CHAPTER II

CONCERNING THE COMPUTATION OF THE CONFUSION * ARISING, WHILE BESIDES LENSES, MIRRORS ALSO ARE USED FOR MAKING OPTICAL INSTRUMENTS

[**i.e. the spherical aberration centred on the focus along the optical axis.*]

PROBLEM 1

23. *If mirrors may be used in place of the first and second lenses, to find the formulas which must be introduced into the calculation found above in Book I, for which evidently the radius of confusion has been found, on account of these two mirrors.*

SOLUTION

We have shown in the first book (§ 91) the diffusion length to arise from two lenses,

$$Gg = \mu\beta^2 x^2 \left\{ \begin{array}{l} +\frac{\alpha^2}{b^2}\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{\nu}{a\alpha}\right) \\ +\frac{b^2}{\alpha^2}\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\lambda'\left(\frac{1}{b}+\frac{1}{\beta}\right)^2+\frac{\nu}{b\beta}\right) \end{array} \right\}$$

which expression on putting $\alpha = Aa,\ \beta = Bb$, while truly also $\frac{A}{1+A} = \mathfrak{A}$ and $\frac{B}{1+B} = \mathfrak{B}$ will be changed into this:

$$Gg = \frac{\mu A^2 B^2 x^2}{a}\left(\frac{\lambda}{\mathfrak{A}^3}+\frac{\nu}{A\mathfrak{A}}+\frac{b}{A^4 a}\left(\frac{\lambda'}{\mathfrak{B}^3}+\frac{\nu}{B\mathfrak{B}}\right)\right),$$

and if here again, as we may use henceforth in the treatment of telescopes we have made, we may put $\frac{\alpha}{b} = \frac{Aa}{b} = -P$, this expression will adopt this form :

$$Gg = \frac{A^2 B^2 x^2}{a}\left(\mu\left(\frac{\lambda}{\mathfrak{A}^3}+\frac{\nu}{A\mathfrak{A}}\right)-\frac{\mu}{A^3 P}\left(\frac{\lambda'}{\mathfrak{B}^3}+\frac{\nu}{B\mathfrak{B}}\right)\right).$$

If now mirrors may be substituted in place of the two lenses, for which the letters $a,\ \alpha,\ b,\ \beta$ shall be similarly related to x , in problem 4 of the preceding chapter (§ 15), we have found the diffusion length to become

$$Gg = \frac{\beta^2}{b^2}\cdot\frac{(a+\alpha)(a-\alpha)^2 x^2}{8a^3\alpha}+\frac{(b+\beta)(b-\beta)^2 x^2}{8\alpha^2 b\beta};$$

which form on putting $\alpha = Aa$, $\beta = Bb$ et $\frac{\alpha}{b} = -P$ will adopt this form:

$$Gg = \frac{A^2B^2x^2}{a}\left(\frac{(1+A)(1-A)^2}{8A^3} - \frac{(1+B)(1-B)^2}{8A^3B^3P}\right),$$

from which we recognise brought together with the above, if in place of the first lens a mirror may be substituted, then in the computation of the confusion, in place of the formula

$$\mu\left(\frac{\lambda}{\mathfrak{A}^3} + \frac{\nu}{A\mathfrak{A}}\right),$$

this must be written:

$$\frac{(1+A)(1-A)^2}{8A^3}$$

and if also in place of the second lens a mirror may be substituted, then in a similar manner in place of the formula

$$\mu\left(\frac{\lambda'}{\mathfrak{B}^3} + \frac{\nu}{B\mathfrak{B}}\right),$$

this must be written :

$$\frac{(1+B)(1-B)^2}{8B^3},$$

and if the circumstances may permit, so that also a mirror may be put in place of the third lens, then in the computation of the confusion in place of the formula

$$\mu\left(\frac{\lambda''}{\mathfrak{C}^3} + \frac{\nu}{C\mathfrak{C}}\right)$$

this formula must be written :

$$\frac{(1+C)(1-C)^2}{8C^3}$$

and from which it is well enough understood above, how the amount of the confusion must be estimated, when mirrors may be used in place of lenses.

COROLLARY 1

24. But in as much as the objective mirror has been bored through by a hole, of which the radius $= y$, in the common factor in place of x^2 hitherto it will be required to write $x^2 - y^2$, thus so that now the expression for the diffusion length found shall be going to become

$$Gg=\frac{A^2B^2\left(x^2-y^2\right)}{a}\left(\frac{(1+A)(1-A)^2}{8A^3}-\frac{(1+B)(1-B)^2}{8A^3B^3P}\right),$$

where it is required to note the formula x^2-y^2 to be proportional to the reflecting surface in the first mirror, to be used just as x^2 was proportional to the refracting surface of the objective lens.

COROLLARY 2

25. And this formula x^2-y^2 also is extended to all the following lenses, however many pairs of mirrors may be used together above ; thus e.g., if two lenses may be used besides the mirrors, the whole diffusion length Ii will be expressed thus :

$$Ii=\frac{A^2B^2C^2D^2\left(x^2-y^2\right)}{a}\left(\begin{array}{c}\frac{(1+A)(1-A)^2}{8A^3}-\frac{(1+B)(1-B)^2}{8A^3B^3P}\\ +\frac{\mu}{A^3B^3PQ}\left(\frac{\lambda''}{\mathfrak{C}^3}+\frac{\nu}{C\mathfrak{C}}\right)-\frac{\mu}{A^3B^3C^3PQR}\left(\frac{\lambda'''}{\mathfrak{D}^3}+\frac{\nu}{D\mathfrak{D}}\right)\end{array}\right)$$

and it is apparent, what must be changed in our general formulas on account of the mirrors.

COROLLARY 3

26. But since our mirrors may be able to be adapted only to telescopes, where there is $a=\infty$, $A=0$ and $Aa=\alpha=p$, from formulas with the denominator A^3 transferred as a common factor held inclusive in a common denominator, and thus this expression will be obtained for the diffusion length arising from two mirrors and with two lenses :

$$Ii=\frac{B^2C^2D^2\left(x^2-y^2\right)}{p}\left(\begin{array}{c}\frac{1}{8}-\frac{(1+B)(1-B)^2}{8B^3P}\\ +\frac{\mu}{B^3PQ}\left(\frac{\lambda''}{\mathfrak{C}^3}+\frac{\nu}{C\mathfrak{C}}\right)-\frac{\mu}{B^3C^3PQR}\left(\frac{\lambda'''}{\mathfrak{D}^3}+\frac{\nu}{D\mathfrak{D}}\right)\end{array}\right)$$

SCHOLIUM 1

27. Moreover since for the second mirror both the letter $B=\frac{\beta}{b}$ as well as $P=-\frac{\alpha}{b}$ no longer can be chosen by us, but the values of these we may see to have been defined before, just as these values shall be required to be introduced into the calculation, and here it will be agreed to establish two cases, just as the smaller mirror is put in place either beyond or within the focal length of the principal mirror. So that which may be able to be expressed more succinctly according to our forms, we may put in general

$y=\varepsilon x$, thus so that there shall become $x^2-y^2=(1-\varepsilon^2)x^2$, clearly where ε may denote the fraction required to define the magnitude of the opening.

I. Therefore in the first place, when the distance of the smaller mirror *AB* is greater than

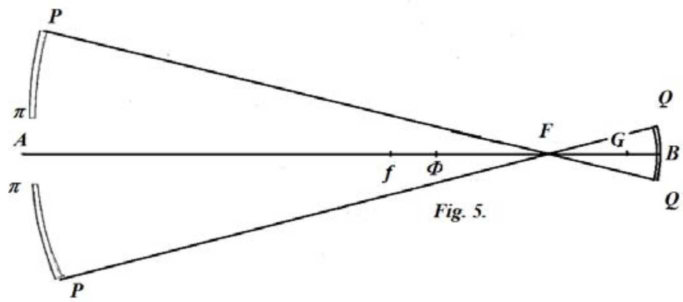

Fig. 5.

the focal length p (Fig. 5) then we have seen (§ 19) this distance to be *AB* or the first interval $=(1+\varepsilon)\alpha=(1+\varepsilon)p$; but since by our general formulas it shall become $=Aa(1-\frac{1}{P})=p(1-\frac{1}{P})$, there will become $\frac{1}{P}=-\varepsilon$. Then truly we have observed also to become $b=\varepsilon p$ and again, if the focal length of the smaller mirror may be put $=q$, there will become $\beta=\frac{bq}{b-q}$, and hence

$$\frac{\beta}{b}=B=\frac{q}{b-q}=\frac{q}{\varepsilon p-q}.$$

But truly we have given these limits for q: $q<\varepsilon p$ and $q>\frac{\varepsilon(1+p)}{1+3\varepsilon}$; with which values substituted that diffusion distance *Ii* will be expressed thus :

$$Ii=\frac{(1-\varepsilon^2)C^2D^2q^2x^2}{(\varepsilon p-q)^2p}\left(\begin{array}{c}\frac{1}{8}+\frac{\varepsilon^2((\varepsilon p-2q))^2p}{8q^3}\\ -\frac{\mu\varepsilon((\varepsilon p-q))^3}{q^3Q}\left(\frac{\lambda''}{\mathfrak{C}^3}+\frac{\nu}{C\mathfrak{C}}\right)+\frac{\mu\varepsilon((\varepsilon p-q))^3}{C^3q^3QR}\left(\frac{\lambda'''}{\mathfrak{D}^3}+\frac{\nu}{D\mathfrak{D}}\right)\end{array}\right).$$

