CHAPTER II

THE DIFFUSION OF THE IMAGE FORMED BY SEVERAL LENSES

PROBLEM 1.

68. *If in place of the object a diffuse image may be present through the interval Ee* (Fig. 4) *and thence may be transmitted by the lens PP of indefinite aperture, to determine the diffusion distance Ff produced by this lens.*

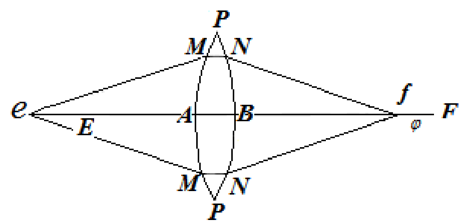

Fig. 4

SOLUTION

In place of a true object here we must consider the image now represented by another lens, which shall be diffuse through the distance *Ee,* thus so that at *E* the principle image shall be formed by the rays close to the axis, but at *e* by an extreme image, evidently formed by the rays transmitted by the preceding lens through the border of the aperture ; which rays make an angle $=\Phi$ with the axis. Whereby only the rays from the point *E* close to the axis shall be emitted into the lens *PP* , but also as far as from the point *e* only the rays of this kind, which shall be inclined to the axis *eA* within the angle $MeA=\Phi$. Now the distance may be put $EA=a$, compared wit which the diffusion length *Ee* may be considered as very small, but the lens *PP* shall be of this kind, so that the principal image of the object present at *E* may refer to the distance $BF=\alpha$, with the thickness of the lens being $AB=d$. On this account, if the radius of the anterior face *AM* may be put $=f$, and of the posterior $BN=g$, with the lens such that it may be considered as convex on both sides, there will be required to be :

$$f=\frac{(n-1)a(k+d)}{k+d+2na} \text{ and } g=\frac{(n-1)\alpha(k-d)}{k-d-2n\alpha},$$

where $n=\frac{31}{20}$ and *k* is an arbitrary quantity. Hence moreover, as we have shown, for the radius of the aperture $=x$, the diffusion length is

$$\frac{n\alpha\alpha xx}{2(n-1)^2}\left\{\left(\frac{k+d}{k-d}\right)^2\left(\frac{n}{a}+\frac{2}{k+d}\right)\left(\frac{1}{a}+\frac{2}{k+d}\right)^2+\left(\frac{k-d}{k+d}\right)^2\left(\frac{n}{\alpha}-\frac{2}{k-d}\right)\left(\frac{1}{\alpha}-\frac{2}{k-d}\right)^2\right\},$$

for which we may write for brevity $P\alpha\alpha xx$. Now of the point E, because thence only if the rays are sent out close to the axis, then the image will be shown at F, so that the distance shall be $BF=\alpha$: therefore we may see, in what direction the image of the point e must fall. And if the rays sent from the point e may be near to the axis, since that is further from the lens than E, its image from the lens will fall closer, consider at φ, so that there shall be, just as we have defined in § 60 ,

$$F\varphi=\frac{\alpha\alpha}{aa}\left(\frac{k+d}{k-d}\right)^2\cdot Ee\ ;$$

evidently the principal image will form at φ, if the object were at e. But because from e only rays eM inclined to the axis at the angle $AeM=\Phi$ are emitted, these occur at the points M of the lens distant from A by the interval $AM=eA\cdot\Phi=a\Phi$, whenever we disregard the interval Ee compared with the distance $AE=a$; and therefore it is likewise, if the aperture of the lens may be considered, of which the radius shall be $=a\Phi$, and the extreme image of the object present at e must be defined, which falls at f, thus so that φf shall be the diffusion length from the object present at e and agreeing with the aperture of the lens, of which the radius $=a\Phi$. Hence therefore the interval will be $\varphi f=Paa\alpha\alpha\Phi\Phi$; and because the image of the point E is shown at F, moreover the image of the point e is shown at f, the diffusion length produced by the lens PP will be :

$$Ff=\frac{\alpha\alpha}{aa}\left(\frac{k+d}{k-d}\right)^2\cdot Ee+Paa\alpha\alpha\Phi\Phi.$$

But in the image the extremity f of the radius Nf thus concurs with the axis, so that there shall be

$$\text{the angle } BfN=\frac{k-d}{k+d}\cdot\frac{a\Phi}{\alpha}.$$

COROLLARY 1.

69. Therefore if the image now diffused through the distance Ee may contain the place of the object with respect to the lens PP, a new Ff is produced by this diffusion space, thus so that the principal image falls at F, truly the extreme at f; and it can happen, that this new diffusion space Ff shall be greater or less than the proposed Ee.

COROLLARY 2

70. But evidently it is sufficient to assume the aperture of the lens PP to be indefinite, provided its radius shall not be less than $a\Phi$. Indeed if it were smaller, the rays sent from

the point *e* plainly will not be found to pass through the lens, nor an image to be expressed of the point *e* , but the extreme point *Ff* of the image will correspond to some image of the intermediate distance *Ee.*

COROLLARY 3

71. If the diameter of the object or rather of the image placed at *E* shall be $=s$, then the diameter of the image formed by the lens *PP* at *F* will be $=\frac{\alpha(k+d)}{a(k-d)}$; which expression, if it shall be positive, likewise declares the position of the image at *F* to be the inverse of that, which is at *E.*

SCHOLION

72. Since in this chapter I am going to consider several lenses, for any of which the determination are seen to be required initially of two determined distances, the one of the object or of the image, from which the lens accepts rays before the lens, the other thence of the image produced by the lens after the lens : which distances will be estimated from principles of imagining. Then the thickness of each lens is introduced into the calculation. In the third place since from these the lens may not yet be determined at once, in addition for each lens a certain arbitrary distance may be added indicated at this point by the letter *k*. Truly in the fourth place an account must be had of the diffusion space, which may be agreed for the given aperture of each lens. Moreover just as from the two determinable distances, with the thickness of the lens and with the thickness of the lens and with that arbitrary quantity *k* with both the two faces of the lens as well as also the diffusion space for the aperture may be defined, the radius of which is put $=x$, has been established in the preceding chapter. Therefore here, since several lenses shall be considered, I will indicate these elements for the individual lenses by the following letters:

For lens	Determinable distance object	imagine	Lens thickness	Arbitrary quantity	Diffusion space for aperture with radius x
first	a	α	v	k	$P\alpha\alpha xx$
second	b	β	v'	k'	$Q\beta\beta xx$
third	c	γ	v''	k''	$R\gamma\gamma xx$
fourth	d	δ	v'''	k'''	$S\delta\delta xx$
fifth	e	ε	v''''	k''''	$T\varepsilon\varepsilon xx$

But if the refractive index may differ in the individual lenses, for the first lens we may put that $=n$, for the second $=n'$, for the third $=n''$ etc.

But for me here the first lens will be always that, which is nearest to the object, and thence by receding the remaining lenses by an ordered number: from which likewise the distances of the lenses will be had, clearly the second from the first $=\alpha+b$, the third from the second $=\beta+c$, the fourth from the third $=\gamma+d$,the fifth from the fourth

$= \delta + e$ etc., which distances must be positive by their nature, even if the individual determinate distances besides the first *a,* evidently which by necessity for the object itself is required to be put in place before the first lens, shall be sometimes negative. The thickness of this lens I designate by the letter v , because the letter *d* is found between the determinate distances, if the number of lenses may rise above three. But for the thickness and for an arbitrary quantity I make use of the same letters, inscribed with distinguishing commas on account of the lack of different letters. It remains to be observed by me to assume all the lenses as above to be set out on a common axis.

PROBLEM 2

73. *If the rays sent from the object Es* (Fig. 5) *may be transmitted through the two lenses PP and QQ, to define the spatial diffusion arising from the given aperture Gg of the first lens, and so that the magnitude of the principle image may be shown at G.*

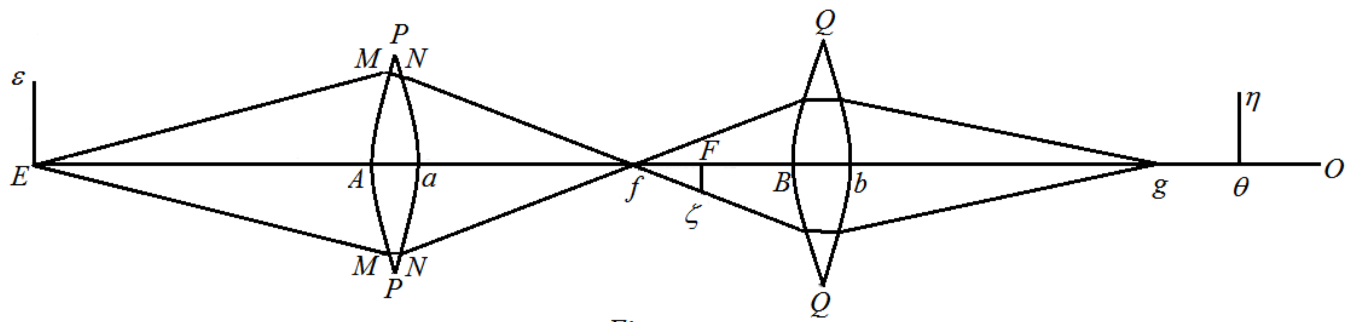

Fig. 5

SOLUTION

The magnitude of the object shall be $Ee = z$, and its principal image may be projected through the first lens *PP* at $F\zeta$, moreover by both at $G\eta$, will be the distances sought to be determined for the first lens *PP* of the object $EA = a$, and with the image $aF = \alpha$, for the second lens *QQ* of the object $FB = b$,with the image $bG = \beta$. Then there shall be

for the lens *PP* with thickness $Aa = v$, the arbitrary quantity $= k$,
for the lens *QQ* with thickness $Bb = v'$, the arbitrary quantity $= k'$.

Finally for the aperture on the anterior face of each lens, the radius of which shall be $= x$,

the diffusion interval of the first lens $PP = P\alpha\alpha xx$
the diffusion interval of the second lens $QQ = Q\beta\beta xx$.

With these in place, $F\zeta$ will be the image produced by the first lens, and its magnitude will be $F\zeta = \frac{\alpha(k+v)}{a(k-v)}z$ by being an inverse image, if this expression were positive. Then if the radius of the aperture of first lens *PP* at the anterior face may be put $= x$, the diffusion length will be $Ff = P\alpha\alpha xx$ and the inclination of the rays at *f* to the axis $= \frac{k-v}{k+v} \cdot \frac{x}{\alpha}$; then truly on the posterior face of the lens *PP* the radius of the aperture cannot be less than $\frac{k-v}{k+v}x$. Now the this whole image diffused through the interval *Ff* must be considered as the object with respect to the other lens *QQ*, of which the representation *Gg* hence will be determined by the preceding proposition.
But here there will be $\Phi = \frac{k-v}{k+v} \cdot \frac{x}{\alpha}$, and there the expressed distance $Ee = P\alpha\alpha xx$; then truly for *a, α, k, d* and *P* here it will be required to write *b, β, k', v'* and *Q,* from which the diffusion space sought :

$$Gg = \frac{\beta\beta}{bb}\left(\frac{k'+v'}{k'-v'}\right)^2 P\alpha\alpha xx + \frac{bb\beta\beta}{\alpha\alpha}\left(\frac{k-v}{k+v}\right)^2 Qxx$$

or

$$Gg = \frac{\beta\beta}{bb}\left(\frac{k'+v'}{k'-v'}\right)^2 P\alpha\alpha xx + \frac{bb}{\alpha\alpha}\left(\frac{k-v}{k+v}\right)^2 Q\beta\beta xx.$$

Again the inclination of the rays incident to the axis at *g* is $\left(\frac{k-v}{k+v}\right)\left(\frac{k'-v'}{k'+v'}\right)\frac{bx}{\alpha\beta}$.
Finally since there shall be $F\zeta = \frac{\alpha(k+v)}{a(k-v)}z$, the magnitude of the image at *G* as considered to be elicited :

$$G\eta = \frac{\alpha\beta}{ab}\left(\frac{k+v}{k-v}\right)\left(\frac{k'+v'}{k'-v'}\right)z.$$

COROLLARY 1

74. So that the image may be restrained to the aperture of the face of the lens *QQ*, that must be greater than the transition distance of the rays ; hence therefore the radius of the aperture

of the anterior face $> \left(\frac{k-v}{k+v}\right)\frac{bx}{a}$; of the posterior face $> \left(\frac{k-v}{k+v}\right)\left(\frac{k'-v'}{k'+v'}\right)\frac{bx}{a}$.

COROLLARY 2

75. If the radius of the anterior face for the first lens PP may be put $=f$ and of the posterior $=g$, there will be on making $n=\frac{31}{20}$,

$$f=\frac{(n-1)a(k+v)}{k+v+2na} \text{ and } g=\frac{(n-1)\alpha(k-v)}{k-v-2n\alpha}.$$

And in a similar manner for the second lens QQ the radius of the anterior face may be put $=f'$, and of the posterior $=g'$, there will be

$$f'=\frac{(n-1)b(k'+v')}{k'+v'+2nb} \text{ and } g'=\frac{(n-1)\beta(k'-v')}{k'-v'-2n\beta},$$

clearly with all the faces considered as convex.

COROLLARY 3

76. But for the diffusion space produced by each lens there will be, as we have found:

$$P=\frac{n}{2(n-1)^2}\left\{\left(\frac{k+v}{k-v}\right)^2\left(\frac{n}{a}+\frac{2}{k+v}\right)\left(\frac{1}{a}+\frac{2}{k+v}\right)^2+\left(\frac{k-v}{k+v}\right)^2\left(\frac{n}{\alpha}-\frac{2}{k-v}\right)\left(\frac{1}{\alpha}-\frac{2}{k-v}\right)^2\right\}$$

and in a similar manner

$$Q=\frac{n}{2(n-1)^2}\left\{\left(\frac{k'+v'}{k'-v'}\right)^2\left(\frac{n}{b}+\frac{2}{k'+v'}\right)\left(\frac{1}{b}+\frac{2}{k'+v'}\right)^2+\left(\frac{k'-v'}{k'+v'}\right)^2\left(\frac{n}{\beta}-\frac{2}{k'-v'}\right)\left(\frac{1}{\beta}-\frac{2}{k'-v'}\right)^2\right\}.$$

There is no need, that the same refractive index $n:1$ may be attributed to all the lenses, but in a similar manner for several lenses there can be put n, n', n'', n''' etc., as now we have warned above, from which no other distinction may arise, except that in the formulas found for f' and g' in place of n there may be written n' and Q may be put in place

$$=\frac{n'}{2(n'-1)^2}\left\{\left(\frac{k'+v'}{k'-v'}\right)^2\left(\frac{n'}{b}+\frac{2}{k'+v'}\right)\left(\frac{1}{b}+\frac{2}{k'+v'}\right)^2+\left(\frac{k'-v'}{k'+v'}\right)^2\left(\frac{n'}{\beta}-\frac{2}{k'-v'}\right)\left(\frac{1}{\beta}-\frac{2}{k'-v'}\right)^2\right\}.$$

COROLLARY 4

77. But if the thickness of the lens may vanish and for the arbitrary quantity *k, k′* we may we may introduce the number *λ, λ′,* there will be

for the lens *PP*

$$f=\frac{a\alpha}{\rho\alpha+\sigma\alpha\pm\tau(a+\alpha)\sqrt{(\lambda-1)}},\quad g=\frac{a\alpha}{\rho\alpha+\sigma\alpha\mp\tau(a+\alpha)\sqrt{(\lambda-1)}}$$

$$P=\mu\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)\left(\lambda\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)^{2}+\tfrac{\nu}{\alpha\alpha}\right)$$

and for the lens *QQ*

$$f'=\frac{b\beta}{\rho\beta+\sigma b\pm\tau(b+\beta)\sqrt{(\lambda'-1)}},\quad g'=\frac{b\beta}{\rho b+\sigma\beta\mp\tau(b+\beta)\sqrt{(\lambda'-1)}}$$

$$Q=\mu\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)\left(\lambda'\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)^{2}+\tfrac{\nu}{b\beta}\right).$$

But in the expressions found the formulas $\left(\frac{k-\nu}{k+\nu}\right)$ and $\left(\frac{k'-\nu'}{k'+\nu'}\right)$ become one.

