NOTES ON EULER'S OPTICS

First let us recall that in Euler's algebraic scheme for treating lens problems, there are three kinds of ratios involved, which allow us to move from element to element via triangles involving some known common ratio or distance, more or less as we do geometrically with ray diagrams, or by using simple matrices multiplied together:

1. For the first lens/mirror element, the object distance is a, while the image distance is α; for the second lens/mirror, the object distance is b, while the image distance is β, etc., these being measured positive according to some convention, which becomes negative for inverted real images, but stay the same for virtual images. The object magnification of the individual elements is defined by A, B, C, etc, so that for the first element, the magnification $A=\frac{\alpha}{a}$, for the second element this becomes $B=\frac{\beta}{b}$, for the third $C=\frac{\gamma}{c}$, etc., arising from similar triangles. As we have observed many times, if the object is very distant, then the first image falls at the distance p, the first focal distance, so that $\alpha = p$; subsequent focal distances of the elements are designated by q, r, s, t, etc.

2. Within the intervals between consecutive elements, cross-element ratios are defined by $P, Q, R,$ etc., expressed in terms of the image distance of the first element to the object distance of the second element, such as (α, b), (β, c) for the second interval, etc., so that the ratio of the image distance of the first element: object distance of the second element becomes $P=\frac{\alpha}{b}$, for the second $Q=\frac{\beta}{c}$, $R=\frac{\gamma}{d}$ for the third , etc. Again, a minus sign may be included as required.

3. A third set of ratio parameters is introduced, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc., $\mathfrak{A}=\frac{A}{A+1}$, $\mathfrak{B}=\frac{B}{B+1}$, etc. such that $\mathfrak{A}a$, $\mathfrak{B}b$, $\mathfrak{C}c$, $\mathfrak{D}d$, $\mathfrak{E}e$, $\mathfrak{F}f$ etc. represent the focal lengths p, q, r, s, t, u, etc. of the elements

For here we have $\mathfrak{A}a=\frac{aA}{A+1}=\frac{a\frac{\alpha}{a}}{\frac{\alpha}{a}+1}=\frac{a\alpha}{\alpha+a}=p$, or $\frac{1}{p}=\frac{1}{a}+\frac{1}{\alpha}$,
in agreement with the customary thin lens formula.

Again, if $a\to\infty$, $\mathfrak{A}\to 0$, so that $\mathfrak{A}a\to p$.

With the aid of these ratios or operators, it is possible to give simple formulas for the development of the field of view progressively from the image produced by the first objective lens, by means of the diaphragms or stops inserted before each lens in turn. For as the magnification of the first image increases from lens to lens, the outer stray rays are successively removed, and the central part of the image remaining is enlarged, provided the radius of the stop is greater than some fraction of the associated focal length, so that finally we are observing only the central region greatly enlarged at a moderately large angle; essentially the telescope has given us a closer view of the object, now seen at an apparently greater angle of view.

Now the radius of the aperture of the first objective lens shall be $= x$, truly for the remaining lenses the angles subtended in radians by the successive apertures may be expressed clearly by a rule of thumb, so that it will be required to take the radius of each aperture to be greater than certain fractions π, π', π'', π''', π'''' etc. of the focal lengths of the relevant lenses, [Ch. 7, Book I]:

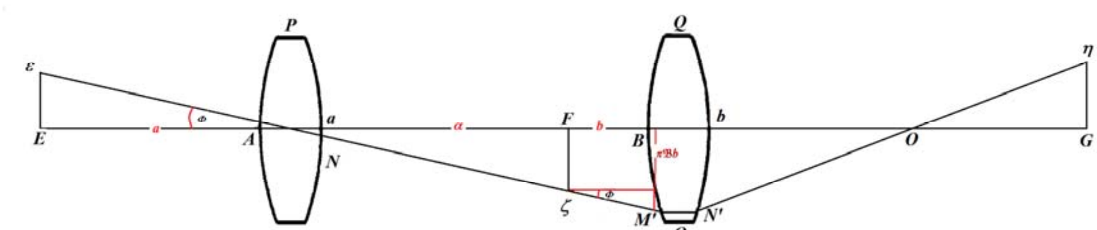


FIG. 13.

We may assume here initially that a ray passing through the centre of the first lens is considered to be undeviated, as the thickness of the lens is negligible; the angle of this ray to the axis is Φ: in which case $\Phi = \frac{z}{a}$. This ray proceeds undeviated to the position of the image at the distance α further along the axis, where the height of the image become $\alpha\Phi$. This ray continues to the second lens from the point ζ of this image by the horizontal distance b and the vertical distance $b\Phi$ where it encounters the aperture or stop which has been inserted to transmit central rays only; Hence, $\pi\mathfrak{B}b = F\zeta + b\Phi$; $b = \frac{F\zeta}{\pi\mathfrak{B}-\Phi} = \frac{\alpha\Phi}{\pi\mathfrak{B}-\Phi} = \frac{Aa\Phi}{\pi\mathfrak{B}-\Phi}$. The following images alternate between being inverted and erect and proceed as in the table below

radius of the aperture				
of the second lens				$> \pi\ \mathfrak{B}b$
"	"	third	"	$> \pi'\ \mathfrak{C}c$
"	"	fourth	"	$> \pi''\mathfrak{D}d$.
"	"	fifth	"	$> \pi'''\ \mathfrak{E}e$
"	"	sixth	"	$> \pi''''\ \mathfrak{F}f$
			etc.	

We now want to examine critically some of Euler's ideas concerning light; this includes some of the information derived from the *O.O.* introduction to the work translated from the German original.

I. COLOR DISPERSION

A major advance of Euler in optics is that he has pointed out the importance of the error introduced by the dispersion of colors.

An object generates different colored images of various magnifications V in the scope of Gaussian optics. Suppose there are the three colors, e.g. violet, yellow and red, and associated with them the refractive indices *nv, ny, nr*, so there corresponds to these the three colored images of magnitudes $V + dV$, V and $V - dV$ .The color magnification error is corrected for a particular position of the eye where the three images are viewed at the same angle of the eye and so overlap completely. This is not entirely the modern view of color vision, which still does not seem to have been unraveled completely. In this case these images do not show any colored fringes. It is to Euler's merit to have pointed out the importance of this angular size, the balance of which compensates for the chromatic aberration of the images along the optic axis and views them with the same magnification.

Since before Euler only the dependence of the width of the intersection was examined, which meant that Euler's method was a decisive step forward, despite the fact that he often neglected the previously studied dependence on the average value.

In contrast, Euler's dispersion formula is incorrect. Newton would have known from the fact that crown glass and water have different refractive indices, namely 1.56 and 1.33, but would have the same dispersion $v = \frac{n-1}{\Delta n}$.

In this introduction, by dispersion we mean the *v* value, i.e. the reciprocal of the magnitude used by Euler. This value was introduced by E. Abbe and is used in all kinds of glass to designate the dispersion.

Having drawn the wrong conclusion that all optical substances must have the same dispersion, Euler took it for granted to assume that the dispersion to be a monotonic function of the index. He concluded that it was is not possible to obtain large deviations in the dispersion. This is demonstrably wrong. Crown glass and flint glass differ little in the index, but the well-defined *v*-value for crown glass (and water) is nearly twice as large as for flint glass.

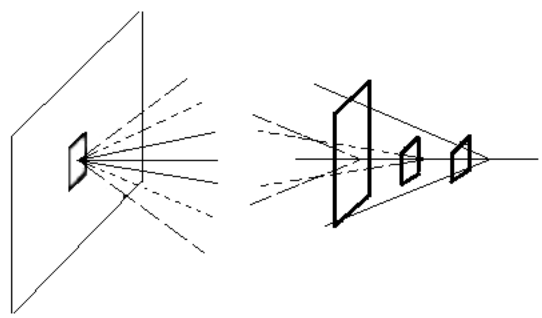
The insight of the English amateur and lawyer, Chester Moor Hall, who produced the first achromatic lenses for use in a telescope of his own making around 1729, was justified by the excellent lenses produced by the English optician Dollond. If Newton (and Euler) had been right, we could not be able to correct optical systems for color aberration. Newton drew this conclusion and believed that because of this impossibility one could not use optical systems with refracting surfaces, without introducing colored fringes.

Spherical Aberration

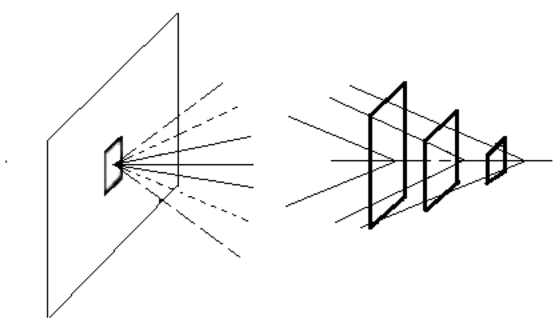
Euler has developed elegant formulas to correct for spherical aberration in optical systems. An analysis of the optical systems calculated by him shows that he succeeded in doing this very well. In these works he generally uses only his formulas for the third order of spherical deviation, but he has also calculated formulas for the fifth order. He deliberately did not know how to plot the points near the axis. He says on page 11 of the Dioptric, and we follow here the translation of H. Boegehold:

"As it stands with the refraction of the rays when the light source lies off the axis, it is not only a very difficult question, but also to be connected with such a reckoning that it can hardly be completed. However, when lenses are used, the rays will never arrive at points that are far from the axis, so you have to be content when things on the axis are accurately reproduced; the indistinctness of points close to the axis may then not be noticeable: if, in fact, the reproduction of an axis point E does not deviate with marginal rays, then it is possible hardly a noticeable indistinctness may appear, if the point be a little away from the axis."

Boegehold added: " Here Euler is mistaken." The right explanation was given only a hundred years later, when E. Abbe showed the meaning of the sine condition and F. Staeble and E. Lihotzky showed the isoplanic condition for that. On the contrary to Euler, his contemporaries, especially the French encyclopaedists J. d' Alembert and A. C. Clairaut, were in possession of formulas that allowed them to improve the correction of neighbouring points, and the objective Dollond used also showed a good correction of the coma. And yet, it's actually the mindset that Euler used in analyzing the color magnification error that ultimately led to the understanding of coma error:



A Isoplanatic condition not fulfilled



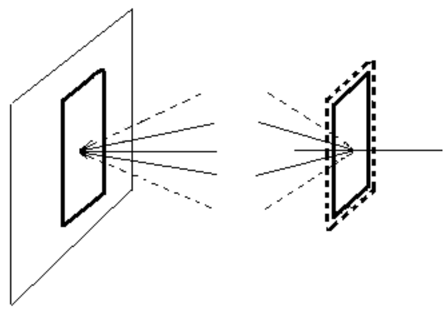
B Isoplanatic condition fulfilled

Let us consider the rays from an axis point of an optical system. The rays, which lie on the object side on a circular cone around the axis, are also on the image side on a circular cone around the axis. While the circle cones on the object side have the same vertex, the cones on the image side have different vertices. The deviation of the vertices from the point where the near-axis rays unite is called the aperture error (or spherical deviation). If one were to attach a circular aperture on the object side which transmits only rays from the angle u about the axis, and if these rays were combined at the angle u', then the system would reproduce in the point of union an image whose magnification would be given

$$V = \frac{\sin u}{\sin u'} \qquad (1),$$

when both the object and the image are in air. In this way, we got a number of images of the object in different places with different constructions, just as we got in the color deviation for different colors images of different magnifications in different places.

Euler had thought it desirable that all these images in different colors be offered to the eye at the same angle. It therefore makes sense to demand that the images pictured by the zones at different locations from the eye appear to us at the same angle with different magnification, i.e. that $\frac{dV}{V} = \frac{ds}{s}$ (2), where s indicates the distance from the eye. (In place of the eye can you also look at the exit pupil, that the "eye" of the optical System represents.) Formula (2) represents the so-called iso-planatic condition free from the coma produced by spherical aberration (Fig. A, B).
See, e.g. W.T. Welford's *Aberrations of Optical Systems* for a discussion of this concept.

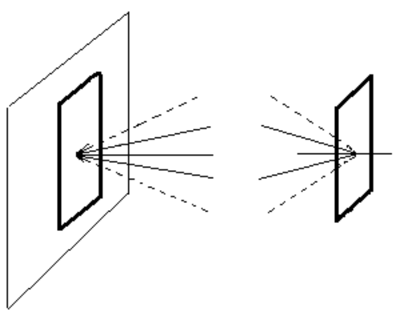


C Sine condition not corrected

The spherical aberration is fixed, so that the iso-planatic condition goes over into the sine condition . In that case all the images in the same place must be the same size. The enlargement must be the same for all the rays,

$$\text{i.e. } V = \frac{\sin u}{\sin u'} = V_0 \qquad (3),$$

is constant for the coma-free condition. (3) is the Abbe sine condition (Fig. C, D).



D Sine condition corrected

For an infinite object with the eye at the principal point, these formulas transform into $\Delta f = \Delta s$, in which

$$f = \frac{h}{\sin u'} \quad (\text{Isoplanatic condition}),$$

and,

$$\frac{h}{\sin u'} = f_0 \quad (\text{Sine condtition}).$$

Comparing authors, we see that in Dollond's telescope lenses spherical aberrations, the sine and color conditions are better corrected than in those systems calculated by Euler, of which only one is correct. Even better were the systems which Clairaut and d'Alembert had calculated.

End of Note.

