CHAPTER IX

CONCERNNG THE USE OF THE DIFFERENTIAL CALCULUS IN RESOLVING EQUATIONS

227. The setting up of equations able to be reduced to an account of functions has been shown well enough now above. Indeed *y* may denote some function of *x;* if there may be put $y = 0$, all finite equations are understood entirely in this form, whether they be algebraic or transcending. Moreover the equation $y = 0$ is said to be resolved, if that value of x may be defined, which substituted into the equation *y* actually may reduce that to zero. Generally several values of this kind are given for *x*, which are called the roots of the equation $y = 0$. Therefore if we may put the numbers *f, g, h, i* etc. to be the roots of the equation $y = 0$, the function *y* will have been prepared, so that if in that in place of *x*, either *f*, *g* or *h* etc. may be substituted, there becomes actually $y = 0$.

228. Therefore because the function *y* vanished, if in place of *x* there may be put *f* or $x + (f - x)$ with the root of the equation $y = 0$ present f, there will be by that, which we have demonstrated above (in § 48) concerning functions,

$$0 = y + \frac{(f-x)dy}{dx} + \frac{(f-x)^2 ddy}{2dx^2} + \frac{(f-x)^3 d^3y}{6dx^3} + \text{etc.},$$

[At this stage, the function notation *y* = *F*(*x*) had not been developed, of which the above is a straight-forwards application.]

from which equation the value of the root *f* may be defined thus, so that, whatever for *x* were put in place and thence the values of the quantities $y, \frac{dy}{dx}, \frac{ddy}{2dx^2}$, etc. substituted, the same equation may result showing the true value of *f*. So that this may be seen more clearly, we may put in place to be

$$y = x^3 - 2x^2 + 3x - 4$$

there will be

$$\frac{dy}{dx} = 3xx - 4x + 3, \quad \frac{ddy}{2dx^2} = 3x - 2 \quad \text{and} \quad \frac{d^3y}{6dx^3} = 1.$$

With which values substituted there will arise

$$0 = x^3 - 2x^2 + 3x - 4 + (f - x)(3xx - 4x + 3) + (f - x)^2 (3x - 2x) + (f - x)^3$$

or with the multiplications actually carried out

$$f^3 - 2ff + 3f - 4 = 0;$$

evidently a similar equation arises similar to that proposed, which therefore contains the same roots.

229. But though in this manner it does not arrive at a new equation, from which the value of the root *f* may be able to be defined easily, yet hence huge aids can be given towards the finding of roots to be deduced. For if a close value were assumed now for *x* falling near to a certain root of the equation, thus so that $f - x$ shall be a very small quantity, then the terms of the equation

$$0 = y + \frac{(f-x)dy}{dx} + \frac{(f-x)^2 ddy}{2dx^2} + \frac{(f-x)^3 d^3y}{6dx^3} + \text{etc.}$$

converge strongly and on account of this reason will not differ much from the truth value, if besides the two initial terms the rest may be rejected. Therefore there will be, if now for *x* the value were assumed approximately equal to some root of the equation $y = 0$, approximately

$$0 = y + \frac{(f-x)dy}{dx} \text{ or } f = x - \frac{ydx}{dy},$$

from which formula even if not true, yet a value may be found very close to the value of the root f, which then in place of *x* substituted anew a much nearer value for *f* will be supplied at this stage and the true value of the root *f* may be approached.
[This first order method of approximation is often called Newton's method, who derived it geometrically.]

230. Hence therefore in the first place the roots of all the powers are able to be extracted for any numbers. Indeed let the proposed number be $a^n + b$, from which the root of the power *n* may be required to be extracted. There is put $x^n = a^n + b$ or $x^n - a^n - b = 0$, so that there shall be $y = x^n - a^n - b$; there will be

$$\frac{dy}{dx} = nx^{n-1}, \quad \frac{ddy}{2dx^2} = \frac{n(n-1)}{1 \cdot 2} x^{n-2}, \quad \frac{d^3y}{6dx^3} = \frac{n(n-1)(n-2)x^{n-2}}{1 \cdot 2 \cdot 3} x^{n-3} \quad \text{etc.}$$

Hence, if the root sought is put $= f$, so that there shall be $f = \sqrt[n]{(a^n + b)}$, there will be

$$0 = x^n - a^n - b + n(f - x)x^{n-1} + \frac{n(n-1)}{1 \cdot 2}(f - x)^2 x^{n-2} + \text{etc.}$$

Therefore if now for x there is established a number lying close to the value of the root sought f, which comes about on putting $x = a$, if indeed b shall be an extremely small number, so that $a^n + b < (a+1)^n$, there will be $b = na^{n-1}(f - a)$ and thus approximately

$$f = a + \frac{b}{na^{n-1}}$$

from which the value of the root will become known much closer. But if at this point we may wish to assume the third term, so that there shall be

$$b = na^{n-1}(f - x) + \frac{n(n-1)}{1\cdot 2}a^{n-2}(f - x)^2,$$

there becomes

$$(f - a)^2 = -\frac{2a}{n-1}(f - a) + \frac{2b}{n(n-1)a^{n-2}}$$

and thus

$$f = a - \frac{a}{n-1} \pm \sqrt{\left(\frac{aa}{(n-1)^2} + \frac{2b}{n(n-1)a^{n-2}}\right)}$$

or

$$f = \frac{(n-2)a + \sqrt{\left(aa + 2(n-1)b : na^{n-2}\right)}}{n-1}$$

Whereby with the aid of the extraction of the square root, the value of the root f will be found closer.

EXAMPLE

We seek the square root from some number c or there shall be $xx - c = y$.

Therefore we may put the approximate number of the root $= a$ and $b = c - aa$; on account of $aa + b = c$ and because there is $n = 2$, the first formula makes $f = a + \frac{c-aa}{2a} = \frac{c+aa}{2a}$ truly the other gives $f = \sqrt{c}$, which is the root sought itself. Therefore since the root shall be approximately $= \frac{c+aa}{2a}$, here the value for a may be written and the root more closely will be $f = \frac{cc+6aac+a^4}{4a(c+aa)}$. For argument's sake let $c = 5$, from the previous formula there will be $f = \frac{5}{2a} + \frac{a}{2}$. Therefore there is put $a = 2$, there will be $f = 2{,}25$; now there is put $a = 2{,}25$, there becomes $f = 2{,}236111$; again there is put $a = 2{,}236111$, there will be $f = 2{,}2360679$, which value disagrees minimally now from the true value.

231. Moreover in a similar manner the root of any equation can be found approximately with the help of the equation $f = x - \frac{ydx}{dy}$, clearly after the assumed value for x had been disagreeing a little from some root of an equation. Indeed towards finding a value for x of this kind in turn various values are substituted for x and between these that may be selected which indicates the smallest value of the function y that is approximately zero. Thus if there shall be

$$y = x^3 - 2x^2 + 3x - 4,$$

on putting $x = 0$ there becomes $y = -4$
on putting $x = 1$ there becomes $y = -2$
on putting $x = 2$ there becomes $y = +2$,

from which that a root is present between the values 1 and 2 of x. Therefore since there shall be $\frac{dy}{dx} = 3xx - 4x + 3$, this equation will be had for the root of the equation f $y = x^3 - 2x^2 + 3x - 4$ required to be found,

$$f = x - \frac{x^3 - 2x^2 + 3x - 4}{3xx - 4x + 3}.$$

Therefore if $x = 1$; it becomes $f = 1 + \frac{2}{2} = 2$. Now there may be put $x = 2$; it makes $f = 2 - \frac{2}{7} = \frac{12}{7}$. Therefore let there be $x = \frac{12}{7}$; there will be $f = \frac{12}{7} - \frac{104}{1701} = \frac{2812}{1701} = 1{,}653$. If we may wish to progress further, we will use logarithms more conveniently.
Therefore there may be put $x = 1{,}653$ and there shall be

lx = 0,2182729	x = 1,653000
lx^2 = 0,4365458	x^2 = 2,732409
lx^3 = 0,6548187	x^3 = 4,516673
x^3 = 4,516673	
$3x$ = 4,959000	
$x^3 + 3x$ = 9,475673	$3xx + 3$ = 11,197227
$2xx + 4$ = 9,464818	$4x$ = 6,612000
num.= 0,010855	den. = 4,585227
l num.= 8,0356298	
l den.= 0,6613608	x = 1,653000
l fraction = 7,3742690	fraction = 0,002367
	f = 1,650633

which now lies nearest to the true value.

232. But we will be able to deduce the approximations more quickly from the general expression. For since on putting some function $y=0$, if the root of this equation were $x=f$, we will have found to be

$$0=y+\frac{(f-x)dy}{dx}+\frac{(f-x)^2 ddy}{2dx^2}+\frac{(f-x)^3 d^3y}{6dx^3}+\text{etc.},$$

let $f-x=z$, thus so that the root shall be $f=x+z$, and there may be put

$$\tfrac{dy}{dx}=p,\quad \tfrac{dp}{dx}=q,\quad \tfrac{dq}{dx}=r,\quad \tfrac{dr}{dx}=s\quad \text{etc.};$$

there will be

$$0=y+zp+\frac{z^2q}{2}+\frac{z^3r}{6}+\frac{z^4s}{24}+\frac{z^5t}{120}+\text{etc.};$$

in which equation on taking some value for *x*, from which likewise *y, p, q, r, s* etc. are determined, the quantity *z* must be found, with which found the root of the proposed equation $y=0$ will be had $f=x+z$. Therefore it is required to concentrate on that, so that the unknown value *z* may be elicited from this most conveniently.

233. We may devise this converging series for *z* series

$$z=A+B+C+D+E+\text{etc}.$$

and with the substitution made there will be

$$\begin{aligned}
y &= y\\
pz &= Ap+Bp+Cp+Dp+Ep+\text{etc.}\\
\tfrac{1}{2}qz^2 &= +\tfrac{1}{2}A^2q+ABq+ACq+ADq+\text{etc.}\\
&\qquad +\tfrac{1}{2}BBq+BCq+\text{etc.}\\
\tfrac{1}{6}rz^3 &= \tfrac{1}{6}A^3r+\tfrac{1}{2}A^2Br+\tfrac{1}{2}A^2Cr+\text{etc.}\\
&\qquad +\tfrac{1}{2}AB^2r+\text{etc.}\\
\tfrac{1}{24}sz^4 &= \tfrac{1}{24}A^4s+\tfrac{1}{6}A^3Bs+\text{etc.}\\
\tfrac{1}{120}tz^5 &= \tfrac{1}{120}A^5t+\text{etc.}
\end{aligned}$$

From which the following sequences are obtained

$$A = -\frac{y}{p}$$
$$B = -\frac{yyq}{2p^3}$$
$$C = -\frac{y^3qq}{2p^5} + \frac{q^3r}{6p^4}$$
$$D = -\frac{5y^4q^3}{8p^7} + \frac{5y^4qr}{12p^6} - \frac{y^4s}{24p^5}$$
etc.

and thus there will be

$$z = -\frac{y}{p} - \frac{y^2q}{2p^3} - \frac{y^3qq}{2p^5} + \frac{q^3r}{6p^4} - \frac{5y^4q^3}{8p^7} + \frac{5y^4qr}{12p^6} - \frac{y^4s}{24p^5} - \text{etc.}$$

EXAMPLE

Let this equation proposed $x^5 + 2x - 2 = 0$.

Therefore there will be

$$y = x^5 + 2x - 2, \quad \frac{dy}{dx} = p = 5x^4 - 2, \quad \frac{dp}{dx} = q = 20x^3,$$
$$\frac{dq}{dx} = r = 60x^2, \quad \frac{dr}{dx} = s = 120x \quad \text{etc.}$$

But now there may be put $x = 1$, because this value differs slightly from the root, there will be

$$y = 1, \quad p = 7, \quad q = 20, \quad r = 60, \quad s = 120,$$

from which there becomes

$$z = -\frac{1}{7} - \frac{10}{7^3} - \frac{200}{7^5} + \frac{10}{7^4} - \frac{5\cdot 1000}{7^7} + \frac{500}{7^6} - \frac{5}{7^5} + \text{etc.}$$

or

$$z = -\frac{1}{7} - \frac{10}{7^3} - \frac{130}{7^5} - \frac{1745}{7^7} - \text{etc.},$$

and therefore there will be $z = -0{,}18$ and the root $f = 0{,}82$; which value, if it may be substituted in place of x anew, will produce a root very close to the true root.

234. Therefore we have found an infinite series, which will express the root of any equation; but this works with this disadvantage, as then the law of the progression may not be apparent, then it shall be exceedingly complex and not convenient enough for use. Therefore we may undertake the same matter in another way and we may investigate a more regular series expressing the root of some proposed equation.