Truly the nature of the numbers $\mu,\ \nu,\ \rho,\ \sigma,\ \tau$ has been established in § 55.

If the refraction of the lenses may differ, also the letters $\mu,\ \nu,\ \rho,\ \sigma,\ \tau$ will be sorted out into values for the individual lenses; which letters if also they may be distinguished by dashes as previously, so that there shall be

$$\rho'=\frac{1}{2(n'-1)}+\frac{1}{n'+2}-1,\quad \sigma'=1+\frac{1}{2(n'-1)}-\frac{1}{n'+2},$$

$$\tau'=\tfrac{1}{3}\left(\frac{1}{2(n'-1)}+\frac{1}{n'+2}\right)\sqrt{(4n'-1)},$$

for the second lens there will be

$$f'=\frac{b\beta}{\rho'\beta+\sigma'b\pm\tau'(b+\beta)\sqrt{(\lambda'-1)}},\quad g'=\frac{b\beta}{\rho'b+\sigma'\beta\mp\tau'(b+\beta)\sqrt{(\lambda'-1)}},$$

$$Q=\mu\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)\left(\lambda'\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)^{2}+\tfrac{\nu}{b\beta}\right).$$

and by the same reasoning, there will be

$$\mu'=\frac{1}{4(n'+2)}+\frac{1}{4(n'-1)}+\frac{1}{8(n'-1)^{2}}\ \text{and}\ \nu'=\frac{4(n'-1)^{2}}{4n'-1},$$

$$\tau'=\left(\frac{1}{2(n'-1)}+\frac{1}{n'+2}\right)\sqrt{(4n'-1)},$$

and which is understood for the following lenses, if perhaps they were provided with a different law of refraction.

SCHOLIUM

78. In order that we may contract more the formulas found, so that, since we will progress to more lenses, they may emerge more succinct, we may put

$$\tfrac{k-v}{k+v}=i \quad \text{and} \quad \tfrac{k'-v'}{k'+v'}=i'$$

thus so that these numbers i and i' may become unity with the thickness of the lenses v and v' vanishing. But then the diffusion length will become

$$Gg=\tfrac{1}{i'i'}\cdot\tfrac{\beta\beta}{bb}\cdot P\alpha\alpha xx+ii\cdot\tfrac{bb}{\alpha\alpha}\cdot Q\beta\beta xx.$$

and the inclination of the rays at g to the axis $=ii'\cdot\tfrac{bx}{\alpha\beta}$; again the magnitude of the image $G\eta=\tfrac{1}{ii'}\cdot\tfrac{\alpha\beta}{ab}z$. And for the aperture of the single faces there will be as follows:

Radius	of the anterior face	of the posterior face
of the first lens *PP*	x	ix
of the second lens *QQ*	$i\cdot\tfrac{bx}{\alpha}$	$ii'\cdot\tfrac{bx}{\alpha}$

but the apertures of these must be assigned greater than the first; for the values assigned may be sufficient, if the object is a single point placed on the axis ; but when it has a magnitude, the rays from its tip may be ingressing more, and may be diverging more widely, and may need a greater aperture for the following faces.

PROBLEM 3

79. *If rays sent from the object Ee* (Fig. 6) *may be refracted by the three lenses PP, QQ and RR, to define the diffusion interval m arising from the given aperture of the first lens, and so that the magnitude of the principal image may be shown at H.*

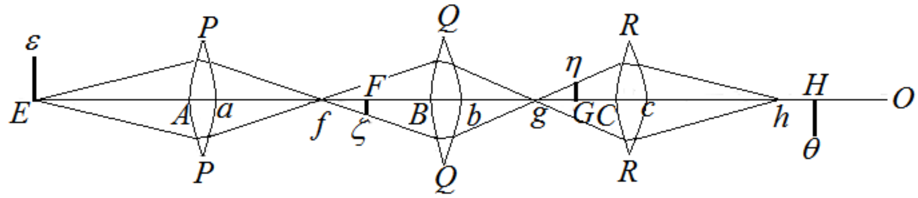

Fig. 6

SOLUTION

With the magnitude of the object $E\varepsilon = z$ put in place, its first principal image through the first lens *PP* falls at $F\zeta$, then through the second *QQ* at $G\eta$, then truly through the third *RR* at $H\theta$. Therefore the distances required to be determined :

for the object $EA = a$ of the lens *PP,* of the image $aF = \alpha$,
for the object $FB = b$ of the lens *QQ,* of the image $bG = \beta$,
for the object $GC = c$ of the lens *RR,* of the image $cH = \gamma$.

Then there shall be

the thickness $Aa = v$ for the lens *PP,* the arbitrary quantity $= k$,
the thickness $Bb = v'$ for the lens *QQ,* the arbitrary quantity $= k'$,
the thickness $Cc = v''$ for the lens *RR,* the arbitrary quantity $= k''$,

and there may be put for the sake of brevity:

$$\tfrac{k-v}{k+v} = i, \quad \tfrac{k'-v'}{k'+v'} = i', \quad \tfrac{k''-v''}{k''+v''} = i''.$$

Finally for any lens, if the radius of its aperture shall be $= x$, there shall be

the diffusion length of the first lens $PP = P\alpha\alpha xx$
" " of the second lens $QQ = Q\beta\beta xx$
" " of the third lens $RR = R\gamma\gamma xx.$

Now in the preceding problem we have found the diffusion length to be produced by the two first lenses

$$Gg = \frac{1}{i'i'} \cdot \frac{\beta\beta}{bb} \cdot P\alpha\alpha xx + ii \cdot \frac{bb}{\alpha\alpha} \cdot Q\beta\beta xx.$$

and the inclination of the rays to the axis concurring at $g = ii' \cdot \frac{bx}{\alpha\beta}$, and the magnitude $G\eta = \frac{1}{ii'} \cdot \frac{\alpha\beta}{ab} z.$ of the image at G. Therefore these if they may be transferred to problem 1, there in place of

$$Ee,\ \frac{\alpha\alpha}{aa},\ \left(\frac{k+d}{k-d}\right)^2,\ Paaaa \text{ and } \Phi$$

there must be written

$$Gg,\ \frac{\gamma\gamma}{cc},\ \frac{1}{i''i''},\ Rcc\gamma\gamma \text{ and } ii' \cdot \frac{bx}{\alpha\beta},$$

and hence the diffusion length for three lenses will arise:

$$Hh = \frac{1}{i'i' \cdot i''i''} \cdot \frac{\beta\beta\gamma\gamma}{bbcc} \cdot P\alpha\alpha xx + \frac{ii}{i''i''} \cdot \frac{bb\gamma\gamma}{\alpha\alpha cc} \cdot Q\beta\beta xx + ii \cdot i'i' \cdot \frac{bbcc}{\alpha\alpha\beta\beta} \cdot R\gamma\gamma xx;$$

but the inclination of the rays to the axis concurring at h will be $= ii'i'' \cdot \frac{bcx}{\alpha\beta\gamma}$. Finally the magnitude of the image formed at H will be $H\theta = \frac{1}{ii'i''} \cdot \frac{\alpha\beta\gamma}{abc} z$ related inversely.

COROLLARY 1

80. So that the apertures of the individual faces may be considered, it is agreed to present these in the following manner:

Radius of the lens aperture	radius of ant. face	radius of post. face
first PP	x	ix
second QQ	$i.\frac{bx}{\alpha}$	$ii'.\frac{bx}{\alpha}$
third RR	$ii'.\frac{bcx}{\alpha\beta}$	$ii'i''.\frac{bcx}{\alpha\beta}$

Clearly the radii must not be smaller than these values.

COROLLARY 2

81. If the inclination of the rays to the axis concurring at h may be called $=\Phi$, then $\Phi = ii'i''.\frac{bcx}{\alpha\beta\gamma}$ and $H\theta = \frac{1}{ii'i''}.\frac{\alpha\beta\gamma}{abc}$; there will be $\Phi \cdot H\theta = \frac{xz}{a}$; which property is present for any number of lenses, however great.

COROLLARY 3

82. Just as the radii of the individual faces shall be required to be determined, it is readily apparent from the preceding. Namely, there will be :

	Radius of the face	
	anterior	posterior
first lens PP	$\frac{(n-1)a(k+v)}{k+v+2na}$	$\frac{(n-1)\alpha(k-v)}{k-v-2n\alpha}$
second lens QQ	$\frac{(n-1)b(k'+v')}{k'+v'+2nb}$	$\frac{(n-1)\beta(k'-v')}{k'-v'-2n\beta}$
third lens RR	$\frac{(n-1)c(k''+v'')}{k''+v''+2nc}$	$\frac{(n-1)\gamma(k''-v'')}{k''-v''-2n\gamma}$.

For the glass present, $n = \frac{31}{20}$.

COROLLARY 4

83. Then truly the values of the letter P, Q, R thus themselves will be had :

$$P = \frac{n}{2(n-1)^2}\left\{\frac{1}{ii}\left(\frac{n}{a}+\frac{2}{k+v}\right)\left(\frac{1}{a}+\frac{2}{k+v}\right)^2 + ii\left(\frac{n}{\alpha}-\frac{2}{k-v}\right)\left(\frac{1}{\alpha}-\frac{2}{k-v}\right)^2\right\}$$

$$Q = \frac{n}{2(n-1)^2}\left\{\frac{1}{i'i'}\left(\frac{n}{b}+\frac{2}{k'+v'}\right)\left(\frac{1}{b}+\frac{2}{k'+v'}\right)^2 + i'i'\left(\frac{n}{\beta}-\frac{2}{k'-v'}\right)\left(\frac{1}{\beta}-\frac{2}{k'-v'}\right)^2\right\}$$

$$R = \frac{n}{2(n-1)^2}\left\{\frac{1}{i''i''}\left(\frac{n}{c}+\frac{2}{k''+v''}\right)\left(\frac{1}{c}+\frac{2}{k''+v''}\right)^2 + i''i''\left(\frac{n}{\gamma}-\frac{2}{k''-v''}\right)\left(\frac{1}{\gamma}-\frac{2}{k''-v''}\right)^2\right\}$$

from which values the diffusion intervals are defined.
If the refraction may differ:

to the letters	P	Q	R,
the refraction may be attributed	n	n'	n''.

COROLLARY 5

84. But it will help also the diffusion lengths, being produced by one, two or three lenses, to be compared between each other, which thus can be seen most conveniently :

$$Ff = \alpha\alpha xx \cdot P, \quad Gg = \beta\beta xx\left\{\tfrac{1}{i'i'} \cdot \tfrac{\alpha\alpha}{bb} \cdot P + ii \cdot \tfrac{bb}{\alpha\alpha} \cdot Q\right\}$$

$$Hh = \gamma\gamma xx\left\{\tfrac{1}{i'i' \cdot i''i''} \cdot \tfrac{\alpha\alpha\beta\beta}{bbcc} \cdot P + \tfrac{ii}{i''i''} \cdot \tfrac{bb\beta\beta}{\alpha\alpha cc} \cdot Q + ii \cdot i'i' \cdot \tfrac{bbcc}{\alpha\alpha\beta\beta} \cdot R\right\}.$$

SCHOLIUM

85. Hence it is seen easily, how the determination of the diffusion length may be extended to more lenses ; from which I will adapt at once to the general problem of any number of lenses, and that for any thickness of lens.

PROBLEM 4

86. *If rays from the object Ee may be refracted by a number of lenses PP, QQ, RR, SS etc. set out on a common axis, to define the spatial diffusion arising from the given first apertures of the given lens, as well as the magnitude of the image represented.*

SOLUTION

With the magnitude of the object put $E\varepsilon = z$ we will consider its principal images: its first principal image through the first lens *PP* therefore will fall at $F\zeta$, then by the second *QQ* at $G\eta$, then by the third *RR* at $H\theta$, again by the fourth *SS* at $I\iota$, by the fifth *TT* at $K\chi$ etc. Hence for the individual lenses we will have the distances to be determined, which thus we will indicate by the letters :

For the lens *PP* : the object $EA = a$, the image $aF = \alpha$, the thickness $Aa = v$,
For the lens *QQ*: the object $FB = b$, the image $bG = \beta$, the thickness $Bb = v'$,
For the lens *RR*: the object $GC = c$, the image $cH = \gamma$, the thickness $Cc = v''$,
For the lens *SS*: the object $HD = d$, the image $dI = \delta$, the thickness $Dd = v'''$,
For the lens *TT*: the object $IE = e$, the image $eK = \varepsilon$, the thickness $Ee = v''''$
etc.

Then since the determination of any lens not only may involve these distances to be determined with its thickness but also a certain arbitrary quantity, on which each diffusion distance will depend, we may put, if any lens may be on its own and the radius of its aperture $= x$:

For the first	The arb. quant.	diffusion space
lens *PP,*	k	$P\alpha\alpha xx$
second *QQ,*	k'	$Q\beta\beta xx$
third *RR,*	k''	$R\gamma\gamma xx$
fourth *SS,*	k'''	$S\delta\delta xx$
fifth *TT,*	k''''	$T\varepsilon\varepsilon xx$
	etc.	

Therefore the construction of each lens hence thus will be had on putting $n=\frac{31}{20}$; or if the refraction may differ, to each lens its value may be attributed, the first n, the second n', the third n'' etc.

