CHAPTER VIII

CONCERNING THE USE OF DIFFERENTIAL CALCULUS IN FORMING SERIES

198. At this point we will put on record one use of the differential calculus in the theory of series, which rests in the formation of those series themselves and to which we have now called upon above, since the question shall be about a fraction, the denominator of which shall be any powers of some function, requiring to be expand out in a series. But such a method is similar to that, which we have used now a number of times, while the function to be converted into a series is set equal to some series having the individual coefficients undefined, which then may be determined from the equality put in place. But this determination may be greatly aided on many occasions, if before that may be undertaken, both the first and sometimes also the second equation may be reduced to differential forms. Which method we may explain more carefully here, since it shall be of the greatest use in the integral calculus.

199. Therefore in the first place we may repeat briefly, which above we have reported on the expansion of fractions into series without the aid of the differential calculus. Let some proposed fraction be

$$\frac{A+Bx+Cx^2+Dx^3+\text{etc.}}{\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4+\text{etc.}}=s,$$

which it may be required to be converted into a series following the preceding powers of x. There may be devised an indeterminate series for s

$$s=\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3+\mathfrak{E}x^4+\mathfrak{F}x^5+\mathfrak{G}x^6+\text{etc.}$$

Therefore since with the fraction raised by multiplication there shall be

$$\begin{aligned}&A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+Gx^6+\text{etc.}\\&=s\left(\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4+\zeta x^5+\eta x^6+\text{etc.}\right),\end{aligned}$$

if for s the series devised may be substituted, the following equation will be produced

$$\begin{aligned} &A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.} \\ &= \mathfrak{A}\alpha + \mathfrak{B}\alpha x + \mathfrak{C}\alpha x^2 + \mathfrak{D}\alpha x^3 + \mathfrak{E}\alpha x^4 + \mathfrak{F}\alpha x^5 + \text{etc.} \\ &\quad + \mathfrak{A}\beta + \mathfrak{B}\beta + \mathfrak{C}\beta + \mathfrak{D}\beta + \mathfrak{E}\beta + \text{etc.} \\ &\quad + \mathfrak{A}\gamma + \mathfrak{B}\gamma + \mathfrak{C}\gamma + \mathfrak{D}\gamma + \text{etc.} \\ &\quad + \mathfrak{A}\delta + \mathfrak{B}\delta + \mathfrak{C}\delta + \text{etc.} \\ &\quad + \mathfrak{A}\varepsilon + \mathfrak{B}\varepsilon + \text{etc.} \\ &\quad + \mathfrak{A}\zeta + \text{etc.} \end{aligned}$$

Therefore the equality between the individual terms, which the same powers of x contain, can be put in place

$$\begin{aligned} &\mathfrak{A}\alpha - A = 0 \\ &\mathfrak{B}\alpha + \mathfrak{A}\beta - B = 0 \\ &\mathfrak{C}\alpha + \mathfrak{B}\beta + \mathfrak{A}\gamma - C = 0 \\ &\mathfrak{D}\alpha + \mathfrak{C}\beta + \mathfrak{B}\gamma + \mathfrak{A}\delta - D = 0 \\ &\mathfrak{E}\alpha + \mathfrak{D}\beta + \mathfrak{C}\gamma + \mathfrak{B}\delta + \mathfrak{A}\varepsilon - E = 0 \\ &\qquad\qquad \text{etc.,} \end{aligned}$$

from which equations the devised coefficients $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. may be determined, and thus the infinite series is come upon

$$\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \text{etc.}$$

equal to the proposed fraction s. And in this form, if both the numerator and the denominator of the fraction s may depend on a finite number of terms, all the recurring series are understood, from which now the above argument is to be treated further.

200. But if either the numerator or the denominator or each were raised to some power, then the series may be more difficult to obtain in this manner, because the calculation therefore, unless the function raised shall be a binomial, shall become extremely laborious. But with the differential calculation this labour is able to be avoided. At first only the numerator shall be present and there shall be

$$s = \left(A + Bx + Cxx\right)^n,$$

from which the preceding series may be sought following the powers of x equal to the power of this trinomial; as indeed it is agreed to be finite, if the exponent n should be a whole positive integer. Again for s there may be devised an indefinite series

$$s=\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3+\mathfrak{E}x^4+\mathfrak{F}x^5+\mathfrak{G}x^6+\text{etc.},,$$

the first term of which $\mathfrak{A}$ is agreed to be $=A^n$*;* for if there is put $x=0$, from the former form proposed there is made $s=A^n$, but from the assumed form $s=\mathfrak{A}$. But this determination of the first term is to be desired, if we wish to descend to differentials, because the first term hence may not be determined, as soon will become apparent.

201. Since there shall be $s=\left(A+Bx+Cx^2\right)^n$ with the logarithms taken there will be

$$ls=nl\left(A+Bx+Cx^2\right)$$

and hence with the differentials taken there will be had

$$\tfrac{ds}{s}=\tfrac{nBdx+2nCxdx}{A+Bx+Cx^2}\quad\text{or}\quad\left(A+Bx+Cx^2\right)\tfrac{ds}{dx}=ns\left(B+2Cx\right).$$

But from the devised series there is

$$\tfrac{ds}{dx}=\mathfrak{B}+2\mathfrak{C}x+3\mathfrak{D}x^2+4\mathfrak{E}x^3+5\mathfrak{F}x^4+\text{etc.}$$

Therefore if this series may be substituted in place of $\tfrac{ds}{dx}$ and for *s* the devised series itself may be substituted, the following equation will be substituted

$$\begin{aligned}
&A\mathfrak{B}+2A\mathfrak{C}x+3A\mathfrak{D}x^2+4A\mathfrak{E}x^3+5A\mathfrak{F}x^4+\text{etc.}\\
&\qquad+B\mathfrak{B}\;+2B\mathfrak{C}\;+3B\mathfrak{D}\;+4B\mathfrak{E}\;+\text{etc.}\\
&\qquad\qquad+C\mathfrak{B}\;+2C\mathfrak{C}\;+3C\mathfrak{D}\;+\text{etc.}\\
=&\\
&nB\mathfrak{A}+nB\mathfrak{B}x+nB\mathfrak{C}x^2+nB\mathfrak{D}x^3+nB\mathfrak{E}x^4+\text{etc.}\\
&\qquad+2nC\mathfrak{A}\;+2nC\mathfrak{B}\;+2nC\mathfrak{C}\;+2nC\mathfrak{D}\;+\text{etc.}
\end{aligned}$$

Therefore with the equality here between the terms of the same power of *x* agreed upon there will be

$$\mathfrak{B} = \frac{nB\mathfrak{A}}{A}$$
$$\mathfrak{C} = \frac{(n-1)\mathfrak{B}B + 2nC\mathfrak{A}}{2A}$$
$$\mathfrak{D} = \frac{(n-2)B\mathfrak{C} + (2n-1)C\mathfrak{B}}{3A}$$
$$\mathfrak{E} = \frac{(n-3)B\mathfrak{D} + (2n-2)C\mathfrak{C}}{4A}$$
$$\mathfrak{F} = \frac{(n-4)B\mathfrak{E} + (2n-3)C\mathfrak{D}}{5A}$$
etc.

