CONSTRUCTION OF OBJECTIVE LENSES FROM TWO GLASSES,

which give rise neither to confusion from the spherical shapes nor to dispersion of the colors.

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DISSERTATION

composed for the Petersburg Imperial Academy of Sciences prize regarding the proposed question of the perfection of telescopes.

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1762.

1. Now initially it is to be observed that objective lenses, which generally are accustomed to be used for telescopes, work with two kinds of glasses. For in the first place on account of the spherical figure, which is produced on each face, (since for other figures, which may become more useful, the practice of figuring the glass has not yet been accomplished), another image may be formed from the extreme rays and from the middle ones, from which there becomes, that where a greater part of the aperture of the lens may be conceded, there will become a greater confusion in the image. But the other fault, where rays of different colors may not be transmitted equally by refraction through the glass, and thus they are dispersed, not only the lenses are affected, while images formed from different colors, therefore they become more separated from each other, where the distance of these from the lens were greater. And both these faults thus are closely connected with the nature of the glass and the spherical shape, so that in no manner may they be able to be separated from the lenses.

2. But here I speak about these simple lenses made from glass; but indeed if we wish to join two or more lenses together, so that they may refer as if to a single lens, now formerly I have established a way, by which each defect of the glass can be avoided separately. For I have shown with two or more lenses joined together, how the individual lenses may be required to be joined together, so that the confusion arising on account of the spherical shape may be removed completely, as also may be seen to be present in practical use. But the other fault, concerning the dispersion of colors by a number of lenses not only remains, but indeed to be increased, if indeed the lenses may be prepared from similar transparent materials.

3. Indeed then I had believed all glass to be prepared likewise with [the same] refractive index, and thus it came to mind lenses to be prepared from glass and water, or from any other clear fluid, thus so that a fluid may be enclosed between two glass lenses. And with

an examination made of the lenses found at any rate thus able to be put in place, so that the dispersion of the colors may be limited. I have done many experiments concerned with this matter, from which lenses of this kind ought to be taken to be free of dispersion, truly these to be disturbed to such an extent by the other fault, that plainly no use thence might be considered to be expected. For even if it were able for the first aim to be satisfied in an infinite number of ways, among which with some more, others less, may still be affected by the other fault, yet partially the prolixity of the calculation, that this investigation demands, has prevented me from investigating into more suitable figures of the lenses, truly I have despaired of such figures, if which I may have found, to be able to be obtained in practice as the most accurate. But I have decided to pursue this business with more diligence for another time.

4. Moreover the theory of refraction, on which I have constructed that of lenses filled with water, even if it may be founded from the most certain principles, plainly was new, and was opposed to the Newtonian theory, in which it is assumed that at no time by any way can the dispersion of colors be diminished, however many different mediums also may be called into use. Even if moreover the great man had never demonstrated this principle, but had assumed that rather precariously, lest he might become involved in more intricate investigations, which might pertain less to its being established; yet on this account I have adopted the reasoning of the adversary, the sharp-witted Englishman Dollond, who has disagreed completely with my principle, as far as concerning the discrepancy with Newton, who has considered it required to be completely overturned, which gave me the opportunity of confirming my theory by the most thorough reasoning, and for it to be freed from all doubts.

5. In so far as my principle is lacking in proof, as my defense may be seen to be displeasing to that most distinguished man, since the Newtonian principle had been investigated experimentally with all rigor. Indeed at once with two prisms joined together, with the one glass and the other water, it is not possible to sustain that view attributed to Newton, while he had observed refraction the dispersion of colors had not been removed and vice versa, so that the experiment serves wonderfully well for my theory being corroborated. (Phil.Transact. vol. 50, part II*, 1758.) Thereafter truly also he subjected various kinds of glass to be examined in a like manner, and thence from the various triangular prisms prepared he noted thus different kinds of glass to be endowed with different strengths of refraction and the dispersion of the colors not to depend on the magnitude of the refraction. From which it was concluded correctly it was certainly possible with two lenses made from different kinds of glass joined together, that the dispersion of the colors might be removed completely. Towards this end it was necessary to have made one of the two lenses convex, thus so that the given convexity and concavity will maintain a ratio determined from the difference of the refractive indices.

*Dollond was in the process of obtaining a patent for this invention, and thus was not providing information for others either by a detailed theory of the construction, or perhaps by a 'cookbook' or recipe approach gleaned from many experiments. See Hutton's *Mathematical & Philosophical Dictionary* Vol. I, p. 25 for a fuller account. Dollond's paper is attached to the end of this translation as an Appendix.

6. Since only the distance of the focus of each lens may be defined by this ratio, it is with the supposed merit that a fabrication of this kind between these two lenses be found in an infinite number of cases, which also may be freed from the other source of confusion from the spherical nature of the figures. But no calculation is provided, nor any theory, for this investigation to be used; but rather the distinguished man has investigated with an elaboration on several different lenses of this kind, two taken together, in which case the least confused images, and thus in no way may it be going to become productive. It may be declared that a construction of this kind to have elicited itself in this manner, such a composed lens may perform better freed from each defect; which without doubt is the greatest discovery, which indeed may be desired in Dioptrics, and it is adorned with the highest praise by the distinguished Short.

7. But least any of the glory may be taken from this illustrious invention, this proof, according to my habits, I will handle by theory alone, from which it may become more apparent, how experiments may agree with theory, and whether perhaps this construction of combined lenses may be able to be raised to the greatest degree of perfection. Therefore I assume to be given two kinds of glass, and I am going to enquire in the solution of this problem, *thence in what manner two lenses may be required to be made, which joined together may represent fully distant objects both without any dispersion of color, as well as without any confusion arising from the spherical shape.* Here therefore four spherical faces occur, of which the radii of the individual faces must be determined, whether they shall be either convex or concave, in order that they may satisfy this twofold condition ; from which it is apparent, even if the distance of the focus of the composite lens may be prescribed, yet three magnitudes remain requiring to be determined, and thus from only two conditions required to be satisfied, at this point there is room for an infinite number of solutions, from which henceforth that will be allowed to be eligible, which will be seen to be the most convenient in practice.

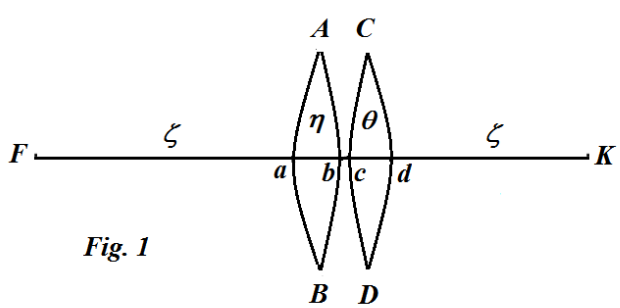

Fig. 1

8. Therefore (Fig. 1) *AB* and *CD* shall be two lenses set up for the common axis *FK*, I will consider each as convex, and of which that one *AB* may be pointed towards the object. For rays of light of the mean kind the ratio of the refraction from air into glass, from which the first lens *AB* has been made, shall be as ζ to η [*i.e.* the refractive index *n* shall be in the ratio of the small angles ζ and η or $n=\frac{\zeta}{\eta}$.] ; but the ratio of refraction from air into glass, on which the latter lens *CD* depends, shall be as ζ to ϑ, again the individual faces of the lenses shall be spherical, and indeed :

$$\text{for the first lens } AB \text{ the radius of the } \begin{cases} \text{anterior face shall be } & AaB = a \\ \text{posterior face shall be } & AbB = b \end{cases}$$

for the other lens *CD* the radius of the $\begin{cases}\text{anterior face shall be } CcD = c \\ \text{posterior face shall be } CdD = d.\end{cases}$

Moreover I put both lenses to be adjoining together, and the thickness of each as minimal, so that it may be neglected in the calculation.

9. Again the focal distance of the first lens $AB = p$, truly of the second lens the distance of the focus $= q$, and from the principles of optics there will be [the modern Lens Makers' Formula is $\frac{1}{f} = (n-1)\left(\frac{1}{a} - \frac{1}{b}\right)$, where *f* is the focal length, *n* the *r.i.* of the glass, and *a, b* the radii of curvature, convex and concave, for a thin lens] :

$$p = \frac{\eta ab}{(\zeta-\eta)(a+b)} \text{ and } q = \frac{\vartheta cd}{(\zeta-\vartheta)(c+d)}$$

or

$$\frac{1}{p} = \left(\frac{\zeta}{\eta} - 1\right)\left(\frac{1}{a} + \frac{1}{b}\right) \text{ and } \frac{1}{q} = \left(\frac{\zeta}{\vartheta} - 1\right)\left(\frac{1}{c} + \frac{1}{d}\right),$$

which values indeed prevail for rays of light of the average nature [*i.e.* middle of the spectrum in color], and only with these which are crossing near the axis. Now so that if the distance of the object before the lens *AB* were $= f$,the image lies at the distance *g,* so that there shall be

$$g = \frac{fp}{f-p}, \text{ or } \frac{1}{g} = \frac{1}{p} - \frac{1}{f},$$

and thus with the value restored for *p,* there will be

$$\frac{1}{f} + \frac{1}{g} = \left(\frac{\zeta}{\eta} - 1\right)\left(\frac{1}{a} + \frac{1}{b}\right).$$

In a similar manner if the distance of the object before the other lens *CD* shall be $= h$, that image may be found after that at the distance $= k$, so that there shall be

$$\frac{1}{h} + \frac{1}{k} = \left(\frac{\zeta}{\vartheta} - 1\right)\left(\frac{1}{c} + \frac{1}{d}\right).$$

10. Therefore with these lenses joined together, and with the distance of the object remaining as before $= f$, since the image from that projected in turn carries on to become the object with respect to the second lens *CD,* there will be $h = -g$, and hence the image by the double lens falls at the distance *k,* so that there shall be

$$\frac{1}{f} + \frac{1}{k} = \left(\frac{\zeta}{\eta} - 1\right)\left(\frac{1}{a} + \frac{1}{b}\right) + \left(\frac{\zeta}{\vartheta} - 1\right)\left(\frac{1}{c} + \frac{1}{d}\right)$$

or

$$\frac{1}{f}+\frac{1}{k}=\frac{1}{p}+\frac{1}{q} ;$$

thus so that, with the distance f assumed infinite, the focal distance of the double lens shall be

$$k=\frac{pq}{p+q}.$$

Which if it were given, it is apparent not only the focal lengths of each lens p and q are able to be varied in an infinite number of ways, thus so that, with the one given, the other may be able to be defined, but also each lens with its focal length given can take an infinitude of variations with regard to the faces, thus so that, with the one face given, the other always may be allowed to be determined.

11. With these it may be observed generally in the first place we may satisfy this condition, so that no dispersion of the color may arise, or the image distance k of the double lens may not be allowed any variation, even if the ratios of refraction $\zeta:\eta$ and $\zeta:\vartheta$ on account of the nature of the rays of light may have been able to be changed. But since this change shall be as if infinitely small, it will be able to satisfy this condition by differentiation. Therefore we may put for the mean nature of the rays:

$$\frac{\zeta}{\eta}=m \text{ and } \frac{\zeta}{\vartheta}=n$$

thus so that we may have this equation:

$$\frac{1}{f}+\frac{1}{k}=(m-1)\left(\frac{1}{a}+\frac{1}{b}\right)+(n-1)\left(\frac{1}{c}+\frac{1}{d}\right),$$

in which the letters a, b, c, d, f and k are required to be considered as constant magnitudes, truly the letters m and n as variables, thus so that this equation thence may continue, even if that may be differentiated. Truly by differentiation there will be produced:

$$dm\left(\frac{1}{a}+\frac{1}{b}\right)+dn\left(\frac{1}{c}+\frac{1}{d}\right)=0.$$

12. Therefore since the focal lengths p and q of each lens are required to be introduced, on account of

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{(m-1)p} \text{ and } \frac{1}{c}+\frac{1}{d}=\frac{1}{(n-1)q},$$

there shall be

$$\frac{dm}{(m-1)p}+\frac{dn}{(n-1)q}=0 \text{ or } q=-\frac{p(m-1)dn}{(n-1)dm},$$

hence the ratio between the focal lengths of each lens p and q is determined, so that diverse kinds of rays may bear no color dispersion, but the images may be united from light rays of all kinds. And if such a ratio may be established between the focal lengths, so that there shall be

$$q=-\frac{(m-1)dn}{(n-1)dm}p,$$

there will be for rays of all kinds

$$\frac{1}{f}+\frac{1}{k}=\frac{1}{p}+\frac{1}{q},$$

nor is this determination restricted to infinitely distant objects, for whatever were the distance of the object *f,* the image may be discerned to be at the distance $=k$, without any dispersion of the color after the double lens.

13. But if both lenses might be made from the same kind of glass, so that there shall be $n=m$, and therefore $dn=dm$, there may become $q=-p$, and the focal length for the composite lens will emerge to be infinite, which is the ratio, since from glass of the same nature, also however many lenses may be used, the dispersion of the color in no manner may be able to be destroyed. But with two kinds of glass allowed, so that the numbers *m* and *n* shall be unequal, so far only the dispersion of colors can be removed, as long as there is not

$$\frac{(m-1)dn}{(n-1)dm}=1,$$

since then the above inconvenience will recur. Whereby if there might be

$$dm:dn=m-1:n-1,$$

as Newton is regarded to have ascribed, certainly lenses equally composite and small simple lenses might be able to be freed from this fault. But since this belief not only shall be overthrown by the strongest of arguments by me, but also refuted sufficiently by the experiments the Cel. Dollond, the destruction of color on account of this ratio is required to be considered.

14. Because dm and dn express the variation of the refraction of the dissimilar rays from the mean nature, thus truly the ratio is had, so that there shall be

$$dm:dn=mlm:nln,$$

just as I have shown sufficiently in another place [See E118, § 32] . Therefore from this established account from the difference of the glass the ratio of the focal distances *p:q* thus will be had, so that there shall be

$$q=-\frac{(m-1)nln}{(n-1)mlm}p.$$

Therefore we may put in place for the sake of brevity:

$$\frac{(m-1)nln}{(n-1)mlm}=\lambda,$$

so that there shall be $q=-\lambda p$, nor may λ be equal to unity, the focal length of the composite lens will be

$$k=-\tfrac{\lambda pp}{p-\lambda p} \text{ or } k=\tfrac{\lambda p}{\lambda-1}.$$

Therefore with the given focal distance k, the focal length if each simple lens must be defined thus, so that there shall be

$$p=\tfrac{\lambda-1}{\lambda}k \text{ and } q=-(\lambda-1)k.$$

15. Therefore if the number λ were greater than unity, the first lens *AB* will be convex, or the focal distance will be taken positive, truly the posterior *CD* concave, or the focal length will be taken negative ; but if λ shall be a fraction less than unity, the opposite will happen. But so that the nature of the number λ may be understood more easily, since the numbers m and n do not exceed unity by much, there will be by the series:

$$lm=l(1+m-1)=m-1-\tfrac{1}{2}(m-1)^2+\tfrac{1}{3}(m-1)^3-\tfrac{1}{4}(m-1)^4+\text{etc.}$$
$$ln=l(1+n-1)=\ n-1-\tfrac{1}{2}(n-1)^2+\tfrac{1}{3}(n-1)^3-\tfrac{1}{4}(n-1)^4+\text{etc.}$$

and hence the number λ may be defined thus :

$$\lambda=\frac{n\left(1-\frac{1}{2}(n-1)+\frac{1}{3}(n-1)^2-\frac{1}{4}(n-1)^3+\text{etc.}\right)}{m\left(1-\frac{1}{2}(m-1)+\frac{1}{3}(m-1)^2-\frac{1}{4}(m-1)^3+\text{etc.}\right)}$$

or by writing

$$m=1+m-1 \text{ and } n=1+n-1$$

there will become

$$\lambda=\frac{1+\frac{1}{2}(n-1)-\frac{1}{6}(n-1)^2+\frac{1}{12}(n-1)^3-\text{etc.}}{1+\frac{1}{2}(m-1)-\frac{1}{6}(m-1)^2+\frac{1}{12}(m-1)^3-\text{etc.}}$$

from which it is evident, if $n>m$, there becomes $\lambda>1$, but if $n<m$, there becomes $\lambda<1$.