II. But if the distance of the second mirror *AB* were less than p

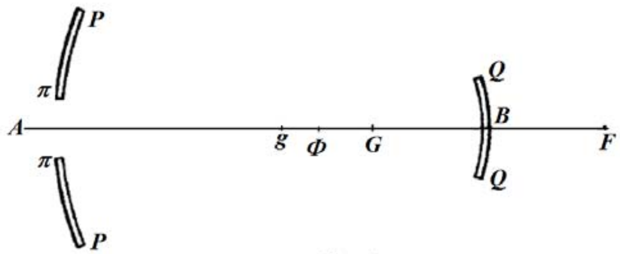

Fig. 6.

(Fig. 6), then the first will be this same distance $=(1-\varepsilon)p$; which since it stall become $=p(1-\frac{1}{P})$, there will be $\frac{1}{P}=\varepsilon$ e. Then the distance $b=-\varepsilon p$, and since the second mirror must be convex, on putting $q=-\mathfrak{q}$ there will become

$$\frac{\beta}{b}=B=\frac{-\mathfrak{q}}{\mathfrak{q}-\varepsilon p};$$

truly we will give these limits for $\mathfrak{q}$: $\mathfrak{q}>\varepsilon p$ and $\mathfrak{q}<\frac{\varepsilon(1-\varepsilon)p}{1-3\varepsilon}$; with which values substituted that diffusion length will be expressed thus :

$$Ii=\frac{(1-\varepsilon^2)C^2D^2q^2x^2}{(\mathfrak{q}-\varepsilon p)^2 p}\left(\begin{array}{c}\frac{1}{8}-\frac{\varepsilon^2(2\mathfrak{q}-\varepsilon p)^2 p}{8\mathfrak{q}^3}\\ -\frac{\mu\varepsilon(\mathfrak{q}-\varepsilon p)^3}{\mathfrak{q}^3 Q}\left(\frac{\lambda''}{\mathfrak{C}^3}+\frac{\nu}{C\mathfrak{C}}\right)+\frac{\mu\varepsilon(\mathfrak{q}-\varepsilon p)^3}{\mathfrak{q}^3C^3QR}\left(\frac{\lambda'''}{\mathfrak{D}^3}+\frac{\nu}{D\mathfrak{D}}\right)\end{array}\right).$$

But if a lens may be put in place in the opening of the objective mirror itself, then in addition a second interval is given, certainly equal to the first, and indeed the first case will become $=(1+\varepsilon)p$. Which since by the general formulas it shall become

$$=-\frac{ABa}{P}\left(1-\frac{1}{Q}\right)=\frac{\varepsilon pq}{\varepsilon p-q}\left(1-\frac{1}{Q}\right),$$

hence there is found :

$$\frac{1}{Q}=1-\frac{(\varepsilon p-q)(1+\varepsilon)}{\varepsilon q}$$

or

$$\frac{1}{Q}=\frac{(2\varepsilon+1)-\varepsilon(1+\varepsilon)p}{\varepsilon q}$$

and hence

$$\frac{1}{PQ}=\frac{\varepsilon(1+\varepsilon)p-(2\varepsilon+1)q}{q},$$

where it is required to be noted Q cannot become positive unless q may be contained within these two limits :

$$q < \frac{\varepsilon(1+\varepsilon)p}{1+2\varepsilon} \text{ and } q > \frac{\varepsilon(1+\varepsilon)p}{1+3\varepsilon}.$$

Clearly these values prevail for the first case; truly for the latter case there is found

$$\frac{1}{Q} = \frac{\varepsilon(1-\varepsilon)p-(1-2\varepsilon)\mathfrak{q}}{\varepsilon\mathfrak{q}}$$

and

$$\frac{1}{PQ} = \frac{(2\varepsilon-1)\mathfrak{q}+\varepsilon(1-\varepsilon)p}{\mathfrak{q}}$$

where equally it will be observed Q to become negative, if $\mathfrak{q}$ may be taken within these limits:

$$\mathfrak{q} > \frac{\varepsilon(1-\varepsilon)p}{1-2\varepsilon} \text{ and } \mathfrak{q} < \frac{\varepsilon(1-\varepsilon)p}{1-3\varepsilon},$$

but truly Q to become positive, if it may be taken between these limits:

$$\mathfrak{q} < \frac{\varepsilon(1-\varepsilon)p}{1-2\varepsilon} \text{ and } \mathfrak{q} > \varepsilon p.$$

SCHOLIUM 2

27. What we have advanced here, relates to the diffusion lengths arising from some number of lenses and mirrors. Truly the conclusion, which hence in the above book has been deduced for the radius of confusion requiring to be determined, also here undergoes a certain change. Indeed since we have inferred the radius of confusion from the final image, it is required also to note this final diffusion distance to be truncated by its principal image. Since indeed no principal image arises from the first mirror on account of the lack of rays in the vicinity of the axis, also the following diffusion lengths, will be lacking in principal images, however many lenses there were; from which since the final of these spaces shall be less because of this truncation, thence also a lesser confusion will arise in the eyepiece, as on account of this cause also the radius of the confusion, just as we have defined in the first book, will be granted a smaller value ; which we will undertake to investigate in the following problem.

PROBLEM 2

28. *With the diffusion of the final image given, which is formed both from the two mirrors as well as all the following lenses, to find the confusion arising thence in the eyepiece itself, clearly by which vision is affected immediately .*

SOLUTION

The distance $L\lambda l$ may represent the final diffusion length formed by both by the mirrors as well as by the following lenses, which is the object of distinct near vision,

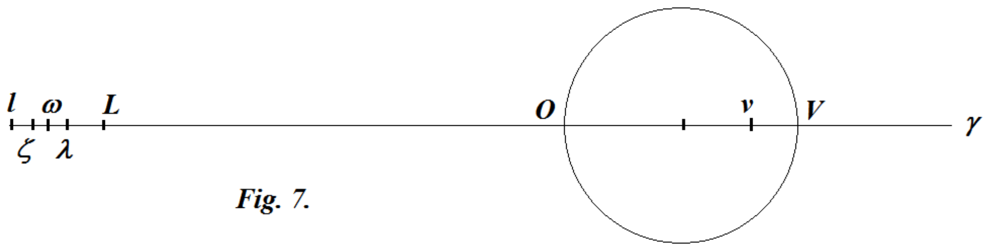

Fig. 7.

from which the rays at once may enter the eye ; in which interval the point L will denote the position of the principal image, where the rays close to the axis will concur, if the objective mirror were whole ; but on account of the hole of this mirror this same principal image will be lacking and a diffuse image now will begin at the point λ , where the rays from around the opening of the hole are reflected and transmitted by all the lenses concur, truly the other boundary shall be at l, where the rays reflected from the edge of the objective mirror and transmitted by the lenses are united. Since now initially it pertains to a magnitude of this distance λl , we have seen above that to be proportional to the formula $xx-yy$ or on putting $y=\varepsilon x$ to this, $(1-\varepsilon\varepsilon)xx$, from which we may put in place:

$$\lambda l = V(1-\varepsilon\varepsilon)xx.$$

From thence in terms of the rays concurring obliquely with the axis from the terminus λ , as above we have seen above to be proportional to y itself, it may be put $=\mathfrak{B}y=\varepsilon\mathfrak{B}x$; truly the obliqueness of the extreme rays concurring at the point l will be $=\mathfrak{B}x$, where the letters V and $\mathfrak{B}$ have these same values, which we have assigned in the first book (Ch. 4, § 165).