Therefore since, as we have seen before, there shall be $\mathfrak{A} = A^n$, there will be $\mathfrak{B} = nA^{n-1}B$ and hence all the remaining coefficients are determined successively. Moreover the law, which these themselves follow, may be apparent easily from these formulas, which might have remained exceedingly obscure, if we had wanted actually to expand out the trinomial.

202. This same method follows, if some polynomial must be raised to a certain power. Let

$$s = \left(A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}\right)^n$$

and let there be devised

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \text{etc.};$$

there will be $\mathfrak{A} = A^n$, which value is deduced if there may be put $x = 0$. Now with the logarithms of these taken as before there will be found

$$\frac{ds}{s} = \frac{nBdx + 2nCxdx + 3nDx^2dx + 4nEx^3dx + \text{etc.}}{A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}}$$

or

$$\left(A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}\right)\frac{ds}{dx} = s\left(nB + 2nCx + 3nDx^2 + 4nEx^3 + \text{etc.}\right).$$

Therefore since there shall be

$$\frac{ds}{dx} = \mathfrak{B} + 2\mathfrak{C}x + 3\mathfrak{D}x^2 + 4\mathfrak{E}x^3 + 5\mathfrak{F}x^4 + \text{etc.}$$

with these series substituted for s and $\frac{ds}{dx}$ there will be [noting the detached coefficients,]

$$\begin{array}{llllll}
A\mathfrak{B}+ & 2A\mathfrak{C}x+3A\mathfrak{D}x^2 & +4A\mathfrak{E}x^3 & +5A\mathfrak{F}x^4 & +\text{etc.} \\
+ & B\mathfrak{B} \quad +2B\mathfrak{C} & +3B\mathfrak{D} & +4B\mathfrak{E} & +\text{etc.} \\
 & \quad\quad +C\mathfrak{B} & +2C\mathfrak{C} & +3C\mathfrak{D} & +\text{etc.} \\
 & & +D\mathfrak{B} & +2C\mathfrak{C} & +\text{etc.} \\
 & & & +E\mathfrak{B} & +\text{etc.}
\end{array}$$

$$=$$

$$\begin{array}{llllll}
nB\mathfrak{A}+ & nB\mathfrak{B}x & +nB\mathfrak{C}x^2 & +nB\mathfrak{D}x^3+ & nB\mathfrak{E}x^4 & +\text{etc.} \\
 & +2nC\mathfrak{A} & +2nC\mathfrak{B} & +2nC\mathfrak{C} & +2nC\mathfrak{D} & +\text{etc.} \\
 & & +3nD\mathfrak{A} & +3nD\mathfrak{B} & +3nD\mathfrak{C} & +\text{etc.} \\
 & & & +4nE\mathfrak{A} & +4nE\mathfrak{B} & +\text{etc.} \\
 & & & & +5nF\mathfrak{A} & +\text{etc.}
\end{array}$$

From which the following determinations may be derived

$$\begin{aligned}
A\mathfrak{B} &= nB\mathfrak{A} \\
2A\mathfrak{C} &= (n-1)B\mathfrak{B}+2nC\mathfrak{A} \\
3A\mathfrak{D} &= (n-2)B\mathfrak{C}+(2n-1)C\mathfrak{B}+3nD\mathfrak{A} \\
4A\mathfrak{E} &= (n-3)B\mathfrak{D}+(2n-2)C\mathfrak{C}+(3n-1)D\mathfrak{B}+4nE\mathfrak{A} \\
5A\mathfrak{F} &= (n-4)B\mathfrak{E}+(2n-3)C\mathfrak{D}+(3n-2)D\mathfrak{C}+(4n-1)E\mathfrak{B}+5nF\mathfrak{A} \\
&\quad \text{etc.},
\end{aligned}$$

from which it may become most splendidly apparent, since there shall be $\mathfrak{A}=A^n$, that the supposed coefficients $\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D}$ etc. may depend on these in turn, and hence may be determined.

203. Because, if the quantity $A+Bx+Cx^2+Dx^3+\text{etc.}$ may depend on a finite number of terms and the number n were a positive integer, also any power must depend on a finite number of terms, and it is evident in this case the formulas found in this way must finally vanish, since all the terms present ought to vanish, so that if the first one were to vanish, likewise all the following must vanish. We may put the proposed formula $A+Bx+Cx^2$ to be a trinomial and the cube of this to be sought, so that there shall be $n=3$; there will be

$$\begin{aligned}
\mathfrak{A} &= A^3 \\
A\mathfrak{B} &= 3B\mathfrak{A} \\
2A\mathfrak{C} &= 2B\mathfrak{B}+6C\mathfrak{A} \\
3A\mathfrak{D} &= 1B\mathfrak{C}+5C\mathfrak{B}
\end{aligned}
\qquad \text{and thus} \qquad
\begin{aligned}
\mathfrak{A} &= A^3 \\
\mathfrak{B} &= 3A^2B \\
\mathfrak{C} &= 3AB^2+3A^2C \\
\mathfrak{D} &= B^3+6ABC
\end{aligned}$$

$$4A\mathfrak{E} = \quad 0 \quad + 4C\mathfrak{C} \qquad \mathfrak{E} = 3B^2C + 3AC^2$$
$$5A\mathfrak{F} = -B\mathfrak{E} + 3C\mathfrak{D} \qquad \mathfrak{F} = 3BC^2$$
$$6A\mathfrak{G} = -2B\mathfrak{F} + 2C\mathfrak{E} \qquad \mathfrak{G} = C^3$$
$$7A\mathfrak{H} = -3B\mathfrak{G} + 1C\mathfrak{F} \qquad \mathfrak{H} = 0$$
$$8A\mathfrak{I} = -4B\mathfrak{H} + 0 \qquad \mathfrak{I} = 0.$$

Therefore because two equations are $= 0$ and any of the following depend on the two preceding equations, it is apparent that all the following equations must vanish equally. And on that account the law, by which these coefficients in turn themselves may depend have been found, therefore is more worthy of note.