16. Moreover with these focal distances p and q found, the faces of each lens allow an infinitude of ways of determining these, so that by which it may appear easier, we may introduce two new indeterminate numbers μ and ν, and since there must become

$$\tfrac{1}{a}+\tfrac{1}{b}=\tfrac{1}{(m-1)p} \text{ and } \tfrac{1}{c}+\tfrac{1}{d}=\tfrac{1}{(n-1)q}$$

we may put:

$$a=\tfrac{m-1}{\mu}p,\quad b=\tfrac{m-1}{1-\mu}p,\quad c=\tfrac{n-1}{\nu}q,\quad d=\tfrac{n-1}{1-\nu}q.$$

For just as these numbers μ and ν may be taken, the lenses always will obtain the prescribed focal lengths, and on taking $p=\frac{\lambda-1}{\lambda}k$ and $q=-(\lambda-1)k$, the two adjoining lenses not only will have the focal lengths prescribed by k, but also the images of any objects will be described without any color dispersion. Therefore it remains, so that we may define the numbers μ and ν thus, so that also a lens composed from another glass, evidently may be free from any confusion arising from the spherical shape of the faces of the lenses.

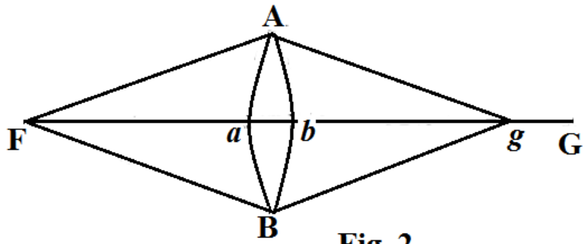

Fig. 2

17. Therefore at first it will be required to show clearly, how the image may be formed by the extreme rays, may differ from the image, which may be formed by the rays passing through the middle of the lens. Therefore *AB* shall be the proposed lens (Fig. 2) lens, of which the radius of the face *AaB* shall be $=a$, truly of the face *AbB*, $=b$; as I consider each as convex, moreover the ratio of refraction from air to glass, from which this lens is composed, shall be $\zeta:\eta$ or $m:1$, on putting $\frac{\zeta}{\eta}=m$. Hence its focal length will be

$$p=\frac{ab}{(m-1)(a+b)},$$

$$\frac{1}{p}=(m-1)\left(\frac{1}{a}+\frac{1}{b}\right),$$

from which as before we may put

$$a=\frac{m-1}{\mu}p \text{ and } b=\frac{m-1}{1-\mu}p.$$

Now if the distance of the object F from this lens shall be $aF=f$, for paraxial rays the image will be produced at G, so that the distance $bG=\frac{fp}{f-p}$; but a closer image may be manifest at g by the extreme rays FA, FB, and the small space Gg is accustomed to be called the interval of confusion, which therefore it will be required to investigate carefully.

18. Let x denote the radius of the aperture, and with the calculation not a little tedious, which therefore I do not wish to repeat here, that diffusion space is found to be expressed thus:

$$\frac{\begin{array}{c}((m+2)\mu\mu-m(2m+1)\mu+m^3)ff\\+(m-1)\big(4(m+1)\mu-m(3m+1)\big)fp\\+(m-1)^2(3m+2)pp\end{array}}{2m(m-1)^2(f-p)^2}\cdot\frac{xx}{p}=Gg.$$

From which it is apparent, if the distance f of the object were infinite, this diffusion space to become :

$$Gg = \frac{(m+2)\mu\mu - m(2m+1)\mu + m^3}{2m(m-1)^2} \cdot \frac{xx}{p},$$

which cannot vanish, whatever value may be given to μ. But the minimum emerges, if there may be taken $\mu = \frac{m(2m+1)}{2(m+2)}$; in which case, where the diffusion of the image in focus shall be a minimum, Huygens has now elicited.

19. We may return now to our double lens described before, and we may put the distance of the object to be infinite, and there will be (Fig. 3) for the rays passing through the middle of the lens the image distance G, from the projection through the first lens

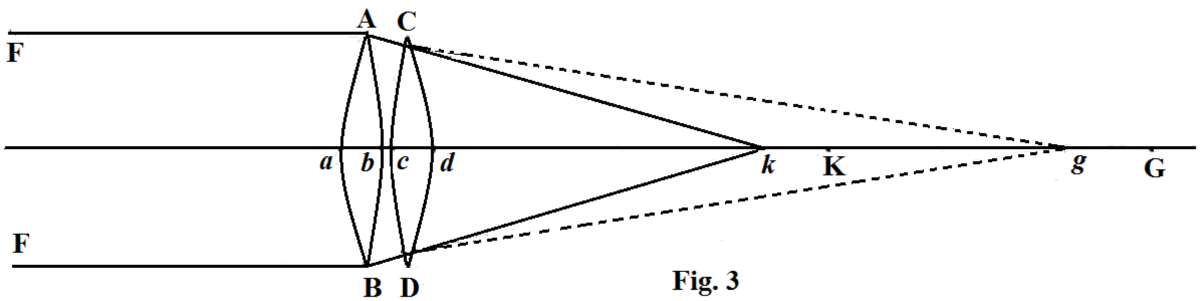

Fig. 3

$dG = p$, and projected through both lenses $dK = \frac{pq}{p+q}$. Truly for the extreme rays, with the radius of the aperture put $= x$, through the first lens alone the image will be at g, so that there shall be $dg = p - M \cdot \frac{xx}{p}$ with there being

$$M = \frac{(m+2)\mu\mu - m(2m+1)\mu + m^3}{2m(m-1)^2},$$

which image may maintain the position of the object for the accounting of the posterior lens, and this distance may be put negative on taking $-p + M \cdot \frac{xx}{p} = g$; and the distance of the image projected through the second lens to k will be $dk = \frac{gq}{q-g} - N \cdot \frac{xx}{q}$, if indeed there may be put:

$$N = \frac{\left((n+2)vv - n(2n+1)v + n^3\right)gg + (n-1)\left(4(n+1)v - n(3n+1)\right)gq + (n-1)^2(3n+2)qq}{2n(n-1)^2(g-q)^2}.$$

20. Now since there shall be $g = -p + M \cdot \frac{xx}{p}$, there will become

$$\frac{gq}{g-q} = \frac{-q\left(p - M \cdot \frac{xx}{p}\right)}{-p - q + M \cdot \frac{xx}{p}},$$

where since the small part $\frac{xx}{p}$ may be allowed to be considered as very small, it will be precise enough to put

$$\frac{gq}{g-q} = \frac{pq}{p+q} - \frac{qq}{(p+q)^2} M \cdot \frac{xx}{p}.$$

On account of which the image shown through the double lens from the extreme rays falls into *k,* so that there shall be:

$$dk = \frac{pq}{p+q} - \frac{qq}{(p+q)^2} M \cdot \frac{xx}{p} - N \cdot \frac{xx}{q},$$

but in the value of the number *N,* because it affects the small value $\frac{xx}{q}$, in place of g it is allowed to write the approximate value $-p,$ thus so that there shall become :

$$N = \frac{((n+2)vv - n(2n+1)v + n^3)pp - (n-1)(4(n+1)v - n(3n+1))pq + (n-1)^2(3n+2)qq}{2n(n-1)^2(p+q)^2}.$$

and the diffusion space is

$$Kk = (\frac{qq}{p(p+q)^2} M + \frac{1}{q} N)\, xx.$$

21. Therefore the whole affair has been reduced thus, so that this diffusion space may be returned towards nothing, or so that there may become :

$$Mq^3 + Np(p+q)^2 = 0,$$

which equation multiplied by $2n(n-1)^2$ produces:

$$\left.\begin{aligned} &\tfrac{n(n-1)^2}{m(m-1)^2}((m+2)\mu\mu - m(2m+1)\mu + m^3)q^3 \\ &+\left((n+2)vv - n(2n+1)v + n^3\right)p^3 \\ &-(n-1)\left(4(n+1)v - n(3n+1)\right)ppq \\ &+(n-1)^2(3n+2)pqq \end{aligned}\right\} = 0$$

Therefore since the first condition gives $q = -\lambda p$, with there being

$$\lambda = \frac{(m-1)nln}{(n-1)mlm},$$

we will have:

$$\frac{n(n-1)^2}{m(m-1)^2}\lambda^3\left((m+2)\mu\mu-m(2m+1)\mu+m^3\right)$$
$$=(n+2)\nu\nu-n(2n+1)\nu+n^3+\lambda(n-1)(4(n+1)\nu-n(3n+1)$$
$$+\lambda\lambda(n-1)^2(3n+2),$$

from which equation without doubt the two numbers μ and ν really are able to be defined in an infinite number of ways.

22. Since this equation shall be exceedingly complex, we may reduce that to a simpler form, by putting:

$$\mu=\frac{m(2m+1)+my}{2(m+2)} \quad \text{and} \quad \nu=\frac{n(2n+1)-4\lambda(nn-1)+nz}{2(n+2)}$$

and with the substitution made it will come to this equation:

$$\frac{m(n-1)^2(n+2)}{n(m-1)^2(m+2)}\lambda^3(yy+4m-1)yy=zz-4\lambda(\lambda-1)(n-1)^2+4n-1.$$

For the sake of brevity there may be put:

$$\frac{m(n-1)^2(n+2)}{n(m-1)^2(m+2)}\lambda^3=A,$$

which number generally will differ little from unity, and there will become :

$$zz=A(4m-1)-(4n-1)+4\lambda(\lambda-1)(n-1)^2+Ayy;$$

therefore unless there were $4n-1>A(4m-1)$, all numbers may be taken for y, and from the individual values the number z can be defined ; but if $4n-1>A(4m-1)$, only smaller values of y are excluded.

23. But for practical use at first it is required to make sure so that the numbers μ and ν may be contained between the limits 0 and 1, or at least these may not transgress much, that which generally will happen, when it will have been allowed to give the minimum values to the numbers y and z. Then truly it is required to observe also, whatever the numbers may be satisfying for y and z, these can be taken equally positive and negative, from which the squares of the numbers for μ and ν will be obtained, and thus the number of solutions may be increased much more. Therefore provided that the two kinds of glass may have notably different ratios of refraction, it will be easy to show the constructions of an infinite number of double lenses of this kind, which may be without each fault, by which ordinary lenses are affected, and from those ones these, which may be seen to be the most fitting in practice, it may be agreed to select, and thus there will be no doubt, why perfect lenses of this kind may not be made and may be able to brought into use.

24. The most illustrious Dollond indeed has not defined the ratio of refraction for each kind of glass, which is usual, nor therefore will it be allowed for me to recall for the calculation the double lenses made there. But from the formulas themselves of the calculation it is evident each law of refraction, containing the numbers m and n, must be known most exactly, since with the smallest error a huge difference may be allowed to be introduced in the making of the lenses. Deliberating therefore for any case offered to form some hypothesis for the numbers m and n, which may depart from the truth in each part, so that it may be apparent, thence how great a difference may arise in the construction of the lens. Therefore initially it will be necessary, for the calculation to be performed, to put several experiments in place, and thus with the aid of the theory we may reach the chosen end. Why not also deservedly to hope in this manner not only the usual lenses will be allowed to be improved much more, but also able plainly to reach the highest degree of perfection, in which matter neither the labour nor the cost is seen to be taken into account.

25. Therefore since it may not yet be allowed to be applied to certain kind so glass, I will fashion two kinds of glass [available] proposed to me, the one denser, for which the ratio of the refraction of the rays entering from the air shall be 31:20, truly the other kind shall be a little less dense, thus so that the ratio of refraction shall be as 3:2. The kind of this glass, which is commonly accustomed to be used for dioptric instruments, truly perhaps may contain this cheaper glass. But even if the kinds may not be given, yet the setting out of this case may be shown clearly enough by us, how two lenses, prepared from denser and rarer glass, may be formed, in order that they show the desired effect. Moreover it will be appropriate to set out this investigation in two ways, according as the first lens may be made either from the rarer or denser glass; therefore initially I may inquire into the construction of perfect lenses, when there is put $m=\frac{3}{2}$ and $n=\frac{31}{20}$; then truly vice versa I may put $m=\frac{31}{20}$ and $n=\frac{3}{2}$; therefore for each case there will not be much disagreement.

I. The case, in which $m=\frac{\zeta}{\eta}=\frac{3}{2}$ and $n=\frac{\zeta}{\vartheta}=\frac{31}{20}$.

26. Therefore here initially for the individual radii of the faces we have :

$$a=\frac{1}{2\mu}p,\quad b=\frac{1}{2(1-\mu)}p,\quad c=\frac{11}{20\nu}q,\quad d=\frac{11}{20(1-\nu)}q,$$

then truly the number is sought $\lambda=\frac{31ln}{33lm}$. Truly there is :

$$\begin{array}{l} ln = 0{,}1903317, \quad lln = 9{,}2795111, \quad llm = 9{,}2457380 \\ lm = 0{,}1760913, \quad l31 = \underline{1{,}4913617}, \quad l33 = \underline{1{,}5185139} \\ \qquad\qquad\qquad\qquad 0{,}7708728 \qquad\quad 0{,}7642519 \; 0{,}7642519 \\ \qquad\qquad\qquad\qquad \underline{0{,}7642519} \\ \qquad\qquad\quad l\lambda = 0{,}0066209. \end{array}$$

Therefore $\lambda = 1{,}015315$ and $q = -1{,}015315p$; then truly if the focal length of the composite lens must be $= k$, there will be :

$$p = 0{,}015083k \text{ and } q = -0{,}015315k,$$

and thus the prior lens will be convex, truly the posterior concave.

Again there is put $A = \frac{25773}{21700}\lambda^3$ $\quad l\lambda^3 = 0{,}0198627$

$$\begin{array}{rl} l25773 = & \underline{4{,}4111650} \\ & 4{,}4310277 \\ l21700 = & \underline{4{,}3364597} \\ lA = & 0{,}0945680 \\ \text{hence } A = & 1{,}24328 \end{array}$$

and thus the equation required to be resolved

$$zz = 5A - 5{,}2 + 0{,}018815 + Ayy,$$

or

$$zz = 1{,}035215 + 1{,}24328yy.$$

27. Moreover the numbers μ and ν thus depend on the numbers y and z, so that there shall be :

$$\mu = \frac{12+3y}{14} \text{ and } \nu = \frac{132{,}24+310z}{1420}.$$

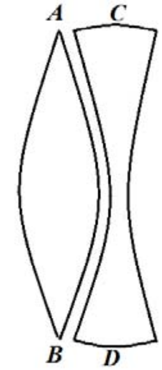

Fig. 4

These values are able to be produced almost to the mean of the limits 0 and 1, by taking $y = -\frac{5}{3}$, from which there becomes exactly $\mu = \frac{1}{2}$; then truly there will be:

$$zz = 1{,}035215 + 3{,}453555 = 4{,}488770$$

and

$$z = 2{,}11867, \text{ and hence } \nu = \frac{789{,}0277}{1420} = 0{,}55565;$$

from which we deduce

$$a = p, \quad b = p, \quad c = \frac{0{,}55}{0{,}5556}q, \quad d = \frac{0{,}55}{0{,}4444} = q.$$

Now since there shall be

$$p = 0,015083k \text{ and } q = -0,015315k,$$

thus the radii of the faces of the lenses themselves will be had :

Fig. 5

$$a = 0,015083k, \quad c = -0,015159k,$$
$$b = 0,015083k, \quad d = -0,018956k.$$

In fig. 4 and 5, I show such a double lens for the case $k = 10\ 000$ scruples of feet, or 10 feet. Indeed the former is extended to greater arcs, which are scarcely admitted able to be seen ; the latter does not involve exceedingly great arcs, but truly also of sufficient apertures, such as the focal length of 10 feet demands, which is not large. So that truly if lenses of this kind may be constructed for greater focal lengths, a large enough aperture will be required to be taken.

