}}=\frac{1}{\sqrt{(3\lambda^{(2)}+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}}.$$

Denique numerus $\lambda^{(4)}$ ideoque et spatium diffusionis *Ii* minimum efficietur sumendo

$$h=\frac{\sqrt{(3\lambda'''+v)}}{\sqrt{(3\lambda^{(3)}+v)}+\sqrt{(3\lambda'''+v)}} \text{ seu } \frac{1}{h}=1+\sqrt{\frac{3\lambda^{(3)}+v}{3\lambda'''+v}},$$

unde obtinetur

$$\frac{1}{\sqrt{(3\lambda^{(4)}+v)}}=\frac{1}{\sqrt{(3\lambda^{(3)}+v)}}+\frac{1}{\sqrt{(3\lambda'''+v)}}.$$

Quod si hic valores ante inventos substituamus, nanciscemur

$$\frac{1}{\sqrt{(3\lambda^{(4)}+v)}}=\frac{1}{\sqrt{(3\lambda+v)}}+\frac{1}{\sqrt{(3\lambda'+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}}+\frac{1}{\sqrt{(3\lambda'''+v)}}.$$

Pro praecedentibus vero erit

$$\frac{1}{\sqrt{(3\lambda^{(3)}+v)}}=\frac{1}{\sqrt{(3\lambda+v)}}+\frac{1}{\sqrt{(3\lambda'+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}}$$

et

$$\frac{1}{\sqrt{(3\lambda^{(2)}+v)}}=\frac{1}{\sqrt{(3\lambda+v)}}+\frac{1}{\sqrt{(3\lambda'+v)}}.$$

Tum vero ex his porro consequimur:

$$f=\sqrt{\frac{3\lambda^{(2)}+v}{3\lambda+v}},\ g=\sqrt{\frac{3\lambda^{(3)}+v}{3\lambda^{(2)}+v}},\ h=\sqrt{\frac{3\lambda^{(4)}+v}{3\lambda^{(3)}+v}}.$$

Superest ergo, ut constructionem singularum lentium luculentius exponamus, et earum distantias determinatrices per solas propositas a et δ exprimamus; erit igitur

$$\frac{1}{\gamma}=\frac{-1+h}{a}+\frac{h}{\delta},\ \frac{1}{\beta}=\frac{-1+gh}{a}+\frac{gh}{\delta},\ \frac{1}{\alpha}=\frac{-1+fgh}{a}+\frac{fgh}{\delta},$$
$$\frac{1}{d}=\frac{1-h}{a}-\frac{h}{\delta},\quad \frac{1}{c}=\frac{1-gh}{a}-\frac{gh}{\delta},\quad \frac{1}{b}=\frac{1-fgh}{a}-\frac{fgh}{\delta}.$$

Ex superioribus vero formulis colligitur:

$$h=\sqrt{\frac{3\lambda^{(4)}+v}{3\lambda^{(3)}+v}},\quad g=\sqrt{\frac{3\lambda^{(3)}+v}{3\lambda^{(2)}+v}},\quad f=\sqrt{\frac{3\lambda^{(2)}+v}{3\lambda+v}}$$

unde fit

$$gh=\sqrt{\frac{3\lambda^{(4)}+v}{3\lambda^{(2)}+v}}\ \text{ et }\ fgh=\sqrt{\frac{3\lambda^{(4)}+v}{3\lambda+v}};$$

et superiores valores ita exprimi poterunt

$$\frac{1}{\alpha}=-\frac{1}{b}=-\frac{\sqrt{(3\lambda^{(4)}+v)}}{a}\left(\frac{1}{\sqrt{(3\lambda'+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}}+\frac{1}{\sqrt{(3\lambda'''+v)}}\right)+\frac{\sqrt{(3\lambda^{(4)}+v)}}{\delta}\cdot\frac{1}{\sqrt{(3\lambda+v)}}$$
$$\frac{1}{\beta}=-\frac{1}{c}=-\frac{\sqrt{(3\lambda^{(4)}+v)}}{a}\left(\frac{1}{\sqrt{(3\lambda''+v)}}+\frac{1}{\sqrt{(3\lambda'''+v)}}\right)+\frac{\sqrt{(3\lambda^{(4)}+v)}}{\delta}\left(\frac{1}{\sqrt{(3\lambda+v)}}+\frac{1}{\sqrt{(3\lambda'+v)}}\right)$$
$$\frac{1}{\gamma}=-\frac{1}{d}=-\frac{\sqrt{(3\lambda^{(4)}+v)}}{a}\frac{1}{\sqrt{(3\lambda'''+v)}}+\frac{\sqrt{(3\lambda^{(4)}+v)}}{\delta}\left(\frac{1}{\sqrt{(3\lambda+v)}}+\frac{1}{\sqrt{(3\lambda'+v)}}+\frac{1}{\sqrt{(3\lambda''+v)}}\right).$$

