DIOPTRICS

The First Part Containing Book One,

Concerning the Explanation of the Principles, from which the Construction both of Telescopes as well as Microscopes is Desired.

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CHAPTER I

CONCERNING THE SPREADING OF THE IMAGE SHOWN BY A SINGLE LENS

DEFINITION 1

1. *The principal image is called that, which is represented by the rays refracted by the lens closest to the axis of the lens.*

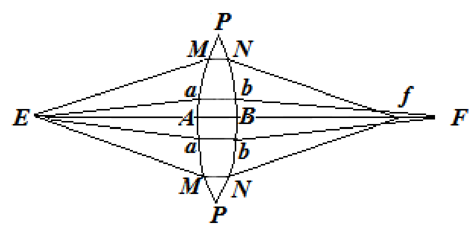

Fig. 1

Evidently if the axis of the lens *PP* (Fig. 1) shall be *EF* and in that only a minimum distance *aAa* shall be open, through which a passage for the rays is allowed, the rays emitted by a single light point *E* may be gathered by one point *F*, which point is called the principal image.

COROLLARY 1

2. Truly since the opening *aAa* shall be a minimum, all the rays, which are incident on that from any point, thus are allowed equal refraction, so that again they may be gathered at one point, that which is to be understood not only from the point light source *E* situated on the axis of the lens, but also from any other placed beyond the axis.

COROLLARY 2

3. Therefore if the aperture of the lens were a minimum, the individual points of any object after refraction again will refer to individual points, and thus the principal image will be distinct and without diffusion: since diffusion then finally arises, when the rays emitted from one point are not again gathered together in a single point.

COROLLARY 3

4. And as long as the aperture of the lens *aAa* is a minimum, there is no concern, according to how the faces of the lens may be elaborated on; for whatever the figure of these were, since only a minimal part enters into the computation, that will always be regarded as spherical.

COROLLARY 4

5. Therefore at first, the position of the principal image depends on the place of the point of light E, whether that be situated on or beyond the axis; then on the refraction of each spherical face *aAa* and *bBb* ; thirdly on the distance *AB* between these or the thickness of the lens, and fourthly, on account of the refraction, that the rays experience in passing through the lens.

SCHOLION 1

6. The refraction of rays by the minimum apertures of lenses of this kind and thus the determination of the principal image is usually found to be treated well enough in the elements of dioptrics, which matter therefore I will not pursue further; but here I have decided rather to enquire into that reasoning of the refraction, when the aperture of the lens is of a moderate amount, which in the first place is required to be considered for the figure of each face. Moreover I always assume here the spherical figures of the lenses, therefore this figure is accustomed usually to be indicated for lenses, unless perhaps the lens may have been finished more carefully. Certainly in practice until now no figures other than spherical figures will be able to be attributed more accurately and conveniently for lenses, and thus for a spherical figure, even if especially in practice it has been convenient often to be modified by craftsmen . But now such faults generally may be avoided happily enough by the more clever, from which thus there will be nothing to fear, lest that, what may be elicited from the hypothesis of spherical figures by calculation, may not be going to agree with experiment. On this account here I demand always, that both faces of lenses shall be carefully made to be exactly according to a spherical figure.

SCHOLION 2

7. But the spherical figure labors under this inconvenience, and which at once may be attributed to the greater aperture of the lens, no longer do all the rays, which indeed have been sent from a point of the object, after refraction, no longer may be directed towards a single point and the rays *EM* further from the axis of transmission of the lens no longer may concur in the point *F*. From which therefore it is necessary more diffusion to have arisen, for these rays which are more distant will be deflected more than these which pass near the axis ; and since the origin of such a deflection is deduced from the spherical

figure, from that it will go forth greater, where a greater aperture may be attributed to the lens. Therefore I have set up this chapter to define in any case how great the diffusion is going to become ; which with everything else being equal shall depend on the size of the aperture, here I always remind myself a circular aperture to be assumed for any lens, through the centre of which the axis of the lens may pass : thus so that the radius of this circle may likewise show a measure of this aperture. Thus if on the surface of the lens *PP* the aperture *MAM* may be left, with the remaining part *MP* covered over with a dark cloth, the distance *MM* of its extreme points will show the diameter of the aperture and half of that the radius.

DEFINITION 2

8. *The extreme image is that, which the rays transmitted by the extremities of the aperture show.*

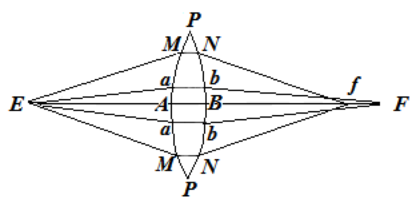

Fig. 1

Thus if *MM* (Fig.1) shall be the aperture of the lens and the rays *EM, EM* transmitted from the light point *E* around the edge of the aperture to be transmitted may meet at the point *f,* the extreme image will be at this point *f* .

COROLLARY 1

9. If the point of light *E* is on the axis of the lens itself, there is no doubt, why the rays thence passing through the circular edge *MM* may not again concur at a point *f* on the axis and represent a distinct image, which is called the extreme image.

COROLLARY 2

10. Truly if the point of light shall not be on the axis of the lens, this [above] by no means may happen, and the rays transmitted by the edge of that circle may no longer be gathered into a single point; from which in this case the extreme image will be more confused, as the point of light were moved away further from the axis.

SCHOLION

11. Just as the refraction of the rays may be had, when the light point is situated beyond the axis of the lens, but not only is the question more difficult, thus also is involved with more prolix calculations, so that hence hardly anything may be able to be concluded.

Meanwhile in the use, to which the lenses are adapted, at no time may the objects more distant from the axis be accustomed to be observed, and it requires us to be content, as long as objects may be represented distinctly on the axis of the lens itself; nor also the diffusion be able to be perceived, with which objects close to the axis are affected : for since the extreme image of the point *E* shall be the point *f* situated on the axis of the lens itself, with no unwanted diffusion, even if that may be moved off axis a little, scarcely any perceptible diffusion will be able to be introduced. On account of which reasons I restrict myself only to objects situated on the axis of the lens itself in the following investigation.

DEFINITION 3

12. *The interval between the principal image and the extreme intercept may be called the diffusion space.* [Now called depth of field, etc.]

Thus if the principal image shall be at *F* (Fig.1), the extreme truly at *f*, the interval *Ff* will be the diffusion distance.

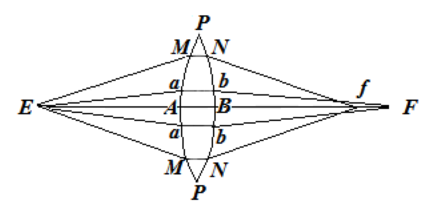

Fig. 1

COROLLARY 1

13. Therefore if the aperture of the lens *MM* may vanish, likewise the diffusion distance vanishes, then indeed only the rays nearest to the axis are transmitted, by which the distinct image at *F* is fashioned. From which it can be understood, where the greater were the aperture of the lens, there the greater to become the space of diffusion *Ff*.

COROLLARY 2

14. Since the image at *F* shall be formed from the rays nearest the axis, but at *f* the image from the rays around the edge of the circle of transmission *MM* : if we may consider the whole aperture of the lens divided into an infinitude of concentric circles, the rays transmitted through the individual circles will show the intermediate images, from which the interval *Ff* will be filled.

COROLLARY 3

15. If indeed the first null interval may be put in place, then truly by continually increasing, the extreme image at first will agree with the principal; then truly continually to differ more from that, and thus, since it would have increased as far as to MM, all these images now also will stop and fill up the space Ff.

SCHOLIUM I

16. This spreading interval contains the source of the confusion, by which the representation of the image is disturbed; indeed since an infinitude of images of the same point of light E may be shown set out by the interval Ff, the mixture of which shall produce the spreading, therefore by necessity that will become greater, as the spreading distance Ff will have increased. Indeed just as it is required for a distinct representation, so that all the rays emitted from the same point of the object may be gathered in a single point, thus, if these rays may join together at more points and several may refer to the same points of the image, as these more or less may disagree with each other, thence more or less confusion will arise. But just as this spreading may be required to be estimated, then finally it will be allowed to explain, as before we will have demonstrated how to define the spreading distance accurately: concerning which in this chapter, since the lens proposed will have been defined by some kind of spherical glass faces, for some bright point E some distance from that and with whatever aperture to be set up to investigate the spatial spreading Ff. With the same accomplished it will be agreed to carry out the same investigation for two or more lenses taken together, so that thence finally we may prevail to assign the spreading in whatever dioptric instruments.

SCHOLIUM 2

17. Therefore the principal question of this chapter is concerned with this, in order that with the proposed lens PP with the light point E on its axis, some incident ray EM may be considered and its refraction by the lens may be defined, in order that the point f, where it may meet the axis again, may be able to be assigned. Since indeed at the same point f all the rays transmitted by the whole periphery of the circle MM concur, from these the image of a certain point E is expressed at f, which will be the extreme, if the circle MM may determine its aperture in the lens; but if it may be taken smaller, a certain intermediate image will be had. But since here a twofold refraction may take place, the one advances into the glass to the radius EM, the other going out of the same glass, to the extent that its direction in each may be changed, is required to be investigated separately; from which two as if preliminary problems arise, from the combination of which successively the matter will be resolved. Truly in order that this problem may be able to be expedited by a more convenient calculation, certain lemmas applying to small angles will be required to be set out first.

LEMMA 1

18. *If the sine of the angle* Φ *may not exceed thirty degrees, its sine accurately enough will be* $\sin.\Phi = \Phi - \frac{1}{6}\Phi^3$ *, in indeed in a circle of which the radius is* $=1$ *, the arc* Φ *, which may be had for a measure of that angle, may be expressed by fractions of the radius.*

DEMONSTRATION

However large were the angle Φ, it is known its sine to be expressed by this infinite series :

$$\sin.\Phi = \Phi - \tfrac{1}{6}\Phi^3 + \tfrac{1}{120}\Phi^5 - \tfrac{1}{5040}\Phi^7 + \text{etc.}$$

therefore with only the first two terms taken, an error is committed equal to the remaining terms ignored; therefore if we may put in place $\sin.\Phi = \Phi - \frac{1}{6}\Phi^3$ and hence for the various angles Φ we may deduce the sine of these, the errors of the sines will be shown by a comparison with a table of sines. Thus if there may be taken $\Phi = 30°$, because for the angle $180°$, the arc 3,14159265 prevails, there will be for the parts of the radius $\Phi = 0{,}52359877$ and $\frac{1}{6}\Phi^3 = 0{,}0239246$, and hence

$$\begin{array}{l} \Phi - \tfrac{1}{6}\Phi = 0{,}4996741. \text{ But actually there is} \\ \underline{\sin.\Phi = 0{,}5000000,} \text{ from which there is had} \\ \text{the error} = 0{,}0003259, \end{array}$$

which therefore does not exceed the $\frac{1}{3000}$th part of the radius. But if the angle Φ may be taken twice as small, evidently $\Phi = 15°$, there will be found

$$\begin{array}{l} \Phi - \tfrac{1}{6}\Phi = 0{,}2588088. \text{ But this is actually} \\ \underline{\sin.\Phi = 0{,}2588190,} \text{ from which there is had} \\ \text{the error present} = 0{,}0000102, \end{array}$$

which is three times smaller than in the preceding case. Therefore since in practice, the error part thus the three thousandths part of the radius may be able to be allowed, much more, if the angle Φ were less than $30°$, truly the expression $\sin.\Phi = \Phi - \frac{1}{6}\Phi^3$ will be considered to represent its sine.

LEMMA 2

19. *In turn if the sine of an angle less than thirty degrees may be given* $=s$, *from that the angle itself may be defined approximately, so that in the circle, of which the radius* $=1$ *the arc itself may be put* $=s+\frac{1}{6}s^3$.

DEMONSTRATION

If indeed Φ may designate this same angle, of which the sine is put $=s$, likewise we see to satisfy exactly $s=\Phi-\frac{1}{6}\Phi^3$; but hence by conversion there arises $\Phi=s+\frac{1}{6}s^3$ approximately, which expression may err somewhat in the angle of thirty degrees as we may see, if $s=\frac{1}{2}$, there becomes $\Phi=s+\frac{1}{6}s^3=0{,}5208333$. But since the number 3,14159265 may correspond to the angle $180°$, this number 0,5208333 provides the angle $29°50'30''$: but the true angle is $30°$, thus so that the error shall be $9'30''$. But if there shall be $s=\frac{1}{4}$, the angle corresponding to which sine $=14°28'39''$, there will be

$$s+\frac{1}{6}s^3=0{,}2526041=14°28'23'',$$

thus so that in this case the error does not exceed $16''$. Therefore it is apparent, while the angle shall be less than $30°$, in this manner the angle can be found well enough from the given sine .

PROBLEM 1

20. *If from the light point E* (Fig. 2), *the ray EM may be incident on the spherical surface arising from the revolution of the circular arc AM about the axis EC, to define, after it has been refracted, its concurrence with the axis EC.*

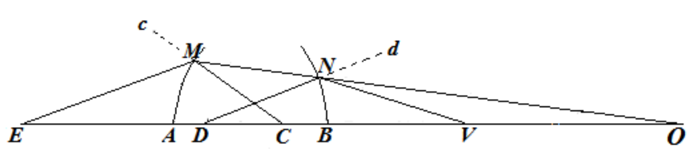

Fig. 2

SOLUTION

Let C be the centre of the spherical surface, and its radius $CA=CM=f$, and the distance of the light point E from the refracting surface $EA=a$, then truly the ratio of the refraction of the rays from air into glass shall be $=n:1$. The angle for the incident ray CM may be called $AEM=\Phi$, and the sine of this angle will be $=\Phi-\frac{1}{6}\Phi^3$. Now for the sake of brevity there may be put the distance $EC=a+f=c$, and because in the triangle ECM

the sides may be given $CM = f$, $EC = c$ with the angle $CEM = \Phi$, there becomes with *CM* produced to c

$$CM(f) : \sin.CEM = CE(c) : \sin.EMc$$

and thus

$$\sin.EMc = \tfrac{c}{f}\sin.\Phi = \tfrac{c}{f}\left(\Phi - \tfrac{1}{6}\Phi^3\right);$$

from which the angle *EMc* itself may be elicited with the powers of Φ higher than the third ignored:

$$EMc = \tfrac{c}{f}\Phi - \tfrac{c}{6f}\Phi^3 + \tfrac{c^3}{6f^3}\Phi^3 = \tfrac{c}{f}\Phi + \tfrac{c(cc-ff)}{6f^3}\Phi^3;$$

which since it shall be $= ECM + CEM$, will become

$$ECM = \tfrac{c-f}{f}\Phi + \tfrac{c(cc-ff)}{6f^3}\Phi^3.$$

Truly the angle *EMc* is the angle of incidence, and if *MO* may refer to the angle of refraction, *CMO* shall be the angle of refraction, and thus by the hypothesis

$$\sin.EMc : \sin.CMO = n : 1,$$

from which it is deduced :

$$\sin.CMO = \tfrac{c}{nf}\left(\Phi - \tfrac{1}{6}\Phi^3\right)$$

and hence the angle itself

$$CMO = \tfrac{c}{nf}\Phi + \tfrac{c(cc-nnff)}{6n^3f^3}\Phi^3,$$

with which taken from the angle *ECM*, the angle is left

$$COM = \tfrac{(n-1)c-nf}{nf}\Phi + \tfrac{c((n^3-1)cc-nn(n-1)ff)}{6n^3f^3}\Phi^3,$$

of which the sine therefore will be

$$\sin.COM = \tfrac{(n-1)c-nf}{nf}\Phi + \tfrac{3(n-1)c^3+3(n-1)^2ccf-4n(n-1)cff+nnf^3}{6nnf^3}\Phi^3.$$

Now truly on account of $\sin.COM : CM(f) = \sin.CMO : CO$ there will be obtained

$$CO=\frac{\frac{c}{n}-\frac{c}{6n}\Phi\Phi}{\frac{(n-1)c-nf}{nf}+\frac{3(n-1)c^3+3(n-1)^2ccf-4n(n-1)cff+nnf^3}{6nnf^3}\Phi^2},$$

of which formula expanded out provides :

$$CO=\frac{cf}{(n-1)c-nf}-\frac{cf\Phi\Phi}{6((n-1)c-nf)}-\frac{c(3(n-1)c^3+3(n-1)^2ccf-4n(n-1)cff+nnf^3)}{6nf((n-1)c-nf)^2}\Phi\Phi,$$

which again is reduced to this form:

$$CO=\frac{cf}{(n-1)c-nf}-\frac{(n-1)cc(c-f)(c+nf)}{2nf((n-1)c-nf)^2}\Phi\Phi,$$

if we may add to which $CA=f$, there will be produced:

$$AO=\frac{nf(c-f)}{(n-1)c-nf}-\frac{(n-1)cc(c-f)(c+nf)}{2nf((n-1)c-nf)^2}\Phi\Phi,$$

And hence the position of the refracted ray *MO* is defined at first for the interval *AO* in the manner found, then truly in addition for the angle *AOM,* which will be :

$$AOM=\frac{(n-1)c-nf}{nf}\Phi-\frac{c((n^3-1)cc-nn(n-1)ff)}{6n^3f^3}\Phi^3,$$

or

$$AOM=\frac{(n-1)c-nf}{nf}\Phi-\frac{(n-1)c((nn+n+1)cc-nnff)}{6n^3f^3}\Phi^3.$$

COROLLARY 1

21. Since we have put $c=a+f$, there will be

$$c-f=a \text{ and } (n-1)c-nf=(n-1)a-f,$$

And

$$(nn+n+1)cc-nnff=(nn+n+1)aa+2(nn+n+1)af+(n+1)ff,$$

with which values substituted we will have

$$AO=\frac{naf}{(n-1)a-f}-\frac{(n-1)a(a+f)^2(a+(n+1)f)}{2nf((n-1)a-f)^2}\Phi\Phi$$

and

$$AOM=\frac{(n-1)a-f}{nf}\Phi+\frac{(n-1)(a+f)((nn+n+1)a(a+2f)+(n+1)ff)}{6n^3f^3}\Phi^3.$$

COROLLARY 2

22. If M shall be the edge of the aperture, the radius of the aperture will be $=f\sin.ECM=a\Phi$ well enough: for neither is it necessary to know the aperture so exactly. From which if the radius of the aperture may be put $=x$, there will be approximately $x=a\Phi$ and thus $\Phi=\frac{x}{a}$.

COROLLARY 3

23. If we wish to define the matter more accurately, since there is

$$\sin.ECM=\frac{c-f}{f}\Phi+\frac{(c-f)(3c-f)}{6ff}\Phi^3=\frac{a}{f}\Phi+\frac{a(3a+2f)}{6ff}\Phi^3,$$

there will be

$$x=\frac{a}{f}\Phi+\frac{a(3a+2f)}{6f}\Phi^3 \text{ and thus } \Phi=\frac{x}{a}-\frac{(3a+2f)x^3}{6a^3f};$$

but since we do not rise beyond the second dimension of Φ in the value of AO, we do not need this expression.

COROLLARY 4

24. Even if these expressions are only approximately true, yet in practice they will be able to be used without error, as long as the angles, which have entered into the calculation, remain within 30°. Therefore not only is it necessary, that the angle $AEM=\Phi$, but also the angle EMc or $\frac{c}{f}\Phi=\Phi+\frac{a}{f}\Phi$ shall be less than 30 degrees.