II. Case, where $m = \frac{31}{20}$ *and* $n = \frac{3}{2}$

28. Here at once there shall be for the radii of the faces :

$$a = \frac{11}{20\mu} p, \quad b = \frac{11}{20(1-\mu)} p, \quad c = \frac{1}{2\nu} q, \quad d = \frac{1}{2(1-\nu)} q,$$

truly the value of the number λ shall become the reciprocal of the preceding, evidently :

$$\lambda = 0,984685 \text{ and } l\lambda = 9,9933791$$

and hence

$$p = -0,015315k \text{ and } q = 0,015083k,$$

where the ratio between the convexity and concavity is the same as before. So that also the value of A is the reciprocal of the preceding, and thus A = 0,804326, from which this equation is obtained :

$$zz = 4,182495 - 5 - 0,014855 + 0,804326yy$$

or

$$zz = -0,832360 + 0,804326yy \ ;$$

then truly there becomes

$$\mu = \frac{1271+310y}{1420} \text{ and } \nu = \frac{2,15312+3z}{14};$$

if there may be taken

$$y = \tfrac{3}{2}, \text{ there may be taken } zz = 0,977373 \text{ and } z = \pm 0,98860,$$

but if

$$y = 2, \text{ there is found } zz = 2,384944 \text{ and } z = \pm 1,54432.$$

29. It is evident in the latter case, if z may take a positive value and of y truly a negative value, to reduce the numbers μ and ν approximately to $\frac{1}{2}$; therefore truly :

$$\mu = \tfrac{651}{1420} = 0,45845 \text{ and } \nu = \tfrac{6,78608}{14} = 0,48472,$$

and hence

$$a = \tfrac{0,55}{0,45845}p, \quad b = \tfrac{0,55}{0,54155}p, \quad c = \tfrac{q}{0,96944}, \quad d = \tfrac{q}{1,03056},$$

which values are expressed for the focal length of the composite lens k :

$$a = -0,018373k, \quad c = +0,015558k$$
$$b = -0,015554k, \quad d = +0,014636k.$$

Therefore this lens scarcely differs from the preceding, if it may inverted, and the concave lens may be directed towards the object. Yet meanwhile the calculation shows that such an inversion cannot always be found; and from which in this case the inversion prevails only approximately, and it is required to be considered to happen fortuitously. But it is required to be observed in practice to be maintained for the measures elicited with the greatest zeal ; if indeed we have departed even a little from these, the lens thence constructed may easily collect huge faults.

30. But the number λ is required to be observed most accurately, since the slightest aberration may scarcely diminish the dispersion of the colors, which can be shown in this manner: The simple lens of focal length $= k$ may be considered , and on account of the different refrangibility of the rays the focus may be differentiated by the small distance dk, so that there shall be

$$-\tfrac{dk}{kk} = \tfrac{dm}{(m-1)k}$$

[From § 9 : for the single lens, $\frac{1}{h} + \frac{1}{k} = \left(\frac{\zeta}{\vartheta} - 1\right)\left(\frac{1}{c} + \frac{1}{d}\right)$, on making the object distance h infinite, then $\frac{1}{k} = (m-1)\left(\frac{1}{c} + \frac{1}{d}\right)$; whence $-\frac{dk}{kk} = dm\left(\frac{1}{c} + \frac{1}{d}\right) = \frac{dm}{(m-1)k}$.]

Now truly for our composite lens the small diffusion length dk is expressed thus, so that there shall be

$$-\tfrac{dk}{kk} = \tfrac{dm}{(m-1)p} + \tfrac{dn}{(n-1)q},$$

which we may put, not to be returned to zero, but must be equal to the fraction $\frac{1}{\alpha}$ of the small distance $\frac{dm}{(m-1)k}$; so that there shall be on account of $\frac{1}{k} = \frac{1}{p} + \frac{1}{q}$:

$$\tfrac{dm}{(m-1)p} + \tfrac{dn}{(n-1)q} = \tfrac{dm}{\alpha(m-1)p} + \tfrac{dm}{\alpha(m-1)q},$$

from which there becomes

$$\frac{q}{p}=-\frac{\alpha(m-1)dn}{(\alpha-1)(n-1)dm}+\frac{1}{(\alpha-1)}=-\frac{\alpha\lambda}{\alpha-1}+\frac{1}{\alpha-1}$$

and thus

$$q=-\left(\lambda+\tfrac{\lambda-1}{\alpha-1}\right)p.$$

Therefore provided the dispersion of the colors shall be required to be diminished moderately, there becomes approximately

$$q=-\lambda p\,;$$

from which it is understood to be minimally reduced from the true value of λ.

31. Since in simple lenses, which must have a huge focal length, the magnitudes of the spheres, the faces of these requiring to be determined, usually to be made in practice with the greatest difficulty, here conveniently it arises in use, that also for the greatest focal lengths the matters may be resolved by exceedingly small spheres; just as if the focal distance k must be 100 feet, from the first case the radii of the faces thus will be had :

$$a=1{,}5083\text{ ft.}\quad c=-1{,}5159\text{ ft.}$$
$$b=1{,}5083\text{ ft.}\quad d=-1{,}8956\text{ ft.}$$

and thus there is no need of a sphere, the radius of which may exceed two feet ; and the lens readily admits an aperture of 5 inches, however great a focal length of 100 ft. may require, an aperture of 10 inches also will be allowed, from which the tube constructed can be equally enlarged, and commonly of 400 ft. focal length, which in addition shall be free from dispersion of the rays. Truly for small telescopes of this kind composite lenses present no outstanding use, since they are incapable of the due apertures, just as now I have observed for focal lengths of ten feet, where the apertures scarcely can exceed one inch.

32. The account of this inconvenience clearly lies in this, because the two kinds of glass may differ by an exceedingly small refractive index; if another transparent material may be given, differing much more from the ratio of the glass, there is no doubt, why the duplicate goal may not be reached much more easily. Indeed water is endowed with a much smaller grade of refractive, but on account of its fluid nature it breaks the figures of the lens ; but enclosed in a glass envelope the refraction of the glass to be added, and disturbs the situation. Truly this inconvenience can be removed, if the envelope shall be the same thickness everywhere, or may appear in the form of meniscus curves, in which the radius of convexity shall be exactly equal to the radius of the concavity, and likewise the thickness shall be a minimum. Therefore such lenses will be allowed to be seen as water, and it is worthwhile to set out the construction of lens composed of this kind, where one shall be glass, and the other water.

III. The case, where the lens AB is water and the lens GD glass.

33. Therefore for the first lens there is $m=\frac{4}{3}$ and for the posterior $n=\frac{31}{20}$, hence there becomes

$$a=\tfrac{1}{3\mu}p,\quad b=\tfrac{1}{3(1-\mu)}p,\quad c=\tfrac{11}{20\nu}q,\quad d=\tfrac{11}{20(1-\nu)}q,$$

then truly

$$\lambda=\tfrac{31ln}{44lm}=1,073304 \text{ and } l\lambda=0,0307228 \text{ and } A=\tfrac{77319}{31000}\lambda^3=3,08385,$$

and

$$p=+0,06830k,\quad q=-0,07330k$$

then

$$\mu=\tfrac{11+3y}{15} \text{ and } \nu=\tfrac{6,6753+31z}{142}.$$

Now truly it is required to define the numbers y and z from this equation :

$$zz=8,25855+3,08385yy$$

Thus we may take y, so that there may become $\mu=\frac{1}{2}$, and there will be $y=-\frac{7}{6}$, and

$$zz=12,45601, \text{ and thus } z=\pm3,52930.$$

The value may be taken positive, and there will become $\nu=0,81749$, and hence

$$a=b=\tfrac{2}{3}p,\ c=\tfrac{0,55}{0,81749}q \text{ and } d=\tfrac{0,55}{0,18251}q,$$

from which the radii of the faces thus themselves will be had :

$$a=0,04553k,\quad c=-0,049315k,$$
$$b=0,04553k,\quad d=-0,220892k,$$

if $y=-\frac{1}{2}$, there is found $\mu=\frac{19}{30}$, $\nu=0,70301$,

$$a=0,035947,\quad c=-0,057346,$$
$$b=0,062091,\quad d=-0,135745,$$

if $y=-\frac{7}{8}$, there becomes $b=-c$ [approx.], which case is to be observed.

34. In figure opposite (Fig. 6) I have taken $k=5$ ft. or 50 dig. of the Rhenish measure, thus so that in digits there shall be : [We may assume 10 digits or fingers is equal to a Rhenish foot.]

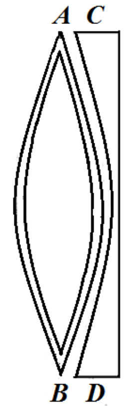

Fig. 6

for the lens *AB* the radius of each face of the convex face $=-\ 2,276$ dig.
but for the lens *CD* the radius of the anterior face $=-\ 2,466$ dig.
and for the radius *CD* of the posterior face $=-11,045$ dig.,

where I have represented the anterior water lens with the glass envelope included, which consists of two meniscus faces, of which both the convex as well as the concave radius shall be 2,276 dig. Therefore a composite lens arises from joined together, having a focal length of 5 ft, which if it may admit an aperture of 2 dig., may be able to be used to multiply sixty times. But if the measures assigned may be doubled, so that the composite lens may follow to have a focal length of 10 ft., certainly that admits an aperture more than 3 digits, and thus the object can be multiplied more than a hundred times.

35. Here with the ratio of refraction put to be 1,55:1, I have assumed for the ratio of refraction from air into water to be $=1,33:1$, which if perhaps it may depart from the truth, the construction of the composite lens described here will respond minimally with that desired, if indeed, as we have seen, it can fail all our expectations for the minimum aberration. But it may be seen that a remedy may be able to be brought for this inconvenience, if a mixture may be prepared thus from water and spirit of wine [brandy], so that the difference of the refraction may be agreed more accurately with that, which was assumed in the calculation. Therefore since the ratio of refraction from air into spirit of wine shall be almost $=1,37:1$, we may put the mixture of become of this kind, in which the ratio of refraction shall be 1,35:1, which certainly will be able to be obtained, and with the refraction of the glass retained $=1,55:1$, we may seek the determinations of each lens for this case, where certainly we may put the anterior lens to be made from water, truly the posterior to be made from glass.

IV. Case, where m = 1,35 *and n* = 1,55

36. Therefore there will be

$$a=\frac{0,35}{\mu}p,\quad b=\frac{0,35}{1-\mu}p,\quad c=\frac{0,55}{\nu}q,\quad d=\frac{0,55}{1-\nu}q,$$

then

$$\lambda=\frac{(m-1)nln}{(n-1)mlm}=1,06698 \quad \text{et} \quad l\lambda=0,0281572$$

and

$$p=\left(1-\tfrac{1}{\lambda}\right)k=0,06278k,\quad q=-\left(\lambda-1\right)k=-\ 0,06698k.$$

Again,

$$A=\frac{m(n-1)^2(n+2)}{n(m-1)^2(m+2)}\lambda^3=2,76215.$$

Now we may put $\mu=\frac{1}{2}$, and there becomes $my=2-2mm,$ hence $y=-1,218518$ and the equation, from which z must be defined, is

$$zz=12,15346-5,2+0,08648+4,11065$$

or

$$zz=11,15059 \text{ and } z=3,33925,$$

from which

$$v=\frac{n(2n+1)+4\lambda(nn-1)+nz}{2(n+2)}=0,78099.$$

Therefore

$$a=b=0,7p,\quad c=\frac{0,55q}{0,78099},\quad d=\frac{0,55q}{0,21901},$$

from which the construction of the lens itself will be had thus :

$$a=0,043946k,\quad c=-0,04717k,$$
$$b=0,043946k,\quad d=-0,16807k.$$

37. The form of this lens differs a little from the preceding, and so that the focal distance k may become 50 dig., it will be necessary to take the convex radius of each face of the water lens AB

$$=2,1973 \text{ dig.},$$

but for the other glass lens CD there must be taken

$$\text{radius of the }\begin{cases}\text{anterior face } =-\ 2,3585 \text{ dig.}\\ \text{posterior face} =-8,4035 \text{ dig.}\end{cases}$$

with each face made concave. But if such a lens were made with all care, then various mixtures from water and spirit of wine may taken pains over following different proportions and it may be investigated experimentally, which of these with the two meniscus envelopes included produce the optimum effect : and this method is seen of obtaining the most suitable absolute lens most conveniently from all these numbers and to be most suitable in practice. Indeed it is agreed not to assume the focal distance to be exceedingly small, since such a lens may not allow a sufficient aperture, and for this reason is seen not to be taken less than 5 feet; but where a greater may be put in place, there more intense will be the light gained before simple lenses of the same focal length.

38. We may gather from these cases presented, if two transparent materials of different natures may be at hand, and from that the anterior lens or the one looking towards the object may be constructed, which uses the smaller refraction, that each side can become

equally convex ; but then the other must be made from a more refracting material, certainly each side concave, but unequally. Therefore it will be worth the trouble, if the radius of each face of the anterior lens *AB* may be given, set out to be seen, how the radii of each face of the other concave lens may be defined. Therefore since in the cases set out the ratio of the refraction shall be $n=\frac{31}{20}$ for the posterior lens, on putting $a=b=1$, the radii c and d will themselves be had thus :

$$\begin{matrix} a=1 \\ b=1 \end{matrix}\left|\begin{matrix} \text{if } m=1,55 \\ c=-1,0000 \\ d=-1,0000 \end{matrix}\right|\begin{matrix} \text{if } m=1,50 \\ c=-1,00506 \\ d=-1,25680 \end{matrix}\left|\begin{matrix} \text{if } m=1,35 \\ c=-1,07336 \\ d=-3,82758 \end{matrix}\right|\begin{matrix} \text{if } m=1,33 \\ c=-1,08314 \\ d=-4,85157 \end{matrix}\Bigg|,$$

from which also it is permitted to infer the values c and d for the mean cases .

APPENDIX

CONCERNED WITH THE CORRECTION OF COMMON OBJECTIVE LENSES,

39. Either if such a lens cannot be prepared from two kinds of glass, of which indeed the ratio of refraction shall be diverse enough, or if the use of water may succeed minimally with that desired, the above calculation set out with the greatest usefulness will be able to be used for the correction of common lenses, with the first condition ignored, from which the dispersion of colors occurs. Now in the first place the confusion arising from the multiplication of lenses from spherical figures to be minimized and thus I have shown to be reduced to nothing, truly where I have assumed the individual lenses themselves to be prepared with minimum confusion, which case was examined for the maximum suitability in practice; nor then would this aim be allowed to be attained with two lenses. But if we may wish to allow some lenses, all the confusion may be removed with two joined together, but where it is to be properly understood much more skill to be required for these requiring to be formed by the artisan.