As before let the proposed equation be $y=0$ with some function y of x present and the question returns to this, that the value of x may be defined, which substituted in place of x may return the function y equal to zero. But since y shall be a function of x, in turn x can be regarded as a function of y and with this consideration used the value is sought of the function x, which it adopts, when the function y vanishes. Therefore if f is put to designate that value of x, which will be the root of the equation $y=0$, because x will change into f, if there may be put $y=0$, there will be by that which has been shown above in § 67,

$$f=x-\frac{ydx}{dy}+\frac{y^2ddx}{2dy^2}-\frac{y^3d^3x}{6dy^3}+\frac{y^4d^4x}{24dy^4}-\text{etc.},$$

In which equation the differential dy is placed constant. Therefore if there is put

$$\frac{dx}{dy}=p,\quad \frac{dp}{dy}=q,\quad \frac{dq}{dy}=r,\quad \frac{dr}{dy}=s\quad \text{etc.},$$

there will be from these values introduced, so that the consideration of constant differentials may be laid aside,

$$f=x-py+\tfrac{1}{2}qy^2-\tfrac{1}{6}ry^3+\tfrac{1}{24}sy^4-\tfrac{1}{120}ty^5+\text{etc.}$$

235. Therefore with some value attributed to x itself the values of y and of the quantities p, q, r, s etc. will be determined and from these found an infinite series will be had expressing the value of the root f. But if the equation $y=0$ may allow several roots, then these will be produced, if different values are taken for x; because indeed y is able to adopt the same value, even if different values are given to x, it is no wonder the same series can accommodate several values generally. Therefore so that in these cases the ambiguity may be removed and likewise the series may be returned converging, a value must be assumed for x now approaching close to the value of that root which is sought. For in this manner the value of y becomes very small and the terms of the series decrease exceedingly, thus so that with a few terms requiring to be taken the correct value for f may be found. Therefore if then this value may be substituted in place of x, a much smaller quantity y may emerge and the series converges much more and in this manner the root f at once becomes known exactly, so that the error shall become minimal. And hence it is evidently clear that the sum of this expression be demanded before that, which we have elicited before.

236. We may put in place the root of the power n requiring to be extracted from some number N. Therefore with the nearest power of the exponent taken of the exponent n the proposed number may be resolved easily in this form $N=a^n+b$. Therefore there will be

$$x^n=a^n+b\quad\text{and}\quad y=x^n-a^n-b,$$

from which there becomes

$$dy = nx^{n-1}dx \qquad \text{et} \quad \tfrac{dx}{dy} = p = \tfrac{1}{nx^{n-1}}$$

$$dp = -\tfrac{(n-1)dx}{nx^{n}} \qquad \text{et} \quad \tfrac{dp}{dy} = q = -\tfrac{n-1}{nnx^{2n-1}}$$

$$dq = \tfrac{(n-1)(2n-1)dx}{nnx^{2n}} \qquad \text{et} \quad \tfrac{dq}{dy} = r = \tfrac{(n-1)(2n-1)}{n^3x^{3n-1}}$$

$$dr = -\tfrac{(n-1)(2n-1)(3n-1)dx}{n^3x^{3n}} \qquad \text{et} \quad \tfrac{dr}{dy} = s = -\tfrac{(n-1)(2n-1)(3n-1)}{n^4x^{4n-1}}$$

etc.

Now there may be put $x = a$ and there shall be $y = -b$ and the root sought $f = \sqrt[n]{\left(a^n + b\right)}$ may be expressed in this manner

$$f = a + \tfrac{b}{na^{n-1}} - \tfrac{(n-1)bb}{n\cdot 2na^{2n-1}} + \tfrac{(n-1)(2n-1)b^3}{n\cdot 2n\cdot 3na^{3n-1}} - \tfrac{(n-1)(2n-1)(3n-1)b^4}{n\cdot 2n\cdot 3n\cdot 4na^{4n-1}} + \text{etc.}$$

and thus the same series will be produced, which commonly is accustomed to be elicited from the expansion of the binomial $\left(a^n + b\right)^{\frac{1}{n}}$.

237. Therefore in the actual extraction of the root after a had been found approximately and likewise the remaining part b had been found, then to the above root it is required to add the value of the fraction $\tfrac{b}{na^{n-1}}$, from which truly a closer root may be obtained. But there will be

$$a^{n-1} = \tfrac{N-b}{n}$$

on account of $N = a^n + b$. But truly in this just manner the root may be found greater, because the third term must be subtracted. So that therefore by division of the remainder b the root may be found approaching much nearer to the true value, a suitable divisor must be sought, which may be devised to be

$$na^{n-1} + \alpha b + \beta bb + \gamma b^3 + \text{etc.}$$

Therefore since there must be

$$\tfrac{b}{na^{n-1}+\alpha b+\beta bb+\gamma b^3+\text{etc.}}$$
$$= \tfrac{b}{na^{n-1}} - \tfrac{(n-1)bb}{2n^2a^{2n-1}} + \tfrac{(n-1)(2n-1)b^3}{6n^3a^{3n-1}} - \tfrac{(n-1)(2n-1)(3n-1)b^4}{24n^4a^{4n-1}} + \text{etc.},$$

there becomes established on multiplication $na^{n-1}+\alpha b+\beta bb+\gamma b^3+\text{etc.}$

$$\begin{aligned} b = b &- \frac{(n-1)bb}{2na^n} + \frac{(n-1)(2n-1)b^3}{6n^2a^{2n}} - \frac{(n-1)(2n-1)(3n-1)b^4}{24n^3a^{3n}} + \text{etc.} \\ &+ \frac{\alpha b^2}{na^{n-1}} - \frac{(n-1)\alpha b^3}{2n^2a^{2n-1}} + \frac{(n-1)(2n-1)\alpha b^4}{6n^3a^{3n-1}} \\ &+ \frac{\beta b^3}{na^{n-1}} - \frac{(n-1)\beta b^4}{2n^2a^{2n-1}} \\ &+ \frac{\gamma b^4}{na^{n-1}} \end{aligned}$$

Hence the following determinations may be deduced

$$\alpha = \frac{n-1}{2a}$$

$$\beta = \frac{(n-1)\alpha}{2na^n} - \frac{(n-1)(2n-1)}{6na^{n+1}} = -\frac{(n-1)(n+1)}{12na^{n+1}}$$

$$\gamma = \frac{(n-1)\beta}{2na^n} - \frac{(n-1)(2n-1)\alpha}{6nna^{2n}} + \frac{(n-1)(2n-1)(3n-1)}{24n^2a^{2n+1}} = -\frac{(n-1)(n+1)}{24na^{2n+1}}.$$

Therefore the fraction will be required to be added to the root *a* now found above

$$\frac{b}{na^{n-1}+\frac{(n-1)b}{2a}-\frac{(nn-1)bb}{12na^{n+1}}-\frac{(nn-1)b^2}{24na^{2n+1}}-\text{etc.}}$$

238. But if therefore the square root must be extracted from the number *N* and now the root shall be found approximately $=a$ with the remainder $=b$, to the root found above there must be added the quotient which arises, if the remainder *b* may be divided by

$$2a+\frac{b}{2a}-\frac{bb}{8a^3}+\frac{b^3}{16a^5}-\text{etc.}$$

But if the cube root should be extracted, then the remainder *b* must be divided by

$$3a^2+\frac{b}{a}-\frac{2bb}{9a^4}+\frac{b^3}{9a^7}-\text{etc.},$$

the use of which formulas we will indicate by these examples.

EXAMPLE 1

The square root is to be extracted from the number 200.

There may be put $N = 200$, and since the nearest square shall be 196, there will be $a = 14$ and the remainder $b = 4$, which therefore will be divided by

$$28 + \tfrac{1}{7} - \tfrac{1}{7\cdot 196} + \tfrac{1}{7\cdot 196\cdot 98},$$

and therefore there will be the divisor $= 28{,}142135$; by which if 4 may be divided, there will be obtained the decimal fraction to be added to 14, which will be correct to 10 figures and beyond.

EXAMPLE 2

The cube root is to be extracted from the number $N = 10$.

The nearest cube is 8 and the remainder $= 2$, from which $a = 2$ and $b = 2$ and the divisor $= 12 \;+ 1 - \tfrac{1}{18} = 12{,}9444.$ Whereby the cube root sought will be approximately

$$= 2\tfrac{2}{12{,}9444} = 2\tfrac{10000}{64722}.$$

239. The series for the root found can also be considered as recurring, arising from a certain fraction; for in this manner several more terms of the series to many fewer will be recalled, which constitute the numerator and denominator of the fraction. Thus with a little attention there may be seen to be given approximately

$$(a+b)^n = a^n \cdot \frac{a + \frac{n+1}{2}b}{a - \frac{n-1}{2}b}$$

and still closer

$$(a+b)^n = a^n \cdot \frac{aa + \frac{n+2}{2}ab + \frac{(n+1)(n+2)}{12}bb}{a - \frac{n-2}{2}ab + \frac{(n-1)(n-2)}{12}bb}$$

In a similar manner by introducing more terms still more accurate fractions can be obtained:

$$(a+b)^n = a^n \cdot \frac{a^3 + \frac{n+3}{2}a^2b + \frac{(n+3)(n+2)}{10}ab^2 + \frac{(n+3)(n+2)(n+1)}{120}b^3}{a^3 - \frac{n-3}{2}a^2b + \frac{(n-3)(n-2)}{10}ab^2 - \frac{(n-3)(n-2)(n-1)}{120}b^3}$$

There is no reason why also a general form of this kind cannot be shown, to which there shall be expressed conveniently

$$A=\frac{m(n+m)}{1\cdot 2m} \qquad \mathfrak{A}=\frac{m(n-m)}{1\cdot 2m}$$
$$B=\frac{(m-1)(n+m-1)}{2(2m-1)}A \qquad \mathfrak{B}=\frac{(m-1)(n-m+1)}{2(2m-1)}\mathfrak{A}$$
$$C=\frac{(m-2)(n+m-2)}{3(2m-2)}B \qquad \mathfrak{C}=\frac{(m-2)(n-m+2)}{3(2m-2)}\mathfrak{B}$$
$$D=\frac{(m-3)(n+m-3)}{4(2m-3)}C \qquad \mathfrak{D}=\frac{(m-3)(n-m+3)}{4(2m-3)}\mathfrak{C}$$
$$\text{etc.} \qquad \text{etc.}$$

But with these values determined there will be

$$(a+b)^n=a^n\cdot\frac{a^m+Aa^{m-1}b+Ba^{m-2}b^2+Ca^{m-3}b^3+\text{etc.}}{a^m-\mathfrak{A}a^{m-1}b+\mathfrak{B}a^{m-2}b^2-\mathfrak{C}a^{m-3}b^3+\text{etc.}}$$

240. If therefore here for *n* there may be substituted a fractional number, these formulas will be very well adapted to the extraction of roots. Thus if the some root of the power *n* must be extracted from the form a^n+b ,the following formulas are able to be called into use

$$(a^n+b)^{\frac{1}{n}}=a\cdot\frac{2na^n+(n+1)b}{2na^n+(n-1)b}$$
$$(a^n+b)^{\frac{1}{n}}=a\cdot\frac{12n^2a^{2n}+6n(2n+1)a^nb+(2n+1)(n+1)bb}{12n^2a^{2n}+6n(2n-1)a^nb+(2n-1)(n-1)bb}.$$

But if there may be put $a^n+b=N$, so that there shall be $a^n=N-b$, there will be

$$(a^n+b)^{\frac{1}{n}}=a\cdot\frac{2nN-(n-1)b}{2nN-(n+1)b}$$
$$(a^n+b)^{\frac{1}{n}}=a\cdot\frac{12n^2N^2-6n(2n-1)Nb+(2n-1)(n-1)bb}{12n^2N^2-6n(2n+1)Nb+(2n+1)(n+1)bb}.$$

241. Therefore the general formula for finding the root of any equation, which depends on several terms, performs the same use, which the usual rule of the binomial is accustomed to bring to the resolution of the pure equation $x^n=c$, and so in this case it will change into that rule itself. But if the equation contained several terms or even should be transcending, our general expression on being called into use always succeeds equally and produces an infinite series, which shows the value of the roots. On account of which since the great strength of this general formula rests on this matter, we may show the use of this a little further. Therefore let this equation be proposed depending on three constant terms

$$x^n + cx = N$$

with c and N denoting some given quantities. There may be put $x^n + cx - N = y$; there will be $dy = \left(nx^{n-1} + c\right)dx$ and hence there becomes $p = \frac{1}{nx^{n-1}+c}$; then there is [from §236]

$$dp = -\frac{n(n-1)x^{n-2}dx}{\left(nx^{n-1}+c\right)^2} \quad \text{and} \quad q = -\frac{n(n-1)x^{n-2}}{\left(nx^{n-1}+c\right)^3}.$$

In a similar manner on account of $r = \frac{dq}{dy}$, $s = \frac{dr}{dy}$; etc. there will be found

$$r = \frac{n^2(n-1)(2n-1)x^{2n-4} - n(n-1)(n-2)cx^{n-3}}{\left(nx^{n-1}+c\right)^5}$$

$$s = \frac{-n^2(n-1)(2n-1)(3n-1)x^{3n-6} + 4n^2(n-1)(n-2)(2n-1)cx^{2n-5} - n(n-1)(n-2)(n-3)c^2x^{n-4}}{\left(nx^{n-1}+c\right)^7}$$

$$t = \frac{\left\{\begin{array}{l} n^4(n-1)(2n-1)(3n-1)(4n-1)x^{4n-8} - n^3(n-1)(n-2)(2n-1)(29n-11)cx^{3n-7} \\ +n^2(n-1)(n-2)(2n-1)(11n-29)c^2x^{2n-6} - n(n-1)(n-2)(n-3)(n-4)c^3x^{n-5} \end{array}\right\}}{\left(nx^{n-1}+c\right)^9}$$

etc.

From which values found the root of the proposed equation will be

$$f = x - py + \tfrac{1}{2}qyy - \tfrac{1}{6}ry^3 + \tfrac{1}{24}sy^4 - \tfrac{1}{120}ty^5 + \text{etc.}$$

indeed for whatever values may be substituted for x, from which likewise the letters y, p, q, r etc. adopt determined values, the sum of the series will be equal to the value of one root.