[Recall the diagram:

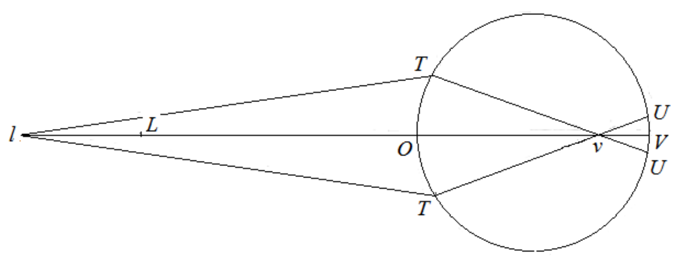

Fig. 10

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From these premises we may seek that position of the eye, from which this image with the least confusion may be observed. Finally we may adopt this certain middle point in the image ζ from which the eye shall be removed to its distance of distinct near vision $=l$, thus so that there shall become $\zeta O = l$ and the rays sent from this point ζ will be gathered together at the point V of the retina. Hence therefore the points will be represented on either side of ζ either further towards λ or placed closer towards l and not to be on the retina V, but either after that at γ or before that at v, and the rays at these points and the rays at these points crossing each other on the retina itself will be referred to either as larger or smaller rings ; and now the whole business is reduced to this, so that these small rings may arise as small as possible, since in this manner the smallest confusion may be produced in the eye. Therefore at first it is required to be seen, how great a quantity of this kind of ring may arise on the retina from the points situated between ζ and λ and which of these shall be going to become the maximum ; for since these rings depend partially on the distance from the point ζ and partially on the obliqueness of the rays, which will increase by moving continually from λ towards ζ, it is easily understood the maximum ring to arise from a certain middle point, for example ω, since both from the point L itself, where the obliqueness is zero, as well as no such ring may be arising from the point ζ . Then by regressing continually from ζ towards l greater rings of this kind may be arising, thus so that the rays sent from this same point l may give rise to the greatest ring ; from which it is evident, if the point ζ were assumed thus, so that the greatest rings may be able to arise in the said manner equal to each other from the points ω and l , then the confusion arising seen in that same point to become a minimum. For if the point ζ were moved closer to ω , then surely the ring arising from this side will become smaller, but truly that from the point l shall emerge so much greater; and the opposite will come about, if the point ζ may be taken closer to l . Therefore so that now we may investigate both the location of the point ζ as well as the point ω corresponding to that, we may introduce the whole distance Ll into our calculation, even if in our case with the part Ll truncated, and we may put for the sake of brevity $Ll = f$, and there will be from the principles set out above :

$$f = Vx^2 \text{ and } L\lambda = Vy^2 = \varepsilon^2 Vx^2,$$

from which there becomes, as we have mentioned at the start,

$$\lambda l = (1-\varepsilon^2)Vx^2.$$

Truly in addition we will call the distance $L\zeta = \zeta$ and $L\omega = \omega$, and since the rays sent out from this point ω are going to produce the largest ring on the retina, for this required to be found it will be required to know the obliqueness of the rays at this point ω. But since the obliqueness at L is zero, at l truly $= \mathfrak{B}x$ and at $\lambda = \varepsilon\mathfrak{B}x$, it is evident the obliqueness to increase in the inverse square ratio of the distance from the point L; from which the obliqueness of the rays at ω will be $= \mathfrak{B}x\sqrt{\frac{\omega}{f}}$.

Therefore the rays passed out from ω will met at the point γ past the eye, thus so that there shall be by the principles established well enough $V\gamma = \frac{uu}{ll} \cdot \zeta\omega$ with u denoting the depth of the eye OV. Moreover the obliqueness of the rays concurring at this point γ will be from the same principles

$$=\tfrac{l}{u}\cdot\mathfrak{B}x\sqrt{\tfrac{\omega}{f}},$$

from which two effects it is concluded the radius of the ring formed at the retina to become

$$=\tfrac{u}{l}\cdot\zeta\omega\cdot\mathfrak{B}x\sqrt{\tfrac{\omega}{f}},$$

and since $\zeta\omega=\zeta-\omega$, the radius of this ring

$$=\tfrac{u}{l}\cdot\mathfrak{B}x\left(\zeta-\omega\right)\sqrt{\tfrac{\omega}{f}};$$

which therefore so that the maximum may emerge, thus it is required to assume the distance ω, so that there may become $(\zeta-\omega)\sqrt{\omega}=$maximum, which arises by assuming $\omega=\frac{1}{3}\zeta$; on account of which the radius of this greatest ring will be

$$=\tfrac{u}{l}\cdot\mathfrak{B}x\cdot\tfrac{2}{3}\zeta\sqrt{\tfrac{\omega}{3f}}.$$

Now truly the rays from the other point l from the other side will be considered incident on the eye, which will be gathered together in front of the retina at the point v with the interval distance being $Vv=\frac{uu}{ll}\cdot\zeta l=\frac{uu}{ll}\left(f-\zeta\right)$, where the obliqueness of the rays will be $=\frac{l}{u}\mathfrak{B}x$; from which the radius of the ring depicted on the retina will be $=\frac{u}{l}\left(f-\zeta\right)\mathfrak{B}x$, which consequently must be placed equal to the first radius of the ring found; from which this equation is obtained:

$$f-\zeta=\tfrac{2}{3}\zeta\sqrt{\tfrac{\zeta}{3f}},$$

from which the distance ζ is required to be found. But with the squares taken we will have

$$f^2-2f\zeta+\zeta^2=\tfrac{4}{27}\cdot\tfrac{\zeta^3}{f}$$

or

$$f^3-2f^2\zeta+f\zeta^2-\tfrac{4}{27}\cdot\zeta^3=0,$$

which on careful examination soon will become apparent to be divisible by $f-\frac{1}{3}\zeta$; with which division performed there is produced

$$f^2 - \tfrac{5}{3} f\zeta + \tfrac{4}{9}\zeta^2 = 0,$$

which again divisible by $f - \tfrac{1}{3}\zeta$ produces

$$f - \tfrac{4}{3}\zeta = 0;$$

truly since these two first factors cannot be used here , since $\zeta = 3f$ is absurd, truly the last factor provides the distance for us

$$L\zeta = \zeta = \tfrac{3}{4} f,$$

thus so that there shall be $l\zeta = \tfrac{1}{4} f$ and $L\omega = \omega = \tfrac{1}{4} f$; from these values found the radius of the minimum ring described in the eye will be $= \tfrac{u}{4l} f\mathfrak{B}x$, and since there shall be $f = Vx^2$, this same radius will be $= \tfrac{u}{4l} V\mathfrak{B}x^3$. Now truly if we may look at a circle in the heavens, of which the apparent radius $= \Phi$, its image upon the retina also shall be a circle, of which the radius $= u\Phi$; therefore for this circle to be equal to that there may be put $\Phi = \frac{V\mathfrak{B}x^3}{4l}$ and the individual points of our image may be discerned from the eye as if circular spots, of which the apparent radius shall be $= \frac{V\mathfrak{B}x^3}{4l}$, as we have called the above expression the radius of confusion [Book I, §193, 194].

COROLLARY 1

29. In this solution we have assumed the point ω to fall between λ and l ; for if the boundary L shall be closer than the point λ, because the image has diffused only through the distance Ll , that same point ω in short may not be able to enter into the calculation, but the greatest ring will be arising in the eye from this part from the same point λ ; and for this case a special solution will be required, as we are going to give soon.

COROLLARY 2

30. Moreover since there shall become $L\omega = \tfrac{1}{3} L\zeta = \tfrac{1}{4} Ll$, and for the term λ there shall become $Ll = \varepsilon\varepsilon \cdot Ll$, the point ω lies between the terms l and λ, as often as there were $L\omega > L\lambda$ and thus as often as there became $\varepsilon < \tfrac{1}{2}$, on account of which, since in practice ε always is assumed to be $< \tfrac{1}{2}$, the solution of the problem in practice generally is applied.

COROLLARY 3

31. Whenever there were $\varepsilon < \tfrac{1}{2}$, then certainly it is allowed to agree on account of the hole, by which the mirror has been bored, the confusion in no way to be diminished, but to be so great always, as if the mirror were whole and its whole surface were reflecting

rays, and thus the general equation found above will prevail for the radius of confusion for mirrors, but only if, as we have now found above, in place of the formulas for lenses pertaining to lenses, the formulas designated here in § 23 may be substituted.

COROLLARY 4

32. And hence we know also, if the telescope were dependent on pure lenses, even if the objective lens were covered over around the middle, the confusion at no time to be diminished, just as some authors have urged, but the best remedy required for diminishing the confusion certainly depends on this, that the objective lens may be covered over around the margin, certainly with that agreed on the radius of that same aperture x is diminished and the confusion thus is diminished by a factor of three, since on the contrary, if the lens were covered around the middle, a minimum diminution of the confusion cannot be expected, unless perhaps half of the whole lens may remain covered up, but with which agreed on the clarity of the image will be diminished excessively.