COROLLARIUM 1

153. Si pro lentibus simplicibus numeri $\lambda,\ \lambda',\ \lambda''$ et λ''' sumantur inter se aequales, fiet pro minimo spatio diffusionis:

$$\sqrt{(3\lambda^{(2)}+v)}=\tfrac{1}{2}\sqrt{(3\lambda+v)},\quad \lambda^{(2)}=\tfrac{3\lambda-1\cdot 3v}{3\cdot 4}\ ;$$
$$\sqrt{(3\lambda^{(3)}+v)}=\tfrac{1}{3}\sqrt{(3\lambda+v)},\quad \lambda^{(3)}=\tfrac{3\lambda-2\cdot 4v}{3\cdot 9}\ ;$$
$$\sqrt{(3\lambda^{(4)}+v)}=\tfrac{1}{4}\sqrt{(3\lambda+v)},\quad \lambda^{(4)}=\tfrac{3\lambda-3\cdot 5v}{3\cdot 16}\ ;$$

hinc $f=\frac{1}{2}$, $g=\frac{2}{3}$, $h=\frac{3}{4}$ et pro constructione lentium simplicium:

$$\frac{1}{\alpha}=-\frac{3}{4a}+\frac{1}{4\delta},\quad \frac{1}{\beta}=-\frac{2}{4a}+\frac{2}{4\delta},\quad \frac{1}{\gamma}=-\frac{1}{4a}+\frac{3}{4\delta},$$
$$\frac{1}{b}=+\frac{3}{4a}-\frac{1}{4\delta},\quad \frac{1}{c}=+\frac{2}{4a}-\frac{2}{4\delta},\quad \frac{1}{d}=+\frac{1}{4a}-\frac{3}{4\delta}.$$

COROLLARIUM 2

154. Hinc ex § 91 sequens quatuor lentium simplicium constructio obtinetur:

Radius of face

first lens PP
$$\begin{cases}\text{anterior}=\dfrac{4a\delta}{(4\rho-3\sigma)\delta+\sigma a\pm\tau(a+\delta)\sqrt{(\lambda-1)}}\\ \text{posterioris}=\dfrac{4a\delta}{(4\sigma-3\rho)\delta+\rho a\mp\tau(a+\delta)\sqrt{(\lambda-1)}}\end{cases}$$

second lens QQ
$$\begin{cases}\text{anterior}=\dfrac{4a\delta}{(3\rho-2\sigma)\delta+(2\sigma-\rho)a\pm\tau(a+\delta)\sqrt{(\lambda-1)}}\\ \text{posterior}=\dfrac{4a\delta}{(3\sigma-2\rho)\delta+(2\rho-\sigma)a\mp\tau(a+\delta)\sqrt{(\lambda-1)}}\end{cases}$$

third lens RR
$$\begin{cases}\text{anterior}=\dfrac{4a\delta}{(2\rho-\sigma)\delta+(3\sigma-2\rho)a\pm\tau(a+\delta)\sqrt{(\lambda-1)}}\\ \text{posterior}=\dfrac{4a\delta}{(2\sigma-\rho)\delta+(3\rho-2\sigma)a\mp\tau(a+\delta)\sqrt{(\lambda-1)}}\end{cases}$$

fourth lens SS
$$\begin{cases}\text{anterior}=\dfrac{4a\delta}{\rho\delta+(4\sigma-3\rho)a\pm\tau(a+\delta)\sqrt{(\lambda-1)}}\\ \text{posterior}=\dfrac{4a\delta}{\sigma\delta+(4\rho-3\sigma)a\mp\tau(a+\delta)\sqrt{(\lambda-1)}}.\end{cases}$$

COROLLARIUM 3

155. Si praeterea numeri λ, λ', λ'' et λ''' unitati aequales statuantur, qui est valor minimus, quem recipere . possunt, erit ob $v=0,232692$

$$\lambda^{(2)}=\tfrac{3-3v}{3\cdot 4}=0{,}191827,\ \lambda^{(3)}=\tfrac{3-8v}{3\cdot 9}=0{,}042165,\ \lambda^{(4)}=\tfrac{3-15v}{3\cdot 16}=-0{,}010216;$$

sicque pro lente quadruplicata valor numeri $\lambda^{(4)}$ adeo infra, nihilum deprimitur.

COROLLARIUM 4

156. Maiorem ergo valorem numeris λ, λ', λ'' et λ''' tribuendo effici poterit, ut valor ipsius $\lambda^{(4)}$ praecise nihilo aequalis prodeat; quippe hoc fiet sumendo $\lambda = 5v = 1{,}163460$. Hinc habebitur $\lambda - 1 = 0{,}163460$ et $\tau\sqrt{(\lambda-1)} = 0{,}365947$, unde singulae lentes simplices duplici modo per formulas exhibitas construi poterunt.