204. If *n* were a negative number, thus so that *s* may become equal to a fraction, the series may extend to infinity. Therefore let there be

$$s = \frac{1}{\left(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.}\right)^n}$$

this series may be devised for the value of this

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \text{etc.}$$

And if in the above formulas there may be put $\alpha,\ \beta,\ \gamma,\ \delta$ etc. for the letters A, *B, C, D* etc. and likewise *n* is made negative, the following determinations of the coefficients $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. will be produced

$$\mathfrak{A} = \alpha^{-n} = \frac{1}{\alpha^n}$$
$$\alpha\mathfrak{B} + n\beta\mathfrak{A} = 0$$
$$2\alpha\mathfrak{C} + (n+1)\beta\mathfrak{B} + 2n\gamma\mathfrak{A} = 0$$
$$3\alpha\mathfrak{D} + (n+2)\beta\mathfrak{C} + (2n+1)\gamma\mathfrak{B} + 3n\delta\mathfrak{A} = 0$$
$$4\alpha\mathfrak{E} + (n+3)\beta\mathfrak{D} + (2n+2)\gamma\mathfrak{C} + (3n+1)\delta\mathfrak{B} + 4n\varepsilon\mathfrak{A} = 0$$
$$5\alpha\mathfrak{F} + (n+4)\beta\mathfrak{E} + (2n+3)\gamma\mathfrak{D} + (3n+2)\delta\mathfrak{C} + (4n+1)\varepsilon\mathfrak{B} + 5n\zeta\mathfrak{A} = 0$$
etc.

Which formulas follow the same law of these coefficients of the numbers, which we have observed now above in the *Introductione,* and so the truth of this now finally is allowed to be demonstrated rigorously.

205. Thus these equations themselves may be obtained, if the numerator of the fraction were unity or also some power of x, such as x^m; for in the latter case it will be required only to multiply the first series found $\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3+\text{etc.}$ by x^m. But if the numerator may depend on two or more terms, then indeed we will not observe the above law of the progression; on account of which we may investigate that here by differentiation. Therefore let there be

$$s=\frac{A+Bx+Cx^2+Dx^3+\text{etc.}}{\left(\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4+\text{etc.}\right)^n}$$

and the following series may be devised for the value of this fraction

$$s=\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3+\mathfrak{E}x^4+\mathfrak{F}x^5+\text{etc.};$$

the first term of which $\mathfrak{A}$ may be defined thus, there may be put $x=0$ and there will be from the first expression $s=\frac{A}{\alpha^n}$, and from the assumed $s=\mathfrak{A}$, from which it is necessary, that there shall be $\mathfrak{A}=\frac{A}{\alpha^n}$. From which determined term the remaining terms may become known by differentiation.

206. With logarithms taken there will be

$$ls=l\left(A+Bx+Cx^2+Dx^3+\text{etc.}\right)-nl\left(\alpha+\beta x+\gamma x^2+\gamma x^3+\varepsilon x^4+\text{etc.}\right)$$

and hence by differentiation there may be found

$$\frac{ds}{s}=\frac{Bdx+2Cxdx+3Dx^2dx+\text{etc.}}{A+Bx+Cx^2+Dx^3+\text{etc.}}$$
$$-\frac{n\beta dx+2n\gamma xdx+3n\delta x^2dx+\text{etc.}}{\alpha+\beta x+\gamma x^2+\delta x^3+\text{etc.}}$$

and with the fractions removed by multiplication there will be

$$\left.\begin{array}{l} A\alpha + A\beta x + A\gamma x^2 + A\delta x^3 + \text{etc.} \\ \quad + B\alpha \quad + B\beta \quad + B\gamma \quad + \text{etc.} \\ \qquad\qquad C\alpha \quad + C\beta \quad + \text{etc.} \\ \qquad\qquad\qquad + D\alpha + \text{etc.} \end{array}\right\}\frac{ds}{dx}$$

$$= \left\{\begin{array}{l} B\alpha + B\beta x + B\gamma x^2 + B\delta x^3 + \text{etc.} \\ \quad + 2C\alpha + 2C\beta + 2C\gamma + \text{etc.} \\ \qquad\qquad 3D\alpha + 3D\beta + \text{etc.} \\ \qquad\qquad\qquad + 4E\alpha + \text{etc.} \end{array}\right\} s$$

$$- \left\{\begin{array}{l} A\beta + 2A\gamma x + 3A\delta x^2 + 4A\varepsilon x^3 + \text{etc.} \\ \quad + B\beta + 2B\gamma + 3B\delta + \text{etc.} \\ \qquad + C\beta + 2C\gamma + \text{etc.} \\ \qquad\qquad + D\beta + \text{etc} \end{array}\right\} ns.$$

Now since there shall be $\frac{ds}{dx} = \mathfrak{B} + 2\mathfrak{C}x + 3\mathfrak{D}x^2 + 4\mathfrak{E}x^3 + \text{etc.}$, there will be with the substitutions made

$$\left.\begin{array}{r} A\alpha\mathfrak{B} + nA\beta\mathfrak{A} \\ - B\alpha\mathfrak{A} \end{array}\right\} = 0$$

$$\left.\begin{array}{r} 2A\alpha\mathfrak{C} + (n+1)A\beta\mathfrak{B} + 2nA\gamma\mathfrak{A} \\ + 0B\alpha\mathfrak{B} + (n-1)B\beta\mathfrak{A} \\ - 2C\alpha\mathfrak{A} \end{array}\right\} = 0$$

$$\left.\begin{array}{r} 3A\alpha\mathfrak{D} + (n+2)A\beta\mathfrak{C} + (2n+1)A\gamma\mathfrak{B} + 3nA\delta\mathfrak{A} \\ + B\alpha\mathfrak{C} + nB\beta\mathfrak{B} + (2n-1)B\gamma\mathfrak{A} \\ - C\alpha\mathfrak{B} + (n-2)C\beta\mathfrak{A} \\ - 3D\alpha\mathfrak{A} \end{array}\right\} = 0$$

$$\left.\begin{array}{r} 4A\alpha\mathfrak{E}+(n+3)A\beta\mathfrak{D}+(2n+2)A\gamma\mathfrak{C}+(3n+1)A\delta\mathfrak{B}+4nA\varepsilon\mathfrak{A} \\ +2B\alpha\mathfrak{D}+(n+1)B\beta\mathfrak{C}+2nB\gamma\mathfrak{B}+(3n-1)B\delta\mathfrak{A} \\ +0C\alpha\mathfrak{C}+(n-1)C\beta\mathfrak{B}+(2n-2)C\gamma\mathfrak{A} \\ -2D\alpha\mathfrak{B}+(n-3)D\beta\mathfrak{A} \\ -4E\alpha\mathfrak{A} \end{array}\right\}=0.$$

etc.