EXAMPLE 1

Let this equation be proposed $x^3 + 2x = 2$.

There will be $c = 2$, $N = 2$ and $n = 3$ and $y = x^3 + 2x - 2$. There may be put $x = 1$; there will be $y = 1$ and

$$p = \frac{1}{5}, \quad q = -\frac{6}{5^3}, \quad r = \frac{78}{5^5}, \quad s = -\frac{16\cdot 90}{5^7} \text{ etc.}$$

[The value for r has been corrected in the *O.O.* edition from the original $\frac{84}{5^5}$ to $\frac{78}{5^5}$.]

and the [amended] root of the equation will be

$$f = 1 - \tfrac{1}{5} - \tfrac{3}{5^3} - \tfrac{13}{5^5} - \tfrac{60}{5^7} - \text{etc.} = 0{,}771072.$$

Now there may be put $x = 0{,}77$, and because there is $y = x^3 + 2x - 2$,

$$p = \tfrac{1}{3xx+2}, \quad q = -6p^3x, \quad r = 90xxp^5 - 12p^5$$

and

$$s = -2160p^7x^3 + 720p^7x,$$

there will be had on using logarithms

$lx = 9{,}8864907$	$x = 0{,}77$
$lx^2 = 9{,}7729814$	$x^2 = 0{,}5929$
$lx^3 = 9{,}6594721$	$x^3 = 0{,}456533$
	$2x = 1{,}54$
	$x^3 + 2x = 1{,}996533$

Hence $y = -\,0{,}003467$

$l(-y) = 7{,}5399538$	$3xx + 2 = 3{,}7787$
$lp = 9{,}4226575$	$l(3xx + 2) = 0{,}5773424$
$l(-py) = 6{,}9626113$	$-py = 0{,}000917511$
$lp^3 = 8{,}2679725$	
$lx = 9{,}8864907$	
$l3 = 0{,}4771213$	
$ly^2 = 5{,}0799076$	
$l\left(-\tfrac{1}{2}qyy\right) = 3{,}7114921$	$-\tfrac{1}{2}qyy = 0{,}000000514.$

Therefore the root $f = 0{,}770916997$, which scarcely varies from the truth in the final figure.

EXAMPLE 2

Let the proposed equation be $x^4 - 2xx + 4x = 8$.

There may be put $y = x^4 - 2xx + 4x - 8$; there will be $dy = 4dx\left(x^3 - x + 1\right)$,

$$p = \tfrac{1}{4\left(x^3 - x + 1\right)}, \quad \tfrac{dp}{dx} = \tfrac{-3xx+1}{4\left(x^3 - x + 1\right)^2}.$$

Therefore

$$q=\frac{-3xx+1}{16\left(x^3-x+1\right)^3},\quad \frac{dq}{dx}=\frac{21x^4-12xx-6x+3}{16\left(x^3-x+1\right)^4}\quad\text{and}\quad r=\frac{21x^4-12xx-6x+3}{64\left(x^3-x+1\right)^5}\quad\text{etc.,}$$

from which the root of the proposed equation will be

$$f=x-\frac{y}{4\left(x^3-x+1\right)}-\frac{(3xx-1)yy}{32\left(x^3-x+1\right)^3}-\frac{\left(7x^4-4xx-2x+1\right)y^3}{128\left(x^3-x+1\right)^5}-\text{etc.}$$

Therefore it may be required for a suitable value of x to be given, so that this series becomes converging. But in the first place it is evident, if such a value may be attributed to *x*, so that it may become x^3-x+1, then all the terms of the series beyond the first go off to infinity so nor thence can anything be concluded. Therefore it is agreed a value of such a kind be assigned to *x*, so that both y becomes exceedingly small and x^3-x+1 not very small. Let $x=1$; there will be $y=-5$ and

$$f=1+\tfrac{5}{4}-\tfrac{25}{16}+\tfrac{125}{64}-\text{etc.};$$

where since the three terms $\frac{5}{4}-\frac{25}{16}+\frac{125}{64}$ are similar to a geometric progression, the sum of which is $\frac{5}{9}$, there will be approximately $f=\frac{14}{9}$. Therefore we may put $x=\frac{3}{2}$; there will be

$$y=-\tfrac{23}{16}\quad\text{and}\quad x^3-x+1=\tfrac{23}{8},$$

from which there becomes [again with small corrections]

$$f=\tfrac{3}{2}+\tfrac{1}{8}-\tfrac{1}{64}+\tfrac{391}{256\cdot 529}-\text{etc.}=1{,}61.$$

Now there is put *x* = 1,61; there will be

$lx = 0{,}2068259$	$x = 1{,}61$	let $x^3-x+1=z$
$lx^2 = 0{,}4136518$	$x^2 = 2{,}5921$	
$lx^3 = 0{,}6204777$	$x^3 = 4{,}173281$	
$lx^4 = 0{,}8273036$	$x^4 = 6{,}718983$	
	hence	
$l(-y) = 8{,}4016934$	$y = -0{,}025217$	
$lz = 0{,}5518502$	$z = 3{,}563281$	
$l\frac{-y}{z} = 7{,}8498432$		
$l4 = 0{,}6020600$		
$l\frac{-y}{4z} = 7{,}2477832$	$-\frac{y}{4z} = 0{,}0017692$	
$l(3xx-1) = 0{,}8309926$	$3xx-1 = 6{,}7763$	

$$
\begin{array}{rl|l}
ly^2 = & \underline{6{,}8033868} & \\
 & 7{,}6343794 & \\
lz^3 = & 1{,}6555506 & \\
 & 5{,}9788288 & \\
l32 = & \underline{1{,}5051500} & \frac{(3xx-1)y^2}{32z^3} = 0{,}000002976 \\
 & 4{,}4736788 &
\end{array}
$$

Hence $f = 1{,}6117662$.

242. This method of finding the approximate roots of equations may be extended equally to transcending quantities. We may seek the number *x*, the logarithm of which taken from some table, to the given number itself may have the ratio as 1 to *n*, and this equation will be had $x - nlx = 0$; but the measure of these logarithms shall be *k*, thus so that these logarithms may themselves be obtained, if the hyperbolic logarithms may be multiplied by *k;* there will be $d.lx = \frac{kdx}{x}$. Therefore there may be put $x - nlx = y$ and *f* shall be the value of *x* sought itself, which may return *x* = *nlx*. Therefore since there shall be $y = x - nlx$, there will be

$$dy = dx - \frac{kndx}{x} = \frac{dx(x-kn)}{x}$$

and

$$\frac{dx}{dy} = p = \frac{x}{(x-kn)}, \qquad \text{from which} \quad dp = -\frac{knxdx}{(x-kn)^2}$$

therefore

$$\frac{dp}{dy} = q = -\frac{knx}{(x-kn)^3}, \qquad dq = \frac{2knxdx + k^2n^2dx}{(x-kn)^4}$$

$$\frac{dq}{dy} = r = \frac{knx(2x+kn)}{(x-kn)^5} \quad \text{etc.}$$

Whereby there becomes

$$f = x - \frac{xy}{x-kn} - \frac{knxyy}{2(x-kn)^3} - \frac{knxy^3(2x+kn)}{6(x-kn)^5} - \text{etc.}$$

Moreover below [§ 272] we will show this problem does not admit a solution, unless there shall be $kn > e$ with the number *e* present, the hyperbolic logarithm of which is = 1, or there must be $kn > 2{,}7182818$.

EXAMPLE

The number is sought beyond 10, *the logarithm of which from tables is equal to the tenth part of the number itself.*

Because the question is set up from tables of logarithms, there will be $k = 0{,}43429448190325$ and on account of $n = 10$ there will be had $kn = 4{,}3429448190325$. Now on making $x = 1$ there will be $y = 1$ and there becomes

$$f = 1 + \tfrac{1}{3{,}3429} + \tfrac{2{,}1714724}{(3{,}3429)^3} - \text{etc.}$$

and thus there will be approximately $f = 1{,}37$. Therefore there may be put in place $x = 1{,}37$; there will be $lx = 0{,}136720567156406$
and on account of $y = x - 10lx$ there will be

$$y = 0{,}00279432843594 \text{ and } -x + kn = 2{,}9729448190325.$$

Therefore there is made

$$\begin{aligned} lx &= 0{,}1367205 \\ ly &= \underline{7{,}4462773} \\ &\ 7{,}5829978 \\ l(kn - x) &= \underline{0{,}4731866} \\ &\ 7{,}1098112 \end{aligned}$$

$$\tfrac{-xy}{x-kn} = 0{,}00128769.$$

Then since there shall be the term $-\frac{knxyy}{2(x-kn)^3} = \frac{kny}{2(x-kn)^2} \cdot \frac{-xy}{x-kn}$, there will be

$$\begin{aligned} l\tfrac{-xy}{x-kn} &= 7{,}1098112 \\ ly &= 7{,}4462773 \\ lkn &= \underline{0{,}6377842} \\ &\ 5{,}1938727 \end{aligned}$$

$$\begin{aligned} l(kn - x)^2 &= \underline{0{,}9463732} \\ &\ 4{,}2474995 \\ l2 &= \underline{0{,}3010300} \\ l\text{third term.} &= 3{,}9464695 \\ 1.\ \text{term.}x &= 1{,}37 \end{aligned}$$

II. term. = 0,00128769
Ill. term. = 0,00000088
f = 1,37128857
lf = 0,137128857

243. If the equation were exponential, that will be able to be reduced to logarithms ; thus if the value of x is sought, so that there shall be $x^x = a$, there will be $xlx = la$. Whereby on putting $y = xlx - la$ there becomes

$$dy = dxlx + dx \text{ and } \frac{dx}{dy} = p = \frac{1}{1+lx}$$

and then

$$dp = \frac{-dx}{x(1+lx)^2} \text{ and } \frac{dp}{dy} = q = \frac{-1}{x(1+lx)^3},$$

$$dq = \frac{dx}{xx(1+lx)^3} + \frac{3dx}{xx(1+lx)^4} \text{ and thus } \frac{dq}{dy} = r = \frac{1}{xx(1+lx)^4} + \frac{3}{xx(1+lx)^5};$$

again there will be

$$dr = \frac{-2dx}{x^3(1+lx)^4} - \frac{10dx}{x^3(1+lx)^5} - \frac{15dx}{x^3(1+lx)^6},$$

therefore

$$s = \frac{-2}{x^3(1+lx)^5} - \frac{10dx}{x^3(1+lx)^6} - \frac{15}{x^3(1+lx)^7},$$

and

$$t = \frac{6}{x^4(1+lx)^6} + \frac{40}{x^4(1+lx)^7} + \frac{105}{x^4(1+lx)^8} + \frac{105}{x^4(1+lx)^9},$$

$$u = \frac{-24}{x^5(1+lx)^7} - \frac{196}{x^5(1+lx)^8} - \frac{700}{x^5(1+lx)^9} - \frac{1260}{x^5(1+lx)^{10}} - \frac{945}{x^5(1+lx)^{11}}.$$

Hence therefore, if the true value of x shall be $= f$, thus so that there shall be $f^f = a$, there will be

$$\begin{aligned} f = x - \frac{y}{1+lx} - \frac{yy}{2x(1+lx)^3} - \frac{y^3}{2xx(1+lx)^5} - \frac{5y^4}{8x^3(1+lx)^7} - \frac{7y^5}{8x^4(1+lx)^9} \\ - \frac{y^3}{6x^2(1+lx)^4} - \frac{5y^4}{12x^3(1+lx)^6} - \frac{7y^5}{8x^4(1+lx)^8} \\ - \frac{y^4}{12x^3(1+lx)^5} - \frac{y^5}{3x^4(1+lx)^7} \\ - \frac{y^5}{20x^4(1+lx)^6} \end{aligned}$$

etc.

Therefore this expression continued to infinity, whatever value for x may be put in place, on taking $y = xlx - la$ will give the true value of f. Thus if there is put $x = 1$, there will be $y = -la$ and

$$f = 1 + la - \frac{(la)^2}{2} + \frac{2(la)^3}{3} - \frac{9(la)^4}{8} + \frac{32(la)^5}{15} - \frac{625(la)^6}{144} - \text{etc.},$$

where it is to be noted that la is the hyperbolic logarithm of a.

EXAMPLE

There is sought the number f, so that there shall be $f^f = 100$.

Since there shall be

$$a = 100 \quad \text{and} \quad y = xlx - la = xlx - l100,$$

because it is apparent that $f > 3$ and < 4, there may be put in place $x = \frac{7}{2}$ and there will be

$$\begin{aligned} lx &= \ \ 1,25276296849 \\ xlx &= \ \ 4,38467038972 \\ l100 &= \ \ \underline{4,60517018599} \\ y &= -0,22049979627 \\ 1 + lx &= \ \ 2,25276296849 \end{aligned}$$

Hence there will be with common logarithms used

$$\begin{aligned} l(-y) &= 9,3434083 \\ l(1+lx) &= 0,3527156 \\ &\ \ \ \ 8,9906927 \qquad \frac{-y}{1+lx} = 0,0978797 \\ ly^2 &= 8,6868166 \\ 3l(1+lx) &= 1,0581468 \\ &= 7,6286698 \\ 12x = l7 &= 0,8450980 \\ &\ \ \ \ 6,7835718 \qquad \frac{y^2}{2x(1+lx)^3} = 0,0006075. \end{aligned}$$

Therefore there will be approximately $f = 3,5972722$; truly with the above following terms taken there will be $f = 3,5972852$.

