APPENDIX

CONCERNING

THE CONSTRUCTION OF REFRACTING/REFLECTING TELESCOPES

[As in numerous concepts and ideas of the day, Greek words were Latinized and used: here we have in the original title the use of :

το κατοπτρικος : *mirror* in Greek ; as well as *speculum* in Latin;

το διοπτρος : the science of dioptrics (*lit*. to see through) in Greek .]

CHAPTER I.

CONCERNED WITH THE

IMAGES FORMED BY SPHERICAL MIRRORS

AND THEIR DIFFUSION.

Problem I.

1.

If rays close to the axis may be incident on the mirror from a bright point established on the axis, to find the location of the image.

PAP shall be a spherical mirror properly placed in position with centre O and with the

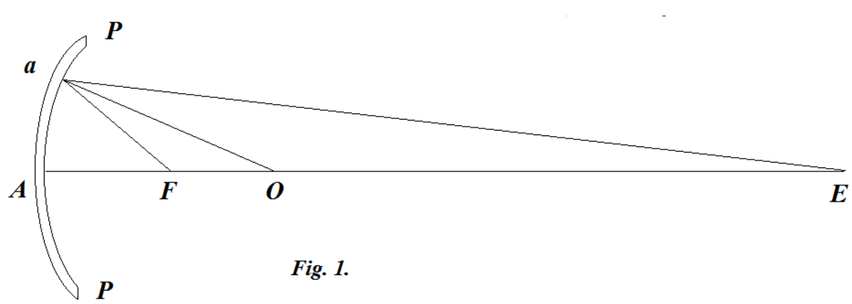


Fig. 1.

radius $OA = f$ described, of which the axis shall be the right line *AOE*, at the point *E* of which the bright point shall be put in place and its distance may be put $EA = a$, from which rays fall on the whole surface of the mirror, but from which we will consider here only these, which shall be close to the axis or which are incident on points close to the centre of the mirror A, therefore *EA* shall be such an incident ray and the radius $Oa = f$ may be drawn to the point *a* from the centre *O* which since it shall be normal to the mirror, E*aO* will be the angle of incidence, to which the angle *OaF* from the other side of the right line *Oa* may be taken equal, and the right line *a*F will be the reflected ray crossing with the axis at the point *F*, at which point therefore all the rays close to the axis emitted from the point *E* will concur, if indeed also the ray *EA* itself emitted following the axis is reflected through the point *F*, thus, so that the point *F* shall be the image of the light point *E* formed by reflection, and since it will be formed from the rays nearest the axis, the principal image will be at this point, just as we have called that in the treatment of lenses. Therefore for the position of this point *F* requiring to be found we will consider the triangle *EaF*, of which the angle E*a*F is bisected by the line *Oa,* from which the known geometric theorem provides this proportion $Ea : EO = Fa : FO$, consequently since in triangle *EaO* the angles for *E* and *a* are indefinitely small, moreover in the triangle *OaF* the angles for *O* and *a;* there will be $\mathrm{E}a = \mathrm{EO} + f$; and $Fa = f - \mathrm{OF}$ from which this proportion will go into this

$$EO + f : EO = f - OF : OF$$

and by addition:

$$2EO + f : EO = f : OF$$

Now since there shall be $EO = EA - AO = a - f$ there will become

$$2a - f : a - f = f : OF$$

and hence

$$OF = \frac{(a-f)f}{2a-f};$$

and thus the position of the point *F* becomes known, the distance of which from the point *A* will become

$$AF = f - FO = \frac{af}{2a-f}. \quad q.\ e.\ d.$$

[N.B. Euler's habit of using the same letter for a point and a length, which occasionally leads to confusion.]

Corollary I.

2. Therefore from the given distance of the light point *Ea* to the mirror $EA = a$, we have found the distance of the principal image on the axis *AF*, just as we will have designated for lenses by the letter α , here we may use the same letter, thus so that there shall become $\alpha = \frac{af}{2a-f}$.

Corollary 2.

3. Here we have observed the mirror as if concave, the radius of which shall be $AO = f$ from which positive values of this letter f will denote concave mirrors, truly negative values convex mirrors. Then truly also the distances *a,* in as much as it is had positive, will indicate the distance of the image before the lens; but if that may be produced negative, by that indication the image will lie after the mirror and that to be

virtual, since the present one shall be real. Moreover hence it is understood, the image to be real, if there were $\alpha > \frac{1}{2}f$, if indeed there shall be $f > 0$; but if there shall be $f < 0$ or the mirror convex; then the image will lie always after the mirror, and that virtual, not real.

Corollary 3.

4. If the distance of the light point $AE = a$ were infinite; then the distance of the principal image from the mirror will be $AF = \frac{1}{2}f$, so that this distance $AF = \frac{1}{2}f$ shall be required to be had for the focal length of the mirror; hence, if we may put the focal length $= p$, the radius of the mirror will be $f = 2p$. Then truly in general the distances a and α thus will depend on that in turn, so that there shall become

$$\alpha = \frac{ap}{a-p} \text{ and hence } p = \frac{\alpha a}{a+p}$$

and

$$\frac{1}{p} = \frac{1}{a} + \frac{1}{\alpha}$$

just as we have seen above to arise in use with lenses.

Scholium.

5. Here from observation it is especially noteworthy, that the three distances a, α and p evidently depend on each other in the same way as with lenses ; from which it is clear in the account of the calculation, mirrors and lenses can be treated likewise, the agreement of which calculation will be illustrated more in the following. Only here it will help to have observed, concave mirrors correspond to convex lenses ; as indeed we have attributed positive focal lengths to convex lenses, certainly the foci of which are real, thus concave mirrors and convex lenses have a real focus there and exert an equally strong burning force ; yet a distinction is put in place there, since the focus shall lie before that with concave mirrors, since with convex lenses it will be formed after these, and in a similar manner convex mirrors can be referred to concave lenses while in each only a virtual image is given, in which clearly the rays actually are not going to converge. Therefore when the talk is concerned with mirrors, the focal length of a concave mirror is always positive ; truly a negative focal length will indicate a convex mirror, and if the focal length may emerge infinite, the mirror is will be plane, in a similar manner, where a lens having an infinite focal distance is plane. Truly besides it will also help to observe, if as we have done in dioptrics, we may put $\alpha = Aa$ and $\mathfrak{A} = \frac{A}{A+1}$, then also there will become $p = \mathfrak{A}a$.

Problem 2.

6. *If E may no longer be a light source, but an object $E\varepsilon$ may be placed perpendicular to the axis, to define its image, which will be represented inverted situated at the point F.*

Solution.

Again the distance of this object from the mirror $EA = a$, and the magnitude of this shall be $Ee = \zeta$, evidently for which denomination we have used the above for lenses, thus, so that ζ shall be very small always with respect to the distance $EA = a$, or the angle $EA\varepsilon$ as if infinitely small.

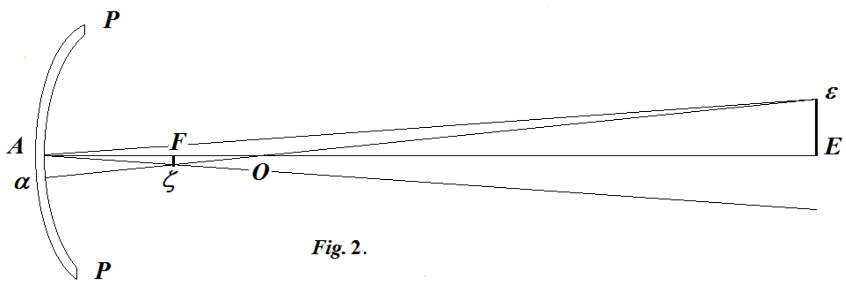


Fig. 2.

Then as before the radius of the mirror shall be $OA = f$, of which the focal length $= p$, thus, so that there shall be $f = 2p$, and the distance of the principal image from the mirror $AF = \alpha$ thus, so that there shall become $\alpha = \frac{af}{2a-f}$. With these in place it is easily understood, the image sought to be formed at F and to be directed towards the other side of the axis ; indeed the right line εA drawn refers to the incident ray, to which the reflected ray $A\zeta$ agrees, which therefore will have to pass through the end of the image ; so that if at the point F the right line $F\zeta$ may be drawn normally to the axis, terminated by the reflected ray $A\zeta$, this right line $F\zeta$ will show the principal image of the object, the magnitude of which therefore thus will be defined from the similar triangles $AE\varepsilon$ and $AF\zeta$, so that there shall become :

$$F\zeta = \frac{AF.E\varepsilon}{AE} = \frac{\alpha.\zeta}{a},$$

as the same can be shown also in the following way. From the point ε the incident ray $\varepsilon O\alpha$ is drawn through the centre of the mirror O also, which since it shall be normal, its reflection falls on the same and will be transmitted through the point ζ also, from which the similar triangles will give $OE\varepsilon$ and $OF\zeta$ will give

$$F\zeta = \frac{OF.E\varepsilon}{OE}.$$

Truly there is $OF = f - \alpha$ and $OE = a - f$, from which there becomes

$$F\zeta = \frac{(f-\alpha)\zeta}{a-f}.$$

Since from the above problem there shall become

$$\alpha = \frac{af}{2a-f} \text{ and hence } f = \frac{2a\alpha}{a+\alpha}$$

there will become

$$f - \alpha = \frac{(a-\alpha)}{a+\alpha} \text{ and } a - f = \frac{(a-\alpha)a}{a+\alpha}$$

and hence with these values substituted there will become

$$F\zeta = \frac{\alpha \cdot \zeta}{a},$$

precisely as before ; from which it shall itself be confirmed the right line $F\zeta$ be drawn at right angles to the normal.

COROLLARY 1

7. Here therefore also the magnitude of the principal image plainly may be determined in the same manner from the magnitude of the object, where we have shown above in the Dioptrics how that can come about ; from which, just as we have done there, if we may put $\alpha = Aa$, here also we will have

$$F\zeta = A \cdot \zeta.$$

[Note again the possible confusion of having the object length ζ as well as being an end point of the image.]

COROLLARY 2

8. Since our figure refers to a concave mirror, its analogy with convex lenses is discerned here also; for just as convex lenses produce [real] inverted images after themselves, thus concave mirrors likewise return real inverted images before themselves; as we have observed now, which occur after the lenses.