Truly there will be,	The radius of the	
for	anterior face	posterior face
the first lens *PP,*	$\frac{(n-1)a(k+v)}{k+v+2na}$	$\frac{(n-1)\alpha(k-v)}{k-v-2n\alpha}$
the second lens *QQ,*	$\frac{(n-1)b(k'+v')}{k'+v'+2nb}$	$\frac{(n-1)\beta(k'-v')}{k'-v'-2n\beta}$
the third lens *RR,*	$\frac{(n-1)c(k''+v'')}{k''+v''+2nc}$	$\frac{(n-1)\gamma(k''-v'')}{k''-v''-2n\gamma}$
the fourth lens *SS,*	$\frac{(n-1)d(k'''+v''')}{k'''+v'''+2nd}$	$\frac{(n-1)\delta(k'''-v''')}{k'''-v'''-2n\delta}$
the fifth lens *TT,*	$\frac{(n-1)e(k''''+v'''')}{k''''+v''''+2ne}$	$\frac{(n-1)\varepsilon(k''''-v'''')}{k''''-v''''-2n\varepsilon}$

and if for brevity we may put :

$$\frac{k-v}{k+v}=i,\quad \frac{k'-v'}{k'+v'}=i',\quad \frac{k''-v''}{k''+v''}=i'',\quad \frac{k'''-v'''}{k'''+v'''}=i''',\quad \frac{k''''-v''''}{k''''+v''''}=i''''\ \text{ etc.},$$

we will have these values for the diffusion lengths :

$$P=\frac{n}{2(n-1)^2}\left\{\frac{1}{ii}\left(\frac{n}{a}+\frac{2}{k+v}\right)\left(\frac{1}{a}+\frac{2}{k+v}\right)^2+ii\left(\frac{n}{\alpha}-\frac{2}{k-v}\right)\left(\frac{1}{\alpha}-\frac{2}{k-v}\right)^2\right\}$$

$$Q=\frac{n}{2(n-1)^2}\left\{\frac{1}{i'i'}\left(\frac{n}{b}+\frac{2}{k'+v'}\right)\left(\frac{1}{b}+\frac{2}{k'+v'}\right)^2+i'i'\left(\frac{n}{\beta}-\frac{2}{k'-v'}\right)\left(\frac{1}{\beta}-\frac{2}{k'-v'}\right)^2\right\}$$

$$R=\frac{n}{2(n-1)^2}\left\{\frac{1}{i''i''}\left(\frac{n}{c}+\frac{2}{k''+v''}\right)\left(\frac{1}{c}+\frac{2}{k''+v''}\right)^2+i''i''\left(\frac{n}{\gamma}-\frac{2}{k''-v''}\right)\left(\frac{1}{\gamma}-\frac{2}{k''-v''}\right)^2\right\}$$

$$S=\frac{n}{2(n-1)^2}\left\{\frac{1}{i'''i'''}\left(\frac{n}{d}+\frac{2}{k'''+v'''}\right)\left(\frac{1}{d}+\frac{2}{k'''+v'''}\right)^2+i'''i'''\left(\frac{n}{\delta}-\frac{2}{k'''-v'''}\right)\left(\frac{1}{\delta}-\frac{2}{k'''-v'''}\right)^2\right\}$$

$$T=\frac{n}{2(n-1)^2}\left\{\frac{1}{i''''i''''}\left(\frac{n}{e}+\frac{2}{k''''+v''''}\right)\left(\frac{1}{e}+\frac{2}{k''''+v''''}\right)^2+i''''i''''\left(\frac{n}{\varepsilon}-\frac{2}{k''''-v''''}\right)\left(\frac{1}{\varepsilon}-\frac{2}{k''''-v''''}\right)^2\right\}$$

etc.

With these in place we will have for the magnitude of the individual images:

the image for one lens,	$F\zeta=\frac{1}{i}\cdot\frac{\alpha}{a}z$	inverted
for two lenses,	$G\eta=\frac{1}{ii'}\cdot\frac{\alpha\beta}{ab}z$	erect
for three lenses,	$H\theta=\frac{1}{ii'i''}\cdot\frac{\alpha\beta\gamma}{abc}z$	inverted
for four lenses,	$Ii=\frac{1}{ii'i''i'''}\cdot\frac{\alpha\beta\gamma\delta}{abcd}z$	erect
for five lenses,	$K\chi=\frac{1}{ii'i''i'''i''''}\cdot\frac{\alpha\beta\gamma\delta\varepsilon}{abcde}z$	inverted

Finally while the apertures of the lenses shall not be smaller, as the following formulas show :

Radius of the lens aperture	of the anterior face,	of the posterior face
first *PP*,	x	$i\cdot x$
second *QQ*,	$i\cdot\frac{bx}{\alpha}$	$ii'\cdot\frac{bx}{\alpha}$
third *RR*,	$ii'\cdot\frac{bcx}{\alpha\beta}$	$ii'i''\cdot\frac{bcx}{\alpha\beta}$
fourth *SS*,	$ii'i''\cdot\frac{bcdx}{\alpha\beta\gamma}$	$ii'i''i'''\cdot\frac{bcdx}{\alpha\beta\gamma}$
fifth *TT*,	$ii'i''i'''\cdot\frac{bcdex}{\alpha\beta\gamma\delta}$	$ii'i''i'''i''''\cdot\frac{bcdex}{\alpha\beta\gamma\delta}$

etc.,

there will be, as follows, for any number of lenses :

I. For one lens
the diffusion length: $Ff = \alpha\alpha xx \cdot P,$
with the inclination of the rays to the axis concurrent at $f \;\; = i \cdot \frac{x}{\alpha}.$

II. For two lenses
the diffusion length: $Gg = \beta\beta xx\left(\frac{1}{i'i'} \cdot \frac{\alpha\alpha}{bb} \cdot P + ii \cdot \frac{bb}{\alpha\alpha} \cdot Q\right)$
and the inclination of the rays to the axis concurrent at $g \;\; = ii' \cdot \frac{bx}{\alpha\beta}.$

III. For three lenses
the diffusion length:

$$Hh = \gamma\gamma xx\left(\frac{1}{i'i' \cdot i''i''} \cdot \frac{\alpha\alpha\beta\beta}{bbcc} \cdot P + \frac{ii}{i''i''} \cdot \frac{bb\beta\beta}{\alpha\alpha cc} \cdot Q + ii \cdot i''i'' \cdot \frac{bbcc}{\alpha\alpha\beta\beta} \cdot R\right),$$

and the inclination of the rays to the axis concurring at $h \;\; = ii'i'' \cdot \frac{bcx}{\alpha\beta\gamma}.$

IV. For four lenses
the diffusion length :

$$Ii = \delta\delta xx\left\{\begin{array}{l} \frac{1}{i'i' \cdot i''i'' \cdot i'''i'''} \cdot \frac{\alpha\alpha\beta\beta\gamma\gamma}{bbccdd} \cdot P + \frac{ii}{i''i'' \cdot i'''i'''} \cdot \frac{bb\beta\beta\gamma\gamma}{\alpha\alpha ccdd} \cdot Q \\ + \frac{ii \cdot i'i'}{i'''i'''} \cdot \frac{bbcc\gamma\gamma}{\alpha\alpha\beta\beta dd} \cdot R + ii \cdot i'i' \cdot i'''i''' \cdot \frac{bbccdd}{\alpha\alpha\beta\beta\gamma\gamma} \cdot S \end{array}\right\}$$

and the inclination of the rays to the axis concurring at $i \; = ii'i''i''' \cdot \frac{bcdx}{\alpha\beta\gamma\delta}.$

V. For five lenses
the diffusion length:

$$Kk = \varepsilon\varepsilon xx\left\{\begin{array}{ll} \frac{1}{i'i' \cdot i''i'' \cdot i'''i''' \cdot i''''i''''} \cdot \frac{\alpha\alpha\beta\beta\gamma\gamma\delta\delta}{bbccddee} \cdot P + & \frac{ii}{i''i'' \cdot i'''i''' \cdot i''''i''''} \cdot \frac{bb\beta\beta\gamma\gamma\delta\delta}{\alpha\alpha ccddee} \cdot Q \\ + \frac{ii \cdot i'i'}{i'''i''' \cdot i''''i''''} \cdot \frac{bbcc\gamma\gamma\delta\delta}{\alpha\alpha\beta\beta ddee} \cdot R & + \frac{ii \cdot i'i' \cdot i'''i'''}{i''''i''''} \cdot \frac{bbccdd\delta\delta}{\alpha\alpha\beta\beta\gamma\gamma ee} \cdot S \\ ii \cdot i'i' \cdot i'''i''' \cdot i'''i''' \cdot \frac{bbccddee}{\alpha\alpha\beta\beta\gamma\gamma\delta\delta} \cdot T & \end{array}\right\}$$

and the inclination of the rays to the axis concurring at $k \; = ii'i''i'''i'''' \cdot \frac{bcdex}{\alpha\beta\gamma\delta\varepsilon}.$

From which the progression of these formulas to still further lenses is clear enough. If the ratio of the refraction may be different in the lenses and may be indicated in this order of the letters *n, n', n'', n'''* etc., this diversity will be readily adapted to the formulas found here. Indeed this correction occurs first in the formulas for the rays of the lens, thus so that, just as the formulas f and g for the first lens involve the number n, thus for the second lens the number n' may be introduced, for the third n'' and thus so on. Also the values of the letters P, Q, R, S etc. require a similar correction and in place of the letter n, in which the value P occurs, in the values Q, R, S etc. it will be required to write n', n'', n''' etc.

COROLLARY 1

87. If the object shall be only a point placed on the axis it suffices, that the apertures of the lenses shall be quantities as large as we have assigned ; but if the object may have a certain magnitude, then the apertures besides the first must therefore overcome the assigned measures more, by which the greater were the magnitude of the object z.

COROLLARY 2

88. In the expression of the diffusion length the square of the radius of the aperture of the first face xx in the first case is multiplied by the square of the distance of the posterior image from the furthest part of the lens : which if it were infinite, the diffusion length also shall be infinite.

COROLLARY 3

89. Therefore with all else equal, however many lenses there were, the diffusion length always is proportional to the square of the radius of the first face, that is to this aperture itself. From which on reducing the diameter of the aperture of the first face by half, the diffusion length becomes smaller by four times.

SCHOLIUM

90. We will consider here at once the positions of the individual principal images as if given, and from these we will determine the structure of each lens by introducing an arbitrary quantity. Because if truly these lenses were given, thus so that both the radiii of both faces may be known as well as the thickness of each together with the intervals between them to be determined, then with the aid of the formulas shown the distances will become known in turn. Clearly the radii of the anterior and posterior faces of the first lens PP shall be f, g, of the second lens QQ f', g', of the third lens RR f'', g'' etc., with the thickness of these being v, v', v'' etc., then truly the distances will be given $aB = F$; $bC = G$; $cD = H$ etc . But besides the distance of the object before the first lens shall be $AE = a$, and in the following manner all the elements for the above problem necessarily will be elucidated:

1. $\frac{n-1}{f} = \frac{1}{a} + \frac{2n}{k+v}$, hence k is found ; 2. $\frac{n-1}{g} = \frac{1}{\alpha} - \frac{2n}{k-v}$, hence truly α ;

3. $F = \alpha + b$, from which $b = F - \alpha$;

4. $\frac{n-1}{f'} = \frac{1}{b} + \frac{2n}{k'+v'}$, hence there is found k'; 5. $\frac{n-1}{g'} = \frac{1}{\beta} - \frac{2n}{k'-v'}$, hence truly again β;

6. $G = \beta + c$, from which $c = G - \beta$;

7. $\frac{n-1}{f''} = \frac{1}{c} + \frac{2n}{k''+v''}$, hence there is found k''; 8. $\frac{n-1}{g''} = \frac{1}{\gamma} - \frac{2n}{k''-v''}$, hence truly γ;

9. $H = \gamma + d$, from which $d = \gamma - H$,

etc.

Therefore whatever lenses may have been given set out on the common axis, if before these the object may be put in place at a given distance $AE = a$, thence the individual determinable distances $\alpha, b, \beta, c, \gamma$ etc. with the arbitrary k, k', k'', k''' etc. are readily defined, and again from these the diffusion distances with the remaining phenomena will be recalled in the solution to the problem. But there will be a need for the case, where the thickness of the lenses may be considered as vanishing, to be set out more carefully.

PROBLEM 5

91. *If the thickness of a lens may vanish and some number of lenses of this kind may have been set out on the common axis, before which the object $E\varepsilon$ is present, to define the diffusion distance, through which the image will be spread out, as well as the magnitude of the image.*

SOLUTION

The magnitude of the object shall be $E\varepsilon = z$, of which the principal images fall successively at $F\zeta, G\eta, H\theta, Ii, K\chi$ etc., and hence for the individual lenses the following distances will have to be determined, since the image through any lens represented with respect to any the following lenses will have to be carried out in turn:

For the lens *PP* the distance of the object $EA = a$, the distance of the image $aF = \alpha$,
for the lens *QQ* the distance of the object $FB = b$, the distance of the image $bG = \beta$,
for the lens *RR* the distance of the object $GC = c$, the distance of the image $cH = \gamma$,
for the lens *SB* the distance of the object $HD = d$, the distance of the image $dI = \delta$,
for the lens *TT* the distance of the object $IE = e$, the distance of the image $eK = \varepsilon$
etc.

But again the numbers determining the figure of each lens shall be arbitrary numbers greater than unity, λ for the lens *PP,* λ' for *QQ,* λ'' for *RR,* λ''' for *SS,* λ'''' for *TT* etc., thus so that on putting for the sake of brevity

$$\rho = 0{,}190781,\ \sigma = 1{,}627401,\ \tau = 0{,}905133$$

$$\text{for the lens } PP \text{ the radius } \begin{cases} \text{of the ant. face} = \frac{a\alpha}{\rho\alpha+\sigma a\pm\tau(a+\alpha)\sqrt{(\lambda-1)}} \\ \text{of the post. face} = \frac{a\alpha}{\rho\alpha+\sigma a\mp\tau(a+\alpha)\sqrt{(\lambda-1)}} \end{cases}$$

$$\text{for the lens } QQ \text{ the radius } \begin{cases} \text{of the ant. face} = \frac{b\beta}{\rho\beta+\sigma b\pm\tau(b+\beta)\sqrt{(\lambda'-1)}} \\ \text{of the post. face} = \frac{b\beta}{\rho b+\sigma\beta\mp\tau(b+\beta)\sqrt{(\lambda'-1)}} \end{cases}$$

$$\text{for the lens } RR \text{ the radius } \begin{cases} \text{of the ant. face} = \frac{c\gamma}{\rho\gamma+\sigma c\pm\tau(c+\gamma)\sqrt{(\lambda''-1)}} \\ \text{of the post. face} = \frac{c\gamma}{\rho c+\sigma\gamma\mp\tau(c+\gamma)\sqrt{(\lambda''-1)}} \end{cases}$$

$$\text{for the lens } SS \text{ the radius } \begin{cases} \text{of the ant. face} = \frac{d\delta}{\rho\delta+\sigma d\pm\tau(d+\delta)\sqrt{(\lambda'''-1)}} \\ \text{of the post. face} = \frac{d\delta}{\rho d+\sigma\delta\mp\tau(d+\delta)\sqrt{(\lambda'''-1)}} \end{cases}$$

$$\text{for the lens } TT \text{ the radius } \begin{cases} \text{of the ant. face} = \frac{e\varepsilon}{\rho\varepsilon+\sigma e\pm\tau(e+\varepsilon)\sqrt{(\lambda''''-1)}} \\ \text{of the post. face} = \frac{e\varepsilon}{\rho e+\sigma\varepsilon\mp\tau(e+\varepsilon)\sqrt{(\lambda''''-1)}} \end{cases}$$

etc.