CHAPTER III

CONCERNING CATA-DIOPTRIC TELESCOPES CONSTRUCTED WITH A SMALLER CONCAVE MIRROR

PROBLEM 1

36. *If the smaller concave mirror QBQ of which the focal distance* $q=\varepsilon p$ (Fig. 8)*, may be put in place at the distance* $AB=(1+\varepsilon)p$ *before the principal mirror PP [of focal length p], with the hole bored of diameter* $\pi\pi$ *, to define the two lenses C and D, thus so that any object may be represented distinctly.*

SOLUTION

Here p denotes the focal length of the larger mirror, the radius [of the opening] of which $AP=x$ and the radius of the opening $A\pi=y=\varepsilon x$, thus so that the radius of curvature

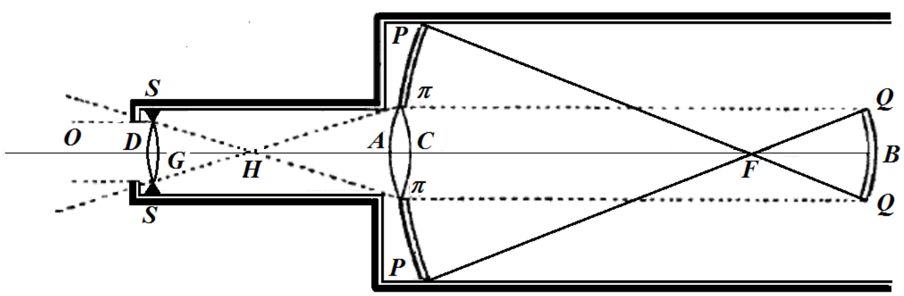


Fig. 8

of the mirror $=2p$. Therefore the principal image of objects will be represented at F by this mirror, so that there shall be $AF=\alpha=p$, therefore the distance of this from the smaller mirror must be $FB=\varepsilon p$, as has been shown before, and the radius [of the opening] of this mirror $BQ=y=\varepsilon x$. Therefore since the focal length of this mirror shall be $=q=\varepsilon p=FB$, hence the reflected rays will be made parallel to each other, while they are incident on the lens G ; therefore according to our general formulas there will be $\frac{1}{P}=-\varepsilon$ and $FB=b=\varepsilon p$, from which certainly on account of $P=-\frac{\alpha}{b}$ there becomes $P=-\frac{1}{\varepsilon}$. Then, since [the position of the second image from the smaller mirror] shall become $\beta = \frac{bq}{b-q} = \infty$ and hence $B = \frac{\beta}{b} = \infty$, now the second interval in general is

$$= -\frac{ABa}{P}\left(1 - \frac{1}{Q}\right) = -\frac{Bp}{P}\left(1 - \frac{1}{Q}\right)$$

and since this is required to be put equal to the first interval, there will become $Q = 1$, but thus still, so that there shall be $B\left(1 - \frac{1}{Q}\right) = \frac{1+\varepsilon}{\varepsilon}$; but from the general formulas this second interval $= \beta + c = (1+\varepsilon)p$;from which on account of $\beta = \infty$ there becomes

$$c = (1+\varepsilon)p - \beta = -\infty \text{ and thus } C = \frac{\gamma}{c} = 0 \text{ and } \mathfrak{C} = 0.$$

Whereby with the establishment of the lens in the opening, the focal length of which $= r$ will become :

$$r = \frac{B\mathfrak{C}}{PQ}p = -\varepsilon B\mathfrak{C}p;$$

from which, since there shall be $B = \infty$ and $\mathfrak{C} = 0$, there is deduced in turn

$$B\mathfrak{C} = BC = \frac{-r}{\varepsilon p}$$

and hence we will have the focal length $s = -\frac{r}{R}$ for the fourth lens *SDS* and the interval $CD = r\left(1 - \frac{r}{R}\right)$. Therefore so that the final lens may become convex, the letter *R* must be negative or a real image lies at the point *H* in the interval *CD,* and from the given magnification *m* the general formulas present the magnification $PQR = m$, since on account of the two real images the representation will be erect. Hence therefore the equation will become $R = \frac{m}{PQ} = -\varepsilon m$, thus so that now there shall be $s = \frac{r}{\varepsilon m}$ and the interval $CD = r(1 + \frac{1}{\varepsilon m}) = r + s$; since here from the nature of the telescope there becomes $CH = r$ and $HD = s$.

Now we will consider the apparent field of view and following our general formulas we may attribute the letter $\mathfrak{q}$ [for the increased angle for the field of view due in radians] to the second mirror, the letter $\mathfrak{r}$ for the lens *C,* and the letter $\mathfrak{s}$ for the lens *D* , and the radius [*i.e.* angle in radians] of the field of view will be

$$\Phi = \frac{\mathfrak{q}+\mathfrak{r}+\mathfrak{s}}{m-1}\xi$$

with ξ taken for the fraction $\frac{1}{4}$; moreover the sum of the letters $\mathfrak{q}$, $\mathfrak{r}$ and $\mathfrak{s}$ can be made equal to unity. Truly we have put [Lib. II, § 265] for the sake of brevity

$$\frac{\mathfrak{q}+\mathfrak{r}+\mathfrak{s}}{m-1} = M,$$

so that there shall become

$$\Phi = M\xi,$$

and our general formulas supply these equations:

$$\mathfrak{B}\mathfrak{q} = (P-1)M, \quad \mathfrak{C}\mathfrak{r} = (PQ-1)M - \mathfrak{q},$$

which on account of the values now found

$$\mathfrak{B} = 1, \ \mathrm{Q} = 1 \ \text{and} \ \mathfrak{C} = 0$$

present both

$$\mathfrak{q} = (P-1)M = -\left(1+\tfrac{1}{\varepsilon}\right)M.$$

But hence we have found the distance of the eye past the lens *D,* clearly

$$DO = O = \tfrac{\mathfrak{s}s}{Mm};$$

which distance since it shall be positive, as nothing prevents it, even if no positive value of this $\mathfrak{s}$ may be given equal to unity, we may remove the colored fringe, if on account of $N' = 0$ and $N'' = N'''$ (since our two lenses are prepared from the same glass) we may satisfy this equation [i.e., the angles before magnification must cancel]:

$$0 = \tfrac{\mathfrak{r}}{PQ} + \tfrac{\mathfrak{s}}{PQR},$$

which therefore is reduced to this:

$$0 = \mathfrak{r} - \tfrac{\mathfrak{s}}{\varepsilon m},$$

from which there is deduced:

$$\mathfrak{r} = \tfrac{\mathfrak{s}}{\varepsilon m};$$

whereby, since there shall be $\mathfrak{q} = -\left(1+\tfrac{1}{\varepsilon}\right)M$, there will become

$$\mathfrak{q} + \mathfrak{r} + \mathfrak{s} = -\left(1+\tfrac{1}{\varepsilon}\right)M + \tfrac{\mathfrak{s}}{\varepsilon m} + \mathfrak{s} = M(m-1),$$

from which there follows

$$M = \tfrac{\mathfrak{s}}{m},$$

thus so that now the radius of the apparent field of view shall be

$$\Phi = \tfrac{\mathfrak{s}}{m}\cdot\xi.$$

But now we understand with regard to the lens *C*, it shall be possible to write unity for $\mathfrak{s}$ here, of which the aperture has been prescribed by us, and of which the radius $= y = \varepsilon x$. Now by our formulas this diameter must be

$$\mathfrak{t}r\xi + \tfrac{x}{PQ} = \tfrac{\mathfrak{s}r}{\varepsilon m}\cdot\xi + \varepsilon x,$$

where it suffices to use the greater member, from which it follows there must be $\frac{\mathfrak{s}r}{\varepsilon m}\cdot\xi < \varepsilon x$, from which, if we may put $\mathfrak{s}=1$ and $\xi=\frac{1}{4}$, it is necessary, that there shall be $r < 4s^2mx$; therefore if we may assume $r > 4s^2mx$, then $\mathfrak{s}$ must be taken less than one, from which the apparent field will be diminished in the same ratio. But here it will be required to attend also the to final lens, for which there is $s=\frac{r}{\varepsilon m}$, thus so that there may be $s < 4\varepsilon x$ or $s < 4y$, from which it is apparent a very small opening cannot be put in place.

But we will be able to remove the whole confusion arising from the differing refrangibilities of the rays with the aid of this equation:

$$0=\frac{N''}{P^2Q^2}\cdot\frac{1}{r}+\frac{N'''}{P^2Q^2R^2}\cdot\frac{1}{s},$$

which will change into this :

$$0=N''+\frac{N'''}{sm};$$

which since in no manner shall it be done, even if we may wish to use different glass, this confusion, which is extremely small always, must be tolerated.

The main point concerning these observations depends on the radius of confusion, which it will be agreed to render insignificant with the aid of this equation:

$$\frac{1}{k^3}=\frac{mx^3}{p^3}\left(\frac{1}{8}+\frac{\varepsilon}{8}+\frac{\mu\varepsilon^4p^3}{r^3}\lambda''+\frac{\mu\varepsilon^3p^3}{r^3m}\lambda'''\right),$$

which equation will be changed into this form :

$$p\sqrt[3]{\left(\frac{1}{k^3mx^3}-\frac{\mu\varepsilon^4\lambda''}{r^3}-\frac{\mu\varepsilon^3\lambda'''}{mr^3}\right)}=\frac{1}{2}\sqrt[3]{(1+\varepsilon)},$$

from which equation p is found; it will be agreed to assume only the amount x, so that thence a sufficient order of clarity may be obtained. But according to the principles of telescopes, for sufficient clarity we have assumed $x=\frac{m}{50}$ in.; which there was x or $\sqrt{x^2}$, but this for us is $\sqrt{\left(1-\varepsilon^2\right)x^2}$, thus so that here we may have

$$x\sqrt{\left(1-\varepsilon^2\right)}=\frac{m}{50}\ \text{in.},$$

of indeed we may wish to enjoy the same degree of clarity ; from which there becomes $x=\frac{m}{50\sqrt{(1-\varepsilon^2)}}$ in. and thus $x>\frac{m}{50}$ in. Truly since mirrors not only reflect rays to the same extent as lenses transmit, lest indeed we may only come upon the degree of clarity as in common telescopes. But if we may be content with a lesser degree of clarity and we may put $x=\frac{m}{50}$ in. and so that we may assume there $k=50$, our equation will become

$$p\sqrt[3]{\left(\frac{1}{m^4}-\frac{\mu\varepsilon^4\lambda''}{r^3}-\frac{\mu\varepsilon^3\lambda'''}{mr^3}\right)}=\frac{1}{2}\sqrt[3]{(1+\varepsilon)},$$

where clearly $\frac{\varepsilon^4}{r^3}$ must be much smaller than the first term $\frac{1}{m^4}$ or $r^3>\varepsilon^4m^4$ and thus r much greater than $\varepsilon m\sqrt[3]{\varepsilon m}$; truly above we have seen that there must become $r<4\varepsilon^2mx$; so that which may be able to be done, $4\varepsilon^2mx$ must be much greater than $\varepsilon m\sqrt[3]{\varepsilon m}$ or $4\varepsilon m>50\sqrt[3]{\varepsilon m}$ and thus $\varepsilon>\frac{44}{m}$, so that it may be able to effect the greatest magnifications.

But if this condition may not be observed, the effect is consistent with that, so that , so that no longer shall there be $\mathfrak{s}=1$ and hence the apparent field may be present much smaller than

$$\Phi=\frac{\xi}{m} \text{ or } \Phi=\frac{859}{m} \text{ minutes.}$$

COROLLARY 1

37. Since with telescopes this shall always require to be effected initially, that the length of these and hence especially the focal length p may be reduced as far as possible, in the final equation the confusion arising from the lenses must be diminished so much, so that it may as if vanish before the confusion of the mirrors ; whereby, since in this same formula it may be seen the fraction $\frac{1}{8}$ may arise from the first mirror, from the second truly $\frac{\varepsilon}{8}$, it is necessary, in order that the following parts arising from the lenses may become much smaller, from which the letter r must be much greater than $2\varepsilon p$ and thus r again will be taken less than p.

COROLLARY 2

38. Therefore since if we may put $r=p$ in place, since ε, as we have seen, may be accustomed to be less than $\frac{1}{8}$, this equation will be allowed to be used safely for defining the confusion:

$$\frac{1}{k^3}=\frac{mx^3(1+\varepsilon)}{8p^3},$$

from which we deduce

$$p=\frac{kx}{2}\sqrt[3]{m(1+\varepsilon)};$$

from which, if for the given order of clarity and distinctiveness there may be taken $kx = m$ in., which quantity will be around twice as small as in common dioptric telescopes, thus so that in this way the total length may be reduced to almost a quarter part of the former.

COROLLARY 3

39. Moreover on taking $r = p$ for the field requiring to be defined the letter $\mathfrak{s}$ cannot be taken greater than so that there may become

$$\frac{\mathfrak{s}p}{4\varepsilon m} = \varepsilon x;$$

hence therefore for the special example, where $\varepsilon = \frac{1}{4}$ and $m = 100$, there it will be deduced

$$\mathfrak{s} = \tfrac{1}{5} \text{ approx},$$

from which it is apparent in this case to become a field of view five times smaller than if it were allowed to take $\mathfrak{s} = 1$, and thus in general it is apparent in this way exceedingly small fields to be obtained.

COROLLARY 4

40. Moreover on taking $r = p$ for the construction of this kind of telescope, the focal lengths themselves will be obtained in the following way :

$$p = \tfrac{1}{2} m \sqrt[3]{m(1+\varepsilon)}, \quad q = \varepsilon p, \quad r = p \text{ and } s = \frac{p}{\varepsilon m};$$

then truly the separation of the lenses or mirrors

$$AB = (1+\varepsilon)p = BC, \; CD = r + s = p(1 + \tfrac{1}{\varepsilon m})$$

and the distance of the eye

$$O = s = \frac{p}{\varepsilon m},$$

from which it is apparent the tube of the container requiring to be added to be long, in which the mirrors are held.

SCHOLIUM

41. Truly besides the inconvenience, which now we have indicated here, telescopes of this kind therefore labor under a great defect, because the rays incident on the lens *C* are parallel to each other; then indeed the stray rays, which are incident directly on the same lens from the neighborhood of the object, since they also are parallel to each other, and are refracted in a similar manner in passing through the lenses and both the proper rays as

well as these likewise will be sent to the eye, and since these stray rays are stronger than the proper ones, if indeed these have passed through a twofold reflection, will completely destroy the impression of these other on the eye. Yet meanwhile, since the stray rays are more oblique to the axis and also remain more oblique on refraction, they will be able to be excluded from passing into the eye with the aid of a small opening [*i.e.* stop], to which the eye may be applied; nevertheless in this manner not only may the clarity be allowed to be diminished exceedingly, but also the above field of view will be restricted ; on this account especially with telescopes of this kind there is need for precautions to be taken, lest stray rays around the small mirror besides those slipping through the entrance of the chamber C , in no way may be allowed a similar refraction with the proper rays. So that it will be better, if only proper rays were incident on the lens C either diverging or converging, so that after refraction the stray rays may be grouped together into the other focus ; indeed then the diaphragm with this opening put in place at this focus will exclude easily the stray rays from progressing further to the eye. But it is evident, so that this remedy may be more sure to succeed, that either the convergent or the divergent must be known well enough, or is required to be effected, so that by the refraction of this lens C the image formed from the stray rays will be much removed from the image formed from the proper rays, which will be used in the following cases.