SCHOLION

25. Indeed we have been able to define the distance EO geometrically with the greatest rigor, nor was there a need to take recourse in approximations: clearly if we had placed the angle $ECM=\omega$ and the distance $CO=u$; we would have arrived at this determination

$$\tfrac{cf}{u} = -c\cos.\omega + \sqrt{(nn(c\cos.\omega - f)^2 + (nn-1)cc\sin^2.\omega)}\ ,$$

and there would become

$$\cos.\omega = \tfrac{c}{f}\sin^2.\Phi + \cos.\Phi\sqrt{\left(1-\tfrac{cc}{ff}\sin^2.\Phi\right)}$$

and

$$\sin.\omega = \tfrac{c}{f}\sin.\Phi\cos.\Phi - \sin.\Phi\sqrt{\left(1-\tfrac{cc}{ff}\sin^2.\Phi\right)}$$

and with these values substituted

$$\begin{aligned}\tfrac{cf}{u} &= -\tfrac{cc\sin^2.\Phi}{f} - c\cos.\Phi\sqrt{\left(1-\tfrac{cc}{ff}\sin^2.\Phi\right)}\\ &+\left(c\cos.\Phi - \sqrt{\left(ff - cc\sin^2.\Phi\right)}\right)\sqrt{\left(nn-\tfrac{cc}{ff}\sin^2.\Phi\right)}\end{aligned}$$

from which on putting the angle Φ very small by the above approximation the above expression will be elicited. Truly the above preceding analysis is seen to be more beneficial for the present proceedings. Meanwhile if it were wished to approach closer to the truth, truly this formula shown will be arrived at,

$$\begin{aligned}\tfrac{cf}{u} &= (n-1)c\cos.\Phi - nf + \tfrac{(n-1)cc}{2nff}((n-1)f + c\cos.\Phi)\sin^2.\Phi\\ &\tfrac{c^4}{8n^3f^4}((nn-1)^2 f + (n^2-1)c\cos.\Phi)\ \sin.^4\Phi\end{aligned}$$

or

$$\begin{aligned}\tfrac{cf}{u} &= (n-1)c - nf + \tfrac{(n-1)c(c-f)(c+nf)}{2nff}\Phi\Phi\\ &+\tfrac{(n-1)c\left((c-f)(3(nn+n+1)c^3 + 3nn(n+2)ccf + nn(3n-4)cff) - n^3f^3c\right)}{24n^3f^4}\Phi^4.\end{aligned}$$

PROBLEM 2

26. *If, just as we have found in the preceding problem, the ray MO* (Fig. 2) *sent into the glass through the spherical surface BN , emerges again into the air, to define the point V, where it crosses the axis.*

SOLUTION

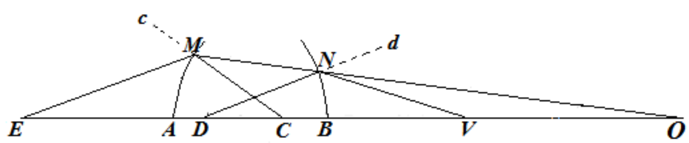

Fig. 2

The interval may be put $AB = d$ and the centre of the spherical surface *BN* shall be at *D* and its radius $DB = DN = g$. Therefore since the position of the incident ray *MNO* is given, we may put $BO = b$ and the angle $BON = \Psi$ and for brevity therefore the interval $DO = b + g = e$. Now since in triangle *DON* the sides may be given $DN = g,\ DO = e$ since the angle $BON = \psi$, there is found

$$\sin.DNM = \tfrac{e}{g}\sin.\Psi = \tfrac{e}{g}\Psi - \tfrac{e}{6g}\Psi^3$$

and hence

$$DNM = \tfrac{e}{g}\Psi = \tfrac{e}{g}\Psi + \tfrac{e(ee-gg)}{6g^3}\Psi^3 \text{ and } ODN = \tfrac{e-g}{g}\Psi = \tfrac{e}{g}\Psi + \tfrac{e(ee-gg)}{6g^3}\Psi^3$$

But this, sin. *DNM* is the sine of the angle of incidence, to which the angle of refraction *VNd* corresponds, the sine of which therefore is to that as $n:1$, from which there becomes

$$\sin.VNM = \tfrac{ne}{g}\Psi - \tfrac{ne}{6g}\Psi^3$$

and hence

$$VNd = \tfrac{ne}{g}\Psi + \tfrac{ne(nnee-gg)}{6g^3}\Psi^3,$$

from which with the angle *ODN* taken the angle is left

$$DVN = \frac{(n-1)e+g}{g}\Psi + \frac{e((n^3-1)ee-(n-1)gg)}{6g^3}\Psi^3,$$

therefore

$$\sin.DVN = \frac{(n-1)e+g}{g}\Psi + \frac{3n(n-1)e^3-3(n-1)^2eeg-4(n-1)egg-g^3}{6g^3}\Psi^3.$$

Now since there shall be $\sin.DVN : g = \sin.VNd : DV$, there will be

$$\frac{g}{DV}=\frac{(n-1)e+g+\frac{1}{6gg}(3n(n-1)e^3-3(n-1)^2eeg-4(n-1)egg-g^3)\Psi^2}{ne-\frac{1}{6}ne\Psi^2},$$

which expression is reduced to this form:

$$\frac{g}{DV}=\frac{(n-1)e+g}{ne}+\frac{(n-1)(e-g)(ne+g)\Psi^2}{2ngg},$$

from which the inverse arises

$$DV=\frac{neg}{(n-1)e+g}+\frac{n(n-1)ee(e-g)(ne+g)}{2g((n-1)e+g)^2}\Psi^2$$

and

$$BV=\frac{g(e-g)}{(n-1)e+g}+\frac{n(n-1)ee(e-g)(ne+g)}{2g((n-1)e+g)^2}\Psi^2.$$

But with the point V found it is to be noted the angle

$$BVN=\frac{(n-1)e+g}{g}\Psi+\frac{(en(n^3-1)ee-(n-1)gg)}{6g^3}\Psi^3.$$

COROLLARY 1

27. Since there shall be $c=b+g$, there will be $(n-1)e+g=(n-1)b+ng$, so that with the value restored we will have

$$BV=\frac{bg}{(n-1)b+ng}-\frac{n(n-1)b(b+g)^2(nb+(n+1)g)}{2g((n-1)b+ng)^2}\Psi^2$$

and

$$BVN=\frac{(n-1)b+ng}{g}\Psi+\frac{((b+g)(n^3-1)(b+g)^2-(n-1)gg)}{6g^3}\Psi^3.$$

COROLLARY 2

28. These formulas also can be elicited from the preceding, if for the letters *n, a, f* there may be written $\frac{1}{n}$, $-b$ and $-g$, since in this manner the case of the preceding problem is reduced to this. Besides moreover, what angle there was Φ, here is Ψ.

COROLLARY 3

29. Since in the preceding problem we have found both the line *AO* as well as the angle *AOM,* from this problem it will be applied to that refracted ray *MO*

$$BO = b = \frac{naf}{(n-1)a-f} - d - \frac{(n-1)a(a+f)^2(a+(n+1)f)}{2nf\left((n-1)a-f\right)^2}\Phi\Phi$$

and

$$\Psi = \frac{(n-1)a-f}{nf}\Phi + \frac{(n-1)(a+f)((nn+n+1)a(a+2f)+(n+1)ff)}{6n^3f^3}\Phi^3.$$

PROBLEM 3

30. *With the proposed glass lens MABN* (Fig. 2) *terminated by the spherical faces AM and BN, if the ray EM may be incident on that from some point E placed on its axis, to define the point V, at which after twin refractions it again shall be coinciding with the axis of the lens.*

SOLUTION

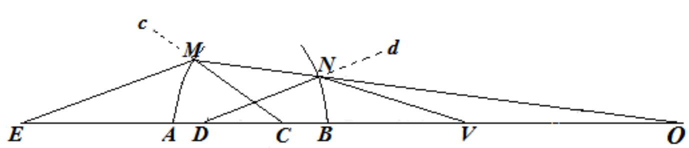

Fig. 2

We may consider the lens as convex on both sides, and the radius of the anterior face *AM* shall be $AC = f$, truly of the posterior *BN,* the radius shall be $BD = g$, moreover, the thickness of the lens $AB = d$. Again the lens shall be glass, thus so that, if a ray of light may be incident on that in air, the sine of the incident ray to the sine of the refracted ray shall be as n to 1. Now a right line joining the centres of each face *C* and *D* will be the axis of the lens, on which there may be found the point of light *E* at the distance $AE = a$ before the lens, from which under the angle $AEM = \Phi$ the ray *EM* is incident on the lens, which thus first is bent by refraction, so that produced it crosses the axis at *O.* So that if now from these, which have been found from the first problem, we may put

$$BO = \frac{naf}{(n-1)a-f} - d - \frac{(n-1)a(a+f)^2(a+(n+1)f)}{2nf((n-1)a-f)^2}\Phi\Phi = b$$

and

$$\text{ang. } BON = \frac{(n-1)a-f}{nf}\Phi = \Psi;$$

indeed in the value of the angle Ψ it is allowed to ignore the term involving Φ^3, since we have extended the calculation only as far as to the second power of Ψ; with these in place from the second problem we have found to become

$$BV = \frac{bg}{(n-1)b+ng} - \frac{n(n-1)b(b+g)^2(nb+(n+1)g))}{2g((n-1)b+ng)^2}\Psi\Psi$$

and

$$BVN = \frac{(n-1)b+ng}{g}\Psi.$$

Therefore the whole procedure returns to this, so that here we may substitute the same values assigned for b and Ψ, which so that it may be able to be made easier, we may put

$$b = P - Q\Phi\Phi \text{ and } \Psi = R\Phi,$$

so that there shall become

$$P = \frac{naf}{(n-1)a-f} - d - \frac{naf-(n-1)ad+df}{(n-1)a-f}$$

$$Q = \frac{(n-1)a(a+f)^2(a+(n+1)f)}{2nf((n-1)a-f)^2} \text{ and } R = \frac{(n-1)a-f}{nf}.$$

Hence there will be:

$$\frac{bg}{(n-1)b+ng} = \frac{Pg-Qg\Phi\Phi}{(n-1)P+ng-(n-1)Q\Phi\Phi} = \frac{Pg}{(n-1)P+ng} - \frac{nQgg\Phi\Phi}{((n-1)P+ng)^2};$$

but for the other part it will suffice to write P for b : from which we will obtain

$$BV = \frac{Pg}{(n-1)P+ng} - \frac{nQgg\Phi\Phi}{((n-1)P+ng)^2} - \frac{n(n-1)PRR(b+g)^2(nP+(n+1)g))}{2g((n-1)P+ng)^2}\Phi\Phi$$

and

$$BVN = \frac{(n-1)P+ng}{g}R\Phi.$$

For with these substituted it is required to be noted:

$$(n-1)P + ng = \frac{n(n-1)a(f+g)-nfg-(n-1)^2ad+(n-1)df}{(n-1)a-f}$$

$$P + g = \frac{naf+(n-1)ag-fg-(n-1)ad+df}{(n-1)a-f}$$

$$nP + (n+1)g = \frac{nnaf+(nn-1)ag-(n+1)fg-n(n-1)ad+ndf}{(n-1)a-f},$$

from which it is concluded:

$$BV = \frac{nafg - n(n-1)adg + dfg}{n(n-1)a(f+g) - nfg - (n-1)^2 ad + (n-1)df}$$
$$- \frac{(n-1)agg(a+f)^2(a+(n+1)f)}{2f(n(n-1)a(f+g) - nfg - (n-1)^2 ad + (n-1)df)^2}\Phi\Phi$$
$$- \frac{(n-1)(naf-(n-1)ad+df)(naf+(n-1)ag-fg-(n-1)ad+df)^2(nnaf+(nn-1)ag-(n+1)fg-n(n-1)ad+ndf)}{2nffg(n(n-1)a(f+g)-nfg-(n-1)^2 ad+(n-1)df)^2}\Phi\Phi$$

and

$$BVN = \frac{n(n-1)a(f+g) - nfg + (n-1)df - (n-1)^2 ad}{nfg}\Phi.$$

COROLLARY 1

31. Certainly if the angle Φ may vanish, the point V lies on the principal image, of which if the distance from the lens may be said to be $= \alpha$, there will be

$$\alpha = \frac{nafg - (n-1)adg + dfg}{n(n-1)a(f+g) - nfg - (n-1)^2 ad + (n-1)df},$$

which if it may be seen as given, the relation between f and g is defined by this equation, so that this principal distance may arise.

COROLLARY 2

32. Whereby if the distance of the object before the lens shall be $= a$, its image after the lens must be projected to a distance $= \alpha$, the equation of which will be required to satisfy :

$$n(n-1)a\alpha(f+g) - nafg - n\alpha fg + (n-1)\alpha dg + (n-1)adf - (n-1)^2 a\alpha d - dfg = 0.$$

COROLLARY 3

33. But by introducing this principal distance of the image α into the calculation, our expressions will be contracted considerably. For since there shall be

$$\alpha = \frac{Pg}{(n-1)P+ng}, \text{ hence there will be } P = \frac{n\alpha g}{g-(n-1)\alpha},$$

and again,

$$(n-1)P + ng = \frac{ngg}{g-(n-1)\alpha}, \quad P + g = \frac{gg+\alpha g}{g-(n-1)\alpha} = \frac{g(a+g)}{g-(n-1)\alpha},$$

$$nP + (n+1)g = \frac{g(\alpha+(n+1)g)}{g-(n-1)\alpha},$$

from which substitution made we will follow with the following simpler determinations :

$$BV = \alpha - \frac{(n-1)a(a+f)^2(g-(n+1)\alpha)^2(a+(n+1)f)}{2nnffg((n-1)a-f)^2}\Phi\Phi$$
$$- \frac{(n-1)\alpha(\alpha+g)^2(g(n-1)a-f)^2(\alpha+(n+1)g)}{2nnffg(g-(n-1)\alpha)^2}\Phi\Phi$$

and

$$BVN = \frac{(n-1)a-f)g}{(g-(n-1)\alpha)f}\Phi.$$

SCHOLIUM

34. Not only are these formulas much briefer and neater than the first found, but also a transparent order may be observed in these, by which the letters α and g are thus connected with the letters a and f, as they are allowed to be interchanged. Therefore there is no doubt, why if we may have introduced the distance α into the calculation, a shorter way may not be able to arrive at once for these. Moreover since these latter formulas no further involve the thickness of the lens $AB = d$, it is evident these arise from the first found, if with the distance α introduced, the thickness d may be eliminated, or it may be replaced by this value :

$$d = \frac{n(n-1)a\alpha(f+g)-n(a+\alpha)fg}{(n-1)^2a\alpha-(n-1)(ag+\alpha f)+fg} = n\frac{(n-1)a\alpha(f+g)-(a+\alpha)fg}{((n-1)a-f)((n-1)\alpha-g)},$$

but which labour may not merit to be undertaken, except that now before selection we may be made more sure of its use. Therefore in the following we may use these more elegant formulas, so that we will have learned to expedite the operation more succinctly than hitherto.

PROBLEM 4

35. *For any proposed lens terminated by spherical faces, if an object may be put in place at a given distance from that, for a given aperture to assign the interval of the diffusion.*

SOLUTION

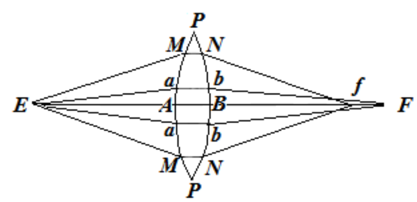

Fig. 1

We may take the lens convex on both sides, and the radius of the anterior face shall be *MAM* (Fig. 1) $= f$, that of the posterior $NBN = g$ and with the thickness of the lens $AB = d$. Again *MM* shall be the aperture of this lens, of which the radius shall be $= x$; and on the axis of the lens the object shall be set out, or perhaps a point of light *E,* the distance of which from the lens may be put to be $AE = a$. Now at first we seek its principal image, which falls at *F,* and we have found above (§ 31) to become

$$BF = \frac{nafg-(n-1)adg+dfg}{n(n-1)a(f+g)-nfg-(n-1)^2ad+(n-1)df},$$

Therefore we may call this distance $= \alpha$, and if the rays *EM* may pass through the mouth of the aperture, and angle $AEM = \Phi = \frac{x}{a}$, with which value substituted into the above formulas, the distance of the extreme image *f* from the lens will be produced, clearly :

$$Bf = \alpha - \frac{(n-1)a(a+f)^2(g-(n-1)\alpha)^2(a+(n+1)f)}{2nnfgg((n-1)a-f)^2}\cdot\frac{xx}{aa}$$
$$-\frac{(n-1)\alpha(\alpha+g)^2((n-1)a-f)^2(\alpha+(n+1)g)}{2nnfgg(g-(n-1)\alpha)^2}\cdot\frac{xx}{aa},$$

from which the diffusion interval sought is deduced:

$$Ff = \begin{cases} +\dfrac{(n-1)a(a+f)^2(g-(n-1)\alpha)^2(a+(n+1)f)}{2nnfgg((n-1)a-f)^2}\cdot\dfrac{xx}{aa} \\ +\dfrac{(n-1)\alpha(\alpha+g)^2((n-1)a-f)^2(\alpha+(n+1)g)}{2nnfgg(g-(n-1)\alpha)^2}\cdot\dfrac{xx}{aa}, \end{cases}$$

and besides the angle BfN will be

$$BfN = \frac{((n-1)a-f)g}{(g-(n-1)\alpha)f} \cdot \frac{x}{a}.$$

COROLLARY 1

36. Therefore as often as the diffusion distance expressed in this manner is positive, the extreme image falls closer to the lens than the principal image, of there is $Bf < BF$. But on the contrary if this expression may obtain a negative value, the extreme image will be more distant than the principal image.

COROLLARY 2

37. Hence it is apparent also the diffusion space thus increases with the aperture, so that it shall be proportional to the square of the radius; thus it follows in the same ratio as the area of the aperture.

COROLLARY 3

38. The space of the diffusion can be expressed in this way also

$$Ff = \begin{cases} +\dfrac{(n-1)a(1+\frac{a}{f})^2(1-\frac{(n-1)\alpha}{g})^2(n+1+\frac{a}{f})}{2nn(\frac{(n-1)a}{f}-1)^2} \cdot \dfrac{xx}{aa} \\ +\dfrac{(n-1)\alpha(1+\frac{\alpha}{g})^2(\frac{(n-1)\alpha}{f}-1)^2(n+1+\frac{\alpha}{g})}{2nn(1-\frac{(n-1)\alpha}{f})^2} \cdot \dfrac{xx}{aa}. \end{cases}$$

and

$$\text{the angle } BfN = \frac{\frac{(n-1)a}{f}-1}{1-\frac{(n-1)\alpha}{g}} \cdot \frac{x}{a}.$$

COROLLARY 4

39. Since these formulas have been returned simpler with the inverse values of the letters introduced, in this manner the equation (§ 32) shown may be treated, which divided by $a\alpha dfg$ will change into this form

$$n(n-1)\frac{1}{d}(\frac{1}{f}+\frac{1}{g}) - n\frac{1}{d}(\frac{1}{a}+\frac{1}{\alpha}) + (n-1)(\frac{1}{\alpha f}+\frac{1}{ag}) - (n-1)^2\frac{1}{fg} - \frac{1}{a\alpha} = 0$$

or

$$\frac{n}{d}((n-1)a\alpha(\frac{1}{f}+\frac{1}{g}) - a - \alpha) = \left(\frac{(n-1)a}{f}-1\right)\left(\frac{(n-1)\alpha}{g}-1\right)$$

which will be more convenient in defining the relation between a and α rather than between f and g.

SCHOLIUM

40. Evidently if the proposed object for the lens may be set at various distances, and if we wish to define the principal images for the individual distances, the equation may be referred to in this manner:

$$\frac{1}{a\alpha}-(n-1)(\frac{1}{\alpha f}+\frac{1}{ag})+n(\frac{1}{da}+\frac{1}{d\alpha})-n(n-1)(\frac{1}{df}+\frac{1}{dg})+(n-1)^2\frac{1}{fg}=0,$$

which thus it will be able to set out by factors

$$(\frac{1}{a}-\frac{(n-1)}{f}+\frac{n}{d})(\frac{1}{\alpha}-\frac{(n-1)}{g}+\frac{n}{d})=\frac{nn}{dd}.$$

From which it will be apparent the product from these two factors always to be the same. So that the resolution of which may be put in place more easily, the numbers μ and ν may be taken thus, so that the sum of these shall be $=1$, evidently $\mu+\nu=1$, and there may be established,

$$\frac{1}{a}-\frac{(n-1)}{f}+\frac{n}{d}=\frac{-\mu n}{\nu d},\quad \text{there will be}\quad \frac{1}{a}=\frac{n-1}{f}-\frac{n}{\nu d},$$
$$\frac{1}{\alpha}-\frac{(n-1)}{g}+\frac{n}{d}=\frac{-\nu n}{\mu d},\quad \text{there will be}\quad \frac{1}{\alpha}=\frac{n-1}{g}-\frac{n}{\mu d}.$$

Whereby if the distance of the object may be taken thus $AE=a$, so that there becomes $\frac{1}{a}=\frac{n-1}{f}-\frac{n}{\nu d}$, the distance of the principal image will be had thus $BF=\alpha$, so that there shall be $\frac{1}{\alpha}=\frac{n-1}{g}-\frac{n}{\mu d}$.

In the same manner if the two distances may be given $AE=a$ and $BF=\alpha$ with the thickness of the lens $AB=d$, the radii of the faces f and g must be prepared thus, so that there shall be

$$\frac{1}{f}=\frac{1}{(n-1)a}+\frac{n}{\nu(n-1)d}\quad \text{et}\quad \frac{1}{g}=\frac{1}{(n-1)\alpha}+\frac{n}{\mu(n-1)d},$$

that which can be performed in an infinite number of ways, since the numbers μ and ν may be able to be taken at will, provided the sum of these $\mu+\nu$ may be equal to unity. If in this way also in the formulas found for the interval of diffusion all the letters in the denominators may be removed, there will be found:

$$Ff=\frac{(n-1)\alpha\alpha xx}{2nn}\left\{\begin{array}{l}+\frac{(\frac{1}{a}+\frac{1}{f})^2(\frac{1}{\alpha}-\frac{(n-1)}{g})^2(\frac{n+1}{a}+\frac{1}{f})}{(\frac{n-1}{f}-\frac{1}{a})^2}\\ +\frac{(\frac{1}{\alpha}+\frac{1}{g})^2(\frac{n-1}{f}-\frac{1}{a})^2(\frac{n+1}{\alpha}+\frac{1}{g})}{(\frac{1}{\alpha}-\frac{(n-1)}{g})^2}\end{array}\right\}$$

and

$$\text{the angle } BfN=\frac{\frac{n-1}{f}-\frac{1}{a}}{\frac{1}{\alpha}-\frac{(n-1)}{g}}\cdot\frac{x}{\alpha}=\frac{\frac{n}{vd}}{\frac{-n}{\mu d}}\cdot\frac{x}{\alpha}=-\frac{\mu}{v}\cdot\frac{x}{\alpha}.$$

But since there shall be $\frac{n-1}{f}-\frac{1}{a}=\frac{n}{vd}$ and $\frac{1}{\alpha}-\frac{(n-1)}{g}=-\frac{n}{\mu d}$, there will be

$$Ff=\frac{(n-1)\alpha\alpha xx}{2nn}\left(\frac{vv}{\mu\mu}(\frac{1}{a}+\frac{1}{f})^2(\frac{n+1}{a}+\frac{1}{f})+\frac{\mu\mu}{vv}(\frac{1}{\alpha}+\frac{1}{g})^2(\frac{n+1}{\alpha}+\frac{1}{g})\right)$$

and

$$\text{the angle } BfN=-\frac{\mu x}{v\alpha}=\frac{\mu x}{(\mu-1)\alpha}.$$

PROBLEM 5

41. *With the distances given before the lens* $AE=a$ (Fig. 1) *and with the principal image after the lens* $BF=\alpha$ *together with the thickness of the lens* $AB=d$, *to define all the satisfying lenses, and likewise for the individual diffusion lengths Ff.*

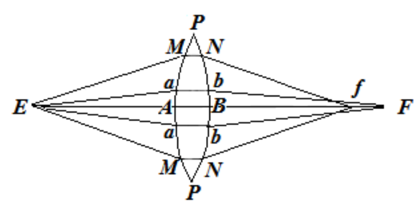

Fig. 1.