Hence the law, by which these formulas are progressing, is seen easily; for the first line of each equation follows the same law, as we had in § 204. Then truly the coefficients of the second lines arise, if $n+1$ may be subtracted from the above coefficients, and in a similar manner from the second line the third line may be formed, and the following continually by subtracting $n+1$ from the above coefficients ; but the components of any terms of these letters may be formed by inspection alone.

207. But if the numerator also were some power or the fraction, evidently

$$s=\frac{(A+Bx+Cx^2+Dx^3+\text{etc.})^m}{(\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4+\text{etc.})^n},$$

and there may be devised

$$s=\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3+\mathfrak{E}x^4+\text{etc.},$$

there will be $\mathfrak{A}=\frac{A^m}{\alpha^n}$; truly the remaining coefficients will be determined from the following formulas

$$\left.\begin{array}{r} A\alpha\mathfrak{B}+nA\beta\mathfrak{A} \\ -mB\alpha\mathfrak{A} \end{array}\right\}=0$$

$$\left.\begin{array}{r} 2A\alpha\mathfrak{C}+(n+1)A\beta\mathfrak{B}+2nA\gamma\mathfrak{A} \\ -(m-1)B\alpha\mathfrak{B}+(n-m)B\beta\mathfrak{A} \\ -2mC\alpha\mathfrak{A} \end{array}\right\}=0$$

$$\left.\begin{array}{r} 3A\alpha\mathfrak{D}+(n+2)A\beta\mathfrak{C}+(2n+1)A\gamma\mathfrak{B}+3nA\delta\mathfrak{A} \\ -(m-2)B\alpha\mathfrak{C}+(n-m+1)B\beta\mathfrak{B}+(2n-m)B\gamma\mathfrak{A} \\ -(2m-1)C\alpha\mathfrak{B}+(n-2m)C\beta\mathfrak{A} \\ -3mD\alpha\mathfrak{A} \end{array}\right\}=0$$

$$\left.\begin{array}{r} 4A\alpha\mathfrak{E}+(n+3)A\beta\mathfrak{D}+(2n+2)A\gamma\mathfrak{C}+(3n+1)A\delta\mathfrak{B}+4nA\varepsilon\mathfrak{A} \\ -(m-3)B\alpha\mathfrak{D}+(n-m+1)B\beta\mathfrak{C}+(2n-m+1)B\gamma\mathfrak{B}+(3n-m)B\delta\mathfrak{A} \\ -(2m-2)C\alpha\mathfrak{C}+(n-2m+1)C\beta\mathfrak{B}+(2n-2m)C\gamma\mathfrak{A} \\ -(3m-1)D\alpha\mathfrak{B}+(n-3m)D\beta\mathfrak{A} \\ -4mE\alpha\mathfrak{A} \end{array}\right\}=0.$$

etc.

The law, by which these formulas may be continued further, may be apparent from inspection, which can be described in words. For on descending the coefficients may be diminished by the difference $n+m$; but by progressing horizontally they may be increased continually by the difference $n-1$.

208. Therefore the principles concerning recurring series may be enlarged upon further in this manner, while we have supplied that weakness and defined the law of the coefficients, if not only the denominator of the fraction were some power, but also the numerator may depend on any terms, for which law it may not be sufficient to be uncovered by induction alone. But besides many uses of recurring series, which we have now established, also bring the greatest use to the sums of any series requiring to be found approximately, an example of which we have shown in the first chapter of this section, while we have transformed the series into another by the substitution $x=\frac{y}{1+ny}$, which depends on many occasions on a finite number of terms. And that method may be possible to be extended further, if other functions were substituted for *x*. Truly because then the law of the progression of the series, which must be put in place of the powers of *x*, may not be in agreement clearly enough, in that place the enlargement is seen to be reserved, since the law now mentioned must be thoroughly uncovered. Yet meanwhile we have learned from careful consideration of the thing that the same calculation can be expedited without the aid of this law of the progression [being known explicitly], by calling upon the method only, which we have used here in the investigation of this law.

209. Therefore let some proposed series be

$$s=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\text{etc.},$$

which it may be required to be transformed into another series, the individual terms of which may be fractions, the denominators of which may proceed following the powers of a formula of this kind

$$\alpha+\beta x+\gamma x^2+\delta x^3+\text{etc.}$$

Therefore so that we may begin from the most simple, we may put the series to be

$$s=\frac{\mathfrak{A}}{\alpha+\beta x}+\frac{\mathfrak{B}x}{(\alpha+\beta x)^2}+\frac{\mathfrak{C}x^2}{(\alpha+\beta x)^3}+\frac{\mathfrak{D}x^3}{(\alpha+\beta x)^4}+\text{etc.};$$

the equal of that series with this expression put in place multiplied everywhere by $\alpha+\beta x$ and there is made

$$\left.\begin{array}{l} A\alpha+B\alpha x+C\alpha x^2+D\alpha x^3+\text{etc.} \\ \quad +A\beta \;\; +B\beta \quad\;\; +C\beta \quad\;\; +\text{etc.}\end{array}\right\}=\mathfrak{A}+\frac{\mathfrak{B}x}{\alpha+\beta x}+\frac{\mathfrak{C}x^2}{(\alpha+\beta x)^2}+\text{etc.}$$

There may be put $\mathfrak{A}=A\alpha$ and there is made

$$\begin{gathered} A\beta+B\alpha=A' \\ B\beta+C\alpha=B' \\ C\beta+D\alpha=C' \\ D\beta+E\alpha=D' \\ \text{etc.;}\end{gathered}$$

with division by x put in place there will be

$$A'+B'x+C'x^2+D'x^3+\text{etc.}=\frac{\mathfrak{B}}{\alpha+\beta x}+\frac{\mathfrak{C}x}{(\alpha+\beta x)^2}+\frac{\mathfrak{D}x^2}{(\alpha+\beta x)^3}+\text{etc.}$$