PROBLEM 2

42. *If the small concave mirror QBQ, of which the focal length* $q=\frac{\varepsilon(1+\varepsilon)}{1+2\varepsilon}\cdot p$ *, may be put in place at a distance* $AB=(1+\varepsilon)p$ *before the main mirror PP* (Fig. 8) *with the hole drilled ππ, to define the two lenses C and D, so that any object thus may be represented distinctly.*

SOLUTION

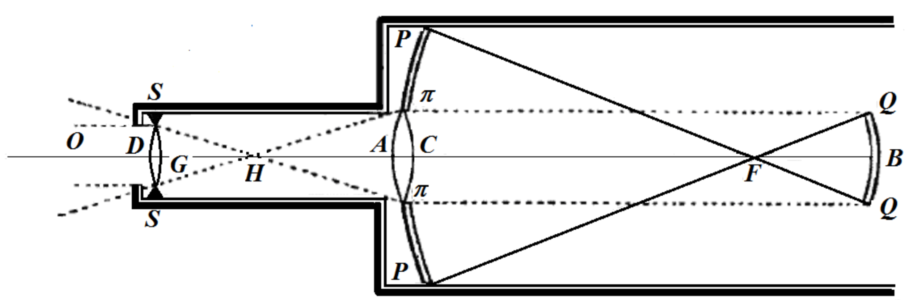


Fig. 8

Here as before therefore the distance $AF=\alpha=p$ and $FB=b=\varepsilon p$ and hence $\frac{1}{P}=-\varepsilon$ on account of $AB=(1+\varepsilon)p$. Truly since here there is $q=\frac{\varepsilon(1+\varepsilon)}{1+2\varepsilon}\cdot p$, there will become

$$\frac{\beta}{b} = \frac{q}{b-q} = \frac{1+\varepsilon}{\varepsilon} = B,$$

thus so that now there shall become $\beta = (1+\varepsilon)p$, which is equal to the second interval itself *BC*, and thus the second image is incident on the lens *C* itself, from which there will become $c = 0$; from which, since we will have put $\frac{\beta}{c} = -Q$, this will become

$$Q = -\infty$$

then truly for the third image there will become

$$\gamma = \frac{cr}{c-r} = 0,$$

thus so that there shall become

$$C = -1 \text{ and } \mathfrak{C} = \infty.$$

Whereby, since there shall become

$$r = \frac{B\mathfrak{C}}{PQ} \cdot p = -\frac{(1+\varepsilon)\mathfrak{C}}{Q} \cdot p,$$

in turn it will be apparent to become

$$\frac{\mathfrak{C}}{Q} = -\frac{r}{(1+\varepsilon)p}.$$

With these found the focal lengths will be

$$p = p, \quad q = \frac{\varepsilon(1+\varepsilon)}{1+2\varepsilon}, \quad r = r \text{ and } s = \frac{B}{PQR} \cdot p = \frac{1+\varepsilon}{\varepsilon m} \cdot p \text{ on account of } PQR = m.$$

Truly the intervals will be expressed thus:

$$AB = (1+\varepsilon)p = BC, \quad CD = \frac{1+\varepsilon}{\varepsilon m} \cdot p = s,$$

as the nature of the matter demands, since the final image remains to be put in place in the lens *C* itself. Moreover it is apparent here two real images occur, the one at *F,* the other at *C,* and thus to be represented by images placed erect and rightly we may assume $PQR = m$.

For the field requiring to be decided there will be

$$M = \frac{\mathfrak{q}+\mathfrak{r}+\mathfrak{s}}{m-1},$$

from which there becomes $\Phi = M\xi$; then truly there must be

$$\mathfrak{B}\mathfrak{q} = (P-1)M,$$

hence

$$\mathfrak{q} = -\frac{(1+2\varepsilon)}{\varepsilon}\cdot M$$

and

$$\mathfrak{C}\mathfrak{r} = (PQ-1)M - \mathfrak{q},$$

hence

$$\mathfrak{r} = \left(\frac{PQ}{\mathfrak{C}} - \frac{1}{\mathfrak{C}}\right)M - \frac{\mathfrak{q}}{\mathfrak{C}}.$$

Since truly

$$\mathfrak{C} = \infty \quad \text{and} \quad \frac{\mathfrak{C}}{Q} = \frac{-r}{(1+\varepsilon)p};$$

there will be

$$\mathfrak{r} = \frac{PQM}{\mathfrak{C}} = \frac{(1+\varepsilon)p}{\varepsilon r}\cdot M;$$

hence therefore there becomes

$$\mathfrak{q} + \mathfrak{r} + \mathfrak{s} = \left(\frac{(1+\varepsilon)p}{\varepsilon r} - \frac{(1+2\varepsilon)}{\varepsilon}\right)M + \mathfrak{s} = (m-1)M,$$

from which there is found

$$M = \frac{\varepsilon r \mathfrak{s}}{m\varepsilon r - (1+\varepsilon)(p-r)};$$

but concerning $\mathfrak{s}$ nothing is defined at this point, but since the radius of the aperture of the lens C actually shall be $= \varepsilon x$, by our formulas it must be

$$= \frac{1}{4}\mathfrak{r}r = \frac{(1+\varepsilon)pM}{4\varepsilon},$$

or with the field itself introduced this radius will be $= \frac{(1+\varepsilon)p\Phi}{\varepsilon}$, which since it may not exceed εx, this is not required to be a source of concern, unless there shall be $\Phi = \frac{\varepsilon^2 x}{(1+\varepsilon)p}$ or greater.

Now so that the colored margin may vanish, there must be

$$0 = \frac{\mathfrak{r}}{PQ} + \frac{\mathfrak{s}}{PQR};$$

and thus there must be

$$\frac{\mathfrak{s}}{m} = 0;$$

from which it is apparent the colored margin cannot be removed in this manner, but for that to become a minimum and scarcely sensible on account of the maximum denominators PQ and PQR.

Therefore with the letter $\mathfrak{s}$ taken, to be used as circumstances permit, for the place of the eye we will have

$$O = \frac{\mathfrak{s}s}{Mm}.$$

Finally the condition for the removal of confusion gives rise to this equation:

$$\frac{1}{k^3} = \frac{mx^3}{p^3}\left(\frac{1}{8} + \frac{(1+2\varepsilon)\varepsilon}{8(1+\varepsilon)^3}\right)$$

with the following parts vanishing at once, thus so that there may be put in place :

$$p = \frac{1}{2}m\sqrt[3]{m}\left(\frac{(1+\varepsilon)^3+\varepsilon(1+2\varepsilon)}{(1+\varepsilon)^3}\right) \text{ in.}$$

COROLLARY 1

43. Since the radius of the aperture of the lens put in place at C must be $= \frac{1}{4}\mathfrak{r}r$, truly that must actually be $= \varepsilon x$, hence it is deduced that $\mathfrak{r} = \frac{4\varepsilon x}{r}$. Truly before, we have found $\mathfrak{r} = \frac{(1+\varepsilon)p}{r\varepsilon}\cdot M$; therefore with these two values equated there arises

$$4\varepsilon^2 x = (1+\varepsilon)\, pM,$$

from which , if there shall be $\mathfrak{r} = 1$, there becomes $r = 4\varepsilon x$, then truly $r = \frac{M(1+\varepsilon)p}{\varepsilon}$; since truly there is

$$M = \frac{\varepsilon r\mathfrak{s}}{m\varepsilon r-(1+\varepsilon)(p-r)},$$

we will now have for r with this value substituted:

$$M = \frac{4\varepsilon^2\mathfrak{s}x}{4m\varepsilon^2 x-(1+\varepsilon)(p-4\varepsilon x)},$$

which value substituted into that equation will give :

$$\mathfrak{s} = \frac{4m\varepsilon^2 x-(1+\varepsilon)(p-4\varepsilon x)}{(1+\varepsilon)p}.$$

COROLLARY 2

44. But since this value $\mathfrak{s}$ cannot be greater than unity, with this value put equal to one there will arise

$$4m\varepsilon^2 x = 2(1+\varepsilon)p - 4\varepsilon(1+\varepsilon)x$$

and hence

$$m = \frac{(1+\varepsilon)p - 2\varepsilon(1+\varepsilon)x}{2\varepsilon^2 x},$$

which equation cannot be sustained, unless the magnification *m* may exceed some thousands, so that in practice this cannot be done.

SCHOLIUM

45. Truly telescopes of this kind labour under two defects ; indeed the first, since the lens *C* is situated in the position of the image itself, unless the lens shall be made from the purest glass, the image presented will be greatly disturbed, as we have observed now often; then also hardly a small fault consists in that the colored margin cannot be reduced to zero ; as on account of that reason it would be superfluous to pursue these telescopes further, but rather we have established a case of this kind, in which the following image after the lens *C* at the same time happily may fall with the colored margin removed. Whereby, since the equation may be had for this outstanding property

$$0 = \frac{\mathfrak{r}}{PQ} + \frac{\mathfrak{s}}{PQR},$$

it is necessary, that there may become $\mathfrak{r} + \frac{\mathfrak{s}}{R} = 0,$ which can be done most conveniently if there were put $R = -1$; indeed since then there will become $\mathfrak{r} = \mathfrak{s}$, we will be able to arrive at the maximum field of view, if it may be allowed to take $\mathfrak{t} = \mathfrak{s} = 1$; for then there becomes

$$M = \frac{\mathfrak{q}+2}{m-1},$$

and as long as $\mathfrak{q}$ shall be a negative fraction, hence the field of view will arise large enough; so that truly *R* may become a negative number, the second image must lie in the interval *CD*, thus so that *Q* may remain a positive quantity, and since by multiplication gives $m = PQR$, on account of $P = -\frac{1}{\varepsilon}$, if we may assume $R = -1$, it becomes necessary that $Q = \varepsilon m$; from which, since there shall become $Q = -\frac{\beta}{c}$, there will become

$$c = -\frac{\beta}{\varepsilon m};$$

but truly the second interval $BC = \beta + c$; which since it must be equal to the first $(1+\varepsilon)p$, we elicit

$$\beta = (1+\varepsilon)p - c = \frac{\varepsilon(1+\varepsilon)mp}{\varepsilon m - 1}.$$

Truly since there shall be

$$\beta = \frac{bq}{b-q} \quad \text{and} \quad b = \varepsilon p,$$

since also

$$\frac{1}{q}=\frac{1}{b}+\frac{1}{\beta}$$

hence there will be

$$\frac{1}{q}=\frac{1+2\varepsilon}{\varepsilon(1+\varepsilon)p}-\frac{1}{\varepsilon(1+\varepsilon)mp};$$

then truly there will become

$$B=\frac{\beta}{b}=\frac{(1+\varepsilon)m}{\varepsilon m-1} \text{ and hence } \mathfrak{B}=\frac{(1+\varepsilon)m}{(1+2\varepsilon)m-1},$$

from which there becomes $q=\mathfrak{B}b$.

Again truly, since there shall be $r=\mathfrak{C}c$, there will be

$$\mathfrak{C}=\frac{r}{c}=\frac{-(\varepsilon m-1)r}{(1+\varepsilon)p} \text{ and hence } \mathrm{C}=\frac{-(\varepsilon m-1)r}{(1+\varepsilon)p+(\varepsilon m-1)r}.$$

But for the other interval *CD,* which is $\gamma+d=\gamma+s$, since there becomes

$$\gamma=\frac{cr}{c-r}=\frac{(1+\varepsilon)pr}{(1+\varepsilon)p+(\varepsilon m-1)r},$$

truly since also there must become $R=-\frac{\gamma}{d}=-1$, hence there will be $s=\gamma$ and thus the interval $CD=2\gamma=2s=\left[\frac{2(1+\varepsilon)pr}{(1+\varepsilon)p+(\varepsilon m-1)r}\right]$.

But since again there is $M=\frac{\mathfrak{q}+2}{m-1}$, evidently on taking $\mathfrak{r}+\mathfrak{s}=1$, there will become

$$\mathfrak{q}=\frac{-((1+2\varepsilon)m-1)}{\varepsilon m}\cdot M$$

and hence

$$\mathfrak{q}+2=\frac{-((1+2\varepsilon)m-1)M+2\varepsilon m}{\varepsilon m}=M(m-1);$$

from which there follows

$$M=\frac{2\varepsilon m}{\varepsilon m^2+(1+\varepsilon)m-1},$$

from which in turn we conclude

$$\mathfrak{q}=\frac{-2((1+2\varepsilon)m-1)}{\varepsilon m^2+(1+\varepsilon)m-1}.$$

Truly moreover at this stage this equation is had $\mathfrak{C}=(PQ-1)\mathrm{M}-\mathfrak{q}$, which will be changed into this :

$$\mathfrak{C}=-(m+1)\mathrm{M}-\mathfrak{q}$$

or with the values substituted :

$$\frac{-(\varepsilon m-1)r}{(1+\varepsilon)p}=\frac{-2\varepsilon m(m+1)+2(1+2\varepsilon)m-2}{\varepsilon m^2+(1+\varepsilon)m-1}=\frac{-2\varepsilon m^2+2(1+\varepsilon)m-2}{\varepsilon m^2+(1+\varepsilon)m-1},$$

from which we conclude to become

$$r=\frac{2\left(\varepsilon m^2-(1+\varepsilon)m+1\right)(1+\varepsilon)p}{(\varepsilon m-1)\left(\varepsilon m^2+(1+\varepsilon)m-1\right)}=\frac{2(m-1)(1+\varepsilon)}{\varepsilon m^2+(1+\varepsilon)m-1}\cdot p;$$

hence, since there shall be $\frac{1}{\gamma}=\frac{1}{r}-\frac{1}{c}$, there will be found

$$\gamma=\frac{2(m-1)(1+\varepsilon)}{3\varepsilon m^2-(1+\varepsilon)m+1}\cdot p=s,$$

from which again the distance of the eye is concluded to be

$$O=\frac{\mathfrak{s}s}{Mm}=\frac{\varepsilon m^2+(1+\varepsilon)m-1}{2\varepsilon m^2}\cdot s=\frac{1}{2}s\left(1+\frac{1+\varepsilon}{\varepsilon m}-\frac{1}{\varepsilon m^2}\right)=\frac{1}{2}s \text{ approx.}$$

But so that it may be extended to the apparent field, since we have assumed $\mathfrak{r}=\mathfrak{s}=1$, it is required to be considered, whether also it may be allowed to put $\xi=\frac{1}{4}$. But this will be apparent from the lens *C*, of which the radius of the aperture $=\xi r$ cannot exceed εx; therefore on putting $\xi r=\varepsilon x$ there is deduced :

$$\xi=\frac{\varepsilon x\left(\varepsilon m^2+(1+\varepsilon)m-1\right)}{2(m-1)(1+\varepsilon)p},$$

which value if it were put less than $\frac{1}{4}$, it will be required to be used there, thus so that then there shall be $\Phi=M\xi$; but if that may produce a value greater than $\frac{1}{4}$, nevertheless there must be taken $\xi=\frac{1}{4}$.
Just as if an example may be taken

$$\varepsilon=\frac{1}{4},\quad m=100,\quad x=\frac{5}{2}\text{in. and } p=25 \text{ in.},$$

actually $\xi=\frac{1}{4}$ arises, thus so that putting this $\xi=\frac{1}{4}$ may be seen to differ a little from practice; from which it will be worth the effort to show these determinations jointly.