SOLUTION

Just as we have seen, if the radius of the foremost face of the lens *AM* may be put $=f$, and of the latter $BN=g$, so that each with the lens may be considered as convex, these two rays must be prepared thus , so that there shall be $\frac{n-1}{f}=\frac{1}{a}+\frac{n}{vd}$ and $\frac{(n-1)}{g}=\frac{1}{\alpha}+\frac{n}{\mu d}$, with any numbers taken for μ and v, so that there shall be $\mu+v=1$; from which sought indefinitely many satisfying lenses are obtained. Thence if the radius of the aperture of the lens may be put $=x$, the diffusion distance Ff can be expressed thus, so that there shall be

$$Ff=\frac{\alpha\alpha xx}{2nn(n-1)^2}\left\{\begin{array}{l}+\frac{\nu\nu}{\mu\mu}(\frac{n-1}{a}+\frac{n-1}{f})^2(\frac{nn-1}{a}+\frac{n-1}{f})\\+\frac{\mu\mu}{\nu\nu}(\frac{n-1}{\alpha}+\frac{n-1}{g})^2(\frac{nn-1}{\alpha}+\frac{n-1}{g})\end{array}\right\}$$

and

$$\text{the angle } BfN=\frac{\mu x}{(\mu-1)\alpha}.$$

But if now here for $\frac{n-1}{f}$ and $\frac{n-1}{g}$ the assigned values may be substituted, the diffusion length will be

$$Ff=\frac{n\alpha\alpha xx}{2(n-1)^2}\left(\frac{\nu\nu}{\mu\mu}(\frac{1}{a}+\frac{1}{\nu d})^2(\frac{n}{a}+\frac{1}{\nu d})+\frac{\mu\mu}{\nu\nu}(\frac{1}{\alpha}+\frac{1}{\mu d})^2(\frac{n}{\alpha}+\frac{1}{\mu d})\right)$$

or

$$Ff=\frac{n\alpha\alpha xx}{2(n-1)^2}\left\{\begin{array}{l}+\frac{\nu\nu}{\mu\mu}(\frac{n}{a^3}+\frac{2n+1}{\nu aad}+\frac{n+2}{\nu\nu add}+\frac{1}{\nu^3d^3})\\+\frac{\mu\mu}{\nu\nu}(\frac{n}{\alpha^3}+\frac{2n+1}{\mu\alpha\alpha d}+\frac{n+2}{\mu\mu\alpha dd}+\frac{1}{\mu^3d^3})\end{array}\right\}.$$

If the thickness of the lens were very small, lest the confusion may become very large, it is necessary for μ to ν to be taken as very large numbers, evidently with the one positive and the other negative. Therefore since there shall become $\mu d+\nu d=d$, there may be put

$$\mu d=\frac{d-k}{2} \quad\text{and}\quad \nu d=\frac{d+k}{2},$$

so that there shall be

$$\frac{n-1}{f}=\frac{1}{a}+\frac{2n}{k+d} \quad\text{and}\quad \frac{n-1}{g}=\frac{1}{\alpha}+\frac{2n}{k-d}$$

or

$$f=\frac{(n-1)a(k+d)}{2na+k+d} \quad\text{and}\quad g=\frac{(n-1)\alpha(k-d)}{k-d-2n\alpha},$$

and hence the diffusion length is obtained

$$Ff=\frac{n\alpha\alpha xx}{2(n-1)^2}\left\{\begin{array}{l}+(\frac{k+d}{k-d})^2(\frac{n}{a}+\frac{2}{k+d})(\frac{1}{a}+\frac{2}{k+d})^2\\+(\frac{k-d}{k+d})^2(\frac{n}{\alpha}-\frac{2}{k+d})(\frac{1}{\alpha}-\frac{2}{k+d})^2\end{array}\right\}$$

in which formulas the smallness of the thickness of the lens $AB=d$ has performed no function; then truly we have

$$\text{the angle } BfN=\frac{k-d}{k+d}\cdot\frac{x}{\alpha}.$$

COROLLARY 1

42. Therefore with the two distances $AE=a$ and $BF=\alpha$ proposed together with the thickness of the lens $AB=d$, suitable lenses can be prepared in an indefinite number of ways, since either positive or negative quantities may be assumed for k as it pleases.

COROLLARY 2

43. Since the radius of the anterior aperture x may be considered and the angle BfN or BFN shall be $=\frac{k-d}{k+d}\cdot\frac{x}{\alpha}$, with the distance present $BF=\alpha$, it is evident the radius of the latter aperture must be $=\frac{k-d}{k+d}\cdot x$ or at least not smaller.

COROLLARY 3

44. But if the thickness of the lens shall be so very small, so that besides k it may be ignored, our formulas become simpler

$$f=\frac{(n-1)ak}{k+2na} \text{ and } g=\frac{(n-1)\alpha k}{k-2n\alpha}$$

and the diffusion distance

$$Ff=\frac{n\alpha\alpha xx}{2(n-1)^2}\left((\tfrac{n}{a}+\tfrac{2}{k})(\tfrac{1}{a}+\tfrac{2}{k})^2+(\tfrac{n}{\alpha}-\tfrac{2}{k})(\tfrac{1}{\alpha}-\tfrac{2}{k})^2\right)$$

and the angle $BfN=\frac{x}{\alpha}$.

SCHOLIUM

45. This may suffice to have dealt with the diffusion distance in general, whatever the thickness of the lens, since without the aid from these neat enough transformations the calculation would be exceedingly troublesome, if we may wish to have an account of the thickness of the lens in the determination of the diffusion length. But it will be allowed, just as we have seen, the thickness of the lens not only may be neglected, when that itself is exceedingly small, truly also provided first the quantity k were very small. And hence also in the following we will be able to judge, whether or not the thickness of the lens may be disregarded in the calculation, for in whatever case an assumed quantity will be considered for k, which if it were many times greater than d, no error will be required to cause concern; truly on the other hand if k does not exceed d by very much, it will be changed markedly, however small this same thickness were by itself.

So that Ff may become a minimum, it will be convenient to define the value of k in this manner.

Putting $\frac{k-d}{k+d}=z$, and for the minimum this equation is arrived at:

$$\begin{aligned}0=2z^4(\tfrac{n}{\alpha}+\tfrac{1}{d})(\tfrac{1}{\alpha}+\tfrac{1}{d})^2-\tfrac{2nz^3}{\alpha d}(\tfrac{1}{\alpha}+\tfrac{1}{d})-\tfrac{z^3}{d}(\tfrac{1}{\alpha}+\tfrac{1}{d})^2\\+\tfrac{2nz}{ad}(\tfrac{1}{a}+\tfrac{1}{d})+\tfrac{z}{d}(\tfrac{1}{a}+\tfrac{2}{d})^2-2(\tfrac{1}{a}+\tfrac{1}{d})(\tfrac{1}{a}+\tfrac{1}{d})^2,\end{aligned}$$

from which the value of z must be found.

PROBLEM 6

46. *With the thickness of the lens ignored, if there may be given with the distance of the object before the lens $AE = a$ then with the distance of the image after the lens $BF = \alpha$, to define the lens, which for a given aperture may produce the minimum diffusion.*

SOLUTION

With the radius of the foremost face $AM = f$ and of the posterior $BN = g$, so that each may be considered convex, we have seen all the lenses with the distances a and α to be determined agreeing with these formulas, if evidently for k we may write $2k$:

$$\frac{n-1}{f} = \frac{1}{a} + \frac{n}{k} \text{ and } \frac{n-1}{g} = \frac{1}{\alpha} - \frac{n}{k}$$

with k denoting some quantity. But then the diffusion distance found expressed thus

$$Ff = \frac{n\alpha\alpha xx}{2(n-1)^2}\left((\tfrac{n}{a} + \tfrac{2}{k})(\tfrac{1}{a} + \tfrac{1}{k})^2 + (\tfrac{n}{\alpha} - \tfrac{1}{k})(\tfrac{1}{\alpha} - \tfrac{1}{k})^2 \right)$$

which expression is reduced to this form:

$$Ff = \frac{n\alpha\alpha xx}{2(n-1)^2}(\tfrac{1}{a} + \tfrac{1}{\alpha})\left(n(\tfrac{1}{aa} - \tfrac{1}{a\alpha} + \tfrac{1}{\alpha\alpha}) + \tfrac{2n+1}{k}(\tfrac{1}{a} - \tfrac{1}{\alpha}) + \tfrac{n+2}{kk} \right).$$

Therefore the question here is reduced to this, so that the quantity k may be defined, by which the minimum value of this expression may be acquired ; for which indeed the required value satisfies

$$\frac{1}{k} = \frac{-(2n+2)}{2(n+2)}(\tfrac{1}{a} - \tfrac{1}{\alpha}) \quad \text{or} \quad \frac{1}{k} = \frac{2n+1}{2(n+2)}(\tfrac{1}{\alpha} - \tfrac{1}{a})$$

with which satisfied there will be found :

$$\frac{n-1}{f} = \frac{4+n-2nn}{2(n+2)a} + \frac{n(2n+1)}{2(n+2)\alpha} \quad \text{and} \quad \frac{n-1}{g} = \frac{4+n-2nn}{2(n+2)\alpha} + \frac{n(2n+1)}{2(n+2)a}$$

and the diffusion distance, which is a minimum:

$$Ff = \frac{n\alpha\alpha xx}{2(n-1)^2}(\tfrac{1}{a} + \tfrac{1}{\alpha})\left(n(\tfrac{1}{aa} - \tfrac{1}{a\alpha} + \tfrac{1}{\alpha\alpha}) - \tfrac{(2n+1)^2}{4(n+2)}(\tfrac{1}{a} - \tfrac{1}{\alpha})^2 \right).$$

But there is

$$n(\tfrac{1}{aa}-\tfrac{1}{a\alpha}+\tfrac{1}{\alpha\alpha})=n(\tfrac{1}{a}-\tfrac{1}{\alpha})^2+\tfrac{n}{a\alpha}$$

and thus

$$Ff=\tfrac{n\alpha\alpha xx}{2(n-1)^2}(\tfrac{1}{a}+\tfrac{1}{\alpha})\left(\tfrac{4n-1}{4(n+2)}(\tfrac{1}{a}-\tfrac{1}{\alpha})^2+\tfrac{n}{a\alpha}\right),$$

which expression also is changed into this form

$$Ff=\tfrac{n\alpha\alpha xx}{2(n-1)^2}(\tfrac{1}{a}+\tfrac{1}{\alpha})\left(\tfrac{4n-1}{4(n+2)}(\tfrac{1}{a}+\tfrac{1}{\alpha})^2+\tfrac{(n-1)^2}{(n+2)a\alpha}\right)$$

or

$$Ff=\tfrac{n(4n-1)\alpha\alpha xx}{8(n-1)^2(n+2)}(\tfrac{1}{a}+\tfrac{1}{\alpha})\left((\tfrac{1}{a}+\tfrac{1}{\alpha})^2+\tfrac{4(n-1)^2}{(4n-1)a\alpha}\right).$$

COROLLARY 1

47. Therefore so that the lens may produce the minimum spatial diffusion, thus it is necessary that to formed, so that there shall be

$$f=\tfrac{2(n-1)(n+2)a\alpha}{n(2n+1)a+(4+n-nn)\alpha}\text{ and }g=\tfrac{2(n-1)(n+2)a\alpha}{n(2n+1)\alpha+(4+n-nn)a},$$

clearly with the thickness of the lens ignored, which indeed is ignored correctly, only if it were very small in comparison with the quantity $k=\tfrac{2(n+2)a\alpha}{(2n+1)(a-\alpha)}$.

COROLLARY 2

48. Therefore if there were $a=\alpha$ or $BF=AE$, the thickness of the lens, however great it were, disturbs nothing in the diffusion distance. But for this case there will be $f=g=(n-1)a$ and the diffusion distance itself $Ff=\tfrac{nnxx}{(n-1)^2a}$. But where the distances a and α disagree more between themselves or the smaller were the quantity $\tfrac{a\alpha}{a-\alpha}$, thus the more this determination of the thickness of the lens will become erroneous.

COROLLARY 8

49. But this minimum diffusion space Ff can be shown in a number of ways, among which the most suitable is agreed to be chosen ; these are in particular:

I. $Ff = \frac{n(4n-1)\alpha\alpha xx}{8(n-1)^2(n+2)}(\frac{1}{a}+\frac{1}{\alpha})\left((\frac{1}{a}+\frac{1}{\alpha})^2+\frac{4(n-1)^2}{(4n-1)a\alpha}\right).$

II. $Ff = \frac{n(4n-1)\alpha\alpha xx}{8(n-1)^2(n+2)}(\frac{1}{a}+\frac{1}{\alpha})\left((\frac{1}{a}-\frac{1}{\alpha})^2+\frac{4n(n+2)}{(4n-1)a\alpha}\right).$

III. $Ff = \frac{n(4n-1)\alpha\alpha xx}{8(n-1)^2(n+2)}(\frac{1}{a}+\frac{1}{\alpha})\left((\frac{1}{aa}+\frac{1}{\alpha\alpha})+\frac{2(2nn+1)}{(4n-1)a\alpha}\right).$

IV. $Ff = \frac{n(4n-1)\alpha\alpha xx}{8(n-1)^2(n+2)}(\frac{1}{a}+\frac{1}{\alpha})\left((\frac{1}{aa}-\frac{1}{a\alpha}+\frac{1}{\alpha\alpha})+\frac{(2n+1)^2}{(4n-1)a\alpha}\right).$

COROLLARY 4

50. Therefore since this diffusion distance shall be a minimum, if some other figure may be attributed to the lens, thus still, so that for the distance of the remote object $AE = a$ it may show the principal image at the distance $BF = \alpha$, the diffusion distance will be more than we have found here.

SCHOLIUM

51. Because $n{:}1$ denotes the ratio of the refraction from air into glass, as this is variable from the nature of the rays, it will be agreed to assume a mean value for *n*, which is $n = \frac{31}{20}$ hence therefore there will be

$$n-1=\tfrac{11}{20};\ n+2=\tfrac{71}{20};\ 2n+1=\tfrac{41}{10};\ 4+n-2nn=\tfrac{140}{200};$$

and hence

$$\frac{n(2n+1)}{2(n-1)(n+2)}=\frac{1271}{781}=1{,}627401\ ;\quad \frac{4+n-2nn}{2(n-1)(n+2)}=\frac{149}{781}=0{,}190781;$$

from which the lens producing the minimum diffusion may be defined thus

$$\frac{1}{f}=\frac{149}{781a}+\frac{1271}{781\alpha}=\frac{0{,}190781}{a}+\frac{1{,}627401}{\alpha},\ \frac{1}{g}=\frac{149}{781\alpha}+\frac{1271}{781a}=\frac{0{,}190781}{\alpha}+\frac{1{,}627401}{a}$$

or

$$f=\frac{781a\alpha}{149\alpha+1271a}=\frac{a\alpha}{0{,}190781\alpha+1{,}627401a},$$

and

$$g=\frac{781a\alpha}{149a+1271\alpha}=\frac{a\alpha}{0{,}190781a+1{,}627401\alpha}.$$

But for the diffusion length requiring to be defined on account of $4n-1=\frac{26}{5}$ there will be

$$\frac{n(4n-1)}{8(n-1)^2(n+2)}=\frac{8060}{8591}=0{,}938191 \text{ and } \frac{4(n-1)^2}{(4n-1)}=\frac{121}{520}=0{,}232692$$

and hence

$$Ff = 0,938191\,\alpha\alpha xx(\tfrac{1}{a}+\tfrac{1}{\alpha})\Big((\tfrac{1}{a}+\tfrac{1}{\alpha})^2+\tfrac{0,232692}{a\alpha}\Big)$$

from which the remaining formulas are deduced easily. Moreover here it is to be advised, because the value $n=\frac{31}{20}$ has been concluded from experiments, nor may it prevail for all the different kinds of rays, it being superfluous to observe that in practice these numbers found are exceedingly difficult to observe ; so that also by its very nature some small aberration is allowed. Truly by nothing less it is observed that these decimal fractions be produced to so many figures, so that it may be able to be discerned more easily, how great the experimental outcome differs from this hypothesis.

PROBLEM 7

52. *With the thickness of the lens ignored, if with the distance of the object before the lens given $AE = a$, then the distance of the principle image after the lens will be $BF = \alpha$, to define that lens, which for a given non-minimum aperture, but may bear a given distance of diffusion Ff.*

SOLUTION

With the lens considered as convex on both sides, the radius of the anterior face $= f$, of the posterior $= g$; and so that with the given distance of the object $AE = a$, the given distance of the principle image may arise $= \alpha$, it is necessary, in general there shall be

$$\tfrac{n-1}{f}=\tfrac{1}{a}+\tfrac{n}{k} \quad \text{and} \quad \tfrac{n-1}{g}=\tfrac{1}{\alpha}-\tfrac{n}{k},$$

from which the diffusion distance shall become

$$Ff=\tfrac{n\alpha\alpha xx}{2(n-1)^2}(\tfrac{1}{a}+\tfrac{1}{\alpha})\Big(n(\tfrac{1}{aa}-\tfrac{1}{a\alpha}+\tfrac{1}{\alpha\alpha})+\tfrac{2n+1}{k}(\tfrac{1}{a}-\tfrac{1}{\alpha})+\tfrac{n+2}{kk}\Big).$$

But since the minimum diffusion distance shall be found :

$$\tfrac{n(4n-1)\alpha\alpha xx}{8(n-1)^2(n+2)}(\tfrac{1}{a}+\tfrac{1}{\alpha})\Big((\tfrac{1}{a}+\tfrac{1}{\alpha})^2+\tfrac{4(n-1)^2}{(4n-1)a\alpha}\Big)$$

it is necessary, that another shall be greater. Therefore, there may be put:

$$Ff=\tfrac{n(4n-1)\alpha\alpha xx}{8(n-1)^2(n+2)}(\tfrac{1}{a}+\tfrac{1}{\alpha})\Big((\tfrac{1}{a}+\tfrac{1}{\alpha})^2+\tfrac{4(n-1)^2}{(4n-1)a\alpha}+S\Big)$$

and this same equation will be had :

$$4n(n+2)(\tfrac{1}{aa}-\tfrac{1}{a\alpha}+\tfrac{1}{\alpha\alpha})+\tfrac{4(n+2)(2n+1)}{k}\left(\tfrac{1}{a}-\tfrac{1}{\alpha}\right)+\tfrac{4(n+2)^2}{kk}=(4n-1)\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)^2+\tfrac{4(n-1)^2}{a\alpha}+(4n-1)S,$$

which is reduced to this form :

$$(2n+1)^2(\tfrac{1}{a}-\tfrac{1}{\alpha})^2+\tfrac{4(n+2)(2n+1)}{k}\left(\tfrac{1}{a}-\tfrac{1}{\alpha}\right)+\tfrac{4(n+2)^2}{kk}=(4n-1)S,$$

from which with the root extracted there shall become

$$(2n+1)(\tfrac{1}{a}-\tfrac{1}{\alpha})+\tfrac{2(n+2)}{k}=\sqrt{(4n-1)S},$$

and

$$\tfrac{1}{k}=\tfrac{-(2n+1)}{2(n+2)}(\tfrac{1}{a}-\tfrac{1}{\alpha})+\tfrac{\sqrt{(4n-1)S}}{2(n+2)}.$$

Whereby there shall be

$$\tfrac{n-1}{f}=\tfrac{1}{a}-\tfrac{n(2n+1)}{2(n+2)}(\tfrac{1}{a}-\tfrac{1}{\alpha})+\tfrac{n}{2(n+2)}\sqrt{(4n-1)S}$$

$$\tfrac{n-1}{g}=\tfrac{1}{\alpha}+\tfrac{n(2n+1)}{2(n+2)}(\tfrac{1}{a}-\tfrac{1}{\alpha})-\tfrac{n}{2(n+2)}\sqrt{(4n-1)S}$$

or

$$f=\frac{2(n-1)(n+2)a\alpha}{(4+n-2nn)\alpha+n(2n+1)a+na\alpha\sqrt{(4n-1)S}}$$

$$g=\frac{2(n-1)(n+2)a\alpha}{(4+n-2nn)a+n(2n+1)\alpha-na\alpha\sqrt{(4n-1)S}}.$$

Therefore it will be required for a positive quantity to be taken for *S*, and so that with the remaining part, to which it may be joined, it shall be homogeneous, we may put

$$S=(\lambda-1)(\tfrac{1}{a}+\tfrac{1}{\alpha})^2$$

where λ denotes a number greater than one ; and it will be effected, so that the diffusion distance may be expressed thus

$$Ff=\tfrac{n(4n-1)\alpha\alpha xx}{8(n-1)^2(n+2)}(\tfrac{1}{a}+\tfrac{1}{\alpha})\left(\lambda(\tfrac{1}{a}+\tfrac{1}{\alpha})^2+\tfrac{4(n-1)^2}{(4n-1)a\alpha}\right)$$

which since it may be considered given, the number λ can be assumed as given, and the lens hence producing this effect will be determined thus :

$$f = \frac{2(n-1)(n+2)a\alpha}{(4+n-2nn)\alpha+n(2n+1)a+n(a+\alpha)\sqrt{(4n-1)(\lambda-1)}}$$
$$g = \frac{2(n-1)(n+2)a\alpha}{(4+n-2nn)a+n(2n+1)\alpha-n(a+\alpha)\sqrt{(4n-1)(\lambda-1)}}.$$

which therefore, since it involves the root with both its signs, can arise in two ways.