The equation may be multiplied anew by $\alpha+\beta x$ and on putting

$$\begin{gathered} A'\beta+B'\alpha=A'' \\ B'\beta+C'\alpha=B'' \\ C'\beta+D'\alpha=C'' \\ \text{etc.}\end{gathered}$$

there becomes

$$A'+B''x+C''x^2+D''x^3+\text{etc.}=\mathfrak{B}+\frac{\mathfrak{C}x}{\alpha+\beta x}+\frac{\mathfrak{D}x^2}{(\alpha+\beta x)^2}+\text{etc.}$$

Therefore let $\mathfrak{B} = A'\alpha$; and the operation as before being put in place, if there is made

$$\begin{array}{ll} A''\beta + B''\alpha = A''' & A'''\beta + B'''\alpha = A'''' \\ B''\beta + C''\alpha = B''' & B'''\beta + C'''\alpha = B'''' \\ C''\beta + D''\alpha = C''' & C'''\beta + D'''\alpha = C'''' \\ \text{etc.} & \text{etc.,} \end{array}$$

there will be $\mathfrak{C} = A''\alpha$, $\mathfrak{D} = A'''\alpha$, $\mathfrak{E} = A''''\alpha$ etc.; from which the sum of the proposed series may be expressed in this manner, so that there shall be

$$s = \frac{A\alpha}{\alpha+\beta x} + \frac{A'\alpha x}{(\alpha+\beta x)^2} + \frac{A''\alpha x^2}{(\alpha+\beta x)^3} + \frac{A'''\alpha x^3}{(\alpha+\beta x)^4} + \text{etc.}$$

Which same series may have arisen from the substitution

$$\frac{x}{\alpha+\beta z} = y \quad \text{or} \quad x = \frac{ay}{1-\beta y}.$$

210. This transformation may be used with the most success, if the proposed series $A + Bx + Cx^2 + \text{etc}$. were prepared thus, so that finally it may be combined with a recurring or rather geometric series arising from the fraction $\frac{P}{\alpha+\beta x}$. Then indeed the values A', B', C', D' etc. may vanish finally, and hence the much larger letters [*i.e.* constants] A'', A''', A'''' etc. may establish an especially convergent series.

But in a similar manner we will be able to use trinomial and whatever polynomial denominators, which will have an excellent use, if the proposed series finally may be combined with the recurring series. Therefore for the proposed series

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

there may be put in place

$$s = \frac{\mathfrak{A}+\mathfrak{B}x}{\alpha+\beta x+\gamma x^2} + \frac{\mathfrak{A}'x^2+\mathfrak{B}'x^3}{(\alpha+\beta x+\gamma x^2)^2} + \frac{\mathfrak{A}''x^4+\mathfrak{B}''x^5}{(\alpha+\beta x+\gamma x^2)^3} + \frac{\mathfrak{A}'''x^6+\mathfrak{B}'''x^7}{(\alpha+\beta x+\gamma x^2)^4} + \text{etc.}$$

This may be multiplied everywhere by $\alpha + \beta x + \gamma x^2$ and on putting

$$\begin{array}{lcl} A\gamma + B\beta + C\alpha = A' & & \\ B\gamma + C\beta + D\alpha = B' & \text{and} & \mathfrak{A} = \alpha A, \quad \mathfrak{B} = A\beta + B\alpha, \\ C\gamma + D\beta + E\alpha = C' & & \\ \text{etc.} & & \end{array}$$

an equation may arise similar to the first, with the division by *xx* put in place,

$$A' + B'x + C'x^2 + D'x^3 + E'x^4 + \text{etc.}$$
$$= \frac{\mathfrak{A}' + \mathfrak{B}'x}{\alpha+\beta x+\gamma xx} + \frac{\mathfrak{A}''x^2 + \mathfrak{B}''x^3}{(\alpha+\beta x+\gamma xx)^2} + \frac{\mathfrak{A}'''x^4 + \mathfrak{B}'''x^5}{(\alpha+\beta x+\gamma xx)^3} + \text{etc.}$$

Therefore if the operation may be put in place as before by making

$$\begin{aligned} A'\gamma + B'\beta + C'\alpha &= A'' \\ B'\gamma + C'\beta + D'\alpha &= B'' \\ C'\gamma + D'\beta + E'\alpha &= C'' \\ &\text{etc.} \end{aligned} \qquad \text{and} \qquad \mathfrak{A}' = \alpha A', \quad \mathfrak{B}' = A'\beta + B'\alpha$$

and again,

$$\begin{aligned} A''\gamma + B''\beta + C''\alpha &= A''' \\ B''\gamma + C''\beta + D''\alpha &= B''' \\ C''\gamma + D''\beta + E''\alpha &= C''' \\ &\text{etc.} \end{aligned} \qquad \text{and} \qquad \mathfrak{A}'' = \alpha A'', \quad \mathfrak{B}'' = A''\beta + B''\alpha$$

and thus further values being investigated in a similar manner, there will be

$$s = \frac{A\alpha + (A\beta + \beta A)x}{\alpha+\beta x+\gamma xx} + \frac{(A'\alpha + (A'\beta + B'\alpha)x)x^2}{(\alpha+\beta x+\gamma xx)^2} + \frac{(A''\alpha + (A''\beta + B''\alpha)x)x^4}{(\alpha+\beta x+\gamma xx)^3} + \text{etc.}$$

211. If there may be put $x = 1$, with which put in place there can be no change to the magnitude, since α, β, γ may be taken as it pleases, and there may become

$$s = A + B + C + D + E + F + G + \text{etc.},$$

since the following values may considered successively

$$\begin{aligned} A\gamma + B\beta + C\alpha &= A' \\ B\gamma + C\beta + D\alpha &= B' \\ C\gamma + D\beta + E\alpha &= C' \\ &\text{etc.} \end{aligned} \qquad \begin{aligned} A'\gamma + B'\beta + C'\alpha &= A'' \\ B'\gamma + C'\beta + D'\alpha &= B'' \\ C'\gamma + D'\beta + E'\alpha &= C'' \\ &\text{etc.} \end{aligned} \qquad \text{and thus henceforth,}$$

therefore indeed above there may be put for brevity

$$\alpha+\beta+\gamma=m,$$

the sum of the proposed series will be obtained expressed in this manner

$$s=\begin{cases}(\alpha+\beta)\left(\frac{A}{m}+\frac{A'}{m^2}+\frac{A''}{m^3}+\frac{A'''}{m^4}+\text{etc.}\right)\\ \quad+\ \alpha\left(\frac{B}{m}+\frac{B'}{m^2}+\frac{B''}{m^3}+\frac{B'''}{m^4}+\text{etc.}\right)\end{cases}$$