EXAMPLE OF A CATA-DIOPTRIC TELESCOPE

46. The main elements of this telescope will be had in the manner brought forwards thus :

$$a=\infty, \quad b=\varepsilon p, \quad c=\frac{-(1+\varepsilon)}{\varepsilon m-1}p, \quad d=\gamma,$$

$$\alpha=p, \quad \beta=\frac{\varepsilon(1+\varepsilon)mp}{\varepsilon m-1}, \quad \gamma=\frac{2(m-1)(1+\varepsilon)p}{3\varepsilon m^2-(1+\varepsilon)m+1}, \quad \delta=\infty.$$

From which the following values are deduced:

$$B=\frac{(1+\varepsilon)m}{\varepsilon m-1}, \quad \mathfrak{B}=\frac{(1+\varepsilon)m}{(1+2\varepsilon)m-1},$$

$$C=\frac{-2(m-1)(\varepsilon m-1)}{3\varepsilon m^2-(1+\varepsilon)m+1}, \quad \mathfrak{C}=\frac{-2(m-1)(\varepsilon m-1)}{\varepsilon m^2+(1+\varepsilon)m-1},$$

$$P=-\frac{\alpha}{b}=-\frac{1}{\varepsilon}, \quad Q=-\frac{\beta}{c}=\varepsilon m, \quad R=-\frac{\gamma}{d}=-1.$$

Truly from these the focal distances are determined

$$p=p, \quad q=\mathfrak{B}b=\frac{s(1+\varepsilon)m}{(1+2\varepsilon)m-1}\cdot p,$$

$$r=\mathfrak{C}c=\frac{-2(m-1)(1+\varepsilon)}{\varepsilon m^2+(1+\varepsilon)m-1}, \quad \text{and} \quad s=d=\gamma$$

$$P=-\frac{\alpha}{b}=-\frac{1}{\varepsilon}, \quad Q=-\frac{\beta}{c}=\varepsilon m, \quad R=-\frac{\gamma}{d}=-1.$$

and for the apertures of these :

$$\mathfrak{q}=\frac{-2((1+2\varepsilon)m-1)}{\varepsilon m^2+(1+\varepsilon)m-1}, \quad \mathfrak{r}=1, \quad \mathfrak{s}=1$$

and hence

$$\mathfrak{q}+\mathfrak{r}+\mathfrak{s}=\frac{2\varepsilon m(m-1)}{\varepsilon m^2+(1+\varepsilon)m-1}$$

and thus

$$M=\frac{2\varepsilon m}{\varepsilon m^2+(1+\varepsilon)m-1},$$

from which the radius of the apparent field $\Phi=M\xi$, and if it may be allowed to take $\xi=\frac{1}{4}$, there will become

$$\Phi=\frac{1718\varepsilon m}{\varepsilon m^2+(1+\varepsilon)m-1} \text{ minut.};$$

but for the location of the eye we find

$$O=\tfrac{1}{2}s\left(1+\tfrac{1+\varepsilon}{\varepsilon m}-\tfrac{1}{\varepsilon m^2}\right).$$

It remains therefore, that the focal length p may be defined from the condition of the confusion, which is found to be

$$p=kx\sqrt[3]{m}\left(\begin{array}{c}\tfrac{1}{8}+\dfrac{\varepsilon(m+1)^2\left((1+2\varepsilon)m-1\right)}{8(1+\varepsilon)^3m^3}\\+\dfrac{\mu\left(\varepsilon m^2+(1+\varepsilon)m-1\right)^3}{8m^4(m-1)^3(1+\varepsilon)^3}\left(\lambda''+v\mathfrak{C}\left(1-\mathfrak{C}\right)\right)+\dfrac{\mu\left(3\varepsilon m^2-(1+\varepsilon)m+1\right)^3\lambda'''}{8m^4(m-1)^3(1+\varepsilon)^3}\end{array}\right)^{\frac{1}{3}}$$

where if we may desire great clarity, such as we have attributed to the above telescopes, there must be taken $x=\frac{m}{50}$ in., and for the order of distinctness $k=50$, so that there shall be $kx=m$.

But if we may be content with a smaller degree of clarity, it will suffice perhaps to put $x=\frac{m}{100}$ in. , or thus $x=\frac{m}{200}$ in.

CONSTRUCTION OF A TELESCOPE OF THIS KIND FOR A MAGNIFICATION $m=100$ BY TAKING $\varepsilon=\frac{1}{4}$

47. Therefore for the larger mirror, the radius of which shall be $=x$, the radius of the opening will be $=\frac{1}{4}x$, of which truly the focal length in general may be put $=p$; from which the following focal lengths will be defined thus :

$$q=\tfrac{125}{596}p=0,2097p,$$

$$r=\tfrac{5\cdot 99}{5248}p=0,0934p,$$

$$s=\tfrac{5\cdot 99}{14752}p=0,03355p.$$

Moreover the intervals will be defined in the following manner :

1. $AB=\frac{5}{4}p=1,25p,$
2. $BC=\frac{5}{4}p=1,25p,$
3. $CD=2\mathrm{s}\ =\ 0,0671p,$
4. $O=0,5248s=\left[0,0176p\right].$

Besides the radius of the aperture of the larger mirror to be used $= x$, thus the radius of the smaller will be $= \frac{1}{4}x$, to which also the aperture of the lens *C* shall be equal; truly the radius of the aperture of the eyepiece lens *D* will be able to be taken $= \frac{1}{4}s$, from which the radius of the apparent field of view will be around $\Phi = 16,368$ minutes, which field has a place unless there shall be $\frac{1}{4}x < \frac{1}{4}r$ or $x < r$; for if this may happen, so that there shall be $x < r$, then the field will be diminished in the same ratio and it will be agreed the aperture of the lens to be diminished in the same ratio *D* .

But truly for requiring the focal length *p* to be defined this same equation will be had:

$$p = \frac{1}{2}kx\left(100 + 19,45 + 0,00951\mu\left(\lambda'' - 5v\right) + 0,211\mu\lambda'''\right)^{\frac{1}{3}},$$

[This is not the original notation used by Euler for the cube root sign, which my equation editor is unable to reproduce.]

where, since the parts arising from the two lenses scarcely may agree to being diminished, this whole magnitude of the root certainly will not exceed 5, thus so that there may safely be taken $p = \frac{5}{2}kx$; moreover above we have found that $k = 50$ approx.

SCHOLIUM 1

48. So that if here we may put $k = 50$ and $x = 2$ in., the focal length of the objective mirror arising from this formula will be $p = 250$ in. and thus to be greater than twenty feet, which will be viewed with the greatest wonder, since such telescopes may be carried around, in which *p* may not exceed 24 in. and thus there is found with *x* greater than two inches and which nevertheless maybe multiplied by hundreds ; therefore the cause of this phenomenon is required to be scrutinized. But in the first place it is evident here this cannot be allowed, since we have assumed the number *k* to be exceedingly great ; indeed even for microscopes we may be accustomed to be content with the value $k = 20$, yet we must accept the confusion then to be perceptible enough, yet such as we have not come across in these telescopes, and although besides we may accept $k = 20$, yet at this stage it produces $p = 100$ in. Therefore it is clear by necessity the cause must be allowed in that, so that after the cube root sign not only shall the two first parts related to the mirrors be much smaller than we have assumed here, but thus must be put equal to zero. Yet meanwhile it is certain, if these mirrors have a spherical figure, as we have assumed in our calculation, thence the parts influencing the confusion not to become smaller, as have been defined here ; from which we may safely conclude in these instruments the mirrors not to be taken pains over for a spherical mirror, but to be induced by the artificer to a parabolic figure, according to which the celebrated Short boasts to have found the very way of elaborating mirrors to a parabolic figures, to which discovery with doubt the smaller value of the letter *p* must be attributed [James Short was a famous telescope maker of the times from Edinburgh originally, who specialized in Gregorian telescopes];

but even if we may omit the two first parts after the root sign, the whole value of this root formula on taking $\lambda'' = \lambda''' = 1$ on account of $\mu = \frac{9}{10}$ is reduced beyond $\frac{3}{5}$; but with this value taken there follows to become almost exactly $p = 30$ in., just as experiment may testify; for it is easy to allow k to assume a value less than 50; then truly also the other constructions can be advanced, in which these two latter members at this stage may be allocated smaller coefficients. Therefore so that both our mirrors may have had a parabolic figure and it may be allowed to assume $p = 30$ in., with x being $= 2$ in., so that there will be $r = 2,829$ in. and the radius of its aperture , as evidently the aperture supplies, $= \varepsilon x = \frac{1}{2}$ in., from which certainly it will not be assumed that $\xi = \frac{1}{4}$, but only $\xi = \frac{3}{17}$, and the above field found must be diminished in the ratio $\frac{1}{4} : \frac{3}{17}$ or as $17 : 12$ or just about by a third part, thus so that at this point its radius shall be $\Phi = 11$ minutes. Just as if the focal length p were to be assumed greater, then at this stage a smaller value would be found for ξ.

SCHOLIUM 2

49. But the common telescopes of this kind do not differ greatly from the measures described above, from which it will be worth the effort to examine the measures of such a telescope more accurately, what will be had for excellence, to be examined more accurately. Nevertheless the focal length of the large mirror was of two feet or $p = 24$ in., the radius of which was $x = 2\frac{1}{2}$ in., truly the radius of the opening $y = \frac{1}{2}$ in., from which the fraction $\varepsilon = \frac{1}{5}$ follows . Truly the small mirror will be separated from the larger one by the interval $AB = 27\frac{1}{3}$ in., from which, since there shall be $AF = p = 24$ dig., it follows that the distance $FB = b = 3\frac{1}{3}$ in. (Fig. 8). Whereby, since we will have put $b = \varepsilon p$, hence there will no longer become $\varepsilon = \frac{1}{5}$, but only $\varepsilon = \frac{5}{36}$, thus so that in the received practice a smaller mirror will be located closer, than the account of the hole may demand. Truly the ratios do not abandon the rule established above on being rescinded. Indeed we may thus put in place this above smaller mirror, which also in practice may be put equal to the opening, so that not only may it receive all the rays parallel to the axis, which are reflected by the greater mirror, but also as if it may be filled completely by these. But since on account of the apparent field also rays oblique to the axis may be reflected by the greater mirror, more of these may be preceding beyond the smaller mirror in our construction, certainly it will be decided to move this same mirror a little closer, so that it may be able to receive these rays also. On this account it will be agreed to attribute a twofold value to the letter ε, the one demanded on account of the hole, the other truly from the position of the smaller mirror, lest we may be confused between these, in the latter we may put $y = \delta x$, but truly $b = \varepsilon p$, thus so that in this case there is going to become $\delta = \frac{1}{5}$ and $\varepsilon = \frac{5}{36}$. Nor truly hence may any other change be introduced

in our formulas, except as in the places, where the formula εx or y occurs, in its place we may write δx, which indeed occurs only, where mention has been made with regard to the magnitude of the hole and of the smaller mirror ; truly in all the remaining formulas, where ε is joined to the letter p, no change is made, thus so that our general formulas may prevail here also.

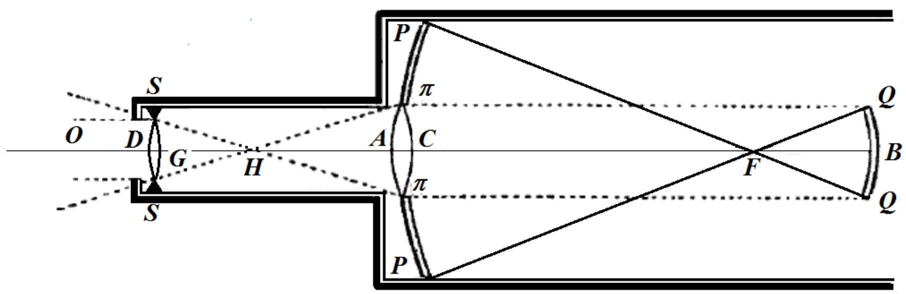


Fig. 8

Truly so that we may return to this same telescope, the focal length of the smaller mirror was $q = 3$ in., from which it is concluded the distance $BG = \beta = \frac{bq}{b-q} = 30$ and hence $CG = 2\frac{2}{3}$. But this is required to be observed properly, even if the smallest change may be made in the position of the smaller mirror, then a significant change may arise in the in the interval CG ; for if in place of $3\frac{1}{3}$ there may be taken $FB = b = 3\frac{3}{8}$, so that there shall become $BC = 27\frac{3}{8}$, $BG = \beta = 27$ is found and hence $CG = -\frac{3}{8}$. For this reason, the small mirror is accustomed to be set up thus, so that its position may only be changed very little at a snail's pace. Moreover in this same example the smaller mirror is required to be established thus, so that thence there may be produced $CG = 1\frac{1}{3}$ in. From which in turn the true value of b will be able to be defined ; for since there becomes $BG = \frac{3b}{b-3}$, on account of $CB = 24 + b$ there will become $CG = \frac{3b}{3-b} - b - 24$; so that which distance may become $= \frac{4}{3}$ in., there will be deduced

$$b = \frac{\sqrt{1525} - 29}{3} = 3{,}35041,$$

which distance exceeds the assumed value $3\frac{1}{3}$ by the small amount $\frac{1}{60}$ only , thus so that in the remaining calculation it shall be possible to assume $b = 3\frac{1}{3}$.