Then if any lens may be considered separately with its two determinable distances, and the radius of the aperture shall become $= x,$ on putting $\mu = 0,938191$ and $v = 0,232692$ the diffusion length of the lenses :

$$PP = \mu\alpha\alpha xx\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)\left(\lambda\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)^2+\tfrac{v}{a\alpha}\right)$$

$$QQ = \mu\beta\beta xx\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)\left(\lambda'\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)^2+\tfrac{v}{b\beta}\right)$$

$$RR = \mu\gamma\gamma xx\left(\tfrac{1}{c}+\tfrac{1}{\gamma}\right)\left(\lambda''\left(\tfrac{1}{c}+\tfrac{1}{\gamma}\right)^2+\tfrac{v}{c\gamma}\right)$$

$$SS = \mu\delta\delta xx\left(\tfrac{1}{d}+\tfrac{1}{\delta}\right)\left(\lambda'''\left(\tfrac{1}{d}+\tfrac{1}{\delta}\right)^2+\tfrac{v}{d\delta}\right)$$

$$TT = \mu\varepsilon\varepsilon xx\left(\tfrac{1}{e}+\tfrac{1}{\varepsilon}\right)\left(\lambda''''\left(\tfrac{1}{e}+\tfrac{1}{\varepsilon}\right)^2+\tfrac{v}{e\varepsilon}\right)$$

etc.

With these established we will have for the magnitude of the individual images :

for one lens $F\zeta = \frac{\alpha}{a}z$ placed inversely,

for two lenses $G\eta = \frac{\alpha\beta}{ab}z$ placed erect,

for three lenses $H\theta = \frac{\alpha\beta\gamma}{abc}z$ placed inversely,

for four lenses $Ii = \frac{\alpha\beta\gamma\delta}{abcd}z$ placed erect,

for five lenses $K\chi = \frac{\alpha\beta\gamma\delta\varepsilon}{abcde}z$ placed inversely,

etc.

But if the radius of the first lens *PP* may be put $= x$, it is necessary that the apertures of the remaining lenses shall be greater :

Radius of aperture

of second lens $QQ > \frac{b}{\alpha}x$, of third lens $RR > \frac{bc}{\alpha\beta}x$,

of fourth lens $SS > \frac{bcd}{\alpha\beta\gamma}x$, of fifth lens $TT > \frac{bcde}{\alpha\beta\gamma\delta}x$

etc.

Hence the diffusion distance for any number of lenses thus will be had :

I. For one lens,

the diffusion length: $Ff = \mu\alpha\alpha xx\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{\nu}{a\alpha}\right)$

and the inclination to the axis of the rays concurring at f, $= \frac{x}{\alpha}$.

II. For two lenses,

the diffusion length:

$$Gg = \mu\beta\beta xx\left\{\begin{array}{l} +\frac{\alpha\alpha}{bb}\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{\nu}{a\alpha}\right) \\ +\frac{bb}{\alpha\alpha}\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\lambda'\left(\frac{1}{b}+\frac{1}{\beta}\right)^2+\frac{\nu}{b\beta}\right) \end{array}\right\}$$

and the inclination to the axis of the rays concurring at *g*, $= \frac{bx}{\alpha\beta}$.

III. For three lenses

the diffusion length:

$$Hh=\mu\gamma\gamma xx\left\{\begin{aligned}&+\tfrac{\alpha\alpha\beta\beta}{bbcc}\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)\left(\lambda\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)^{2}+\tfrac{\nu}{a\alpha}\right)\\&+\tfrac{bb\beta\beta}{\alpha\alpha cc}\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)\left(\lambda'\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)^{2}+\tfrac{\nu}{b\beta}\right)\\&+\tfrac{bbcc}{\alpha\alpha\beta\beta}\left(\tfrac{1}{c}+\tfrac{1}{\gamma}\right)\left(\lambda''\left(\tfrac{1}{c}+\tfrac{1}{\gamma}\right)^{2}+\tfrac{\nu}{c\gamma}\right)\end{aligned}\right\}$$

and the inclination to the axis of the rays concurring at h, $=\frac{bcx}{\alpha\beta\gamma}$.

IV. For four lenses,

the diffusion length:

$$Ii=\mu\delta\delta xx\left\{\begin{aligned}&+\tfrac{\alpha\alpha\beta\beta\gamma\gamma}{bbccdd}\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)\left(\lambda\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)^{2}+\tfrac{\nu}{a\alpha}\right)\\&+\tfrac{bb\beta\beta\gamma\gamma}{\alpha\alpha ccdd}\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)\left(\lambda'\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)^{2}+\tfrac{\nu}{b\beta}\right)\\&+\tfrac{bbcc\gamma\gamma}{\alpha\alpha\beta\beta dd}\left(\tfrac{1}{c}+\tfrac{1}{\gamma}\right)\left(\lambda''\left(\tfrac{1}{c}+\tfrac{1}{\gamma}\right)^{2}+\tfrac{\nu}{c\gamma}\right)\\&+\tfrac{bbccdd}{\alpha\alpha\beta\beta\gamma\gamma}\left(\tfrac{1}{d}+\tfrac{1}{\delta}\right)\left(\lambda'''\left(\tfrac{1}{d}+\tfrac{1}{\delta}\right)^{2}+\tfrac{\nu}{d\delta}\right)\end{aligned}\right\}$$

and the inclination to the axis of the rays concurring at i, $=\frac{bcdx}{\alpha\beta\gamma\delta}$.

V. For five lenses,

the diffusion length:

$$Kk=\mu\varepsilon\varepsilon xx\left\{\begin{array}{l}
+\frac{\alpha\alpha\beta\beta\gamma\gamma\delta\delta}{bbccddee}\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{\nu}{a\alpha}\right)\\
+\frac{bb\beta\beta\gamma\gamma\delta\delta}{\alpha\alpha ccddee}\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\lambda'\left(\frac{1}{b}+\frac{1}{\beta}\right)^2+\frac{\nu}{b\beta}\right)\\
+\frac{bbcc\gamma\gamma\delta\delta}{\alpha\alpha\beta\beta ddee}\left(\frac{1}{c}+\frac{1}{\gamma}\right)\left(\lambda''\left(\frac{1}{c}+\frac{1}{\gamma}\right)^2+\frac{\nu}{c\gamma}\right)\\
+\frac{bbccdd\delta\delta}{\alpha\alpha\beta\beta\gamma\gamma ee}\left(\frac{1}{d}+\frac{1}{\delta}\right)\left(\lambda'''\left(\frac{1}{d}+\frac{1}{\delta}\right)^2+\frac{\nu}{d\delta}\right)\\
+\frac{bbccddee}{\alpha\alpha\beta\beta\gamma\gamma\delta\delta}\left(\frac{1}{e}+\frac{1}{\varepsilon}\right)\left(\lambda''''\left(\frac{1}{e}+\frac{1}{\varepsilon}\right)^2+\frac{\nu}{e\varepsilon}\right)
\end{array}\right\}$$

and the inclination to the axis of the rays concurring at *i* , $=\frac{bcdex}{\alpha\beta\gamma\delta\varepsilon}$; nor therefore no case, in which more lenses occur, will labor under any more difficulty.

If the lenses may differ in refractive index, and the letters *n, n', n'', n'''* etc. shall refer to these, the formulas found in this problem can be adapted in an easy way to this more general case. Clearly in the first place in the formulas for the radii of the faces found, the letters $\rho,\ \sigma$ et τ pertain to the first lens only, and in place of these for the second lens it is required to write $\rho',\ \sigma'$ et τ', moreover for the third $\rho'',\ \sigma''$ et τ'' and so forth. Truly in addition the diffusion lengths hence require some change, and the diffusion length will become :

I. For one lens

$$\alpha\alpha xx\left(\frac{1}{a}+\frac{1}{\alpha}\right)\mu\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{\nu}{a\alpha}\right).$$

II. For two lenses

$$\beta\beta xx\left\{\frac{\mu\alpha\alpha}{bb}\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{\nu}{a\alpha}\right)+\frac{\mu'bb}{\alpha\alpha}\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\lambda'\left(\frac{1}{b}+\frac{1}{\beta}\right)^2+\frac{\nu'}{b\beta}\right)\right\}$$

III. For three lenses

$$\gamma\gamma xx\left\{\begin{array}{l}\frac{\mu\alpha\alpha\beta\beta}{bbcc}\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{\nu}{a\alpha}\right)\\ +\frac{\mu'bb\beta\beta}{\alpha\alpha cc}\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\lambda'\left(\frac{1}{b}+\frac{1}{\beta}\right)^2+\frac{\nu'}{b\beta}\right)\\ +\frac{\mu''bbcc}{\alpha\alpha\beta\beta}\left(\frac{1}{c}+\frac{1}{\gamma}\right)\left(\lambda''\left(\frac{1}{c}+\frac{1}{\gamma}\right)^2+\frac{\nu''}{c\gamma}\right)\end{array}\right\}$$

IV. For four lenses

$$\delta\delta xx\left\{\begin{array}{l}\frac{\mu\alpha\alpha\beta\beta\gamma\gamma}{bbccdd}\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{\nu}{a\alpha}\right)\\ +\frac{\mu'bb\beta\beta\gamma\gamma}{\alpha\alpha ccdd}\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\lambda'\left(\frac{1}{b}+\frac{1}{\beta}\right)^2+\frac{\nu'}{b\beta}\right)\\ +\frac{\mu''bbcc\gamma\gamma}{\alpha\alpha\beta\beta dd}\left(\frac{1}{c}+\frac{1}{\gamma}\right)\left(\lambda''\left(\frac{1}{c}+\frac{1}{\gamma}\right)^2+\frac{\nu''}{c\gamma}\right)\\ +\frac{\mu'''bbccdd}{\alpha\alpha\beta\beta\gamma\gamma}\left(\frac{1}{d}+\frac{1}{\delta}\right)\left(\lambda''\left(\frac{1}{d}+\frac{1}{\delta}\right)^2+\frac{\nu'''}{d\delta}\right)\end{array}\right\}$$

but the values of these letters μ', ν', μ'', ν'' etc. we have now defined above in § 77.

COROLLARY 1

92. If the radius of the anterior face of the first lens PP may be put $=f$ and of the posterior $=g$, there will be.

$$\frac{1}{f}=\frac{\rho}{a}+\frac{\sigma}{\alpha}\pm\tau\left(\frac{1}{a}+\frac{1}{\alpha}\right)\sqrt{(\lambda-1)},\quad \frac{1}{g}=\frac{\rho}{\alpha}+\frac{\sigma}{a}\mp\tau\left(\frac{1}{a}+\frac{1}{\alpha}\right)\sqrt{(\lambda-1)},$$

from which, if the distance of the object may be given $EA=a$, initially α is found from the equation

$$\frac{1}{f}+\frac{1}{g}=(\rho+\sigma)\left(\frac{1}{a}+\frac{1}{\alpha}\right)=1,818182\left(\frac{1}{a}+\frac{1}{\alpha}\right)=\frac{20}{11}\left(\frac{1}{a}+\frac{1}{\alpha}\right).$$

But with the distance α found, λ is found from this equation

$$\frac{1}{f}-\frac{1}{g}=(\sigma-\rho)\left(\frac{1}{a}-\frac{1}{\alpha}\right)=2\tau\left(\frac{1}{a}+\frac{1}{\alpha}\right)\sqrt{(\lambda-1)}.$$

COROLLARY 2

93. Consequently if the distance of the second lens from the first shall be $=F$, since $F=\alpha+b$ there is had $b=F-\alpha$; with the distance b known, if the radius of the anterior face of the second lens may be given f' and of the posterior g', again these two equations will be had

$$\tfrac{1}{f'}=\tfrac{\rho}{b}+\tfrac{\sigma}{\beta}\pm\tau(\tfrac{1}{b}+\tfrac{1}{\beta})\sqrt{(\lambda'-1)},\quad \tfrac{1}{g'}=\tfrac{\rho}{\beta}+\tfrac{\sigma}{b}\mp\tau(\tfrac{1}{b}+\tfrac{1}{\beta})\sqrt{(\lambda''-1)},$$

from which both the distance β as well as the number λ' can be defined. And in a similar manner from the form of the following lenses and their distances the remaining elements will become known.

COROLLARY 3

94. If the individual lenses were adapted for the minimum diffusion length, there will be $\lambda=1,\ \lambda'=1,\ \lambda''=1,\ \lambda'''=1$ etc.; but if these lenses were provided with other forms, these numbers will be greater than one.

SCHOLIUM

95. Where there will have been more lenses, it will be agreed therefore with more diffusion lengths produced from these members; nor yet therefore with an increase in the number of lenses by necessity is the number of diffusion lengths increased. Indeed since the quantities $\alpha,\ b,\ \beta,\ c,\ \gamma,\ d,\ \delta$ etc. may be able to receive negative values, while the sum of two $\alpha+b,\ \beta+c,\ \gamma+d,\ \delta+e$ etc. etc. in order that the separation of the lenses may remain positive, it can happen, that one or more members may become negative and hence the diffusion distance may be diminished, why not also sometime evidently it may vanish, in which case without doubt the representation will be the most perfect. Truly in dioptric instruments such as microscopes and telescopes constructed as an aid to vision not only this, that we have defined, of the diffusion length as must arise to be confusing to vision itself ; but which, even if it may differ from having much spatial diffusion, yet from that it can be defined, as we will explain soon. But before it may be appropriate to consider composite or multiples, lenses of whatever kind may arise, if two or more lenses at once may be joined together, the thickness of which is so very small that it may be ignored, where indeed they can be seen to take the place of simple lenses ; truly the conjunction of such can be effected, so that the diffusion length may become much smaller, as if it were brought to become a simple lens, and thus may vanish and the value of the number λ of this kind of composite lens shall be going to be produced less than unity, from which the maximum suitability for diminishing the confusion will be obtained.