SCHOLIUM 1

33. But if the radius of the hole $y=\varepsilon x$ may exceed half of the whole aperture x, thus so that the point ω may lie between L and λ, our problem demands another solution. Indeed since now from the part $\zeta\lambda$ the maximum ring may arise in the eye from the same point λ and there shall become $Ll=\varepsilon\varepsilon f$ on account of $Ll=f$ and hence the distance $\zeta\lambda=\zeta-\varepsilon\varepsilon f$, the distance past the eye will become $V\gamma=\frac{uu}{ll}\left(\zeta-\varepsilon\varepsilon f\right)$ and the obliqueness of the rays will become $=\frac{l}{u}\varepsilon\mathfrak{B}x$, hence the radius of the ring formed on the retina will be $=\frac{u}{l}\varepsilon\left(\zeta-\varepsilon\varepsilon f\right)\mathfrak{B}x$. But from the other part a ring arises in the retina from the limit l, of which the radius $=\frac{u}{l}\left(f-\zeta\right)\mathfrak{B}x$; which two radii on account of the ratios put up before are required to be set equal to each other, from which we follow

$$f-\zeta=\varepsilon\zeta-\varepsilon^3 f$$

and hence

$$\zeta=\frac{f\left(1+\varepsilon^3\right)}{1+\varepsilon}=f\left(1-\varepsilon+\varepsilon^2\right);$$

hence therefore there will become:

$$f-\zeta=\varepsilon\left(1-\varepsilon\right)f$$

and thus the radius of the ring on the retina will become

$$=\frac{u}{l}\varepsilon\left(1-\varepsilon\right)f\mathfrak{B}x=\frac{u}{l}\varepsilon\left(1-\varepsilon\right)V\mathfrak{B}x^3.$$

Consequently in the case, where $\varepsilon > \frac{1}{2}$, the radius of the confusion will be $= \frac{\varepsilon(1-\varepsilon)}{l} V\mathfrak{B}x^3$, which in the preceding case, where $\varepsilon < \frac{1}{2}$, was $= \frac{V\mathfrak{B}}{4l} \cdot x^3$; therefore while $\varepsilon < \frac{1}{2}$, the formula will prevail always $\frac{V\mathfrak{B}}{4l} \cdot x^3$, which even now has a place, if $\varepsilon = \frac{1}{2}$, and truly if at once there becomes $\varepsilon > \frac{1}{2}$, then at last the confusion begins to diminish and finally in short it vanishes if there may become $\varepsilon = 1$. But since the clarity also may be diminished and finally vanish, hence plainly in practice it will not be worth the effort to be carried to excess ; for whoever may still doubt, whether in place of a solid lens, of which the radius shall be *p,* it may not be possible to use the glass outer edge of part of the glass surface, of which the exterior radius shall be $= q$ and the interior $= \varepsilon q$, thus so that there shall become $p^2 = q^2(1-\varepsilon\varepsilon)$ and the confusion of this part may arise smaller, this doubt now will be easy to resolve ; indeed from a solid lens the confusion will arise as $\frac{1}{4} p^3$, but from an outer portion as $\varepsilon(1-\varepsilon)q^3$; from which on account of $p = q\sqrt{(1-\varepsilon\varepsilon)}$ the confusion arising from the solid lens to the confusion arising from the outer portion to be as $(1+\varepsilon)\sqrt{(1-\varepsilon\varepsilon)} : 4\varepsilon$; whereby, since by hypothesis there shall be $\varepsilon > \frac{1}{2}$ (since for the other case $\varepsilon < \frac{1}{2}$, no doubt indeed can emerge), the latter member 4ε evidently shall be greater than 2; but since likewise $\varepsilon < 1$, there will become $1+\varepsilon < 2$ and thus much more $(1+\varepsilon)\sqrt{(1-\varepsilon\varepsilon)} < 2$, from which it is evident the first member shall be much less than the second always, or the confusion of the portion will greatly exceed the confusion of the solid lens.

SCHOLIUM 2

34. Moreover since in practical use we may be able to assume $\varepsilon < \frac{1}{2}$ without risk, where in the case of the perforated objective mirror the confusion shall arise equally great, and if it may be whole, if it may be shown above for telescopes generally, for which the radius of confusion is expressed, in place of the two first lenses we may introduce our
mirrors, hence the equation will be obtained in the following manner :

$$\frac{1}{k^3} = \frac{mx^3}{p^3} \left(\begin{array}{c} \frac{1}{8} - \frac{(1+B)(1-B)^2}{8B^3P} \\ + \frac{\mu}{B^3PQ}\left(\frac{\lambda''}{\mathfrak{C}^3} + \frac{\nu}{C\mathfrak{C}}\right) - \frac{\mu}{B^3C^3PQR}\left(\frac{\lambda'''}{\mathfrak{D}^3} + \frac{\nu}{D\mathfrak{D}}\right) + \text{etc.} \end{array} \right),$$

where it will be agreed to note, if perhaps lenses may be made from various kinds of glass after the mirrors, then for any lens it must be able to assume the kind of glass from the letters μ and ν, from which the lens was made.

But the remaining general precepts require no change on account of the mirrors for the construction of telescopes, with only these formulas excepted, by which the colored margin may be removed, as well as all the confusion arising from the different refrangibility of the rays to be returned to zero. For since in these formulas we have introduced the letters *N, N', N'', N'''* etc. for the individual lenses, which letters have been taken proportional to the differential formulas $\frac{dn}{n-1}$, $\frac{dn'}{n'-1}$ etc., if in place of the two first lenses we may substitute the mirrors, on account of the absence of refraction these two first letters *N* and *N'* are agreed to be considered equal to zero; with which observed for all these general formulas likewise we will be able to use for mirrors, and has been done in the second book, provided that both the focal lengths of the mirrors and the two earlier intervals in the preceding chapter brought forth, will be observed properly.

SCHOLIUM 3

35. But reflecting telescopes of this kind refer at once to two general principles, if indeed we have seen above the second mirror to be put in place either beyond or within the first focus, and in the first case the second mirror to be concave, truly in the other case convex. Then since for the first case we will have found these bounds for the focal length q of the second mirror

$$q < \varepsilon p \text{ and } q > \frac{\varepsilon(1+\varepsilon)p}{1+3\varepsilon}$$

with the first interval present $=(1+\varepsilon)p$, to which the second must be equal, then truly

$$b = \varepsilon p \text{ and } \frac{\beta}{b} = \frac{q}{b-q} = B,$$

where duly we may set out this first kind, it will be appropriate to put in place three cases : evidently for the first we may take $q = \varepsilon p$, for the second $q = \frac{\varepsilon(1+\varepsilon)p}{1+2\varepsilon}$ and for the third $q = \frac{\varepsilon(1+\varepsilon)p}{1+3\varepsilon}$. Truly for the other kind the second mirror was arranged within the first focal length, thus so that

$$b = -\varepsilon p \text{ and } \frac{\beta}{b} = \frac{q}{b-q} = \frac{-q}{\varepsilon p+q} = B,$$

and there since the focal length q in this case may emerge negative, by putting $q = -\mathfrak{q}$ we deduce these bounds in that place :

$$\mathfrak{q} > \varepsilon p \text{ and } \mathfrak{q} < \frac{\varepsilon(1-\varepsilon)}{1-3\varepsilon},$$

from which again we may establish these three cases : for the first evidently we may take $q = -\varepsilon p$, for the second $q = \frac{-\varepsilon(1-\varepsilon)p}{1-2\varepsilon}$, and for the third $q = \frac{-\varepsilon(1-\varepsilon)p}{1-3\varepsilon}$; but in this case the first interval will be $=(1-\varepsilon)p$, to which also the second must be equal.

Finally in the former generally there was $\frac{1}{P}=-\varepsilon$, thus so that in the first at once a real image may be found; truly in the other there was $\frac{1}{P}=\varepsilon$, thus so that no real image may occur in the first interval ; in addition truly, as we have mentioned now, we have taken this always to be $\varepsilon<\frac{1}{3}$, from which the latter case will merit a mention at this point, where evidently there shall be $\varepsilon=\frac{1}{3}$, since then a plane mirror will be allowed to be accepted; whereby we are going to be treating these cata-dioptric telescopes following these seven cases.