We may proceed now in our examination, and since the focal distance of the lens at C was $= 4$ in. $= r$, on account of $c = -\frac{4}{3}$ in. there will become $CH = \gamma = 1$ in. Then truly there was the distance $CD = 3$ in. and the focal length of the eyepiece lens D was $s = 2$ in. and thus the distance $HD = d = 2$ in. will be produced and thus $d = s$, as the nature of the telescope demands. On account of which the individual elements of this telescope will themselves be had thus :

$$\alpha = 24, \quad b = 3,35041, \quad c = -1,33333, \quad d = 2, \quad \beta = 28,68374, \quad \gamma = 1$$

and the focal lengths

$$p = 24, \quad q = 3, \quad r = 4 \quad \text{and} \quad s = 2 \text{ in.},$$

truly the intervals

$$AB = BC = 27,35041, \quad \mathrm{CD} = 3 \text{ in.}$$

Hence truly our remaining letters will be found

$$B = \frac{\beta}{b} = 8,5613, \quad \mathfrak{B} = 0,89541,$$

$$C = \frac{\gamma}{c} = -\ 0,75, \quad \mathfrak{C} = -3$$

and

$$\varepsilon = \frac{b}{p} = 0,13960 = \frac{1}{7,1633}$$

and finally

$$P = -\frac{\alpha}{b} = -7,1633, \qquad Q = -\frac{\beta}{c} = 21,51281, \qquad R = -\frac{\gamma}{d} = -\frac{1}{2}.$$

From these values found the properties of this telescope will be able to be defined in the following manner:

1. So that it may be extended to magnification, since there is $m = PQR$, it will become $m = 77,05$.

2. So that now we may define the apparent field of view also, at first from the aperture of the lens C, of which the radius $y = \frac{1}{2}$ in., on taking $\xi = \frac{1}{4}$ [the radius or angle of the apparent field of view of the lens C in radians] there will be $\frac{1}{4}\mathfrak{r}r = \frac{1}{2}$ in. and thus $\mathfrak{r} = \frac{1}{2}$; then truly there is

$$\mathfrak{q} = \frac{(P-1)M}{\mathfrak{B}} = -9,1168M$$

and in a similar manner

$$\mathfrak{C}\mathfrak{r} = \left(PQ-1\right)M - \mathfrak{q} = -143,99M$$

and hence $\mathfrak{r} = 47,99M$.

Therefore since before there was $\mathfrak{r} = \frac{1}{2}$, hence it is concluded

$$M = \frac{1}{96{,}00} = \frac{\mathfrak{q}+\mathfrak{r}+\mathfrak{s}}{77{,}05},$$

from which there is deduced

$$\mathfrak{s} = \frac{8517}{9600} - 0{,}5 = 0{,}3871,$$

which value, since it shall be less than unity, shall be agreeing with the true value ; for if it were produced greater than one, then we would have to attribute a value of the letter $\mathfrak{r}$ less than a half. On account of which the radius of apparent field will be

$$\Phi = M\xi = \tfrac{1}{4}M = 859M \text{ min.} = 8'\,57'',$$

or the radius of the actual field of view will be $= 17'\,54''$.

3. We may see, whether or not by this telescope the colored margin may be cancelled also; which condition since it will demand

$$0 = \frac{\mathfrak{r}}{PQ} + \frac{\mathfrak{s}}{PQR} \quad \text{or} \quad \mathfrak{r} = 2\mathfrak{s},$$

which since it may not differ much from the truth, the margin will be negligible everywhere; yet meanwhile the colored margin may be removed perfectly, if there were produced $\mathfrak{r} = 2\mathfrak{s}$ exactly; which indeed may be able to be done by the slightest change.

But finally it will remain, so that also we may investigate , how the equation containing the radius of confusion here may be fulfilled exactly, or since here now we may know the letters $m,\ x,\ p,\ B,\ C$ together with $\mu,\ \nu$ and λ from the nature of the glass and the shapes of the lenses, thence we may define the letter *k,* as we know it can scarcely be allowed beyond 50. But first we may assume both mirrors to be elaborated to a spherical figure, since it will be easy to return the two first terms by a calculation, when we know these mirrors to be parabolic. But from the general form given above in § 34 it is apparent to become

$$\frac{1}{k} = 0{,}222\left(1 + \frac{\varepsilon(1+B)(1-B)^2}{B^3} - \frac{4\mu}{mB^3\mathfrak{C}^3}\left(\lambda'' + \nu\mathfrak{C}\left(1-\mathfrak{C}\right)\right) - \frac{8\mu}{mB^3C^3}\lambda'''\right)^{\frac{1}{3}}$$

on account of

$$x = \tfrac{5}{2}, \quad p = 24 \quad \text{and} \quad m = 77{,}05.$$

Then since there shall be

$$\varepsilon = 0{,}1396, \quad B = 8{,}5613, \quad \mathfrak{B} = 0{,}89541, \quad \mathfrak{C} = -3 \quad \text{and} \quad C = -\tfrac{3}{4},$$

these individual terms will be changed into number thus :

$$\tfrac{1}{k}=0,222\left(1+0,1216-0,000003\mu\left(\lambda''-12\nu\right)+0,00039\mu\lambda'''\right)^{\frac{1}{3}}$$

hence therefore we deduce, if the first mirror shall be spherical, certainly there is going to be produced :

$$\tfrac{1}{k}>0,222,\ \text{that is}\ \ \tfrac{1}{k}>\tfrac{2}{9}\ \ \text{and thus}\ \ k<\tfrac{9}{2},$$

from which certainly a great confusion will arise ; so that when nevertheless it must happen, that the first mirror shall be parabolic or at least approximately, so that the first term may vanish. Again if the lesser mirror may be spherical, according to this there will be produced $\tfrac{1}{k}>0,111$ or $k<9$, from which an intolerable amount of confusion will arise accordingly, from which we conclude also no confusion to arise from the second mirror.
Therefore with the two first terms reinstated there will be had

$$\tfrac{1}{k}=0,222\left(0,000003\mu\left(\lambda''-12\nu\right)+0,00039\mu\lambda'''\right)^{\frac{1}{3}}$$

where at once it is apparent only the latter member to come into the calculation ; from which therefore, since it may be possible to assume $\mu\lambda'''=1$, there is produced

$$\tfrac{1}{k}=0,222\cdot 0,073=\tfrac{2}{9}\cdot\tfrac{1}{13}\ \text{or}\ k=59,$$

which value now is so great, that there will be no need to be concerned about confusion, and hence we understand now much more the whole expertise required to make telescopes of this kind, which if it can be expected from the craftsman, there is no doubt, why telescopes of this kind may be preferred by far as opposed to those which are found everywhere. Therefore in the above paragraph 46, so that we may adapt that to the manner of examining telescopes, there will be taken ε, whether it may refer to p, $=\tfrac{1}{7}$, or rather to x , $=\tfrac{1}{5}$, so that there may become $y=\tfrac{1}{5}x$, from which for some magnification telescopes of this kind will be able to be formed, which certainly will reveal a much greater field and likewise remove completely the colored margin. Truly if the smaller mirror may become convex, a much greater convenience thence will be allowed to be wished, as we will show in the following chapter. For the case, which may be desired at this stage, where a real image may fall in the interval BC , indeed we may not attain, since both an exceedingly small field will be produced, as well as suffering greatly from the flaw of a colored margin. For since then there shall become $R>0$, the equation for removing the margin $0=\mathfrak{r}+\tfrac{\mathfrak{s}}{R}$ may not be able to remain in place, unless $\mathfrak{r}$ may become negative, and since $\mathfrak{q}$ also is negative, the field will be reduced almost to zero.

CAPUT III

DE TELESCOPIIS CATADIOPTRICIS MINORE SPECULO CONCAVO INSTRUCTIS

PROBLEMA 1

36. *Si ante speculum principale PP* (Fig. 8) *foramine ππ pertusum ad distantiam* $AB=(1+\varepsilon)p$ *constituatur minus speculum concavum QBQ, cuius distantia focalis* $q=\varepsilon p$, *definire binas lentes C et D, ita ut quaevis obiecta distincte repraesententur.*

SOLUTIO

Hic denotat p distantiam focalem maioris speculi, cuius semidiameter $AP=x$ eiusque foraminis $A\pi=y=\varepsilon x$, ita ut radius curvaturae huius

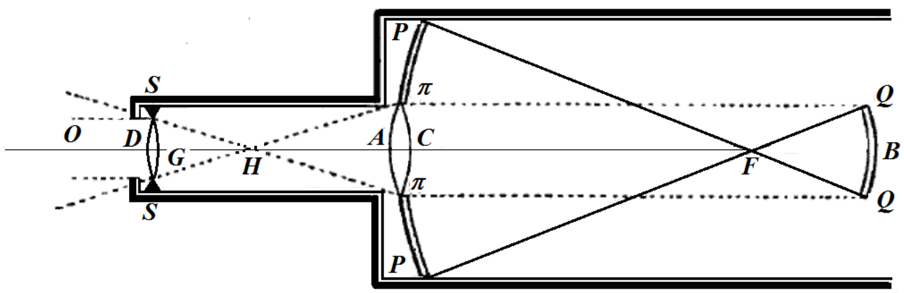


Fig. 8

speculi $=2p$. Obiectorum igitur imago principalis ab hoc speculo repraesentabitur in *F*, ut sit $AF=\alpha=p$, cuius ergo distantia a minore speculo debet esse, uti ante est ostensum, $FB=\varepsilon p$, et semidiameter huius speculi $BQ=y=\varepsilon x$. Cum igitur distantia focalis huius speculi sit $=q=\varepsilon p=FB$, radii hinc reflexi inter se fient paralleli, donec in lentem *G* incidant; pro formulis ergo nostris generalibus erit $\frac{1}{P}=-\varepsilon$ et $FB=b=\varepsilon p$, unde utique ob $P=-\frac{\alpha}{b}$ fit $P=-\frac{1}{\varepsilon}$. Deinde, cum fiat $\beta=\frac{bq}{b-q}=\infty$ hincque $B=\frac{\beta}{b}=\infty$, iam quia intervallum secundum in genere est

$$=-\frac{ABa}{P}\left(1-\frac{1}{Q}\right)=-\frac{Bp}{P}\left(1-\frac{1}{Q}\right)$$

hocque primo intervallo aequale est ponendum, fiet $Q=1$, sed ita tamen, ut sit $B\left(1-\frac{1}{Q}\right)=\frac{1+\varepsilon}{\varepsilon}$; per formulas autem generales hoc secundum intervallum $=\beta+c=(1+\varepsilon)p$; unde ob $\beta=\infty$ fit

$$c=\left(1+\varepsilon\right)p-\beta=-\infty \text{ ideoque } C=\frac{\gamma}{c}=0 \text{ et } \mathfrak{C}=0.$$

Quare posita lentis in foramine constitutae distantia focali $=r$ erit

$$r=\frac{B\mathfrak{C}}{PQ}p=-\varepsilon B\mathfrak{C}p;$$

unde, cum sit $B=\infty$ et $\mathfrak{C}=0$, vicissim colligitur

$$B\mathfrak{C}=BC=\frac{-r}{\varepsilon p}$$

atque hinc pro quarta lente *SDS* habebimus distantiam focalem $s=-\frac{r}{R}$ et intervallum $CD=r\left(1-\frac{r}{R}\right)$. Ut ergo postrema lens fiat convexa, littera *R* debet esse negativa sive in intervallum *CD* incidit imago realis in puncto *H* atque ex data multiplicatione *m* formulae generales praebent $PQR=m$, quoniam ob binas imagines reales repraesentatio erit erecta. Hinc ergo fiet $R=\frac{m}{PQ}=-\varepsilon m$, ita ut nunc sit $s=\frac{r}{\varepsilon m}$ et intervallum $CD=r(1+\frac{1}{\varepsilon m})=r+s$; quandoquidem hic fit ex natura rei $CH=r$ et $HD=s$.

Contemplemur nunc campum apparentem et secundum formulas nostras generales secundo speculo tribuamus litteram $\mathfrak{q}$, lenti *C* litteram $\mathfrak{r}$ et lenti *D* litteram $\mathfrak{s}$ et semidiameter campi apparentis erit

$$\Phi=\frac{\mathfrak{q}+\mathfrak{r}+\mathfrak{s}}{m-1}\xi$$

sumto ξ pro fractione $\frac{1}{4}$; litterae autem $\mathfrak{q}$, $\mathfrak{r}$ et $\mathfrak{s}$ ad summum unitati aequales fieri possunt. Posuimus vero [Lib. II, § 265] brevitatis gratia

$$\frac{\mathfrak{q}+\mathfrak{r}+\mathfrak{s}}{m-1}=M,$$

ut sit

$$\Phi=M\xi,$$

atque formulae nostrae generales has suppeditant aequationes:

$$\mathfrak{B}\mathfrak{q}=\left(P-1\right)M, \quad \mathfrak{C}\mathfrak{r}=\left(PQ-1\right)M-\mathfrak{q},$$

quae ob valores iam inventos

$$\mathfrak{B}=1,\ \mathrm{Q}=1 \text{ et } \mathfrak{C}=0$$

praebent ambae

$$\mathfrak{q} = (P-1)M = -\left(1+\tfrac{1}{\varepsilon}\right)M.$$

Hinc autem invenimus distantiam oculi post lentem *D,* scilicet

$$DO = O = \tfrac{\mathfrak{s}s}{Mm};$$

quae distantia cum sit positiva, quandoquidem nihil impedit, quominus ipsi $\mathfrak{s}$ valor positivus detur isque unitati aequalis, marginem coloratum tollemus, si ob $N' = 0$ et $N'' = N'''$ (quandoquidem nostrae duae lentes ex eodem vitro parantur) huic aequationi satisfaciamus:

$$0 = \tfrac{\mathfrak{r}}{PQ} + \tfrac{\mathfrak{s}}{PQR},$$

quae ergo reducitur ad hanc:

$$0 = \mathfrak{r} - \tfrac{\mathfrak{s}}{\varepsilon m},$$

unde colligitur

$$\mathfrak{r} = \tfrac{\mathfrak{s}}{\varepsilon m};$$

quare, cum sit $\mathfrak{q} = -\left(1+\tfrac{1}{\varepsilon}\right)M$, erit

$$\mathfrak{q} + \mathfrak{r} + \mathfrak{s} = -\left(1+\tfrac{1}{\varepsilon}\right)M + \tfrac{\mathfrak{s}}{\varepsilon m} + \mathfrak{s} = M(m-1),$$

unde sequitur

$$M = \tfrac{\mathfrak{s}}{m},$$

ita ut iam sit semidiameter campi apparentis

$$\Phi = \tfrac{\mathfrak{s}}{m}\cdot\xi.$$

Num autem hic pro $\mathfrak{s}$ unitas scribi queat, intelligemus ex lente *C,* cuius apertura nobis est praescripta et cuius semidiameter $= y = \varepsilon x$. Iam par formulas nostras haec semidiameter esse debet

$$\mathfrak{r}r\xi + \tfrac{x}{PQ} = \tfrac{\mathfrak{s}r}{\varepsilon m}\cdot\xi + \varepsilon x,$$

ubi sufficit maiori membro uti, ex quo sequitur esse debere $\tfrac{\mathfrak{s}r}{\varepsilon m}\cdot\xi < \varepsilon x$, unde, si statuamus $\mathfrak{s} = 1$ et $\xi = \tfrac{1}{4}$, necesse est, ut sit $r < 4s^2 mx$; si igitur velimus sumere $r > 4s^2 mx$, tum $\mathfrak{s}$ unitate minus accipi debet, ex quo campus apparens in eadem ratione diminuetur. Hic autem inprimis quoque ad ultimam lentem attendi oportet, pro qua est $s = \tfrac{r}{\varepsilon m}$, ita ut esse debeat $s < 4\varepsilon x$ sive $s < 4y$, unde patet foramen non nimis exiguum statui posse.