244. But besides the differential calculus has a conspicuous use in the resolution of equations, if a certain relation were known, which lies between the roots. Let the proposed equation be $y = 0$, in which y shall be some function of x. If now for argument's sake the two roots of this equation agree to differ between themselves by a given quantity a, these two roots may be found easily in the following manner. Of these two roots, x may denote the smaller of the roots ; the greater will be $=x+a$; whereby since the function y may vanish, if x should signify one root from the roots of the equation $y=0$, y may also vanish, if in place of x there may be put $x+a$. On account of which there will be

$$0=y+\frac{ady}{dx}+\frac{a^2ddy}{2dx^2}+\frac{a^3d^3y}{6dx^3}+\text{etc.}$$

From which since there shall be $y=0$, there will also be

$$0=\frac{dy}{dx}+\frac{addy}{2dx^2}+\frac{a^2d^3y}{6dx^3}+\frac{a^3d^4y}{24dx^4}+\text{etc.},$$

which two equations taken together by the method of elimination will give the value of this root x, which the other root exceeds by the quantity a.

EXAMPLE

Let this equation be proposed $x^5-24x^3+49xx-36=0$, as from which it may be agreed in some manner to have two roots different by unity.

On putting $y=x^5-24x^3+49xx-36$ there will be

$$\frac{dy}{dx}=5x^4-72x^2+98x$$
$$\frac{ddy}{2dx^2}=10x^3-72x+49$$
$$\frac{d^3y}{6dx^3}=10x^2-24$$
$$\frac{d^4y}{24dx^4}=5x$$
$$\frac{d^5y}{120dx^5}=1.$$

Now on account of $a=1$ there will be

$$A \quad . \quad . \quad . \quad 5x^4+10x^3-62x^2+31x+26=0.$$

But there is

$$B \quad . \quad . \quad . \quad x^5-24x^3+49x-36=0.$$

The former may be multiplied by x and the latter by 5 and with the one taken from the other there will remain

$$10x^4+\ 58x^3-214x^2+26x+180=0$$

or

$$C \quad . \quad . \quad . \quad 5x^4+29x^3-107x^2+13x+90=0,$$

from which with the first subtracted there will be left

$$
\begin{array}{lllll}
D & \ldots & 19x^3 - 45x^2 & -18x + & 64 = 0. \\
D \cdot 5x & \ldots & 95x^4 - 225x^3 - & 90x^2 + 320x = 0 & \\
A \cdot 19 & \ldots & 95x^4 + 190x^3 - 1178x^2 & + 589x + & 494 = 0 \\ \hline
E & \ldots\ \ldots & 415x^3 - 1088x^2 & + 269x + & 494 = 0 \\
D \cdot 415 & \ldots\ . & 7885x^3 - 18675x^2 & - 7470x + 26560 & = 0 \\
E \cdot 19 & \ldots\ . & 7885x^3 - 20672x^2 & + 5111x + 9386 & = 0 \\ \hline
F & \ldots\ \ldots\ \ldots & & 1997x^2 - 12581x + 17174 = 0 &
\end{array}
$$

$$
\begin{array}{ll}
D \cdot 247. \ . & 4693x^3 - 11115x^2 - 4446x + 15808 = 0 \\
E \cdot 32 \ . & 13280x^3 - 34816x^2 + 8608x + 15808 = 0 \\ \hline
 & 8587x^3 - 23701x^2 + 13054x = 0 \\
G. \ldots\ldots & 8587x^2 - 23701x + 13054 = 0
\end{array}
$$

$$
\begin{array}{ll}
F \cdot 8587. \ . & .17148239x^2 - 108033047x + 147473138 = 0 \\
G \cdot 1997. \ . & .17148239x^2 - 47330897x + 26068838 = 0 \\ \hline
 & 60702150x - 121404390 = 0.
\end{array}
$$

From which equation it follows that $x = 2$ and therefore also a root of the equation will be t $x = 3$, each value of which satisfies the equation.

245. But this operation can be completed without the aid of the differential calculus, therefore so that the same equation is produced, as has been made available by the differential calculus, if in the proposed equation itself there may be put $x + a$ in place of *x*. Moreover this method is exceedingly labourous to be carried out, and if the equations were of higher grades, the labour would become completely insurmountable ; from which it is much less able to have a place in transcendental equations. But if we may put the two roots of the proposed equation $y = 0$ equal to each other, then on account of $a = 0$, the differential equation will change into this $\frac{dy}{dx} = 0$. Therefore as often as a certain equation $y = 0$ will have two equal roots, so also there will be $\frac{dy}{dx} = 0$ and these two equations taken together will give that value of *x*, to which the two roots are equal. From which in turn, if both the equations $y = 0$ and $\frac{dy}{dx} = 0$ may have a common root, that will be a double root of the equation $y = 0$. But here it may come about, if after the quantity *x* were completely removed finally with the aid of these two equations $y = 0$ et $\frac{dy}{dx} = 0$, we may arrive at the same equation. Thus if the equation may be proposed

$$x^3 - 2xx - 4x + 8 = 0,$$

there will also be $3xx - 4x - 4 = 0$, the double of which added to that gives

$$x^3 + 4xx - 12x = 0 \text{ or } xx + 4x - 12 = 0,$$

the triple of which is

$$\begin{array}{r} 3xx + 12x - 36 = 0 \\ 3xx - \ \ 4x - \ \ 4 = 0 \\ \hline 16x - 32 = 0 \\ x - 2 = 0. \end{array}$$

there is on taking

Therefore since $x = 2$ will have been produced, this value may be substituted into the one preceding $3xx - 4x - 4 = 0$ and the same equation will be produced $12 - 8 - 4 = 0$, from which it is deduced the proposed equation $x^3 - 2xx - 4x + 8 = 0$, to have two equal roots, evidently 2.

246. If therefore an algebraic equation of some dimension may be had

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \text{etc.} = 0,$$

which may have roots equal to each other, there will be also

$$nx^{n-1} + (n-1)Ax^{n-2} + (n-2)Bx^{n-3} + (n-3)Cx^{n-4} + (n-4)Dx^{n-5} + \text{etc.} = 0,$$

Clearly the double root of the former equation will likewise be a root of the latter equation. The former may be multiplied by *n* and subtracted from the latter multiplied by *x* and this new equation will be produced

$$Ax^{n-1} + 2Bx^{n-2} + 3Cx^{n-3} + 4Dx^{n-4} + \text{etc.} = 0.$$

Now the first equation multiplied by *a* and this one multiplied by *b* may be added; there will be

$$ax^n + (a+b)Ax^{n-1} + (a+2b)Bx^{n-2} + (a+3b)Cx^{n-3} + \text{etc.} = 0,$$

which equation taken together with that proposed will show equal roots, if the proposed equation has such roots. Therefore since the quantities a and b may be able to be assumed as it pleases, the coefficients $a, a+b, a+2b$ etc. represent some arithmetic progression. On account of which if some equation may have two equal roots, these may be found, if the individual terms of the proposed equation may be multiplied respectively by the terms of some arithmetical progression ; for the new equation resulting in this way also will contain that root, which is present twice in the proposed equation. Thus the equation

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \text{etc.} = 0,$$

if the terms of this may be multiplied by this arithmetical progression

$$a,\ a+b,\ a+2b,\ a+3b,\ a+4b\ \text{ etc.},$$

this new equation will be produced

$$ax^n + (a+b)Ax^{n-1} + (a+2b)Bx^{n-2} + (a+3b)Cx^{n-3} + \text{etc.} = 0,$$

which will show equal roots with that. And this is the rule known well enough required for finding the equal roots of some equation.

247. If the equation $y=0$ may have three equal roots, not only will there be $\frac{dy}{dx}=0$, but also there will be $\frac{ddy}{dx^2}=0$, if a certain value of this root may be put in place for x, which is present three time in the equation $y=0$. Towards showing this we may put the equation $y=0$ to have roots of this kind $x,\ x+a$ and $x+b,$ which in the first place with finite intervals a and b disagree with each other in turn ; and because y vanishes, if in place of x both $x+a$ as well as $x+b$ may be written, there will be

$$y=0$$
$$y+\frac{ady}{dx}+\frac{a^2ddy}{2dx^2}+\frac{a^3d^3y}{6dx^3}+\frac{a^4d^4y}{24dx^4}+\text{etc.}=0$$
$$y+\frac{bdy}{dx}+\frac{b^2ddy}{2dx^2}+\frac{b^3d^3y}{6dx^3}+\frac{b^4d^4y}{24dx^4}+\text{etc.}=0;$$

from which if the first equation may be subtracted from the two latter equations, there will be

$$\frac{dy}{dx}+\frac{addy}{2dx^2}+\frac{a^2ddy}{6dx^3}+\frac{a^3d^4y}{24dx^4}+\text{etc.}=0$$
$$\frac{dy}{dx}+\frac{bddy}{2dx^2}+\frac{b^2ddy}{6dx^3}+\frac{b^3d^4y}{24dx^4}+\text{etc.}=0$$

These also may be subtracted from each other in turn and on division by $a-b$ there will be made

$$\frac{ddy}{2dx^2}+\frac{(a+b)d^3y}{6dx^3}+\frac{(aa+ab+bb)d^4y}{24dx^4}+\text{etc.}=0.$$

Now there may be put $a=0$ and $b=0$, thus so that these three roots shall be equal to each other, and there will be on account of the vanishing terms

$$y=0,\quad \frac{dy}{dx}=0 \quad \text{and} \quad \frac{ddy}{dx^2}=0.$$

248. Therefore as often as the equation $y=0$ may have three equal roots, for example *f, f, f*, then that quantity *f* will be also a root not only of this equation $\frac{dy}{dx}=0$, but also of this $\frac{ddy}{dx^2}=0$. Hence it is evident, since *f* shall be a root of the common equations $\frac{dy}{dx}=0$ and of its differential $\frac{ddy}{dx^2}=0$, that must be present twice in the equation $\frac{dy}{dx}=0$ by that which we have shown before concerning two equal roots. Whereby if the equation

$$x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+Dx^{n-4}+\text{etc.}=0,$$

may contain the three equal roots *f, f, f,* if the terms of this may be multiplied by the terms of some arithmetical progression, then the resulting equation will have the two equal roots *f* and *f*; on account of which that will be able to be multiplied again by some arithmetical progression, so that an equation may be produced including the same root *f* once. Therefore the three equations will be obtained having the common root *f* , from the combination of which this root will be extracted easily. For if such arithmetical progressions of this kind may be chosen, of which either their first or last term shall be $=0$, then an equation will be produced inferior by one degree and thus an easier elimination will emerge from that.

249. It will be shown in a similar manner, if the equation $y=0$ may have four equal roots *f, f, f, f* then on putting $x=f$ not only does there become $y=0$, $\frac{dy}{dx}=0$ and $\frac{ddy}{dx^2}=0$, but also there shall be $\frac{d^3y}{dx^3}=0$. Evidently as the equation $y=0$ may contain the root $x=f$ four times, thus the equation $\frac{dy}{dx}$ will include the same root three times, truly the equation $\frac{ddy}{dx^2}=0$ twice and the equation $\frac{d^3y}{dx^3}=0$ once. This also may be seen easier, if we may consider the function *y* in this case must have a form of this kind t$(x-f)^4 X$ with X denoting some function of *x*. This form assumed will have

$$\frac{dy}{dx}=(x-f)^3\left(4X+\frac{(x-f)dX}{dx}\right)$$

and thus on division by $(x-f)^3$. Similarly again $\frac{ddy}{dx^2}$ will be have the factor $(x-f)^2$ and $\frac{d^3y}{dx^3}$ the factor $x-f$; from which it is evident, if the root $x=f$ may be present four times in the equation $y=0$, that must be present three times in the equation $\frac{dy}{dx}=0$, twice in the equation $\frac{ddy}{dx^2}=0$, and once in $\frac{d^3y}{dx^3}=0$.

CAPUT IX

DE USU CALCULI DIFFERENTIALIS IN AEQUATIONIBUS RESOLVENDIS

227. Constitutionem aequationum ad functionum rationem reduci posse supra iam satis ostensum est. Denotet enim y functionem quamcunque ipsius x; si ponatur $y=0$, in hac forma omnes omnino aequationes finitae, sive sint algebraicae sive transcendentes, comprehenduntur. Aequatio autem $y=0$ resolvi dicitur, si is ipsius x valor definiatur, qui in functione y substitutus eam actu nihilo aequalem reddat. Plerumque autem plures eiusmodi valores pro x dantur, qui aequationis $y=0$ radices vocantur. Si igitur ponamus numeros f, g, h, i etc. esse radices aequationis $y=0$, functio y ita erit comparata, ut, si in ea loco x vel f vel g vel h etc. substituatur, fiat revera $y=0$.