212. In the same manner the denominators from several constant terms can be taken, and because the operation may be easily seen from the preceding, here we will set out the case for the quadrinomial only. Therefore let there be

$$s=A+B+C+D+E+F+G+\text{etc}.$$

The following values may be sought

$$\begin{gathered}A\delta+B\gamma+C\beta+D\alpha=A'\\ B\delta+C\gamma+D\beta+E\alpha=B'\\ C\delta+D\gamma+E\beta+F\alpha=C'\\ \text{etc.}\end{gathered}$$

$$\begin{gathered}A'\delta+B'\gamma+C'\beta+D'\alpha=A''\\ B'\delta+C'\gamma+D'\beta+E'\alpha=B''\\ C'\delta+D'\gamma+E'\beta+F'\alpha=C''\\ \text{etc.}\end{gathered}$$

$$\begin{gathered}A''\delta+B''\gamma+C''\beta+D''\alpha=A'''\\ B''\delta+C''\gamma+D''\beta+E''\alpha=B'''\\ C''\delta+D''\gamma+E''\beta+F''\alpha=C'''\\ \text{etc.}\end{gathered}$$

Truly there shall then be $\alpha+\beta+\gamma=m$ and there shall be

$$s=\begin{cases}(\alpha+\beta+\gamma)\left(\frac{A}{m}+\frac{A'}{m^2}+\frac{A''}{m^3}+\frac{A'''}{m^4}+\text{etc.}\right)\\ \quad+(\alpha+\beta)\left(\frac{B}{m}+\frac{B'}{m^2}+\frac{B''}{m^3}+\frac{B'''}{m^4}+\text{etc.}\right)\\ \quad+\qquad\alpha\left(\frac{C}{m}+\frac{C'}{m^2}+\frac{C''}{m^3}+\frac{C'''}{m^4}+\text{etc.}\right)\end{cases}$$

from which a similar progression is evident clearly, if at this point more parts may be attributed to the denominator *m*.

213. Nor indeed is there any need that the denominators of the fractions, to which we reduce the sum of the series, shall be powers of the same formula

$$\alpha+\beta x+\gamma x^2+\text{etc.},$$

but this itself is able to be varied into individual terms. So that this may appear clearer, we may take only two terms in the first place, and the series may be devised

$$s=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\text{etc.}$$

to be converted into this series of fractions

$$s=\frac{\mathfrak{A}}{\alpha+\beta x}+\frac{\mathfrak{A}'x}{(\alpha+\beta x)(\alpha'+\beta'x)}+\frac{\mathfrak{A}''x^2}{(\alpha+\beta x)(\alpha'+\beta'x)(\alpha''+\beta''x)}+\text{etc.}$$

First each side may be multiplied by $\alpha+\beta x$ and there may be put

$$\begin{aligned}A\beta+B\alpha&=A'\\ B\beta+C\alpha&=B'\quad\text{and}\quad \mathfrak{A}=A\alpha\\ C\beta+D\alpha&=C'\\ &\text{etc.}\end{aligned}$$

and there is made on division by *x*

$$A'+B'x+C'x^2+D'x^3+\text{etc.}=\frac{\mathfrak{A}'}{\alpha'+\beta'x}+\frac{\mathfrak{A}''x}{(\alpha'+\beta'x)(\alpha''+\beta''x)}+\text{etc.}$$

Then in a similar manner on multiplying by $\alpha'+\beta'x$ and then by $\alpha''+\beta''x$ and thus henceforth, if there may be put in place

$$\begin{array}{lll} A'\beta'+B'\alpha'=A'' & A''\beta''+B''\alpha''=A''' & A'''\beta'''+B'''\alpha'''=A'''' \\ B'\beta'+C'\alpha'=B'' & B''\beta''+C''\alpha''=B''' & B'''\beta'''+C'''\alpha'''=B'''' \quad \text{etc.} \\ C'\beta'+D'\alpha'=C'' & C''\beta''+D''\alpha''=C''' & C'''\beta'''+D'''\alpha'''=C'''' \\ \text{etc.} & \text{etc.} & \text{etc.} \end{array}$$

there becomes

$$\mathfrak{A}'=A'\alpha', \quad \mathfrak{A}''=A''\alpha'', \quad \mathfrak{A}'''=A'''\alpha''', \quad \text{etc.}$$

and hence the proposed series may be converted into this

$$s=\frac{A\alpha}{\alpha+\beta x}+\frac{A'\alpha' x}{(\alpha+\beta x)(\alpha'+\beta' x)}+\frac{A''\alpha'' x^2}{(\alpha+\beta x)(\alpha'+\beta' x)(\alpha''+\beta'' x)}+\text{etc.}$$

where the values $\alpha, \beta, \alpha', \beta', \alpha'', \beta'', \alpha''', \beta'''$ etc. are arbitrary, but in whatever case they are able to be taken thus, so that the new series may converge more.

214. We may apply this also to trinomial factors and let any proposed series be

$$s=A+B+C+D+E+F+G+\text{etc.}$$

$$\begin{array}{ll} A\gamma+B\beta+C\alpha=A' & A'\gamma'+B'\beta'+C'\alpha'=A'' \\ B\gamma+C\beta+D\alpha=B' & B'\gamma'+C'\beta'+D'\alpha'=B'' \\ C\gamma+D\beta+E\alpha=C' & C'\gamma'+D'\beta'+E'\alpha'=C'' \\ \text{etc.} & \text{etc.} \end{array}$$

$$\begin{array}{ll} A''\gamma''+B''\beta''+C''\alpha''=A''' & A'''\gamma'''+B'''\beta'''+C'''\alpha'''=A'''' \\ B''\gamma''+C''\beta''+D''\alpha''=B''' & B'''\gamma'''+C'''\beta'''+D'''\alpha'''=B'''' \\ C''\gamma''+D''\beta''+E''\alpha''=C' & C'''\gamma'+D'''\beta'''+E'''\alpha'=C'''' \\ \text{etc.} & \text{etc.} \end{array}$$