Totam autem confusionem ex diversa radiorum refrangibilitate oriundam tolleremus ope huius aequationis:

$$0 = \tfrac{N''}{P^2Q^2}\cdot\tfrac{1}{r} + \tfrac{N'''}{P^2Q^2R^2}\cdot\tfrac{1}{s},$$

quae abit in hanc:

$$0 = N'' + \frac{N'''}{sm};$$

quod cum nullo modo fieri possit, etiamsi diverso vitro uti vellemus, hanc confusionem, quae semper est valde exigua, tolerari oportet.

His observatis cardo rei versabitur in semidiametro confusionis, quam insensibilem reddi convenit ope huius aequationis:

$$\frac{1}{k^3} = \frac{mx^3}{p^3}\left(\frac{1}{8} + \frac{\varepsilon}{8} + \frac{\mu\varepsilon^4 p^3}{r^3}\lambda'' + \frac{\mu\varepsilon^3 p^3}{r^3 m}\lambda'''\right),$$

quae aequatio abit in hanc formam:

$$p\sqrt[3]{\left(\frac{1}{k^3 mx^3} - \frac{\mu\varepsilon^4\lambda''}{r^3} - \frac{\mu\varepsilon^3\lambda'''}{mr^3}\right)} = \frac{1}{2}\sqrt[3]{(1+\varepsilon)},$$

ex qua aequatione reperitur *p;* verum quantitatem *x* tantam assumi convenit, ut inde sufficiens claritatis gradus obtineatur. In doctrina de telescopiis autem pro sufficiente claritatis gradu sumsimus $x = \frac{m}{50}$ dig.; quod autem ibi erat *x* seu $\sqrt{x^2}$, hic nobis est $\sqrt{\left(1-\varepsilon^2\right)x^2}$, ita ut hic habeamus

$$x\sqrt{\left(1-\varepsilon^2\right)} = \frac{m}{50} \text{ dig.},$$

siquidem eodem claritatis gradu frui velimus; unde foret $x = \frac{m}{50\sqrt{\left(1-\varepsilon^2\right)}}$ dig. ideoque $x > \frac{m}{50}$ dig. Quia vero specula non tantum radiorum reflectunt, quantum lentes transmittunt, ne hoc quidem modo tantum claritatis gradum adipiscemur quam in telescopiis vulgaribus. Sin autem minori claritatis gradu contenti esse velimus atque statuamus $x = \frac{m}{50}$ dig. sumamusque ut ibi $k = 50$, aequatio nostra erit

$$p\sqrt[3]{\left(\frac{1}{m^4} - \frac{\mu\varepsilon^4\lambda''}{r^3} - \frac{\mu\varepsilon^3\lambda'''}{mr^3}\right)} = \frac{1}{2}\sqrt[3]{(1+\varepsilon)},$$

ubi manifesto debet esse $\frac{\varepsilon^4}{r^3}$ multo minus quam prius membrum $\frac{1}{m^4}$ sive $r^3 > \varepsilon^4 m^4$ ideoque *r* multo maius quam $\varepsilon m\sqrt[3]{\varepsilon m}$; supra vero vidimus esse debere $r < 4\varepsilon^2 mx$*;* quod ut fieri possit, debet esse $4\varepsilon^2 mx$ multo maius quam $\varepsilon m\sqrt[3]{\varepsilon m}$ sive $4\varepsilon m > 50\sqrt[3]{\varepsilon m}$ ideoque $\varepsilon > \frac{44}{m}$, quod in magnis multiplicationibus effici posset.

At si haec conditio non observetur, effectus in eo consistet, ut non amplius sit $\mathfrak{s} = 1$ hincque campus apparens multo minor existat quam

$$\Phi = \frac{\xi}{m} \text{ sive } \Phi = \frac{859}{m} \text{ min.}$$

COROLLARIUM 1

37. Cum in telescopiis id semper inprimis sit efficiendum, ut eorum longitudo hincque praecipue distantia focalis p quam minima reddatur, in aequatione ultima confusio a lentibus oriunda tantopere diminui debet, ut prae confusione speculorum quasi evanescat; quare, cum in ista formula ex primo speculo nascatur portio $\frac{1}{8}$, ex secundo vero $\frac{\varepsilon}{8}$, necesse est, ut portiones sequentes ex lentibus oriundae multo fiant minores, ex quo littera r multo maior esse debet quam $2\varepsilon p$ ideoque r vix minus capi poterit quam p.

COROLLARIUM 2

38. Quodsi igitur statuamus $r = p$, cum ε, uti vidimus, minus esse soleat quam $\frac{1}{8}$, pro confusione definienda tuto uti licebit hac aequatione:

$$\frac{1}{k^3} = \frac{mx^3(1+\varepsilon)}{8p^3},$$

unde colligimus

$$p = \frac{kx}{2}\sqrt[3]{m(1+\varepsilon)};$$

unde, si pro dato claritatis et distinctionis gradu capiatur $kx = m$ dig., erit quae quantitas circiter duplo minor est quam in telescopiis dioptricis communibus, ita ut hoc modo tota longitudo fere ad partem quartam reducatur.

COROLLARIUM 3

39. Sumto autem $r = p$ pro campo definiendo littera $\mathfrak{s}$ maior accipi nequit, quam ut fiat

$$\frac{\mathfrak{s}p}{4\varepsilon m} = \varepsilon x;$$

hinc ergo pro exemplo speciali, quo $\varepsilon = \frac{1}{4}$ et $m = 100$, colligetur

$$\mathfrak{s} = \frac{1}{5},$$

ex quo patet hoc casu fore campum quinquies minorem, quam si capere liceret $\mathfrak{s} = 1$, sicque in genere patet hoc modo nimis exiguum campum obtineri.

COROLLARIUM 4

40. Sumto autem $r = p$ pro constructione huiusmodi telescopii distantiae focales sequenti modo se habebunt:

$$p=\tfrac{1}{2}m\sqrt[3]{m\left(1+\varepsilon\right)},\quad q=\varepsilon p,\quad r=p \text{ et } s=\tfrac{p}{\varepsilon m};$$

tum vero intervalla lentium seu speculorum

$$AB=(1+\varepsilon)p=BC,\ CD=r+s=p(1+\tfrac{1}{\varepsilon m})$$

et distantia oculi

$$O=s=\tfrac{p}{\varepsilon m},$$

unde patet tubum arcae, in qua specula continentur, adiungendum admodum fore longum.

SCHOLION

41. Praeter incommoda vero, quae hic iam commemoravimus, huiusmodi telescopia maximo vitio laborarent, propterea quod radii in lentem *C* incidentes inter se sunt paralleli; tum enim radii peregrini, qui ab obiectis vicinis directe in eandem lentem incidunt, quia etiam sunt paralleli inter se, in transitu per lentes simili modo refringentur ac radii proprii ideoque cum iis simul ad oculum deferentur, et quoniam hi radii peregrini multo sunt fortiores quam proprii, siquidem hi duplicem reflexionem iam sunt passi, in oculo impressionem istorum penitus extinguent. Interim tamen, quia radii peregrini ad axem magis sunt obliqui atque etiam in refractione maiorem obliquitatem conservant, ab egressu in oculum excludi possent ope exigui foraminuli, cui oculus adplicatur; hoc autem modo non solum claritas nimium detrimentum pateretur, sed etiam campus insuper restringeretur; quam ob causam in huiusmodi telescopiis inprimis cavendum est, ne radii peregrini, qui circa minus speculum praeter labentes ab introitu in *C* arceri nullo modo possunt, cum radiis propriis similem refractionem patiantur. Quod praestari poterit, si modo radii proprii in lentem *C* incidentes fuerint vel divergentes vel convergentes, ut post refractionem in alio foco congregentur ac peregrini; tum enim diaphragma debito foramine in isto foco constitutum facile radios peregrinos ab ulteriori progressu ad oculum excludet. Perspicuum autem est, quo hoc remedium certius succedat, illam sive convergentiam sive divergentiam satis notabilem esse debere, sive efficiendum est, ut per refractionem huius lentis *C* imago a radiis peregrinis formata multum distet ab imagine a radiis propriis formata, id quod in sequentibus casibus usu veniet.

PROBLEMA 2

42. *Si ante speculum principale PP* (Fig. 8) *foramine ππ pertusum ad distantiam* $AB=(1+\varepsilon)p$ *constituatur minus speculum concavum QBQ, cuius distantia focalis* $q=\frac{\varepsilon(1+\varepsilon)}{1+2\varepsilon}\cdot p$ *, definire binas lentes C et D, ita ut quaevis obiecta distincte repraesententur.*

SOLUTIO

Hic ergo ut ante est distantia $AF = \alpha = p$ et $FB = b = \varepsilon p$ hincque $\frac{1}{P} = -\varepsilon$ ob $AB = (1+\varepsilon)p$. Quia vero hic est $q = \frac{\varepsilon(1+\varepsilon)}{1+2\varepsilon} \cdot p$, fiet

$$\frac{\beta}{b} = \frac{q}{b-q} = \frac{1+\varepsilon}{\varepsilon} = B,$$

ita ut iam sit $\beta = (1+\varepsilon)p$, quae distantia ipsi secundo intervallo BC est aequalis, sicque secunda imago in ipsam lentem C incidet, unde fiet $c = 0$; unde, cum posuerimus $\frac{\beta}{c} = -Q$, fiet hic

$$Q = \infty$$

tum vero pro tertia imagine erit

$$\gamma = \frac{cr}{c-r} = 0,$$

ita ut sit

$$C = -1 \text{ et } \mathfrak{C} = \infty.$$

Quare, cum sit

$$r = \frac{B\mathfrak{C}}{PQ} \cdot p = -\frac{(1+\varepsilon)\mathfrak{C}}{Q} \cdot p,$$

vicissim adparet fore

$$\frac{\mathfrak{C}}{Q} = -\frac{r}{(1+\varepsilon)p}.$$

His inventis distantiae focales erunt

$$p = p, \quad q = \frac{\varepsilon(1+\varepsilon)}{1+2\varepsilon}, \quad r = r \text{ et } s = \frac{B}{PQR} \cdot p = \frac{1+\varepsilon}{\varepsilon m} \cdot p \text{ ob } PQR = m.$$

Intervalla vero ita erunt expressa:

$$AB = (1+\varepsilon)p = BC, \quad CD = \frac{1+\varepsilon}{\varepsilon m} \cdot p = s,$$

uti rei natura postulat, quandoquidem ultima imago in ipsa lente C manet constituta. Ceterum patet hic duas occurrere imagines reales, alteram in F, alteram in C, ideoque imagines situ erecto repraesentari et recte nos assumsisse $PQR = m$.

Pro campo diiudicando erit

$$M = \frac{\mathfrak{q}+\mathfrak{r}+\mathfrak{s}}{m-1},$$

unde fit $\Phi = M\xi$; tum vero esse debet

$$\mathfrak{B}\mathfrak{q} = (P-1)M,$$

hinc

$$\mathfrak{q} = -\frac{(1+2\varepsilon)}{\varepsilon}\cdot M$$

et

$$\mathfrak{C}\mathfrak{r} = (PQ-1)M - \mathfrak{q},$$

hinc

$$\mathfrak{r} = \left(\frac{PQ}{\mathfrak{C}} - \frac{1}{\mathfrak{C}}\right)M - \frac{\mathfrak{q}}{\mathfrak{C}}.$$

Quia vero

$$\mathfrak{C} = \infty \quad \text{et} \quad \frac{\mathfrak{C}}{Q} = \frac{-r}{(1+\varepsilon)p};$$

erit

$$\mathfrak{r} = \frac{PQM}{\mathfrak{C}} = \frac{(1+\varepsilon)p}{\varepsilon r}\cdot M;$$

hinc ergo fit

$$\mathfrak{q}+\mathfrak{r}+\mathfrak{s} = \left(\frac{(1+\varepsilon)p}{\varepsilon r} - \frac{(1+2\varepsilon)}{\varepsilon}\right)M + \mathfrak{s} = (m-1)M,$$

unde reperitur

$$M = \frac{\varepsilon\mathfrak{r}\mathfrak{s}}{m\varepsilon r-(1+\varepsilon)(p-r)};$$

circa $\mathfrak{s}$ autem nihil adhuc definitur, sed cum lentis C semidiameter aperturae revera sit $=\varepsilon x,$ per formulas autem nostras esse debeat

$$= \frac{1}{4}\mathfrak{r}r = \frac{(1+\varepsilon)pM}{4\varepsilon},$$

sive ipso campo introducto haec semidiameter erit $= \frac{(1+\varepsilon)p\Phi}{\varepsilon},$ quae cum excedere nequeat $\varepsilon x,$ hoc non est verendum, nisi esset $\Phi = \frac{\varepsilon^2 x}{(1+\varepsilon)p}$ vel maius.

Iam ut margo coloratus evanescat, debet esse

$$0 = \frac{\mathfrak{r}}{PQ} + \frac{\mathfrak{s}}{PQR};$$

ideoque esse deberet

$$\frac{\mathfrak{s}}{m} = 0;$$

unde patet hoc modo marginem coloratum evitari non posse, sed tamen eum fore minimum et vix sensibilem ob denominatores PQ et PQR maximos.

Sumta porro littera $\mathfrak{s}$, uti circumstantiae permittunt, pro loco oculi habebimus

$$O = \frac{\mathfrak{s}s}{Mm}.$$

Denique conditio confusionis tollendae praebet hanc aequationem:

$$\frac{1}{k^3} = \frac{mx^3}{p^3}\left(\frac{1}{8} + \frac{(1+2\varepsilon)\varepsilon}{8(1+\varepsilon)^3}\right)$$

sequentibus partibus sponte evanescentibus, ita ut statui possit

$$p = \frac{1}{2}m\sqrt[3]{m}\left(\frac{(1+\varepsilon)^3+\varepsilon(1+2\varepsilon)}{(1+\varepsilon)^3}\right) \text{ dig.}$$

COROLLARIUM 1

43. Cum lentis in *C* positae semidiameter aperturae esse debeat $= \frac{1}{4}\mathfrak{r}r$, ea vero revera debeat $= \varepsilon x$, hinc collititur $\mathfrak{r} = \frac{4\varepsilon x}{r}$. Verum ante invenimus $\mathfrak{r} = \frac{(1+\varepsilon)p}{r\varepsilon}\cdot M$; his ergo duobus valoribus aequatis prodit

$$4\varepsilon^2 x = (1+\varepsilon)\, pM,$$

unde, si esset $\mathfrak{r} = 1$, foret $r = 4\varepsilon x$, tum vero $r = \frac{M(1+\varepsilon)p}{\varepsilon}$; quia vero est

$$M = \frac{\varepsilon r\mathfrak{s}}{m\varepsilon r-(1+\varepsilon)(p-r)},$$

habebimus nunc substituto pro *r* illo valore

$$M = \frac{4\varepsilon^2\mathfrak{s}x}{4m\varepsilon^2 x-(1+\varepsilon)(p-4\varepsilon x)},$$

qui valor in illa aequatione substitutus dabit

$$\mathfrak{s} = \frac{4m\varepsilon^2 x-(1+\varepsilon)(p-4\varepsilon x)}{(1+\varepsilon)p}.$$

COROLLARIUM 2

44. Quia autem $\mathfrak{s}$ unitate maius esse nequit, hoc valore unitati aequali posito prodibit

$$4m\varepsilon^2 x = 2(1+\varepsilon)\, p - 4\varepsilon(1+\varepsilon)x$$

hincque

$$m = \frac{(1+\varepsilon)p-2\varepsilon(1+\varepsilon)x}{2\varepsilon^2 x},$$

quae aequatio subsistere nequit, nisi multiplicatio *m* aliquot millia excedat, quod in praxi nunquam locum habere potest.