40. Therefore so that we may provide satisfaction to this only, I assume both lenses prepared from the same kind of glass, so that there shall be $n=m$; and with the focal distance of the first *AB* put $=p$, of the second *CD* truly $=q$, the faces of the lenses themselves will be had thus, so that the radii of these shall be :

$$a=\tfrac{m-1}{\mu}p,\ b=\tfrac{m-1}{1-\mu}p,\ c=\tfrac{m-1}{\nu}q \text{ and } d=\tfrac{m-1}{1-\nu}q.$$

Now since there shall be no question of removing the color dispersion, the number λ may be assumed arbitrarily, or there shall be $q=-\lambda p$, from which the focal distance k of the composite lens will be

$$p=\left(1-\tfrac{1}{\lambda}\right)k \text{ and } q=-(\lambda-1)k.$$

Then truly we may put

$$\mu = \frac{m(2m+1)+my}{2(m+2)} \text{ and } \nu = \frac{m(2m+1)-4\lambda(mm-1)+mz}{2(m+2)},$$

on account of the quantity $A = \lambda^3$ the general determination for the resolution of this equation is led to :

$$zz = (4m-1)(\lambda^3 - 1) + 4(m-1)^2 \lambda(\lambda - 1) + \lambda^3\, yy,$$

where it is apparent the individual values for y and z requiring to be found can be taken both to be positive as well as negative.

41. Therefore we may put in place for the glass, from which both the lenses are to be prepared,

$$m = \tfrac{31}{20} = 1{,}55,$$

with which value substituted we will have :

$$a = \frac{0{,}55}{\mu} p,\; b = \frac{0{,}55}{1-\mu} p,\; c = \frac{0{,}55}{\nu} q,\;\; d = \frac{0{,}55}{1-\nu} q.$$

then truly

$$p = (1 - \tfrac{1}{\lambda})k \text{ and } q = -(\lambda - 1)k.$$

Again,

$$\mu = \frac{1271+310y}{1420} \text{ and } \nu = \frac{1271-1122\lambda+310z}{1420},$$

from which the equation requiring to be resolved will be :

$$zz = (\tfrac{26}{5} + yy)\lambda^3 + 1{,}21\lambda\lambda - 1{,}21\lambda - \tfrac{26}{5},$$

where it is clear for y all the numbers can be assumed as it pleases, since a positive number may be produced always for zz, then there shall be $\lambda > 1$. But if $\lambda < 1$, on account of the negative terms prevailing, the number y must exceed a certain limit. Truly a negative value cannot be allowed to be assume generally for λ. But it is desired to beware always, lest the numbers μ and ν may pass beyond the bounds 0 and 1, certainly since they may arise in the case of the meniscus, in which the lightest error can be calamitous.

42. Therefore each side of the first lens AB may be equally convex, or $\mu = \frac{1}{2}$ and there must be assumed that

$$310y = 710 - 1271 \text{ or } y = -\tfrac{561}{310} \text{ and } yy = 3{,}27493,$$

from which our equation will be:

$$zz = 8{,}47493\lambda^3 + 1{,}21\lambda\lambda - 1{,}21\lambda - 5{,}2$$

and we will have

$$a = b = 1{,}1p = \tfrac{11}{10}(1 - \tfrac{1}{\lambda})k,$$

if

$$\lambda = 2,\ \ a = b = 0{,}55k,\ c = -\tfrac{55k}{176{,}033},\ \ d = \tfrac{55k}{76{,}033}$$

or

$$c = -\,0{,}31244k,\ \ d = +\,0{,}72337k.$$

Here I observe at first, if there were $\lambda = 1$, there will become $zz = yy$, but in this case both focal distances p and q will vanish. Then if λ were a little greater than unity, the quantities p and q will become very small, which in practice is not a little inconvenient. In Case III above we will have $\lambda = 1{,}073$, so that here therefore we may assume to have a greater value. Nor truly is there a need, that we may rise above the value $\lambda = 2$, because then the number v may transgress the bounds 0 and 1. Therefore we shall present the following cases.

Case I

43. Let $\lambda = \frac{11}{10}$, there will be

$$p = \tfrac{1}{11}k,\ \ q = -\tfrac{1}{10}k \ \text{ and } \ a = b = \tfrac{1}{10}k,$$

then truly $zz = 6{,}21323$ and $z = \pm\, 2{,}49263$, with the value of which taken positive there becomes

$$v = \tfrac{809{,}5155}{1420},\ \text{therefore } c = \tfrac{0{,}55q}{0{,}57008} \ = 0 \ \text{ and } \ d = \tfrac{0{,}55q}{0{,}42992}.$$

From which the radii of the two faces of the lenses will be :

$$\begin{aligned} a &= 0{,}10000k,\ \ c = -0{,}09649k,\\ b &= 0{,}10000k,\ \ d = -0{,}12796k, \end{aligned}$$

therefore this lens admits the greatest aperture, the diameter of which $= \frac{64}{1000}k = \frac{1}{16}k$; which therefore can be used for the smallest focal lengths.

Case II

44. Let $\lambda = \frac{12}{10} = 1{,}2$; there will be $p = \frac{1}{6}k$ and $q = -\frac{1}{5}k$, and also

$$a = b = \tfrac{11}{60}k = 0{,}18333k.$$

Then truly there is found:

$$zz = 9{,}73507 \ \text{ and } \ z = 3{,}12011,$$

and hence

$$v = \frac{1271-1346+967,231}{1420} = 0,62805$$

therefore

$$c = \frac{0,55q}{0,62805}, \text{ et } d = \frac{0,55q}{0,37195},$$

from which the radii for the lenses required to be constructed thus provide :

$$a = 0,18333k, \quad c = -0,17514k,$$
$$b = 0,18333k, \quad d = -0,29573k,$$

which lens allows an aperture $\frac{1}{9}k$ in diameter.

Case III

45. Let $\lambda = \frac{13}{10} = 1,3,$ there will be $p = \frac{3}{13}k, \ \ q = -\frac{3}{13}k$ and

$$a = b = \frac{33}{130}k = 0,253846k,$$

then truly there is produced:

$$zz = 13,89130 \text{ and } z = 3,72710$$

$$v = \frac{967,801}{1420} = 0,68155, \ c = \frac{0,55q}{0,68155}, \text{ and } d = \frac{0,55q}{0,31846},$$

from which the radii for each lens thus provide :

$$a = 0,25385k, \quad c = -0,24209k,$$
$$b = 0,25385k, \quad d = -0,51813k,$$

the diameter of th aperture can be $\frac{1}{6}k$.

Case IV

46. Let $\lambda = \frac{14}{10} = 1,4,$ there will be $p = \frac{2}{7}k$ and $q = -\frac{2}{5}k,$ and $a = b = \frac{22}{70}k,$
then truly there arises :

$$zz = 18,73277 \text{ and } z = 4,32814$$

hence

$$v = \frac{1041,923}{1420} = 0,73375.$$

Therefore

$$c = \frac{0,55q}{0,73375} \text{ et } d = \frac{0,55q}{0,26625},$$

from which the radii of each face will be :

$$a = 0,31429k,\quad c = -0,29981k,$$
$$b = 0,31429k,\quad d = -0,82625k,$$

here the diameter of the aperture can be $\frac{1}{5}k$.

Case V

47. Let $\lambda = \frac{3}{2} = 1,5,$ there will be $p = \frac{1}{3}k,\quad q = -\frac{1}{2}k,$ and $a = b = \frac{11}{330}k,$ then truly there is produced:

$$zz = 24,31033 \text{ and } z = 4,93056$$

and hence

$$v = \frac{1116,4706}{1420} = 0,78624.$$

Therefore

$$0,55q\ c = \frac{0,55q}{0,78624} \text{ and } d = \frac{0,55q}{0,21376},$$

from which the radii of each face of our lenses are :

$$a = 0,36666k,\quad c = -0,34976k,$$
$$b = 0,36666k,\quad d = -1,28650k,$$

the diameter of which aperture can rise almost to $\frac{1}{4}k$, if indeed we may assume in lenses of this kind the diameter of the aperture can be equal to two thirds of the minimum radius ; so that no arc in the aperture may be entered greater than 40 degrees.

48. Therefore since likewise for the first lens *AB* an infinite number of concave lenses are possible to be adjoined, just as some or other focal length is required. Thus if each radius of the lens *AB* may be expressed by unity, the radii of each of the other concave lenses *CD* thus will be established according to the five following cases :

	Case I	Case II	Case III	Case IV	Case V
$a = 1,$	$c = -0,9649$	$-0,9553$	$-0,9537$	$-0,9539$	$-0,9539$
$b = 1,$	$d = -1,2796$	$-1,6131$	$-2,0411$	$-2,6290$	$-3,5086$

where in the first place it is noteworthy that in the cases II, III, IV and V the value of c scarcely may change. So that if the radius on both sides of a convex lens therefore may be put $= 1$, truly of the other concave lens thus another face may be formed, so that its radius shall be $= 0,9538$, its other face is left to our choice, provided it shall be concave, and its radius may be contained within the limits 2 and 4 ; Which in practice presents the greatest convenience, since it may not be necessary, that the artisan may not be so concerned in fashioning this latter face.

XCVIII. *An Account of Experiments concerning the different Refrangibility of Light. By Mr. John Dollond. With a Letter from* James Short, *M.A. F.R.S., Acad. Reg. Suec. Soc. To the Rev. Dr.* Birch, *Secret. R. S.*

Dear Sir,

Read June 8, 1758. I Have received the inclosed paper from Mr. Dollond, which he desires may be laid before the Royal Society. It contains the theory of correcting the errors arising from the different refrangibility of the rays of light in the object-glasses of refracting telescopes ; and I have found, upon examination, that telescopes made according to this theory are entirely free from colors, and are as distinct as reflecting telescopes. I am,

Dear Sir,

Your most obedient humble Servant,

Ja. Short.

Surrey-street,

8th June, 1758.

IT is well known, that a ray of light, refracted by passing thro' mediums of different densities, is at the same time proportionally divided or spread into a number of parts, commonly called homogeneal rays, each of a different colour ; and that these, after refraction, proceed diverging: a proof, that they are differently refracted, and that light consists of parts that differ in degrees of refrangibility.

Every ray of light passing from a rarer into a denser medium, is refracted towards the perpendicular; but from a denser into a rarer one, from the perpendicular ; and the sines of the angles of incidence and refraction are in a given ratio. But light consisting of parts, which are differently refrangible, each part of an original or compound ray has a ratio peculiar to itself; and therefore the more a heterogene ray is refracted, the more will the colours diverge, since the ratios of the sines of the homogene rays are constant; and equal refractions produce equal divergencies.

That this is the case when light is refracted by one given medium only, as suppose any particular sort of glass, is out of all dispute, being indeed selfevident; but that the divergency of the colours will be the same under equal refractions, whatsoever mediums the light may be refracted by, tho generally supposed, does not appear quite so clearly.

However, as no medium is known, which will refract light without diverging the colours, and as difference of refrangibility seems thence to be a property inherent in light itself, Opticians have, upon that consideration, concluded, that equal refractions must produce equal divergencies in every sort of medium : whence it should also follow, that equal and contrary refractions must not only destroy each other, but that the divergency of the colours from one refraction would likewise be correded by the other; and there could be no possibility of producing any such thing as refraction, which would not be affected by the different refrangibilty of light or, in other words, that however a ray of light might be refracted backwards and forwards by different mediums, as water, glass, etc. provided it was so done, that the emergent ray should be parallel to the incident one, it would ever after be white ; and, conversely, if it should come out inclined to the incident, it would diverge, and ever after be coloured. From which it was natural to infer, that all spherical object-glasses of telescopes must be equally affected by the different refrangibility of light, in proportion to their apertures, whatever material they may be formed of.

But it seems worthy of consideration, that notwithftanding this notion has been generally adopted as an incontestable truth, yet it does not seem to have been hitherto so confirmed by evident experiment, as the nature of so important a matter justly demands ; and this it was that determined me to attempt putting the thing to issue by the following experiment.

I cemented together two plates of parallel glass at their edges, so as to form a prismatic or wedge-like vessel, when stopped at the ends or bases ; and its edge being turned downwards, I placed therein a glass prism with one of its edges upwards, and filled up the vacancy with clear water thus the refraction of the prism was contrived to be contrary to that of the water, so that a ray of light transmitted thro' both these refracting mediums would be refracted by the difference only between the two refractions. Wherefore, as I found the water to refract more or less than the glass prism, I diminished or increased the angle between the glass plates, till I found the two contrary refractions to be equal ; which I discovered by viewing an object thro' this double prism; which, when it appeared neither raised nor depressed, I was satisfied, that the refractions were equal, and that the emergent rays were parallel to the incident.

Now, according to the prevailing opinion, the object should have appeared thro' this double prism quite of its natural colour; for if the difference of refrangibility had been equal in the two equal refractions, they would have rectified each other: but the experiment fully proved the fallacy of this received opinion, by shewing the divergency of the light by the prism to be almost double of that by the water; for the object, tho' not at all refracted, was yet as much infected with prismatic colours, as if it had been seen thro' a glass wedge only, whose refracting angle was near 30 degrees.

*N. B.*This experiment will be readily perceived to be the same as that which Sir Iaaac Newton mentions [Optics : Book I. Part *ii*. Prop. 3. Expt.8.]; but how it comes to differ so very remarkably in the result, I shall not take upon me to account for ; but will only add, that I used all possible precaution and care in the process, and that I keep the apparatus by me to evince the truth of what I write, whenever I may be properly required so to do.

I plainly saw then, that if the refracting angle of the water-vessel could have admitted of a sufficient increase, the divergency of the coloured rays would have been greatly diminished, or intirely rectified; and there would have been a very great refraction without colour, as now I had a great discolouring without refraction: but the inconveniency of so large an angle, as that of the vessel must have been, to bring the light to an equal divergency with that of the glass prism, whose angle was about 60 degrees, made it necessary to try some experiments of the same kind, by smaller angles.

I ground a wedge of common plate glass to an angle of somewhat less than 9 degrees, which refracted the mean rays about 5 degrees. I then made a wedge-like vessel, as in the former experiment, and filling it with water, managed it so, that it refracted equally with the glass wedge **;** or, in other words, the difference of their refractions was nothing, and objects viewed thro' them appeared neither raised nor depressed. This was done with an intent to observe the same thing over again in these small angles, which I had seen in the prism : and it appeared indeed the same in proportion, or as near as I could judge; for notwithstanding the refractions were here also equal, yet the divergency of the colours by the glass was vastly greater than that by the water; for objeds seen by these two refractions were very much discoloured. Now this was a demonstration, that the divergency of the light, by the different refrangibility, was far from being equal in these two refractions. I also saw, from the position of the colours, that the excess of divergency was in the glass; so that I increased the angle of the waterwedge, by different trials, till the divergency of the light by the water was equal to that by the glass ; that is, till the objed, tho' considerably rcfracted, by the excefs of the refraction of the water, appeared nevertheless quite free from any colours proceeding from the different refrangibiiity of light; and, as near as I could then measure, the refraction by the water was about $\frac{5}{4}$ of that by the glass. Indeed 1 was not very exact in taking the measures, because my business was not at that time about the proportions, so much as to shew, that the divergency of the colours, by different substances, was by no means in proportion to the refractions ; and that there was a possibility of refraction without any divergency of the light at all.

Having, about the beginning of the year 1757, tried these experiments, 1 soon after set about grinding telescopic object-glasses upon the new principles of refractions, which I had gathered from them ; which object-glasses were compounded of two spherical glasses with water between them. These glasses I had the satisfaction to find, as I had expected, free from the errors arising from the different refrangibiiity of light : for the refractions, by which the rays were brought to a focus, were every-where the differences between two contrary refractions, in the same manner, and in the same proportions, as in the experiment with the wedges.

However, the images formed at the foci of these object-glasses were still very far from being so distinct as might have been expected from the removal of so great a disturbance ; and yet it was not very difficult to guess at the reason, when I considered, that the radii of the spherical surfaces of those glasses were required to be so short, in order to make the refractions in the required proportions, that they must produce aberrations, or errors, in the image, as great, or greater, than those from the different refrangibility of light. And therefore, seeing no method of getting over that difficulty, I gave up all hopes of succeeding in that way.