CAPUT II

DE DIFFUSIONE IMAGINIS

PER PLURES LENTES REPRAESENTATAE

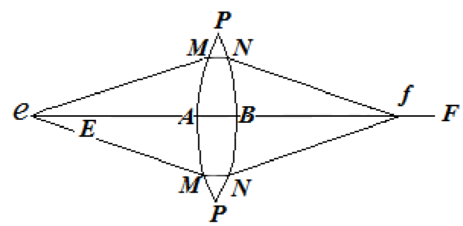

Fig. 4

PROBLEMA 1

68. *Si loco obiecti adsit imago per spatium Ee* (Fig. 4) *diffusa indeque radii per lentem PP aperturae indefinitae transmittantur, determinare spatium diffusionis Ff per hanc lentem productum.*

SOLUTIO

Loco obiecti veri hic consideramus imaginem iam per aliam lentem repraesentatam, quae sit diffusa per spatium *Ee,* ita ut in *E* sit imago principalis per radios axi proximos formata, in *e* autem imago extrema, radiis scilicet per marginem aperturae lentis praecedentis transmissis formata; qui radii cum axe constituant angulum $=\Phi$. Quare a puncto *E* nonnisi radii axi proximi in lentem *PP* emittuntur, a puncto *e* autem eiusmodi tantum radii, qui ad axem *eA* sub angulo $MeA=\Phi$ sint inclinati. Ponatur iam distantia $EA=a$, prae qua spatium diffusionis *Ee* ut valde parvum spectetur, lens autem *PP* sit eiusmodi, ut obiecti in *E* existentis imaginem principalem referat in distantia $BF=\alpha$, existante lentis crassitie $AB=d$. Hanc ob rem, si ponatur faciei anterioris *AM* radius $=f$, posterioris $BN=g$, lente ut convexa utrinque spectata, oportet esse:

$$f=\frac{(n-1)a(k+d)}{k+d+2na} \quad \text{et} \quad g=\frac{(n-1)\alpha(k-d)}{k-d-2n\alpha},$$

ubi est $n=\frac{31}{20}$ et *k* quantitas arbitraria. Hinc autem, uti demonstravimus, pro aperturae semidiametro $=x$ nasceretur spatium diffusionis

$$\frac{n\alpha\alpha xx}{2(n-1)^2}\left\{\left(\frac{k+d}{k-d}\right)^2\left(\frac{n}{a}+\frac{2}{k+d}\right)\left(\frac{1}{a}+\frac{2}{k+d}\right)^2+\left(\frac{k-d}{k+d}\right)^2\left(\frac{n}{\alpha}-\frac{2}{k-d}\right)\left(\frac{1}{\alpha}-\frac{2}{k-d}\right)^2\right\}$$

pro quo scribamus brevitatis gratia $P\alpha\alpha xx$. Iam puncti *E,* quia inde nonnisi radii axi proximi ad lentem emittuntur, imago exhibebitur in *F,* ut sit distantia $BF = \alpha$: videamus ergo, quorsum imago puncti *e* debeat cadere. Ac si radii ex puncto *e* emissi essent axi proximi, quia id a lente magis est remotum quam *E,* eius imago propius ad lentem caderet, puta in φ, ut esset, sicut § 60 definivimus,

$$F\varphi = \frac{\alpha\alpha}{aa}\left(\frac{k+d}{k-d}\right)^2 .Ee \ ;$$

in φ scilicet caderet imago principalis, si obiectum esset in *e.* Sed quia ab *e* tantum radii *eM* ad axem angulo $AeM = cp$ inclinati emittuntur, hi lenti in punctis *M* ab *A* intervallo $AM = eA\cdot\Phi = a\Phi$ remotis occurrent, quandoquidem intervallum *Ee* prae distantia $AE = a$ contemnimus; perinde igitur est, ac si lenti apertura tribueretur, cuius semidiameter esset $= a\Phi$, et obiecti in *e* existentis imago extrema definiri deberet, quae cadat in *f,* ita ut φf sit spatium diffusionis obiecto in *e* existenti et aperturae lentis, cuius semidiameter $= a\Phi$, conveniens. Hinc ergo erit intervallum $\varphi f = Paa\alpha\alpha\Phi\Phi$; et quia puncti *E* imago in *F,* puncti *e* autem imago in *f* exhibetur, erit spatium diffusionis per lentem *PP* productum

$$Ff = \frac{\alpha\alpha}{aa}\left(\frac{k+d}{k-d}\right)^2 .Ee \ + Paa\alpha\alpha\Phi\Phi.$$

In imagine autem extrema *f* radii *Nf* ita cum axe concurrent, ut sit

$$\text{angulus } BfN = \frac{k-d}{k+d}\cdot\frac{a\Phi}{\alpha}.$$

COROLLARIUM 1.

69. Si ergo imago iam diffusa per spatium *Ee* locum obiecti teneat respectu lentis *PP,* per hanc spatium diffusionis novum producitur *Ff,* ita ut imago principalis cadat in *F,* extrema vero in *f;* fierique poterit, ut hoc novum spatium diffusionis *Ff* maius sit vel minus proposito *Ee.*

COROLLARIUM 2

70. Aperturam lentis *PP* ut indefinitam assumsi, manifestum autem est sufficere, dum eius semidiameter non sit minor quam $a\Phi$. Si enim esset minor, radii ex puncto *e* emissi plane non per lentem transitum invenirent, neque imago puncti *e* exprimeretur, sed imaginis *Ff* punctum extremum responderet imagini cuipiam intermediae spatii *Ee.*

COROLLARIUM 3

71. Si diameter obiecti seu potius imaginis in E sitae sit $= s$, tum imaginis inde per lentem PP in F formatae diameter erit $= \frac{\alpha(k+d)}{a(k-d)}$; quae expressio, si sit positiva, simul declarat situm imaginis in F esse inversum respectu eius, quae est in E.

SCHOLION

72. Cum in hoc capite plures lentes sim consideraturus, pro cuiuslibet determinatione spectandae sunt primo binae distantiae determinatrices, altera obiecti seu imaginis, a qua lens radios accipit ante lentem, altera imaginis inde per lentem repraesentatae post lentem: quae distantiae ex imaginibus principalibus aestimentur. Deinde cuiusque lentis crassities in computum est ducenda. Tertio cum his lens nondum prorsus determinetur, insuper pro quaque lente accedit distantia quaedam arbitraria hactenus per litteram k indicata. Quarto vero imprimis ratio haberi deberet spatii diffusionis, quod cuique lenti pro data apertura conveniat. Quemadmodum autem ex binis distantiis determinatricibus, crassitie lentis et quantitate illa arbitraria k cum binae lentis facies tum etiam spatium diffusionis pro apertura, cuius semidiameter ponitur $= x$, definiatur, in praecedente capite fusius est expositum. Hic igitur, cum plures lentes sint considerandae, pro singulis haec elementa sequentibus litteris indicabo:

Pro lente	Distantiae determinatrices		Crassities lentis	Quantitas arbitraria	Spatium diffusionis pro aperturae semidiametro x
	obiecti	imaginis			
prima	a	α	v	k	$P\alpha\alpha xx$
secunda	b	β	v'	k'	$Q\beta\beta xx$
tertia	c	γ	v''	k''	$R\gamma\gamma xx$
quarta	d	δ	v'''	k'''	$S\delta\delta xx$
quinta	e	ε	v''''	k''''	$T\varepsilon\varepsilon xx$

Sin autem ratio refractionis in singulis lentibus discrepet, pro prima lente eam ponamus $= n$, pro secunda $= n'$, pro tertia $= n''$ etc.

Prima autem lens hic mihi perpetuo erit ea, quae obiecto est proxima, indeque recedendo reliquas lentes ordine numero: ex quo simul habebuntur distantiae lentium, scilicet secundae a prima $= \alpha + b$, tertiae a secunda $= \beta + c$, quartae a tertia $= \gamma + d$, quintae a quarta $= \delta + e$ etc., quae distantiae ex sua natura debent esse positivae, etiamsi singulae distantiae determinatrices praeter primam a, quippe quae ad ipsum obiectum necessario ante lentem primam constituendum refertur, sint quandoque negativae. Crassitiem lentium hic littera v designo, quia littera d inter distantias determinatrices, si numerus lentium ultra ternarium assurgat, reperitur. Pro crassitie autem et quantitate arbitraria iisdem litteris utor, commatibus inscriptis distinguendis ob penuriam litterarum diversarum. Ceterum observandum est me omnes lentes tanquam super communi axe dispositas assumere.

PROBLEMA 2

73. *Si radii ab obiecto Es* (Fig. 5) *emissi per duas lentes PP et QQ transmittantur, definire spatium diffusionis Gg a data apertura primae lentis oriundum, ut et magnitudinem imaginis principalis in G exhibitae.*

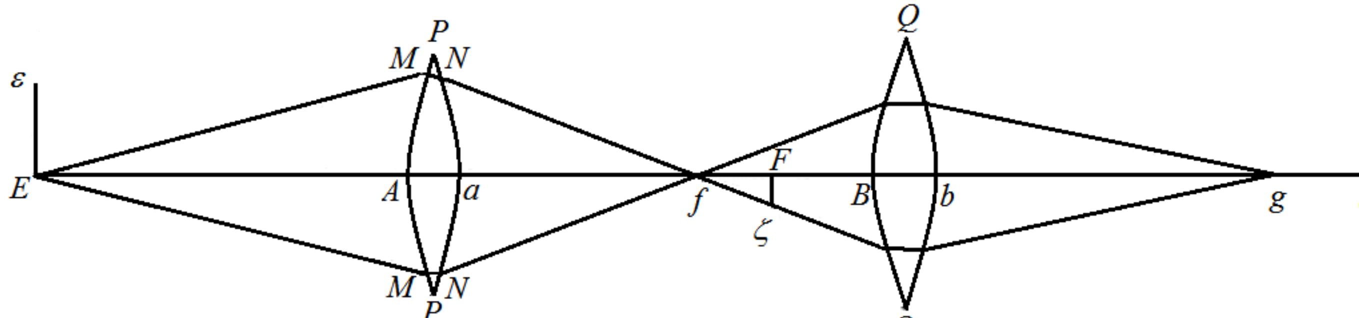

Fig. 5

SOLUTIO

Sit obiecti magnitudo $Ee = z$, eiusque imago principalis per primam lentem *PP* proiiciatur in $F\zeta$ at per ambas in $G\eta$, erunt distantiae determinatrices pro lente prima *PP* obiecti $EA = a$, imaginis $aF = \alpha$, pro lente secunda *QQ* obiecti $FB = b$, imaginis $bG = \beta$. Deinde sit

pro lente *PP* crassities $Aa = v$, quantitas arbitraria $= k$,
pro lente *QQ* crassities $Bb = v'$, quantitas arbitraria $= k'$.

Denique pro apertura in anteriori facie utriusque lentis, cuius semidiameter sit $= x$,

spatium diffusionis primae lentis $PP = P\alpha\alpha xx$
spatium diffusionis secundae lentis $QQ = Q\beta\beta xx$.

His positis pro imagine per primam lentem proiecta $F\zeta$ erit eius magnitudo $F\zeta = \frac{\alpha(k+v)}{a(k-v)}z$ pro inversa habenda, si haec expressio fuerit positiva.
Deinde si lentis primae *PP* semidiameter aperturae in anteriori facie ponatur $= x$, erit spatium diffusionis $Ff = P\alpha\alpha xx$ et inclinatio radiorum in *f* ad axem $= \frac{k-v}{k+v}\cdot\frac{x}{\alpha}$; tum vero in facie lentis *PP* posteriori semidiameter aperturae non minor esse debet quam $\frac{k-v}{k+v}x$.
Tota iam haec imago diffusa per spatium *Ff* respectu alterius lentis *QQ* tanquam obiectum spectari debet, cuius proinde repraesentatio *Gg* per propositionem praecedentem determinabitur.

Erit autem hic $\Phi=\frac{k-v}{k+v}\cdot\frac{x}{\alpha}$, et spatium ibi expressum $Ee=P\alpha\alpha xx$; tum vero pro *a, α, k, d* et *P* hic scribi oportet *b, β, k′, v′* et *Q,* unde fiet spatium diffusionis quaesitum:

$$Gg=\frac{\beta\beta}{bb}\left(\frac{k'+v'}{k'-v'}\right)^2P\alpha\alpha xx+\frac{bb\beta\beta}{\alpha\alpha}\left(\frac{k-v}{k+v}\right)^2Qxx$$

sive

$$Gg=\frac{\beta\beta}{bb}\left(\frac{k'+v'}{k'-v'}\right)^2P\alpha\alpha xx+\frac{bb}{\alpha\alpha}\left(\frac{k-v}{k+v}\right)^2Q\beta\beta xx.$$

Radiorum porro in *g* incidentium inclinatio ad axem est $\left(\frac{k-v}{k+v}\right)\left(\frac{k'-v'}{k'+v'}\right)\frac{bx}{\alpha\beta}$.

Denique cum sit $F\zeta=\frac{\alpha(k+v)}{a(k-v)}z$, erit magnitudo imaginis in *G* ut erutae consideratae:

$$G\eta=\frac{\alpha\beta}{ab}\left(\frac{k+v}{k-v}\right)\left(\frac{k'+v'}{k'-v'}\right)z.$$

COROLLARIUM 1

74. Quod ad aperturam facierum lentis *QQ* attinet, ea maior esse debet spatio transitus radiorum; hinc ergo erit semidiameter aperturae

faciei anterioris $>\left(\frac{k-v}{k+v}\right)\frac{bx}{a}$; faciei posterioris $>\left(\frac{k-v}{k+v}\right)\left(\frac{k'-v'}{k'+v'}\right)\frac{bx}{a}$.

COROLLARIUM 2

75. Si ponatur pro lente prima *PP* radius faciei anterioris $=f$ et posterioris $=g$, erit posito $n=\frac{31}{20}$

$$f=\frac{(n-1)a(k+v)}{k+v+2na}\ \text{ et }\ g=\frac{(n-1)\alpha(k-v)}{k-v-2n\alpha}.$$

Similique modo si pro lente altera *QQ* radius faciei anterioris ponatur $=f'$ et posterioris $=g'$, erit

$$f'=\frac{(n-1)b(k'+v')}{k'+v'+2nb}\ \text{ et }\ g'=\frac{(n-1)\beta(k'-v')}{k'-v'-2n\beta},$$

omnibus scilicet faciebus ut convexis consideratis.

COROLLARIUM 3

76. Pro spatio autem diffusionis ab utraque lente productae erit, quemadmodum invenimus:

$$P=\frac{n}{2(n-1)^2}\left\{\left(\frac{k+v}{k-v}\right)^2\left(\frac{n}{a}+\frac{2}{k+v}\right)\left(\frac{1}{a}+\frac{2}{k+v}\right)^2+\left(\frac{k-v}{k+v}\right)^2\left(\frac{n}{\alpha}-\frac{2}{k-v}\right)\left(\frac{1}{\alpha}-\frac{2}{k-v}\right)^2\right\}$$

similique modo

$$Q=\frac{n}{2(n-1)^2}\left\{\left(\frac{k'+v'}{k'-v'}\right)^2\left(\frac{n}{b}+\frac{2}{k'+v'}\right)\left(\frac{1}{b}+\frac{2}{k'+v'}\right)^2+\left(\frac{k'-v'}{k'+v'}\right)^2\left(\frac{n}{\beta}-\frac{2}{k'-v'}\right)\left(\frac{1}{\beta}-\frac{2}{k'-v'}\right)^2\right\}.$$

Non opus est, ut omnibus lentibus eadem refractionis ratio $n:1$ tribuatur, sed simili modo pro pluribus lentibus poni potest n, n', n'', n''' etc., ut iam supra monuimus, unde nullum aliud discrimen nascitur, nisi quod in formulis pro f' et g' inventis loco n scribi debeat n' et Q statui debeat

$$=\frac{n'}{2(n'-1)^2}\left\{\left(\frac{k'+v'}{k'-v'}\right)^2\left(\frac{n'}{b}+\frac{2}{k'+v'}\right)\left(\frac{1}{b}+\frac{2}{k'+v'}\right)^2+\left(\frac{k'-v'}{k'+v'}\right)^2\left(\frac{n'}{\beta}-\frac{2}{k'-v'}\right)\left(\frac{1}{\beta}-\frac{2}{k'-v'}\right)^2\right\}.$$

COROLLARIUM 4

77. Sin antem crassities lentium evanescat et pro quantitate arbitraria introducamus numerum λ, λ', erit

pro lente PP

$$f=\frac{a\alpha}{\rho\alpha+\sigma\alpha\pm\tau(a+\alpha)\sqrt{(\lambda-1)}},\quad g=\frac{a\alpha}{\rho\alpha+\sigma\alpha\mp\tau(a+\alpha)\sqrt{(\lambda-1)}}$$

$$P=\mu\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{v}{\alpha\alpha}\right)$$

at pro lente QQ

$$f'=\frac{b\beta}{\rho\beta+\sigma b\pm\tau(b+\beta)\sqrt{(\lambda'-1)}},\quad g'=\frac{b\beta}{\rho b+\sigma\beta\mp\tau(b+\beta)\sqrt{(\lambda'-1)}}$$

$$Q=\mu\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\lambda'\left(\frac{1}{b}+\frac{1}{\beta}\right)^2+\frac{v}{b\beta}\right).$$

In expressionibus autem inventis formulae $\left(\frac{k-v}{k+v}\right)$ et $\left(\frac{k'-v'}{k'+v'}\right)$ abeunt in unitatem.