PROBLEM 3

9. *If the rays from a light point E on the axis of a mirror may be incident at the extremity of the mirror P, to investigate the concurrence of these with the axis at the point f and thence to determine the diffusion interval.*

SOLUTION

Again the distance shall be $EA = a$ (Fig. 3), the radius of the mirror $OA = OP = f = 2p$ with p denoting the focal length of the mirror. Now the mirror shall be so great, that the angle shall be $AOP = \omega$, and since the perpendicular PX will denote the radius of the of

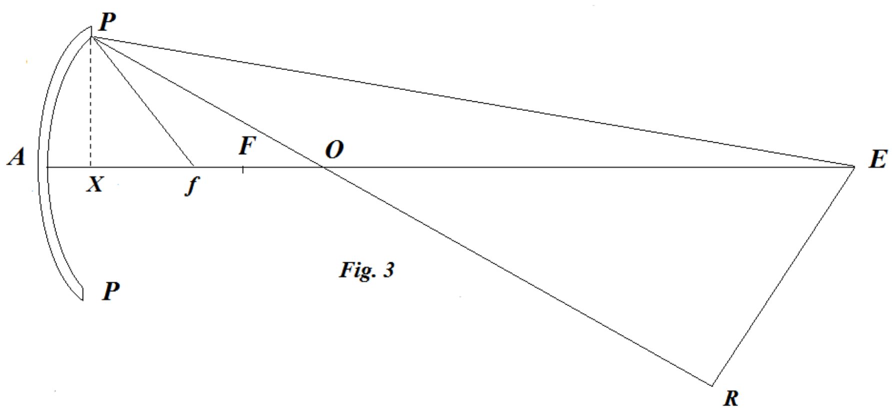


Fig. 3

the aperture of the mirror, this line shall become $PX = x$, and there will be $x = f\sin.\omega$. Now with the light sent from the light point E, with the radius PO produced perpendicular to ER on account of $EO = a - f$ and the angle $EOR = \omega$ there will become

$$ER = (a-f)\sin.\omega \text{ and } OR = (a-f)\cos.\omega$$

and hence

$$PR = f + (a-f)\cos.\omega;$$

from which there is found :

$$EP = \sqrt{(PR^2 + ER^2)} = \sqrt{(a^2 - 2af + 2f^2 + 2f(a-f)\cos.\omega)},$$

which for the sake of brevity shall be $= v$, and hence the angle of incidence will be EPO and thus also the sine of the angle of reflection OPf

$$\frac{ER}{EP} = \frac{(a-f)\sin.\omega}{v}$$

and the cosine

$$= \frac{f + (a-f)\cos.\omega}{v}.$$

Now since in the triangle OPf the angle OPf may be given together with the angle $POf = \omega$ and with the side $OP = f$, if the angle may be called $AfP = \psi$, on account of $\psi = \omega + OPf$, there will become

$$\sin.\psi = \frac{f sin.\omega + 2(a-f)\sin.\omega \cos.\omega}{v}$$

and hence from the nature of the triangle there will be $\sin.\psi : OP = \sin.OPf : Of$, from which by analogy there is deduced :

$$Of = \frac{f(a-f)}{f+2(a-f)\cos.\omega}$$

and hence the interval

$$Af = \frac{f^2 + f(a-f)(2\cos.\omega - 1)}{f+2(a-f)\cos.\omega},$$

and this is the general solution of our problem.

But since in practice the angle AOP at no time may be assumed so great, that it may not be allowed to ignore the squares and higher powers of the angle ω, the expression found conveniently will be reduced to the following simpler form. Since there shall become:

$$\cos.\omega = \sqrt{(1-\sin^2.\omega)} = 1 - \tfrac{1}{2}\sin^2.\omega,$$

on account of $\sin.\omega = \frac{x}{f}$ there will be

$$\cos.\omega = 1 - \frac{x^2}{2f^2}$$

and hence that denominator $f + 2(a-f)\cos.\omega$ will become

$$= 2a - f - \frac{(a-f)x^2}{f2},$$

from which equally there will be approximately

$$\frac{1}{f+2(a-f)cos.\omega} = \frac{1}{2a-f-\frac{(a-f)x^2}{f^2}} = \frac{1}{2a-f} + \frac{(a-f)x^2}{f^2(2a-f)^2}.$$

From which the interval shall be found as follows

$$Of = \frac{f(a-f)}{2a-f} + \frac{(a-f)^2 x^2}{f(2a-f)^2}$$

and hence the interval, which we seek chiefly,

$$Af = \frac{af}{2a-f} - \frac{(a-f)^2 x^2}{f(2a-f)^2}.$$

Whereby, since thus we shall have found the position F of the main image before, so that there shall be $AF = \frac{af}{2a-f}$, now the diffusion interval becomes known

$$Ff = \frac{(a-f)^2 x^2}{f(2a-f)^2}.$$

Besides since for the most part the angle ψ in between is known, by which the reflected rays Pf are inclined to the axis, from the above formula found we may deduce likewise approximately

$$\psi = \frac{(2a-f)x}{af}.$$

Indeed since we may neglect the powers of x greater than the square, where the numerator found becomes $\frac{(2a-f)x}{f}$ and in the denominator, where now the square x^2 may be ignored, it becomes more simply $= a$.

COROLLARY 1

10. So that we may adapt this to the given formulas for lenses, where we have introduced only the two distances a and α into the calculation, on account of $\alpha = \frac{af}{2a-f}$ we will have $f = \frac{2a\alpha}{a+\alpha}$; from which there becomes

$$a - f = \frac{(a-\alpha)a}{a+\alpha} \quad \text{and} \quad 2a - f = \frac{2a^2}{a+\alpha},$$

and hence the diffusion interval will become

$$Ff = \frac{(a+\alpha)(a-\alpha)^2 x^2}{8a^3\alpha},$$

therefore so that, as it arises in the use of lenses, to be proportional to the square of the radius of the aperture x^2; also there is no reason why this same interval Ff may not be produced as in lenses.

COROLLARY 2

11. Also in a similar manner we will be able to express the angle of the obliqueness ψ by the distances a and α and likewise x; for there will be produced $\psi = \frac{x}{\alpha}$. But we have defined this angle above in the calculation established concerned with lenses.

SCHOLIUM

12. When the question may be concerned with lenses and with the greatest apertures of these, which may be able to be taken, we have assumed x equal to the fourth part of the radius of curvature; so that if here we may follow the same procedure and we may assume $x = \frac{1}{4} f$, hence the angle $\omega = 14° \ 30'$ will be found, thus so that the whole arc *PAP* within 30° must be taken. But when we sustain this mirror in place of the objective lens, its aperture demands another long determination, which clearly must be defined from the measurement of the confusion, from which the aperture of this mirror will be reduced to a much smaller order, as we will demonstrate in the following. But now also there is a need, as we may show, in what way the rays reflected from our mirror and forming a diffuse again may be reflected from another mirror and what kind of diffusion of the image then they are going to produce. Finally, it will be appropriate to consider these two following lemmas.

LEMMA 1

13. *If the distance of the object from the mirror $EA = a$ may be moved by a small amount da further from the mirror, then the principal image, of which the distance from the mirror was $AF = \alpha$, will approach closer to the mirror by the small amount $d\alpha$, thus so that there shall be $d\alpha = \frac{-\alpha^2 da}{a^2}$.*

DEMONSTRATION

Indeed since there shall be

$$\frac{1}{a} + \frac{1}{\alpha} = \frac{1}{p} = \frac{2}{f}$$

and the radius f shall remain the same, in whatever manner the distances a and α may be varied between themselves, differentiation will give

$$\frac{da}{a^2} + \frac{d\alpha}{\alpha^2} = 0, \text{ from which } d\alpha = -\frac{\alpha^2 da}{a^2}.$$

LEMMA 2

14. *If the rays incident on the mirror shall be inclined to the axis at the angle* $=\Phi$ *, to find the angle* ψ*, under which the reflected rays will be inclined to the axis of the mirror.*

SOLUTION

Therefore the angle $AEP=\Phi$ (Fig. 3), by which the incident rays *EP* shall be inclined to the axis, and there will be approximately $\Phi=\frac{x}{a}$ and thus $x=a\Phi$. Then truly we have seen the angle, by which the reflected rays shall be inclined to the same axis, to become $\psi=\frac{x}{\alpha}$; on account of which there will be $\psi=\frac{a\Phi}{\alpha}$ or there will be

$$\Phi:\psi=\alpha:a$$

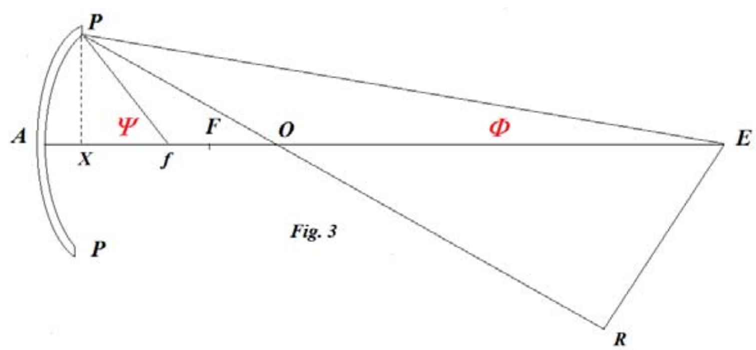


Fig. 3

or inversely as the distances from the mirror.