Then for the sake of brevity there may be put

$$\begin{aligned} \alpha+\beta+\gamma &= m \\ \alpha'+\beta'+\gamma' &= m' \\ \alpha''+\beta''+\gamma'' &= m'' \\ \alpha'''+\beta'''+\gamma''' &= m''' \\ &\text{etc.} \end{aligned}$$

and the sum of the proposed series will be

$$s = \frac{\alpha(A+B)}{m} + \frac{\alpha'(A'+B')}{mm'} + \frac{\alpha''(A''+B'')}{mm'm''} + \frac{\alpha'''(A'''+B''')}{mm'm''m'''} + \text{etc.}$$
$$+ \frac{\beta A}{m} + \frac{\beta' A'}{mm'} + \frac{\beta'' A''}{mm'm''} + \frac{\beta''' A'''}{mm'm''m'''} + \text{etc.}$$

215. Because these may appear so hidden, so that the use may be less clearly understood, we may restrict ourselves to discussing the transformation § 213 and let there be $x = -1$, to that this series may be had

$$s = A - B + C - D + E - F + G - \text{etc.},$$

and there may be put in place

$$\begin{array}{llll} B - A = A' & B' - 2A' = A'' & B'' - 3A'' = A''' & B''' - 4A''' = A'''' \\ C - B = B' & C' - 2B' = B'' & C'' - 3B'' = B''' & C''' - 4B''' = B'''' \\ D - C = C' & D' - 2C' = C'' & D'' - 3C'' = C''' & D''' - 4C''' = C'''' \\ E - D = D' & E' - 2D' = D'' & E'' - 3D'' = D''' & E''' - 4D''' = D'''' \\ \text{etc.} & \text{etc.} & \text{etc.} & \text{etc.} \end{array}$$

With which values found the sum of the series proposed will be equal to the following series

$$s = \frac{A}{2} - \frac{A'}{2\cdot 3} + \frac{A''}{2\cdot 3\cdot 4} - \frac{A'''}{2\cdot 3\cdot 4\cdot 5} + \frac{A''''}{2\cdot 3\cdot 4\cdot 5\cdot 6} - \text{etc.}$$

Therefore in a similar manner any proposed series can be transformed into innumerable others equal to itself, between which without doubt there may be found maximally converging series, with the aid of which the proposed sum may be able to be investigated approximately.

216. But we may revert to the discovery of series, of which the differential calculus may indicate the law of progression. Therefore since now there shall be excellence with algebraic quantities, we may progress to transcending quantities, and the series may be sought equal to the logarithm of this

$$s = l\left(1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.}\right);$$

this series may be devised to satisfy what is sought

$$s = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \mathfrak{F}x^6 + \text{etc.}$$

Therefore since from the differentiation of this equation, it may follow

$$\frac{ds}{dx}=\frac{\alpha+2\beta x+3\gamma x^2+4\delta x^3+5\varepsilon x^4+\text{etc.}}{1+\alpha x+\beta x^2+\gamma x^3+\delta x^4+\varepsilon x^5+\text{etc.}},$$

there will be

$$\left(1+\alpha x+\beta x^2+\gamma x^3+\delta x^4+\varepsilon x^5+\text{etc.}\right)\frac{ds}{dx}=\alpha+2\beta x+3\gamma x^2+4\delta x^3+5\varepsilon x^4+\text{etc.}$$

Truly since from the equation devised there is

$$\frac{ds}{dx}=\mathfrak{A}+2\mathfrak{B}x+3\mathfrak{C}x^2+4\mathfrak{D}x^3+5\mathfrak{E}x^4+\text{etc.},$$

with this substitution made this equation arises

$$\begin{array}{llllll}
\mathfrak{A}+ & 2\mathfrak{B}x+ & 3\mathfrak{C}x^2+ & 4\mathfrak{D}x^3+ & 5\mathfrak{E}x^4 & +\text{etc.}\\
 & +\mathfrak{A}\alpha & +2\mathfrak{B}\alpha & +3\mathfrak{C}\alpha & +4\mathfrak{D}\alpha & +\text{etc.}\\
 & & +\mathfrak{A}\beta & +2\mathfrak{B}\beta & +3\mathfrak{C}\beta & +\text{etc.}\\
 & & & +\mathfrak{A}\gamma & +2\mathfrak{B}\gamma & +\text{etc.}\\
 & & & & +\mathfrak{A}\delta & +\text{etc.}\\
=\alpha+ & 2\beta x+ & 3\gamma x^2 & +4\delta x^3 & +5\varepsilon x^4 & +\text{etc.}
\end{array}$$

From which the following determinations are obtained

$$\begin{aligned}
\mathfrak{A}&=\alpha\\
\mathfrak{B}&=-\tfrac{1}{2}\mathfrak{A}\alpha+\beta\\
\mathfrak{C}&=-\tfrac{2}{3}\mathfrak{B}\alpha-\tfrac{1}{3}\mathfrak{A}\beta+\gamma\\
\mathfrak{D}&=-\tfrac{3}{4}\mathfrak{C}\alpha-\tfrac{2}{4}\mathfrak{B}\beta-\tfrac{1}{4}\mathfrak{A}\gamma+\delta\\
\mathfrak{E}&=-\tfrac{4}{5}\mathfrak{D}\alpha-\tfrac{3}{5}\mathfrak{C}\beta-\tfrac{2}{5}\mathfrak{B}\gamma-\tfrac{1}{5}\mathfrak{A}\delta+\varepsilon\\
&\text{etc.}
\end{aligned}$$

217. Now let the proposed quantity be the exponential

$$s=e^{\alpha x+\beta x^2+\gamma x^3+\delta x^4+\varepsilon x^5+\text{etc.}}$$

in which *e* may denote the number, the hyperbolic logarithm of which is = 1, and the series sought may be devised

$$s=1+\mathfrak{A}x+\mathfrak{B}x^2+\mathfrak{C}x^3+\mathfrak{D}x^4+\mathfrak{E}x^5+\text{etc.}$$

Now indeed from the case $x = 0$ it is apparent that the first term must be equal to unity. Therefore since on taking the logarithms there shall be

$$ls = \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \zeta x^6 + \text{etc.}$$

there will be with the differentials taken

$$\tfrac{ds}{dx} = s\left(\alpha + 2\beta x + 3\gamma x^2 + 4\delta x^3 + 5\varepsilon x^4 + \text{etc.}\right)$$