Numerorum vero $\mu, v, \rho, \sigma, \tau$ indoles in § 55 exposita est.

Si refractio lentium differat, etiam litterae $\mu, v, \rho, \sigma, \tau$ diversos valores pro singulis lentibus sortientur; quae litterae si etiam commatibus a prioribus distinguantur, ut sit

$$\rho'=\frac{1}{2(n'-1)}+\frac{1}{n'+2}-1,\quad \sigma'=1+\frac{1}{2(n'-1)}-\frac{1}{n'+2},$$

$$\tau'=\frac{1}{3}\left(\frac{1}{2(n'-1)}+\frac{1}{n'+2}\right)\sqrt{(4n'-1)},$$

pro secunda lente erit

$$f' = \frac{b\beta}{\rho'\beta + \sigma' b \pm \tau'(b+\beta)\sqrt{(\lambda'-1)}}, \quad g' = \frac{b\beta}{\rho' b + \sigma'\beta \mp \tau'(b+\beta)\sqrt{(\lambda'-1)}},$$

$$Q = \mu\left(\tfrac{1}{b} + \tfrac{1}{\beta}\right)\left(\lambda'\left(\tfrac{1}{b} + \tfrac{1}{\beta}\right)^2 + \tfrac{\nu}{b\beta}\right).$$

et ob eandem rationem erit

$$\mu' = \frac{1}{4(n'+2)} + \frac{1}{4(n'-1)} + \frac{1}{8(n'-1)^2} \text{ et } \nu' = \frac{4(n'-1)^2}{4n'-1},$$

$$\tau' = \left(\frac{1}{2(n'-1)} + \frac{1}{n'+2}\right)\sqrt{(4n'-1)},$$

quod et de sequentibus lentibus est intelligendum, si forte diversa refractionis lege fuerint praeditae.

SCHOLION

78. Quo formulas in problemate inventas magis contrahamus, ut, cum ad plures lentes processerimus, succinctiores evadant, ponamus

$$\tfrac{k-v}{k+v} = i \quad \text{et} \quad \tfrac{k'-v'}{k'+v'} = i'$$

ita ut hi numeri i et i' abeant in unitatem evanescente lentium crassitie v et v'. Tum autem erit spatium diffusionis

$$Gg = \tfrac{1}{i'i'} \cdot \tfrac{\beta\beta}{bb} \cdot P\alpha\alpha xx + ii \cdot \tfrac{bb}{\alpha\alpha} \cdot Q\beta\beta xx.$$

et radiorum in g incidentium inclinatio ad axem $= ii' \cdot \frac{bx}{\alpha\beta}$; porro magnitudo imaginis $G\eta = \frac{1}{ii'} \cdot \frac{\alpha\beta}{ab} z$. Ac pro apertura singularum facierum erit, ut sequitur:

Semidiameter aperturae lentis	faciei anterioris	faciei posterioris
primae PP	x	ix
secundae QQ	$i.\frac{bx}{\alpha}$	$ii'.\frac{bx}{\alpha}$

aperturae autem hae praeter primam maiores esse debent assignatis: valores enim assignati sufficerent, si obiectum esset merum punctum in axe positum; quando autem habet magnitudinem, radii ab eius terminis per primam faciem ingressi latius divagantur et ad sequentes facies maiorem aperturam exigunt.

PROBLEMA 3

79. *Si radii ob obiecto Ee* (Fig. 6) *emissi per tres lentes PP, QQ et RR refringantur, definire spatium diffusionis m ob datam aperturam lentis primae oriundum, ut et magnitudinem imaginis principalis in H exhibitae.*

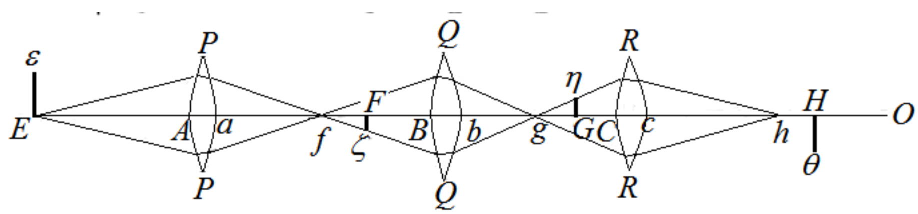

Fig. 6

SOLUTIO

Posita obiecti magnitudine $E\varepsilon = z$ cadat eius imago principalis per lentem primam PP in $F\zeta$, dehinc per secundam QQ in $G\eta$, tum vero per tertiam RR in $H\theta$. Erunt ergo distantiae determinatrices

pro lente PP obiecti $EA = a$, imaginis $aF = \alpha$,
pro lente QQ obiecti $FB = b$, imaginis $bG = \beta$,
pro lente BR obiecti $GC = c$, imaginis $cH = \gamma$.

Deinde sit

pro lente PP crassities $Aa = v$, quantitas arbitraria $= k$,
pro lente QQ crassities $Bb = v'$, quantitas arbitraria $= k'$,
pro lente RR crassities $Cc = v''$, quantitas arbitraria $= k''$,

ac ponatur brevitatis gratia:

$$\frac{k-v}{k+v} = i, \quad \frac{k'-v'}{k'+v'} = i', \quad \frac{k''-v''}{k''+v''} = i''.$$

Denique pro quavis lente, si esset singularis eiusque aperturae semidiameter esset $= x$, sit

spatium diffusionis primae lentis $PP = P\alpha\alpha xx$
" " secundae lentis $QQ = Q\beta\beta xx$
" " tertiae lentis $RR = R\gamma\gamma xx.$

Iam in praecedente problemate invenimus fore spatium diffusionis per duas lentes priores productum

$$Gg = \frac{1}{i'i'} \cdot \frac{\beta\beta}{bb} \cdot P\alpha\alpha xx + ii \cdot \frac{bb}{\alpha\alpha} \cdot Q\beta\beta xx.$$

et radiorum in g concurrentium inclinationem ad axem $= ii' \cdot \frac{bx}{\alpha\beta}$, imaginisque in G magnitudinem $G\eta = \frac{1}{ii'} \cdot \frac{\alpha\beta}{ab} z.$ Haec igitur si ad problema 1 transferantur, ibi loco

$$Ee,\ \frac{\alpha\alpha}{aa},\ \left(\frac{k+d}{k-d}\right)^2,\ Paaaa \text{ et } \Phi \text{ scribi debebit}$$
$$Gg,\ \frac{\gamma\gamma}{cc},\ \frac{1}{i''i''},\ Rcc\gamma\gamma \text{ et } ii'.\frac{bx}{\alpha\beta},$$

hincque spatium diffusionis par tres lentes productum orietur:

$$Hh = \frac{1}{i'i' \cdot i''i''} \cdot \frac{\beta\beta\gamma\gamma}{bbcc} \cdot P\alpha\alpha xx + \frac{ii}{i''i''} \cdot \frac{bb\gamma\gamma}{\alpha\alpha cc} \cdot Q\beta\beta xx + ii \cdot i'i' \cdot \frac{bbcc}{\alpha\alpha\beta\beta} \cdot R\gamma\gamma xx;$$

at radiorum in h concurrentium inclinatio ad axem erit $= ii'i'' \cdot \frac{bcx}{\alpha\beta\gamma}$. Denique imaginis in H formatae magnitudo erit $H\theta = \frac{1}{ii'i''} \cdot \frac{\alpha\beta\gamma}{abc} z$ ad situm inversum relata.

COROLLARIUM 1

80. Quod ad aperturas singularum facierum attinet, eas sequenti modo comparatas esse convenit:

Semidiameter aperturae lentis	faciei anterioris	faciei posterioris
primae PP	x	ix
secundae QQ	$i.\frac{bx}{\alpha}$	$ii'.\frac{bx}{\alpha}$
tertia RR	$ii'.\frac{bcx}{\alpha\beta}$	$ii'i''.\frac{bcx}{\alpha\beta}$

his scilicet valoribus non debent esse minores.

COROLLARIUM 2

81. Si inclinatio radiorum in h concurrentium ad axem vocetur $= \Phi$, cum $\Phi = ii'i''.\frac{bcx}{\alpha\beta\gamma}$ et $H\theta = \frac{1}{ii'i''}.\frac{\alpha\beta\gamma}{abc}$; erit $\Phi \cdot H\theta = \frac{xz}{a}$; quae proprietas pro lentium numero quantumvis magno locum habet.

COROLLARIUM 3

82. Quemadmodum radii singularum facierum determinandi sint, ex praecedentibus facile liquet:

Erit nempe	Radius faciei	
pro	anterioris	posterioris
lente prima *PP*	$\frac{(n-1)a(k+v)}{k+v+2na}$	$\frac{(n-1)\alpha(k-v)}{k-v-2n\alpha}$
lente secunda *QQ*	$\frac{(n-1)b(k'+v')}{k'+v'+2nb}$	$\frac{(n-1)\beta(k'-v')}{k'-v'-2n\beta}$
lente tertia *RR*	$\frac{(n-1)c(k''+v'')}{k''+v''+2nc}$	$\frac{(n-1)\gamma(k''-v'')}{k''-v''-2n\gamma}$

existente pro vitri $n=\frac{31}{20}$.

COROLLARIUM 4

83. Tum vero valores literarum *P, Q, R* ita se habebunt

$$P=\frac{n}{2(n-1)^2}\left\{\frac{1}{ii}\left(\frac{n}{a}+\frac{2}{k+v}\right)\left(\frac{1}{a}+\frac{2}{k+v}\right)^2+ii\left(\frac{n}{\alpha}-\frac{2}{k-v}\right)\left(\frac{1}{\alpha}-\frac{2}{k-v}\right)^2\right\}$$

$$Q=\frac{n}{2(n-1)^2}\left\{\frac{1}{i'i'}\left(\frac{n}{b}+\frac{2}{k'+v'}\right)\left(\frac{1}{b}+\frac{2}{k'+v'}\right)^2+i'i'\left(\frac{n}{\beta}-\frac{2}{k'-v'}\right)\left(\frac{1}{\beta}-\frac{2}{k'-v'}\right)^2\right\}$$

$$R=\frac{n}{2(n-1)^2}\left\{\frac{1}{i''i''}\left(\frac{n}{c}+\frac{2}{k''+v''}\right)\left(\frac{1}{c}+\frac{2}{k''+v''}\right)^2+i''i''\left(\frac{n}{\gamma}-\frac{2}{k''-v''}\right)\left(\frac{1}{\gamma}-\frac{2}{k''-v''}\right)^2\right\}$$

quibus valoribus spatia diffusionis definiuntur.
Si refractio differat:

litteris	*P*	*Q*	*R*
tribuatur refractio	n	n'	n''.

COROLLARIUM 5

84. Iuvabit etiam spatia diffusionis, prout per unam, duas ac tres lentes producuntur, inter se comparasse, quod ita commodissime fieri videtur:

$$Ff=\alpha\alpha xx\cdot P,\quad Gg=\beta\beta xx\left\{\frac{1}{i'i'}\cdot\frac{\alpha\alpha}{bb}\cdot P+ii\cdot\frac{bb}{\alpha\alpha}\cdot Q\right\}$$

$$Hh=\gamma\gamma xx\left\{\frac{1}{i'i'\cdot i''i''}\cdot\frac{\alpha\alpha\beta\beta}{bbcc}\cdot P+\frac{ii}{i''i''}\cdot\frac{bb\beta\beta}{\alpha\alpha cc}\cdot Q+ii\cdot i'i'\cdot\frac{bbcc}{\alpha\alpha\beta\beta}\cdot R\right\}.$$

SCHOLlON

85. Hinc facile perspicitur, quomodo determinatio spatii diffusionis ad plures lentes extendi debeat; unde problema generale statim ad numerum lentium quemcunque accommodabo, idque pro quacunque lentium crassitie.

PROBLEMA 4

86. *Si radii ab obiecto Ee emissi per lentes quotcunque PP, QQ, RR, SS etc. super communi axe dispositas refringantur, definire spatium diffusionis a data apertura lentis primae oriundum, ut et magnitudinem imaginis repraesentatae.*

SOLUTIO

Posita obiecti magnitudine $E\varepsilon = z$ imagines eius principales contemplemur: cadat igitur eius imago principalis per lentem primam *PP* in $F\zeta$, deinde per secundam *QQ* in $G\eta$, tum per tertiam *RR* in $H\theta$, porro per quartam *SS* in $I\iota$, per quintaro *TT* in $K\chi$ etc. Hinc pro singulis lentibus habebimus distantias determinatrices, quas ita litteris indicerons :

Pro lente *PP* obiecti $EA = a$, imaginis $aF = \alpha$, crassitiem $Aa = v$,
pro lente *QQ* obiecti $FB = b$, imaginis $bG = \beta$, crassitiem $Bb = v'$,
pro lente *RR* obiecti $GC = c$, imaginis $cH = \gamma$, crassitiem $Cc = v''$,
pro lente *SS* obiecti $HD = d$, imaginis $dI = \delta$, crassitiem $Dd = v'''$,
pro lente *TT* obiecti $IE = e$, imaginis $eK = \varepsilon$, crassitiem $Ee = v''''$
etc.

Deinde cum determinatio cuiusvis lentis non solum has distantias determinatrices cum crassitie cuiusque sed etiam quantitatem quandam arbitrariam involvat, a qua spatium diffusionis cuiusque pendet, ponamus, si quaelibet lens esset solitaria eiusque aperturae semidiameter $= x$:

Pro lente Spatium	Quant. arbitr.	diffussionis
prima *PP*	k	$P\alpha\alpha xx$
secunda *QQ*	k'	$Q\beta\beta xx$
tertia *RR*	k''	$R\gamma\gamma xx$
quarta *SS*	k'''	$S\delta\delta xx$
quinta *TT*	k''''	$T\varepsilon\varepsilon xx$
	etc.	

Hinc ergo constructio cuiusque lentis ita se habebit posito $n=\frac{31}{20}$; vel si refractio differat, cuique lenti suus tribuatur valor, primae *n*, secundae *n'*, tertiae *n''* etc.