228. Quoniam igitur functio y evanescit, si in ea loco x ponatur f seu $x+(f-x)$ existente f radice aequationis $y=0$, erit per ea, quae supra § 48 de functionibus demonstravimus,

$$0=y+\frac{(f-x)dy}{dx}+\frac{(f-x)^2ddy}{2dx^2}+\frac{(f-x)^3d^3y}{6dx^3}+\text{etc.},$$

ex qua aequatione valor radicis f ita definitur, ut, quicquid pro x fuerit positum indeque valores quantitatum $y,\frac{dy}{dx},\frac{ddy}{2dx^2}$; etc. substituti, semper resultet aequatio verum valorem ipsius f exhibens. Quo hoc clarius percipiatur, ponamus esse

$$y=x^3-2x^2+3x-4$$

erit

$$\frac{dy}{dx}=3xx-4x+3,\quad \frac{ddy}{2dx^2}=3x-2\quad \text{et}\quad \frac{d^3y}{6dx^3}=1.$$

Quibus valoribus substitutis oritur

$$0=x^3-2x^2+3x-4+(f-x)(3xx-4x+3)+(f-x)^2(3x-2x)+(f-x)^3$$

seu multiplicationibus actu institutis

$$f^3-2ff+3f-4=0;$$

oritur scilicet aequatio similis ipsi propositae, quae propterea easdem continet radices.

229. Quanquam autem hoc modo ad novam aequationem non pervenitur, ex qua valor radicis *f* facilius definiri queat, tamen hinc ingentia subsidia ad inventionem radicum deduci possunt. Si enim pro *x* assumtus fuerit valor iam proxime ad quampiam radicem aequationis accedens, ita ut $f-x$ sit quantitas valde parva, tum termini aequationis

$$0=y+\frac{(f-x)dy}{dx}+\frac{(f-x)^2ddy}{2dx^2}+\frac{(f-x)^3d^3y}{6dx^3}+\text{etc.}$$

vehementer convergent hancque ob causam non multum a veritate aberrabitur, si praeter binos terminos initiales reliqui reiiciantur. Erit ergo, si pro *x* iam valor cuipiam aequationis $y=0$ radici prope aequalis fuerit assumtus, proxime

$$0=y+\frac{(f-x)dy}{dx}\ \text{seu}\ f=x-\frac{ydx}{dy},$$

ex qua formula etsi non verus, tamen admodum propinquus radicis *f* valor reperietur, qui deinceps denuo loco *x* substitutus multo adhuc propiorem valorem pro *f* suppeditabit sicque continuo propius ad verum radicis *f* valorem accedetur.

230. Hinc igitur primum radices omnium dignitatum ex quibuscunque numeris extrahi possunt. Sit enim propositus numerus a^n+b, ex quo radicem potestatis *n* extrahi oporteat. Ponatur $x^n=a^n+b$ seu $x^n-a^n-b=0$, ut sit $y=x^n-a^n-b$; erit

$$\frac{dy}{dx}=nx^{n-1},\quad \frac{ddy}{2dx^2}=\frac{n(n-1)}{1\cdot 2}x^{n-2},\quad \frac{d^3y}{6dx^3}=\frac{n(n-1)(n-2)x^{n-2}}{1\cdot 2\cdot 3}x^{n-3}\quad \text{etc.}$$

Hinc, si radix quaesita ponatur $=f$, ut sit $f=\sqrt[n]{(a^n+b)}$, erit

$$0=x^n-a^n-b+n(f-x)x^{n-1}+\frac{n(n-1)}{1\cdot 2}(f-x)^2x^{n-2}+\text{etc.}$$

Si igitur pro *x* iam statuatur numerus ad valorem radicis quaesitae *f* prope accedens, quod fiet ponendo $x=a$, si quidem *b* sit numerus tam parvus, ut $a^n+b<(a+1)^n$, erit $b=na^{n-1}(f-a)$ proxime ideoque

$$f=a+\frac{b}{na^{n-1}}$$

unde valor radicis multo propius cognoscetur. Sin autem adhuc tertium terminum assumera velimus, ut sit

$$b = na^{n-1}(f-x) + \frac{n(n-1)}{1\cdot 2}a^{n-2}(f-x)^2,$$

fiat

$$(f-a)^2 = -\frac{2a}{n-1}(f-a) + \frac{2b}{n(n-1)a^{n-2}}$$

ideoque

$$f = a - \frac{a}{n-1} \pm \sqrt{\left(\frac{aa}{(n-1)^2} + \frac{2b}{n(n-1)a^{n-2}}\right)}$$

seu

$$f = \frac{(n-2)a + \sqrt{\left(aa + 2(n-1)b : na^{n-2}\right)}}{n-1}$$

Quare ope extractionis radicis quadratae valor radicis *f* adhuc propius reperietur.

EXEMPLUM

Quaeramus radicem quadratam ex numero quocunque c seu sit $xx - c = y$.

Ponatur ergo numerus radici proximus $= a$ et $b = c - aa$; ob $aa + b = c$ et quia est $n = 2$, fiet prior formula $f = a + \frac{c-aa}{2a} = \frac{c+aa}{2a}$ altera vero dat $f = \sqrt{c}$, quae est ipsa radix quaesita. Cum igitur sit proxima radix $= \frac{c+aa}{2a}$, hic ipse valor pro *a* scribatur eritque propius radix $f = \frac{cc+6aac+a^4}{4a(c+aa)}$. Sit verbi gratia $c = 5$, erit ex priori formula $f = \frac{5}{2a} + \frac{a}{2}$. Ponatur ergo $a = 2$, erit $f = 2{,}25$; nunc ponatur $a = 2{,}25$, fiet $f = 2{,}236111$; statuatur porro $a = 2{,}236111$, erit $f = 2{,}2360679$, qui valor iam minime a vero discrepat.

231. Simili autem modo radix cuiuscunque aequationis inveniri potest proxime ope aequationis $f = x - \frac{ydx}{dy}$, postquam scilicet pro *x* assumtus fuerit valor parum a quapiam aequationis radice discrepans. Ad huiusmodi vero valorem pro *x* inveniendum substituantur successive pro *x* varii valores inter eosque is eligatur, qui functionis *y* minimum, hoc est cyphrae proximum valorem indicat. Sic si sit

$$y = x^3 - 2x^2 + 3x - 4,$$

posito $x = 0$ fit $y = -4$
posito $x = 1$ fit $y = -2$
posito $x = 2$ fit $y = +2$,

unde videmus radicem contineri inter valores 1 et 2 ipsius x. Cum ergo sit $\frac{dy}{dx}=3xx-4x+3$, habebitur pro radice f aequationis $y=x^3-2x^2+3x-4$ invenienda haec aequatio

$$f=x-\frac{x^3-2x^2+3x-4}{3xx-4x+3}.$$

Si ergo $x=1$; fiet $f=1+\frac{2}{2}=2$. Nunc ponatur $x=2$; fiet $f=2-\frac{2}{7}=\frac{12}{7}$. Sit ergo $x=\frac{12}{7}$; erit $f=\frac{12}{7}-\frac{104}{1701}=\frac{2812}{1701}=1{,}653$. Si ulterius progredi velimus, logarithmis commodius utemur.

Ponatur ergo $x=1{,}653$ eritque

lx = 0,2182729	x = 1,653000
lx^2 = 0,4365458	x^2 = 2,732409
lx^3 = 0,6548187	x^3 = 4,516673
x^3 = 4,516673	
$3x$ = 4,959000	
x^3+3x = 9,475673	$3xx+3$ = 11,197227
$2xx+4$ = 9,464818	$4x$ = 6,612000
num.= 0,010855	den. = 4,585227
l num.= 8,0356298	
l den.= 0,6613608	x = 1,653000
l fract = 7,3742690	fractio = 0,002367
	f = 1,650633

qui valor iam proxime ad verum accedit.

232. Citiores autem approximationes ex generali expressione deducere poterimus. Cum enim posita functione quacunque $y=0$, si radix huius aequationis fuerit $x=f$, invenerimus esse

$$0=y+\frac{(f-x)dy}{dx}+\frac{(f-x)^2ddy}{2dx^2}+\frac{(f-x)^3d^3y}{6dx^3}+\text{etc.},$$

sit $f-x=z$, ita ut sit radix $f=x+z$, atque ponatur

$$\frac{dy}{dx}=p,\quad \frac{dp}{dx}=q,\quad \frac{dq}{dx}=r,\quad \frac{dr}{dx}=s\quad \text{etc.};$$

erit

$$0=y+zp+\frac{z^2q}{2}+\frac{z^3r}{6}+\frac{z^4s}{24}+\frac{z^5t}{120}+\text{etc.};$$

in qua aequatione sumto pro x valore quocunque, ex quo simul y, p, q, r, s etc. determinantur, inveniri debet quantitas z, qua inventa habebitur aequationis propositae

$y = 0$ radix $f = x + z$. In id ergo est incumbendum, ut quam commodissime ex hac aequatione valor incognitae z eruatur.

233. Fingatur pro z series convergens haec

$$z = A + B + C + D + E + \text{etc}.$$

atque facta substitutione erit

$$\begin{aligned}
y &= y \\
pz &= Ap + Bp + Cp + Dp + Ep + \text{etc.} \\
\tfrac{1}{2}qz^2 &= + \tfrac{1}{2}A^2q + ABq + ACq + ADq + \text{etc.} \\
&\qquad + \tfrac{1}{2}BBq + BCq + \text{etc.} \\
\tfrac{1}{2}rz^3 &= \tfrac{1}{6}A^3r + \tfrac{1}{2}A^2Br + \tfrac{1}{2}A^2Cr + \text{etc.} \\
&\qquad + \tfrac{1}{2}AB^2r + \text{etc.} \\
\tfrac{1}{24}sz^4 &= \tfrac{1}{24}A^4s + \tfrac{1}{6}A^3Bs + \text{etc.} \\
\tfrac{1}{120}tz^5 &= \tfrac{1}{120}A^5t + \text{etc.}
\end{aligned}$$

Unde obtinentur sequentes aequationes

$$\begin{aligned}
A &= -\frac{y}{p} \\
B &= -\frac{yyq}{2p^3} \\
C &= -\frac{y^3qq}{2p^5} + \frac{q^3r}{6p^4} \\
D &= -\frac{5y^4q^3}{8p^7} + \frac{5y^4qr}{12p^6} - \frac{y^4s}{24p^5} \\
&\text{etc.}
\end{aligned}$$

ideoque erit

$$z = -\frac{y}{p} - \frac{y^2q}{2p^3} - \frac{y^3qq}{2p^5} + \frac{q^3r}{6p^4} - \frac{5y^4q^3}{8p^7} + \frac{5y^4qr}{12p^6} - \frac{y^4s}{24p^5} - \text{etc.}$$

EXEMPLUM

Sit proposita haec aequatio $x^5 + 2x - 2 = 0$.

Erit ergo

$$y = x^5 + 2x - 2, \quad \frac{dy}{dx} = p = 5x^4 - 2, \quad \frac{dp}{dx} = q = 20x^3,$$
$$\frac{dq}{dx} = r = 60x^2, \quad \frac{dr}{dx} = s = 120x \quad \text{etc.}$$

Ponatur autem nunc $x = 1$, quia hic valor parum a radice discrepat; erit

$$y = 1, \quad p = 7, \quad q = 20, \quad r = 60, \quad s = 120,$$

unde fiet

$$z = -\frac{1}{7} - \frac{10}{7^3} - \frac{200}{7^5} + \frac{10}{7^4} - \frac{5\cdot 1000}{7^7} + \frac{500}{7^6} - \frac{5}{7^5} + \text{etc.}$$

seu

$$z = -\frac{1}{7} - \frac{10}{7^3} - \frac{130}{7^5} - \frac{1745}{7^7} - \text{etc.},$$

eritque ergo $z = -0{,}18$ et radix $f = 0{,}82$; qui valor, si denuo loco x substitueretur, prodiret radix maxime verae propinqua.

234. Invenimus ergo seriem infinitam, quae cuiusvis aequationis radicem exprimit; ea autem hoc laborat incommodo, ut tum lex progressionis non pateat, tum ipsa nimis sit perplexa atque ad usum non satis accommodata. Alio igitur modo idem negotium suscipiamus seriemque magis regularem investigemus cuiuscunque aequationis propositae radicem exprimentem.

Sit ut ante proposita aequatio $y = 0$ existente y functione quacunque ipsius x et quaestio huc redit, ut valor ipsius x definiatur, qui loco x substitutus functionem y reddat nihilo aequalem. Cum autem y sit functio ipsius x, vicissim x tanquam functio spectari poterit ipsius y atque hac consideratione adhibita quaerendus est valor ipsius functionis x, quem induit, cum quantitas y evanescit. Si igitur f ponatur designare istum ipsius x valorem, qui erit radix aequationis $y = 0$, quoniam x abit in f, si statuatur $y = 0$, erit per ea, quae supra § 67 sunt demonstrata,

$$f = x - \frac{ydx}{dy} + \frac{y^2 ddx}{2dy^2} - \frac{y^3 d^3x}{6dy^3} + \frac{y^4 d^4x}{24dy^4} - \text{etc.},$$

In qua aequatione statuitur differentiale dy constans. Si igitur ponatur

$$\frac{dx}{dy} = p, \quad \frac{dp}{dy} = q, \quad \frac{dq}{dy} = r, \quad \frac{dr}{dy} = s \quad \text{etc.},$$

erit his valoribus introductis, ut consideratio differentialis constantis exuatur,

$$f = x - py + \frac{1}{2}qy^2 - \frac{1}{6}ry^3 + \frac{1}{24}sy^4 - \frac{1}{120}ty^5 + \text{etc.}$$

235. Tributo ergo ipsi x quocunque valore simul valores ipsius y atque quantitatum p, q, r, s etc. determinabuntur hisque inventis habebitur series infinita valorem radicis f exprimens. Sin autem aequatio $y = 0$ plures admittat radices, tum eae prodibunt, si pro x diversi valores assumantur; quia enim y eundem valorem induere potest, etiamsi ipsi x diversi valores tribuantur, mirum non est eandem seriem saepenumero plures valores suppeditare posse. Quo igitur his casibus ambiguitas tollatur simulque series convergens reddatur, pro x assumi debet valor iam prope ad valorem eius radicis, quae quaeritur, accedens. Hoc enim modo valor ipsius y fiet admodum parvus serieique termini vehementer decrescent, ita ut paucis terminis sumendis iam satis iustus valor pro f inveniatur. Hic igitur valor si deinceps loco x substituatur, quantitas y multo minor evadet seriesque multo magis converget hocque modo statim radix f tam exacte innotescet, ut error futurus sit minimus. Hincque summa huius expressionis praerogativa prae ea, quam ante elicueramus, manifesto perspicitur.