CAPUT II

DE COMPUTO CONFUSIONIS DUM PRAETER LENTES ETIAM SPECULA AD INSTRUMENTA DIOPTRICA CONFICIENDA ADHIBENTUR

PROBLEMA 1

23. *Si loco primae et secundae lentis specula usurpentur, invenire formulas, quae ob haec duo specula in expressionem supra in Libro 1 inventam, qua scilicet semidiameter confusionis est inventa, introduci in calculum debent.*

SOLUTIO

In primo libro (§ 91) ostendimus a duabus lentibus oriri spatium diffusionis

$$Gg=\mu\beta^2x^2\left\{\begin{array}{l}+\frac{\alpha^2}{b^2}\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\lambda\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{\nu}{a\alpha}\right)\\+\frac{b^2}{\alpha^2}\left(\frac{1}{b}+\frac{1}{\beta}\right)\left(\lambda'\left(\frac{1}{b}+\frac{1}{\beta}\right)^2+\frac{\nu}{b\beta}\right)\end{array}\right\}$$

quae expressio ponendo $\alpha=Aa,\ \beta=Bb$, tum vero etiam $\frac{A}{1+A}=\mathfrak{A}$ et $\frac{B}{1+B}=\mathfrak{B}$ abit in hanc:

$$Gg=\frac{\mu A^2B^2x^2}{a}\left(\frac{\lambda}{\mathfrak{A}^3}+\frac{\nu}{A\mathfrak{A}}+\frac{b}{A^4a}\left(\frac{\lambda'}{\mathfrak{B}^3}+\frac{\nu}{B\mathfrak{B}}\right)\right),$$

atque si hic porro, uti deinceps in tractatu de telescopiis fecimus, ponamus $\frac{\alpha}{b}=\frac{Aa}{b}=-P$, ista expressio induet hanc formam:

$$Gg=\frac{A^2B^2x^2}{a}\left(\mu\left(\frac{\lambda}{\mathfrak{A}^3}+\frac{\nu}{A\mathfrak{A}}\right)-\frac{\mu}{A^3P}\left(\frac{\lambda'}{\mathfrak{B}^3}+\frac{\nu}{B\mathfrak{B}}\right)\right).$$

Si nunc loco duarum harum lentium duo substituantur specula, ad quae litterae $a,\ \alpha,\ b,\ \beta$ cum x similiter sint relatae, in problemate 4 capitis praecedentis (§ 15) invenimus fore spatium diffusionis

$$Gg=\frac{\beta^2}{b^2}\cdot\frac{(a+\alpha)(a-\alpha)^2x^2}{8\alpha^3\alpha}+\frac{(b+\beta)(b-\beta)^2x^2}{8\alpha^2b\beta};$$

quae forma posito $\alpha=Aa$, $\beta=Bb$ et $\frac{\alpha}{b}=-P$ induet hanc formam:

$$Gg=\frac{A^2B^2x^2}{a}\left(\frac{(1+A)(1-A)^2}{8A^3}-\frac{(1+B)(1-B)^2}{8A^3B^3P}\right),$$

ex qua cum superiori collata cognoscimus, si loco primae lentis speculum substituatur, tum in computo confusionis loco formulae

$$\mu\left(\frac{\lambda}{\mathfrak{A}^3}+\frac{\nu}{A\mathfrak{A}}\right)$$

scribi debere hanc:

$$\frac{(1+A)(1-A)^2}{8A^3}$$

ac si etiam loco lentis secundae speculum substituatur, tum simili modo loco formulae

$$\mu\left(\frac{\lambda'}{\mathfrak{B}^3}+\frac{\nu}{B\mathfrak{B}}\right)$$

scribi debere hanc:

$$\frac{(1+B)(1-B)^2}{8B^3}$$

ac si circumstantiae permitterent, ut etiam loco tertiae lentis speculum simile substitueretur, tum in computo confusionis loco formulae

$$\mu\left(\frac{\lambda''}{\mathfrak{C}^3}+\frac{\nu}{C\mathfrak{C}}\right)$$

scribi deberet haec formula:

$$\frac{(1+C)(1-C)^2}{8C^3}$$

unde satis superque intelligitur, quomodo quantitas confusionis aestimari debeat, quando loco lentium specula adhibentur.

COROLLARIUM 1

24. Quatenus autem speculum obiectivum foramine est pertusum, cuius radius $=y$, eatenus in factore communi loco x^2 scribi oportet x^2-y^2, ita ut iam expressio pro spatio diffusionis inventa futura sit

$$Gg=\frac{A^2B^2(x^2-y^2)}{a}\left(\frac{(1+A)(1-A)^2}{8A^3}-\frac{(1+B)(1-B)^2}{8A^3B^3P}\right),$$

ubi notandum est formulam $x^2 - y^2$ proportionalem esse supeficiei reflectenti in primo speculo, prorsus uti x^2 proportionale erat superficiei refringenti lentis obiectivae.

COROLLARIUM 2

25. Atque haec formula $x^2 - y^2$ etiam extenditur ad omnes lentes sequentes, quotquot binis speculis insuper adiunguntur; ita, ex. gr., si duae lentes praeter specula adhibeantur, totum spatium diffusionis Ii ita exprimitur:

$$Ii = \frac{A^2B^2C^2D^2(x^2-y^2)}{a}\left(\begin{array}{c}\frac{(1+A)(1-A)^2}{8A^3} - \frac{(1+B)(1-B)^2}{8A^3B^3P} \\ + \frac{\mu}{A^3B^3PQ}\left(\frac{\lambda''}{\mathfrak{C}^3} + \frac{\nu}{C\mathfrak{C}}\right) - \frac{\mu}{A^3B^3C^3PQR}\left(\frac{\lambda'''}{\mathfrak{D}^3} + \frac{\nu}{D\mathfrak{D}}\right)\end{array}\right)$$

unde patet, quid propter specula in nostris formulis generalibus immutari debeat.

COROLLARIUM 3

26. Cum autem nostra specula tantum ad telescopia accommodari queant, ubi est $a = \infty$, $A = 0$ et $Aa = \alpha = p$, ex formulis vinculo inclusis denominator A^3 in factorem communem transfertur sicque pro spatio diffusionis a binis speculis et duabus lentibus orto habebitur haec expressio:

$$Ii = \frac{B^2C^2D^2(x^2-y^2)}{p}\left(\begin{array}{c}\frac{1}{8} - \frac{(1+B)(1-B)^2}{8A^3B^3P} \\ + \frac{\mu}{B^3PQ}\left(\frac{\lambda''}{\mathfrak{C}^3} + \frac{\nu}{C\mathfrak{C}}\right) - \frac{\mu}{B^3C^3PQR}\left(\frac{\lambda'''}{\mathfrak{D}^3} + \frac{\nu}{D\mathfrak{D}}\right)\end{array}\right)$$

SCHOLION 1

27. Quoniam autem pro secundo speculo tam littera $B = \frac{\beta}{b}$ quam $P = -\frac{\alpha}{b}$ non amplius ab arbitrio nostro pendet, sed earum valores iam ante sunt definiti, videamus, quomodo isti valores in computum sint introducendi, atque hic duos casus evolvi conveniet, prouti minus speculum sive ultra focum speculi principalis constituitur, sive citra. Quod quo ad nostras formas succinctius exprimi possit, ponamus in genere $y = \varepsilon x$, ita ut sit $x^2 - y^2 = (1-\varepsilon^2)x^2$, ubi scilicet ε denotat fractionem foraminis magnitudinem definientem.

I. Primo igitur, quando distantia minoris speculi AB maior est quam

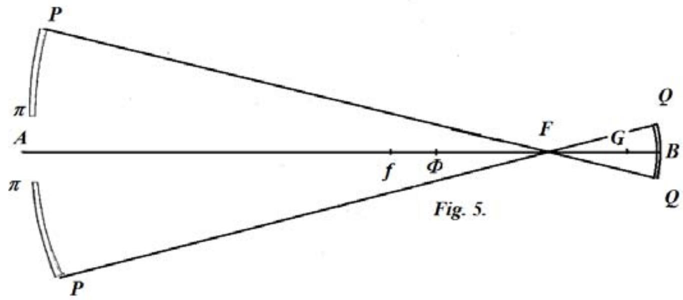

Fig. 5.

distantia focalis p (Fig. 5) tum vidimus (§ 19) esse hanc distantiam AB seu primum intervallum $=(1+\varepsilon)\alpha=(1+\varepsilon)p$; quod cum per formulas nostras generales sit $=Aa(1-\frac{1}{P})=p(1-\frac{1}{P})$, erit $\frac{1}{P}=-\varepsilon$. Deinde vero etiam vidimus esse $b=\varepsilon p$ et porro, si distantia focalis minoris speculi ponatur $=q$, erit $\beta=\frac{bq}{b-q}$, hincque

$$\frac{\beta}{b}=B=\frac{q}{b-q}=\frac{q}{\varepsilon p-q}.$$

At vero pro q hos dedimus limites: $q<\varepsilon p$ et $q>\frac{\varepsilon(1+p)}{1+3\varepsilon}$; quibus valoribus substitutis spatium illud diffusionis Ii ita exprimetur:

$$Ii=\frac{(1-\varepsilon^2)C^2D^2q^2x^2}{(\varepsilon p-q)^2 p}\left(\begin{array}{c}\frac{1}{8}+\frac{\varepsilon^2((\varepsilon p-2q))^2 p}{8q^3}\\ -\frac{\mu\varepsilon((\varepsilon p-q))^3}{q^3Q}\left(\frac{\lambda''}{\mathfrak{C}^3}+\frac{\nu}{C\mathfrak{C}}\right)+\frac{\mu\varepsilon((\varepsilon p-q))^3}{C^3q^3QR}\left(\frac{\lambda'''}{\mathfrak{D}^3}+\frac{\nu}{D\mathfrak{D}}\right)\end{array}\right).$$

II. Sin autem distantia secundi speculi AB minor fuerit quam p

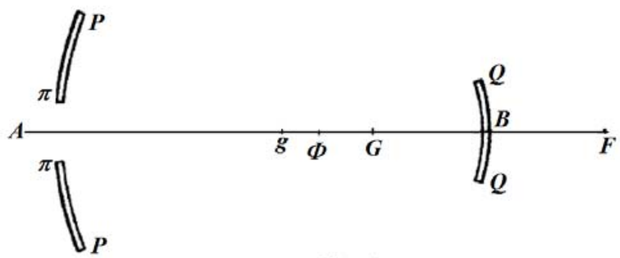

Fig. 6.