COROLLARY 1

53. Therefore all the lenses, which for the distance $AE = a$ of the object represent the principal removed image at the distance $BF = \alpha$, hence have the property, that there shall be :

$$\frac{n-1}{f} + \frac{n-1}{g} = \frac{1}{a} + \frac{1}{\alpha} \text{ or } \frac{fg}{f+g} = \frac{(n-1)a\alpha}{a+\alpha},$$

and since there shall be $n = \frac{31}{20}$, there will be

$$\frac{fg}{f+g} = \frac{11a\alpha}{20(a+\alpha)} = \frac{0,55a\alpha}{a+\alpha}.$$

COROLLARY 2

54. Therefore if the lens shall have each side equally convex or $f = g$, there will be required to be

$$f = g = \frac{2(n-1)a\alpha}{a+\alpha} = \frac{11a\alpha}{10(a+\alpha)};$$

but if the lens may be required to be plano-convex, so that there shall be $f = \infty$, there must be taken

$$g = \frac{(n-1)a\alpha}{a+\alpha} = \frac{11a\alpha}{20(a+\alpha)} = \frac{0,55a\alpha}{a+\alpha}.$$

But if the lens may be convexo-planar or $g = \infty$, it will be required to take

$$f = \frac{(n-1)a\alpha}{a+\alpha} = \frac{11a\alpha}{20(a+\alpha)} = \frac{0,55a\alpha}{a+\alpha}.$$

SCHOLIUM 1

55. We may substitute for n the value agreeing to that $\frac{31}{20}$, and now we see to become:

$$\frac{n(4n-1)}{8(n-1)^2(n+2)}=\frac{8060}{8591}=0,938191;\quad \frac{4(n-1)^2}{(4n-1)}=\frac{121}{520}=0,232692;$$

$$\frac{4+n-2nn}{2(n-1)(n+2)}=\frac{149}{781}=0,190781\,;\frac{n(2n+1)}{2(n-1)(n+2)}=\frac{1271}{781}=1,627401;$$

now truly it is required to be noted :

$$\frac{n\sqrt{(4n-1)}}{2(n-1)(n+2)}=\frac{62\sqrt{130}}{781}=0,905133.$$

Whereby if the lens were constructed thus, so that there shall be

$$f=\frac{a\alpha}{0,190781\alpha+1,62740a\pm 0,905133(a+\alpha)\sqrt{(\lambda-1)}}$$
$$g=\frac{a\alpha}{0,190781a+1,62740\alpha\mp 0,905133(a+\alpha)\sqrt{(\lambda-1)}},$$

the diffusion distance will be for the aperture of this, of which the radius is $=x$,

$$Ff=0,938191\alpha\alpha xx(\tfrac{1}{a}+\tfrac{1}{\alpha})\left(\lambda(\tfrac{1}{a}+\tfrac{1}{\alpha})^2+\tfrac{0,232692}{a\alpha}\right).$$

But since in the following these numbers occur most frequently, in place of these we may make use of certain characters for abbreviation; therefore we may put

$$\frac{n(4n-1)}{8(n-1)^2(n+2)}=\frac{8060}{8591}=0,938191=\mu,\quad \frac{4(n-1)^2}{(4n-1)}=0,232692=\nu;$$
$$\frac{4+n-2nn}{2(n-1)(n+2)}=\frac{149}{781}=0,190781=\rho,\quad \frac{n(2n+1)}{2(n-1)(n+2)}=\frac{1271}{781}=1,627401=\sigma,$$
$$\frac{n\sqrt{(4n-1)}}{2(n-1)(n+2)}=0,905133=\tau$$

or

$$\mu=\frac{1}{4(n+2)}+\frac{1}{4(n-1)}+\frac{1}{8(n-1)^2},\quad \rho=\frac{1}{2(n-1)}+\frac{1}{n+2}-1,$$
$$\sigma=1+\frac{1}{2(n-1)}-\frac{1}{n+2},\qquad \tau=\tfrac{1}{3}\left(\frac{1}{2(n-1)}+\frac{1}{n+2}\right)\sqrt{(4n-1)},$$

from which for any other transparent medium, these values can be calculated out, and thus so that the diffusion space

$$Ff = \mu\alpha\alpha xx\left(\tfrac{1}{a}+\tfrac{1}{\alpha}\right)\left(\lambda(\tfrac{1}{a}+\tfrac{1}{\alpha})^2+\tfrac{\nu}{a\alpha}\right);$$

the construction of the lens will be

$$f = \frac{a\alpha}{\rho\alpha+\sigma a\pm\tau(a+\alpha)\sqrt{(\lambda-1)}}, \quad g = \frac{a\alpha}{\rho a+\sigma\alpha\mp\tau(a+\alpha)\sqrt{(\lambda-1)}}.$$

Therefore provided λ were a positive number not less than unity, such a lens will be able to be made in two ways. But in the case $\lambda = 1$, where the diffusion distance is a minimum, a single lens satisfying the proposed can be constructed.

COROLLARY 3

56. If therefore the lens shall be equally convex on both sides and thus

$$f = g = \frac{2(n-1)a\alpha}{a+\alpha} = \frac{11}{10}\cdot\frac{a\alpha}{a+\alpha},$$

in our expression for the diffusion length found the value of λ will be defined from the equation between f and g put in place, from which there becomes

$$(\sigma-\rho)(a-\alpha) = 2\tau(a+\alpha)\sqrt{(\lambda-1)}$$

and hence

$$\lambda = 1+\frac{0{,}629795\alpha\alpha-1{,}259589a\alpha+0{,}629795aa}{(a+\alpha)^2}.$$

COROLLARY 4

57. If the lens may be taken to be plano-convex, so that there shall be:

$$f = \infty \text{ and } g = \frac{(n-1)a\alpha}{a+\alpha} = \frac{11}{20}\cdot\frac{a\alpha}{a+\alpha},$$

there will be found for the diffusion length :

$$\rho\alpha+\sigma a = \mp\tau(a+\alpha)\sqrt{(\lambda-1)},$$

from which it may be deduced in numbers :

$$\lambda = 1 + \frac{0{,}044427\alpha\alpha + 0{,}757940a\alpha + 3{,}232692aa}{(a+a)^2}.$$

COROLLARY 5

58. Finally if the lens may be had convexo-planar, so that there shall be :

$$g = \infty \text{ and } f = \frac{(n-1)a\alpha}{a+\alpha} = \frac{11}{20} \cdot \frac{a\alpha}{a+\alpha},$$

for the diffusion length requiring to be found it will be required to put

$$\rho a + \sigma\alpha = \pm\tau(a+\alpha)\sqrt{(\lambda-1)},$$

from which we gather in numbers:

$$\lambda = 1 + \frac{3{,}232692\alpha\alpha + 0{,}7757940a\alpha + 0{,}044427aa}{(a+\alpha)^2}.$$

SCHOLIUM 2

59. Because it pertains to the aperture, now initially we have observed in these greater arcs are not to be considered, than those which conform to the principles of stability. Evidently, in order that no angles above 30° may occur, the angles *ACM* (Fig. 2) and *BDN* certainly must be less than 30 degrees, since the angles *EMc* and *VNd* themselves shall be greater,

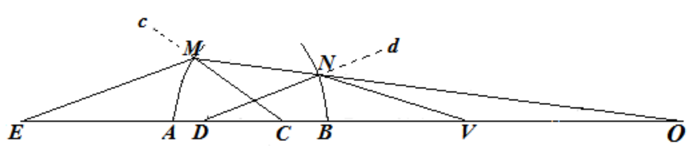

Fig. 2

of which the former since whenever it may be able to rise to both, we will have this rule in place, so that the radius x of the aperture will exceed neither $\frac{1}{4}f$ nor $\frac{1}{4}g$. Truly in whatever case for these angles *EMc* and *VNd,* which are the maximum, it will be agreed to note, and how great an aperture it will be able to allow, from which neither of these angles may arise greater than 30 degrees; if we may wish to proceed more cautiously, also we will be able to avoid angles greater than 20 , with which agreed on the aperture will be more restricted.

PROBLEM 8

60. *Without neglecting the thickness of the lens AB* (Fig. 3), *if for the distance* $AE = a$ *the principal distance of the image may be given* $BF = \alpha$ *, but the object may be moved a little further to e , to define the principal position of the image f.*

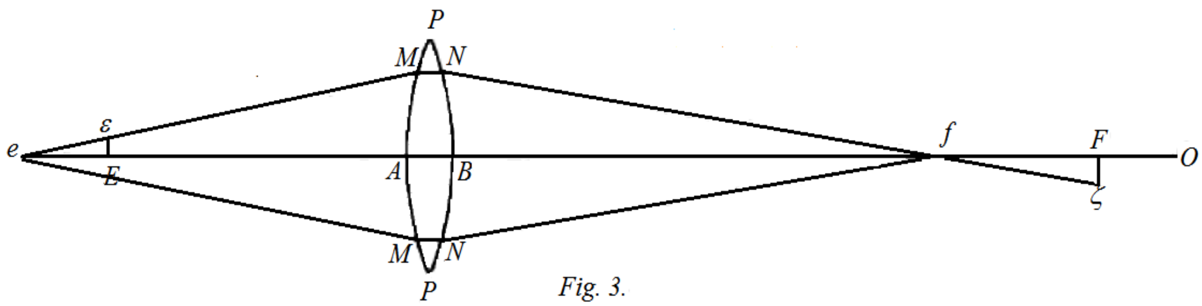

Fig. 3.

SOLUTION

With the thickness of the lens $AB = d$ and with the radius of the anterior face $AM = f$ and of the posterior $BN = g$, we have seen above these rays must depend on the two distances a and α and with the thickness of the lens d, so that there shall be

$$\frac{n-1}{f} = \frac{1}{a} + \frac{2n}{k+d} \quad \text{and} \quad \frac{n-1}{g} = \frac{1}{\alpha} - \frac{2n}{k-d}$$

with k denoting some quantity. Hence therefore since there shall be

$$\frac{k+d}{2n} = \frac{af}{(n-1)a-f} \quad \text{and} \quad \frac{k-d}{2n} = \frac{\alpha g}{g-(n-1)\alpha},$$

by eliminating k there shall be

$$\frac{d}{n} = \frac{af}{(n-1)a-f} - \frac{\alpha g}{g-(n-1)\alpha}.$$

Now we may put the distance $AE = a$ to increase by a little amount $Ee = da$, and by differentiation we will find, how much the image distance $BF = a$ may be changed; evidently we will have :

$$\frac{-ffda}{((n-1)a-f)^2} - \frac{ggd\alpha}{(g-(n-1)\alpha)^2} = 0,$$

from which we deduce

$$d\alpha = \frac{-ff(g-(n-1)\alpha)^2}{gg((n-1)a-f)^2} da = \frac{-\alpha\alpha}{aa} da \left(\frac{k+d}{k-d}\right)^2.$$

Whereby with the object *E* moved further back through the minimum interval *Ee* from the lens, the principal image will be moved from *F* towards the lens through the minimum interval *Ff,* thus so that there shall be

$$Ff = \frac{\alpha\alpha}{aa}\left(\frac{k+d}{k-d}\right)^2 .Ee = \frac{ff(g-(n-1)\alpha)^2}{gg((n-1)a-f)^2}.Ee.$$

COROLLARY 1

61. Because the quantity $\frac{\alpha\alpha}{aa}\left(\frac{k+d}{k-d}\right)^2$ by necessity is positive, it is evident, if the object may be moves further from the lens, the image always to be moved closer to the lens. Therefore also on the other hand, if the object may approach closer to the lens, the image recedes further from that.

COROLLARY 2

62. If the thickness of the lens *d* may vanish, there will be $Ff = \frac{\alpha\alpha}{aa}.Ee$; but if this may not happen, it can happen, so that if there shall be either $Ff > \frac{\alpha\alpha}{aa}.Ee$ or $Ff < \frac{\alpha\alpha}{aa}.Ee$; the first will happen, if k shall be a positive quantity, the latter if it shall be negative. But if there shall be either $k = \infty$ or $k = 0$, in each case there will be $Ff = \frac{\alpha\alpha}{aa}.Ee$, even if the thickness of the lens may not vanish.

COROLLARY 3

63. Since at the distances a et α minimal changes may be considered to happen, the diffusion distance thence is agreed to undergo no variation : therefore either the object may be found either at *E* or *e* and the radius of the aperture of the lens on the anterior face would be $= x$, the diffusion distance will be :

$$Ff = \frac{n\alpha\alpha xx}{2(n-1)^2}\left\{\begin{array}{l} +\left(\frac{k+d}{k-d}\right)^2\left(\frac{n}{a}+\frac{2}{k+d}\right)\left(\frac{1}{a}+\frac{2}{k+d}\right)^2 \\ +\left(\frac{k-d}{k+d}\right)^2\left(\frac{n}{\alpha}-\frac{2}{k+d}\right)\left(\frac{1}{\alpha}-\frac{2}{k+d}\right)^2 \end{array}\right\}$$

but in the posterior face the radius of the aperture must be $= \frac{k-d}{k+d}.x.$

PROBLEM 9

64. *To define the ratio, which the magnitude of the image has to the magnitude of the object without ignoring the thickness of the lens.*

SOLUTION

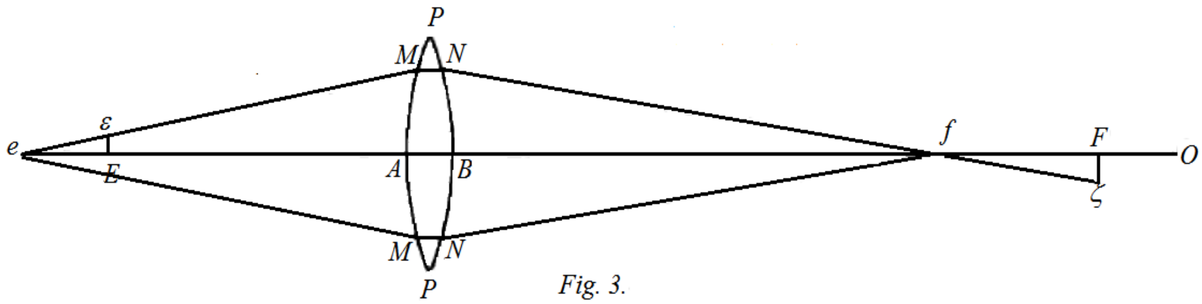

Fig. 3.

The distance of the object before the lens shall be $AE = a$ (Fig. 3), truly of the image $BF = \alpha$, with the thickness of the lens $= d$; where indeed we may only consider neglecting the diffusion length of the principal image. Now $E\varepsilon$ shall be the object, to which we may attribute a magnitude as small as $E\varepsilon = z$, standing normally to the axis of the lens, and of which the image will be shown at $F\zeta$; the magnitude of which $F\zeta$ is sought. Therefore since the point ζ may arise from the point ε, thus so that the rays emitted from ε may be gathered at ζ, some ray εM may be considered, which produced crosses with the axis at e : and thence it is as if this ray came from the point e on the axis. Whereby this will meet the axis at f after refraction, so that there shall be $Ff = \frac{\alpha\alpha}{aa}\left(\frac{k+d}{k-d}\right)^2 .Ee$, and thence will go to ζ : from which the magnitude $F\zeta$ will be allowed to be defined. To this end, there may be put $AM = x$, there will be $BN = \frac{k-d}{k+d}.x$; and hence we will deduce these proportions :

$$Ee : E\varepsilon = eA : AM = a : x, \quad Ff : F\zeta = fB : BN = \alpha : \tfrac{k-d}{k+d}x,$$

on account of the small distances Ee and Ff taken as minimal: from which we will have

$$\frac{Ff}{Ee} : \frac{F\zeta}{E\varepsilon} = \frac{\alpha}{a} : \frac{k-d}{k+d} = \frac{\alpha\alpha}{aa}\left(\frac{k+d}{k-d}\right)^2 : \frac{F\zeta}{E\varepsilon}.$$

Therefore it is concluded $\frac{F\zeta}{E\varepsilon} = \frac{\alpha}{a} : \frac{k-d}{k+d}$, from which, since the magnitude of the object $E\varepsilon$ shall be put $= z$, the magnitude of the image will be $F\zeta = \frac{\alpha(k+d)}{a(k-d)} z$.

COROLLARY 1

65. Therefore according to this account, the diameter of the object is changed. Where it is to be observed, if the expression $\frac{\alpha(k+d)}{a(k-d)}$ may have a positive value, the image of the object to be represented placed inverted, but on the other hand to be placed erect , if $\frac{\alpha(k+d)}{a(k-d)}$ arrives at a negative value.

COROLLARY 2

66. If the thickness of the lens may vanish, there shall become $F\zeta = \frac{\alpha}{a}z,$ therefore in which case a right line joining the extreme points ε and ζ passes through the centre of the lens. But if the thickness may be introduced into the calculation, the right line $\varepsilon\zeta$ sometimes will be able to cut within the lens, at other times beyond the lens.

SCHOLIUM

67. Thus if therefore we have arranged everything, which will be required to know about some lens, so that also we will have taken into account the thickness of the lens. And indeed in the first place it is agreed for two distances from the lens to be considered as if determinable, which are the distance of the object before the lens $AE = a$ and of the principal distance after the lens $BF = \alpha$, to which must be added the thickness of the lens $AB = d$. Moreover infinitely many lenses satisfy these conditions ; if indeed the radius of the anterior face AM may be called $= f$, and that of the posterior $BN = g$, as long as each may be considered as convex, the construction by the lens is contained in these formulas :

$$\frac{n-1}{f} = \frac{1}{a} + \frac{2n}{k+d} \quad \text{and} \quad \frac{n-1}{g} = \frac{1}{\alpha} - \frac{2n}{k-d}$$

or

$$f = \frac{(n-1)a(k+d)}{k+d+2na} \quad \text{and} \quad g = \frac{(n-1)\alpha(k-d)}{k-d-2n\alpha},$$

with $n = \frac{31}{20}$ being present; where k is an arbitrary quantity, and hence innumerable lenses are obtained satisfying the sought quantities.

So that if now the diameter of the object may be put $= z$, the diameter of the principal image represented will be $= \frac{\alpha(k+d)}{a(k-d)}z$, in as much as it is considered to be in place inverted.

Then if the radius of the aperture in the anterior face shall be $= x$, the diffusion length, in as much as it is stretched out from the principal image for the lens is expressed thus, so that there shall be :

$$\frac{n\alpha\alpha xx}{2(n-1)^2}\left\{\left(\frac{k+d}{k-d}\right)^2\left(\frac{n}{a}+\frac{2}{k+d}\right)\left(\frac{1}{a}+\frac{2}{k+d}\right)^2+\left(\frac{k-d}{k+d}\right)^2\left(\frac{n}{\alpha}-\frac{2}{k-d}\right)\left(\frac{1}{\alpha}-\frac{2}{k-d}\right)^2\right\}$$

but in the posterior face the radius of the aperture must be $=\frac{k-d}{k+d}\cdot x$ or at least not smaller. Since the diffusion distance has the factor $\alpha\alpha xx,$ for the sake of brevity we will indicate that by $P\alpha\alpha xx$. And these in general may contain also an account given of the thickness of the lens. But if the thickness of the lens may vanish, it will be permitted to establish more formulas, clearly if for brevity there may be put

$$\mu=0{,}938191,\ \rho=0{,}190781,\ \tau=0{,}905133,$$
$$\nu=0{,}232692,\ \sigma=1{,}627401,\ \lambda>1 \text{ arbitr.}$$

and for the formation of the lens there may be taken :

$$f=\frac{a\alpha}{\rho\alpha+\sigma a\pm\tau(a+\alpha)\sqrt{(\lambda-1)}},\ g=\frac{a\alpha}{\rho a+\sigma\alpha\mp\tau(a+\alpha)\sqrt{(\lambda-1)}},$$

the diffusion length will be for apertures with the radius x

$$\mu\alpha\alpha xx\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{\nu}{a\alpha}\right)$$

and with the diameter of the object present $=z$ the diameter of the image will be $=\frac{\alpha}{a}z$. Therefore with these determined for one lens, we may see, how in a combination of two or more lenses the diffusion length of the lenses may be defined : so that hence all in general to define the diffusion length in all kinds of dioptric instruments either for telescopes or microscopes and in that manner we may be able to investigate how that may be diminishing.

DIOPTRICAE

PARS PRIMA

CONTINENS

LIBRUM PRIMUM,

DE EXPLICATIONE PRINCIPIORUM,

EX QUIBUS

CONSTRUCTIO TAM TELESCOPIORUM

QUAM

MICROSCOPIORUM

EST PETENDA.

AUCTORE

LEONHARDO EULERO

ACAD. SCIENT. BORUSSIAE DIRECTORE VICENNALI ET SOCIO
ACAD. PETROP. PARISIN. ET LOND.

PETROPOLI
Impensis Academiae Imperialis Scientiarum
1769

LIBER PRIMVS CONTINENS EXPLICATIONEM PRINCIPIORUM, EX QUIBUS CONSTRUCTIO TAM TELESCOPIORUM QUAM MICROSCOPIORUM EST PETENDA.

CAPUT I

DE DIFFUSIONE IMAGINIS

PER UNICAM LENTEM REPRAESENTATAE

DEFINITIO 1

1. *Imago principalis vocatur ea, quae a radiis axi lentis proximis per lentem refractis repraesentatur*

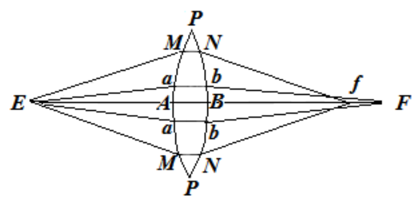

Fig. 1.

Scilicet si lentis *PP* (Fig. 1) axis sit *EF* in eaque tantum spatium minimum *aAa* sit apertum, per quod radiis transitus concedatur, radii a puncto lucido *E* emissi in uno puncto *F* colligentur, quod punctum imago principalis vocatur.

COROLLARIUM 1

2. Cum nempe apertura *aAa* sit minima, omnes radii, qui a puncto quocunque in eam incidunt, ita aequabilem patiuntur refractionem, ut omnes iterum in unum punctum colligantur, id quod non solum de puncto lucido *E* in axe lentis sito, sed etiam de quibusvis aliis extra axem positis est intelligendum.