PROBLEM 4

15. *If the rays, after they will have formed a diffuse image reflected from the first mirror, are incident on another mirror in place on the same axis, to determine both the principal image as well as it diffusion, which the rays will show reflected from the second mirror.*

SOLUTION

Since $F\zeta$ (Fig. 4) shall be the principal image formed by the first mirror, as we have found $F\zeta = \frac{\alpha \cdot \zeta}{a}$, its distance from the second mirror shall be $FB = b$ and this mirror itself shall be prepared thus, so that its reflection $G\eta$ will be formed from the principal image, and thus the distance $BG = \beta$ and, as we have seen now, there will be found

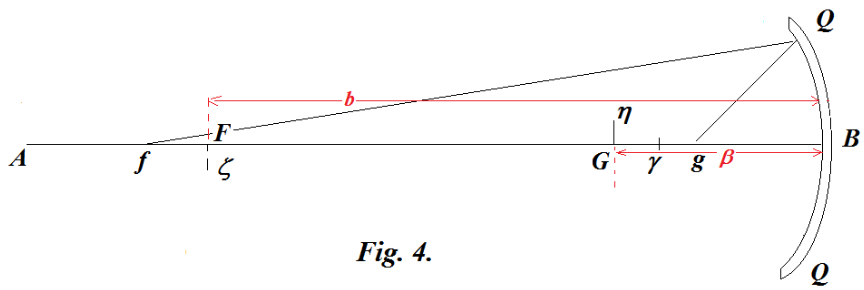


Fig. 4.

$$G\zeta = \frac{\beta}{b} \cdot F\zeta = \frac{\alpha\beta}{ab}\zeta$$

which image will be erect again and formed from rays close to the axis.

[Recall that for thin lenses, Euler calls the object and image distances for successive coaxial lenses a and α, b and β, c and γ, etc., with the respective focal lengths p, q, r, etc. The magnifying powers of these lenses are called A, B, C, etc., while $\mathfrak{A}a$, $\mathfrak{B}b$, $\mathfrak{C}c$, $\mathfrak{D}d$, $\mathfrak{E}e$, $\mathfrak{F}f$ etc. are the focal lengths of the lenses p, q, r, etc. For here we have, for example $\mathfrak{A}a = \frac{aA}{A+1} = \frac{a\frac{\alpha}{a}}{\frac{\alpha}{a}+1} = \frac{a\alpha}{\alpha+a}$, or $\frac{1}{\mathfrak{A}a} = \frac{1}{a} + \frac{1}{\alpha} = \frac{1}{p}$, a useful form of the customary thin lens formula.]

Now also we will consider the extremity of the given diffusion length f, from where the emitted rays may make an angle $= \psi = \frac{x}{\alpha}$ with the axis; truly before we may have an account of this obliquity, we may consider the point f also to emit rays close to the axis, and since that point f shall be further away from the mirror B than the point F, its rays will concur in a point γ nearer to this mirror, with which requiring to be found, this will

be referred to $db = Ff$ and $d\beta = -G\gamma$; from which there is deduced $G\gamma = \frac{\beta^2}{b^2} \cdot Ff$.
Whereby, if the object truly were to be set up at *f*, its principal image would fall at γ; but with which in place no other rays would be emitted from f, except those which may make an angle $= \psi$ with the axis, again these reflected incident on the axis at the point *g* themselves at this point closer to the mirror *B* then γ, thus so that here the case shall be similar to the preceding problem, where the point *f* will correspond to the point *E,* the point γ to the point *F* and the point *g* to the point *f,* with this difference only, so that, since before there was *a* and α, now there shall be *b* and β; for it will be allowed generally here to take $BF = b$ for the distance Bf, and to take β for the distance $B\gamma$; hence therefore by the formula found above, if in place of *x* here there may be written *y,* there will become

$$\gamma g = \frac{(b+\beta)(b-\beta)^2 y^2}{8b^3\beta}$$

But now *y* shall be defined most easily from the angle ψ. For with the ray *fQ* taken under the angle $BfQ = \psi = \frac{x}{\alpha}$, y will be the radius of the aperture of this mirror *QBQ* and thus

$$y = Bf \cdot \psi = \frac{bx}{\alpha};$$

with which value substituted there is produced

$$\gamma g = \frac{(b+\beta)(b-\beta)^2 x^2}{8\alpha^2 b\beta}.$$

On which account the whole diffusion length now will be

$$Gg = \frac{\beta^2}{b^2} \cdot Ff + \gamma g$$

or

$$Gg = \frac{\beta^2}{b^2} \cdot \frac{(a+\alpha)(a-\alpha)^2 x^2}{8\alpha^3 \alpha} + \frac{(b+\beta)(b-\beta)^2 x^2}{8\alpha^2 b\beta}.$$

But now after the second reflection the angle, under which the extreme rays will be inclined to the axis, is deduced from lemma 2

$$= \frac{b\psi}{\beta} = \frac{b}{\alpha\beta} \cdot x.$$

SCHOLIUM 1

16. Therefore since the mirror, to which the two distances a and α and the radius of its aperture $=x$ are being referred, give rise to the diffusion length

$$Ff=\frac{(a+\alpha)(a-\alpha)^2x^2}{8a^3\alpha},$$

we will compare this interval with that, which a lens produces under the same circumstances, and we have seen in the first book (§ 49) for such a lens the diffusion interval to become

$$Ff=\frac{n(4n-1)\alpha^2x^2}{8(n-1)^2(n+2)}\left(\frac{1}{a}+\frac{1}{\alpha}\right)\left(\left(\frac{1}{a}+\frac{1}{\alpha}\right)^2+\frac{4(n-1)^2}{(4n-1)a\alpha}\right),$$

which certainly is a minimum now, which generally can be put in place for these distances a and α relative to the aperture, the radius of which is x. But as we may wish to put in place an easier comparison, we may put the distance of the object a to be infinite on both sides, and from the mirror the diffusion distance will emerge

$$Ff=\frac{x^2}{8\alpha};$$

but what arises from the lens, will be

$$Ff=\frac{n(4n-1)x^2}{8(n-1)^2(n+2)\alpha},$$

where $n:1$ will denote the ratio of the refraction, and by taking $n=1,55$ here the distance found is $Ff=0,938191\cdot\frac{x^2}{\alpha}$. From which it will be apparent much less diffusion to arise from the mirror than from the lens, since this will be to that as $\frac{1}{8}:0,938191$, that is almost as 1: 7,505528, or as $1:7\frac{1}{2}$; therefore which proportion since it may have a special place with regard to the objectives for mirrors and lenses, hence the particular reason will become known, why mirrors substituted in place of objective lenses will have provided much shorter telescopes, since on account of a much shorter diffusion length a shorter focal length may be able to be accepted, to which it is agreed, since with these reflecting telescopes the rays incident on the objective mirror in the first place may be reflected to the other mirror, from which again they are reflected in the same way, before they pass through the eyepiece lenses, thus so that the distances of both mirrors will require to be computed twice and thus the length of the instrument may again be reduced by almost half. Therefore by this convenience mirrors will be outstanding also without any regard had to the quality of these, by which the rays of diverse colors are not spread out by reflection, as happens with refraction. But still here also a conspicuous inconvenience of mirrors is not to be ignored, consisting in this, as a mirror even with the maximum polish always shall reflect fewer rays, than will be transmitted by a lens of the same magnitude. And this is the reason, why reflecting telescopes generally may provide a much smaller degree of clarity.

SCHOLIUM 2

17. Just as we have resolved this latter problem and we have also defined the diffusion of the image arising from the second mirror, thus it may be possible for the same investigation to be adapted to several mirrors, unless by the nature of the thing it will be restricted to the use of two mirrors. On account of which we have gathered together the rays reflected from two mirrors to be directed to glass lenses, through which finally they will be directed to the eye, and on this account this same reasoning the objective mirror itself must be perforated around the middle, so that a passage may be allowed for the rays reflected by the second mirror through this opening, where likewise they may be excepted from lenses. Whereby, since until now we have been able to consider the objective mirror as a whole, now it remains, that also we shall have to give an account of the hole in the calculation, by which it is perforated, where likewise it will be required to be investigated, how the second mirror must be prepared with regard to this hole, lest it may not intercept an excessive supply of the rays and yet it may suffice for all the rays required to be reflected by the primary mirror to be removed; and therefore we will consider these concerns more accurately in the following problem.

PROBLEM 5

18. *If, in a telescope, a concave mirror PπAπP may be used in place of the objective lens* (Fig. 5) *with a hole πAπ drilled out in the middle, the centre of which shall be on the axis AB, on which as if at an infinite distance an object or point of light may be considered, from which the rays incident on that same mirror PππP may be directed parallel to the axis, and thence they may be reflected by a smaller mirror QBQ placed normally to the same axis, from which again they may be reflected to a glass lens close to the opening ππ likewise placed normally on the same axis, to determine the images formed by the twofold reflection and the diffusion of these.*

SOLUTION

The radius of the whole objective mirror shall be $AP = x$ and the radius of the opening $A\pi = y$, truly the radius of curvature of the mirror $= f$ and thus the focal length $p = \frac{1}{2} f$, then truly the focal length of the smaller mirror QBQ shall be $= q$ and the separation of these mirrors $AB = k$. With these in place, since the distance on the axis AB to the remote object may be considered to be infinite, thence rays parallel to the axis will arrive at the objective mirror PP; which therefore in order that the whole of its reflecting surface $P\pi$ may be filled up in whatever manner, the mirror QBQ must not be greater than the opening $\pi\pi$ nor either may it be agreed to be less, since otherwise the rays from the object will be entering directly into the opening and into the lens placed

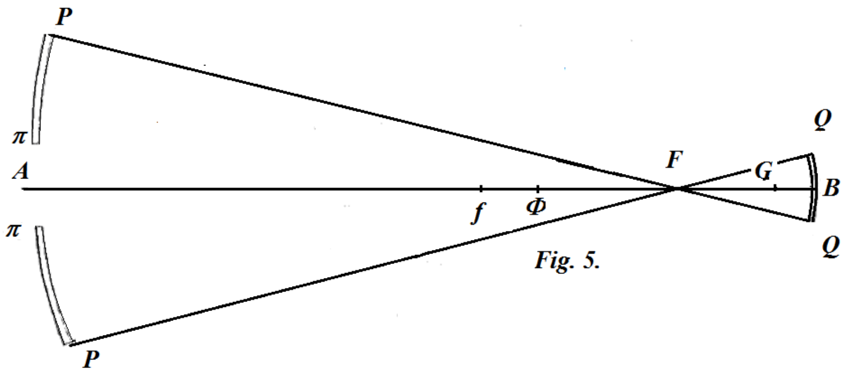


Fig. 5.