Erit nempe	Radius faciei	
pro	anterioris	posterioris
lente prima *PP*	$\frac{(n-1)a(k+v)}{k+v+2na}$	$\frac{(n-1)\alpha(k-v)}{k-v-2n\alpha}$
lente secunda *QQ*	$\frac{(n-1)b(k'+v')}{k'+v'+2nb}$	$\frac{(n-1)\beta(k'-v')}{k'-v'-2n\beta}$
lente tertia *RR*	$\frac{(n-1)c(k''+v'')}{k''+v''+2nc}$	$\frac{(n-1)\gamma(k''-v'')}{k''-v''-2n\gamma}$
lente quarta *SS*	$\frac{(n-1)d(k'''+v''')}{k'''+v'''+2nd}$	$\frac{(n-1)\delta(k'''-v''')}{k'''-v'''-2n\delta}$
lente quinta *TT*	$\frac{(n-1)e(k''''+v'''')}{k''''+v''''+2ne}$	$\frac{(n-1)\varepsilon(k''''-v'''')}{k''''-v''''-2n\varepsilon}$

at si ponamus brevitatis gratia:

$$\frac{k-v}{k+v}=i,\quad \frac{k'-v'}{k'+v'}=i',\quad \frac{k''-v''}{k''+v''}=i'',\quad \frac{k'''-v'''}{k'''+v'''}=i''',\quad \frac{k''''-v''''}{k''''+v''''}=i'''' \text{ etc.,}$$

pro spatiis diffusionis habebimus hos valores:

$$P=\frac{n}{2(n-1)^2}\left\{\frac{1}{ii}\left(\frac{n}{a}+\frac{2}{k+v}\right)\left(\frac{1}{a}+\frac{2}{k+v}\right)^2+ii\left(\frac{n}{\alpha}-\frac{2}{k-v}\right)\left(\frac{1}{\alpha}-\frac{2}{k-v}\right)^2\right\}$$

$$Q=\frac{n}{2(n-1)^2}\left\{\frac{1}{i'i'}\left(\frac{n}{b}+\frac{2}{k'+v'}\right)\left(\frac{1}{b}+\frac{2}{k'+v'}\right)^2+i'i'\left(\frac{n}{\beta}-\frac{2}{k'-v'}\right)\left(\frac{1}{\beta}-\frac{2}{k'-v'}\right)^2\right\}$$

$$R=\frac{n}{2(n-1)^2}\left\{\frac{1}{i''i''}\left(\frac{n}{c}+\frac{2}{k''+v''}\right)\left(\frac{1}{c}+\frac{2}{k''+v''}\right)^2+i''i''\left(\frac{n}{\gamma}-\frac{2}{k''-v''}\right)\left(\frac{1}{\gamma}-\frac{2}{k''-v''}\right)^2\right\}$$

$$S=\frac{n}{2(n-1)^2}\left\{\frac{1}{i'''i'''}\left(\frac{n}{d}+\frac{2}{k'''+v'''}\right)\left(\frac{1}{d}+\frac{2}{k'''+v'''}\right)^2+i'''i'''\left(\frac{n}{\delta}-\frac{2}{k'''-v'''}\right)\left(\frac{1}{\delta}-\frac{2}{k'''-v'''}\right)^2\right\}$$

$$T=\frac{n}{2(n-1)^2}\left\{\frac{1}{i''''i''''}\left(\frac{n}{e}+\frac{2}{k''''+v''''}\right)\left(\frac{1}{e}+\frac{2}{k''''+v''''}\right)^2+i''''i''''\left(\frac{n}{\varepsilon}-\frac{2}{k''''-v''''}\right)\left(\frac{1}{\varepsilon}-\frac{2}{k''''-v''''}\right)^2\right\}$$

etc.

His constitutis pro magnitudine singularum imaginum habebimus:

pro una lente imaginem	$F\zeta=\frac{1}{i}\cdot\frac{\alpha}{a}z$	ad situm inversum
pro duabus lentibus	$G\eta=\frac{1}{ii'}\cdot\frac{\alpha\beta}{ab}z$	ad situm erectum
pro tribus lentibus	$H\theta=\frac{1}{ii'i''}\cdot\frac{\alpha\beta\gamma}{abc}z$	ad situm inversum

pro quatuor lentibus	$Ii = \frac{1}{ii'i''i'''} \cdot \frac{\alpha\beta\gamma\delta}{abcd} z$	ad situm erectum
pro quinque lentibus	$K\chi = \frac{1}{ii'i''i'''i''''} \cdot \frac{\alpha\beta\gamma\delta\varepsilon}{abcde} z$	ad situm inversum

Denique dum aperturae lentium non sint minores, quam sequentes formulae exhibent:

Semidiameter aperturae lentis	faciei anterioris	faciei posterioris
primae PP	x	$i \cdot x$
secundae QQ	$i \cdot \frac{bx}{\alpha}$	$ii' \cdot \frac{bx}{\alpha}$
tertiae RR	$ii' \cdot \frac{bcx}{\alpha\beta}$	$ii'i'' \cdot \frac{bcx}{\alpha\beta}$
quartae SS	$ii'i'' \cdot \frac{bcdx}{\alpha\beta\gamma}$	$ii'i''i''' \cdot \frac{bcdx}{\alpha\beta\gamma}$
quintae TT	$ii'i''i''' \cdot \frac{bcdex}{\alpha\beta\gamma\delta}$	$ii'i''i'''i'''' \cdot \frac{bcdex}{\alpha\beta\gamma\delta}$

etc.,

erit, ut sequitur, pro quolibet lentium numero:

I. Pro una lente

spatium diffusionis: $Ff = \alpha\alpha xx \cdot P$,

inclinatio radiorum in f concurrentium ad axem $= i \cdot \frac{x}{\alpha}$.

II. Pro duabus lentibus

spatium diffusionis: $Gg = \beta\beta xx\left(\frac{1}{i'i'} \cdot \frac{\alpha\alpha}{bb} \cdot P + ii \cdot \frac{bb}{\alpha\alpha} \cdot Q\right)$

et radiorum in g concurrentium inclinatio ad axem $= ii' \cdot \frac{bx}{\alpha\beta}$.

III. Pro tribus lentibus

spatium diffusionis:

$$Hh = \gamma\gamma xx\left(\frac{1}{i'i' \cdot i''i''} \cdot \frac{\alpha\alpha\beta\beta}{bbcc} \cdot P + \frac{ii}{i''i''} \cdot \frac{bb\beta\beta}{\alpha\alpha cc} \cdot Q + ii \cdot i''i'' \cdot \frac{bbcc}{\alpha\alpha\beta\beta} \cdot R\right),$$

et radiorum in h concurrentium inclinatio ad axem $= ii'i'' \cdot \frac{bcx}{\alpha\beta\gamma}$.

IV. Pro quatuor lentibus

spatium diffusionis:

$$Ii = \delta\delta xx\left\{\begin{array}{l} \frac{1}{i'i' \cdot i''i'' \cdot i'''i'''} \cdot \frac{\alpha\alpha\beta\beta\gamma\gamma}{bbccdd} \cdot P + \frac{ii}{i''i'' \cdot i'''i'''} \cdot \frac{bb\beta\beta\gamma\gamma}{\alpha\alpha ccdd} \cdot Q \\ + \frac{ii \cdot i'i'}{i'''i'''} \cdot \frac{bbcc\gamma\gamma}{\alpha\alpha\beta\beta dd} \cdot R + ii \cdot i'i' \cdot i'''i''' \cdot \frac{bbccdd}{\alpha\alpha\beta\beta\gamma\gamma} \cdot S \end{array}\right\}$$

et radiorum in *i* concurrentium inclinatio ad axem $= ii'i''i''' \cdot \frac{bcdx}{\alpha\beta\gamma\delta}$.

V. Pro quinque lentibus

spatium diffusionis:

$$Kk = \varepsilon\varepsilon xx \left\{ \begin{array}{l} \frac{1}{i'i' \cdot i''i'' \cdot i'''i''' \cdot i''''i''''} \cdot \frac{\alpha\alpha\beta\beta\gamma\gamma\delta\delta}{bbccddee} \cdot P + \frac{ii}{i''i'' \cdot i'''i''' \cdot i''''i''''} \cdot \frac{bb\beta\beta\gamma\gamma\delta\delta}{\alpha\alpha ccddee} \cdot Q \\ + \frac{ii \cdot i'i'}{i'''i''' \cdot i''''i''''} \cdot \frac{bbcc\gamma\gamma\delta\delta}{\alpha\alpha\beta\beta ddee} \cdot R \qquad + \frac{ii \cdot i'i' \cdot i'''i'''}{i''''i''''} \cdot \frac{bbccdd\delta\delta}{\alpha\alpha\beta\beta\gamma\gamma ee} \cdot S \\ ii \cdot i'i' \cdot i'''i''' \cdot i'''i''' \cdot \frac{bbccddee}{\alpha\alpha\beta\beta\gamma\gamma\delta\delta} \cdot T \end{array} \right\}$$

et radiorum in *k* concurrent inclinatio ad axem $= ii'i''i'''i'''' \cdot \frac{bcdex}{\alpha\beta\gamma\delta\varepsilon}$.

Unde progressio harum formularum ad plures adhuc lentes satis est manifesta. Si in lentibus ratio refractionis sit diversa atque ad singulas lentes ordine his litteris indicetur n, n', n'', n''' etc., haec diversitas facile ad formulas hic inventas accommodabitur. Primum enim haec correctio occurrit in formulis pro radiis lentium, ita ut, quemadmodum formulae f et g pro prima lente numerum n involvunt, ita pro secunda lente numerus n', pro tertia n'' et ita porro introducatur. Similem correctionem etiam requirunt valores litterarum P, Q, R, S etc., et loco litterae n, quae in valore P occurrit, in valoribus Q, R, S etc. scribi oportet n', n'', n''' etc.

COROLLARIUM 1

87. Si obiectum sit tantum punctum in axe positum sufficit, ut lentium aperturae sint tantae, quantas assignavimus; sin autem obiectum habeat quandam magnitudinem, tum aperturae praeter primam eo magis mensuras assignatas superare debent, quo maior fuerit obiecti magnitudo z.

COROLLARIUM 2

88. In expressione spatii diffusionis quadratum semidiametri aperturae primae faciei xx primo multiplicatur per quadratum distantiae postremae imaginis ab ultima lentium: quae ergo si fuerit infinita, etiam spatium diffusionis fit infinitum.

COROLLARIUM 3

89. Ceteris ergo paribus, quotcunque fuerint lentes, spatium diffusionis semper est proportionale quadrato diametri aperturae primae faciei, hoc est ipsi huic aperturae. Unde diametro aperturae primae faciei ad semissem redacto, spatium diffusionis quadruplo fiet minus.

SCHOLION

90. Consideravimus hic statim loca singularum imaginum principalium tanquam data, ex iisque structuram cuiusque lentis quantitatem arbitrariam introducendo determinavimus. Quod si vero ipsae lentes fuerint datae, ita ut tam radii ambarum facierum cuiusque quam crassities una cum earum intervallis cognoscantur, tum ope formularum exhibitarum vicissim distantiae determinatrices innotescunt. Sint scilicet radii facierum anterioris et posterioris primae lentis *PP* f, g, secundae lentis *QQ* f', g', tertiae lentis *RR* f'',g'' etc., crassitie earum existante v, v', v'' etc., tum vero dentur distantiae $aB = F$; $bC = G$; $cD = H$ etc . Praeterea autem distantia obiecti ante lentem primam sit $AE = a$, ac sequenti modo omnia elementa ad problema superius necessaria elicientur:

1. $\frac{n-1}{f} = \frac{1}{a} + \frac{2n}{k+v}$, hinc reperitur k ; 2. $\frac{n-1}{g} = \frac{1}{\alpha} - \frac{2n}{k-v}$, hinc vero α ;

3. $F = \alpha + b$, unde $b = F - \alpha$;

4. $\frac{n-1}{f'} = \frac{1}{b} + \frac{2n}{k'+v'}$, hinc reperitur k' ; 5. $\frac{n-1}{g'} = \frac{1}{\beta} - \frac{2n}{k'-v'}$, hinc vero porro β;

6. $G = \beta + c$, unde $c = G - \beta$;

7. $\frac{n-1}{f''} = \frac{1}{c} + \frac{2n}{k''+v''}$, hinc reperitur k'' ; 8. $\frac{n-1}{g''} = \frac{1}{\gamma} - \frac{2n}{k''-v''}$, hinc vero γ;

9. $H = \gamma + d$, unde $d = \gamma - H$,

etc.

Utcunque ergo lentes datae fuerint dispositae super axe communi, si ante eas constituatur obiectum in data distantia $AE = a$, inde singulae distantiae determinatrices α, b, β, c, γ etc. cum arbitrariis k, k', k'', k''' etc. facile definiuntur, ex iisque porro spatium diffusionis cum reliquis phaenomenis in solutione problematis commemoratis assignabitur. Operae pretium autem erit casum, quo crassities lentium ut evanescens spectatur, accuratius evolvisse.

PROBLEMA 5

91. *Si crassities lentium evanescat et quotcunque huiusmodi lentes super communi axe fuerint dispositae, ante quas existat obiectum Eε, definire spatium diffusionis, per quod imago erit dissipata, ut et magnituitinem imaginis.*

SOLUTIO

Sit obiecti magnitudo $E\varepsilon = z$, cuius imagines principales successive cadant in $F\zeta$, $G\eta$, $H\theta$, Ii, $K\chi$ etc., hincque pro singulis lentibus sequentes habebimus distantias determinatrices, cum imago per quamvis lentem repraesentata respectu lentis sequentis vicem obiecti gerat:

Pro lente *PP* distantia obiecti $EA = a$, distantia imaginis $aF = \alpha$,

pro lente *QQ* distantia obiecti $FB = b$, distantia imaginis $bG = \beta$,

pro lente *RR* distantia obiecti $GC = c$, distantia imaginis $cH = \gamma$,
pro lente *SB* distantia obiecti $HD = d$, distantia imaginis $dI = \delta$,
pro lente *TT* distantia obiecti $IE = e$, distantia imaginis $eK = \varepsilon$
etc.

Porro autem sint numeri arbitrarii unitate maiores cuiusque lentis figuram determinantes, λ pro lente *PP,* λ' pro *QQ,* λ'' pro *RR,* λ''' pro *SS,* λ'''' pro *TT* etc.,
ita ut ponendo brevitatis gratia

$$\rho = 0{,}190781,\ \sigma = 1{,}627401,\ \tau = 0{,}905133$$

$$\text{pro lente } PP \text{ radius faciei } \begin{cases} \text{anterioris} = \dfrac{a\alpha}{\rho\alpha+\sigma a\pm\tau(a+\alpha)\sqrt{(\lambda-1)}} \\ \text{posterioris} = \dfrac{a\alpha}{\rho\alpha+\sigma a\mp\tau(a+\alpha)\sqrt{(\lambda-1)}} \end{cases}$$

$$\text{pro lente } QQ \text{ radius faciei } \begin{cases} \text{anterioris} = \dfrac{b\beta}{\rho\beta+\sigma b\pm\tau(b+\beta)\sqrt{(\lambda'-1)}} \\ \text{posterioris} = \dfrac{b\beta}{\rho b+\sigma\beta\mp\tau(b+\beta)\sqrt{(\lambda'-1)}} \end{cases}$$

$$\text{pro lente } RR \text{ radius faciei } \begin{cases} \text{anterioris} = \dfrac{c\gamma}{\rho\gamma+\sigma c\pm\tau(c+\gamma)\sqrt{(\lambda''-1)}} \\ \text{posterioris} = \dfrac{c\gamma}{\rho c+\sigma\gamma\mp\tau(c+\gamma)\sqrt{(\lambda''-1)}} \end{cases}$$

$$\text{pro lente } SS \text{ radius faciei } \begin{cases} \text{anterioris} = \dfrac{d\delta}{\rho\delta+\sigma d\pm\tau(d+\delta)\sqrt{(\lambda'''-1)}} \\ \text{posterioris} = \dfrac{d\delta}{\rho d+\sigma\delta\mp\tau(d+\delta)\sqrt{(\lambda'''-1)}} \end{cases}$$

$$\text{pro lente } TT \text{ radius faciei } \begin{cases} \text{anterioris} = \dfrac{e\varepsilon}{\rho\varepsilon+\sigma e\pm\tau(e+\varepsilon)\sqrt{(\lambda''''-1)}} \\ \text{posterioris} = \dfrac{e\varepsilon}{\rho e+\sigma\varepsilon\mp\tau(e+\varepsilon)\sqrt{(\lambda''''-1)}} \end{cases}$$

etc.