COROLLARIUM 2

3. Quodsi ergo lentis apertura fuerit minima, singula cuiusvis obiecti puncta post refractionem iterum singulis punctis referentur, sicque imago principalis erit distincta et confusione carebit: siquidem confusio tum demum oritur, quando radii ex uno puncto emissi non iterum in uno puncto colliguntur.

COROLLARIUM 3

4. Et quamdiu apertura lentis *aAa* est minima, nihil interest, secundum quamnam figuram facies lentis sint elaboratae; quaecumque enim earum figura fuerit, quoniam tantum portiuncula minima in computum venit, ea semper ut sphaerica spectari poterit.

COROLLARIUM 4

5. Pendet ergo locus imaginis principalis primum a loco puncti lucidi *E,* sive id in axe sive extra axem fuerit situm; deinde a sphaericitate utriusque faciei *aAa* et *bBb* refringentis; tertio ab earum distantia *AB* seu lentis crassitie, et quarto a ratione refractionis, quam radii in transitu per lentem patiuntur.

SCHOLION 1

6. Refractio radiorum per huiusmodi lentium aperturas minimas transmissorum ideoque determinatio imaginum principalium satis accurate in elementis dioptricis tradi solet, quod igitur negotium hic fusius non prosequar; sed potius in eam refractionis rationem hic inquirere constitui, quando lentium apertura est modicae quantitatis, in quo imprimis ad utriusque faciei figuram est spectandum. Hic autem perpetuo lentium figuras sphaericas assumo, propterea quod haec figura vulgo lentibus induci vel saltem, nisi forte accurate successerit, intendi solet. In praxi certe nulla adhuc alia figura lentibus commode et accurate tribui poterit, atque adeo a sphaerica figura, etsi ad praxin maxime est accommodata, ab artificibus frequenter aberrari solet. A sollertioribus autem talia vitia iam plerumque satis feliciter evitantur, unde non adeo erit verendum, ne ea, quae per calculum ex hypothesi sphaericae figuras elicientur, experientiae non consentanea sint futura. Hanc ob rem hic perpetua postulo, ut ambae facies lentium exactissime secundum sphaericam figuram sint elaboratae.

SCHOLION 2

7. Sphaerica autem figura hoc laborat incommodo, quod statim ac lenti maior apertura tribuatur, non amplius omnes radii, qui quidem ab uno obiecti puncto sunt profecti, post refractionem ad unum punctum dirigantur radiique *EM* longius ab axe lentis transmissi non amplius in puncto *F* concurrant. Unde confusionem eo maiorem nasci necesse est, quo magis hi radii remotiores ab iis, qui prope axem transeunt, declinaverint; et quoniam talis declinatio a figura sphaerica originem ducit, eo maior evadet, quo maior lenti apertura tribuatur. Quanta igitur quovis casu futura sit haec confusio, hoc capite definire constitui; quae cum ceteris paribus a quantitate aperturae pendeat, hic in perpetuum moneo me cuiusvis lentis aperturam circularem assumere, per cuius centrum axis lentis transeat: ita ut semidiameter huius circuli simul mensuram aperturae exhibeat. Ita si in superficie lentis *PP* spatium *MAM* apertum relinquatur, reliqua parte *MP* velamine opaco obducta, punctorum extremorum *MM* distantia diametrum aperturae eiusque semissis semidiametrum aperturae praebebit.

DEFINITIO *2*

8. *Imago extrema est ea, quam radii per extremitatem aperturae transmissi exhibent.*

Ita si *MM* (Fig.1, p. 7) sit apertura lentis radiique *EM, EM* a puncto lucido *E* circa oram aperturae transmissi conveniant in puncto *f,* in hoc ipso puncto *f* erit imago extrema.

COROLLARIUM 1

9. Si punctum lucidum *E* est in ipso axe lentis, nullum est dubium, quin radii inde per marginem circularem *MM* transeuntes iterum in uno axis puncto *f* concurrant imaginemque distinctam repraesentent, quae imago extrema vocatur.

COROLLARIUM 2

10. Verum si punctum lucidum non esset in axe lentis, hoc neutiquam eveniet, radiique per marginem illum circularem transmissi non amplius in uno puncto colligentur; unde hoc casu imago extrema eo magis erit confusa, quo magis punctum lucidum ab axe fuerit remotum.

SCHOLION

11. Quomodo se habeat refractio radiorum, quando punctum lucidum extra axem lentis fuerit constitutum, quaestio est non solum difficillima, sed etiam ita prolixis calculis involvitur, ut inde vix quicquam concludi possit. Ceterum in usu, ad quem lentes accommodantur, nunquam obiecta ab axe remotiora spectari solent, atque contentos esse nos oportet, dummodo obiecta in ipso axe lentis sita distincte repraesententur; neque etiam confusio, qua obiecta axi proxima afficiuntur, sensibilis esse potest: nam cum imago extrema puncti *E* in ipso lentis axe siti sit punctum *f,* nulla confusione inquinatum, etiamsi id parumper esset remotum. ab axe, vix sensibilis confusio se immiscere poterit. Quam ob causam investigationes sequentes tantum ad obiecta in ipso lentis axe sita adstringam.

DEFINITIO 3

12. *Spatium diffusionis vocatur intervallum inter imaginem principalem et extremam interceptum.*
Ita si imago principalis sit in *F* (Fig.1), extrema vero in *f,* intervallum *Ff* appellatur spatium diffusionis.

COROLLARIUM 1

13. Si ergo apertura lentis *MM* evanescat, spatium simul diffusionis evanescit, tum enim tantum radii axi proximi transmittuntur, quibus imago distincta in *F* effingitur. Ex quo intelligi licet, quo maior fuerit apertura lentis, eo maius fore spatium diffusionis *Ff.*

COROLLARIUM 2

14. Cum in *F* imago a radiis axi proximis, in *f* autem imago a radiis circa marginem circularem *MM* transmissis formetur: si totam aperturam lentis infinitis circulis concentricis divisam concipiamus, radii per singulos circulos transmissi imagines intermedias exhibebunt, quibus intervallum *Ff* replebitur.

COROLLARIUM 3

15. Si enim apertura primum nulla, tum vero continuo increscens statuatur, imago extrema primum cum principali congruet; tum vero continuo magis ab ea discedet, sicque, cum usque ad *MM* fuerit aucta, omnes illae imagines etiam nunc subsistent spatiumque *Ff* implebunt.

SCHOLION 1

16. Spatium hoc diffusionis causam continet confusionis, qua repraesentatio imaginis perturbatur; cum enim eiusdem puncti lucidi *E* infinitae imagines per intervallum *Ff* dispositae exhibeantur, earum commistio confusionem pariat necesse est, quae eo maior erit, quo maius fuerit spatium diffusionis *Ff.* Quemadmodum enim ad repraesentationem distinctam requiritur, ut omnes radii ex eodem obiecti puncto emissi iterum in unico puncto colligantur, ita, si hi radii in plura puncta coeant pluresque eiusdem puncti imagines referant, prout hae magis minusve inter se discrepant, maior inde minorve confusio nascetur. Quemadmodum autem hanc confusionem aestimari oporteat, deinceps demum explicare licebit, cum ante spatium diffusionis accurate definire docuerimus: quo circa in hoc capite, cum proposita fuerit lens quaecunque vitrea faciebus sphaericis terminata, pro quavis puncti lucidi *E* ab ea distantia et quavis apertura spatium diffusionis *Ff* investigare constitui. Quo facto eandem investigationem pro duabus pluribusve lentibus inter se coniunctis suscipi conveniet, ut tandem inde confusionem in quibusvis instrumentis dioptricis assignare valeamus.

SCHOLION 2

17. Quaestio igitur principalis huius capitis in hoc versatur, ut proposita lente *PP* in eiusque axe puncto lucido *E,* radius quicunque incidens *EM* consideretur eiusque per lentem refractio definiatur, unde punctum *f,* ubi iterum in axem incidat, assignari queat. Cum enim in eodem puncto *f* omnes radii per totam peripheriam circularem *MM* transmissi

concurrant, ab his imago quaepiam puncti *E* in *f* exprimetur, quae erit extrema, si circulus *MM* in lente eius aperturam determinet; sin autem capiatur minor, imago quaedam intermedia habebitur. Cum autem hic duplex refractio eveniat, altera in ingressu radii *EM* in vitrum, altera in egressu eiusdem e vitro, quemadmodum eius directio in utraque inflectatur, seorsim est investigandum; unde duo nascuntur problemata quasi praeliminaria, ex quorum combinatione deinceps negotium conficietur. Verum quo haec problemata commodius calculo expediri queant, quaedam lemmata ex doctrina angulorum petita praemitti oportet.

LEMMA 1

18. *Si angulus* Φ *triginta gradus non excedat, eius sinus satis accurate erit* $\sin.\Phi = \Phi - \frac{1}{6}\Phi^3$, *si quidem in circulo, cuius radius est* $=1$, *arcus* Φ, *qui pro illius anguli mensura habetur, in partibus radii exprimatur.*

DEMONSTRATIO

Quantuscunque fuerit angulus Φ, notum est eius sinum hac serie infinita exprimi:

$$\sin.\Phi = \Phi - \frac{1}{6}\Phi^3 + \frac{1}{120}\Phi^5 - \frac{1}{5040}\Phi^7 + \text{etc.}$$

sumtis igitur tantum binis primis terminis, error committitur reliquis neglectis aequalis; si ergo statuamus $\sin.\Phi = \Phi - \frac{1}{6}\Phi^3$ hincque pro variis angulis Φ sinus colligamus, eorum comparatio cum tabulis sinuum errores manifestabit. Ita si sumatur $\Phi = 30°$, quia arcus $180°$ valet 3,14159265, erit in partibus radii $\Phi = 0,52359877$ et $\frac{1}{6}\Phi^3 = 0,0239246$, hincque

$$\Phi - \tfrac{1}{6}\Phi = 0,4996741. \text{ At est revera}$$
$$\sin.\Phi = 0,5000000, \text{ unde habetur}$$
$$\text{error} = 0,0003259,$$

qui ergo ne ad $\frac{1}{3000}$ partem radii quidem assurgit. At si angulus Φ caperetur duplo minor, scilicet $\Phi = 15°$, reperiretur

$$\Phi - \tfrac{1}{6}\Phi = 0,2588088. \text{ At est revera}$$
$$\sin.\Phi = 0,2588190, \text{ unde habetur}$$
$$\text{errore existante} = 0,0000102,$$

qui tricies bis minor est quam casu praecedente. Cum ergo in praxi error partem adeo termillesimam radii adaequans facile tolerari possit, multo magis, si angulus Φ fuerit minor $30°$, expressio $\sin.\Phi = \Phi - \frac{1}{6}\Phi^3$ verum eius sinum exhibere censenda erit.

LEMMA 2

19. *Vicissim si anguli triginta gradibus minoris detur sinus* $= s$, *ex eo ipse angulus ita proxime definitur, ut sit in circulo, cuius radius* $= 1$, *arcus eum metiens* $= s + \frac{1}{6}s^3$.

DEMONSTRATIO

Si enim Φ designet istum angulum, cuius sinus proponitur $= s$, modo vidimus esse satis exacte $s = \Phi - \frac{1}{6}\Phi^3$; hinc autem per conversionem oritur proxime $\Phi = s + \frac{1}{6}s^3$, quae expressio quantum peccet in angulo triginta graduum ut videamus, sit $s = \frac{1}{2}$, eritque $\Phi = s + \frac{1}{6}s^3 = 0,5208333$. Cum autem numerus 3,14159265 respondeat angulo $180°$, hic numerus 0,5208333 praebet angulum $29°50'30''$: verus autem angulus est $30°$, ita ut error sit $9'30''$. Sit autem $s = \frac{1}{4}$, cui sinui respondet angulus $= 14°28'39''$, erit

$$s + \frac{1}{6}s^3 = 0,2526041 = 14°28'23'',$$

ita ut hoc casu error non excedat $16''$. Patet ergo, dum angulus minor sit $30°$, hoc modo satis exacte ex dato sinu reperiri angulum.

PROBLEMA 1

20. *Si ex puncto lucido E* (Fig. 2) *in superficiem sphaericam revolutione arcus circularis AM circa axem EC genitam incidat radius EM, definire eius, postquam fuerit refractus, concursum cum axe EC.*

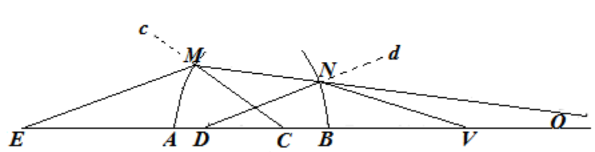

Fig. 2

SOLUTIO

Sit C centrum superficiei sphaericae, eiusque radius $CA = CM = f$, et distantia puncti lucidi E a superficie refringente $EA = a$, tum vero sit ratio refractionis radiorum ex aere in vitrum $= n:1$. Vocetur pro radio incidente CM angulus $AEM = \Phi$, eritque huius anguli sinus $=\Phi-\frac{1}{6}\Phi^3$. Ponatur iam brevitatis gratia distantia $EC=a+f=c$, et quia in triangulo *ECM* dantur latera $CM=f$, $EC=c$ cum angulo $CEM=\Phi$, fiet producta *CM* in *c*

$$CM\left(f\right):\sin.CEM=CE\left(c\right):\sin.EMc$$

ideoque

$$\sin.EMc=\frac{c}{f}\sin.\Phi=\frac{c}{f}\left(\Phi-\frac{1}{6}\Phi^3\right);$$

unde ipse angulus *EMc* elicitur neglectis tertia altioribus potestatibus ipsius Φ:

$$EMc=\frac{c}{f}\Phi-\frac{c}{6f}\Phi^3+\frac{c^3}{6f^3}\Phi^3=\frac{c}{f}\Phi+\frac{c(cc-ff)}{6f^3}\Phi^3;$$

qui cum sit $=ECM+CEM$, erit

$$ECM=\frac{c-f}{f}\Phi+\frac{c(cc-ff)}{6f^3}\Phi^3.$$

Verum angulus *EMc* est angulus incidentiae, ac si *MO* referat radium refractum, erit *CMO* angulus refractionis, sicque per hypothesin

$$\sin.EMc:\sin.CMO=n:1,$$

unde colligitur

$$\sin.CMO=\frac{c}{nf}\left(\Phi-\frac{1}{6}\Phi^3\right)$$

hincque ipse angulus

$$CMO=\frac{c}{nf}\Phi+\frac{c(cc-nnff)}{6n^3f^3}\Phi^3,$$

quo ablato ab angulo *ECM* relinquitur angulus

$$COM=\frac{(n-1)c-nf}{nf}\Phi+\frac{c((n^3-1)cc-nn(n-1)ff)}{6n^3f^3}\Phi^3,$$

cuius sinus propterea erit

$$\sin.COM=\frac{(n-1)c-nf}{nf}\Phi+\frac{3(n-1)c^3+3(n-1)^2ccf-4n(n-1)cff+nnf^3}{6nnf^3}\Phi^3.$$

Iamvero ob $\sin.COM:CM(f)=\sin.CMO:CO$ obtinebitur

$$CO=\frac{\frac{c}{n}-\frac{c}{6n}\Phi\Phi}{\frac{(n-1)c-nf}{nf}+\frac{3(n-1)c^3+3(n-1)^2ccf-4n(n-1)cff+nnf^3}{6nnf^3}\Phi^2},$$

cuius formulae evolutio praebet:

$$CO=\frac{cf}{(n-1)c-nf}-\frac{cf\Phi\Phi}{6((n-1)c-nf)}-\frac{c(3(n-1)c^3+3(n-1)^2ccf-4n(n-1)cff+nnf^3)}{6nf((n-1)c-nf)^2}\Phi\Phi,$$

quae porro reducitur ad hanc formam:

$$CO=\frac{cf}{(n-1)c-nf}-\frac{(n-1)cc(c-f)(c+nf)}{2nf((n-1)c-nf)^2}\Phi\Phi,$$

cui si addamus $CA=f$, prodibit

$$AO=\frac{nf(c-f)}{(n-1)c-nf}-\frac{(n-1)cc(c-f)(c+nf)}{2nf((n-1)c-nf)^2}\Phi\Phi,$$

Atque hinc positio radii refracti *MO* primo per intervallum *AO* modo inventum definitur, tum vero insuper ex angulo *AOM,* qui erit:

$$AOM=\frac{(n-1)c-nf}{nf}\Phi-\frac{c((n^3-1)cc-nn(n-1)ff)}{6n^3f^3}\Phi^3,$$

sive

$$AOM=\frac{(n-1)c-nf}{nf}\Phi-\frac{(n-1)c((nn+n+1)cc-nnff)}{6n^3f^3}\Phi^3.$$

COROLLARIUM 1

21. Cum posuerimus $c=a+f$, erit

$$c-f=a \text{ et } (n-1)c-nf=(n-1)a-f,$$

Atque

$$(nn+n+1)cc-nnff=(nn+n+1)aa+2(nn+n+1)af+(n+1)ff,$$

quibus valoribus substitutis habebimus

$$AO = \frac{naf}{(n-1)a-f} - \frac{(n-1)a(a+f)^2(a+(n+1)f)}{2nf((n-1)a-f)^2}\Phi\Phi$$

et

$$AOM = \frac{(n-1)a-f}{nf}\Phi + \frac{(n-1)(a+f)((nn+n+1)a(a+2f)+(n+1)ff)}{6n^3f^3}\Phi^3.$$

COROLLARY 2

22. Si *M* sit in extremitate aperturae, erit semidiameter aperturae $= f\sin.ECM = a\Phi$ satis exacte: neque enim necesse est aperturam tam exacte nosse. Unde si semidiameter aperturae ponatur $= x$, erit prope $x = a\Phi$ ideoque $\Phi = \frac{x}{a}$.

COROLLARY 3

23. Si accuratius rem definire velimus, quia est

$$\sin.ECM = \frac{c-f}{f}\Phi + \frac{(c-f)(3c-f)}{6ff}\Phi^3 = \frac{a}{f}\Phi + \frac{a(3a+2f)}{6ff}\Phi^3,$$

erit

$$x = \frac{a}{f}\Phi + \frac{a(3a+2f)}{6f}\Phi^3 \text{ ideoque } \Phi = \frac{x}{a} - \frac{(3a+2f)x^3}{6a^3f};$$

sed quia in valore ipsius *AO* non ultra secundam dimensionem ipsius Φ ascendimus, hac expressione non indigemus.

COROLLARY 4

24. Etsi hae expressiones tantum sunt prope verae, tamen in praxi sine errore adhiberi poterunt, dummodo anguli, qui in calculum sunt ingressi, infra 30° subsistant. Non solum ergo necesse est, ut angulus $AEM = \Phi$, sed etiam angulus *EMc* seu $\frac{c}{f}\Phi = \Phi + \frac{a}{f}\Phi$ minor sit 30 gradibus.