there and destroying the quality of the image; from which it is understood the radius of the aperture of this smaller mirror must be $BO = y$, or perhaps not much greater than that. Therefore since here the distance of the object, which above was put $= a$, is infinite in our case, if the rays were able to be incident approximately on the axis, from these the principal image would be formed at *F,* thus so that the distance would be $AF = \alpha = p$. But since the rays near the axis are excluded, no principal image will be formed. Therefore the first image will be formed by the rays reflected around the border of the opening at Φ, thus so that the interval shall be $F\Phi = \frac{y^2}{8\alpha}$, since here there is now *y,* which above was x, and the distance of the object $a = \infty$. But the image extremity will be formed at the point *f* from the rays reflected around the boundary of the mirror *PP* and the interval will become $Ff = \frac{x^2}{8\alpha}$; whereby, since the principal image itself shall be missing here, here the total diffusion length will be only

$$\Phi f = \frac{x^2 - y^2}{8\alpha}.$$

Yet meanwhile the points F, Φ, f will be so close to each other, so that they may be had equal to the same in the calculation. Therefore since all the rays reflected by the greater mirror may be agreed to pass through the point *F*, so that they may be incident on the mirror *QBQ*, its radius *BQ* must be so great, so that there shall be

$$AF : AP = BF : BQ,$$

from which there becomes:

$$BQ = \frac{k-\alpha}{\alpha} \cdot x;$$

which since it must be equal to *y*, we will have

$$y = \frac{k-\alpha}{\alpha} \cdot x \text{ and hence } k = \frac{\alpha(y+x)}{x}.$$

But if a mirror may be put in place between *A* and *F* (Fig. 6), there will be found

$$BQ = \frac{\alpha - k}{\alpha} \cdot x = y \text{ and hence } k = \frac{\alpha(x-y)}{x},$$

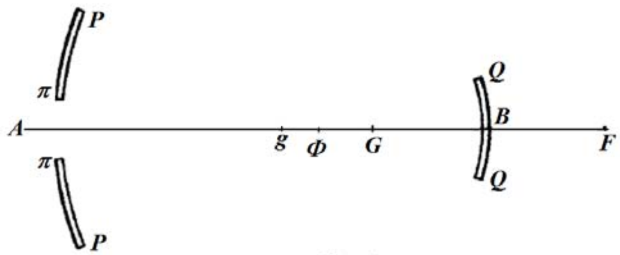


Fig. 6.

which expression contained in the above truly is required to be considered, therefore so that the radius of the opening y is allowed to be taken positive as well as negative. Therefore since now the distance of the first image F from the second mirror shall be $k - \alpha = \frac{\alpha y}{x}$, as above we have called $= b$, thus so that there shall become $b = \frac{\alpha y}{x}$, the second image reflected from the mirror *QBQ* falls at the point *G,* thus so that there shall become $BG = \beta = \frac{bq}{b-q}$ thus so that all the rays reflected from the mirror *QBQ* shall be agreed to pass through this point *G*, if indeed here we may ignore diffusion completely. Now therefore in addition it is required to be effected, so that all these rays may enter through the opening $\pi A \pi$, that which will happen, since there shall be $BQ = A\pi$, but only if the point *G* may lie closer to *A* than towards *B,* or there will have to become $\beta > \frac{1}{2}k$. Therefore we find

$$\beta = \frac{bq}{b-q} = \frac{\alpha yq}{\alpha y - qx} \text{ and } k = \frac{\alpha(y+x)}{x},$$

thus so that now there must become

$$\frac{\alpha yq}{\alpha y - qx} > \frac{\alpha(y+x)}{2x}, \text{ from which there arises } q > \frac{\alpha y(x+y)}{x(3y+x)};$$

from which formula it will be able to define the focal length of the smaller mirror, which therefore will be determined by the radii of the opening and of the larger mirror itself $p = \alpha$, together with the focal length of the greater mirror *;* but if the smaller mirror may be put in place between F and *A,* now we see to be $AB = k = \frac{\alpha(x-y)}{x}$, and since now the distance shall be $b = -\frac{\alpha y}{x}$, the distance $BG = \beta = \frac{\alpha yq}{\alpha y + xq}$; which since it shall be greater than $\frac{1}{2}k$, it is necessary that there shall become $q > \frac{\alpha y(x-y)}{x(3y-x)}$; from which, if $x > 3y$, then q must become negative, thus so that there shall be

$$q > -\frac{\alpha y(x-y)}{x(x-3y)};$$

but if there shall become $x = 3y$, there may be taken $q = \infty$ and thus the smaller mirror will become plane. Finally, so that it may pertain to the diffusion of the second image represented at G, that again will be as if with its principal image truncated ; so that if $G\Phi g$ may be represented by letters according to the similitude of the letters $F\Phi f$, the whole diffusion distance will be agreed to be only $= \Phi g$, the magnitude of which will be found from the preceding formula, if in place of x^2 there may be written $x^2 - y^2$; from which on account of $a = \infty$, here there will become :

$$\Phi g = \frac{\beta^2}{b^2}\frac{\left(x^2-y^2\right)}{8\alpha} + \frac{(b+\beta)(b-\beta)^2\left(x^2-y^2\right)}{8\alpha^2 b\beta},$$

and now the obliquity of the rays to the axis concurrent in Φ will become $= \frac{b}{\alpha\beta}\cdot y$, truly the obliquity of the rays at $g = \frac{b}{\alpha\beta}\cdot x$.

COROLLARY 1

19. Therefore if the smaller mirror may be placed beyond the position of the image F, the distance of that from the first mirror must become

$$AB = \frac{\alpha(x+y)}{x} = \alpha + \frac{y}{x}\cdot\alpha,$$

thus so that there shall become $FB = \frac{\alpha y}{x}$, and therefore in this case the distance AB will be greater than the focal length of the principal mirror ; then truly the focal length of this second mirror must become

$$q > \frac{\alpha y(x+y)}{x(x+3y)}.$$

COROLLARY 2

20. Moreover here clearly the point G is presumed to move away from the point B towards the point A, thus so that the distance β may become positive; if indeed there shall become $q > b$, the point G will fall on the other side of the mirror QBQ and evidently the rays GQ produced may pass beyond the opening. Whereby here for q the other bound will be required to be observed properly, so that there shall be
$q < b$ or if $q < \frac{\alpha y}{x}$, then truly also $q > \frac{\alpha y(x+y)}{x(x+3y)}$.

COROLLARY 3

21. But if the mirror QBQ may be located within the focus F, the distance will be required to become

$$AB=\frac{\alpha(x-y)}{x}=\alpha-\frac{y}{x}\alpha,$$

thus so that there shall be

$$FB=\frac{\alpha y}{x},$$

and with an interval of such a size that the first image may fall past the second mirror and there may become $b=-\frac{\alpha y}{x}$, from which the distance is deduced

$$BG=\beta=\frac{\alpha yq}{\alpha y+qx};$$

which distance is positive always or is directed towards A, unless q perhaps shall be negative ; which since it may be greater than $\frac{1}{2}k$, there must be

$$2xyq>\alpha y(x-y)+x\left(x-y\right)q;$$

therefore there must be

$$2xy>x(x-y)\ \text{ or }\ y>\tfrac{1}{3}x.$$

Whereby if, as always happens in practice, there shall become $y<\frac{1}{3}x$, clearly it is unable for this condition to be satisfied, if the other mirror shall be concave.

COROLLARY 4

22. Therefore in this case it is necessary, that the small mirror shall be convex and its focal length negative. Therefore there may be put $q=-\mathfrak{q}$, so that there may become $\beta=\frac{-b\mathfrak{q}}{b+\mathfrak{q}}$, which value on account of $b=-\frac{\alpha y}{x}$ will be changed into this :

$$\beta=\frac{\alpha y\mathfrak{q}}{\mathfrak{q}x-\alpha y};$$

for which value , so that in the first place it shall be positive, there must become

$$\mathfrak{q}>\frac{\alpha y}{x},$$

then, so that there may become $2\beta>k$, there must be

$$2xy\mathfrak{q}>x\left(x-\ y\right)\mathfrak{q}-ay(x-y),$$

from which

$$ay\left(x-y\right)>x(x-3y)\mathfrak{q},$$

from which for $\mathfrak{q}$ there is elicited either the bound

$$\mathfrak{q} < \frac{\alpha y(x-y)}{x(x-3y)} \text{ or with there being present } \mathfrak{q} > \frac{\alpha y}{x}.$$

COROLLARY 5

23. But if besides it shall be normal practice for the opening to be made, so that there shall be $3y > x$, then the smaller concave mirror will be allowed to be used, provided it focal length shall be $\mathfrak{q} > \frac{\alpha y(x-y)}{x(3y-x)}$, as is evident from corollary 3, and in this case, since the letter q is not restricted to any condition, indeed this mirror will be able to become plane.

SCHOLIUM

24. Thus here we have considered these two mirrors, just as are accustomed to be used in Gregorian telescopes [not quite the case, as a large parabolic mirror and a small elliptical mirror were used having a common focal point, which focused the light through the hole in the large mirror to the other focal point of the ellipse; see *Optica Promota* in this series of translations], and here we have considered only for the midpoint of the object to be situated on the axis of the tube, from which rays parallel to the axis are incident on the principal mirror; thus we have constructed the other mirror, so that it may receive all the reflected rays from the first, and these again may be projected through the hole. But since also parts of the object situated beyond the axis should be offered to be viewed, so that rays within some small oblique angle may be incident on the mirror, the tube, in which these two mirrors are inserted, are able to support a little divergence or, which amounts to the same, it may be convenient to make the tube a little wider than the diameter of the mirror; then on account of the same reasoning the smaller mirror also should be designated to be extended beyond the limits, so that also these oblique rays may be able to be received after reflection; but since it will be of little interest, whether the extremities of the object may be seen to be equally illuminate as its middle or not, we will be able easily to be without this increase, so that this whole obliquity may not exceeded beyond some very small amount in the prescribed magnification. But otherwise the treatment of this matter is going to be had for a long time, if indeed the mirrors may be called into use oblique to the axis of the instrument, which has been done in the invention of this instrument from the beginning by Newton; but since the reflection of incident rays may give rise to a great deal of confusion, by no means will we touch on this argument here.

APPENDIX

CONSTRUCTIONE TELESCOPIORUM CATOPTRICO-DIOPTRICORUM.

CAPUT I.

DE IMAGINlBUS PER SPECULA SPHAERICA FORMATIS EARUMQUE DIFFUSIONE.

Problema I.

1.