SCHOLION 1

157. Si inter se comparemus hos duos casus, quibus est vel $\lambda^{(4)} = -0{,}010216$, vel $\lambda^{(4)} = 0$ videmus, etiamsi discrimen vix partem centesimam unitatis superet, in constructione tamen lentium simplicium satis magnum discrimen deprehendi, cum denominatores formularum (§ 154) sive augeri sive diminui debeant quantitate $0{,}365947(a+\delta)$; quae differentia maior est quam errores, qui forte ab artifice non nimis rudi committi queant. Ex quo vicissim colligimus, etiamsi in constructione harum lentium quadruplicatarum ab artifice leves errores committantur, inde vix perceptibilem effectum in spatio diffusionis vel valore ipsius $\lambda^{(4)}$ esse metuendum, quam ob causam hae lentes imprimis ad praxin accommodatae videntur. Si scilicet opus fuerit lente, pro qua valor ipsius λ in nihilum abeat, multo magis his lentibus quadruplicatis corollario 4 descriptis erit utendum quam triplicatis, quas supra definivi. Quin etiam eas lentes quadruplicatas adhibere licebit, in quibus est $\lambda^{(4)} = -0{,}010216$, quia hic numerus vix nihilo est minor; ac si in constructione a praescriptis mensuris aberretur, ille valor adhuc propius ad nihilum perducatur, ita ut hoc casu adeo errores commissi scopo magis attingendo inserviant. Infra autem videbimus plurimos dari casus, quibus eiusmodi lentibus uti conveniat, pro quibus valor ipsius λ non solum sit nihilo aequalis, sed etiam tantillum infra nihilum deprimatur; tunc optimo cum successu huiusmodi lentes quadruplicatas in usum vocabimus. Sin autem eiusmodi lentes sufficiant, pro quibus valor ipsius λ sit 0,042165 vel aliquantillum excedat, lentes triplicatae erunt commendandae, quarum constructionem supra (§ 136) dedimus; quemadmodum duplicatae negotium conficient, si non opus fuerit minore valore ipsius λ quam 0,191827. Tota autem instrumentorum dioptricorum perfectio in hoc maxime consistit, ut lentes habeantur, pro quibus valor ipsius λ sit quam minimus, cum eae sint aptissimae ad confusionem penitus tollendam; ex quo lentium quadruplicatarum hic descriptarum usus erit amplissimus.

SCHOLION 2

158. Si haec, quae de lentibus quadruplicatis hic tradidimus, attente considerentur, facile patebit, quomodo lentes quintuplicatae magisque multiplicatae ad usum sint accommodandae. Eiusmodi scilicet semper constructione erit opus, quae a natura minimi

parumper recedat, quoniam hoc modo errores in praxi commissi effectum propositum minime perturbant. Cum autem vix unquam usu veniat, ut lentibus opus sit, pro quibus valor numeri λ magis ultra nihilum imminuatur, superfluum foret constructionem lentium quintuplicatarum magisve multiplicatarum ulterius prosequi. Interim tamen iuvabit, si pro huiusmodi lentibus numeri λ. lentium numero convenienter signis $\lambda^{(5)}$, $\lambda^{(6)}$ etc. indicentur, eorum valores, quos ex natura minimi recipiunt, exposuisse. Sumamus omnes numeros λ, λ', λ'', λ''', λ'''', λ''''' etc. lentibus simplicibus respondentes unitati aequales, et eorum, qui lentibus multiplicatis conveniunt ex natura minimi, ita se habebunt:

Pro lente	
solitaria	$\lambda^{(1)} = \frac{3-0v}{3\cdot 1} = 1,000000$
duplicata	$\lambda^{(2)} = \frac{3-3v}{3\cdot 4} = 0,191827$
triplicata	$\lambda^{(3)} = \frac{3-8v}{3\cdot 9} = 0,421653$
quadruplicata	$\lambda^{(4)} = \frac{3-15v}{3\cdot 16} = -0,010216$
quintuplicata	$\lambda^{(5)} = \frac{3-24v}{3\cdot 25} = -0,034461$
sextuplicata	$\lambda^{(6)} = \frac{3-35v}{3\cdot 36} = -0,047632$
septuplicata	$\lambda^{(7)} = \frac{3-48v}{3\cdot 49} = -0,055573$
octuplicata	$\lambda^{(8)} = \frac{3-63v}{3\cdot 64} = -0,060727$
noncuplicata	$\lambda^{(9)} = \frac{3-80v}{3\cdot 81} = -0,064261$
decuplicata	$\lambda^{(10)} = \frac{3-99v}{3\cdot 100} = -0,066788$

ac si huiusmodi lentes in infinitum multiplicentur, valor numeri respondentis $\lambda^{(\infty)}$ erit $-\frac{v}{3} = -0,077564$, ita ut nunquam infra hunc numerum deprimi possit: ex quo patet vix unquam casum existere posse, quo lente saltem quintuplicata opus esset. Ceterum etiam in genere, quicunque alii valores praeter unitatem numeris λ, λ', λ'', λ''' etc. tribuantur, ex formulis superioribus numeri $\lambda^{(5)}$, $\lambda^{(6)}$ etc. facile derivabuntur; quin etiam distantiae determinatrices singularum lentium simplicium indidem sine difficultate definientur, cum lex progressionis satis sit manifesta. Verum per totum hoc caput probe tenendum est ubique crassitiem lentium tanquam evanescentem esse consideratam, in sequente autem capite iterum lentes in genere non neglecta crassitie sumus contemplaturi.