And yet, as these experiments clearly proved, that different substances diverged the light very differently, in proportion to the refraction; I began to suspect, that such variety might possibly be found in different sorts of glass, especially as experience had already shewn, that some made much better object-glasses, in the usual way, than others: and as no satisfactory cause had as yet been assigned for such difference, there was great reason to presume, that it might be owing to the different divergency of the light by their refractions.

Wherefore, the next business to be undertaken, was to grind wedges of different kinds of glass, and apply them together, so that the refractions might be made in contrary diredions, in order to discover, as in the foregoing experiments, whether the refraction and divergency of the colours would vanish together. But a considerable time elapsed before I could set about that work; for tho' I was determined to try it at my leisure, for satisfying my own curiosity, yet I did not exped to meet with a difference sufficient to give room for any great improvement of telescopes ; so that it was not till the latter end of the year that I undertook it, when my first trials convinced me, that this business really deserved my utmost attention and application.

I discovered a difference, far beyond my hopes, in the refractive qualities of different kinds of glass, with respect to their divergency of colours. The yellow or straw-coloured foreign sort, commonly called Venice glass, and the English crown glass, are very near alike in that respect, tho' in general the crown glass seems to diverge the light rather the least of the two. The common plate glass made in England diverges more; and the white crystal or flint English glass, as it is called, most of all.

It was not now my business to examine into the particular qualities of every kind of glass that I could come at, much less to amuse myself with conjectures about the cause, but to fix upon such two sorts as their difference was the greatest; which I soon found to be the crown, and the white flint or crystal. I therefore ground a wedge of white flint of about 25 degrees, and another of crown of about 29 degrees, which refracted nearly alike ; but their divergency of the colours was very different. I then ground several others of crown to different angles, till I got one, which was equal, with respect to the divergency of the light, to that in the white flint: for when they were put together, so as to refract in contrary diredions, the refracted light was intirely free from colour. Then measuring the refractions of each wedge, I found that of the white glass to be to that of the crown nearly as 2 to 3 ; and this proportion would hold very nearly in all small angles. Wherefore any two wedges made in this proportion, and applied together, so as to refract in a contrary direction, would refract the light without any difference of refrangibility.

To make therefore two spherical glasses, that shall refract the light in contrary directions, it is easy to understand, that one must be concave, and the other convex; and as the rays are to converge to a real focus, the excels of refraction must evidently be in the convex; and as the convex is to refract most, it appears from the experiment, that it must be made with crown glass,. and the concave with white flint glass.

And further, as the refractions of spherical glasses are in an inverse ratio of their focal distances; it follows, that the focal distances of the two glasses should be inversely as the ratio's of the fractions of the wedges: for being thus proportioned, every ray of light, that passes thro' this combined glass, at whatever distance it may pass from its axe, will constantly be refracted, by the difference between two contrary refractions, in the proportion required; and therefore the different refrangibility of the light will be intirely removed.

Having thus got rid of the principal cause of the imperfection of refracting telelescopes, there seemed to be nothing more to do, but to go to work upon this principle: but I had not made many attempts, before I found, that the removal of one impediment had introduced another equally detrimental (the same as I had before found in two glasses with water between them): for the two glasses, that were to be combined together, were the segments of very deep spheres; and therefore the aberrations from the spherical surfaces became very considerable, and greatly disturbed the distinctness of the image. Tho' this appeared at first a very great difficulty, yet I was not long without hopes of a remedy: for considering, the surfaces of spherical glasses admit of great variations, tho' the focal distance be limited, and that by these variations their aberrations may be made more or less, almost at pleasure; I plainly saw the possibility of making the aberrations of any two glasses equal; and as in this case the refractions of the two glasses were contrary to each other, their aberrations, being equal, would intirely vanish.

And thus, at last, I obtained a perfect theory for making object-glasses, to the apertures of which I could scarce conceive any limits: for if the practice could come up to the theory, they must certainly admit of very extensive ones, and of course bear very great magnifying powers.

But the difficulties attending the practice are very considerable. In the first place, the focal distances, as well as the particular surfaces, must be very nicely proportioned to the densities or refracting powers of the glasses; which are very apt to vary in the same sort of glass made at different times. Secondly, the centres of the two glasses must be placed truly on the common axis of the telescope, otherwise the desired effect will be in a great measure destroyed. Add to these, that there are four surfaces to be wrought perfectly spherical; and any person, but moderately practifed in optical operations, will allow, that there must be the greatest accuracy throughout the whole work.

Notwithstanding so many difficulties, as I have enumerated, I have, after numerous trials, and a resolute perseverance, brought the matter at last to such an issue, that I can construct refracting telescopes, with such apertures and magnifying powers, under limited lengths, as, in the opinion of the best and undeniable judges, who have experienced them, far exceed any thing that has been hitherto produced, as representing objects with great distinctness, and in their true colours.

John Dollond.

CONSTRUCTIO LENTIUM OBIECTIVARUM

EX DUPLICI VITRO

quae neque confusionem a figura sphaerica oriundam, neque dispersionem colorum pariant,

Auctore

LEONHARDO EULERO

DISSERTATIO

occasione Quaestionis de Perfectione Telescopiorum ab Imperiali Academia Scientiarum Petropolitana pro praemio propositae conscripta.

PETROPOLl
Typis Academiae Scientiarum
1762.

1. Lentes obiectivas, quae vulgo ad Telescopia adhiberi solent, duplici vitio laborare, iam pridem est observatum. Primum enim ob figuram sphaericam, quae utrique faciei inducitur, (quoniam ad alias figuras, quae aptiores essent futurae, praxis vitra poliendi nondum est accommodata), a radiis extremis alia imago efformatur atque a mediis, quo fit, ut, quo maior tali lenti apertura concedatur, eo maior confusio in imaginem redundet. Alterum autem vitium, quo radii diversorum colorum haud pari refractione per vitrum transmittuntur, ideoque disperguntur, non minus lentes infestat, dum imagines a diversis coloribus formatae, eo magis a se invicem divelluntur, quo maior earum a lente fuerit distantia. Atque ambo haec vitia ita arcte cum vitri natura et sphaerica figura sunt coniuncta, ut a lentibus nullo modo separari queant.

2. Loquor hic autem de lentibus simplicibus ex vitro paratis; quodsi enim duas pluresve lentes coniungere velimus, ut unicam quasi lentem referant, iam dudum modum exposui, quo utrumque vitium seorsim evitari potest. Binis enim pluribusve lentibus coniungendis ostendi, quomodo singulas comparatas esse oporteat, ut confusio ob sphaericam figuram oriunda penitus tollatur, quod etiam in praxi usu haud caruisse videtur. Alterum autem vitium, in dispersione colorum positum, multiplicandis lentibus non solum non destruere, sed ne diminuere quidem licet, siquidem lentes ex simili materia pellucida parentur.

3. Tum equidem credideram omne vitrum ratione refractionis perinde esse comparatum, ideoque in mentem mihi venerat lentes ex vitro et aqua, aliave materia fluida pellucida, parare, ita ut fluidum intra binas lentes vitreas includeretur. Atque examine instituto inveni figuram lentium utique ita attemperari posse, ut dispersio colorum coerceatur. Feci

hac de re plurima experimenta, quibus istiusmodi lentes a dispersione colorum immunes deprehendi, verum alterum vitium eas tanto magis inquinabat, ut nullus plane usus inde expectandus videretur. Etsi enim priori scopo infinitis modis satisfieri poterat, inter quos alii magis, alii minus, altero vitio essent infecti, tamen partim prolixitas calculi, quam haec investigatio postulat, me deterruit, ne in figuras lentium aptiores inquirerem, partim vero desperavi tales figuras, si quas invenissem, quam accuratissime per praxin obtineri posse. Alio autem tempore hoc negotium diligentius persequi constitui.

4. Theoria autem refractionis, cui illam lentium aqua repletarum confectionem superstruxi, etsi certissimis principiis innixa, plane erat nova, ac theoriae NEWTONIANAE adversabatur, in qua assumitur nunquam ullo modo dispersionem colorum ne diminui quidem posse, quotcunque etiam diversa media refringentia in usum vocentur. Etsi autem Vir summus hoc principium nusquam demonstraverat, sed id potius precario assumserat, ne intricatioribus investigationibus, quae ad eius institutum minus pertinerent, se implicaret; tamen acrem ob hanc causam adeptus sum adversarium DOLLONDUM Anglum, qui mea principia, utpote a NEWTONIANIS discrepantia, penitus censuit profliganda, quod mihi occasionem dedit theoriam meam firmissimis rationibus confirmandi et ab omnibus dubiis liberandi.

5. Tantum abest, ut defensio mea Viro Clarissimo displicuisse videatur, ut potius omni studio principium illud NEWTONIANUM per experimenta exploraverit. Mox quidem binis prismatibus, altero vitreo, altero aqueo, coniunctis deprehendit opinionem illam NEWTONO tributam subsistere non posse, dum observavit sublata refractione dispersionem colorum non tolli ac vicissim; quod experimentum meae theoriae corroborandae mirifice inservit. (Vid. Transact. vol. L.) Deinceps vero etiam varias vitri species pari modo examini subiecit, paratisque inde variis cuneis animadvertit dispersionem colorum a refractionis quantitate non pendere, ideoque diversas vitri species diversa vi refringendi esse praeditas. Unde recte conclusit binis lentibus ex diversis vitri speciebus factis coniungendis utique fieri posse, ut dispersio colorum penitus tollatur. Hunc in finem necesse erat alteram lentem convexam confici, ita ut convexitas ad concavitatem datam teneret rationem a diversitate refractionis determinatam.

6. Cum hac ratione tantum distantia foci utriusque lentis definiatur, pro eadem vero distantia foci innumerabiles lentes confici possint, merito suspicatus est inter hos infinitos casus eiusmodi binarum lentium fabricam reperiri, quae etiam ab altero vitio confusionis a sphaerica figura natae esset immunis. Non videtur Vir Clarissimus calculo, vel ulla theoria, ad hanc investigationem esse usus; sed potius plurimis huiusmodi lentibus diversae formae elaboratis, binis coniungendis exploravit, quo casu confusio imaginis minima, atque adeo nulla, esset proditura. Hoc modo eiusmodi constructionem se eruisse profitetur, quae lentem talem compositam ab utroque vitio liberam praestaret; quod sine dubio summum est inventum, quod quidem in Dioptrica desiderari queat, ac merito a Cel. SHORTO summis laudibus condecoratur.

7. Minime autem quicquam de gloria Clarissimi Inventoria detracturus, hoc argumentum, meo more, per solam theoriam pertractabo, quo clarius appareat, quomodo experientia cum theoria consentiat, ac num forte haec constructio lentium compositarum ad maiorem perfectionis gradum evehi queat. Assumo igitur dari duplicis generis vitrum, atque in solutionem huius problematis sum inquisiturus, *quomodo inde binas lentes confici oporteat, quae coniunctae obiecta vehementer remota tam sine ulla colorum dispersione, quam sine ulla confusione a figura sphaerica oriunda, repraesentent.* Quatuor ergo hic occurrunt facies sphaericae, quarum singularum, sive sint convexae, sive concavae, radii determinari debent, ut huic duplici conditioni satisfiat; ex quo patet, etiamsi lentis compositae distantia foci praescribatur, tres tamen quantitates determinandas remanere, ideoque ob duas tantum conditiones adimplendas, infinitas adhuc solutiones locum habere posse, ex quibus deinceps eam, quae ad praxin commodissima videbitur, eligere licebit.

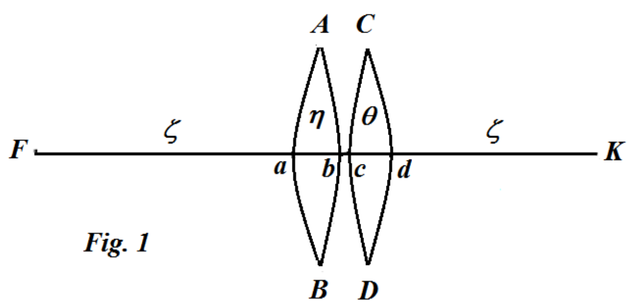

Fig. 1

8. Sint igitur (Fig. 1) *AB* et *CD* binae lentes ad communem axem *FK* constitutae, quas tanquam quam utrinque convexas spectabo, et quarum illa *AB* obiecto versus dirigatur. Pro radiis lucis mediae naturae sit ratio refractionis ex aere in vitrum, ex quo prior lens *AB* est confecta, ut ζ ad η; ratio autem refractionis ex aere in vitrum, ex quo lens posterior *CD* constat, sit ut ζ ad ϑ, singulae facies porro harum lentium sint sphaerioae, et quidem:

pro lente priori *AB* sit radius faciei $\begin{cases}\text{anterioris} \quad AaB = a \\ \text{posterioris } AbB = b\end{cases}$

pro lente altera *CD* sit radius faciei $\begin{cases}\text{anterioris} \quad CcD = c \\ \text{posterioris } CdD = d.\end{cases}$

Ambas autem lentes proxime esse coniunctas pono, et utriusque crassitiem quam minimam, ut eam in calculo negligere liceat.

9. Statuatur porro prioris lentis *AB* distantia focalis $= p$, posterioris vero lentis distantia focalis $= q$, eritque ex principiis dioptricis:

$$p = \frac{\eta ab}{(\zeta-\eta)(a+b)} \quad \text{et} \quad q = \frac{\vartheta cd}{(\zeta-\vartheta)(c+d)}$$

seu

$$\frac{1}{p} = \left(\frac{\zeta}{\eta}-1\right)\left(\frac{1}{a}+\frac{1}{b}\right) \quad \text{et} \quad \frac{1}{q} = \left(\frac{\zeta}{\vartheta}-1\right)\left(\frac{1}{c}+\frac{1}{d}\right),$$

qui valores quidem pro radiis lucis mediae naturae, iisque tantum, qui proxime ad axem transeunt, valent. Quodsi iam pro lente *AB* distantia obiecti ante eam fuerit $=f$, imago post eam cadet ad distantiam *g*, ut sit

$$g=\frac{fp}{f-p}, \text{ seu } \frac{1}{g}=\frac{1}{p}-\frac{1}{f},$$

ideoque pro *p* restituto valore erit

$$\frac{1}{f}+\frac{1}{g}=\left(\frac{\zeta}{\eta}-1\right)\left(\frac{1}{a}+\frac{1}{b}\right).$$

Simili modo si ante alteram lentem *CD* obiecti distantia sit $=h$, imago post eam reperietur ad distantiam $=k$, ut sit

$$\frac{1}{h}+\frac{1}{k}=\left(\frac{\zeta}{\vartheta}-1\right)\left(\frac{1}{c}+\frac{1}{d}\right).$$

10. Iunctis igitur his lentibus, manenteque distantia obiecti ante primam $=f$, quia imago ab ea proiecta vicem gerit obiecti respectu alterius lentis *CD,* erit $h=-g$, hincque imago per lentem duplicatam cadet ad distantiam *k,*
ut sit

$$\frac{1}{f}+\frac{1}{k}=\left(\frac{\zeta}{\eta}-1\right)\left(\frac{1}{a}+\frac{1}{b}\right)+\left(\frac{\zeta}{\vartheta}-1\right)\left(\frac{1}{c}+\frac{1}{d}\right)$$

seu

$$\frac{1}{f}+\frac{1}{k}=\frac{1}{p}+\frac{1}{q}\ ;$$

ita ut, sumta distantia *f* infinita, lentis duplicatae distantia focalis sit

$$k=\frac{pq}{p+q}.$$

Quae si fuerit data, patet non solum utriusque lentis distantias focales *p* et *q* infinitis modis variari posse, ita ut, altera data, altera definiri possit, sed etiam utraque lens pro data eius distantia focali infinitas variationes respectu facierum recipere potest, ita ut, altera facie data, alteram semper determinare liceat.