Si a puncto lucido in axe speculi constituto radii axi proximi in speculam incidant, invenire locum imaginis.

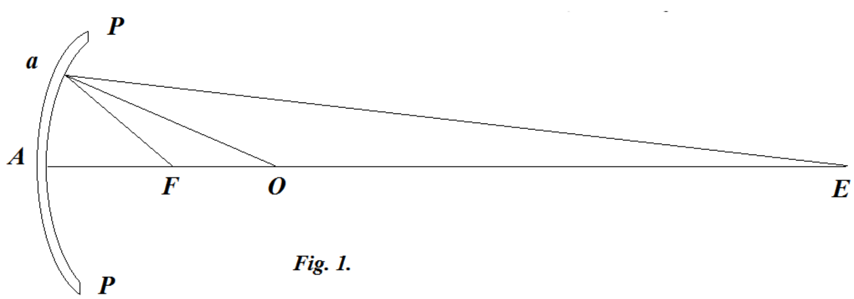


Fig. 1.

Sit *PAP* speculam sphaericum probe positum centro *O* radio $OA = f$ descriptum, cuius axis sit recta *AOE*, in cuius puncto *E* constitutum sit punctum lucidum et ponatur eius distantia $EA = a$, unde radii in totam speculi superficiem incidant, e quibus autem eos tantum hic consideramus, qui axi sint proximi seu qui in puncta a medio puncto speculi A proxima incidant, talis igitur radius incidens sit *EA* et ad punctum *a* ex centro *O* ducatur radius $Oa = r$ qui cum in speculam sit normalis, erit E*aO* angulus incidentiae, cui ab altera parte rectae *Oa* capiatur angulus aequalis *OaF*, eritque recta *a*F radius reflexus cum axe occurrens in puncto *F*, in quo puncto adeo omncs radii axi proximi e puncto *E* emissi concurrent, siquidem etiam radius *EA* secundum ipsum axem emissus in punctum *F* reflectitur, ita, ut punctum *F* sit imago puncti lucidi *E* per reflexionem formata, et cum a radiis axi proximis formetur, in hoc puncto erit imago principalis, uti eam in tractatu de lentibus vocavimus. Ad locum igitur ipsius puncti *F* inveniendum consideretur triangulum *EaF*, cuius angulus E*a*F bisectus est recta *Oa,* unde notum theorema Geometricum praebet hanc proportionem $Ea:EO=Fa:FO$ deinde quia in triangula *EaO* anguli ad *E* et ad *a* sunt infinite parvi in triangulo autem *OaF* anguli ad *O* et *a;* erit $\mathrm{E}a=\mathrm{EO}+f$; et $Fa=f-\mathrm{OF}$ unde illa proportio abit in hanc

$$EO+f:EO=f-OF:OF$$

et componendo

$$2EO+f:EO=f:OF$$

Cum iam sit $EO=EA-AO\ =a-f$ fiet

$$2a-f:a-f=f:OF$$

hincque $OF=\frac{(a-f)f}{2a-f}$

sicque locus puncti *F* innotescit, cuius distantia a puncto *A* erit

$$AF=f-FO=\frac{af}{2a-f}.\ \ q.\ e.\ i.$$

Coroll. I.

2. Ex data ergo distantia puncti lucidi *Ea* speculo $EA=a$, invenimus distantiam imaginis principalis super axe *AF*, quam cum in lentibus littera α destgnaverimus, etiam hic eadem littera utamur, ita ut fit $\alpha=\frac{af}{2a-f}$.

Coroll. 2.

3. Speculam hic tanquam concavum spectavimus, cuius radius esset $AO=f$ unde valores postitivi huius litterae *f* specula concava; valores vero negativi specula convexa denotabunt. Tum vero etiam distantia *a,* quatenus valorem habet positiuum, distantiam imaginis ante speculam indicabit; sin autem prodeat negativa, id indicio erit imaginem post speculum cadere ea atque fore fictam, cum praesens sit realis. Hinc autem intelligitur, imaginem fore realem, si fuerit $\alpha>\frac{1}{2}f$, si quidem sit $f>0$; sin autem sit $f<0$ seu speculam convexum; tum imago semper post speculam cadet, eritque ficta, non realis.

Coroll. 3.

4. Si puncti lucidi distantia $AE=a$ fuerit infinita; tum distantia imaginis principalis a speculo erit $AF=\frac{1}{2}f$, ut haec distantia $AF=\frac{1}{2}f$ pro distantia focali spectuli sit habenda hinc si speculi distantiam foculem ponamus $=p$, erit radius speculi

$f=2p$. Tum vero in genere distantiae a et α ita a se invicem pendebunt, ut sit $\alpha=\frac{ap}{a-p}$ hincque $p=\frac{\alpha a}{a+p}$ et $\frac{1}{p}=\frac{1}{a}+\frac{1}{\alpha}$ prorsus uti in lentibus usu venire supra vidimus.

Scholion.

5. Hic notatu inprimis dignum occurrit, quod tres istae distantiae a, α et p eodem prorsus modo a se invicem pendent, uti in lentibus ; ex quo evidens est ratione calculi specula perinde tractari posse ac lentes, quae calculi convenientia adhuc in sequentibus magis illustrabitur. Hic tantum notasse iuvabit, lentibus convexis respondere specula concava ; uti enim lentibus convexis distantias focales positivas tribuimus, quippe quarum foci sunt reales, ita etiam specula concava realem habent focum ibique aeque vi urendi pollent atque lentes convexae in suis focis; discrimcn tamen in eo situm est, quod in speculis concavis focus ante eo cadat, cum in lentibus convexis post eas formetur atque simili modo specula convexa ad lentes concavas referentur dum in utrisque focus tantum fictus datur, in quo scilicet radii non revera congregentur. Quando ergo de speculis sermo erit, distantia focalis positiva semper speculam concavum; distantia vero focalis negativa speculam convexum indicabit, ac si distantia focalis evadat infinita, speculam erit planum, simili modo, quo lens distantiam focalem habens infinitam est plano plana. Praeterea vero etiam observasse invabit, si uti in Dioptrica fecimus, statuamus $\alpha=Aa$ et $\mathfrak{A}=\frac{A}{A+1}$, tum etiam fore $p=\mathfrak{A}a$.

Problema 2.

6. *Si non amplius lucidum punctum E, sed obiectum Eε axi speculi perpendiculatiter insistat, eius imaginem, quae in puncto F situ inverso repraesentabitur, definire.*

Solutio.

Ponatur iterum distantia huius obiecti a speculo $EA=a$, sitque eius magnitudo $Ee=\zeta$, quippe qua denominatione supra de lentibus sumus usi, ita, ut ζ semper sit quantitas valde parva respectu distantiae $EA=a$, seu angulus $EA\varepsilon$ quasi infinite parvus.

Deinde sit ut ante radius speculi $OA=f$, eius distantia focalis $=p$, ita, ut sit $f=2p$,

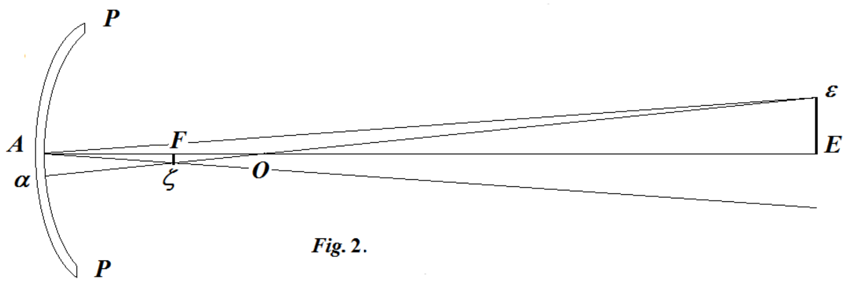


Fig. 2.

et distantia imaginis principalis a speculo $AF=\alpha$ ita, ut sit $\alpha=\frac{af}{2a-f}$. His positis facile intelligitur, imaginem quaesitam in punctum F incidere atque ad contrariam partem axis fore directam; ducta enim recta εA referet radium incidentem, cui convenit radius reflexus $A\zeta$, qui ergo per imaginis extremitatem transire debebit; unde si in puncto F normaliter ad axem ducatur recta $F\zeta$, ad radium reflexum $A\zeta$ terminata, haec recta $F\zeta$ imaginem principalem objecti exhibebit, cuius ergo magnitudo ex similitudine triangulorum $AE\varepsilon$ et $AF\zeta$ ita definietur, ut sit

$$F\zeta=\frac{AF.E\varepsilon}{AE}=\frac{\alpha.\zeta}{a}.$$

quod idem etiam hoc modo ostendi potest. Ex puncto ε per centrum speculi O ducatur etiam radius incidens $\varepsilon O\alpha$, qui cum fit normalis, eius reflexus in ipsum cadet transibitque etiam per punctum ζ unde similitudo triangulorum $OE\varepsilon$ et $OF\zeta$ dabit

$$F\zeta=\frac{OF.E\varepsilon}{OE}.$$

Est vero $OF=f-\alpha$ et $OE=a-f$, ex quo fit

$$F\zeta=\frac{(f-\alpha)\zeta}{a-f}.$$

Cum ex superiori problemate sit

$$\alpha=\frac{af}{2a-f}\ \text{ hincque }\ f=\frac{2a\alpha}{a+\alpha}$$

erit

$$f-\alpha=\frac{(a-\alpha)}{a+\alpha}\ \text{ et }\ a-f=\frac{(a-\alpha)a}{a+\alpha}$$

hincque substitutis his valoribus fiet

$$F\zeta=\frac{\alpha\cdot\zeta}{a},$$

prorsus ut ante; quo ipso confirmatur rectam $F\zeta$ axi recte normalem esse ductam.

COROLLARIUM 1

7. Hic ergo etiam magnitudo imaginis principalis eodem plane modo ex obiecti magnitudine determinatur, quo in Dioptrica id fieri supra ostendimus; unde, si, ut ibi fecimus, statuamus $\alpha=Aa$, habebimus etiam hic

$$F\zeta=A\cdot\zeta.$$

COROLLARIUM 2

8. Quia nostra figura speculum concavum refert, eius analogia cum lentibus convexis etiam hic manifesto cernitur; quemadmodum enim lentes convexae imagines inversas post se repraesentant, ita specula concava imagines itidem inversas ante se referunt; iam enim observavimus, quae post lentes contingunt.