(Fig. 6), tum primo erit haec ipsa distantia $=(1-\varepsilon)p$; quae cum sit $=p(1-\frac{1}{P})$, erit $\frac{1}{P}=\varepsilon$ e. Deinde erit distantia $b=-\varepsilon p$, et quia secundum speculum debet esse convexum, posito $q=-\mathfrak{q}$ fiet

$$\frac{\beta}{b}=B=\frac{-\mathfrak{q}}{\mathfrak{q}-\varepsilon p};$$

verum pro $\mathfrak{q}$ hos dedimus limites: $\mathfrak{q}>\varepsilon p$ et $\mathfrak{q}<\frac{\varepsilon(1-\varepsilon)p}{1-3\varepsilon}$; quibus valoribus substitutis spatium illud diffusionis ita exprimetur:

$$Ii=\frac{(1-\varepsilon^2)C^2D^2q^2x^2}{(\mathfrak{q}-\varepsilon p)^2 p}\left(\begin{array}{c}\frac{1}{8}-\frac{\varepsilon^2(2\mathfrak{q}-\varepsilon p)^2 p}{8\mathfrak{q}^3}\\ -\frac{\mu\varepsilon(\mathfrak{q}-\varepsilon p)^3}{\mathfrak{q}^3 Q}\left(\frac{\lambda''}{\mathfrak{C}^3}+\frac{\nu}{C\mathfrak{C}}\right)+\frac{\mu\varepsilon(\mathfrak{q}-\varepsilon p)^3}{\mathfrak{q}^3C^3QR}\left(\frac{\lambda'''}{\mathfrak{D}^3}+\frac{\nu}{D\mathfrak{D}}\right)\end{array}\right).$$

Quodsi lens in ipso foramine speculi obiectivi constituatur, tum insuper datur intervallum secundum, primo quippe aequale, ac primo quidem casu erit $=(1+\varepsilon)p$. Quod cum per formulas generales sit

$$=-\frac{ABa}{P}\left(1-\frac{1}{Q}\right)=\frac{\varepsilon pq}{\varepsilon p-q}\left(1-\frac{1}{Q}\right),$$

hinc reperitur

$$\frac{1}{Q}=1-\frac{(\varepsilon p-q)(1+\varepsilon)}{\varepsilon q}$$

seu

$$\frac{1}{Q}=\frac{(2\varepsilon+1)-\varepsilon(1+\varepsilon)p}{\varepsilon q}$$

hincque

$$\frac{1}{PQ}=\frac{\varepsilon(1+\varepsilon)p-(2\varepsilon+1)q}{q},$$

ubi notandum est Q fieri non posse positivum nisi q contineatur intra hos limites:

$$q<\frac{\varepsilon(1+\varepsilon)p}{1+2\varepsilon}\ \text{ et }\ q>\frac{\varepsilon(1+\varepsilon)p}{1+3\varepsilon}.$$

Haec scilicet valent pro casu priore; pro casu vero posteriore reperitur

$$\frac{1}{Q}=\frac{\varepsilon(1-\varepsilon)p-(1-2\varepsilon)\mathfrak{q}}{\varepsilon\mathfrak{q}}$$

et

$$\frac{1}{PQ}=\frac{(2\varepsilon-1)\mathfrak{q}+\varepsilon(1-\varepsilon)p}{\mathfrak{q}}$$

ubi pariter notetur Q fieri negativum, si $\mathfrak{q}$ capiatur intra hos limites:

$$\mathfrak{q}>\frac{\varepsilon(1-\varepsilon)p}{1-2\varepsilon} \text{ et } \mathfrak{q}<\frac{\varepsilon(1-\varepsilon)p}{1-3\varepsilon},$$

at vero Q fieri positivum, si capiatur intra hos limites:

$$\mathfrak{q}<\frac{\varepsilon(1-\varepsilon)p}{1-2\varepsilon} \text{ et } \mathfrak{q}>\varepsilon p.$$

SCHOLION 2

27. Quae hic attulimus, ad spatia diffusionis ex speculis et lentibus quotcunque ortae pertinent. Conclusio vero, quae in superiore libro hinc ad semidiametrum confusionis ipsam determinandam est deducta, etiam hic quandam mutationem patitur. Quoniam enim semidiametrum confusionis ex ultimae imaginis diffusione conclusimus, notandum est etiam hoc ultimum spatium diffusionis sua imagine principali fore truncatum. Quoniam enim a primo speculo nulla gignitur imago principalis ob defectum radiorum axi proximorum, etiam sequentia spatia diffusionis, quotcunque fuerint lentes, imagine principali destituentur; unde cum horum spatiorum ultimum minus sit propter ipsam hanc mutilationem, inde etiam minor confusio in oculo orietur, quam ob causam etiam semidiameter confusionis, prouti eam in primo libro definivimus, minorem valorem adipiscetur; quam investigationem sequenti problemate suscipiemus.

PROBLEMA 2

28. *Data ultima imagine diffusa, quae tam per bina specula quam omnes lentes sequentes formatur, invenire confusionem in ipso oculo inde oriundam, qua scilicet visio immediate afficitur.*

SOLUTIO

Repraesentet $L\lambda l$ ultimum spatium diffusionis tam per specula quam omnes sequentes lentes formatum, quippe quod est obiectum immediatum visionis,

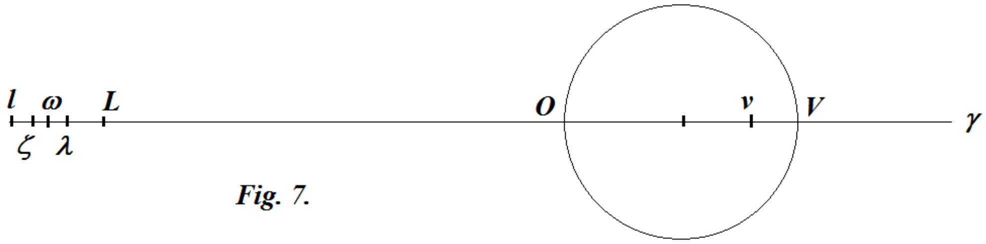

Fig. 7.

unde radii immediate in oculum ingrediuntur; in quo spatio punctum L denotet locum imaginis principalis, ubi radii axi proximi concurrerent, si speculum obiectivum esset integrum; ob foramen autem huius speculi ista imago principalis plane deerit et imago diffusa demum in puncto λ incipiet, ubi radii circa oram foraminis reflexi et per omnes lentes transmissi concurrunt, alter vero terminus sit in l, ubi radii ab extremitate speculi obiectivi reflexi ac per lentes transmissi uniuntur. Quod nunc primo ad magnitudinem huius spatii λl attinet, supra vidimus id esse proportionale formulae $xx-yy$ sive posito $y=\varepsilon x$ huic $(1-\varepsilon\varepsilon)xx$, unde statuamus hoc spatium:

$$\lambda l = V(1-\varepsilon\varepsilon)xx.$$

Deinde radiorum in termino λ cum axe concurrentium obliquitas, quam supra ipsi y proportionalem esse vidimus, ponatur $=\mathfrak{B}y=\varepsilon\mathfrak{B}x$; obliquitas vero radiorum extremorum in puncto l concurrentium erit $=\mathfrak{B}x$, ubi litterae V et $\mathfrak{B}$ eosdem valores habent, quos in primo libro (§ 165) assignavimus.