SCHOLION

25. Summo quidem rigore geometrico distantiam *EO* definire potuissemus, neque opus fuisset ad approximationes confugere: scilicet si posuissemus angulum $ECM = \omega$ et distantiam $CO = u$; devenissemus ad hanc determinationem

$$\frac{cf}{u}=-c\cos.\omega+\sqrt{(nn(c\cos.\omega-f)^2+(nn-1)cc\sin^2.\omega)}\ ,$$

foretque

$$\cos.\omega=\tfrac{c}{f}\sin^2.\Phi+\cos.\Phi\sqrt{\left(1-\tfrac{cc}{ff}\sin^2.\Phi\right)}$$

et

$$\sin.\omega=\tfrac{c}{f}\sin.\Phi\cos.\Phi-\sin.\Phi\sqrt{\left(1-\tfrac{cc}{ff}\sin^2.\Phi\right)}$$

hisque valoribus substitutis

$$\frac{cf}{u}=-\frac{cc\sin^2.\Phi}{f}-c\cos.\Phi\sqrt{\left(1-\tfrac{cc}{ff}\sin^2.\Phi\right)}$$
$$+\left(c\cos.\Phi-\sqrt{\left(ff-cc\sin^2.\Phi\right)}\right)\sqrt{\left(nn-\tfrac{cc}{ff}\sin^2.\Phi\right)}$$

unde ponendo angulo Φ valde parvo per approximationem superior expressio eliceretur. Verum praecedens analysis magis ad praesens institutum videtur accommodata. Interim si quis propius ad veritatem accedere voluerit, ex formula vera hic exhibita adipiscetur

$$\frac{cf}{u}=(n-1)c\cos.\Phi-nf+\frac{(n-1)cc}{2nff}((n-1)f+c\cos.\Phi)\sin^2.\Phi$$
$$\frac{c^4}{8n^3f^4}((nn-1)^2f+(n^2-1)c\cos.\Phi)\ \sin.^4\Phi$$

seu

$$\frac{cf}{u}=\left(n-1\right)c-nf+\frac{(n-1)c\left(c-f\right)\left(c+nf\right)}{2nff}\Phi\Phi$$
$$+\frac{(n-1)c\left((c-f)(3(nn+n+1)c^3+3nn(n+2)ccf+nn(3n-4)cff)-n^3f^3c\right)}{24n^3f^4}\Phi^4$$

PROBLEMA 2

26. *Si, prout in problemate praecedente invenimus, radius MO* (Fig. 2) *in vitrum missus per superficiem sphaericam BN iterum in aerem erumpat, definire punctum V, ubi is cum axe concurret.*

SOLUTIO

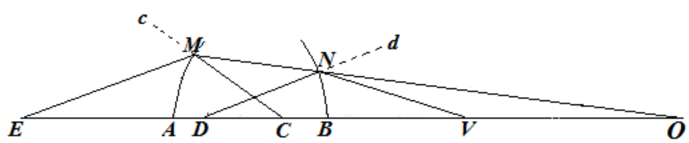

Fig. 2

Ponatur intervallum $AB = d$ sitque superficiei sphaerieae *BN* centrum in *D* eiusque radius $DB = DN = g$. Quoniam igitur positio radii incidentis *MNO* datur, ponamus $BO = b$ et angulum $BON = \Psi$ et brevitatis ergo intervallum $DO = b + g = e$. Cum nunc in triangulo *DON* dentur latera $DN = g$, $DO = e$ cum angulo $BON = \psi$, reperitur

$$\sin .DNM = \tfrac{e}{g}\sin .\Psi = \tfrac{e}{g}\Psi - \tfrac{e}{6g}\Psi^3$$

hincque

$$DNM = \tfrac{e}{g}\Psi = \tfrac{e}{g}\Psi + \tfrac{e(ee-gg)}{6g^3}\Psi^3 \text{ et } ODN = \tfrac{e-g}{g}\Psi = \tfrac{e}{g}\Psi + \tfrac{e(ee-gg)}{6g^3}\Psi^3$$

Sed hic sin. *DNM* est sinus incidentiae, cui respondet angulus refractionis *VNd,* cuius sinus propterea est ad illum ut n:1, unde fit

$$\sin .VNM = \tfrac{ne}{g}\Psi - \tfrac{ne}{6g}\Psi^3$$

hincque

$$VNd = \tfrac{ne}{g}\Psi + \tfrac{ne(nnee-gg)}{6g^3}\Psi^3,$$

a quo ablato angulo *ODN* relinquitur angulus

$$DVN = \frac{(n-1)e+g}{g}\Psi + \frac{e((n^3-1)ee-(n-1)gg)}{6g^3}\Psi^3,$$

ergo

$$\sin .DVN = \frac{(n-1)e+g}{g}\Psi + \frac{3n(n-1)e^3-3(n-1)^2 eeg-4(n-1)egg-g^3}{6g^3}\Psi^3.$$

Cum nunc sit $\sin.DVN : g = \sin.VNd : DV$, erit

$$\frac{g}{DV} = \frac{(n-1)e+g+\frac{1}{6gg}(3n(n-1)e^3-3(n-1)^2eeg-4(n-1)egg-g^3)\Psi^2}{ne-\frac{1}{6}ne\Psi^2},$$

quae expressio reducitur ad hanc formam:

$$\frac{g}{DV} = \frac{(n-1)e+g}{ne} + \frac{(n-1)(e-g)(ne+g)\Psi^2}{2ngg},$$

unde recîproce oritur

$$DV = \frac{neg}{(n-1)e+g} + \frac{n(n-1)ee(e-g)(ne+g)}{2g((n-1)e+g)^2}\Psi^2$$

et

$$BV = \frac{g(e-g)}{(n-1)e+g} + \frac{n(n-1)ee(e-g)(ne+g)}{2g((n-1)e+g)^2}\Psi^2.$$

Puncto autem V invento notandum est esse angulum

$$BVN = \frac{(n-1)e+g}{g}\Psi + \frac{(en(n^3-1)ee-(n-1)gg)}{6g^3}\Psi^3.$$

COROLLARIUM 1

27. Cum sit $c = b + g$, erit $(n-1)e + g = (n-1)b + ng$, quo valore restituto habebimus

$$BV = \frac{bg}{(n-1)b+ng} - \frac{n(n-1)b(b+g)^2(nb+(n+1)g)}{2g((n-1)b+ng)^2}\Psi^2$$

et

$$BVN = \frac{(n-1)b+ng}{g}\Psi + \frac{((b+g)(n^3-1)(b+g)^2-(n-1)gg)}{6g^3}\Psi^3.$$

COROLLARIUM 2

28. Hae formulae etiam ex praecedentibus erui possunt, si pro litteris *n, a, f* scribantur $\frac{1}{n}, -b$ et $-g$, quoniam hoc modo casus praecedentis problematis ad hunc reducitur. Praeterea autem, qui angulus ibi erat Φ, hic est Ψ.

COROLLARIUM 3

29. Quia in praecedente problemate invenimus tam lineam *AO* quam angulum *AOM,* erit hoc problemate ad radium illum refractum *MO* accommodato

$$BO = b = \frac{naf}{(n-1)a-f} - d - \frac{(n-1)a(a+f)^2(a+(n+1)f)}{2nf\left((n-1)a-f\right)^2}\Phi\Phi$$

et

$$\Psi = \frac{(n-1)a-f}{nf}\Phi + \frac{(n-1)(a+f)((nn+n+1)a(a+2f)+(n+1)ff)}{6n^3f^3}\Phi^3.$$

PROBLEMA 3

30. *Proposita lente vitrea MABN* (Fig. 2) *faciebus sphaericis AM et BN terminata, si a puncto quocunque E in eius axe posito incidat in eam radius EM, definire punctum V, in quo is post geminam refractionem iterum cum axe lentis sit concursurus.*

SOLUTIO

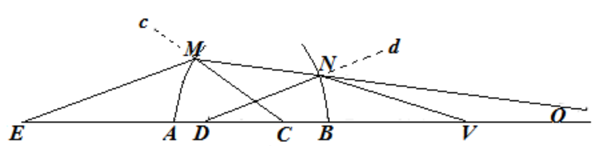

Fig. 2

Consideremus lentem ut utrinque convexam, sitque faciei anterioris *AM* radius $AC = f$, posterioris vero *BN* radius $BD = g$, ipsius lentis autem crassities $AB = d$. Lens porro sit vitrea, ita ut, si in eam radius lucis ex aere incidat, sit sinus incidentiae ad sinum refractionis ut *n* ad 1. Iam recta iungens centra utriusque faciei *a* et *D* erit axis lentis, in quo reperiatur punctum lucidum *E* ante lentem in distantia $AE = a$, unde sub angulo $AEM = \Phi$ in lentem incidat radius *EM,* qui prima refractione ita inflectatur, ut productus cum axe concurrat in *O.* Quodsi iam ex iis, quae problemate primo sunt inventa, ponamus

$$BO = \frac{naf}{(n-1)a-f} - d - \frac{(n-1)a(a+f)^2(a+(n+1)f)}{2nf((n-1)a-f)^2}\Phi\Phi = b$$

et

$$\text{ang. } BON = \frac{(n-1)a-f}{nf}\Phi = \Psi;$$

in valore enim anguli Ψ negligere licet terminum Φ^3 involventem, quoniam calculum tantum ad secundam potestatem ipsius Ψ extendimus; his positis in problemate secundo invenimus fore

$$BV=\frac{bg}{(n-1)b+ng}-\frac{n(n-1)b(b+g)^2(nb+(n+1)g))}{2g((n-1)b+ng)^2}\Psi\Psi$$

et

$$BVN=\frac{(n-1)b+ng}{g}\Psi.$$

Totum ergo negotium huc redit, ut isthic pro b et Ψ valores assignatos substituamus, quod quo facilius fieri possit, statuamus

$$b=P-Q\Phi\Phi \text{ et } \Psi=R\Phi,$$

ut sit

$$P=\frac{naf}{(n-1)a-f}\ -d-\frac{naf-(n-1)ad+df}{(n-1)a-f}$$

$$Q=\frac{(n-1)a(a+f)^2(a+(n+1)f)}{2nf((n-1)a-f)^2} \text{ et } R=\frac{(n-1)a-f}{nf}.$$

Hinc erit:

$$\frac{bg}{(n-1)b+ng}=\frac{Pg-Qg\Phi\Phi}{(n-1)P+ng-(n-1)Q\Phi\Phi}=\frac{Pg}{(n-1)P+ng}-\frac{nQgg\Phi\Phi}{((n-1)P+ng)^2};$$

at in altero membro sufficit pro b scribere P: ex quo obtinebimus

$$BV=\frac{Pg}{(n-1)P+ng}-\frac{nQgg\Phi\Phi}{((n-1)P+ng)^2}-\frac{n(n-1)PRR(b+g)^2(nP+(n+1)g))}{2g((n-1)P+ng)^2}\Phi\Phi$$

et

$$BVN=\frac{(n-1)P+ng}{g}R\Phi.$$

Pro his autem substitutionibus notandum est fore:

$$(n-1)P+ng=\frac{n(n-1)a(f+g)-nfg-(n-1)^2ad+(n-1)df}{(n-1)a-f}$$

$$P+g=\frac{naf+(n-1)ag-fg-(n-1)ad+df}{(n-1)a-f}$$

$$nP+(n+1)g=\frac{nnaf+(nn-1)ag-(n+1)fg-n(n-1)ad+ndf}{(n-1)a-f},$$

unde concluditur:

$$BV = \frac{nafg - n(n-1)adg + dfg}{n(n-1)a(f+g) - nfg - (n-1)^2 ad + (n-1)df}$$
$$- \frac{(n-1)agg(a+f)^2(a+(n+1)f)}{2f(n(n-1)a(f+g) - nfg - (n-1)^2 ad + (n-1)df)^2}\Phi\Phi$$
$$- \frac{(n-1)(naf-(n-1)ad+df)(naf+(n-1)ag-fg-(n-1)ad+df)^2(nnaf+(nn-1)ag-(n+1)fg-n(n-1)ad+ndf)}{2nffg(n(n-1)a(f+g)-nfg-(n-1)^2 ad+(n-1)df)^2}\Phi\Phi$$

et

$$BVN = \frac{n(n-1)a(f+g) - nfg + (n-1)df - (n-1)^2 ad}{nfg}\Phi.$$

COROLLARIUM 1

31. Si angulus Φ prorsus evanescat, punctum V cadet in imaginem principalem, cuius si a lente distantia dicatur $= \alpha$, erit

$$\alpha = \frac{nafg - (n-1)adg + dfg}{n(n-1)a(f+g) - nfg - (n-1)^2 ad + (n-1)df},$$

quae si ut data spectetur, hac aequatione relatio inter f et g definitur, ut haec distantia principalis oriatur.

COROLLARIUM 2

32. Quare si distantia obiecti ante lentem sit $= a$, eiusque imago principalis post lentem ad distantiam $= \alpha$ proiici debeat, huic aequationi satisfieri oportet:

$$n(n-1)a\alpha(f+g) - n\alpha fg - n\alpha fg + (n-1)adg + (n-1)\alpha df - (n-1)^2 a\alpha d - dfg = 0.$$

COROLLARIUM 3

33. Hanc autem imaginis principalis distantiam α in calculum introducendo expressiones nostrae inventae haud mediocriter contrahentur. Cum enim sit

$$\alpha = \frac{Pg}{(n-1)P + ng}, \text{ erit hinc } P = \frac{n\alpha g}{g-(n-1)\alpha},$$

et porro

$$(n-1)P + ng = \frac{ngg}{g-(n-1)\alpha}, \quad P + g = \frac{gg+\alpha g}{g-(n-1)\alpha} = \frac{g(a+g)}{g-(n-1)\alpha},$$

$$nP + (n+1)g = \frac{g(\alpha+(n+1)g)}{g-(n-1)\alpha},$$

unde facta substitutione sequentes determinationes simpliciores assequemur:

$$BV = \alpha - \frac{(n-1)a(a+f)^2(g-(n+1)\alpha)^2(a+(n+1)f)}{2nnffg((n-1)a-f)^2}\Phi\Phi$$
$$- \frac{(n-1)\alpha(\alpha+g)^2(g(n-1)a-f)^2(\alpha+(n+1)g)}{2nnffg(g-(n-1)\alpha)^2}\Phi\Phi$$

et

$$BVN = \frac{(n-1)a-f)g}{(g-(n-1)\alpha)f}\Phi.$$

SCHOLION

34. Non solum hae formulae multo sunt breviores et concinniores quam primo inventae, sed etiam perspicuus ordo in iis observatur, quo litterae α et g cum litteris a et f ita connexae sunt, ut permutationem admittant.
Nullum igitur est dubium, quin si statim distantiam α in calculum introduxissemus, via breviori ad eas pervenire licuisset. Ceterum quia hae formulae posteriores non amplius crassitiem lentis $AB = d$ involvunt, evidens est eas ex primo inventis nasci, si introducta distantia α crassities d eliminetur seu eius loco hic valor surrogetur:

$$d = \frac{n(n-1)a\alpha(f+g)-n(a+\alpha)fg}{(n-1)^2 a\alpha-(n-1)(ag+\alpha f)+fg} = n\frac{(n-1)a\alpha(f+g)-(a+\alpha)fg}{((n-1)a-f)((n-1)\alpha-g)},$$

qui labor autem ne suscipi quidem mereretur, nisi iam ante de eximio eius usu certiores essemus facti. In sequentibus igitur his elegantioribus formulis utamur, quoad negotium adhuc succinctius expedire didicerimus.

PROBLEMA 4

35. *Proposita lente quacunque faciebus sphaericis terminata, si obiectum in data ab ea distantia sit constitutum, pro data lentis apertura spatium diffusionis assignare.*

SOLUTIO

Concipiamus lentem utrinque convexam, sitque faciei anterioris *MAM* radius (Fig. 1) $= f$, posterioris $NBN = g$ lentisque crassities $AB = d$. Sit porro *MM* lentis huius apertura, cuius semidiameter sit $= x$; atque in axe

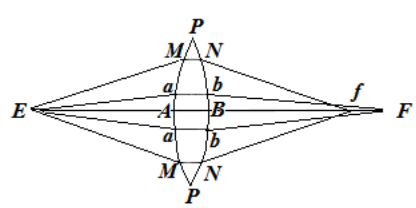

Fig. 1

lentis expositum sit obiectum vel saltem punctum lucidum *E,* cuius a lente ponatur distantia $AE = \alpha$. Iam primo quaeramus eius imaginem principalem, quae cadat in *F,* atque supra (§ 31) invenimus fore

$$BF = \frac{nafg-(n-1)adg+dfg}{n(n-1)a(f+g)-nfg-(n-1)^2 ad+(n-1)df},$$

Vocetur ergo haec distantia $= a$, et si radii *EM* circa oram aperturae transeant, erit angulus $AEM = \Phi = \frac{x}{a}$, quo valore in superioribus formulis substituto prodibit distantia imaginis extremae *f* a lente, scilicet:

$$Bf = \alpha - \frac{(n-1)a(a+f)^2(g-(n-1)\alpha)^2(a+(n+1)f)}{2nnfgg((n-1)a-f)^2}\cdot\frac{xx}{aa}$$
$$-\frac{(n-1)\alpha(\alpha+g)^2((n-1)a-f)^2(\alpha+(n+1)g)}{2nnfgg(g-(n-1)\alpha)^2}\cdot\frac{xx}{aa},$$

ex quo colligitur spatium diffusionis quaesitum:

$$Ff = \begin{cases} +\dfrac{(n-1)a(a+f)^2(g-(n-1)\alpha)^2(a+(n+1)f)}{2nnfgg((n-1)a-f)^2}\cdot\dfrac{xx}{aa} \\ +\dfrac{(n-1)\alpha(\alpha+g)^2((n-1)a-f)^2(\alpha+(n+1)g)}{2nnfgg(g-(n-1)\alpha)^2}\cdot\dfrac{xx}{aa}, \end{cases}$$

ac praeterea angulus *BfN* erit

$$BfN = \frac{((n-1)a-f)g}{(g-(n-1)\alpha)f}\cdot\frac{x}{a}.$$

COROLLARIUM 1

36. Quoties ergo spatium diffusionis hoc modo expressum est positivum, imago extrema propius ad lentem cadit quam principalis, seu est $Bf < BF$. Contra autem si ista expressio valorem obtineat negativum, imago extrema a lente longius erit remota principali.

COROLLARIUM 2

37. Patet hinc etiam spatium diffusionis cum apertura ita crescere, ut sit quadrato semidiametri aperturae proportionale; sequetur ergo ipsam aperturae rationem.

COROLLARIUM 3

38. Spatium diffusionis etiam hoc modo exprimi potest

$$Ff=\begin{cases}+\dfrac{(n-1)a(1+\frac{a}{f})^2(1-\frac{(n-1)\alpha}{g})^2(n+1+\frac{a}{f})}{2nn(\frac{(n-1)a}{f}-1)^2}\cdot\dfrac{xx}{aa}\\[2ex]+\dfrac{(n-1)\alpha(1+\frac{\alpha}{g})^2(\frac{(n-1)\alpha}{f}-1)^2(n+1+\frac{\alpha}{g})}{2nn(1-\frac{(n-1)\alpha}{f})^2}\cdot\dfrac{xx}{aa}.\end{cases}$$

et

$$\text{angulus } BfN=\frac{\frac{(n-1)a}{f}-1}{1-\frac{(n-1)\alpha}{g}}\cdot\frac{x}{a}.$$

COROLLARIUM 4

39. Quia hae formulae introducendis litterarum valoribus reciprocis redditae sunt simpliciores, in hunc modum etiam aequatio (§ 32) exhibita tractetur, quae per $a\alpha dfg$ divisa abit in hanc formam

$$n(n-1)\tfrac{1}{d}(\tfrac{1}{f}+\tfrac{1}{g})-n\tfrac{1}{d}(\tfrac{1}{a}+\tfrac{1}{\alpha})+(n-1)(\tfrac{1}{\alpha f}+\tfrac{1}{ag})-(n-1)^2\tfrac{1}{fg}-\tfrac{1}{a\alpha}=0$$

seu

$$\tfrac{n}{d}((n-1)a\alpha(\tfrac{1}{f}+\tfrac{1}{g})-a-\alpha)=(\tfrac{(n-1)a}{f}-1)(\tfrac{(n-1)\alpha}{g}-1)$$

quae commodior erit tam ad relationem inter a et a quam inter f et g definiendam.

SCHOLION

40. Scilicet si proposita lente obiectum in variis distantiis exponatur ac pro singulis distantiam imaginis principalis definire velimus, aequatio hoc modo referatur:

$$\tfrac{1}{a\alpha}-(n-1)(\tfrac{1}{\alpha f}+\tfrac{1}{ag})+n(\tfrac{1}{da}+\tfrac{1}{d\alpha})-n(n-1)(\tfrac{1}{df}+\tfrac{1}{dg})+(n-1)^2\tfrac{1}{fg}=0,$$

quae per factores ita adornari poterit

$$(\tfrac{1}{a}-\tfrac{(n-1)}{f}+\tfrac{n}{d})(\tfrac{1}{\alpha}-\tfrac{(n-1)}{g}+\tfrac{n}{d})=\tfrac{nn}{dd}.$$

Unde patet productum ex his duobus factoribus semper esse idem. Cuius resolutio quo facilius instituatur, capiantur numeri μ et v ita, ut eorum summa sit $=1$, scilicet $\mu+v=1$, ac statuatur

$$\frac{1}{a}-\frac{(n-1)}{f}+\frac{n}{d}=\frac{-\mu n}{vd},\quad \text{erit}\quad \frac{1}{a}=\frac{n-1}{f}-\frac{n}{vd},$$
$$\frac{1}{\alpha}-\frac{(n-1)}{g}+\frac{n}{d}=\frac{-vn}{\mu d},\quad \text{erit}\quad \frac{1}{\alpha}=\frac{n-1}{g}-\frac{n}{\mu d}.$$

Quare si distantia obiecti $AE=a$ ita capiatur, ut sit $\frac{1}{a}=\frac{n-1}{f}-\frac{n}{vd}$, distantia imaginis principalis $BF=\alpha$ ita se habebit, ut sit $\frac{1}{\alpha}=\frac{n-1}{g}-\frac{n}{\mu d}$.

Eodem modo si dentur binae distantiae $AE=a$ et $BF=\alpha$ cum crassitie lentis $AB=d$, radii facierum f et g ita debent esse comparati, ut sit

$$\frac{1}{f}=\frac{1}{(n-1)a}+\frac{n}{v(n-1)d}\quad \text{et}\quad \frac{1}{g}=\frac{1}{(n-1)\alpha}+\frac{n}{\mu(n-1)d},$$

id quod infinitis modis praestari potest, cum numeri μ et v pro arbitrio accipi queant, dummodo eorum summa $\mu+v$ aequetur unitati. Si hoc modo etiam in formulis pro spatio diffusionis inventis omnes litterae in denominatores detrudantur, reperietur:

$$Ff=\frac{(n-1)\alpha\alpha xx}{2nn}\left\{\begin{array}{l}+\dfrac{(\frac{1}{a}+\frac{1}{f})^2(\frac{1}{\alpha}-\frac{(n-1)}{g})^2(\frac{n+1}{a}+\frac{1}{f})}{(\frac{n-1}{f}-\frac{1}{a})^2}\\[2ex] +\dfrac{(\frac{1}{\alpha}+\frac{1}{g})^2(\frac{n-1}{f}-\frac{1}{a})^2(\frac{n+1}{\alpha}+\frac{1}{g})}{(\frac{1}{\alpha}-\frac{(n-1)}{g})^2}\end{array}\right\}$$

et

$$\text{angulus } BfN=\frac{\frac{n-1}{f}-\frac{1}{a}}{\frac{1}{\alpha}-\frac{(n-1)}{g}}\cdot\frac{x}{\alpha}=\frac{\frac{n}{vd}}{\frac{-n}{\mu d}}\cdot\frac{x}{\alpha}=-\frac{\mu}{v}\cdot\frac{x}{\alpha}.$$

Cum autem sit $\frac{n-1}{f}-\frac{1}{a}=\frac{n}{vd}$ et $\frac{1}{\alpha}-\frac{(n-1)}{g}=-\frac{n}{\mu d}$, erit

$$Ff=\frac{(n-1)\alpha\alpha xx}{2nn}\left(\frac{vv}{\mu\mu}(\frac{1}{a}+\frac{1}{f})^2(\frac{n+1}{a}+\frac{1}{f})+\frac{\mu\mu}{vv}(\frac{1}{\alpha}+\frac{1}{g})^2(\frac{n+1}{\alpha}+\frac{1}{g})\right)$$

et

$$\text{angulus } BfN=-\frac{\mu x}{v\alpha}=\frac{\mu x}{(\mu-1)\alpha}.$$

PROBLEMA 5

41. *Datis distantiis obiecti ante lentem* $AE = a$ (Fig. 1) *et imaginis principalis post lentem* $BF = \alpha$ *una cum lentis crassitie* $AB = d$, *definire omnes lentes satisfacientes, simulque pro singulis spatium diffusionis Ff.*

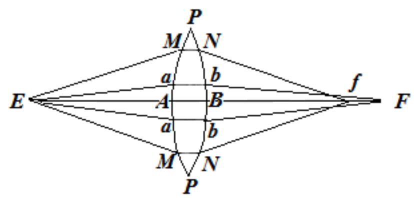

Fig. 1.