11. His in genere animadversis primo satisfaciamus huic conditioni, ut nulla colorum dispersio oriatur, seu lentis duplicatae distantia imaginis *k* nullam variationem patiatur, etiamsi rationes refractionis $\zeta:\eta$ et $\zeta:\vartheta$ ob diversam radiorum lucis naturam immutentur. Cum autem haec immutatio sit quasi infinite parva, differentiando huic conditioni satisfieri poterit. Ponamus ergo pro radiis mediae naturae:

$$\frac{\zeta}{\eta}=m \text{ et } \frac{\zeta}{\vartheta}=n$$

ita ut habeamus hanc aequationem:

$$\frac{1}{f}+\frac{1}{k}=(m-1)\left(\frac{1}{a}+\frac{1}{b}\right)+(n-1)\left(\frac{1}{c}+\frac{1}{d}\right),$$

in qua litterae *a, b, c, d, f* et *k* ut quantitates constantes sunt spectandae, litterae vero *m* et *n* ut variabiles, ita ut haec aequatio perinde subsistat, etiamsi ea differentietur. Differentiatio vero praebet:

$$dm\left(\frac{1}{a}+\frac{1}{b}\right)+dn\left(\frac{1}{c}+\frac{1}{d}\right)=0.$$

12. Cum igitur distantias focales *p* et *q* utriusque lentis introducendo, ob

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{(m-1)p} \quad \text{et} \quad \frac{1}{c}+\frac{1}{d}=\frac{1}{(n-1)q},$$

sit

$$\frac{dm}{(m-1)p}+\frac{dn}{(n-1)q}=0 \quad \text{seu} \quad q=-\frac{p(m-1)dn}{(n-1)dm},$$

hinc ratio inter distantias focales utriusque lentis *p* et *q* determinatur, ut diversa radiorum natura nullam colorum dispersionem pariat, sed ab omnis generis radiis lucis imagines uniantur. Ac si talis statuatur ratio inter distantias focales, ut sit

$$q=-\frac{(m-1)dn}{(n-1)dm}p,$$

erit pro omnis generis radiis

$$\frac{1}{f}+\frac{1}{k}=\frac{1}{p}+\frac{1}{q},$$

neque haec determinatio ad obiecta infinite remota est restricta, sed quaecunque fuerit obiecti distantia *f,* imago sine ulla colorum dispersione post lentem geminatam ad distantiam $=k$ cernetur.

13. Quodsi ambae lentes ex pari vitro conficiantur, ut esset $n=m$, ac propterea $dn=dm$, foret $q=-p$, et distantia foci lentis compositae prodiret infinita, quae ratio est, quod ex vitro eiusdem indolis, quotcunque etiam lentes adhibeantur, dispersio colorum nullo modo destrui possit. Admisso autem duplicis generis vitro, ut numeri *m* et *n* sint inaequales, eatenus tantum dispersio colorum tolli potest, quatenus non est

$$\frac{(m-1)dn}{(n-1)dm}=1,$$

quia tum superius incommodum recurreret. Quare si esset

$$dm:dn=m-1:n-1,$$

uti NEWTONUS statuisse perhibetur, lentes utique compositae aeque parum ab hoc vitio liberari possent, ac simplices. Cum autem haec opinio non solum firmissimis argumentis a me sit profligata, sed etiam per experimenta a Cl. DOLLONDO sufficienter refutata, destructio colorum ob hanc rationem locum habere est censenda.

14. Quoniam dm et dn variationem refractionis radiorum a media natura discrepantium exprimunt, vera ratio ita se habet, ut sit

$$dm : dn = mlm : nln\,,$$

quemadmodum alio loco sufficienter demonstravi. Hac igitur stabilita ratione ex vitri differentia ratio distantiarum focalium $p:q$ ita se habebit, ut sit

$$q = -\frac{(m-1)nln}{(n-1)mlm}\,p.$$

Statuamus ergo brevitatis gratia:

$$\frac{(m-1)nln}{(n-1)mlm} = \lambda,$$

ut sit $q = -\lambda p$, neque λ unitati aequetur, lentis compositae distantia focalis erit

$$k = -\frac{\lambda pp}{p-\lambda p} \quad \text{seu} \quad k = \frac{\lambda p}{\lambda - 1}.$$

Data ergo ista distantia focali k, utriusque lentis simplicis distantia focalis ita debet definiri, ut sit

$$p = \frac{\lambda-1}{\lambda}k \quad \text{et} \quad q = -(\lambda - 1)k.$$

15. Si ergo numerus λ fuerit unitate maior, lens prior AB erit convexa, seu distantiam focalem habebit positivam, posterior vero CD concava, seu distantiam focalem habebit negativam; sin autem λ sit fractio unitate minor, contrarium eveniet. Quo autem indoles numeri λ facilius perspici queat, quia numeri m et n non multum unitatem excedunt, erit per series:

$$lm = l(1+m-1) = m-1-\tfrac{1}{2}(m-1)^2+\tfrac{1}{3}(m-1)^3-\tfrac{1}{4}(m-1)^4+\text{etc.}$$
$$ln = l(1+n-1) \quad = n-1-\tfrac{1}{2}(n-1)^2+\tfrac{1}{3}(n-1)^3-\tfrac{1}{4}(n-1)^4+\text{etc.}$$

hincque numerus λ ita definietur:

$$\lambda = \frac{n\left(1-\frac{1}{2}(n-1)+\frac{1}{3}(n-1)^2-\frac{1}{4}(n-1)^3+\text{etc.}\right)}{m\left(1-\frac{1}{2}(m-1)+\frac{1}{3}(m-1)^2-\frac{1}{4}(m-1)^3+\text{etc.}\right)}$$

seu scribendo

$$m = 1 + m - 1 \text{ et } n = 1 + n - 1$$

erit

$$\lambda = \frac{1+\frac{1}{2}(n-1)-\frac{1}{6}(n-1)^2+\frac{1}{12}(n-1)^3-\text{etc.}}{1+\frac{1}{2}(m-1)-\frac{1}{6}(m-1)^2+\frac{1}{12}(m-1)^3-\text{etc.}}$$

unde patet, si $n > m$, fore $\lambda > 1$, at si $n < m$, fore $\lambda < 1$.

16. Inventis autem distantiis focalibus p et q, facies utriusque lentis infinitas adhuc determinationes admittunt, quod quo facilius perspiciatur, introducamus binos novos numeros indeterminatos μ et v, et cum esse debeat

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{(m-1)p} \quad \text{et} \quad \frac{1}{c} + \frac{1}{d} = \frac{1}{(n-1)q}$$

statuamus:

$$a = \frac{m-1}{\mu}p, \quad b = \frac{m-1}{1-\mu}p, \quad c = \frac{n-1}{v}q, \quad d = \frac{n-1}{1-v}q.$$

Quomodocunque enim hi numeri μ et v accipiantur, lentes semper praescriptas distantias focales obtinebunt, ac sumto $p = \frac{\lambda-1}{\lambda}k$ et $q = -(\lambda - 1)k$, binae lentes coniunctae non solum distantiam focalem praescriptam k habebunt, sed etiam imagines quorumvis obiectorum sine ulla colorum dispersione repraesentabunt. Superest igitur, ut numeros μ et v ita definiamus, ut etiam lens composita ab altero vitio, confusione scilicet a figura sphaerica facierum lentium oriunda, liberetur.

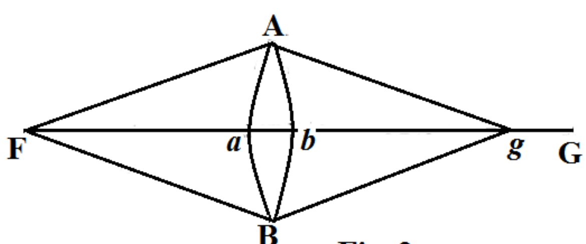

Fig. 2

17. Primum igitur dispiciendum erit, quomodo in unica lente imago a radiis extremis formata, ab imagine, quae a radiis per medium lentis transeuntibus formatur, discrepet. Sit igitur (Fig. 2) proposita lens AB, cuius faciei AaB radius sit $= a$, faciei vero $AbB = b$; quam utramque ut convexam specto, ratio refractionis autem ex aere in vitrum, quo haec lens constat, sit $\zeta : \eta$ seu $m:1$, posito $\frac{\zeta}{\eta} = m$. Hinc eius distantia focalis erit

$$p = \frac{ab}{(m-1)(a+b)},$$

$$\frac{1}{p} = (m-1)\left(\frac{1}{a} + \frac{1}{b}\right),$$

unde ut ante ponamus

$$a = \frac{m-1}{\mu}p \text{ et } b = \frac{m-1}{1-\mu}p.$$

Iam si obiecti F ab hac lente distantia sit $aF = f$, a radiis axi proximis imago referetur in G, ut sit distantia $bG = \frac{fp}{f-p}$; per radios autem extremos FA, FB imago propius repraesentatur in g, et spatiolum Gg intervallum confusionis appellari solet, quod igitur sollicite investigari oportet.

18. Denotet x semidiametrum aperturae, atque calculo non parum taedioso, quem idcirco hic repetere nolo, spatium illud confusionis ita reperitur expressum:

$$\frac{\begin{array}{c}((m+2)\mu\mu-m(2m+1)\mu+m^3)ff\\ +(m-1)\left(4(m+1)\mu-m(3m+1)\right)fp\\ +(m-1)^2(3m+2)pp\end{array}}{2m(m-1)^2(f-p)^2}\cdot\frac{xx}{p}=Gg.$$

Unde patet, si obiecti distantia f fuerit infinita, fore hoc spatium diffusionis:

$$Gg=\frac{(m+2)\mu\mu-m(2m+1)\mu+m^3}{2m(m-1)^2}\cdot\frac{xx}{p},$$

quod evanescere nequit, quicunque valor numero μ tribuatur. Minimum autem evadit, si capiatur $\mu=\frac{m(2m+1)}{2(m+2)}$; quem casum, quo diffusio imaginis in foco fit minima, HUGENIUS iam elicuit.

19. Revertamur iam ad lentem nostram geminatam ante descriptam, ac ponamus obiecti distantiam esse quasi infinitam, eritque (Fig. 3) pro radiis per medium lentis transeuntibus distantia imaginis G, a sola prima lente proiectae

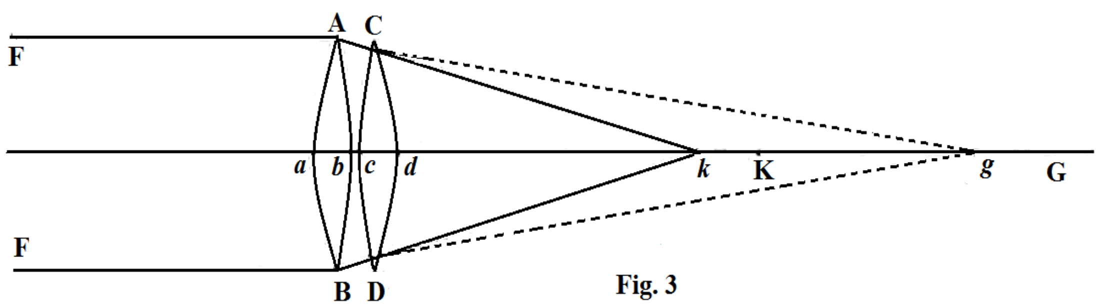

Fig. 3

$dG = p$, ac per lentes ambas proiectae $dK=\frac{pq}{p+q}$. Pro radiis vero extremis, posita aperturae semidiametro $= x$, per solam primam lentem imago erit in g, ut sit $dg = p - M\cdot\frac{xx}{p}$ existente

$$M=\frac{(m+2)\mu\mu-m(2m+1)\mu+m^3}{2m(m-1)^2},$$

quae imago cum locum teneat obiecti ratione posterioris lentis, ponatur haec distantia negative sumta $-p+M\cdot\frac{xx}{p}=g;$ eritque imaginis per alteram lentem proiectae in k distantia $dk=\frac{gq}{q-g}-N\cdot\frac{xx}{q}$, siquidem ponatur:

$$N=\frac{\left((n+2)vv-n(2n+1)v+n^3\right)gg+(n-1)\left(4(n+1)v-n(3n+1)\right)gq+(n-1)^2(3n+2)qq}{2n(n-1)^2(g-q)^2}.$$

20. Cum iam sit $g=-p+M\cdot\frac{xx}{p}$, erit

$$\frac{gq}{g-q}=\frac{-q\left(p-M\cdot\frac{xx}{p}\right)}{-p-q+M\cdot\frac{xx}{p}},$$

ubi cum particulam $\frac{xx}{p}$ ut valde parvam spectare liceat, erit satis exacte

$$\frac{gq}{g-q}=\frac{pq}{p+q}-\frac{qq}{(p+q)^2}M\cdot\frac{xx}{p}.$$

Quam ob rem imago per lentem duplicatam a radiis extremis exhibita cadet in k, ut sit:

$$dk=\frac{pq}{p+q}-\frac{qq}{(p+q)^2}M\cdot\frac{xx}{p}-N\cdot\frac{xx}{q},$$

in valore autem numeri N, quia particulam minimam $\frac{xx}{q}$ afficit, loco g scribere licet valorem vero proximum $-p$, ita ut sit:

$$N=\frac{\left((n+2)vv-n(2n+1)v+n^3\right)pp-(n-1)\left(4(n+1)v-n(3n+1)\right)pq+(n-1)^2(3n+2)qq}{2n(n-1)^2(p+q)^2}.$$

et spatium diffusionis est

$$Kk=\left(\frac{qq}{p(p+q)^2}M+\frac{1}{q}N\right)xx.$$

21. Totum ergo negotium huc reductum est, ut hoc spatium diffusionis ad nihilum redigatur seu ut fiat:

$$Mq^3+Np(p+q)^2=0,$$

quae aequatio per $2n(n-1)^2$ multiplicata praebet:

$$\left.\begin{array}{l}\frac{n(n-1)^2}{m(m-1)^2}((m+2)\mu\mu-m(2m+1)\mu+m^3)q^3\\+((n+2)vv-n(2n+1)v+n^3)p^3\\-(n-1)(4(n+1)v-n(3n+1))ppq\\+(n-1)^2(3n+2)pqq\end{array}\right\}=0$$

Quia ergo prior conditio dedit $q=-\lambda p$, existante

$$\lambda=\frac{(m-1)nln}{(n-1)mlm},$$

habebimus:

$$\begin{aligned}&\frac{n(n-1)^2}{m(m-1)^2}\lambda^3((m+2)\mu\mu-m(2m+1)\mu+m^3)\\&=(n+2)vv-n(2n+1)v+n^3+\lambda(n-1)(4(n+1)v-n(3n+1)\\&\qquad+\lambda\lambda(n-1)^2(3n+2),\end{aligned}$$

ex qua aequatione sine dubio bini numeri μ et v infinitis modis realiter definiri possunt.

22. Cum haec aequatio nimis sit perplexa, eam ad formam simpliciorem reducamus, ponendo:

$$\mu=\frac{m(2m+1)+my}{2(m+2)}\quad\text{et}\quad v=\frac{n(2n+1)-4\lambda(nn-1)+nz}{2(n+2)}$$

ac facta substitutione pervenietur ad hanc aequationem:

$$\frac{m(n-1)^2(n+2)}{n(m-1)^2(m+2)}\lambda^3(yy+4m-1)yy=zz-4\lambda(\lambda-1)(n-1)^2+4n-1.$$

Statuatur brevitatis gratia

$$\frac{m(n-1)^2(n+2)}{n(m-1)^2(m+2)}\lambda^3=A,$$

qui numerus plerumque parum ab unitate differet, eritque:

$$zz=A(4m-1)-(4n-1)+4\lambda(\lambda-1)(n-1)^2+Ayy;$$

nisi igitur fuerit $4n-1>A(4m-1)$, pro y omnes numeros accipere, et ex singulis numerum z definire licet; sin autem $4n-1>A(4m-1)$, minores tantum valores ipsius y excluduntur.