236. Ponamus extrahendam esse radicem potestatis n ex numero quocunque N. Sumta igitur proxima potestate exponentis n numerus propositus facile resolvetur in hanc formam $N = a^n + b$. Erit ergo

$$x^n = a^n + b \quad \text{et} \quad y = x^n - a^n - b,$$

unde fit

$$dy = nx^{n-1}dx \qquad \text{et} \quad \frac{dx}{dy} = p = \frac{1}{nx^{n-1}}$$

$$dp = -\frac{(n-1)dx}{nx^n} \qquad \text{et} \quad \frac{dp}{dy} = q = -\frac{n-1}{nnx^{2n-1}}$$

$$dq = \frac{(n-1)(2n-1)dx}{nnx^{2n}} \qquad \text{et} \quad \frac{dq}{dy} = r = \frac{(n-1)(2n-1)}{n^3x^{3n-1}}$$

$$dr = -\frac{(n-1)(2n-1)(3n-1)dx}{n^3x^{3n}} \qquad \text{et} \quad \frac{dr}{dy} = s = -\frac{(n-1)(2n-1)(3n-1)}{n^4x^{4n-1}}$$

etc.

Ponatur nunc $x = a$ eritque $y = -b$ atque radix quaesita $f = \sqrt[n]{\left(a^n + b\right)}$ hoc modo exprimetur

$$f = a + \frac{b}{na^{n-1}} - \frac{(n-1)bb}{n\cdot 2na^{2n-1}} + \frac{(n-1)(2n-1)b^3}{n\cdot 2n\cdot 3na^{3n-1}} - \frac{(n-1)(2n-1)(3n-1)b^4}{n\cdot 2n\cdot 3n\cdot 4na^{4n-1}} + \text{etc.}$$

sicque prodit eadem series, quae vulgo per evolutionem binomii $\left(a^n + b\right)^{\frac{1}{n}}$ erui solet.

237. Postquam ergo in actuali extractione radix proxime vera a fuerit inventa simulque residuum b fuerit repertum, tum ad radicem insuper addi oportet valorem fractionis $\frac{b}{na^{n-1}}$, quo propius vera radix obtineatur. Erit autem

$$a^{n-1} = \frac{N-b}{n}$$

ob $N = a^n + b$. At vero hoc modo radix iusto maior invenietur, quoniam tertius terminus subtrahi debet. Quo igitur per divisionem residui *b* radix multo propius ad verum accedens inveniatur, idoneus divisor debet investigari, qui fingatur esse

$$na^{n-1} + \alpha b + \beta bb + \gamma b^3 + \text{etc.}$$

Cum igitur debeat esse

$$\frac{b}{na^{n-1} + \alpha b + \beta bb + \gamma b^3 + \text{etc.}}$$
$$= \frac{b}{na^{n-1}} - \frac{(n-1)bb}{2n^2a^{2n-1}} + \frac{(n-1)(2n-1)b^3}{6n^3a^{3n-1}} - \frac{(n-1)(2n-1)(3n-1)b^4}{24n^4a^{4n-1}} + \text{etc.},$$

fiet multiplicatione per $na^{n-1} + \alpha b + \beta bb + \gamma b^3 + \text{etc.}$ instituta

$$\begin{aligned} b = b &- \frac{(n-1)bb}{2na^n} + \frac{(n-1)(2n-1)b^3}{6n^2a^{2n}} - \frac{(n-1)(2n-1)(3n-1)b^4}{24n^3a^{3n}} + \text{etc.} \\ &+ \frac{\alpha b^2}{na^{n-1}} - \frac{(n-1)\alpha b^3}{2n^2a^{2n-1}} + \frac{(n-1)(2n-1)\alpha b^4}{6n^3a^{3n-1}} \\ &+ \frac{\beta b^3}{na^{n-1}} - \frac{(n-1)\beta b^4}{2n^2a^{2n-1}} \\ &+ \frac{\gamma b^4}{na^{n-1}} \end{aligned}$$

Hinc deducuntur sequentes determinationes

$$\alpha = \frac{n-1}{2a}$$
$$\beta = \frac{(n-1)\alpha}{2na^n} - \frac{(n-1)(2n-1)}{6na^{n+1}} = -\frac{(n-1)(2n-1)}{12na^{n+1}}$$
$$\gamma = \frac{(n-1)\beta}{2na^n} - \frac{(n-1)(2n-1)\alpha}{6nna^{2n}} + \frac{(n-1)(2n-1)(3n-1)}{24n^2a^{2n+1}} = -\frac{(n-1)(n+1)}{24na^{2n+1}}.$$

Fractio ergo ad radicem iam inventam *a* insuper addenda erit

$$\frac{b}{na^{n-1} + \frac{(n-1)b}{2a} - \frac{(nn-1)bb}{12na^{n+1}} - \frac{(nn-1)b^2}{24na^{2n+1}} - \text{etc.}}$$

238. Quodsi ergo radix quadrata extrahi debeat ex numero *N* atque inventa iam sit radix proxima $= a$ cum residuo $= b$, ad radicem inventam insuper addi debet quotus, qui oritur, si residuum *b* dividatur per

$$2a + \frac{b}{2a} - \frac{bb}{8a^3} + \frac{b^3}{16a^5} - \text{etc.}$$

Sin autem radix cubica extrahi debeat, tum residuum b dividi debet per

$$3a^2+\frac{b}{a}-\frac{2bb}{9a^4}+\frac{b^3}{9a^7}-\text{etc.},$$

quarum formularum usum in his exemplis declarabimus.

EXEMPLUM 1

Extrahatur radix quadrata ex numero 200.

Ponatur $N=200$, et cum proximum quadratum sit 196, erit $a=14$ et residuum $b=4$, quod propterea dividi debebit per

$$28+\frac{1}{7}-\frac{1}{7\cdot 196}+\frac{1}{7\cdot 196\cdot 98},$$

eritque ergo divisor $=28,142135$; per quem si 4 dividatur, obtinebitur fractio decimalis ad 14 addenda, quae iusta erit ad 10 figuras et ultra.

EXEMPLUM 2

Extrahatur radix cubica ex numero $N = 10$.

Proximus cubus est 8 et residuum $=2$, unde $a=2$ et $b=2$ atque divisor $=12\ +1-\frac{1}{18}=12,9444.$ Quare radix cubica quaesita erit proxime

$$=2\frac{2}{12,9444}=2\frac{10000}{64722}.$$

239. Series pro radice inventa etiam considerari potest tanquam recurrens, orta ex quapiam fractione; hoc enim modo plures termini seriei ad multo pauciores, qui numeratorem et denominatorem fractionis constituant, revocabuntur. Sic levi attentione adhibita perspicietur fore proxime

$$(a+b)^n=a^n\cdot\frac{a+\frac{n+1}{2}b}{a-\frac{n-1}{2}b}$$

atque adhuc propius

$$(a+b)^n=a^n\cdot\frac{aa+\frac{n+2}{2}ab+\frac{(n+1)(n+2)}{12}bb}{a-\frac{n-2}{2}ab+\frac{(n-1)(n-2)}{12}bb}$$

Simili modo plures terminos introducendo fractiones adhuc accuratiores obtineri possunt:

$$\left(a+b\right)^{n}=a^{n}\cdot\frac{a^{3}+\frac{n+3}{2}a^{2}b+\frac{(n+3)(n+2)}{10}ab^{2}+\frac{(n+3)(n+2)(n+1)}{120}b^{3}}{a^{3}-\frac{n-3}{2}a^{2}b+\frac{(n-3)(n-2)}{10}ab^{2}-\frac{(n-3)(n-2)(n-1)}{120}b^{3}}$$

Quin etiam huiusmodi forma generalis exhiberi potest, ad quam commode exprimendam sit

$$\begin{array}{ll}
A=\frac{m(n+m)}{1\cdot 2m} & \mathfrak{A}=\frac{m(n-m)}{1\cdot 2m}\\
B=\frac{(m-1)(n+m-1)}{2(2m-1)}A & \mathfrak{B}=\frac{(m-1)(n-m+1)}{2(2m-1)}\mathfrak{A}\\
C=\frac{(m-2)(n+m-2)}{3(2m-2)}B & \mathfrak{C}=\frac{(m-2)(n-m+2)}{3(2m-2)}\mathfrak{B}\\
D=\frac{(m-3)(n+m-3)}{4(2m-3)}C & \mathfrak{D}=\frac{(m-3)(n-m+3)}{4(2m-3)}\mathfrak{C}\\
\text{etc.} & \text{etc.}
\end{array}$$

His autem valoribus determinatis erit

$$\left(a+b\right)^{n}=a^{n}\cdot\frac{a^{m}+Aa^{m-1}b+Ba^{m-2}b^{2}+Ca^{m-3}b^{3}+\text{etc.}}{a^{m}-\mathfrak{A}a^{m-1}b+\mathfrak{B}a^{m-2}b^{2}-\mathfrak{C}a^{m-3}b^{3}+\text{etc.}}$$

240. Si igitur hic pro *n* substituatur numerus fractus, istae formulae ad extractionem radicum apprime erunt accommodatae. Sic si radix quaecunque potestatis *n* extrahi debeat ex forma $a^{n}+b$, sequentes formulae in usum vocari possunt

$$\left(a^{n}+b\right)^{\frac{1}{n}}=a\cdot\frac{2na^{n}+(n+1)b}{2na^{n}+(n-1)b}$$

$$\left(a^{n}+b\right)^{\frac{1}{n}}=a\cdot\frac{12n^{2}a^{2n}+6n(2n+1)a^{n}b+(2n+1)(n+1)bb}{12n^{2}a^{2n}+6n(2n-1)a^{n}b+(2n-1)(n-1)bb}.$$

Sin autem ponatur $a^{n}+b=N$, ut sit $a^{n}=N-b$, erit

$$\left(a^{n}+b\right)^{\frac{1}{n}}=a\cdot\frac{2nN-(n-1)b}{2nN-(n+1)b}$$

$$\left(a^{n}+b\right)^{\frac{1}{n}}=a\cdot\frac{12n^{2}N^{2}-6n(2n-1)Nb+(2n-1)(n-1)bb}{12n^{2}N^{2}-6n(2n+1)Nb+(2n+1)(n+1)bb}.$$

241. Formula igitur generalis pro radice cuiusque aequationis invenienda in aequationibus, quae ex pluribus terminis constant, eundem praestat usum, quem solita regula binomii ad resolutionem aequationum purarum $x^{n}=c$ afferre solet, atque adeo hoc casu in regulam illam ipsam abit. Sin autem aequatio fuerit affecta vel etiam transcendens, expressio nostra generalis semper aequali successu in usum vocatur seriemque praebet infinitam, quae valorem radicis exhibet. Quamobrem cum in hoc negotio summa vis istius formulae generalis consistat, eius usum hic aliquanto fusius ostendamus. Sit igitur proposita haec aequatio affecta tribus terminis constans

$$x^n + cx = N$$

denotantibus c et N quantitates quascunque datas. Ponatur $x^n + cx - N = y$; erit $dy = \left(nx^{n-1} + c\right)dx$ hincque fiet $p = \frac{1}{nx^{n-1}+c}$; tum est

$$dp = -\frac{n(n-1)x^{n-2}dx}{\left(nx^{n-1}+c\right)^2} \quad \text{et} \quad q = -\frac{n(n-1)x^{n-2}}{\left(nx^{n-1}+c\right)^3}.$$

Simili modo ob $r = \frac{dq}{dy}$, $s = \frac{dr}{dy}$; etc. reperietur

$$r = \frac{n^2(n-1)(2n-1)x^{2n-4} - n(n-1)(n-2)cx^{n-3}}{\left(nx^{n-1}+c\right)^5}$$

$$s = \frac{-n^2(n-1)(2n-1)(3n-1)x^{3n-6} + 4n^2(n-1)(n-2)(2n-1)cx^{2n-5} - n(n-1)(n-2)(n-3)c^2x^{n-4}}{\left(nx^{n-1}+c\right)^7}$$

$$t = \frac{\left\{\begin{array}{l} n^4(n-1)(2n-1)(3n-1)(4n-1)x^{4n-8} - n^3(n-1)(n-2)(2n-1)(29n-11)cx^{3n-7} \\ +n^2(n-1)(n-2)(2n-1)(11n-29)c^2x^{2n-6} - n(n-1)(n-2)(n-3)(n-4)c^3x^{n-5} \end{array}\right\}}{\left(nx^{n-1}+c\right)^9}$$

etc.