SCHOLION

45. Huiusmodi vero telescopia duplici laborant defectu; primo enim, quia lens *C* in ipso imaginis loco constituitur, nisi lens ex purissimo vitro sit confecta, repraesentatio vehementer erit inquinata, uti iam saepius observavimus; deinde etiam haud exiguum vitium in eo consistit, quod marginem coloratum non licuit ad nihilum reducere; quam ob causam haec telescopia superfluum foret uberius prosequi, sed potius eiusmodi casum evolvamus, in quo secunda imago post lentem *C* cadat simulque margo coloratus feliciter tolli queat. Quare, cum pro hoc praestando habeatur aequatio

$$0 = \frac{\mathfrak{r}}{PQ} + \frac{\mathfrak{s}}{PQR},$$

necesse est, ut fieri queat $\mathfrak{r} + \frac{\mathfrak{s}}{R} = 0,$ quod commodissime fieri poterit, si fuerit $R = -1;$ quia enim tum erit $\mathfrak{r} = \mathfrak{s}$, maximum campum adipisci poterimus, si sumere liceat $\mathfrak{t} = \mathfrak{s} = 1$; tum enim fiet

$$M = \frac{\mathfrak{q}+2}{m-1},$$

et quamvis $\mathfrak{q}$ sit fractio negativa, tamen campus hinc orietur satis magnus; ut vero fiat *R* numerus negativus, secunda imago in intervallum *CD* cadere debet, ita ut *Q* maneat quantitas positiva, et quia multiplicatio dat $m = PQR$, ob $P = -\frac{1}{\varepsilon}$, si sumamus $R = -1$, necesse est fiat $Q = \varepsilon m$; unde, cum sit $Q = -\frac{\beta}{c}$ erit

$$c = -\frac{\beta}{\varepsilon m};$$

at vero secundum intervallum $BC = \beta + c$; quod cum primo $(1+\varepsilon)p$ aequale esse debeat, elicimus

$$\beta = (1+\varepsilon)p - c = \frac{\varepsilon(1+\varepsilon)mp}{\varepsilon m-1}.$$

Cum vero sit

$$\beta = \frac{bq}{b-q} \quad \text{et} \quad b = \varepsilon p$$

sive etiam

$$\frac{1}{q} = \frac{1}{b} + \frac{1}{\beta}$$

hinc erit

$$\frac{1}{q} = \frac{1+2\varepsilon}{\varepsilon(1+\varepsilon)p} - \frac{1}{\varepsilon(1+\varepsilon)mp};$$

tum vero erit

$$B = \frac{\beta}{b} = \frac{(1+\varepsilon)m}{\varepsilon m-1} \quad \text{hincque} \quad \mathfrak{B} = \frac{(1+\varepsilon)m}{(1+2\varepsilon)m-1},$$

unde fit $q=\mathfrak{B}b$.

Porro vero, cum sit $r=\mathfrak{C}c,$ erit

$$\mathfrak{C}=\frac{r}{c}=\frac{-(\varepsilon m-1)r}{(1+\varepsilon)p} \quad \text{hincque} \quad \mathrm{C}=\frac{-(\varepsilon m-1)r}{(1+\varepsilon)p+(\varepsilon m-1)r}.$$

Pro intervallo autem *CD,* quod est $\gamma+d=\gamma+s$, quia est

$$\gamma=\frac{cr}{c-r}=\frac{(1+\varepsilon)pr}{(1+\varepsilon)p+(\varepsilon m-1)r},$$

quia vero etiam esse debet $R=-\frac{\gamma}{d}=-1,$ hinc erit $s=\gamma$ sicque intervallum $CD=2\gamma=2s=\left[\frac{2(1+\varepsilon)pr}{(1+\varepsilon)p+(\varepsilon m-1)r}\right].$

Quia autem porro est $M=\frac{\mathfrak{q}+2}{m-1},$ sumto scilicet $\mathfrak{r}+\mathfrak{s}=1$, erit ,

$$\mathfrak{q}=\frac{-((1+2\varepsilon)m-1)}{\varepsilon m}\cdot M$$

hincque

$$\mathfrak{q}+2=\frac{-((1+2\varepsilon)m-1)M+2\varepsilon m}{\varepsilon m}=M(m-1);$$

unde sequitur

$$M=\frac{2\varepsilon m}{\varepsilon m^2+(1+\varepsilon)m-1},$$

ex quo vicissim concludimus

$$\mathfrak{q}=\frac{-2((1+2\varepsilon)m-1)}{\varepsilon m^2+(1+\varepsilon)m-1}.$$

Praeterea vero adhuc habetur haec aequatio $\mathfrak{C}=(PQ-1)\mathrm{M}-\mathfrak{q}$, quae abit in hanc:

$$\mathfrak{C}=-(m+1)\mathrm{M}-\mathfrak{q}$$

seu substitutis valoribus

$$\frac{-(\varepsilon m-1)r}{(1+\varepsilon)p}=\frac{-2\varepsilon m(m+1)+2(1+2\varepsilon)m-2}{\varepsilon m^2+(1+\varepsilon)m-1}=\frac{-2\varepsilon m^2+2(1+\varepsilon)m-2}{\varepsilon m^2+(1+\varepsilon)m-1},$$

unde concludimus fore

$$r=\frac{2(\varepsilon m^2-(1+\varepsilon)m+1)(1+\varepsilon)p}{(\varepsilon m-1)(\varepsilon m^2+(1+\varepsilon)m-1)}=\frac{2(m-1)(1+\varepsilon)}{\varepsilon m^2+(1+\varepsilon)m-1}\cdot p;$$

hinc, cum sit $\frac{1}{\gamma}=\frac{1}{r}-\frac{1}{c}$, reperitur

$$\gamma=\frac{2(m-1)(1+\varepsilon)}{3\varepsilon m^2-(1+\varepsilon)m+1}\cdot p=s,$$

unde porro concluditur distantia oculi

$$O=\frac{\mathfrak{s}s}{Mm}=\frac{\varepsilon m^2+(1+\varepsilon)m-1}{2\varepsilon m^2}\cdot s=\frac{1}{2}s\left(1+\frac{1+\varepsilon}{\varepsilon m}-\frac{1}{\varepsilon m^2}\right)=\frac{1}{2}s \text{ proxime.}$$

Quod autem ad campum apparentem attinet, quoniam sumsimus $\mathfrak{r}=\mathfrak{s}=1$, dispiciendum est, num etiam ponere liceat $\xi=\frac{1}{4}$. Hoc autem patebit ex lente *C,* cuius semidiameter aperturae $=\xi r$ excedere nequit εx; posito igitur $\xi r=\varepsilon x$ colligitur

$$\xi=\frac{\varepsilon x\left(\varepsilon m^2+(1+\varepsilon)m-1\right)}{2(m-1)(1+\varepsilon)p},$$

qui valor si fuerit minor quam $\frac{1}{4}$, eo erit utendum, ita ut tum sit $\Phi=M\xi$; sin autem ille valor prodeat maior quam $\frac{1}{4}$, nihilominus sumi debet $\xi=\frac{1}{4}$.
Si tanquam exemplum sumatur

$$\varepsilon=\frac{1}{4},\quad m=100,\quad x=\frac{5}{2}\text{dig. et } p=25\text{dig.},$$

revera prodit $\xi=\frac{1}{4}$, ita ut haec positio $\xi=\frac{1}{4}$ parum a praxi discrepare videatur; unde operae pretium erit has determinationes coniunctim ob oculos ponere.

EXEMPLUM TELESCOPTI CATADIOPTRICI

46. Ex modo allatis prima elementa huius telescopii ita se habebunt:

$$a=\infty,\quad b=\varepsilon p,\quad c=\frac{-(1+\varepsilon)}{\varepsilon m-1}p,\quad d=\gamma,$$

$$\alpha=p,\quad \beta=\frac{\varepsilon(1+\varepsilon)mp}{\varepsilon m-1},\quad \gamma=\frac{2(m-1)(1+\varepsilon)p}{3\varepsilon m^2-(1+\varepsilon)m+1},\quad \delta=\infty.$$

Ex quibus deducuntur sequentes valores:

$$B=\frac{(1+\varepsilon)m}{\varepsilon m-1},\quad \mathfrak{B}=\frac{(1+\varepsilon)m}{(1+2\varepsilon)m-1},$$

$$C=\frac{-2(m-1)(\varepsilon m-1)}{3\varepsilon m^2-(1+\varepsilon)m+1},\quad \mathfrak{C}=\frac{-2(m-1)(\varepsilon m-1)}{\varepsilon m^2+(1+\varepsilon)m-1},$$

$$P=-\frac{\alpha}{b}=-\frac{1}{\varepsilon},\quad Q=-\frac{\beta}{c}=\varepsilon m,\quad R=-\frac{\gamma}{d}=-1.$$

Ex his vero colliguntur distantiae focales

$$p=p,\quad q=\mathfrak{B}b=\frac{s(1+\varepsilon)m}{(1+2\varepsilon)m-1}\cdot p,$$

$$r=\mathfrak{C}c=\frac{-2(m-1)(1+\varepsilon)}{\varepsilon m^2+(1+\varepsilon)m-1},\quad \text{et}\quad s=d=\gamma$$

$$P=-\frac{\alpha}{b}=-\frac{1}{\varepsilon},\quad Q=-\frac{\beta}{c}=\varepsilon m,\quad R=-\frac{\gamma}{d}=-1.$$

et pro earum aperturis

$$\mathfrak{q}=\frac{-2((1+2\varepsilon)m-1)}{\varepsilon m^2+(1+\varepsilon)m-1},\quad \mathfrak{r}=1,\quad \mathfrak{s}=1$$

hincque

$$\mathfrak{q}+\mathfrak{r}+\mathfrak{s}=\frac{2\varepsilon m(m-1)}{\varepsilon m^2+(1+\varepsilon)m-1}$$

ideoque

$$M=\frac{2\varepsilon m}{\varepsilon m^2+(1+\varepsilon)m-1},$$

ex quo elicitur semidiameter campi apparentis $\Phi=M\xi$, ac si liceat sumere $\xi=\frac{1}{4}$, fiet

$$\Phi=\frac{1718\varepsilon m}{\varepsilon m^2+(1+\varepsilon)m-1}\ \text{minut.};$$

at pro loco oculi invenimus

$$O=\frac{1}{2}s\left(1+\frac{1+\varepsilon}{\varepsilon m}-\frac{1}{\varepsilon m^2}\right).$$

Superest igitur, ut ex conditione confusionis definiatur distantia focalis *p,* quae reperitur

$$p = kx\sqrt[3]{m}\left(\begin{array}{c}\frac{1}{8}+\frac{\varepsilon(m+1)^2((1+2\varepsilon)m-1)}{8(1+\varepsilon)^3m^3}\\ +\frac{\mu(\varepsilon m^2+(1+\varepsilon)m-1)^3}{8m^4(m-1)^3(1+\varepsilon)^3}(\lambda''+\nu\mathfrak{C}(1-\mathfrak{C}))+\frac{\mu(3\varepsilon m^2-(1+\varepsilon)m+1)^3\lambda'''}{8m^4(m-1)^3(1+\varepsilon)^3}\end{array}\right)^{\frac{1}{3}}$$

ubi si tantam claritatem desideremus, qualem supra telescopiis tribuimus, sumi debet $x=\frac{m}{50}$ dig. et pro gradu distinctionis $k=50$, ut sit $kx=m$.

Sin autem minori claritatis gradu contenti esse velimus, fortasse sufficiet ponere $x=\frac{m}{100}$ dig. vel adeo $x=\frac{m}{200}$ dig.

CONSTRUCTIO HUIUSMODI TELESCOPII
PRO MULTIPLICATIONE $m=100$ SUMTO $\varepsilon=\frac{1}{4}$

47. Pro maiori ergo speculo, cuius semidiameter sit $=x$, foraminis semidiameter erit $=\frac{1}{4}x$, eius vero distantia focalis in genere ponatur $=p$; ex qua sequentes distantiae focales ita definientur:

$$q=\frac{125}{596}p=0,2097p,$$

$$r=\frac{5\cdot 99}{5248}p=0,0934p,$$

$$s=\frac{5\cdot 99}{14752}p=0,03355p.$$

Intervalla autem sequenti modo definientur :

1. $AB=\frac{5}{4}p=1,25p,$
2. $BC=\frac{5}{4}p=1,25p,$
3. $CD=2s = 0,0671p,$
4. $O=0,5248s=[0,0176p].$

Praeterea uti speculi maioris semidiameter aperturae est $=x$, ita minoris erit $=\frac{1}{4}x$, cui etiam aequatur apertura lentis *C;* lentis vero ocularis *D* semidiameter aperturae poterit sumi $=\frac{1}{4}s$, unde campi apparentis semidiameter erit circiter $\Phi=16,368$ minut., qui campus locum habet, nisi sit $\frac{1}{4}x<\frac{1}{4}r$ seu $x<r$; hoc enim si evenerit, ut sit $x<r$, tum campus in eadem ratione diminuetur atque in eadem ratione aperturam lentis *D* diminui conveniet.

At vero pro definienda distantia focali *p* habetur ista aequatio:

$$p = \tfrac{1}{2}kx\sqrt[3]{(100 + 19,45 + 0,00951\mu(\lambda'' - 5\nu) + 0,211\mu\lambda''')},$$

ubi, cum partes ex binis lentibus oriundae vix ad dimidium accedant, tota haec quantitas radicalis certe non ad 5 exsurget, ita ut tuto sumi possit $p = \tfrac{5}{2}kx$; supra autem notavimus esse circiter $k = 50$.