23. Ad usum practicum autem inprimis curandum est, ut numeri μ et ν intra limites 0 et 1 contineantur, vel saltem ne multum eos transgrediantur, id quod plerumque eveniet, quando numeris y et z quam minimos valores tribuere licuerit. Tum vero etiam observandum est, quicunque numeri pro y et z satisfaciant, eos aeque affirmative ac negative accipi posse, unde quadruplices numeri pro μ et ν obtinebuntur, sicque solutionum multitudo vehementer augetur. Dummodo ergo duae vitri species ratione refractionis notabiliter diversae habeantur, facile erit infinitas huiusmodi lentium geminatarum constructiones exhibere, quae utroque vitio, quo lentes ordinariae inquinantur, careant, atque ex illis eas, quae ad praxin maxime videantur accommodatae, eligi conveniet, sicque nullum erit dubium, quin huiusmodi lentes perfectae confici atque ad usum transferri queant.

24. Clarissimus DOLLOND quidem rationem refractionis pro utraque vitri specie, qua usus est, non definivit, neque ergo lentes duplicatas ab eo confectas mihi ad calculum revocare licet. Ex ipsis autem calculi formulis perspicuum est utramque refractionis legem, numeris m et n contentam, exactissime cognitam esse debere, cum levis error saepe ingens discrimen in fabricam lentium inducere valeat. Consultum ergo erit pro quovis casu oblato aliquot hypotheses pro numeris m et n fingere, quae a vero in utramque partem declinent, ut appareat, quanta inde differentia in constructione lentium nascatur. Inprimis vero necesse erit, calculo peracto, plura experimenta instituere, sicque theoria adiuti tandem ad optatum finem pertingemus. Quin etiam merito sperare licet hoc modo non solum lentes solitis multo praestantiores, sed etiam plane perfectissimas obtineri posse, in quo negotio certe neque sumtibus neque labori parcendum videtur.

25. Cum igitur ad certas vitri species calculum applicari nondum liceat, fingam binas species vitri mihi proponi, alteram densiorem, pro qua radiorum ex aere intrantium ratio refractionis sit 31:20, altera vero species aliquanto minus sit densa, ita ut ratio refractionis sit ut 3:2. Illa species eius videtur vitri, quod vulgo ad lentes dioptricas adhiberi solet, haec vero fortasse vitrum vilius continet. Etiamsi autem talis species non daretur, tamen evolutio huius casus nobis satis perspicue monstrabit, quomodo binae lentes, ex vitro densiori et rariori paratae, formari debeant, ut effectum desideratum praestent. Duplicem autem hic investigationem institui conveniet, prout lens prior ex vitro, vel rariori, vel densiori, conficiatur; primo igitur in constructionem lentium perfectarum inquiram, quando ponitur $m=\frac{3}{2}$ et $n=\frac{31}{20}$; tum vero vice versa ponam $m=\frac{31}{20}$ et $n=\frac{3}{2}$; pro utroque vero casu calculus haud multum discrepabit.

I. Casus, quo $m=\frac{\zeta}{\eta}=\frac{3}{2}$ et $n=\frac{\zeta}{\vartheta}=\frac{31}{20}$.

26. Hic igitur primo pro radiis singularum facierum habemus:

$$a=\frac{1}{2\mu}p,\quad b=\frac{1}{2(1-\mu)}p,\quad c=\frac{11}{20\nu}q,\quad d=\frac{11}{20(1-\nu)}q,$$

tum vero quaeratur numerus $\lambda = \frac{31ln}{33lm}$. Est vero

$$\begin{array}{lll} ln = 0{,}1903317, & lln = 9{,}2795111, & llm = 9{,}2457380 \\ lm = 0{,}1760913, & l31 = \underline{1{,}4913617}, & l33 = \underline{1{,}5185139} \\ & 0{,}7708728 & 0{,}7642519\ 0{,}7642519 \\ & \underline{0{,}7642519} & \\ & l\lambda = 0{,}0066209. & \end{array}$$

Ergo $\lambda = 1{,}015315$ et $q = -1{,}015315p$; tum vero si lentis compositae distantia focalis esse debeat = k, erit:

$$p = 0{,}015083k \text{ et } q = -0{,}015315k,$$

sicque lens prior erit convexa, posterior vero concava.

Porro reperitur $A = \frac{25773}{21700}\lambda^3$

$$\begin{array}{rl} l\lambda^3 = & 0{,}0198627 \\ l25773 = & \underline{4{,}4111650} \\ & 4{,}4310277 \\ l21700 = & \underline{4{,}3364597} \\ lA = & 0{,}0945680 \\ \text{hinc } A = & 1{,}24328 \end{array}$$

ideoque aequatio resolvenda

$$zz = 5A - 5{,}2 + 0{,}018815 + Ayy,$$

seu

$$zz = 1{,}035215 + 1{,}24328yy.$$

27. A numeris autem y et z ita pendent numeri μ et ν, ut sit:

$$\mu = \frac{12+3y}{14} \quad \text{et} \quad \nu = \frac{132{,}24+310z}{1420}.$$

Hi valores fere ad medium limitum 0 et 1 perduci possunt, sumendo $y = -\frac{5}{3}$, quo fiat exacte $\mu = \frac{1}{2}$; tum vero erit:

$$zz = 1{,}035215 + 3{,}453555 = 4{,}488770$$

et

$$z = 2{,}11867, \text{ hincque } \nu = \frac{789{,}0277}{1420} = 0{,}55565;$$

unde deducimus

$$a = p, \quad b = p, \quad c = \frac{0{,}55}{0{,}5556}q, \quad d = \frac{0{,}55}{0{,}4444} = q.$$

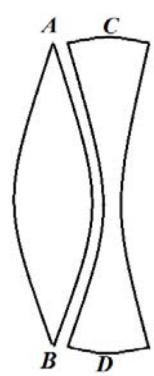

Fig. 4

Cum nunc sit

$$p = 0,015083k \ \text{ et } \ q = -0,015315k,$$

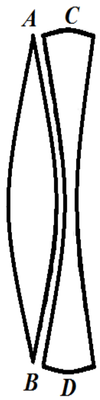

Fig. 5

radii facierum lentis ita se habebunt:

$$a = 0,015083k, \quad c = -0,015159k,$$
$$b = 0,015083k, \quad d = -0,018956k.$$

In fig. 4 et 5 talem lentem duplicatam exhibeo pro casu $k = 10\ 000$ scrup. pedis, seu 10 pedum. Prior quidem ad maiores arcus extenditur, qui vix admitti posse videntur; posterior nimis magnos arcus non involvit, at vero etiam sufficientis aperturae, qualem distantia focalis 10 pedum postulat, non est capax. Quodsi vero huiusmodi lentes ad multo maiores distantias focales construantur, aperturam satis magnam recipere poterunt.

II. Casus, quo $m = \frac{31}{20}$ *et* $n = \frac{3}{2}$

28. Hic statim fit pro facierum radiis:

$$a = \frac{11}{20\mu} p, \quad b = \frac{11}{20(1-\mu)} p, \quad c = \frac{1}{2\nu} q, \quad d = \frac{1}{2(1-\nu)} q,$$

numeri vero λ valor praecedentis fit reciprocus, scilicet:

$$\lambda = 0,984685 \ \text{ et } \ l\lambda = 9,9933791$$

hincque

$$p = -0,015315k \ \text{ et } \ q = 0,015083k\ ,$$

ubi ratio inter convexitatem et concavitatem eadem est atque ante. Quin etiam valor ipsius A praecedentis est reciprocus, ideoque A = 0,804326, unde obtinetur haec aequatio:

$$zz = 4,182495 - 5 - 0,014855 + 0,804326yy$$

seu

$$zz = -0,832360 + 0,804326yy\ ;$$

tum vero est

$$\mu = \frac{1271 + 310y}{1420} \quad \text{et} \quad \nu = \frac{2,15312 + 3z}{14}$$

si capiatur

$$y = \tfrac{3}{2}, \ \text{ reperitur } \ zz = 0,977373 \text{ et } \ z = \pm 0,98860,$$

at si

$$y = 2, \ \text{ reperitur } \ zz = 2,384944 \ \text{ et } \ z = \pm 1,54432.$$

29. Evidens est postremum casum, si ipsius z capiatur valor positivus, ipsius y vero negativus, numeros μ et ν proxime ad $\frac{1}{2}$ reducere; erit ergo:

$$\mu = \frac{651}{1420} = 0,45845 \text{ et } \nu = \frac{6,78608}{14} = 0,48472,$$

hincque

$$a = \frac{0,55}{0,45845}p, \quad b = \frac{0,55}{0,54155}p, \quad c = \frac{q}{0,96944}, \quad d = \frac{q}{1,03056},$$

qui valores per distantiam focalem k lentis ipsius compositae ita exprimuntur:

$$a = -0,018373k, \quad c = +0,015558k$$
$$b = -0,015554k, \quad d = +0,014636k.$$

Haec igitur lens composita vix differt a praecedente, si invertatur, et lens concava obiectum versus dirigatur. Interim tamen calculus non ostendit talem inversionem semper locum habere; unde et hoc casu inversio proxime tantum valet, et fortuito evenire censenda est. Probe autem tenendum est mensuras calculo erutas summo studio in praxi observari oportere; si enim vel minimum ab iis aberraverimus, lens inde confecta ingentia vitia facile contrahet.

30. Inprimis autem numerus λ accuratissime est observandus, cum levissima aberratio dispersionem colorum vix minuat, id quod hoc modo ostendi potest: Concipiatur lens simplex distantiae focalis $= k$, et ob diversam radiorum refrangibilitatem focus diffundetur per spatiolum dk, ut sit [§ 9]

$$-\frac{dk}{kk} = \frac{dm}{(m-1)k}$$

Nunc vero pro lente nostra composita spatiolum diffusionis dk ita exprimitur, ut sit

$$-\frac{dk}{kk} = \frac{dm}{(m-1)p} + \frac{dn}{(n-1)q},$$

quod ponamus non ad nihilum redigi, sed parti $\frac{1}{\alpha}$ illius spatioli $\frac{dm}{(m-1)k}$ aequari debere; ut sit ob $\frac{1}{k} = \frac{1}{p} + \frac{1}{q}$:

$$\frac{dm}{(m-1)p} + \frac{dn}{(n-1)q} = \frac{dm}{\alpha(m-1)p} + \frac{dm}{\alpha(m-1)q},$$

unde fit

$$\frac{q}{p} = -\frac{\alpha(m-1)dn}{(\alpha-1)(n-1)dm} + \frac{1}{(\alpha-1)} = -\frac{\alpha\lambda}{\alpha-1} + \frac{1}{\alpha-1}$$

ideoque

$$q = -\left(\lambda + \frac{\lambda-1}{\alpha-1}\right)p.$$

Dummodo ergo dispersio colorum modice imminuenda sit, proxime fit

$$q=-\lambda p;$$

unde intelligitur a vero valore ipsius λ minime esse recedendum.

31. Cum in lentibus simplicibus, quae ingentem foci distantiam habere debent, magnitudo sphaerarum, facies earum determinantium, in praxi maximam difficultatem creare soleat, hic commode usu venit, ut etiam pro maxima distantia focali admodum exiguae sphaerae negotium absolvant; veluti si foci distantia k debeat esse 100 pedum, ex casu primo radii facierum ita se habebunt:

$$a=1,5083\text{ ped.}\quad c=-1,5159\text{ ped.}$$
$$b=1,5083\text{ ped.}\quad d=-1,8956\text{ ped.}$$

sicque ne sphaera quidem opus est, cuius radius ad duos pedes exsurgat; ac talis lens facile aperturam 5 pollicum admittit, quantam distantia focalis 100 pedum exigit, quin etiam aperturam 10 pollicum admitteret, unde tubus confici posset aeque amplificans, ac vulgaris 400 pedum, qui insuper a dispersione radiorum sit immunis. Verum pro exiguis telescopiis huiusmodi lentes compositae nullum usum praestabunt, quia debitae aperturae sunt incapaces, quemadmodum iam pro distantia focali decem pedum observavi, ubi apertura vix unum pollicem superare potest.

32. Ratio huius incommodi manifesto in hoc est sita, quod binae vitri species nimis parum ratione refractionis discrepant; si alia daretur materia diaphana, multo magis a vitri ratione discrepans, nullum est dubium, quin duplicem scopum multo felicius attingere liceat. Aqua quidem multo minori refractionis gradu est praedita, sed ob fluiditatem figurae lentium refragatur; crustis autem vitreis inclusa etiam vitri refractio accedit, et negotium turbat. Verum hoc incommodum tolli posse videtur, si crustae ubique aequaliter sint crassae, seu instar meniscorum parentur, in quibus radius convexitatis praecise sit radio concavitatis aequalis, simulque crassities sit minima. Tales ergo lentes tanquam aqueas spectare licebit, operaeque pretium erit constructionem eiusmodi lentium compositarum evolvere, ubi altera sit vitrea, altera aquea.

III. Casus, quo lens AB est aquea et lens GD vitrea

33. Pro lente ergo priori est $m=\frac{4}{3}$ et pro posteriori $n=\frac{31}{20}$, hinc fit

$$a=\frac{1}{3\mu}p,\quad b=\frac{1}{3(1-\mu)}p,\quad c=\frac{11}{20\nu}q,\quad d=\frac{11}{20(1-\nu)}q,$$

tum vero

$$\lambda=\frac{31ln}{44lm}=1,073304\ \text{ et } l\lambda=0,0307228\ \text{ et }\ A=\frac{77319}{31000}\lambda^3=3,08385,$$

atque

$$p=+0,06830k,\quad q=-0,07330k$$

tum

$$\mu=\frac{11+3y}{15} \quad \text{et} \quad v=\frac{6,6753+31z}{142}.$$

Nunc vero numeros y et z ex hac aequatione definiri oportet:

$$zz=8,25855+3,08385yy$$

Sumamus y ita, ut fiat $\mu=\frac{1}{2}$, eritque $y=-\frac{7}{6}$, et

$$zz=12,45601, \text{ ideoque } z=\pm 3,52930.$$

Sumatur valor positivus, fietque $v=0,81749$, hincque

$$a=b=\frac{2}{3}p,\ c=\frac{0,55}{0,81749}q \quad \text{et} \quad d=\frac{0,55}{0,18251}q,$$

unde radii facierum ita se habebunt:

$$a=0,04553k,\quad c=-0,049315k,$$
$$b=0,04553k,\quad d=-0,220892k,$$

si $y=-\frac{1}{2}$, reperiur $\mu=\frac{19}{30}$, $v=0,70301$,

$$a=0,035947,\quad c=-0,057346,$$
$$b=0,062091,\quad d=-0,135745,$$

si $y=-\frac{7}{8}$, fit $b=-c$ [proxime], qui casus notandus.

34. In figura apposita (Fig. 6) sumsi $k=5$ ped. vel 50 dig.
mensurae Rhenanae, ita ut in digitis sit:

pro lente AB utriusque faciei radius convex. $=-\ 2,276$ dig.
pro lente autem CD radius faciei anterioris $=-\ 2,466$ dig.
pro lente autem CD radius faciei posterioris $=-11,045$ dig.,

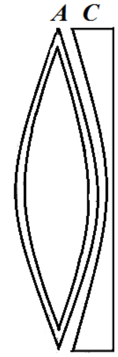

Fig. 6

ubi lentem anteriorem aqueam crustis vitreis inclusam repraesentavi, quae duabus meniscis constat, quarum tam convexitatis quam concavitatis radius sit 2,276 dig. Has igitur lentes iungendo oritur lens composita, distantiam focalem 5 pedum habens, quae si aperturam 2 dig. admitteret, ad multiplicationem sexagies adhiberi posset. Sin autem mensurae assignatae duplicentur, ut lens composita consequatur distantiam focalem 10 pedum, ea certe aperturam ultra 3 digitos admittet, ideoque obiecta plus quam centies multiplicare poterit.