SOLUTIO

Modo vidimus, si lentis faciei anterioris *AM* radius ponatur $= f$, posterioris $BN = g$, lente ut convexa utriusque spectata, hos duos radios ita comparatos esse debere, ut sit $\frac{n-1}{f} = \frac{1}{a} + \frac{n}{\nu d}$ et $\frac{(n-1)}{g} = \frac{1}{\alpha} + \frac{n}{\mu d}$, sumtis pro μ et ν numeris quibuscunque, ut sit $\mu + \nu = 1$; unde infinitae lentes quaesito satisfacientes obtinentur. Deinde si huius lentis semidiameter aperturae ponatur $= x$, spatium diffusionis *Ff* ita exprimi potest, ut sit

$$Ff = \frac{\alpha\alpha xx}{2nn(n-1)^2}\left\{\begin{aligned}&+\tfrac{\nu\nu}{\mu\mu}\left(\tfrac{n-1}{a}+\tfrac{n-1}{f}\right)^2\left(\tfrac{nn-1}{a}+\tfrac{n-1}{f}\right)\\&+\tfrac{\mu\mu}{\nu\nu}\left(\tfrac{n-1}{\alpha}+\tfrac{n-1}{g}\right)^2\left(\tfrac{nn-1}{\alpha}+\tfrac{n-1}{g}\right)\end{aligned}\right\}$$

et

$$\text{angulus } BfN = \frac{\mu x}{(\mu-1)\alpha}.$$

Quod si iam hic pro $\frac{n-1}{f}$ et $\frac{n-1}{g}$ valores assignati substituantur, spatium diffusionis erit

$$Ff = \frac{n\alpha\alpha xx}{2(n-1)^2}\left(\frac{\nu\nu}{\mu\mu}\left(\frac{1}{a}+\frac{1}{\nu d}\right)^2\left(\frac{n}{a}+\frac{1}{\nu d}\right)+\frac{\mu\mu}{\nu\nu}\left(\frac{1}{\alpha}+\frac{1}{\mu d}\right)^2\left(\frac{n}{\alpha}+\frac{1}{\mu d}\right)\right)$$

seu

$$Ff = \frac{n\alpha\alpha xx}{2(n-1)^2}\left\{\begin{aligned}&+\tfrac{\nu\nu}{\mu\mu}\left(\tfrac{n}{a^3}+\tfrac{2n+1}{\nu aad}+\tfrac{n+2}{\nu\nu add}+\tfrac{1}{\nu^3 d^3}\right)\\&+\tfrac{\mu\mu}{\nu\nu}\left(\tfrac{n}{\alpha^3}+\tfrac{2n+1}{\mu\alpha\alpha d}+\tfrac{n+2}{\mu\mu\alpha dd}+\tfrac{1}{\mu^3 d^3}\right)\end{aligned}\right\}.$$

Si crassities lentis fuerit valde parva, ne confusio fiat enormis, necesse est pro μ et ν sumi numeros vehementer magnos, alterum scilicet positivum, alterum negativum. Cum igitur sit $\mu d+\nu d=d$, statuatur

$$\mu d=\tfrac{d-k}{2} \quad \text{et} \quad \nu d=\tfrac{d+k}{2},$$

ut sit

$$\tfrac{n-1}{f}=\tfrac{1}{a}+\tfrac{2n}{k+d} \text{ et } \tfrac{n-1}{g}=\tfrac{1}{\alpha}+\tfrac{2n}{k-d}$$

seu

$$f=\tfrac{(n-1)a(k+d)}{2na+k+d} \text{ et } g=\tfrac{(n-1)\alpha(k-d)}{k-d-2n\alpha},$$

hincque obtinetur spatium diffusionis

$$Ff=\tfrac{n\alpha\alpha xx}{2(n-1)^2}\left\{\begin{array}{l}+(\tfrac{k+d}{k-d})^2(\tfrac{n}{a}+\tfrac{2}{k+d})(\tfrac{1}{a}+\tfrac{2}{k+d})^2\\+(\tfrac{k-d}{k+d})^2(\tfrac{n}{\alpha}-\tfrac{2}{k+d})(\tfrac{1}{\alpha}-\tfrac{2}{k+d})^2\end{array}\right\}$$

in quibus formulis parvitas crassitiei lentis $AB=d$ nullum negotium facessit; tum vero est

$$\text{angulus } BfN=\tfrac{k-d}{k+d}\cdot\tfrac{x}{\alpha}.$$

COROLLARIUM 1

42. Propositis ergo binis distantiis $AE=a$ et $BF=\alpha$ una cum crassitie lentis $AB=d$, infinitis modis lentes idoneae parari possunt, cum pro k quantitates pro lubitu sive positivac sive negativae assumi queant.

COROLLARIUM 2

43. Cum semidiameter aperturae x faciem anteriorem respiciat et angulus BfN seu BFN sit $=\tfrac{k-d}{k+d}\cdot\tfrac{x}{a}$, existente distantia $BF=\alpha$, manifestum est semidiametrum aperturae posterioris faciei esse debere $=\tfrac{k-d}{k+d}\cdot x$ vel saltem non minorem.

COROLLARIUM 3

44. Quodsi lentis crassities tam sit parva, ut ea prae k contemni queat, formulae nostrae fient simpliciores

$$f=\tfrac{(n-1)ak}{k+2na} \quad \text{et} \quad g=\tfrac{(n-1)\alpha k}{k-2n\alpha}$$

et spatium diffusionis

$$Ff=\frac{n\alpha\alpha xx}{2(n-1)^2}\left((\tfrac{n}{a}+\tfrac{2}{k})(\tfrac{1}{a}+\tfrac{2}{k})^2+(\tfrac{n}{\alpha}-\tfrac{2}{k})(\tfrac{1}{\alpha}-\tfrac{2}{k})^2\right)$$

angulusque $BfN=\frac{x}{\alpha}$.

SCHOLION

45. Sufficiat haec de spatio diffusionis in genere, quaecunque fuerit lentis crassities, tradidisse, cum non obstante his satis concinnis transformationibus calculus nimium fieret molestus, si in determinatione spatii diffusionis rationem crassitiei lentis habere vellemus. Licebit autem, ut modo vidimus, crassitiem lentis negligera non solum, quando ipsa est per se valde exigua, verum etiam dummodo prae quantitate *k* fuerit perparva. Atque hinc etiam in sequentibus facile iudicarc poterimus, utrum crassitiem lentis recte in calculo contemserimus, nec ne? quovis enim casu consideretur quantitas pro *k* assumta, quae si multoties fuerit maior quam *d,* error nullus erit pertimescendus; contra vero si *k* non multum superet *d,* multum aberrabitur, quantumvis exigua fuerit crassities ipsa per se.

Ut fiat *Ff* minimum, definiri potest conveniens valor ipsius *k* hoc modo.
Ponatur $\frac{k-d}{k+d}=z$, et pro minima pervenitur ad hanc aequationem:

$$0=2z^4(\tfrac{n}{\alpha}+\tfrac{1}{d})(\tfrac{1}{\alpha}+\tfrac{1}{d})^2-\tfrac{2nz^3}{\alpha d}(\tfrac{1}{\alpha}+\tfrac{1}{d})-\tfrac{z^3}{d}(\tfrac{1}{\alpha}+\tfrac{1}{d})^2$$
$$+\tfrac{2nz}{ad}(\tfrac{1}{a}+\tfrac{1}{d})+\tfrac{z}{d}(\tfrac{1}{a}+\tfrac{2}{d})^2-2(\tfrac{1}{a}+\tfrac{1}{d})(\tfrac{1}{a}+\tfrac{1}{d})^2$$

unde valor ipsius z erui debet.

PROBLEMA 6

46. *Neglecta lentis crassitie, si detur cum obiecti ante lentem distantia $AE=a$ tum imaginis principalis post lentem distantia $BF=\alpha$, eam definire lentem, quae pro data apertura minimam pariat diffusionem.*

SOLUTIO

Positis radiis faciei anterioris $AM=f$ et posterioris $BN=g$, utraque ut convexa spectata, vidimus omnes lentes datis distantiis *a* et α convenientes his formulis determinari, si scilicet pro *k* scribamus 2*k*:

$$\frac{n-1}{f}=\frac{1}{a}+\frac{n}{k}\ \text{ et }\ \frac{n-1}{g}=\frac{1}{\alpha}-\frac{n}{k}$$

denotante *k* quantitatem quamcunque. Tum autem spatium diffusionis ita exprimi est repertum

$$Ff=\frac{n\alpha\alpha xx}{2(n-1)^2}\left((\tfrac{n}{a}+\tfrac{2}{k})(\tfrac{1}{a}+\tfrac{1}{k})^2+(\tfrac{n}{\alpha}-\tfrac{1}{k})(\tfrac{1}{\alpha}-\tfrac{1}{k})^2\right)$$

quae expressio ad hanc reducitur formam:

$$Ff=\frac{n\alpha\alpha xx}{2(n-1)^2}(\tfrac{1}{a}+\tfrac{1}{\alpha})\left(n(\tfrac{1}{aa}-\tfrac{1}{a\alpha}+\tfrac{1}{\alpha\alpha})+\tfrac{2n+1}{k}(\tfrac{1}{a}-\tfrac{1}{\alpha})+\tfrac{n+2}{kk}\right).$$

Quaestio igitur huc reducitur, ut definiatur quantitas *k,* qua huic expressioni valor minimus concilietur; cui quidem requisito satisfacit valor

$$\tfrac{1}{k}=\frac{-(2n+2)}{2(n+2)}(\tfrac{1}{a}-\tfrac{1}{\alpha})\quad\text{seu}\quad\tfrac{1}{k}=\frac{2n+1}{2(n+2)}(\tfrac{1}{\alpha}-\tfrac{1}{a})$$

quo substituto habebitur:

$$\frac{n-1}{f}=\frac{4+n-2nn}{2(n+2)a}+\frac{n(2n+1)}{2(n+2)\alpha}\quad\text{et}\quad\frac{n-1}{g}=\frac{4+n-2nn}{2(n+2)\alpha}+\frac{n(2n+1)}{2(n+2)a}$$

et spatium diffusionis, quod est minimum:

$$Ff=\frac{n\alpha\alpha xx}{2(n-1)^2}(\tfrac{1}{a}+\tfrac{1}{\alpha})\left(n(\tfrac{1}{aa}-\tfrac{1}{a\alpha}+\tfrac{1}{\alpha\alpha})-\frac{(2n+1)^2}{4(n+2)}(\tfrac{1}{a}-\tfrac{1}{\alpha})^2\right).$$

At est

$$n(\tfrac{1}{aa}-\tfrac{1}{a\alpha}+\tfrac{1}{\alpha\alpha})=n(\tfrac{1}{a}-\tfrac{1}{\alpha})^2+\tfrac{n}{a\alpha}$$

ideoque

$$Ff=\frac{n\alpha\alpha xx}{2(n-1)^2}(\tfrac{1}{a}+\tfrac{1}{\alpha})\left(\frac{4n-1}{4(n+2)}(\tfrac{1}{a}-\tfrac{1}{\alpha})^2+\tfrac{n}{a\alpha}\right),$$

quae expressio etiam in hanc formam transfunditur

$$Ff=\frac{n\alpha\alpha xx}{2(n-1)^2}(\tfrac{1}{a}+\tfrac{1}{\alpha})\left(\frac{4n-1}{4(n+2)}(\tfrac{1}{a}+\tfrac{1}{\alpha})^2+\frac{(n-1)^2}{(n+2)a\alpha}\right)$$

seu

$$Ff=\frac{n(4n-1)\alpha\alpha xx}{8(n-1)^2(n+2)}(\tfrac{1}{a}+\tfrac{1}{\alpha})\left((\tfrac{1}{a}+\tfrac{1}{\alpha})^2+\frac{4(n-1)^2}{(4n-1)a\alpha}\right).$$

COROLLARIUM 1

47. Ut igitur lens minimum spatium diffusionis producat, eam ita formari necesse est, ut sit

$$f=\frac{2(n-1)(n+2)a\alpha}{n(2n+1)a+(4+n-nn)\alpha} \text{ et } g=\frac{2(n-1)(n+2)a\alpha}{n(2n+1)\alpha+(4+n-nn)a}$$

neglecta scilicet lentis crassitie, quae quidem recte negligitur, si modo fuerit vehementer parva prae quantitate $k=\frac{2(n+2)a\alpha}{(2n+1)(a-\alpha)}$.

COROLLARIUM 2

48. Si igitur sit $a=\alpha$ seu $BF=AE$, crassities lentis, quantacunque fuerit, nihil turbat in spatio diffusionis. Pro hoc autem casu erit $f=g=(n-1)a$ et spatium diffusionis ipsum $Ff=\frac{nnxx}{(n-1)^2a}$. At quo magis distantiae a et α a se invicem discrepant seu minor fuerit quantitas $\frac{a\alpha}{a-\alpha}$, eo magis haec determinatio ob lentis crassitiem fiet erronea.

COROLLARIUM *8*

49. Spatium autem hoc confusionis minimum *Ff* pluribus modis exhiberi potest, inter quos commodissimum eligi convenit; sunt autem praecipui:

$$\text{I. } Ff=\frac{n(4n-1)\alpha\alpha xx}{8(n-1)^2(n+2)}\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{4(n-1)^2}{(4n-1)a\alpha}\right).$$

$$\text{II. } Ff=\frac{n(4n-1)\alpha\alpha xx}{8(n-1)^2(n+2)}\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\left(\frac{1}{a}-\frac{1}{\alpha}\right)^2+\frac{4n(n+2)}{(4n-1)a\alpha}\right).$$

$$\text{III. } Ff=\frac{n(4n-1)\alpha\alpha xx}{8(n-1)^2(n+2)}\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\left(\frac{1}{aa}+\frac{1}{\alpha\alpha}\right)+\frac{2(2nn+1)}{(4n-1)a\alpha}\right).$$

$$\text{IV. } Ff=\frac{n(4n-1)\alpha\alpha xx}{8(n-1)^2(n+2)}\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\left(\frac{1}{aa}-\frac{1}{a\alpha}+\frac{1}{\alpha\alpha}\right)+\frac{(2n+1)^2}{(4n-1)a\alpha}\right).$$

COROLLARIUM 4

50. Cum ergo hoc spatium diffusionis sit minimum, si lenti alia quaecunque figura tribuatur, ita tamen, ut obiecti ad distantiam $AE=a$ remoti imaginem principalem in distantia $BF=\alpha$ exhibeat, spatium diffusionis erit maius, quam hic invenimus.

SCHOLION

51. Quia $n:1$ denotat rationem refractionis ex aere in vitrum, haec autem pro radiorum natura est variabilis, conveniet pro n medium valorem assumi, qui est $n=\frac{31}{20}$ hinc ergo erit

$$n-1=\tfrac{11}{20};\ n+2=\tfrac{71}{20};\ 2n+1=\tfrac{41}{10};\ 4+n-2nn=\tfrac{140}{200};$$

hincque

$$\frac{n(2n+1)}{2(n-1)(n+2)}=\frac{1271}{781}=1,627401\ ;\quad \frac{4+n-2nn}{2(n-1)(n+2)}=\frac{149}{781}=0,190781;$$

unde lens minimam confusionem pariens ita definietur

$$\frac{1}{f}=\frac{149}{781a}+\frac{1271}{781\alpha}=\frac{0,190781}{a}+\frac{1,627401}{\alpha},\ \frac{1}{g}=\frac{149}{781\alpha}+\frac{1271}{781a}=\frac{0,190781}{\alpha}+\frac{1,627401}{a}$$

seu

$$f=\frac{781a\alpha}{149\alpha+1271a}=\frac{a\alpha}{0,190781\alpha+1,627401a},$$

et

$$g=\frac{781a\alpha}{149a+1271\alpha}=\frac{a\alpha}{0,190781a+1,627401\alpha}.$$

Pro spatio autem diffusionis ipso definiendo ob $4n-1=\frac{26}{5}$ erit

$$\frac{n(4n-1)}{8(n-1)^2(n+2)}=\frac{8060}{8591}=0,938191 \text{ et } \frac{4(n-1)^2}{(4n-1)}=\frac{121}{520}=0,232692$$

hincque

$$Ff=0,938191\alpha\alpha xx(\tfrac{1}{a}+\tfrac{1}{\alpha})\left((\tfrac{1}{a}+\tfrac{1}{\alpha})^2+\frac{0,232692}{a\alpha}\right)$$

unde reliquae formulae facile deducuntur. Ceterum hic monendum est, quoniam valor $n=\frac{31}{20}$ ex experimentis est conclusus neque ipse pro omnibus radiorum generibus valet, superfluum fore in praxi hos numeros inventos nimis studiose observare; quin etiam ipsa natura minimi aliquam aberrationem permittit. Nihilo vero minus has fractiones decimales ad tot figuras producere visum est, quo facilius, quantum ab hac hypothesi aberretur, dignosci queat.

PROBLEMA 7

52. *Neglecta lentis crassitie, si detur cum obiecti ante lentem distantia* $AE = a$ *tum imaginis principalis post lentem distantia* $BF = \alpha$ *, eam definire lentem, quae pro data apertura non minimum, sed datum pariat spatium diffusionis Ff.*

SOLUTIO

Lente utrinque ut convexa spectata sit faciei anterioris radius $= f$, posterioris $= g$; atque ut ex data distantia obiecti $AE = a$ data oriatur imaginis principalis distantia $= \alpha$, necesse est, sit in genere

$$\frac{n-1}{f} = \frac{1}{a} + \frac{n}{k} \quad \text{et} \quad \frac{n-1}{g} = \frac{1}{\alpha} - \frac{n}{k},$$

unde spatium diffusionis fit

$$Ff = \frac{n\alpha\alpha xx}{2(n-1)^2}\left(\frac{1}{a} + \frac{1}{\alpha}\right)\left(n\left(\frac{1}{aa} - \frac{1}{a\alpha} + \frac{1}{\alpha\alpha}\right) + \frac{2n+1}{k}\left(\frac{1}{a} - \frac{1}{\alpha}\right) + \frac{n+2}{kk}\right).$$

Cum autem spatium diffusionis minimum repertum sit

$$\frac{n(4n-1)\alpha\alpha xx}{8(n-1)^2(n+2)}\left(\frac{1}{a} + \frac{1}{\alpha}\right)\left(\left(\frac{1}{a} + \frac{1}{\alpha}\right)^2 + \frac{4(n-1)^2}{(4n-1)a\alpha}\right)$$

necesse est, ut illud sit maius. Statuatur ergo:

$$Ff = \frac{n(4n-1)\alpha\alpha xx}{8(n-1)^2(n+2)}\left(\frac{1}{a} + \frac{1}{\alpha}\right)\left(\left(\frac{1}{a} + \frac{1}{\alpha}\right)^2 + \frac{4(n-1)^2}{(4n-1)a\alpha} + S\right)$$

et habebitur ista aequatio:

$$4n(n+2)\left(\frac{1}{aa} - \frac{1}{a\alpha} + \frac{1}{\alpha\alpha}\right) + \frac{4(n+2)(2n+1)}{k}\left(\frac{1}{a} - \frac{1}{\alpha}\right) + \frac{4(n+2)^2}{kk} = (4n-1)\left(\frac{1}{a} + \frac{1}{\alpha}\right)^2 + \frac{4(n-1)^2}{a\alpha} + (4n-1)S,$$

quae in hanc formam redigitur

$$(2n+1)^2\left(\frac{1}{a} - \frac{1}{\alpha}\right)^2 + \frac{4(n+2)(2n+1)}{k}\left(\frac{1}{a} - \frac{1}{\alpha}\right) + \frac{4(n+2)^2}{kk} = (4n-1)S,$$

unde radice extracta fit

$$(2n+1)\left(\frac{1}{a} - \frac{1}{\alpha}\right) + \frac{2(n+2)}{k} = \sqrt{(4n-1)S},$$

et

$$\frac{1}{k}=\frac{-(2n+1)}{2(n+2)}\left(\frac{1}{a}-\frac{1}{\alpha}\right)+\frac{\sqrt{(4n-1)S}}{2(n+2)}.$$

Quare erit

$$\frac{n-1}{f}=\frac{1}{a}-\frac{n(2n+1)}{2(n+2)}\left(\frac{1}{a}-\frac{1}{\alpha}\right)+\frac{n}{2(n+2)}\sqrt{(4n-1)S}$$

$$\frac{n-1}{g}=\frac{1}{\alpha}+\frac{n(2n+1)}{2(n+2)}\left(\frac{1}{a}-\frac{1}{\alpha}\right)-\frac{n}{2(n+2)}\sqrt{(4n-1)S}$$

sive

$$f=\frac{2(n-1)(n+2)a\alpha}{(4+n-2nn)\alpha+n(2n+1)a+na\alpha\sqrt{(4n-1)S}}$$

$$g=\frac{2(n-1)(n+2)a\alpha}{(4+n-2nn)a+n(2n+1)\alpha-na\alpha\sqrt{(4n-1)S}}.$$

Oportet ergo pro S sumi quantitatem positivam, atque ut cum reliqua parte, cui iungitur, sit homogenea, ponamus

$$S=(\lambda-1)\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2$$

ubi λ denotat numerum unitate maiorem; atque effici poterit, ut spatium diffusionis ita exprimatur

$$Ff=\frac{n(4n-1)\alpha\alpha xx}{8(n-1)^2(n+2)}\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{4(n-1)^2}{(4n-1)a\alpha}\right)$$

quod cum ut datum spectetur, numerus λ pro dato assumi poterit, ac lens hunc effectum producens ita determinabitur:

$$f=\frac{2(n-1)(n+2)a\alpha}{(4+n-2nn)\alpha+n(2n+1)a+n(a+\alpha)\sqrt{(4n-1)(\lambda-1)}}$$

$$g=\frac{2(n-1)(n+2)a\alpha}{(4+n-2nn)a+n(2n+1)\alpha-n(a+\alpha)\sqrt{(4n-1)(\lambda-1)}}.$$

quod ergo, cum signum radicale ambiguo illatum involvat, duplici modo fieri poterit.