Deinde si quaelibet lens cum binis suis distantiis determinatricibus seorsim consideretur eiusque aperturae semidiameter foret $= x$, posito $\mu = 0{,}938191$ et $\nu = 0{,}232692$ esset spatium diffusionis

$$\text{lentis } PP = \mu\alpha\alpha xx\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)\left(\lambda\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)^2+\tfrac{\nu}{a\alpha}\right)$$

$$\text{lentis } QQ = \mu\beta\beta xx\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)\left(\lambda'\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)^2+\tfrac{\nu}{b\beta}\right)$$

$$\text{lentis } RR = \mu\gamma\gamma xx\left(\tfrac{1}{c}+\tfrac{1}{\gamma}\right)\left(\lambda''\left(\tfrac{1}{c}+\tfrac{1}{\gamma}\right)^2+\tfrac{\nu}{c\gamma}\right)$$

$$\text{lentis } SS = \mu\delta\delta xx\left(\tfrac{1}{d}+\tfrac{1}{\delta}\right)\left(\lambda'''\left(\tfrac{1}{d}+\tfrac{1}{\delta}\right)^2+\tfrac{\nu}{d\delta}\right)$$

$$\text{lentis } TT = \mu\varepsilon\varepsilon xx\left(\tfrac{1}{e}+\tfrac{1}{\varepsilon}\right)\left(\lambda''''\left(\tfrac{1}{e}+\tfrac{1}{\varepsilon}\right)^2+\tfrac{\nu}{e\varepsilon}\right)$$

etc.

His constitutis pro magnitudine singularum imaginum habebimus

pro una lente $F\zeta = \frac{\alpha}{a}z$ situ inverso,

pro duabus lentibus $G\eta = \frac{\alpha\beta}{ab}z$ situ erecto,

pro tribus lentibus $H\theta = \frac{\alpha\beta\gamma}{abc}z$ situ inverso,

pro quatuor lentibus $Ii = \frac{\alpha\beta\gamma\delta}{abcd}z$ situ erecto,

pro quinque lentibus $K\chi = \frac{\alpha\beta\gamma\delta\varepsilon}{abcde}z$ situ inverso

etc.

At si semidiameter aperturae primae lentis PP ponatur $= x$, necesse est, ut reliquarum lentium aperturae superent sequentes valores:

Semidiameter aperturae

lentis secundae $QQ > \frac{b}{\alpha}x$, lentis tertiae $RR > \frac{bc}{\alpha\beta}x$,

lentis quartae $SS > \frac{bcd}{\alpha\beta\gamma}x$, lentis quinqae $TT > \frac{bcde}{\alpha\beta\gamma\delta}x$

etc.

Hinc spatium diffusionis pro quolibet lentium numero ita se habebit:

I. Pro una lente

$$\text{spatium diffusionis: } Ff = \mu\alpha\alpha xx\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)\left(\lambda\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)^2+\tfrac{\nu}{a\alpha}\right)$$

radiorum in f concurrentium inclinatio ad axem $= \frac{x}{\alpha}$.

II. Pro duabus lentibus

spatium diffusionis:

$$Gg=\mu\beta\beta xx\left\{\begin{array}{l}+\frac{\alpha\alpha}{bb}\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^{2}+\frac{\nu}{a\alpha}\right)\\+\frac{bb}{\alpha\alpha}\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\lambda'\left(\frac{1}{b}+\frac{1}{\beta}\right)^{2}+\frac{\nu}{b\beta}\right)\end{array}\right\}$$

et radiorum in g concurrentium inclinatio ad axem $=\frac{bx}{\alpha\beta}$.

III. Pro tribus lentibus

spatium diffusionis:

$$Hh=\mu\gamma\gamma xx\left\{\begin{array}{l}+\frac{\alpha\alpha\beta\beta}{bbcc}\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^{2}+\frac{\nu}{a\alpha}\right)\\+\frac{bb\beta\beta}{\alpha\alpha cc}\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\lambda'\left(\frac{1}{b}+\frac{1}{\beta}\right)^{2}+\frac{\nu}{b\beta}\right)\\+\frac{bbcc}{\alpha\alpha\beta\beta}\left(\frac{1}{c}+\frac{1}{\gamma}\right)\left(\lambda''\left(\frac{1}{c}+\frac{1}{\gamma}\right)^{2}+\frac{\nu}{c\gamma}\right)\end{array}\right\}$$

et radiorum in h concurrent inclinatio ad axem $=\frac{bcx}{\alpha\beta\gamma}$.

IV. Pro quatuor lentibus

spatium diffusionis:

$$Ii=\mu\delta\delta xx\left\{\begin{aligned}
&+\tfrac{\alpha\alpha\beta\beta\gamma\gamma}{bbccdd}\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)\left(\lambda\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)^2+\tfrac{\nu}{a\alpha}\right)\\
&+\tfrac{bb\beta\beta\gamma\gamma}{\alpha\alpha ccdd}\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)\left(\lambda'\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)^2+\tfrac{\nu}{b\beta}\right)\\
&+\tfrac{bbcc\gamma\gamma}{\alpha\alpha\beta\beta dd}\left(\tfrac{1}{c}+\tfrac{1}{\gamma}\right)\left(\lambda''\left(\tfrac{1}{c}+\tfrac{1}{\gamma}\right)^2+\tfrac{\nu}{c\gamma}\right)\\
&+\tfrac{bbccdd}{\alpha\alpha\beta\beta\gamma\gamma}\left(\tfrac{1}{d}+\tfrac{1}{\delta}\right)\left(\lambda'''\left(\tfrac{1}{d}+\tfrac{1}{\delta}\right)^2+\tfrac{\nu}{d\delta}\right)
\end{aligned}\right\}$$

et radiorum in i concurrentium inclinatio ad axem $=\frac{bcdx}{\alpha\beta\gamma\delta}$.

V. Pro quinque lentibus

spatium diffusionis:

$$Kk=\mu\varepsilon\varepsilon xx\left\{\begin{aligned}
&+\tfrac{\alpha\alpha\beta\beta\gamma\gamma\delta\delta}{bbccddee}\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)\left(\lambda\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)^2+\tfrac{\nu}{a\alpha}\right)\\
&+\tfrac{bb\beta\beta\gamma\gamma\delta\delta}{\alpha\alpha ccddee}\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)\left(\lambda'\left(\tfrac{1}{b}+\tfrac{1}{\beta}\right)^2+\tfrac{\nu}{b\beta}\right)\\
&+\tfrac{bbcc\gamma\gamma\delta\delta}{\alpha\alpha\beta\beta ddee}\left(\tfrac{1}{c}+\tfrac{1}{\gamma}\right)\left(\lambda''\left(\tfrac{1}{c}+\tfrac{1}{\gamma}\right)^2+\tfrac{\nu}{c\gamma}\right)\\
&+\tfrac{bbccdd\delta\delta}{\alpha\alpha\beta\beta\gamma\gamma ee}\left(\tfrac{1}{d}+\tfrac{1}{\delta}\right)\left(\lambda'''\left(\tfrac{1}{d}+\tfrac{1}{\delta}\right)^2+\tfrac{\nu}{d\delta}\right)\\
&+\tfrac{bbccddee}{\alpha\alpha\beta\beta\gamma\gamma\delta\delta}\left(\tfrac{1}{e}+\tfrac{1}{\varepsilon}\right)\left(\lambda''''\left(\tfrac{1}{e}+\tfrac{1}{\varepsilon}\right)^2+\tfrac{\nu}{e\varepsilon}\right)
\end{aligned}\right\}$$

et radiorum in i concurrentium inclinatio ad axem $=\frac{bcdex}{\alpha\beta\gamma\delta\varepsilon}$; neque ergo casus, quibus plures occurrunt lentes, ulla amplius laborant difficultate.

Si lentes ratione refractionis discrepent ad easque referendae sint litterae *n, n', n'', n'''* etc., formulae in hoc problemate inventae sequenti modo facile ad hunc casum latius patentem adcommodabuntur. Primo scilicet in formulis pro radiis facierum inventis litterae ρ, σ et τ tantum ad primam lentem pertinent, earumque loco pro secunda lente scribi oportet ρ', σ' et τ', pro tertia autem ρ'', σ'' et τ'' et ita porro. Praeterea vero spatia diffusionis hinc aliquam mutationem requirunt, eritque spatium diffusionis :

I. Pro una lente

$$\alpha\alpha xx\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)\mu\left(\lambda\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)^2+\tfrac{\nu}{a\alpha}\right).$$

II. Pro duabus lentibus

$$\beta\beta xx\left\{\frac{\mu\alpha\alpha}{bb}\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{\nu}{a\alpha}\right)+\frac{\mu' bb}{\alpha\alpha}\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\lambda'\left(\frac{1}{b}+\frac{1}{\beta}\right)^2+\frac{\nu'}{b\beta}\right)\right\}$$

III. Pro tribus lentibus

$$\gamma\gamma xx\left\{\begin{aligned}&\frac{\mu\alpha\alpha\beta\beta}{bbcc}\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{\nu}{a\alpha}\right)\\&+\frac{\mu' bb\beta\beta}{\alpha\alpha cc}\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\lambda'\left(\frac{1}{b}+\frac{1}{\beta}\right)^2+\frac{\nu'}{b\beta}\right)\\&+\frac{\mu'' bbcc}{\alpha\alpha\beta\beta}\left(\frac{1}{c}+\frac{1}{\gamma}\right)\left(\lambda''\left(\frac{1}{c}+\frac{1}{\gamma}\right)^2+\frac{\nu''}{c\gamma}\right)\end{aligned}\right\}$$

IV. Pro quatuor lentibus

$$\delta\delta xx\left\{\begin{aligned}&\frac{\mu\alpha\alpha\beta\beta\gamma\gamma}{bbccdd}\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{\nu}{a\alpha}\right)\\&+\frac{\mu' bb\beta\beta\gamma\gamma}{\alpha\alpha ccdd}\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\lambda'\left(\frac{1}{b}+\frac{1}{\beta}\right)^2+\frac{\nu'}{b\beta}\right)\\&+\frac{\mu'' bbcc\gamma\gamma}{\alpha\alpha\beta\beta dd}\left(\frac{1}{c}+\frac{1}{\gamma}\right)\left(\lambda''\left(\frac{1}{c}+\frac{1}{\gamma}\right)^2+\frac{\nu''}{c\gamma}\right)\\&+\frac{\mu''' bbccdd}{\alpha\alpha\beta\beta\gamma\gamma}\left(\frac{1}{d}+\frac{1}{\delta}\right)\left(\lambda''\left(\frac{1}{d}+\frac{1}{\delta}\right)^2+\frac{\nu'''}{d\delta}\right)\end{aligned}\right\}$$

valores autem harum litterarum μ', ν', μ'', ν'' etc. iam supra definivimus § 77.

COROLLARIUM 1

92. Si lentis primae PP ponatur radius faciei anterioris $=f$ et posterioris $=g$, erit

$$\tfrac{1}{f}=\tfrac{\rho}{a}+\tfrac{\sigma}{\alpha}\pm\tau(\tfrac{1}{a}+\tfrac{1}{\alpha})\sqrt{(\lambda-1)},\quad \tfrac{1}{g}=\tfrac{\rho}{\alpha}+\tfrac{\sigma}{a}\mp\tau(\tfrac{1}{a}+\tfrac{1}{\alpha})\sqrt{(\lambda-1)},$$

unde, si detur distantia obiecti $EA=a$, primo invenitur α ex hac aequatione

$$\frac{1}{f}+\frac{1}{g}=(\rho+\sigma)(\frac{1}{a}+\frac{1}{\alpha})=1,818182(\frac{1}{a}+\frac{1}{\alpha})=\frac{20}{11}(\frac{1}{a}+\frac{1}{\alpha}).$$

Inventa autem distantia α numerus λ reperitur ex hac aequatione

$$\frac{1}{f}-\frac{1}{g}=(\sigma-\rho)(\frac{1}{a}-\frac{1}{\alpha})=2\tau(\frac{1}{a}+\frac{1}{\alpha})\sqrt{(\lambda-1)}.$$

COROLLARIUM 2

93. Deinde si distantia secundae lentis a prima sit $=F$, ob $F=\alpha+b$ habetur $b=F-\alpha$; qua distantia b cognita, si pro lente secunda datus sit radius faciei anterioris f' et posterioris g', habebuntur iterum duae aequationes

$$\frac{1}{f'}=\frac{\rho}{b}+\frac{\sigma}{\beta}\pm\tau(\frac{1}{b}+\frac{1}{\beta})\sqrt{(\lambda'-1)},\quad \frac{1}{g'}=\frac{\rho}{\beta}+\frac{\sigma}{b}\mp\tau(\frac{1}{b}+\frac{1}{\beta})\sqrt{(\lambda''-1)},$$

ex quibus cum distantiam β tum numerum λ' definira licet. Similique modo ex forma sequentium lentium earumque distantia reliqua elementa innotescent.

COROLLARIUM 3

94. Si singulae lentes ad minimum spatium diffusionis fuerint accommodatae, erit $\lambda=1,\ \lambda'=1,\ \lambda''=1,\ \lambda'''=1$ etc.; sin autem hae lentes alia forma fuerint praeditae, isti numeri erunt unitate maiores.

SCHOLION

95. Quo plures fuerint lentes, eo pluribus constabit membris spatium diffusionis ab iis productum; neque tamen propterea aucto lentium numero spatium diffusionis necessario augetur. Cum enim quantitates $\alpha,\ b,\ \beta,\ c,\ \gamma,\ d,\ \delta$ etc. valores quoque negativos recipere queant, dummodo binarum summae $\alpha+b,\ \beta+c,\ \gamma+d,\ \delta+e$ etc. etc. utpote lentium distantiae maneant positivae, fieri potest, ut unum vel aliquot membra fiant negativa hincque spatium diffusionis diminuatur, quin etiam interdum prorsus evanescat, quo casu repraesentatio sine dubio erit perfectissima. Verum in instrumentis dioptricis ad visionem instructis veluti Telescopiis ac Microscopiis non tam hoc, quod definivimus, spatium diffusionis quam confusio in ipsa visione orta spectari debet; quae autem, etsi a spatio diffusionis plurimum differt, tamen ex eo definiri potest, uti mox explicabimus. Ante autem conveniet lentes compositas seu multiplicatas considerare, cuiusmodi oriuntur, si duae pluresve lentes, quarum crassities tam est parva, ut negligi queat,

immediate iungantur, quo quidem pacto instar lentium simplicium spectari possunt; verum tali coniunctione effici potest, ut spatium diffusionis multo fiat minus, quam si lens simplex adhiberetur, atque adeo evanescat valorque numeri λ istiusmodi lenti compositae conveniens unitate minor sit proditurus, unde maxima commoda ad confusionem diminuendam obtinebuntur.