35. Hic posita ratione refractionis ex aere in vitrum 1,55:1 , pro ratione refractionis ex aere in aquam assumsi $=1,33:1$, quae si forte a veritate aberret, constructio lentis compositae hic descriptae voto minime respondebit, siquidem, ut vidimus, minima aberratio omnem nostram expectationem fallere potest. Huic autem incommodo remedium afferri posse videtur, si mixtura ex aqua et spiritu vini ita praeparetur, ut

differentia refractionum accuratissime conveniat cum ea, quae in calculo fuerit assumta. Cum igitur ratio refractionis ex aere in spiritum vini sit fere $=1,37:1$, ponamus eiusmodi fieri mixturam, in qua ratio refractionis sit 1,35:1, quae semper certe obtineri poterit, atque retenta vitri refractione $=1,55:1$, determinationes utriusque lentis pro hoc casu quaeramus, ubi quidam lentem anteriorem ex aqua, posteriorem vero ex vitro confici ponamus.

IV. Casus, quo $m = 1,35$ *et* $n = 1,55$

36. Erit ergo

$$a=\tfrac{0,35}{\mu}p,\quad b=\tfrac{0,35}{1-\mu}p,\quad c=\tfrac{0,55}{v}q,\quad d=\tfrac{0,55}{1-v}q,$$

tum

$$\lambda=\tfrac{(m-1)nln}{(n-1)mlm}=1,06698 \quad \text{et} \quad l\lambda=0,0281572$$

et

$$p=\left(1-\tfrac{1}{\lambda}\right)k=0,06278k,\quad q=-(\lambda-1)k=-\ 0,06698k.$$

Porro

$$A=\tfrac{m(n-1)^2(n+2)}{n(m-1)^2(m+2)}\lambda^3=2,76215.$$

Iam statuamus esse $\mu=\frac{1}{2}$, ac fit $my=2-2mm,$ hinc $y=-1,218518$ et aequatio, ex qua z definiri debet, est

$$zz=12,15346-5,2+0,08648+4,11065$$

seu

$$zz=11,15059 \ \text{ et } \ z=3,33925,$$

unde

$$v=\tfrac{n(2n+1)+4\lambda(nn-1)+nz}{2(n+2)}=0,78099.$$

Ergo

$$a=b=0,7p,\quad c=\tfrac{0,55q}{0,78099},\quad d=\tfrac{0,55q}{0,21901},$$

ex quo lentis constructio ita se habebit:

$$\begin{aligned} a&=0,043946k, & c&=-0,04717k,\\ b&=0,043946k, & d&=-0,16807k. \end{aligned}$$

37. Forma huius lentis parum differt a praecedente, et quo distantia focalis k fiat 50 dig., lentis aqueae AB convexae radium utriusque faciei sumi oportet

$$= 2,1973 \text{ dig.},$$

pro altera autem lente vitrea CD statui debet

$$\text{radius faciei}\begin{cases}\text{anterioris} = -\,2,3585 \text{ dig.}\\ \text{posterioris} = -8,4035 \text{ dig.}\end{cases}$$

utraque facta concava. Quodsi talis lens omni cura fuerit elaborata, tum variae praeparentur mixturae ex aqua et spiritu vini secundum diversas proportiones et experimentando exploretur, quaenam earum binis meniscis inclusa optatum effectum producat: haecque methodus lentem omnibus numeris absolutam obtinendi commodissima videtur et ad praxin maxime idonea. Distantiam quidem focalem non nimis parvam assumi convenit, quia talis lens sufficientem aperturam non admitteret et hanc ob causam ea non infra 5 pedes sumenda videtur; quo maior autem statuatur, eo luculentius erit lucrum prae lentibus simplicibus eiusdem foci.

38. Ex his casibus evolutis colligimus, si duae materiae pellucidae diversae indolis praesto sint, ac lens anterior seu obiecta respiciens ex ea, quae minore refractione gaudet, conficiatur, eam utrinque aeque convexam fieri posse; tum autem alteram, ex materia magis refringente factam, utrinque quidam concavam, sed inaequaliter, confici debere. Operae igitur pretium erit, si lentis anterioris *AB* utriusque faciei radius detur, conspectui exponere, quomodo radii utriusque faciei concavae alterius lentis definiantur. Cum igitur in casibus expositis sit ratio refractionis pro lente posteriori $n=\frac{31}{20}$, posito $a=b=1$, radii c et d ita se habebunt:

$$\begin{array}{l|l|l|l|l|}
a=1 & \text{si } m=1,55 & \text{si } m=1,50 & \text{si } m=1,35 & \text{si } m=1,33\\
 & c=-\,1,0000 & c=-1,00506 & c=-1,07336 & c=-1,08314\\
b=1 & d=-1,0000 & d=-1,25680 & d=-3,82758 & d=-4,85157
\end{array}$$

unde etiam pro casibus mediis valores c et d concludere licet.

APPENDIX

DE LENTIUM OBIECTIVARUM VULGARIUM EMENDATIONE

39. Si vel duplicis generis vitrum, cuius quidem refractionis ratio satis sit diversa, comparari nequeat, vel si aquae usus minus ex voto succedat, calculus supra evolutus summa cum utilitate ad emendationem lentium vulgarium adhiberi poterit, neglecta priori conditione, qua dispersioni colorum occurritur. Iam pridem quidem lentium multiplicatione confusionem a figura sphaerica oriundam imminuere atque adeo ad nihilum redigera sum conatus, verum ibi singulas lentes seorsim minimam confusionem parere assumsi, qui casus ad praxin maxime accommodatus videbatur; neque tum hunc scopum binis lentibus attingere licuit. Si autem lentes quascunque admittere velimus, binis coniungendis confusio omnis tolli poterit, ubi autem probe tenendum est multo maiorem sollertiam ad eas efformandas ab artifice requiri.

40. Quo igitur huic tantum conditioni satisfaciamus, ambas lentes ex pari vitri specie confectas assumo, ut sit $n=m$; ac posita prioris *AB* distantia focali $=p$, posterioris *CD* vero $=q$, facies lentium ita se habebunt, ut sint earum radii

$$a=\tfrac{m-1}{\mu}p,\ b=\tfrac{m-1}{1-\mu}p,\ c=\tfrac{m-1}{v}q \text{ et } d=\tfrac{m-1}{1-v}q.$$

Nunc cum de colorum dispersione tollenda non sit quaestio, numerus λ pro arbitrio assumatur, seu sit $q=-\lambda p$, unde posita lentis compositae distantia focali *k,* erit

$$p=\left(1-\tfrac{1}{\lambda}\right)k \text{ et } q=-\left(\lambda-1\right)k.$$

Tum vero si ponamus

$$\mu=\tfrac{m(2m+1)+my}{2(m+2)} \text{ et } v=\tfrac{m(2m+1)-4\lambda(mm-1)\ +mz}{2(m+2)},$$

ob quantitatem $A=\lambda^3$ universa determinatio ad huius aequationis resolutionem perducitur:

$$zz=\left(4m-1\right)(\lambda^3-1)+4(m-1)^2\lambda(\lambda-1)+\lambda^3\ yy,$$

ubi patet singulos valores pro *y* et *z* inveniendos tam affirmative quam negative accipi posse.

41. Statuamus ergo pro vitro, unde ambae lentes parantur,

$$m=\tfrac{31}{20}=1,55,$$

quo valore substituto habebimus:

$$a=\tfrac{0,55}{\mu}p,\ b=\tfrac{0,55}{1-\mu}p,\ c=\tfrac{0,55}{v}q,\ \ d=\tfrac{0,55}{1-v}q.$$

tum vero

$$p=(1-\tfrac{1}{\lambda})k \text{ et } q=-\left(\lambda-1\right)k.$$

Porro

$$\mu=\tfrac{1271+310y}{1420} \text{ et } v=\tfrac{1271-1122\lambda+310z}{1420},$$

unde aequatio resolvenda erit:

$$zz=(\tfrac{26}{5}+yy)\lambda^3+1,21\lambda\lambda-1,21\lambda-\tfrac{26}{5},$$

ubi evidens est pro *y* omnes numeros pro lubitu assumi posse, cum pro *zz* semper numerus positivus prodeat, dum sit $\lambda>1$. At si $\lambda<1$, ob terminos negativos praevalentes, numerus *y* certum limitem superare debet. Verum pro λ valorem negativum omnino assumere non licet. Perpetuo autem cavendum est, ne numeri μ et v extra limites

0 et 1 excurrant, quippe quo casu menisci orientur, in quibus levissimus error funestus esse potest.

42. Statuatur ergo lens prior *AB* utrinque aequaliter convexa, seu $\mu = \frac{1}{2}$ sumique debet

$$310y = 710 - 1271 \text{ seu } y = -\tfrac{561}{310} \text{ et } yy = 3,27493,$$

unde nostra aequatio erit:

$$zz = 8,47493\lambda^3 + 1,21\lambda\lambda - 1,21\lambda - 5,2$$

et habebimus

$$a = b = 1,1p = \tfrac{11}{10}(1 - \tfrac{1}{\lambda})k,$$

si

$$\lambda = 2, \ a = b = 0, \ 55k, \ c = \tfrac{55k}{176,033}, \ d = \tfrac{55k}{76,033}$$

seu

$$c = -0,31244k, \ d = +0,72337k.$$

Hic primo observo, si esset $\lambda = 1$, fieret $zz = yy$, hoc autem casu ambae distantiae focales p et q evanescerent. Tum si λ parum superet unitatem, quantitates p et q nimis fierent exiguae, quod in praxi non parum est incommodum. In superiori Casu III habuimus $\lambda = 1,073$, quo ergo valore hic maiores assumamus. Neque vero opus est, ut ad valorem $\lambda = 2$ ascendamus, quia tum numerus v limites 0 et 1 transgrederetur. Sequentes ergo casus evolvamus.

Casus 1

43. Sit $\lambda = \frac{11}{10}$, erit

$$p = \tfrac{1}{11}k, \ q = -\tfrac{1}{10}k \text{ et } a = b = \tfrac{1}{10}k,$$

tum vero $zz = 6,21323$ et $z = \pm\, 2,49263$, cuius sumto valore positivo fit

$$v = \tfrac{809,5155}{1420}, \text{ ergo } c = \tfrac{0,55q}{0,57008} \ = 0 \ \text{ et } \ d = \tfrac{0,55q}{0,42992}.$$

Ex quo binarum lentium radii facierum erunt:

$$a = 0,10000k, \ c = -0,09649k,$$
$$b = 0,10000k, \ d = -0,12796k,$$

haec ergo lens ad summum admittit aperturam, cuius diameter $= \frac{64}{1000}k = \frac{1}{16}k$;
quae ergo ad satis parvas distantias focales adhiberi potest.

Casus II

44. Sit $\lambda = \frac{12}{10} = 1,2;$ erit $p = \frac{1}{6}k$ et $q = -\frac{1}{5}k$, atque

$$a = b = \frac{11}{60}k = 0,18333k.$$

Tum vero reperitur:

$$zz = 9,73507 \text{ et } z = 3,12011,$$

hincque

$$v = \frac{1271-1346+967,231}{1420} = 0,62805$$

ergo

$$c = \frac{0,55q}{0,62805}, \text{ et } d = \frac{0,55q}{0,37195},$$

unde pro lentibus construendis radii ita se habent:

$$a = 0,18333k, \quad c = -0,17514k,$$
$$b = 0,18333k, \quad d = -0,29573k,$$

quae lens admittit aperturam $\frac{1}{9}k$ in diametro.

Casus III

45. Sit $\lambda = \frac{13}{10} = 1,3,$ erit $p = \frac{3}{13}k,$ $q = -\frac{3}{13}k$ atque

$$a = b = \frac{33}{130}k = 0,253846k,$$

tum vero prodit:

$$zz = 13,89130 \text{ et } z = 3,72710$$

$$v = \frac{967,801}{1420} = 0,68155, \; c = \frac{0,55q}{0,68155}, \text{ et } d = \frac{0,55q}{0,31846},$$

unde radii pro utraque lente ita se habent:

$$a = 0,25385k, \quad c = -0,24209k,$$
$$b = 0,25385k, \quad d = -0,51813k,$$

aperturae diameter esse potest $\frac{1}{6}k.$

Casus IV

46. Sit $\lambda = \frac{14}{10} = 1,4,$ erit $p = \frac{2}{7}k$ et $q = -\frac{2}{5}k,$ atque $a = b = \frac{22}{70}k,$
tum vero prodit:

$$zz = 18,73277 \text{ et } z = 4,32814$$

hinc

$$v = \frac{1041,923}{1420} = 0,73375.$$

Ergo

$$c=\tfrac{0{,}55q}{0{,}73375}\ \text{ et }\ d=\tfrac{0{,}55q}{0{,}26625},$$

unde radii utriusque faciei erunt:

$$a=0{,}31429k,\ \ c=-0{,}29981k,$$
$$b=0{,}31429k,\ \ d=-0{,}82625k,$$

hic diameter aperturae esse potest $\frac{1}{5}k$.

Casus V

47. Sit $\lambda=\frac{3}{2}=1{,}5,$ erit $p=\frac{1}{3}k,\ \ q=-\frac{1}{2}k,$ atque $a=b=\frac{11}{330}k,$
tum vero prodit:

$$zz=24{,}31033\ \text{ et }\ z=4{,}93056$$

hincque

$$v=\tfrac{1116{,}4706}{1420}=0{,}78624.$$

Ergo

$$0{,}55q\ c=\tfrac{0{,}55q}{0{,}78624}\ \text{ et }\ d=\tfrac{0{,}55q}{0{,}21376},$$

unde radii utriusque faciei nostrarum lentium sunt:

$$a=0{,}36666k,\ \ c=-0{,}34976k,$$
$$b=0{,}36666k,\ \ d=-1{,}28650k,$$

cuius diameter aperturae fere ad $\frac{1}{4}k$ exsurgere potest, siquidem sumamus in huiusmodi lentibus diametrum aperturae duabus partibus tertiis radii minimi aequari posse; ut arcus nullus 40 gradibus maior in aperturam ingrediatur.

48. Cum eadem ergo lente priori *AB* infinitae lentes concavae coniungi possunt, prout alia atque alia distantia focalis requiritur. Ita si lentis *AB* uterque radius unitate exprimatur, alterius lentis *CD* radii utriusque concavitatis ita se habebunt secundum quinos casus evolutos:

	Casus I	Casus II	Casus III	Casus IV	Casus V
$a=1,$	$c=-0{,}9649$	$-0{,}9553$	$-0{,}9537$	$-0{,}9539$	$-0{,}9539$
$b=1,$	$d=-1{,}2796$	$-1{,}6131$	$-2{,}0411$	$-2{,}6290$	$-3{,}5086$

ubi imprimis notatu dignum evenit, quod in casibus II, III, IV et V valor ipsius *c* vix mutetur. Quodsi ergo lentis utrinque aequaliter convexae radius ponatur $=1$, alterius vero lentis concavae altera facies ita formetur, ut eius radius sit $=0{,}9538$, altera eius facies arbitrio nostro relinquitur, dummodo sit concava, eiusque radius intra limites 2 et 4 contineatur; id quod in praxi maximum commodum praestat, cum non sit necesse, ut artifex tantopere sit sollicitus in hac postrema facie effingenda.