COROLLARIUM 1

53. Omnes igitur lentes, quae obiecti ad distantiam $AE=a$ remoti imaginem principalem in distantia $BF=\alpha$ repraesentant, hanc habent proprietatem, ut sit:

$$\frac{n-1}{f}+\frac{n-1}{g}=\frac{1}{a}+\frac{1}{\alpha}\quad\text{seu}\quad\frac{fg}{f+g}=\frac{(n-1)a\alpha}{a+\alpha},$$

et cum sit $n=\frac{31}{20}$, erit

$$\frac{fg}{f+g}=\frac{11a\alpha}{20(a+\alpha)}=\frac{0{,}55a\alpha}{a+\alpha}.$$

COROLLARIUM 2

54. Si ergo lens sit utrinque aeque convexa seu $f = g$, oportet esse

$$f = g = \frac{2(n-1)a\alpha}{a+\alpha} = \frac{11a\alpha}{10(a+\alpha)};$$

sin autem lens desideretur plano-convexa, ut sit $f = \infty$, capi debet

$$g = \frac{(n-1)a\alpha}{a+\alpha} = \frac{11a\alpha}{20(a+\alpha)} = \frac{0{,}55a\alpha}{a+\alpha}.$$

At si lens debeat esse convexo-plana seu $g = \infty$, capi oportet

$$f = \frac{(n-1)a\alpha}{a+\alpha} = \frac{11a\alpha}{20(a+\alpha)} = \frac{0{,}55a\alpha}{a+\alpha}.$$

SCHOLION 1

55. Substituamus pro n valorem ipsi convenientem $\frac{31}{20}$, et iam vidimus fore:

$$\frac{n(4n-1)}{8(n-1)^2(n+2)} = \frac{8060}{8591} = 0{,}938191; \quad \frac{4(n-1)^2}{(4n-1)} = \frac{121}{520} = 0{,}232692;$$

$$\frac{4+n-2nn}{2(n-1)(n+2)} = \frac{149}{781} = 0{,}190781\,; \frac{n(2n+1)}{2(n-1)(n+2)} = \frac{1271}{781} = 1{,}627401;$$

nunc vero notari oportet esse:

$$\frac{n\sqrt{(4n-1)}}{2(n-1)(n+2)} = \frac{62\sqrt{130}}{781} = 0{,}905133.$$

Quare si lens ita construatur, ut sit

$$f = \frac{a\alpha}{0{,}190781\alpha+1{,}62740a\pm 0{,}905133(a+\alpha)\sqrt{(\lambda-1)}}$$
$$g = \frac{a\alpha}{0{,}190781a+1{,}62740\alpha\mp 0{,}905133(a+\alpha)\sqrt{(\lambda-1)}},$$

erit pro eius apertura, cuius semidiameter $= x$, spatium diffusionis

$$Ff = 0{,}938191\alpha\alpha xx(\tfrac{1}{a}+\tfrac{1}{\alpha})\left(\lambda(\tfrac{1}{a}+\tfrac{1}{\alpha})^2 + \tfrac{0{,}232692}{a\alpha}\right)$$

Cum autem in posterum hi numeri frequentissime occurrant, eorum loco ad abbreviandum certis characteribus utamur; ponamus ergo

$$\frac{n(4n-1)}{8(n-1)^2(n+2)} = \frac{8060}{8591} = 0,938191 = \mu, \quad \frac{4(n-1)^2}{(4n-1)} = 0,232692 = \nu;$$
$$\frac{4+n-2nn}{2(n-1)(n+2)} = \frac{149}{781} = 0,190781 = \rho, \quad \frac{n(2n+1)}{2(n-1)(n+2)} = \frac{1271}{781} = 1,627401 = \sigma,$$
$$\frac{n\sqrt{(4n-1)}}{2(n-1)(n+2)} = 0,905133 = \tau$$

seu

$$\mu = \frac{1}{4(n+2)} + \frac{1}{4(n-1)} + \frac{1}{8(n-1)^2}, \quad \rho = \frac{1}{2(n-1)} + \frac{1}{n+2} - 1,$$
$$\sigma = 1 + \frac{1}{2(n-1)} - \frac{1}{n+2}, \qquad \tau = \frac{1}{3}\left(\frac{1}{2(n-1)} + \frac{1}{n+2}\right)\sqrt{(4n-1)},$$

unde pro quovis alio medio pellucido hi valores supputari possunt, sicque ut prodeat spatium diffusionis

$$Ff = \mu\alpha\alpha xx\left(\frac{1}{a} + \frac{1}{\alpha}\right)\left(\lambda(\frac{1}{a} + \frac{1}{\alpha})^2 + \frac{\nu}{a\alpha}\right);$$

lentis constructio erit

$$f = \frac{a\alpha}{\rho\alpha + \sigma a \pm \tau(a+\alpha)\sqrt{(\lambda-1)}}, \quad g = \frac{a\alpha}{\rho a + \sigma\alpha \mp \tau(a+\alpha)\sqrt{(\lambda-1)}}.$$

Dummodo igitur λ fuerit numerus positivus unitate non minor, talis lens duplici modo effici poterit. Casu autem $\lambda = 1$, quo spatium diffusionis est minimum, unica lens proposito satisfaciens construi potest.

COROLLARIUM 3

56. Si igitur lens sit utrinque aequaliter convexa ideoque

$$f = g = \frac{2(n-1)a\alpha}{a+\alpha} = \frac{11}{10}\cdot\frac{a\alpha}{a+\alpha},$$

in expressione nostra pro spatio diffusionis inventa valor ipsius λ ex aequalitate inter *f* et *g* statuta definietur, unde fit

$$(\sigma - \rho)(a - \alpha) = 2\tau(a+\alpha)\sqrt{(\lambda-1)}$$

hincque

$$\lambda = 1 + \frac{0,629795\alpha\alpha - 1,259589a\alpha + 0,629795aa}{(a+\alpha)^2}.$$

COROLLARIUM 4

57. Si lens capiatur plano-convexa, ut sit:

$$f=\infty \text{ et } g=\frac{(n-1)a\alpha}{a+\alpha}=\frac{11}{20}\cdot\frac{a\alpha}{a+\alpha},$$

pro spatio diffusionis habebitur

$$\rho\alpha+\sigma a=\mp\tau(a+\alpha)\sqrt{(\lambda-1)},$$

unde in numeris colligitur

$$\lambda=1+\frac{0{,}044427\alpha\alpha+0{,}757940a\alpha+3{,}232692aa}{(a+a)^2}.$$

COROLLARIUM 5

58. Si denique lens adhibeatur convexo-plana, ut sit:

$$g=\infty \text{ et } f=\frac{(n-1)a\alpha}{a+\alpha}=\frac{11}{20}\cdot\frac{a\alpha}{a+\alpha},$$

pro spatio diffusionis inveniendo poni oportet

$$\rho a+\sigma\alpha=\pm\tau(a+\alpha)\sqrt{(\lambda-1)},$$

unde in numeris colligimus:

$$\lambda=1+\frac{3{,}232692\alpha\alpha+0{,}7757940a\alpha+0{,}044427aa}{(a+\alpha)^2}.$$

SCHOLION 2

59. Quod ad aperturam attinet, iam initio animadvertimus in ea maiores arcus comprehendi non debere, quam qui sint principüs stabilitis conformes. Scilicet, ut nullus angulus supra 30° gradus occurrat, anguli *ACM* (Fig. 2) et *BDN* certe minores 30 gradibus esse debent, cum anguli *EMc* et

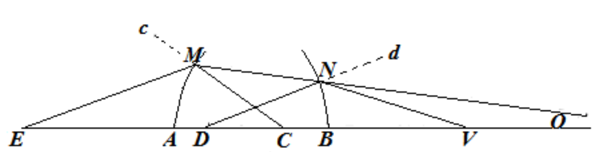

Fig. 2

VNd ipsis sint maiores, quorum alter cum quandoque ad duplum assurgere possit, poterimus hanc regulam statuere, ut aperturae semidiameter x neque $\frac{1}{4}f$ neque $\frac{1}{4}g$ superet. Verum quovis casu ad ipsos angulos *EMc* et *VNd,* qui sunt maximi, attendi conveniet, atque tanta apertura admitti poterit, unde neuter horum angulorum 30 gradus superans oriatur; si cautius procedera velimus, etiam angulos 20 gradibus maiores evitare poterimus, quo pacto apertura magis restringetur.

PROBLEMA 8

60. *Non neglecta lentis crassitie AB* (Fig. 3), *si pro distantia obiecti* $AE = a$ *detur distantia imaginis principalis* $BF = \alpha$ *, obiectum autem parumper longius in e removeatur, definire locum imaginis principalis f.*

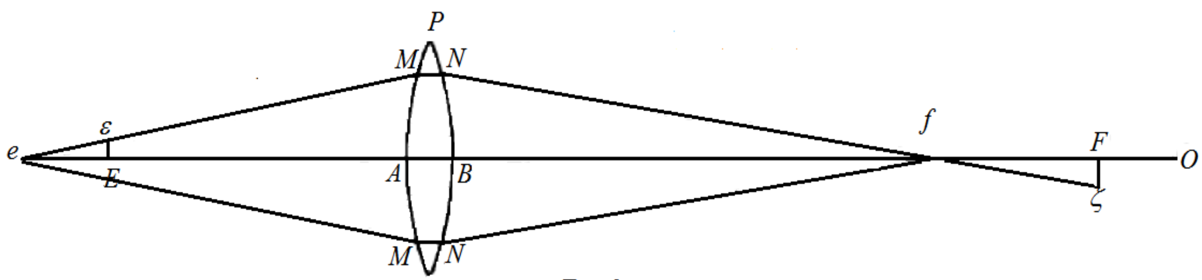

Fig. 3.

SOLUTIO

Posita lentis crassitie $AB = d$ radiisque faciei anterioris $AM = f$ et posterioris $BN = g$, supra vidimus hos radios ita a binis distantüs a et α atque crassitie lentis d pendere debere, ut sit

$$\frac{n-1}{f} = \frac{1}{a} + \frac{2n}{k+d} \quad \text{et} \quad \frac{n-1}{g} = \frac{1}{\alpha} - \frac{2n}{k-d}$$

denotante k quantitatem quamcunque. Hinc ergo cum sit

$$\frac{k+d}{2n} = \frac{af}{(n-1)a-f} \quad \text{et} \quad \frac{k-d}{2n} = \frac{\alpha g}{g-(n-1)\alpha},$$

erit eliminando k

$$\frac{d}{n} = \frac{af}{(n-1)a-f} - \frac{\alpha g}{g-(n-1)\alpha}.$$

Ponamus iam distantiam $AE = a$ crescere particula $Ee = da$, ac per differentiationem inveniemus, quantum inde distantia imaginis $BF = \alpha$ immutetur; habebimus scilicet:

$$\frac{-ffda}{((n-1)a-f)^2}-\frac{ggd\alpha}{(g-(n-1)\alpha)^2}=0,$$

unde elicimus

$$d\alpha=\frac{-ff(g-(n-1)\alpha)^2}{gg((n-1)a-f)^2}da=\frac{-\alpha\alpha}{aa}da\left(\frac{k+d}{k-d}\right)^2.$$

Quare obiecto E per spatiolum minimum Ee longius a lente remoto, imago principalis ex F propius ad lentem admovebitur per spatiolum minimum Ff, ita ut sit

$$Ff=\frac{\alpha\alpha}{aa}\left(\frac{k+d}{k-d}\right)^2.Ee=\frac{ff(g-(n-1)\alpha)^2}{gg((n-1)a-f)^2}.Ee.$$

COROLLARIUM 1

61. Quia quantitas $\frac{\alpha\alpha}{aa}\left(\frac{k+d}{k-d}\right)^2$ necessario est positiva, evidens est, si obiectum longius a lente removeatur, imaginem semper propius ad lentem admoveri. Contra ergo etiam, si obiectum propius ad lentem accedat, imago longius ab ea recedet.

COROLLARIUM 2

62. Si crassities lentis d evanescat, erit $Ff=\frac{\alpha\alpha}{aa}.Ee$; sin autem ea non evanescat, fieri potest, ut sit vel $Ff>\frac{\alpha\alpha}{aa}.Ee$ vel $Ff<\frac{\alpha\alpha}{aa}.Ee$; prius eveniet, si k sit quantitas positiva, posterius, si negativa. At si sit vel $k=\infty$ vel $k=0$, utroque casu erit $Ff=\frac{\alpha\alpha}{aa}.Ee$, etiamsi crassities lentis non evanescat.

COROLLARIUM 3

63. Cum in distantiis a et a mutatio minima fieri concipiatur, spatium diffusionis nullam inde variationem subire censendum est: sive ergo obiectum in E sive e reperiatur ac semidiameter aperturae lentis in facie anteriori fuerit $=x$, erit spatium diffusionis:

$$Ff=\frac{n\alpha\alpha xx}{2(n-1)^2}\begin{Bmatrix}+\left(\frac{k+d}{k-d}\right)^2\left(\frac{n}{a}+\frac{2}{k+d}\right)\left(\frac{1}{a}+\frac{2}{k+d}\right)^2\\+\left(\frac{k-d}{k+d}\right)^2\left(\frac{n}{\alpha}-\frac{2}{k+d}\right)\left(\frac{1}{\alpha}-\frac{2}{k+d}\right)^2\end{Bmatrix}$$

k-d in facie autem posteriori semidiameter aperturae debet esse $=\frac{k-d}{k+d}.x.$.

PROBLEMA 9

64. *Rationem definire, quam habet magnitudo imaginis ad magnitudinem obiecti non neglecta lentis crassitie.*

SOLUTIO

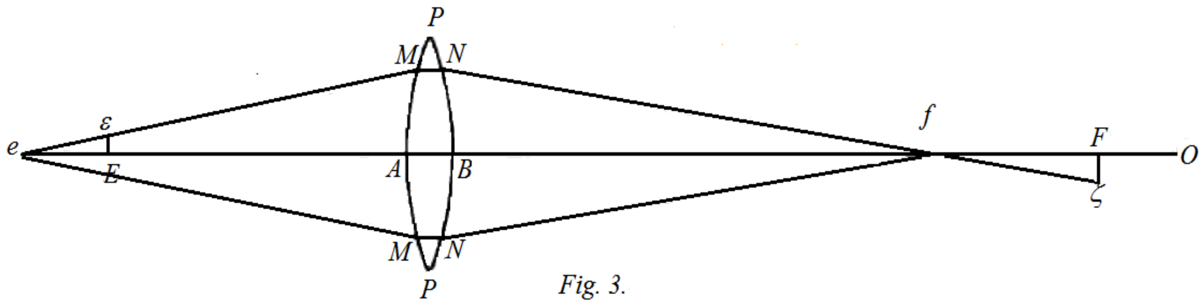

Fig. 3.

Sit obiecti ante lentem distantia $AE = a$ (Fig. 3), imaginis vero $BF = a$, existante lentis crassitie $= d$; ubi quidem tantum imaginem principalem spectamus neglecto spatio diffusionis. Sit iam $E\varepsilon$ obiectum, cui tribuatur magnitudo quam minima $E\varepsilon = z$, normaliter axi lentis insistens, eiusque imago in $F\zeta$ exhibebitur; cuius magnitudo $F\zeta$ quaeritur. Cum igitur punctum ζ a puncto ε oriatur, ita ut radii ab ε emissi in ζ colligantur, consideretur radius quicunque εM, qui productus cum axe in e occurrat: et perinde est, ac si hic radius ex axis puncto e emanaret. Quare is post refractionem cum axe in f concurret, ut sit $Ff = \frac{\alpha\alpha}{aa}\left(\frac{k+d}{k-d}\right)^2 .Ee,$ indeque ad ζ perget: ex quo magnitudinem $F\zeta$ definire licebit. Statuatur in hunc finem $AM = x$, erit $BN = \frac{k-d}{k+d}.x;$ hincque colligemus has proportiones:

$$Ee : E\varepsilon = eA : AM = a : x, \quad Ff : F\zeta = fB : BN = \alpha : \tfrac{k-d}{k+d}x$$

ob spatiola Ee et Ff quam minima: unde habebimus

$$\frac{Ff}{Ee} : \frac{F\zeta}{E\varepsilon} = \frac{\alpha}{a} : \frac{k-d}{k+d} = \frac{\alpha\alpha}{aa}\left(\frac{k+d}{k-d}\right)^2 : \frac{F\zeta}{E\varepsilon}.$$

Concluditur ergo $\frac{F\zeta}{E\varepsilon} = \frac{\alpha}{a} : \frac{k-d}{k+d}$, ex quo, cum magnitudo obiecti $E\varepsilon$ posita sit $= z$, erit magnitudo imaginis $F\zeta = \frac{\alpha(k+d)}{a(k-d)}z.$

COROLLARIUM 1

65. Secundum hanc ergo rationem diameter obiecti immutatur. Ubi notandum est, si expressio $\frac{\alpha(k+d)}{a(k-d)}$ habeat positivum valorem, obiecti imaginem situ inverso repraesentari, contra autem situ erecto, si $\frac{\alpha(k+d)}{a(k-d)}$ negativum valorem adipiscatur.

COROLLARIUM 2

66. Si crassities lentis evanescat, fit $F\zeta = \frac{\alpha}{a}z$, quo ergo casu recta iungens puncta extrema ε et ζ transit per centrum lentis. At si crassities in computum ducatur, recta $\varepsilon\zeta$ modo intra lentem, modo extra axem lentis secare poterit.

SCHOLION

67. Sic igitur omnia, quae circa unam lentem quamcunque nosse oportet, expedivimus, ut etiam crassitiei rationem habuerimus. Ac primo quidem ad binas distantias lentis quasi determinatrices spectari convenit, quae sunt obiecti distantia ante lentem $AE = a$ et imaginis principalis post lentem distantia $BF = \alpha$, quibus addi debet lentis crassities $AB = d$. His autem conditionibus infinitae lentes satisfaciunt; si enim radius faciei anterioris AM dicatur $= f$, et posterioris $BN = g$, utraque tanquam convexa considerata, constructio lentis his continetur formulis:

$$\frac{n-1}{f} = \frac{1}{a} + \frac{2n}{k+d} \quad \text{et} \quad \frac{n-1}{g} = \frac{1}{\alpha} - \frac{2n}{k-d}$$

seu

$$f = \frac{(n-1)a(k+d)}{k+d+2na} \quad \text{et} \quad g = \frac{(n-1)\alpha(k-d)}{k-d-2n\alpha},$$

existante $n = \frac{31}{20}$; ubi k est quantitas arbitraria, hincque lentes innumerabiles quaesito satisfacientes obtinentur.

Quod si iam obiecti diameter ponatur $= z$, erit imaginis principalis repraesentatae diameter $= \frac{\alpha(k+d)}{a(k-d)}z$, quatenus imago situ inverso exhibita consideratur.

Deinde si aperturae in facie anteriori lentis semidiameter sit $= x$, spatium diffusionis, quatenus ab imagine principali ad lentem porrigitur, ita exprimetur, ut sit:

$$\frac{n\alpha\alpha xx}{2(n-1)^2}\left\{\left(\frac{k+d}{k-d}\right)^2\left(\frac{n}{a}+\frac{2}{k+d}\right)\left(\frac{1}{a}+\frac{2}{k+d}\right)^2+\left(\frac{k-d}{k+d}\right)^2\left(\frac{n}{\alpha}-\frac{2}{k-d}\right)\left(\frac{1}{\alpha}-\frac{2}{k-d}\right)^2\right\}$$

in facie autem posteriori aperturae semidiameter debet esse $=\frac{k-d}{k+d}\cdot x$ vel saltem non minor. Quia spatium diffusionis factorem habet $a\alpha xx$, brevitatis gratia id ita $P\alpha\alpha xx$ indicabimus. Haecque in genere teneantur etiam crassitiei lentis ratione habita. At si crassities lentis evanescat, formulas magis evolvere licuit, scilicet si brevitatis gratia ponatur

$$\mu = 0{,}938191,\ \rho = 0{,}190781,\ \tau = 0{,}905133,$$
$$\nu = 0{,}232692,\ \sigma = 1{,}627401,\ \lambda > 1 \text{ arbitr.}$$

sumaturque pro formatione lentis

$$f = \frac{a\alpha}{\rho\alpha+\sigma\alpha\pm\tau(a+\alpha)\sqrt{(\lambda-1)}},\ g = \frac{a\alpha}{\rho\alpha+\sigma\alpha\mp\tau(a+\alpha)\sqrt{(\lambda-1)}},$$

spatium diffusionis erit pro aperturae semidiametro x

$$\mu\alpha\alpha xx(\tfrac{1}{a}+\tfrac{1}{\alpha})\left(\lambda(\tfrac{1}{a}+\tfrac{1}{\alpha})^2+\tfrac{\nu}{a\alpha}\right)$$

et obiecti diametro existente $=z$ imaginis diameter erit $=\frac{\alpha}{a}z$. His ergo pro una lente determinatis, videamus, quomodo in combinatione duarum pluriumve lentium spatium diffusionis definiatur: ut deinceps in omnis generis instrumentis dioptricis sive Telescopiis sive Microscopiis confusionem assignare modumque eam diminuendi investigare possimus.