Quibus valoribus inventis erit aequationis propositae radix

$$f = x - py + \tfrac{1}{2}qyy - \tfrac{1}{6}ry^3 + \tfrac{1}{24}sy^4 - \tfrac{1}{120}ty^5 + \text{etc.}$$

quicquid enim pro x substituatur, unde simul litterae y, p, q, r etc. valores determinatos induunt, summa seriei aequabitur valori unius radicis.

EXEMPLUM 1

Sit proposita haec aequatio $x^3 + 2x = 2$.

Erit $c = 2, N = 2$ et $n = 3$ atque $y = x^3 + 2x - 2$. Ponatur $x = 1$; erit $y = 1$ et

$$p = \tfrac{1}{5}, \quad q = -\tfrac{6}{5^3}, \quad r = \tfrac{78}{5^5}, \quad s = -\tfrac{16\cdot 90}{5^7} \text{ etc.}$$

atque aequationis radix erit

$$f = 1 - \tfrac{1}{5} - \tfrac{3}{5^3} - \tfrac{13}{5^5} - \tfrac{60}{5^7} - \text{etc.} = 0{,}771072.$$

Ponatur nunc $x = 0{,}77$, et quia est $y = x^3 + 2x - 2$,

$$p = \tfrac{1}{3xx+2}, \quad q = -6p^3x, \quad r = 90xxp^5 - 12p^5$$

atque

$$s = -2160p^7x^3 + 720p^7x,$$

habebitur logarithmis adhibendis

$lx = 9{,}8864907$	$x = 0{,}77$
$lx^2 = 9{,}7729814$	$x^2 = 0{,}5929$
$lx^3 = 9{,}6594721$	$x^3 = 0{,}456533$
	$2x = 1{,}54$
	$x^3 + 2x = 1{,}996533$
Ergo	$y = -0{,}003467$

$l(-y) = 7{,}5399538$	$3xx + 2 = 3{,}7787$
$lp = 9{,}4226575$	$l(3xx+2) = 0{,}5773424$
$l(-py) = 6{,}9626113$	$-py = 0{,}000917511$
$lp^3 = 8{,}2679725$	
$lx = 9{,}8864907$	
$l3 = 0{,}4771213$	
$ly^2 = 5{,}0799076$	
$l\left(-\tfrac{1}{2}qyy\right) = 3{,}7114921$	$-\tfrac{1}{2}qyy = 0{,}000000514.$

Ergo radix $f = 0{,}770916997$, quae vix in ultima figura a vero aberrabit.

EXEMPLUM 2

Sit proposita aequatio $x^4 - 2xx + 4x = 8$.

Ponatur $y = x^4 - 2xx + 4x - 8$; erit $dy = 4dx\left(x^3 - x + 1\right)$,

$$p = \frac{1}{4\left(x^3 - x + 1\right)}, \quad \frac{dp}{dx} = \frac{-3xx + 1}{4\left(x^3 - x + 1\right)^2}.$$

Ergo

$$q = \frac{-3xx + 1}{16\left(x^3 - x + 1\right)^3}, \quad \frac{dq}{dx} = \frac{21x^4 - 12xx - 6x + 3}{16\left(x^3 - x + 1\right)^4} \quad \text{et} \quad r = \frac{21x^4 - 12xx - 6x + 3}{64\left(x^3 - x + 1\right)^5} \quad \text{etc.},$$

ex quibus erit radix aequationis propositae

$$f = x - \frac{y}{4\left(x^3 - x + 1\right)} - \frac{(3xx - 1)yy}{32\left(x^3 - x + 1\right)^3} - \frac{\left(7x^4 - 4xx - 2x + 1\right)y^3}{128\left(x^3 - x + 1\right)^5} - \text{etc.}$$

Oportet ergo ipsi x idoneum valorem tribui, quo series ista fiat convergens. Primum autem perspicuum est, si ipsi x tribueretur talis valor, quo fieret $x^3 - x + 1$, tum omnes seriei terminos praeter primum evadere infinitos neque adeo exinde quicquam concludi posse. Convenit ergo ipsi x eiusmodi valorem assignare, quo et y fiat exiguum et $x^3 - x + 1$ non admodum parvum. Sit $x = 1$; erit $y = -5$ et

$$f = 1 + \frac{5}{4} - \frac{25}{16} + \frac{125}{64} - \text{etc.};$$

ubi cum tres termini $\frac{5}{4} - \frac{25}{16} + \frac{125}{64}$ congruant cum progressione geometrica, cuius summa est $\frac{5}{9}$, erit circiter $f = \frac{14}{9}$. Statuamus ergo $x = \frac{3}{2}$; erit

$$y = -\frac{23}{16} \quad \text{et} \quad x^3 - x + 1 = \frac{23}{8},$$

unde fit

$$f = \frac{3}{2} + \frac{1}{8} - \frac{1}{64} + \frac{391}{256 \cdot 529} - \text{etc.} = 1{,}61.$$

Ponatur nunc x = 1,61; erit

lx = 0,2068259	x =	1,61	sit $x^3 - x + 1 = z$
lx^2 = 0,4136518	x^2 =	2,5921	
lx^3 = 0,6204777	x^3 =	4,173281	
lx^4 = 0,8273036	x^4 =	6,718983	

	hinc
$l(-y) = 8,4016934$	$y = -0,025217$
$lz = 0,5518502$	$z = 3,563281$
$l\frac{-y}{z} = 7,8498432$	
$l4 = 0,6020600$	
$l\frac{-y}{4z} = 7,2477832$	$-\frac{y}{4z} = 0,0017692$
$l(3xx-1) = 0,8309926$	$3xx-1 = 6,7763$
$ly^2 = 6,8033868$	
$7,6343794$	
$lz^3 = 1,6555506$	
$5,9788288$	
$l32 = 1,5051500$	$\frac{(3xx-1)y^2}{32z^3} = 0,000002976$
$4,4736788$	

Ergo $f = 1,6117662.$

242. Methodus haec inveniendi radices aequationum proxima aeque patet ad quantitates transcendentes. Quaeramus numerum x, cuius logarithmus ex quocunque canone desumtus ad ipsum numerum datam habeat rationem ut 1 ad n, atque habebitur ista aequatio $x - nlx = 0$; sit autem k modulus horum logarithmorum, ita ut isti logarithmi obtineantur, si logarithmi hyperbolici multiplicentur per k; erit $d.lx = \frac{kdx}{x}$. Ponatur ergo $x - nlx = y$ sitque f valor ipsius x quaesitus, qui reddat $x = nlx$. Cum igitur sit $y = x - nlx$, erit

$$dy = dx - \frac{kndx}{x} = \frac{dx(x-kn)}{x}$$

et

$$\frac{dx}{dy} = p = \frac{x}{(x-kn)}, \qquad \text{unde} \quad dp = -\frac{knxdx}{(x-kn)^2}$$

ergo

$$\frac{dp}{dy} = q = -\frac{knx}{(x-kn)^3}, \qquad dq = \frac{2knxdx + k^2n^2dx}{(x-kn)^4}$$

$$\frac{dq}{dy} = r = \frac{knx(2x+kn)}{(x-kn)^5} \quad \text{etc.}$$

Quare fiet

$$f = x - \frac{xy}{x-kn} - \frac{knxyy}{2(x-kn)^3} - \frac{knxy^3(2x+kn)}{6(x-kn)^5} - \text{etc.}$$

Infra [§ 272] autem ostendemus hoc problema solutionem non admittere, nisi sit $kn > e$ existente e numero, cuius logarithmus hyperbolicus est = 1, seu debet esse $kn > 2,7182818$.

EXEMPLUM

Quaeratur numerus praeter 10, *cuius logarithmus tabularis aequetur decimae parti ipsius numeri.*

Quia de logarithmis tabularibus quaestio instituitur, erit $k = 0,434294481903 25$ atque ob $n = 10$ habebitur $kn = 4,3429448190325$. Facto iam $x = 1$ erit $y = 1$ fietque

$$f = 1 + \frac{1}{3,3429} + \frac{2,1714724}{(3,3429)^3} - \text{etc.}$$

sicque proxime erit $f = 1,37$. Statuatur ergo $x = 1,37$; erit $lx = 0,136720567156406$ et ob $y = x - 10lx$ erit

$$y = 0,00279432843594 \text{ et } -x + kn = 2,9729448190325.$$

Fiat ergo

$$\begin{aligned} lx &= 0,1367205 \\ ly &= \underline{7,4462773} \\ & 7,5829978 \\ l(kn - x) &= \underline{0,4731866} \\ & 7,1098112 \end{aligned}$$

$$\frac{-xy}{x-kn} = 0,00128769.$$

Deinde cum sit terminus $-\frac{knxyy}{2(x-kn)^3} = \frac{kny}{2(x-kn)^2} \cdot \frac{-xy}{x-kn}$, erit

$$\begin{aligned} l\frac{-xy}{x-kn} &= 7,1098112 \\ ly &= 7,4462773 \\ lkn &= \underline{0,6377842} \\ & 5,1938727 \end{aligned}$$

$$
\begin{aligned}
l(kn-x)^2 &= \underline{0,9463732}\\
&\quad\ 4,2474995\\
l2 &= \underline{0,3010300}\\
l\text{tert. term.} &= 3,9464695\\
\text{1. term.}x &= 1,37\\
\text{II. term.} &= 0,00128769\\
\text{Ill. term.} &= \underline{0,00000088}\\
f &= 1,37128857\\
lf &= 0,137128857
\end{aligned}
$$

243. Si aequatio fuerit exponentialis, ea ad logarithmicam reduci poterit; ita si quaeratur valor ipsius *x,* ut sit $x^x = a$, erit $xlx = la$. Quare posito $y = xlx - la$ fiet

$$dy = dxlx + dx \quad \text{et} \quad \tfrac{dx}{dy} = p = \tfrac{1}{1+lx}$$

tumque

$$dp = \frac{-dx}{x(1+lx)^2} \quad \text{et} \quad \frac{dp}{dy} = q = \frac{-1}{x(1+lx)^3},$$

$$dq = \frac{dx}{xx(1+lx)^3} + \frac{3dx}{xx(1+lx)^4} \quad \text{ideoque} \quad \frac{dq}{dy} = r = \frac{1}{xx(1+lx)^4} + \frac{3}{xx(1+lx)^5};$$

porro erit

$$dr = \frac{-2dx}{x^3(1+lx)^4} - \frac{10dx}{x^3(1+lx)^5} - \frac{15dx}{x^3(1+lx)^6},$$

ergo

$$s = \frac{-2}{x^3(1+lx)^5} - \frac{10dx}{x^3(1+lx)^6} - \frac{15}{x^3(1+lx)^7},$$

et

$$t = \frac{6}{x^4(1+lx)^6} + \frac{40}{x^4(1+lx)^7} + \frac{105}{x^4(1+lx)^8} + \frac{105}{x^4(1+lx)^9},$$

$$u = \frac{-24}{x^5(1+lx)^7} - \frac{196}{x^5(1+lx)^8} - \frac{700}{x^5(1+lx)^9} - \frac{1260}{x^5(1+lx)^{10}} - \frac{945}{x^5(1+lx)^{11}}.$$

Hinc ergo, si verus valor ipsius *x* sit $= f$, ita ut sit $f^f = a$, erit

$$f = x - \frac{y}{1+lx} - \frac{yy}{2x(1+lx)^3} - \frac{y^3}{2xx(1+lx)^5} - \frac{5y^4}{8x^3(1+lx)^7} - \frac{7y^5}{8x^4(1+lx)^9}$$
$$- \frac{y^3}{6x^2(1+lx)^4} - \frac{5y^4}{12x^3(1+lx)^6} - \frac{7y^5}{8x^4(1+lx)^8}$$
$$- \frac{y^4}{12x^3(1+lx)^5} - \frac{y^5}{3x^4(1+lx)^7}$$
$$- \frac{y^5}{20x^4(1+lx)^6}$$
etc.

Haec ergo expressio in infinitum continuata, quicunque valor pro x statuatur, sumto $y = xlx - la$ verum ipsius f dabit valorem. Sic si ponatur $x = 1$,
erit $y = -la$ et

$$f = 1 + la - \frac{(la)^2}{2} + \frac{2(la)^3}{3} - \frac{9(la)^4}{8} + \frac{32(la)^5}{15} - \frac{625(la)^6}{144} - \text{etc.},$$

ubi notandum est esse la logarithmum hyperbolicum ipsius a.

EXEMPLUM

Quaeratur numerus f, ut sit $f^f = 100$.

Cum sit
$$a = 100 \quad \text{et} \quad y = xlx - la = xlx - l100,$$

quia patet esse $f > 3$ et < 4, statuatur $x = \frac{7}{2}$ eritque

$$\begin{aligned} lx &= 1{,}25276296849 \\ xlx &= 4{,}38467038972 \\ l100 &= \underline{4{,}60517018599} \\ y &= -0{,}22049979627 \\ 1 + lx &= 2{,}25276296849 \end{aligned}$$

Hinc erit logarithmis vulgaribus adhibendis

$$\begin{aligned} l(-y) &= 9{,}3434083 \\ l(1+lx) &= 0{,}3527156 \\ &\ 8{,}9906927 \qquad \frac{-y}{1+lx} = 0{,}0978797 \\ ly^2 &= 8{,}6868166 \\ 3l(1+lx) &= 1{,}0581468 \\ &= 7{,}6286698 \end{aligned}$$

$12x = l7 = 0,8450980$ $\qquad \frac{y^2}{2x(1+lx)^3} = 0,0006075.$
$6,7835718$

Ergo proxime erit $f = 3,5972722$;
sequentibus vero insuper terminis sumtis erit $f = 3,5972852$.