PROBLEMA 3

9. *Si a puncto lucido E in axe speculi sito radii incidant in extremitatem speculi P, eorum cum axe concursum in puncto f investigare indeque spatium diffusionis determinare.*

SOLUTIO

Sit iterum distantia $EA = a$ (Fig. 3), radius speculi $OA = OP = f = 2p$ denotante p distantiam speculi focalem. Iam tantum sit speculum, ut sit

angulus $AOP = \omega$, et cum perpendiculum PX denotet semidiametrum aperturae speculi, sit haec linea $PX = x$ eritque $x = f\sin.\omega$. Demisso iam ex puncto lucido E in radium PO productum perpendiculo ER ob $EO = a - f$ et angulum $EOR = \omega$ erit

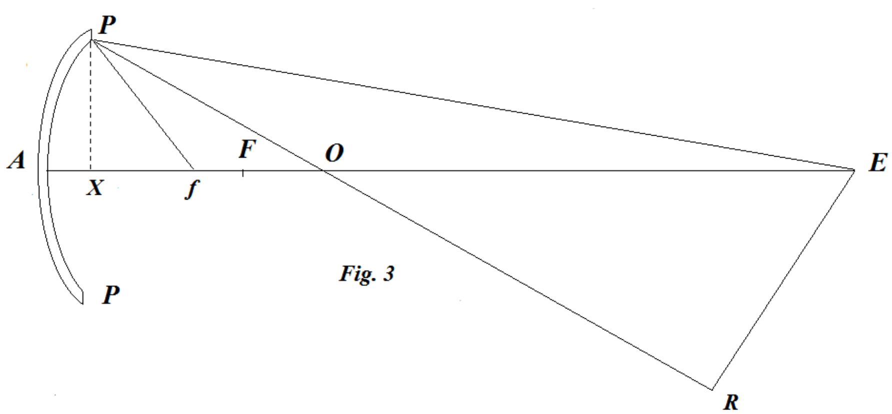


Fig. 3

$$ER = (a - f)\sin.\omega \ \text{ et } \ OR = (a - f)\cos.\omega$$

hincque

$$PR = f + (a - f)\cos.\omega;$$

unde invenitur

$$EP = \sqrt{(PR^2 + ER^2)} = \sqrt{(a^2 - 2af + 2f^2 + 2f(a - f)\cos.\omega)},$$

quae brevitatis gratia sit $= v$, atque hinc erit anguli incidentiac EPO ideoque etiam anguli reflexionis OPf sinus

$$\frac{ER}{EP} = \frac{(a-f)\sin.\omega}{v}$$

et cosinus

$$=\frac{f+(a-f)\cos.\omega}{v}.$$

Cum iam in triangulo *OPf* detur angulus *OPf* una cum angulo $POf=\omega$ et latere $OP=f$, si vocetur angulus $AfP=\psi$, ob $\psi=\omega+OPf$ erit

$$\sin.\psi=\frac{f\sin.\omega+2(a-f)\sin.\ \omega\cos.\omega}{v}$$

atque hinc ex natura trianguli erit $\sin.\psi:OP=\sin.OPf:Of$, ex qua analogia colligitur

$$Of=\frac{f(a-f)}{f+2(a-f)\cos.\omega}$$

hincque intervallum

$$Af=\frac{f^2+f(a-f)(\ 2\cos.\omega-1)}{f+2(a-f)\cos.\omega},$$

haecque est solutio generalis nostri problematis.

Cum autem in praxi angulus *AOP* nunquam tantus assumatur, ut non liceat potestates anguli ω quadratica altiores negligera, expressio inventa commode ad formam simpliciorem sequenti modo reducetur. Cum sit

$$\cos.\omega=\sqrt{(1-\sin^2.\omega)}=1-\tfrac{1}{2}\sin^2.\omega,$$

ob $\sin.\omega=\frac{x}{f}$ erit

$$\cos.\omega=1-\frac{x^2}{2f^2}$$

hincque ille denominator $f+2(a-f)\cos.\omega$ fiet

$$=2a-f-\frac{(a-f)x^2}{f2},$$

ex quo pariter proxime erit

$$\frac{1}{f+2(a-f)cos.\omega}=\frac{1}{2a-f-\frac{(a-f)x^2}{f^2}}=\frac{1}{2a-f}+\frac{(a-f)x^2}{f^2(2a-f)^2}.$$

Unde intervallum modo inventum fit

$$Of=\frac{f(a-f)}{2a-f}+\frac{(a-f)^2x^2}{f(2a-f)^2}$$

atque hinc intervallum, quod potissimum quaerimus,

$$Af=\frac{af}{2a-f}-\frac{(a-f)^2x^2}{f(2a-f)^2}.$$

Quare, cum ante locum imaginis, principalis F ita invenissemus, ut esset $AF=\frac{af}{2a-f}$, nunc innotescit spatium diffusionis

$$Ff=\frac{(a-f)^2x^2}{f(2a-f)^2}.$$

Praeterea cum etiam plurimum intersit angulum ψ nosse, quo radii reflexi Pf ad axem inclinantur, ex formula supra inventa colligemus itidem proxima

$$\psi=\frac{(2a-f)x}{af}.$$

Quoniam enim potestates ipsius x quadrato maiores negligimus, numerator ibi inventus fit $\frac{(2a-f)x}{f}$ et in denominatore, ubi iam ipsum quadratum x^2 negligera licet, fit simpliciter $=a$.

COROLLARIUM 1

10. Quo haec ad formulas pro lentibus datas accommodemus, ubi tantum binas distantias a et α in computum induximus, ob $\alpha=\frac{af}{2a-f}$ habebimus $f=\frac{2a\alpha}{a+\alpha}$; unde fit

$$a-f=\frac{(a-\alpha)a}{a+\alpha} \quad \text{et} \quad 2a-f=\frac{2a^2}{a+\alpha},$$

atque hinc spatium diffusionis erit

$$Ff=\frac{(a+\alpha)(a-\alpha)^2x^2}{8a^3\alpha},$$

quod ergo, perinde ac in lentibus usu venit, quadrato semidiametri aperturae x^2 est proportionale; quin etiam ipsum hoc spatium Ff in eundem sensum cadit ac in lentibus.

COROLlARIUM 2

11. Simili modo poterimus etiam angulum obliquitatis ψ per solas distantias a et α itemque x exprimere; prodibit enim $\psi=\frac{x}{\alpha}$. Hunc autem angulum supra in calculo circa lentes instituto sollicite definivimus.

SCHOLION

12. Cum quaestio esset de lentibus earumque apertura maxima, quam capere possent, sumsimus x aequale parti quartac radii curvaturae; quodsi ergo hic idem institutum

sequamur et sumamus $x = \frac{1}{4}f$, hinc reperietur angulus $\omega = 14°\ 30'$, ita ut totus arcus PAP infra 30° capi debeat. Quando autem hoc speculum locum lentis obiectivac sustinet, eius apertura longe aliam determinationem postulat, quam scilicet ex mensura confusionis definiri oportet, unde huius speculi apertura ad multo pauciores gradus reducetur, uti in sequentibus docebitur. Nunc autem etiam opus est, ut ostendamus, quemadmodum radii a nostro speculo reflexi et imaginem diffusam formantes porro ab alio speculo denuo reftectantur et qualem imaginis diffusionem tum sint producturi. Hunc in finem bina sequentia lemmata perpendi conveniet.

LEMMA 1

13. *Si distantia obiecti a speculo* $EA = a$ *particula minima da ulterius a speculo removeatur, tum imago principalis, cuius distantia a speculo erat* $AF = \alpha$ *, ad speculum propius accedet particula* $d\alpha$*, ita ut sit* $d\alpha = \frac{-\alpha^2 da}{a^2}$.

DEMONSTRATIO

Cum enim sit

$$\frac{1}{a} + \frac{1}{\alpha} = \frac{1}{p} = \frac{2}{f}$$

atque radius f idem maneat, utcunque distantiae a et α inter se varientur, differentiatio dabit

$$\frac{da}{a^2} + \frac{d\alpha}{\alpha^2} = 0, \text{ unde } d\alpha = -\frac{\alpha^2 da}{a^2}.$$

LEMMA 2

14. *Si radii in speculum incidentes ad axem sint inclinati angulo* $= \Phi$ *, invenire angulum* ψ*, sub quo radii reflexi ad axem speculi erunt inclinati.*

SOLUTIO

Sit igitur angulus $AEP = \Phi$ (Fig. 3), quo radii incidentes EP ad axem speculi inclinantur, eritque proxima $\Phi = \frac{x}{a}$ ideoque $x = a\Phi$. Tum vero vidimus angulum, quo radii reflexi ad eundem axem inclinantur, fore $\psi = \frac{x}{\alpha}$; quocirca erit $\psi = \frac{a\Phi}{\alpha}$ seu erit

$$\Phi : \psi = \alpha : a$$

seu reciproce ut distantiae a speculo.

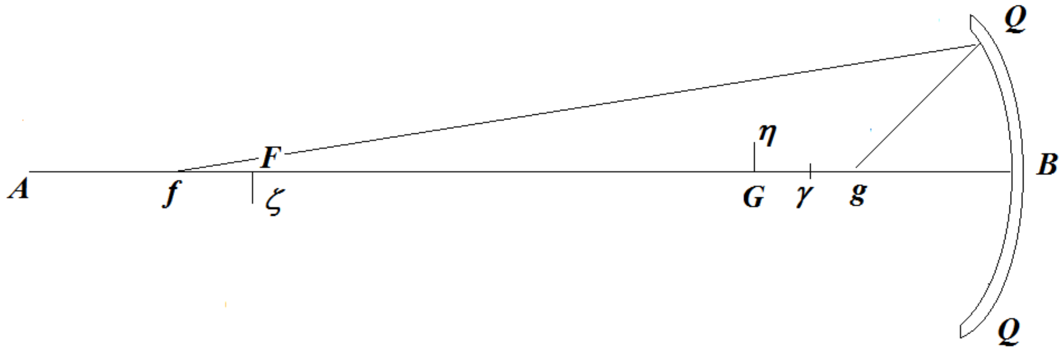
PROBLEMA 4

15. *Si radii, postquam a primo speculo reflexi imaginem diffusam formaverunt, in aliud speculum super eodem axe constitutum incidant, determinare tam imaginem*

principalem quam eius diffusionem, quam radii a secundo speculo reflexi exhibebunt.