SCHOLION 1

48. Quodsi hic statuamus $k = 50$ et $x = 2$ dig., distantia focalis speculi obiectivi ex hac formula prodit $p = 250$ dig. ideoque maius viginti pedibus, quod merito maxime mirum videbitur, cum talia telescopia circumferantur, in quibus p non superat 24 dig. atque x adeo duobus digitis maior reperitur et quae nihilominus centies multiplicant; cuius ergo phaenomeni causam scrutari oportet. Primo autem manifestum est eam non in hoc esse sitam, quod numerum k nimis magnum assumsimus; etsi enim pro microscopiis contenti esse soleamus valore $k = 20$, tamen fateri debemus confusionem tum satis esse sensibilem, qualem tamen in his telescopiis non deprehendimus, et quamvis praeterea sumeremus $k = 20$, tamen adhuc prodiret $p = 100$ dig. Evidens ergo est causam necessario in eo sitam esse debere, quod post signum radicale cubicum binae priores partes ad specula relatae non solum multo sint minores, quam hic assumsimus, sed adeo nihilo aequales poni debeant. Interim tamen certum est, si haec specula haberent figuram sphaericam, uti in calculo nostro assumsimus, partes inde in confusionem influentes minores non fore, quam hic sunt definitae; ex quo tuto concludere possumus in his instrumentis specula non ad figuram sphaericam esse elaborata, sed iis ab artifice figuram parabolicam esse inductam, in quo Cel. Short gloriatur se modum invenisse specula ad figuram parabolicam elaborandi, cui invento sine dubio exiguus valor litterae p tribui debet; quodsi enim post signum radicale binas priores partes omittamus, totus valor huius formulae radicalis sumto $\lambda'' = \lambda''' = 1$ ob $\mu = \tfrac{9}{10}$ circiter reducetur infra $\tfrac{3}{5}$; sumto autem hoc valore sequitur fore $p = 30$ dig. prorsus fere, uti experientia testatur; facile enim licet k assumere minus quam 50; tum vero etiam aliae constructiones proferri possunt, in quibus haec duo membra posteriora adhuc minores sortirentur coefficientes. Quodsi ergo ambo nostra specula figuram habuerint parabolicam sumereque liceat $p = 30$ dig., existante $x = 2$ dig., erit $r = 2,829$ dig. eiusque aperturae semidiameter, quam scilicet foramen suppeditat, $= \varepsilon x = \tfrac{1}{2}$ dig., unde utique sumi non licebit $\xi = \tfrac{1}{4}$, sed tantum $\xi = \tfrac{3}{17}$, et campus supra inventus diminui debet in ratione $\tfrac{1}{4} : \tfrac{3}{17}$ sive 17 : 12 sive suo triente propemodum, ita ut adhuc sit eius semidiameter $\Phi = 11$ minut. Quodsi autem distantia focalis p maior assumi debeat, tum pro ξ adhuc minor valor reperietur.

SCHOLION 2

49. Telescopia autem vulgaria huius generis non mediocriter discrepant a mensuris supra descriptis, unde operae pretium erit mensuras talis telescopii, quod pro excellenti habetur, accuratius examinare. Erat autem speculi maioris distantia focalis duorum pedum seu $p=24$ dig., semidiameter eius $x=2\frac{1}{2}$ dig., foraminis vero semidiameter $y=\frac{1}{2}$ dig., unde sequitur fractio $\varepsilon=\frac{1}{5}$. Verum minus speculum a maiore distabat intervallo $AB=27\frac{1}{3}$ dig., unde, cum sit $AF=p=24$ dig., sequitur distantia $FB=b=3\frac{1}{3}$ dig. (Fig. 8). Quare, cum posuerimus $b=\varepsilon p$, hinc non amplius fiet $\varepsilon=\frac{1}{5}$, sed tantum $\varepsilon=\frac{5}{36}$, ita ut in praxi recepta minus speculum propius collocetur, quam ratio foraminis postulat. Verum rationes non desunt a regula supra stabilita recedendi. Supra enim hoc speculum minus, quod etiam in praxi foramini aequabatur, ita constituimus, ut omnes radios axi parallelos, qui a maiore speculo reflectuntur, non solum reciperet, sed etiam ab iis quasi impleretur. Cum autem ob campum apparentem etiam radii ad axem obliqui a maiori speculo reflectantur, quorum plures in nostra constructione minus speculum praetergrederentur, utique consultum erit istud speculum aliquanto propius admovere, ut etiam hos radios recipere queat. Quamobrem conveniet litterae ε duplicem valorem tribui, alterum ex ratione foraminis petitum, alterum vero ex loco minoris speculi, quos ne inter se confundamus, in posterum statuamus $y=\delta x$, at vero $b=\varepsilon p$, ita ut hoc casu futurum sit $\delta=\frac{1}{5}$ et $\varepsilon=\frac{5}{36}$. Neque vero hinc in nostras formulas alia mutatio inferetur, nisi ut in locis, ubi formula *ex* seu *y* occurrit, eius loco scribamus δx, quod quidem tantum, ubi de quantitate foraminis et minoris speculi sermo est, occurrit; in reliquis vero omnibus formulis, ubi ε cum littera p coniungitur, nulla fit mutatio, ita ut nostrae formulae generales etiam hic valeant.

Verum ut ad istud telescopium revertamur, distantia focalis speculi minoris erat $q=3$ dig., unde concluditur distantia $BG=\beta=\frac{bq}{b-q}=30$ hincque $CG=2\frac{2}{3}$. Hic autem probe notandum est, si vel levissima mutatio in loco minoris speculi fiat, tum in hoc intervallo CG insignem mutationem oriri; si enim loco $3\frac{1}{3}$ sumatur $FB=b=3\frac{3}{8}$, ut sit $BC=27\frac{3}{8}$, reperietur $BG=\beta=27$ hincque $CG=-\frac{3}{8}$. Quam ob causam etiam minus speculum ita constitui solet, ut eius locus ope cochleae tantillum immutari possit. In isto autem exemplo speculum minus ita est constituendum, ut inde prodeat $CG=1\frac{1}{3}$ dig. Unde vicissim verus valor ipsius b definiri poterit; quia enim fit $BG=\frac{3b}{b-3}$, ob $CB=24+b$ erit $CG=\frac{3b}{3-b}-b-24$; quae distantia ut fiat $=\frac{4}{3}$ dig., elicietur

$$b=\frac{\sqrt{1525}-29}{3}=3{,}35041,$$

qui valor assumtum $3\frac{1}{3}$ tantum superat particula $\frac{1}{60}$, ita ut in reliquo calculo sumi possit $b = 3\frac{1}{3}$.

Pergamus nunc in nostro examine, et quia lentis in *C* distantia focalis erat $= 4$ dig. $= r$, ob $c = -\frac{4}{3}$ dig. fiet $CH = \gamma = 1$ dig. Deinde vero erat intervallum $CD = 3$ dig. et lentis ocularis *D* distantia focalis $s = 2$ dig. sicque prodibit distantia $HD = d = 2$ dig. ideoque $d = s$, uti natura telescopii postulat. Quocirca singula huius telescopii elementa ita se habebunt:

$$\alpha = 24, \quad b = 3,35041, \quad c = -1,33333, \quad d = 2, \quad \{\beta = 28,68374, \quad \gamma = 1$$

et distantiae focales

$$p = 24, \quad q = 3, \quad r = 4 \quad \text{et} \quad s = 2 \text{ dig.},$$

intervalla vero

$$AB = BC = 27,35041, \quad \text{CD} = 3 \text{ dig.}$$

Hinc vero reliquae nostrae litterae invenientur

$$B = \frac{\beta}{b} = 8,5613, \quad \mathfrak{B} = 0,89541,$$

$$C = \frac{\gamma}{c} = -\ 0,75, \quad \mathfrak{C} = -3$$

et

$$\varepsilon = \frac{b}{p} = 0,13960 = \frac{1}{7,1633}$$

ac denique

$$P = -\frac{\alpha}{b} = -7,1633, \qquad Q = -\frac{\beta}{c} = 21,51281, \qquad R = -\frac{\gamma}{d} = -\frac{1}{2}.$$

His inventis valoribus proprietates huius telescopii sequenti modo definiri poterunt:

1. Quod ad multiplicationem attinet, quia est $m = PQR$, erit $m = 77,05$.

2. Ut nunc etiam campum apparentem definiamus, primo ex apertura lentis *C*, cuius semidiameter est $y = \frac{1}{2}$ dig., sumto $\xi = \frac{1}{4}$ erit $\frac{1}{4}\mathfrak{r}r = \frac{1}{2}$ dig. ideoque $\mathfrak{r} = \frac{1}{2}$ dig.; tum vero est

$$\mathfrak{q} = \frac{(P-1)M}{\mathfrak{B}} = -9,1168M$$

similique modo

$$\mathfrak{C}\mathfrak{r} = \left(PQ - 1\right)M - \mathfrak{q} = -143,99M$$

hincque $\mathfrak{r} = 47,99M$.

Cum igitur ante esset $\mathfrak{r} = \frac{1}{2}$ dig., hinc concluditur

$$M = \frac{1}{96,00} = \frac{\mathfrak{q}+\mathfrak{r}+\mathfrak{s}}{76,05},$$

unde elicitur

$$\mathfrak{s} = \frac{8517}{9600} - 0,5 = 0,3871,$$

qui valor, cum unitate sit minor, veritati erit consentaneus; si enim unitate maior prodiisset, tum litterae $\mathfrak{r}$ valorem semisse minorem tribuere debuissemus. Quocirca semidiameter campi apparentis erit

$$\Phi = M\xi = \frac{1}{4}M = 859M \text{ min.} = 8'\ 57'',$$

sive diameter campi erit $= 17'\ 54''$.

3. Videamus, an per hoc telescopium etiam margo coloratus destruatur; quae conditio cum postulet

$$0 = \frac{\mathfrak{r}}{PQ} + \frac{\mathfrak{s}}{PQR} \quad \text{sive} \quad \mathfrak{r} = 2\mathfrak{s},$$

quod cum non multum a veritate discrepet, margo utique debebit esse insensibilis; interim tamen perfectius margo coloratus tolleretur, si prodiisset exacte $\mathfrak{r} = 2\mathfrak{s}$; id quod quidam levissima mutatione fieri posset.

Tandem autem restabit, ut etiam investigemus, quam exacte aequatio semidiametrum confusionis complectens hic impleatur, sive cum hic iam cognoscamus litteras $m,\ x,\ p,\ B,\ C$ una cum $\mu,\ \nu$ et λ ex indole vitri et figura lentium, definiemus inde litteram *k,* quam novimus vix infra 50 admitti posse. Sumamus autem primo ambo specula ad figuram sphaericam esse elaborata, quoniam facile erit facto calculo duos terminos priores reiicere, quando noverimus haec specula esse parabolica. Ex forma autem generali supra § 34 data patet fore

$$\frac{1}{k} = 0,222\left(\quad 1 + \frac{\varepsilon(1+B)(1-B)^2}{B^3} - \frac{4\mu}{mB^3\mathfrak{C}^3}\left(\lambda'' + \nu\mathfrak{C}\left(1-\mathfrak{C}\right)\right) - \frac{8\mu}{mB^3C^3}\lambda''' \right)^{\frac{1}{3}}$$

ob

$$x = \frac{5}{2}, \quad p = 24 \quad \text{et} \quad m = 77,05.$$

Deinde cum sit

$$\varepsilon = 0,1396, \quad B = 8,5613, \quad \mathfrak{B} = 0,89541, \quad \mathfrak{C} = -3 \quad \text{et} \quad C = -\frac{3}{4},$$

singuli hi termini ita in numeris evolventur:

$$\frac{1}{k} = 0,222\left(1 + 0,1216 - 0,000003\mu\left(\lambda'' - 12\nu\right) + 0,00039\mu\lambda'''\right)^{\frac{1}{3}}$$

hinc ergo colligimus, si primum speculum esset sphaericum, certe proditurum esse

$$\tfrac{1}{k} > 0,222, \text{ hoc est } \tfrac{1}{k} > \tfrac{2}{9} \text{ ideoque } k < \tfrac{9}{2},$$

unde certe confusio enormis nasceretur; quod cum neutiquam fieri debet, necesse est, ut primum speculum sit parabolicum vel proxima saltem, ut primus terminus evanescat. Si porro speculum minus esset sphaericum, prodiret adhuc $\tfrac{1}{k} > 0,111$ seu $k < 9$, unde confusio adhuc intolerabilis nasceretur, ex quo concludimus etiam a secundo speculo nullam confusionem nasci.

Reiectis ergo binis prioribus terminis habebitur

$$\tfrac{1}{k} = 0,222\left(0,000003\mu\left(\lambda'' - 12\nu\right) + 0,00039\mu\lambda'''\right)^{\frac{1}{3}}$$

ubi statim patet solum postremum membrum in computum venire; unde ergo, cum sumi possit $\mu\lambda''' = 1$, prodit

$$\tfrac{1}{k} = 0,222 \cdot 0,073 = \tfrac{2}{9} \cdot \tfrac{1}{13}$$

sive

$$k = 59,$$

qui valor iam tantus est, ut nulla confusio sit metuenda, atque hinc iam multo magis intelligimus summam sollertiam ad huiusmodi telescopia conficienda requiri, quae si ab artifice exspectari potest, nullum est dubium, quin species telescopiorum a nobis ante exposita his, quae passim reperiuntur, longe sit anteferenda. In paragrapho igitur superiori 46, ut eam ad modo examinatum telescopium accommodemus, sumi poterit ε, quatenus ad p refertur, $= \tfrac{1}{7}$, quatenus autem ad x refertur, $= \tfrac{1}{5}$, ut fiat $y = \tfrac{1}{5}x$, unde pro quavis multiplicatione huiusmodi telescopia formari poterunt, quae certe multo maiorem campum patefacient simulque marginem coloratum perfectius tollent. Verum si speculum minus fiat convexum, multo maiora commoda inde sperare licebit, uti in sequente capite ostendemus. Casum enim, qui hic adhuc desiderari posset, quo imago realis in intervallum BC caderet, ne quidam attingemus, quoniam tam campum nimis parvum produceret, quam vitio marginis colorati vehementer laboraret. Cum enim tum esset $R > 0$, aequatio pro margine tollendo $0 = \mathfrak{r} + \tfrac{\mathfrak{s}}{R}$ subsistere non posset, nisi $\mathfrak{r}$ foret negativum, et quia $\mathfrak{q}$ etiam est negativum, campus fere ad nihilum redigeretur.