His praemissis quaeramus cum oculi locum, unde haec imago diffusa minima cum confusione conspiciatur. Hunc in finem concipiamus punctum quoddam medium in imagine ζ a quo oculus ad distantiam suam iustam $=l$ sit remotus, ita ut sit $\zeta O=l$ radiique ex hoc puncto ζ emissi praecise in puncto retinae V congregentur. Hinc ergo puncta cis et ultra hoc punctum ζ vel λ vel l versus sita non in ipsa retina V, sed vel post eam in γ vel ante eam in v repraesentabuntur radiique in his punctis se decussantes in ipsa retina circellos sive maiores sive minores referent; atque nunc totum negotium huc reducitur, ut hi circelli quam minimi evadant, quia hoc modo in oculo minima confusio producetur. Primum igitur videndum est, quanti huiusmodi circelli a punctis intra ζ et λ sitis in retina oriantur et quinam eorum futurus sit maximus; quoniam enim hi circelli partim a distantia a puncto ζ partim a radiorum obliquitate pendent, quae a λ versus ζ progrediendo continuo crescit, facile intelligitur ex puncto quodam medio, puta ω, maximum circellum oriri, quandoquidem tam ex ipso puncto L, ubi obliquitas est nulla, quam ex puncto ζ nullus talis circellus orietur. Deinde a ζ ad l regrediendo continuo maiores huiusmodi circelli orirentur, ita ut radii ex ipso puncto l emissi ab hac parte maximum circellum gignant; ex quo manifestum est, si punctum ζ ita fuerit assumtum, ut maximi modo dicti circelli ex punctis ω et l orti fiant inter se aequales, tum confusionem in ipsa visione natam omnium fore minimam. Si enim punctum ζ propius ad ω moveretur, tum circellus quidem ab hac parte ortus fieret minor, alter vero ex puncto l ortus tanto maior evaderet; atque contrarium eveniret, si punctum ζ propius versus l

capereretur. Ut igitur nunc tam locum puncti ζ quaro ei respondentis puncti ω investigemus, totum spatium *Ll,* etsi id nostro casu parte *Ll* est truncatum, in computum ducamus ponamusque brevitatis gratia $Ll = f$, eritque ex principiis supra expositis

$$f = Vx^2 \text{ et } L\lambda = Vy^2 = \varepsilon^2 Vx^2,$$

unde fit, uti initio commemoravimus,

$$\lambda l = (1-\varepsilon^2)Vx^2.$$

Praeterea vero vocemus spatia $L\zeta = \zeta$ et $L\omega = \omega,$ et quia radii ex hoc puncto ω egressi super retina maximum circellum producere ponuntur, ad hunc inveniendum obliquitatem radiorum in puncto hoc ω nosse oportet. Quia autem obliquitas in *L* est nulla, in *l* vero $= \mathfrak{B}x$ et in $\lambda = \varepsilon\mathfrak{B}x,$ evidens est obliquitatem crescere in ratione subduplicata distantiae a puncto *L;* unde obliquitas radiorum in ω erit $= \mathfrak{B}x\sqrt{\frac{\omega}{f}}.$

Radii igitur ex ω egressi concurrent post oculum in puncto *γ,* ita ut sit per principia supra satis stabilita $V\gamma = \frac{uu}{ll}\cdot\zeta\omega$ denotante *u* profunditatem oculi *OV.* Radiorum autem in hoc puncto γ concurrentium obliquitas ex iisdem principiis erit

$$= \frac{l}{u}\cdot\mathfrak{B}x\sqrt{\frac{\omega}{f}},$$

ex quibus duobus momentis concluditur circelli in retina depicti radius

$$= \frac{u}{l}\cdot\zeta\omega\cdot\mathfrak{B}x\sqrt{\frac{\omega}{f}},$$

et quia est $\zeta\omega = \zeta - \omega$, erit radius istius circelli

$$= \frac{u}{l}\cdot\mathfrak{B}x\left(\zeta-\omega\right)\sqrt{\frac{\omega}{f}};$$

qui ergo ut maximus evadat, spatium ω ita assumi oportet, ut fiat $(\zeta-\omega)\sqrt{\omega} =$ maximo, quod evenit sumendo $\omega = \frac{1}{3}\zeta;$ quocirca maximi huius circelli erit radius

$$= \frac{u}{l}\cdot\mathfrak{B}x\cdot\frac{2}{3}\zeta\sqrt{\frac{\omega}{3f}}.$$

Nunc vero ex altera parte radii ex altero puncto *l* in oculum incidentes considerentur, qui ante retinam in puncto *v* colligentur existante spatio $Vv = \frac{uu}{ll}\cdot\zeta l = \frac{uu}{ll}\left(f-\zeta\right),$ ibique radiorum obliquitas erit $= \frac{l}{u}\mathfrak{B}x$; unde circelli super retina depicti radius erit

$=\frac{u}{l}\left(f-\zeta\right)\mathfrak{B}x,$ qui consequenter radio prioris circelli inventi aequalis statui debet; ex quo obtinebitur haec aequatio:

$$f-\zeta=\tfrac{2}{3}\zeta\sqrt{\tfrac{\zeta}{3f}},$$

ex qua intervallum ζ definiri oportet. Sumtis autem quadratis habebimus

$$f^2-2f\zeta+\zeta^2=\tfrac{4}{27}\cdot\tfrac{\zeta^3}{f}$$

sive

$$f^3-2f^2\zeta+f\zeta^2-\tfrac{4}{27}\cdot\zeta^3=0,$$

quam perpendenti mox patebit divisibilem esse per $f-\frac{1}{3}\zeta$; divisione autem facta prodit

$$f^2-\tfrac{5}{3}f\zeta+\tfrac{4}{9}\zeta^2=0,$$

quae denuo per $f-\frac{1}{3}\zeta$ divisa praebet

$$f-\tfrac{4}{3}\zeta=0;$$

quia vero bini priores factores hic locum habere nequeunt, quia absurdum foret esse $\zeta=3f$, ultimus factor nobis verum praebet intervallum

$$L\zeta=\zeta=\tfrac{3}{4}f,$$

ita ut sit $l\zeta=\frac{1}{4}f$ et $L\omega=\omega=\frac{1}{4}f$ *;* his valoribus inventis circelli minimi in oculo descripti radius erit $=\frac{u}{4l}f\mathfrak{B}x,$ et cum sit $f=Vx^2$, erit iste radius $=\frac{u}{4l}V\mathfrak{B}x^3.$
Iam vero si in coelo circulum conspiceremus, cuius radius apparens $=\Phi$, eius imago super retina etiam esset circulus, cuius radius $=u\Phi$ *;* hoc ergo circulo illi aequali posito fit $\Phi=\frac{V\mathfrak{B}x^3}{4l}$ et singula imaginis nostrae puncta ab oculo cernentur tanquam maculae circulares, quarum semidiameter apparens sit $=\frac{V\mathfrak{B}x^3}{4l}$, quam expressionem supra [Lib. I, §193, 194] nominavimus semidiametrum confusionis.

COROLLARIUM 1

29. In hac solutione assumsimus punctum ω intra λ et *l* cadere; si enim termino *L* propius esset quam punctum λ, quoniam imago tantum per spatium *Ll* est diffusa, istud

punctum ω prorsus non in computum venire posset, sed maximus circellus in oculo ex hac parte ab ipso puncto λ orirétur; atque pro hoc casu peculiaris solutio requiretur, quam mox sumus daturi.

COROLLARIUM 2

30. Cum autem sit $L\omega = \frac{1}{3}L\zeta = \frac{1}{4}Ll,$ pro termino autem λ sit $Ll = \varepsilon\varepsilon \cdot Ll$, punctum ω intra terminos l et λ cadet, quoties fuerit $L\omega > L\lambda$ ideoque quoties fuerit $\varepsilon < \frac{1}{2}$, quamobrem, quia in praxi ε semper assumitur $< \frac{1}{2}$, solutio problematis ad praxin utique est accommodata.

COROLLARIUM 3

31. Quoties igitur fuerit $\varepsilon < \frac{1}{2}$, tum certo affirmare licet ob foramen, quo speculum est pertusum, confusionem nullo modo imminui, sed semper tantam esse, ac si speculum esset integrum totaque sua superficie radios reflecteret, ideoque aequatio generalis supra inventa pro semidiametro confusionis etiam pro speculis valebit, si modo, ut supra iam invenimus, loco formularum ad lentes pertinentium formulae ibi assignatae § 23 substituantur.

COROLLARIUM 4

32. Atque hinc etiam cognoscimus, si telescopium ex meris lentibus constet, confusionem neutiquam diminui, etiamsi lens obiectiva circa medium obtegatur, quemadmodum nonnulli auctores suaserunt, sed optimum remedium confusionem diminuendi certo in hoc constat, ut lens obiectiva circa marginem obtegatur, quippe quo pacto ipsa semidiameter aperturae x diminuitur et confusio adeo in ratione triplicata minor redditur, cum e contrario, si lens circa medium obtegeretur, ne minima quidem confusionis diminutio sit exspectanda, nisi forte pars obtecta semissem totius lentis superet, quo pacto autem claritas nimium diminueretur.