244. Praeterea autem calculus differentialis insignem habet usum in resolutione aequationum, si quaepiam relatio, quae inter radices intercedit, fuerit cognita. Sit proposita aequatio $y = 0$, in qua sit *y* functio quaecunque ipsius *x*. Si iam verbi gratia constet duas huius aequationis radices inter se differre quantitate data *a*, hae duae radices facile invenientur sequenti modo. Denotet *x* harum duarum radicum minorem; erit maior $= x + a$; quare cum functio *y* evanescat, si *x* significet unam ex radicibus aequationis $y = 0$, evanescet quoque *y*, si loco *x* ponatur $x + a$. Quocirca erit

$$0 = y + \frac{ady}{dx} + \frac{a^2ddy}{2dx^2} + \frac{a^3d^3y}{6dx^3} + \text{etc.}$$

Unde cum sit $y = 0$, erit quoque

$$0 = \frac{dy}{dx} + \frac{addy}{2dx^2} + \frac{a^2d^3y}{6dx^3} + \frac{a^3d^4y}{24dx^4} + \text{etc.},$$

quae duae aequationes simul sumtae per methodum eliminationis dabunt valorem illius radicis *x*, quam alia radix superat quantitate *a*.

EXEMPLUM

Sit proposita haec aequatio $x^5 - 24x^3 + 49xx - 36 = 0$, *quam undecunque constet habere duas radices unitate differentes.*

Posito $y = x^5 - 24x^3 + 49xx - 36$ erit

$$\frac{dy}{dx} = 5x^4 - 72x^2 + 98x$$

$$\frac{ddy}{2dx^2} = 10x^3 - 72x + 49$$

$$\frac{d^3y}{6dx^3} = 10x^2 - 24$$

$$\frac{d^4y}{24dx^4} = 5x$$

$$\frac{d^5y}{120dx^5} = 1.$$

Iam ob $a = 1$ erit

$$A \quad . \quad . \quad . \quad 5x^4 + 10x^3 - 62x^2 + 31x + 26 = 0.$$

At est

$$B \quad . \quad . \quad . \quad x^5 - 24x^3 + 49x - 36 = 0.$$

Multiplicetur superior per x et inferior per 5 alteraque ab altera subtracta relinquet

$$10x^4 + 58x^3 - 214x^2 + 26x + 180 = 0$$

seu

$$C \quad . \quad . \quad . \quad 5x^4 + 29x^3 - 107x^2 + 13x + 90 = 0,$$

a qua prima A subtracta relinquet

$$D \quad . \quad . \quad . \quad 19x^3 - 45x^2 - 18x + 64 = 0.$$
$$D \cdot 5x \quad . \quad . \quad . \quad 95x^4 - 225x^3 - 90x^2 + 320x = 0$$
$$A \cdot 19 \quad . \quad . \quad . \quad 95x^4 + 190x^3 - 1178x^2 + 589x + 494 = 0$$
$$E \quad . \quad . \quad . \quad 415x^3 - 1088x^2 + 269x + 494 = 0$$
$$D \cdot 415 \quad . \quad . \quad . \quad 7885x^3 - 18675x^2 - 7470x + 26560 = 0$$
$$E \cdot 19 \quad . \quad . \quad . \quad 7885x^3 - 20672x^2 + 5111x + 9386 = 0$$
$$F \quad . \quad . \quad . \quad 1997x^2 - 12581x + 17174 = 0$$

$$D \cdot 247. \quad . \quad 4693x^3 - 11115x^2 - 4446x + 15808 = 0$$
$$E \cdot 32 \quad . \quad 13280x^3 - 34816x^2 + 8608x + 15808 = 0$$
$$8587x^3 - 23701x^2 + 13054x = 0$$
$$G. \quad . \quad . \quad . \quad 8587x^2 - 2370\mathrm{l}x + 13054 = 0$$

$$F \cdot 8587. \quad . \quad .17148239x^2 - 108033047x + 147473138 = 0$$
$$G \cdot 1997. \quad . \quad .17148239x^2 - 47330897x + 26068838 = 0$$
$$60702150x - 121404390 = 0.$$

Ex qua aequatione sequitur $x = 2$ ac propterea quoque radix aequationis erit $x = 3$, quorum uterque valor aequationis satisfacit.

245. Potest autem haec operatio absolvi sine subsidio calculi differentialis, propterea quod eadem aequatio, quam calculus differentialis suppeditavit, prodit, si in ipsa aequatione proposita ponatur $x + a$ loco x. Ceterum vero haec methodus eliminandi nimium est operosa, et si aequationes essent altioris gradus, labor penitus foret insuperabilis; ex quo multo minus in aequationibus transcendentibus locum habere potest. Quodsi autem ponamus duas aequationis propositae $y=0$ radices inter se esse aequales, tum ob $a=0$ aequatio differentialis abit in hanc $\frac{dy}{dx}=0$. Quoties ergo quaepiam aequatio $y=0$ habuerit duas radices aequales, toties erit $\frac{dy}{dx}=0$ atque hae duae aequationes coniunctae praebebunt eum ipsius *x* valorem, cui binae radices sunt aequales. Unde vicissim, si ambae aequationes $y=0$ et $\frac{dy}{dx}=0$ communem habeant radicem, ea erit radix duplex aequationis $y=0$. Evenit autem hoc, si, postquam quantitas *x*, ope duarum istarum aequationum $y=0$ et $\frac{dy}{dx}=0$ penitus fuerit eliminata, perveniatur ad aequationern identicam. Sic si proponatur aequatio

$$x^3-2xx-4x+8=0,$$

erit quoque $3xx-4x-4=0$, cuius duplum ad eam additum dat

$$x^3+4xx-12x=0 \text{ seu } xx+4x-12=0,$$

cuius triplum est

$$\begin{array}{rr} & 3xx+12x-36=0 \\ & 3xx-\ \ 4x-\ \ 4=0 \\ \hline \text{subtrahatur} & 16x-32=0 \\ & x-2=0. \end{array}$$

Cum ergo prodierit $x=2$, substituatur hic valor in una praecedentium $3xx-4x-4=0$ et prodibit aequatio identica $12-8-4=0$, unde colligitur aequationem propositam $x^3-2xx-4x+8=0,$ duas habere radices aequales, nempe 2.

246. Si igitur habeatur aequatio algebraica quotcunque dimensionum

$$x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+Dx^{n-4}+\text{etc.}=0,$$

quae duas habeat radices inter se aequales, erit quoque

$$nx^{n-1}+(n-1)Ax^{n-2}+(n-2)Bx^{n-3}+(n-3)Cx^{n-4}+(n-4)Dx^{n-5}+\text{etc.}=0,$$

Scilicet illius aequationis radix duplex simul erit radix istius aequationis. Multiplicetur illa per *n* ab eaque haec per *x* multiplicata subtrahatur prodibitque haec nova aequatio

$$Ax^{n-1} + 2Bx^{n-2} + 3Cx^{n-3} + 4Dx^{n-4} + \text{etc.} = 0.$$

Nunc addantur prima per *a* et haec per *b* multiplicata; erit

$$ax^n + (a+b)Ax^{n-1} + (a+2b)Bx^{n-2} + (a+3b)Cx^{n-3} + \text{etc.} = 0,$$

quae aequatio cum ipsa proposita coniuncta monstrabit radices aequales, si quas habet proposita. Cum igitur quantitates *a* et *b* pro lubitu assumi queant, coefficientes $a, a+b, a+2b$ etc. progressionem quamcunque arithmeticam repraesentant. Quamobrem si aequatio quaecunque habeat duas radices aequales, eae invenientur, si singuli aequationis propositae termini multiplicentur per terminos cuiusvis progressionis arithmeticae respective; nova enim aequatio hoc modo resultans eam radicem, quae in proposita bis inest, quoque continebit. Sic aequatio

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \text{etc.} = 0,$$

si eius termini multiplicentur per progressionem arithmeticam hanc

$$a,\ a+b,\ a+2b,\ a+3b,\ a+4b \ \text{ etc.},$$

prodibit nova aequatio haec

$$ax^n + (a+b)Ax^{n-1} + (a+2b)Bx^{n-2} + (a+3b)Cx^{n-3} + \text{etc.} = 0,$$

quae cum ilIa coniuncta radices aequales ostendet. Haecque est regula satis cognita inveniendi radices aequales cuiuscunque aequationis.

247. Si aequatio $y = 0$ tres habeat radices aequales, non solum erit $\frac{dy}{dx} = 0$, sed etiam erit $\frac{ddy}{dx^2} = 0$, si quidem pro *x* statuatur eius radicis valor, quae in aequatione $y = 0$ ter inest. Ad hoc ostendendum ponamus aequationem $y = 0$ tres habere radices huiusmodi $x,\ x+a$ et $x+b$, quae primum intervallis finitis *a* et *b* a se invicem discrepent; et quia *y* evanescit, si loco *x* tam $x+a$ quam $x+b$ scribatur, erit

$$y = 0$$
$$y + \frac{ady}{dx} + \frac{a^2ddy}{2dx^2} + \frac{a^3d^3y}{6dx^3} + \frac{a^4d^4y}{24dx^4} + \text{etc.} = 0$$
$$y + \frac{bdy}{dx} + \frac{b^2ddy}{2dx^2} + \frac{b^3d^3y}{6dx^3} + \frac{b^4d^4y}{24dx^4} + \text{etc.} = 0;$$

a quibus binis posterioribus si prima subtrahatur, erit

$$\frac{dy}{dx}+\frac{addy}{2dx^2}+\frac{a^2ddy}{6dx^3}+\frac{a^3d^4y}{24dx^4}+\text{etc.}=0$$
$$\frac{dy}{dx}+\frac{bddy}{2dx^2}+\frac{b^2ddy}{6dx^3}+\frac{b^3d^4y}{24dx^4}+\text{etc.}=0$$

Subtrahantur quoque hae a se invicem divisioneque per $a-b$ facta erit

$$\frac{ddy}{2dx^2}+\frac{(a+b)d^3y}{6dx^3}+\frac{(aa+ab+bb)d^4y}{24dx^4}+\text{etc.}=0.$$

Ponatur iam $a=0$ et $b=0$, ita ut tres illae radices inter se sint aequales, eritque ob terminos evanescentes

$$y=0,\quad \frac{dy}{dx}=0 \quad \text{et} \quad \frac{ddy}{dx^2}=0.$$

248. Quoties ergo aequatio $y=0$ tres habeat radices aequales, puta *f, f, f,* tum ista quantitas *f* erit quoque radix non solum huius aequationis $\frac{dy}{dx}=0$, sed etiam huius $\frac{ddy}{dx^2}=0$. Hinc manifestum est, cum *f* sit radix communis aequationis $\frac{dy}{dx}=0$ et eius differentialis $\frac{ddy}{dx^2}=0$, eam in aequatione $\frac{dy}{dx}=0$ bis inesse debere per ea, quae ante de binis radicibus aequalibus ostendimus. Quare si aequatio

$$x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+Dx^{n-4}+\text{etc.}=0,$$

tres contineat radices aequales *f, f, f,* si eius termini per terminos progressionis arithmeticae cuiusvis multiplicentur, tum aequatio resultans binas habebit radices aequales *f* et *f*; quamobrem ea denuo per progressionem arithmeticam quamcunque multiplicari poterit, ut prodeat aequatio eandem radicem *f* semel complectens. Obtinebuntur ergo tres aequationes communem radicem *f* habentes, ex quarum combinatione haec ipsa radix facile elicietur. Si enim eiusmodi progressiones arithmeticae eligantur, quarum vel primus vel ultimus terminus sit $=0$, tum aequatio prodibit uno gradu inferior sicque eliminatio eo facilior evadet.

249. Simili modo ostendetur, si aequatio $y=0$ quatuor habeat radices aequales *f, f, f, f* tum posito $x=f$ non solum fieri $y=0$, $\frac{dy}{dx}=0$ et $\frac{ddy}{dx^2}=0$, sed etiam fore $\frac{d^3y}{dx^3}=0$. Scilicet uti aequatio $y=0$ quater continet radicem $x=f$, ita aequatio $\frac{dy}{dx}$ eandem radicem ter, aequatio vero $\frac{ddy}{dx^2}=0$ bis et aequatio $\frac{d^3y}{dx^3}=0$ semel complectetur. Hoc quoque facilius perspicietur, si perpendamus functionem *y* hoc casu huiusmodi formam $(x-f)^4 X$ habere debere denotante X functionem quamcunque ipsius x. Hac forma assumta erit

$$\frac{dy}{dx}=(x-f)^3\left(4X+\frac{(x-f)dX}{dx}\right)$$

ideoque per $(x-f)^3$ divisibilis. Similiter porro habebit $\frac{ddy}{dx^2}$ factorem $(x-f)^2$ et $\frac{d^3y}{dx^3}$ factorem $x-f$; ex quo perspicuum est, si radix $x=f$ in aequatione $y=0$ quater insit, eam in aequatione $\frac{dy}{dx}=0$ ter, in aequatione $\frac{ddy}{dx^2}=0$ bis atque in $\frac{d^3y}{dx^3}=0$ semel adhuc inesse debere.