SOLUTIO

Cum $F\zeta$ (Fig. 4) sit imago principalis a primo speculo formata, quam invenimus



$F\zeta = \frac{\alpha \cdot \zeta}{a}$, sit eius distantia a secundo speculo $FB = b$ atque ipsum hoc speculum ita sit comparatum, ut ab eius reflexione imago principalis formetur $G\eta$, sitque distantia $BG = \beta$ atque, uti iam vidimus, reperietur

$$G\zeta = \frac{\beta}{b} \cdot F\zeta = \frac{\alpha\beta}{ab}\zeta$$

quae imago iterum erit erecta atque a radiis axi proximis formata. Nunc etiam consideremus in spatio diffusionis dato extremitatem f, unde radii emissi cum axe faciant angulum $= \psi = \frac{x}{\alpha}$; verum antequam huius obliquitatis rationem habeamus, fingamus punctum f etiam radios axi proximos emittere, et cum id a speculo B longius sit remotum quam F, eius radii concurrent in puncto huic speculo propiore γ ad quod inveniendum referet hic $db = Ff$ et $d\beta = -G\gamma$; unde colligitur $G\gamma = \frac{\beta^2}{b^2} \cdot Ff$. Quare, si in f obiectum verum esset constitutum, eius imago principalis caderet in γ; quatenus autem ex f nulli alii radii emittuntur, nisi qui cum axe faciant angulum $= \psi$, ii denuo reflexi incident in axem in puncto g ipsi speculo B adhuc propiore quam γ, ita ut hic casus similis sit praecedenti problemati, quo punctum f respondet puncto E, punctum γ puncto F et punctum g puncto f, hoc solo discrimine, ut, quod ibi erat a et α, hic sit b et β; licebit enim utique hic pro distantia Bf sumere $BF = b$ et pro distantia $B\gamma$ sumere β; hinc ergo per formulam supra inventam, si loco x hic scribatur y, fiet

$$\gamma g = \frac{(b+\beta)(b-\beta)^2 y^2}{8b^3\beta}$$

Quid autem nunc sit y, ex angulo ψ facillime definitur. Ducto enim radio fQ sub angulo $BfQ = \psi = \frac{x}{\alpha}$ erit y semidiameter aperturac huius speculi QBQ ideoque

$$y = Bf \cdot \psi = \frac{bx}{\alpha};$$

quo valore substituto prodit

$$\gamma g = \frac{(b+\beta)(b-\beta)^2 x^2}{8\alpha^2 b\beta}.$$

Quocirca totum spatium diffusionis iam erit

$$Gg = \frac{\beta^2}{b^2} \cdot Ff + \gamma g$$

seu

$$Gg = \frac{\beta^2}{b^2} \cdot \frac{(a+\alpha)(a-\alpha)^2 x^2}{8\alpha^3 \alpha} + \frac{(b+\beta)(b-\beta)^2 x^2}{8\alpha^2 b\beta}.$$

Nunc autem post secundam reflexionem angulus, sub quo radii extremi ad axem erunt inclinati, colligitur ex lemmate 2

$$= \frac{b\psi}{\beta} = \frac{b}{\alpha\beta} \cdot x.$$

SCHOLION 1

16. Cum igitur speculum, ad quod referuntur binae distantiae a et α et cuius semidiameter aperturac est $= x$, gignat spatium diffusionis

$$Ff = \frac{(a+\alpha)(a-\alpha)^2 x^2}{8\alpha^3 \alpha},$$

comparemus hoc spatium cum eo, quod lens sub similibus circumstantiis producit, atque in primo libro (§ 49) vidimus pro tali lente esse spatium diffusionis

$$Ff = \frac{n(4n-1)\alpha\alpha xx}{8(n-1)^2 (n+2)} \left(\frac{1}{a} + \frac{1}{\alpha}\right) \left(\left(\frac{1}{a} + \frac{1}{\alpha}\right)^2 + \frac{4(n-1)^2}{(4n-1)a\alpha} \right),$$

quod quidem iam est minimum, quod a lente ad has distantias a et α relata cum apertura, cuius semidiameter est x, generari potest. Quo autem facilius hanc comparationem instituera valeamus, ponamus utrinque distantiam obiecti a esse infinitam atque e speculo nascetur spatium diffusionis

$$Ff = \frac{x^2}{8\alpha};$$

quod autem a lente nascitur, erit

$$Ff = \frac{n(4n-1)x^2}{8(n-1)^2 (n+2)\alpha},$$

ubi $n:1$ denotat rationem refractionis, et sumto $n=1,55$ hoc spatium inventum est $Ff=0,938191\cdot\frac{x^2}{\alpha}$. Unde patet a speculo multo minorem diffusionem oriri quam a lente, quandoquidem illa erit ad hanc ut $\frac{1}{8}:0,938191$, hoc est propemodum ut 1: 7,505528 seu ut $1:7\frac{1}{2}$; quae ergo proportio cum proprie in speculis vel lentibus obiectivis locum habeat, hinc praecipua causa innotescit, cur specula loco lentium obiectivarum substituta multo breviora telescopia suppeditaverint, quandoquidem ob minorem confusionem distantiam focalem minorem accipere licet, ad quod accedit, quod in his telescopiis catoptricis radii in speculum obiectivum incidentes primo ad alterum speculum reflectantur, unde denuo per eandem viam revertuntur, antequam per lentes oculares transeunt, ita ut distantia amborum speculorum bis sit computanda sicque longitudo instrumenti denuo fere ad semissem reducatur. Hoc ergo commodum specula praestarent etiam sine ullo respectu ad eorum qualitatem habito, qua radii diversorum colorum a reflexione non disperguntur, uti fit in refractione. Verum tamen hic etiam insigne speculorum incommodum non est reticendum, in eo consistens, quod speculum etiam maxime politum semper multo pauciores radios reflectat, quam per lentem eiusdem magnitudinis transmittuntur. Atque haec causa est, quod telescopia catoptrica plerumque multo minorem claritatis gradum largiantur.

SCHOLION 2

17. Quemadmodum hoc postremum problema resolvimus atque etiam diffusionem imaginis a secundo speculo natam definivimus, ita eadem investigatio ad plura specula accommodari posset, nisi ipsa rei natura speculorum usum ad binarium restringeret. Quamobrem coacti sumus radios a secundo speculo reflexos ad lentes vitreas dirigere, per quas demum ad oculum propagentur, atque ob hanc ipsam rationem ipsum speculum obiectivum circa medium perforatum esse debet, ut radiis a secundo speculo reflexis transitus per hoc foramen concedatur, ubi simul a lentibus excipiantur. Quare, cum hactenus speculum obiectivum tanquam integrum simus contemplati, nunc superest, ut etiam foraminis, quo illud est pertusum, in calculo rationem habeamus, ubi simul erit disquirendum, quomodo speculum secundum respectu huius foraminis comparatum esse debeat, ne scilicet nimiam radiorum copiam intercipiat ac tamen sufficiat omnibus radiis a primo speculo reflexis excipiendis; haecque ergo momenta in sequenti problemate accuratius perpendemus.

PROBLEMA 5

18. *Si in telescopio loco lentis obiectivac adhibeatur speculum concavum PπAπP* (Fig. 5) *in medio pertusum foramine πAπ, cuius centrum sit in axe AB, in quo ad distantiam quasi infinitam obiectum seu punctum lucidum concipiatur, ex quo radii axi paralleli in istud speculum PππP incidant indeque reflexi ad speculum minus super eodem axe normaliter positum QBQ dirigantur, unde porro ad lentem vitream prope foramen ππ itidem super*

eodem axe normaliter sitam reflectantur, determinare imagines per duplicem reflexionem formatas earumque diffusionem.

SOLUTIO

Sit semidiameter totius speculi obiectivi $AP = x$ et semidiameter foraminis $A\pi = y$, radius vero curvaturac speculi $= f$ ideoque distantia focalis $p = \frac{1}{2}f$, tum vero speculi minoris *QBQ* sit distantia focalis $= q$ et distantia horum speculorum $AB = k$. His positis, cum obiectum. in axe *AB* ad distantiam infinitam remotum concipiatur, radii inde axi paralleli ad speculum obiectivum *PP* pervenient; qui ergo ut totam eius superficiem reflectentem *Pπ* quaquaversus adimpleant, speculum *QBQ* maius esse non debet quam foramen *ππ* neque etiam id minus esse conveniet, quia alioquin radii ab obiecto directe in foramen lentemque ibi sitam ingrederentur et repraesentationem inquinarent;

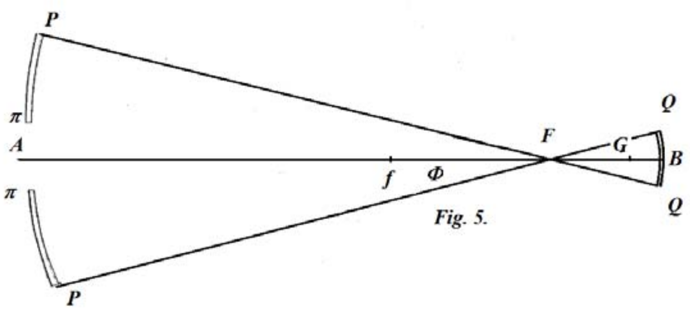


Fig. 5.