SCHOLION 1

33. Sin autem semidiameter foraminis $y = \varepsilon x$ semissem totius aperturae x superet, ita ut punctum ω inter L et λ cadat, problema nostrum aliam solutionem postulat. Cum enim nunc ex parte $\zeta\lambda$ maximus circellus in oculo ab ipso puncto λ oriatur sitque $Ll = \varepsilon\varepsilon f$ ob $Ll = f$ hincque spatium $\zeta\lambda = \zeta - \varepsilon\varepsilon f$, spatiolum post oculum fiet $V\gamma = \frac{uu}{ll}\left(\zeta - \varepsilon\varepsilon f\right)$ ibique radiorum obliquitas $= \frac{l}{u}\varepsilon\mathfrak{B}x$, circelli hinc super retina formati erit radius $= \frac{u}{l}\varepsilon\left(\zeta - \varepsilon\varepsilon f\right)\mathfrak{B}x.$ At ex altera parte a termino l nascitur in retina circellus, cuius radius $= \frac{u}{l}\left(f - \zeta\right)\mathfrak{B}x;$; qui duo radii ob rationes ante allegatas inter se aequales sunt statuendi, ex quo consequimur

$$f-\zeta=\varepsilon\zeta-\varepsilon^{3}f$$

hincque

$$\zeta=\tfrac{f\left(1+\varepsilon^{3}\right)}{1+\varepsilon}=f\left(1-\varepsilon+\varepsilon^{2}\right);$$

hinc ergo erit

$$f-\zeta=\varepsilon\left(1-\varepsilon\right)f$$

sicque semidiameter circellorum in retina erit

$$=\tfrac{u}{l}\varepsilon\left(1-\varepsilon\right)f\mathfrak{B}x=\tfrac{u}{l}\varepsilon\left(1-\varepsilon\right)V\mathfrak{B}x^{3},$$

Consequenter hoc casu, quo $\varepsilon>\frac{1}{2}$, semidiameter confusionis erit $=\frac{\varepsilon(1-\varepsilon)}{l}V\mathfrak{B}x^{3}$, quae casu praecedente, quo $\varepsilon<\frac{1}{2}$, erat $=\frac{V\mathfrak{B}}{4l}\cdot x^{3}$; quamdiu ergo est $\varepsilon<\frac{1}{2}$, semper valet formula $\frac{V\mathfrak{B}}{4l}\cdot x^{3}$, quae etiamnum locum habet, si $\varepsilon=\frac{1}{2}$, verum statim ac fit $\varepsilon>\frac{1}{2}$, tum demum confusio diminui incipit atque tandem prorsus evanescit, si fiat $\varepsilon=1$. Quia antem claritas quoque diminuitur et tandem evanescit, hinc nullum plane lucrum in praxin redundare potest; siquis enim adhuc dubitet, utrum loco lentis solidae, cuius radius sit *p,* non adhiberi possit limbus vitreus paris superficiei, cuius radius exterior sit $=q$ et interior $=\varepsilon q$, ita ut sit $p^{2}=q^{2}(1-\varepsilon\varepsilon)$ atque confusio istius limbi minor evadat, hoc dubium nunc facile erit resolvere; a lente enim solida nascetur confusio ut $\frac{1}{4}p^{3}$, ex limbo autem ut $\varepsilon\left(1-\varepsilon\right)q^{3}$*;* unde ob $p=q\sqrt{(1-\varepsilon\varepsilon)}$ erit confusio ex lente solida nata ad confusionem ex limbo oriundam uti $(1+\varepsilon)\sqrt{(1-\varepsilon\varepsilon)}:4\varepsilon$; quare, cum sit per hypothesin $\varepsilon>\frac{1}{2}$ (quia altero casu $\varepsilon<\frac{1}{2}$, ne dubium quidem exsistere potest), posterius membrum 4ε manifesto erit maius quam 2; at quia simul $\varepsilon<1$, erit $1+\varepsilon<2$ ideoque multo magis $(1+\varepsilon)\sqrt{(1-\varepsilon\varepsilon)}<2$, ex quo perspicuum est prius membrum semper esse multo minus posteriore sive confusionem limbi multum excedera confusionem lentis solidae.

SCHOLION 2

34. Cum autem pro usu practico tuto sumere queamus $\varepsilon<\frac{1}{2}$, quo casu speculum obiectivum perforatum aeque magnam gignit confusionem, ac si esset integrum, si in formula generali supra pro telescopiis exhibita, qua semidiameter confusionis exprimitur, loco duarum priorum lentium nostra specula introducamus, aequatio hinc nata sequenti modo se habebit:

$$\frac{1}{k^3}=\frac{mx^3}{p^3}\left(\begin{array}{c}\frac{1}{8}-\frac{(1+B)(1-B)^2}{8B^3P}\\+\frac{\mu}{B^3PQ}\left(\frac{\lambda''}{\mathfrak{C}^3}+\frac{\nu}{C\mathfrak{C}}\right)-\frac{\mu}{B^3C^3PQR}\left(\frac{\lambda'''}{\mathfrak{D}^3}+\frac{\nu}{D\mathfrak{D}}\right)+\text{etc.}\end{array}\right)$$

ubi notari convenit, si forte lentes post specula adhibitae ex vario vitro conficiantur, tum pro qualibet lente litteras μ et ν ex eo vitri genere sumi debere, ex quo lens fuerit facta.

Reliqua autem praecepta generalia pro constructione telescopiorum nullam mutationem ob specula requirent, exceptis iis tantum formulis, quibus tam margo coloratus tollitur, quam omnis confusio a diversa radiorum refrangibilitate oriunda ad nihilum redigitur. Cum enim in has formulas induxerimus pro singulis lentibus litteras N, N', N'', N''' etc., quae litterae proportionales sunt sumtae formulis differentialibus $\frac{dn}{n-1}$, $\frac{dn'}{n'-1}$ etc., si loco duarum priorum lentium specula substituantur, ob defectum refractionis istae binae litterae priores N et N' nihilo aequales sunt censendae; quo observato omnibus illis formulis generalibus pro speculis perinde uti poterimus, atque in secundo libro est factum, dummodo, quae circa distantias focales speculorum et circa duo intervalla priora in capite praecedente sunt allata, probe observentur.

SCHOLION 3

35. Telescopia autem catadioptrica huius generis sponte ad duo genera principalia revocantur, siquidem supra vidimus secundum speculum vel ultra focum primi constitui posse vel intra eum, atque priori casu secundum speculum fore concavum, altero vero convexum. Deinde cum pro priori casu hos limites pro secundi speculi distantia focali q invenerimus

$$q<\varepsilon p \quad \text{et} \quad q>\frac{\varepsilon(1+\varepsilon)p}{1+3\varepsilon}$$

existante primo intervallo $=(1+\varepsilon)p$, cui secundum debet esse aequale, tum vero

$$b=\varepsilon p \quad \text{et} \quad \frac{\beta}{b}=\frac{q}{b-q}=B,$$

quo hoc prius genus debite evolvamus, tres casus constitui conveniet: primo scilicet sumamus $q=\varepsilon p$, secundo $q=\frac{\varepsilon(1+\varepsilon)p}{1+2\varepsilon}$ et tertio $q=\frac{\varepsilon(1+\varepsilon)p}{1+3\varepsilon}$. Pro altero vero genere secundum speculum intra focum prioris collocabatur, ita ut esset

$$b=-\varepsilon p \quad \text{et} \quad \frac{\beta}{b}=\frac{q}{b-q}=\frac{-q}{\varepsilon p+q}=B,$$

ibique cum distantia focalis q hoc casu evadat negativa, posito $q=-\text{q}$ hos ibidem dedimus limites:

$$q > \varepsilon p \text{ et } q < \frac{\varepsilon(1-\varepsilon)}{1-3\varepsilon},$$

unde iterum tres casus evolvamus: primo scilicet sumamus $q = -\varepsilon p$, secundo $q = \frac{-\varepsilon(1-\varepsilon)p}{1-2\varepsilon}$, tertio $q = \frac{-\varepsilon(1-\varepsilon)p}{1-3\varepsilon}$; ; hoc autem casu erit intervallum primum $= (1-\varepsilon)p$, cui etiam secundum aequale esse debet.

Ceterum in priori genere erat $\frac{1}{P} = -\varepsilon$, ita ut in primo statim intervalla reperiatur imago realis; in altero vero genere erat $\frac{1}{P} = \varepsilon$, ita ut in primo intervallo nulla occurrat imago realis; praeterea vero, uti iam monuimus, sumimus hic semper $\varepsilon < \frac{1}{3}$, unde postremus adhuc casus considerari merebitur, quo scilicet sit $\varepsilon = \frac{1}{3}$, quoniam tum secundum speculum planum accipere licebit; quocirca secundum hos septem casus haec telescopia catadioptrica sumus pertractaturi.