ex quo intelligitur semidiametrum aperturac huius speculi minoris esse debere $BO = y$ vel saltem eo non multo maiorem. Quoniam igitur hic distantia obiecti, quae supra posita est $= a$, nostro casu est infinita, si radii axi proximi in speculum incidere possent, iis formaretur imago principalis in *F,* ita ut esset distantia $AF = \alpha = p$. Quia autem radii axi proximi excluduntur, nulla imago principalis formabitur. Prima ergo imago a radiis circa oram foraminis reflexis formabitur in *Φ*, ita ut sit intervallum $F\Phi = \frac{y^2}{8\alpha}$, quia hic est *y,* quod supra erat *x,* et distantia obiecti $a = \infty$. Imago autem extrema a radiis circa oram speculi *PP* reflexis formetur in puncto *f* eritque intervallum $Ff = \frac{x^2}{8\alpha}$; quare, cum ipsa imago principalis hic desit, totum spatium diffusionis hic tantum erit

$$\Phi f = \frac{x^2 - y^2}{8\alpha}.$$

Interim tamen haec puncta *F, Φ, f* inter se tam erunt propinqua, ut in calculo pro eodem haberi queant. Cum ergo omnes radii a speculo maiore reflexi per punctum *F* transire sint censendi, ut in speculum *QBQ* incidant, eius semidiameter *BQ* tanta esse debet, ut sit

$$AF : AP = BF : BQ,$$

unde fit

$$BQ = \frac{k - \alpha}{\alpha} \cdot x;$$

quae cum ipsi y debeat esse aequalis, habebimus

$$y = \frac{k-\alpha}{\alpha} \cdot x \text{ hincque } k = \frac{\alpha(y+x)}{x}.$$

Sin autem minus speculum intra A et F (Fig. 6) esset constitutum, reperiretur

$$BQ = \frac{\alpha-k}{\alpha} \cdot x = y \text{ hincque } k = \frac{\alpha(x-y)}{x},$$

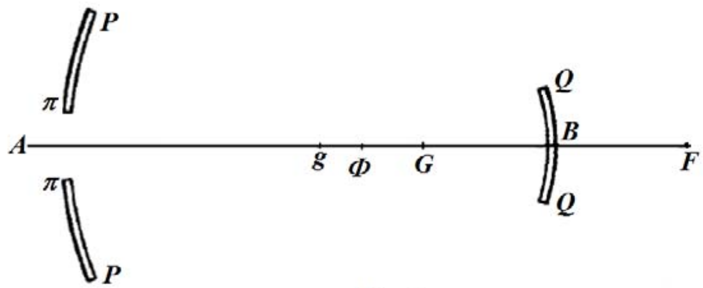


Fig. 6.

quae vero expressio in superiori contenta est censenda, propterea quod radium foraminis y tam positive quam negative capere licet. Cum igitur nunc primae imaginis F distantia a speculo secundo sit $k-\alpha = \frac{\alpha y}{x}$, quam supra vocavimus $= b$, ita ut sit $b = \frac{\alpha y}{x}$, secunda imago a speculo QBQ reflexa cadet in punctum G, ita ut sit $BG = \beta = \frac{bq}{b-q}$ ita ut radii a speculo QBQ reflexi omnes per punctum hoc G transire sint censendi, siquidem hic animum a diffusione imaginis abstrahimus. Nunc igitur insuper efficiendum est, ut isti radii omnes in ipsum foramen $\pi A\pi$ ingrediantur, id quod, cum sit $BQ = A\pi$, eveniet, si modo punctum G propius versus A cadat quam versus B, seu debebit esse $\beta > \frac{1}{2}k$. Invenimus vero

$$\beta = \frac{bq}{b-q} = \frac{\alpha yq}{\alpha y - qx} \text{ et } k = \frac{\alpha(y+x)}{x},$$

ita ut nunc esse debeat

$$\frac{\alpha yq}{\alpha y - qx} > \frac{\alpha(y+x)}{2x}, \text{ unde oritur } q > \frac{\alpha y(x+y)}{x(3y+x)};$$

ex qua ergo formula distantia focalis speculi minoris definiri poterit, quae ergo determinabitur per semidiametros foraminis et ipsius speculi $p = \alpha$ maioris una cum focali distantia speculi maioris ; sin autem speculum minus constituatur intra F et A, iam vidimus fore $AB = k = \frac{\alpha(x-y)}{x}$, et cum nunc sit distantia $b = -\frac{\alpha y}{x}$, distantia $BG = \beta = \frac{\alpha yq}{\alpha y + xq}$; quae ut maior sit quam $\frac{1}{2}k$, necesse est fiat $q > \frac{\alpha y(x-y)}{x(3y-x)}$; unde, si x sit $> 3y$, debebit esse q negativum, ita ut sit

$$q > -\frac{\alpha y(x-y)}{x(x-3y)};$$

at si esset $x = 3y$, capi posset $q = \infty$ sicque speculum minus fieret planum. Quod denique ad diffusionem imaginis secundae in *G* repraesentatae attinet, ea iterum erit quasi truncata sua imagine principali ; quod si litteris $G\Phi g$ repraesentetur ad similitudinem litterarum $F\Phi f$, totum spatium diffusionis tantum erit censendum $= \Phi g$, cuius quantitas ex formula praecedentis problematis reperietur, si loco x^2 scribatur $x^2 - y^2$; unde ob $a = \infty$ erit hic

$$\Phi g = \frac{\beta^2}{b^2}\frac{\left(x^2-y^2\right)}{8\alpha} + \frac{(b+\beta)(b-\beta)^2\left(x^2-y^2\right)}{8\alpha^2 b\beta},$$

atque nunc radiorum in Φ concurrentium obliquitas ad axem erit $= \frac{b}{\alpha\beta}\cdot y$, obliquitas vero radiorum in $g = \frac{b}{\alpha\beta}\cdot x$.

COROLLARIUM 1

19. Si ergo minus speculum ultra locum imaginis *F* collocetur, eius distantia a primo speculo debet esse

$$AB = \frac{\alpha(x+y)}{x} = \alpha + \frac{y}{x}\cdot\alpha,$$

ita ut sit $FB = \frac{\alpha y}{x}$, hocque ergo casu distantia *AB* maior erit quam distantia focalis speculi principalis ; tum vero huius secundi speculi distantia focalis esse debet

$$q > \frac{\alpha y(x+y)}{x(x+3y)}.$$

COROLLARIUM 2

20. Hic autem manifesto supponitur punctum *G* a puncto *B* versus *A* cadere, ita ut distantia β evadat positiva; si enim esset $q > b$, punctum *G* ad alteram partem speculi *QBQ* caderet radiique *GQ* producti manifesto extra foramen praetergrederentur. Quare hic pro *q* alterum limitem probe observari oportet, ut sit $q < b$ sive $q < \frac{\alpha y}{x}$, tum vero etiam $q > \frac{\alpha y(x+y)}{x(x+3y)}$.

COROLLARIUM 3

21. Sin autem speculum *QBQ* intra focum *F* collocetur, oportebit esse distantiam

$$AB = \frac{\alpha(x-y)}{x} = \alpha - \frac{y}{x}\alpha,$$

ita ut sit

$$FB = \frac{\alpha y}{x}$$

tantoque intervallo prima imago post secundum speculum cadat fiatque $b = -\frac{\alpha y}{x}$, unde deducitur distantia

$$BG = \beta = \frac{\alpha yq}{\alpha y + qx};$$

quae distantia semper est positiva seu versus A dirigitur, nisi forte q sit quantitas negativa; quae cum superare debeat $\frac{1}{2}k$, debet esse

$$2xyq > \alpha y(x-y) + x(x-y)q;$$

deberet ergo esse

$$2xy > x(x-y) \text{ seu } y > \frac{1}{3}x.$$

Quare si, ut semper in praxi evenit, sit $y < \frac{1}{3}x$, huic conditioni satisfieri nequit, si scilicet alterum speculum sit concavum.

COROLLARIUM 4

22. Hoc ergo casu necesse est, ut minus speculum sit convexum eiusque distantia focalis negativa. Statuatur ergo $q = -\mathfrak{q}$, ut fiat $\beta = \frac{-b\mathfrak{q}}{b+\mathfrak{q}}$, qui valor ob $b = -\frac{\alpha y}{x}$ abit in hunc :

$$\beta = \frac{\alpha y\mathfrak{q}}{\mathfrak{q}x - \alpha y};$$

qui valor ut primo sit positivus, debet esse

$$\mathfrak{q} > \frac{\alpha y}{x},$$

deinde, ut fiat $2\beta > k$, debet esse

$$2xy\mathfrak{q} > x(x-y)\mathfrak{q} - ay(x-y),$$

ex qua fit

$$ay(x-y) > x(x-3y)\mathfrak{q},$$

unde pro $\mathfrak{q}$ elicitur alter limes

$\mathfrak{q} < \frac{\alpha y(x-y)}{x(x-3y)}$ altera existante $\mathfrak{q} > \frac{\alpha y}{x}$.

COROLLARIUM 5

23. Sin vero praeter consuetudinem foramen tantum fiat, ut sit $3y > x$, tum speculo minori concavo uti licebit, dummodo eius distantia focalis sit $\mathfrak{q} > \frac{\alpha y(x-y)}{x(3y-x)}$, quemadmodum ex corollario 3 est manifestum, atque hoc casu, quoniam littera q nulla alia conditione restringitur, hoc speculum adeo planum fieri poterit.

SCHOLION

24. Haec duo specula ita hic sumus contemplati, quemadmodum in telescopiis GREGORIANIS usurpari solent, atque hic tantum ad obiecti punctum medium in axe tubi situm spectavimus, unde radii axi paralleli in speculum principale incidant; alterum vero speculum ita instruximus, ut omnes radios a priori reflexos recipiat eosque porro in foramen proiiciat. Cum autem etiam partes obiecti extra axem sitae visui offerri debeant, quoniam inde radii sub aliqua exigna obliquitate in speculum incidunt, tubum, in quo haec duo specula inseruntur, aliquantillum divergentem confici oporteret vel, quod eodem redit, tubum aliquanto ampliorem effici conveniet quam est diameter speculi; deinde ob eandem rationem etiam speculum minus ultra limites ipsi assignatos extendi deberet, ut etiam istos radios obliquas post reflexionem recipere posset; sed quoniam parum interest, sive extremitates obiecti pari lumine conspiciantur atque eius medium, sive minore, hac amplificatione facile eo magis carere poterimus, quod tota haec obliquitas non ultra aliquot minuta in magnis praesertim multiplicationibus excrescat. Longe aliter autem se habitura esset huius rei tractatio, si etiam specula ad axem instrumenti oblique posita in usum vocarentur, quem admodum in ipso huius inventionis principio a Neutono est factum; sed quia reflexio radiorum oblique incidentium haud exiguam gignit confusionem, hoc argumentum hic neutiquam attingimus.