But truly from the equation devised there will be

$$\begin{aligned}\tfrac{ds}{dx} &= \mathfrak{A} + 2\mathfrak{B}x + 3\mathfrak{C}x^2 + 4\mathfrak{D}x^3 + 5\mathfrak{E}x^4 + \text{etc.}\\ &= \alpha + \mathfrak{A}\alpha x + \mathfrak{B}\alpha x^2 + \mathfrak{C}\alpha x^3 + \mathfrak{D}\alpha x^4 + \text{etc.}\\ &\quad\ + 2\beta \quad + 2\mathfrak{A}\beta \quad + 2\mathfrak{B}\beta \quad + 2\mathfrak{C}\beta \quad + \text{etc.}\\ &\qquad\qquad + 3\gamma \quad + 3\mathfrak{A}\gamma \quad + 3\mathfrak{B}\gamma \quad + \text{etc.}\\ &\qquad\qquad\qquad + 4\delta \quad + 4\mathfrak{A}\delta \quad + \text{etc.}\\ &\qquad\qquad\qquad\qquad + 5\varepsilon \quad + \text{etc.},\end{aligned}$$

from which the following determinations of the letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. may be produced

$$\begin{aligned}\mathfrak{A} &= \alpha\\ \mathfrak{B} &= \beta + \tfrac{1}{2}\mathfrak{A}\alpha\\ \mathfrak{C} &= \gamma + \tfrac{2}{3}\mathfrak{A}\beta + \tfrac{1}{3}\mathfrak{B}\alpha\\ \mathfrak{D} &= \delta + \tfrac{3}{4}\mathfrak{A}\gamma + \tfrac{2}{4}\mathfrak{B}\beta + \tfrac{1}{4}\mathfrak{C}\alpha\\ \mathfrak{E} &= \varepsilon + \tfrac{4}{5}\mathfrak{A}\delta + \tfrac{3}{5}\mathfrak{B}\gamma + \tfrac{2}{5}\mathfrak{C}\beta + \tfrac{1}{5}\mathfrak{D}\alpha\\ &\text{etc.}\end{aligned}$$

218. Also if the arc, of which the sine or cosine is sought, may be expressed by a binomial or polynomial or also by an infinite series, in this manner also the sine or cosine of this can be expressed by an infinite series. But truly so that this can be made most conveniently, it is not sufficient to proceed with the first differentials, but there is a need, that we may call into help the differentials of the second order. Therefore let there be

$$s = \sin\left(\alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.}\right)$$

and the series is devised, which is sought,

$$s = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \text{etc.}$$

For the first term is agreed to vanish ; because truly it is required to descend to the second differentials, the coefficient $\mathfrak{A}$ also is required to be defined from elsewhere, which comes about, if we may put x infinitely small. Then indeed on account of the arc $= \alpha x$ the sine becomes equal to this itself and therefore there will be $\mathfrak{A} = \alpha$. Now we may put for the sake of brevity

$$z = \alpha x + \beta x^2 + \gamma x^3 + \text{etc.},$$

so that there shall be $s = \sin z$; on differentiation there will be $ds = dz \cos z$ and on differentiation anew $dds = ddz \cos z - dz^2 \sin z$ .Therefore because there is $\sin z = s$ and $\cos z = \frac{ds}{dz}$, there will be

$$dds = \frac{dsddz}{dz} - sdz^2 \quad \text{or} \quad dzdds + sdz^3 = dsddz.$$

219. We may put the arc z to be expressed by a binomial and there is

$$z = \alpha x + \beta x^2;$$

there will be

$$dz = (\alpha + 2\beta x)dx$$

and on putting dx constant

$$ddz = 2\beta dx^2$$

and

$$dz^3 = \left(\alpha^3 + 6\alpha^2 \beta x + 12\alpha\beta^2 x^2 + 8\beta^3 x^3\right)dx^3$$

Then on account of $s = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \text{etc.}$, there will be

$$\frac{ds}{dx} = \mathfrak{A} + 2\mathfrak{B}x + 3\mathfrak{C}x^2 + 4\mathfrak{D}x^3 + \text{etc.}$$

and

$$\frac{dds}{dx^2} = 2\mathfrak{B} + 6\mathfrak{C}x + 12\mathfrak{D}x^2 + \text{etc.}$$

With which values substituted into the second order differential equation [literally, into the differential of the differential equation] there becomes [with detached coefficients again]

$$\begin{array}{llllll}
\frac{dzdds}{dx^3} = 1\cdot 2\mathfrak{B}\alpha & +\ 2\cdot 3\mathfrak{C}\alpha x & +\ 3\cdot 4\mathfrak{D}\alpha x^2 & +\ 4\cdot 5\mathfrak{E}\alpha x^3 & +\ 5\cdot 6\mathfrak{F}\alpha x^4 & +\text{etc.}\\
 & +2\cdot 1\cdot 2\mathfrak{B}\beta & +2\cdot 2\cdot 3\mathfrak{C}\beta & +\ 2\cdot 3\cdot 4\mathfrak{D}\beta & +2\cdot 4\cdot 5\mathfrak{E}\beta & +\text{etc.}\\
\frac{sdz^3}{dx^3} = & +\ \mathfrak{A}\alpha^3 & +\ \mathfrak{B}\alpha^3 & +\ \mathfrak{C}\alpha^3 & +\ \mathfrak{C}\alpha^3 & +\text{etc.}\\
 & & +\ 6\mathfrak{A}\alpha^2\beta & +\ 6\mathfrak{B}\alpha^2\beta & +\ 6\mathfrak{C}\alpha^2\beta & +\text{etc.}\\
 & & & +\ 12\mathfrak{A}\alpha\beta^2 & +12\mathfrak{B}\alpha\beta^2 & +\text{etc.}\\
 & & & & +\ 8\mathfrak{A}\beta^3 & +\text{etc.}\\
\frac{dsddz}{dx^3} = 2\mathfrak{A}\beta & +\ 4\mathfrak{B}\beta & +\ 6\mathfrak{C}\beta & +\ 8\mathfrak{D}\beta & +\ 10\mathfrak{E}\beta & +\text{etc}
\end{array}$$

From which the coefficients may be defined in the following manner :

$$\begin{aligned}
&\mathfrak{B} = \frac{2\mathfrak{A}\beta}{2\alpha}\\
&\mathfrak{C} = 0 - \frac{\mathfrak{A}\alpha^2}{2\cdot 3}\\
&\mathfrak{D} = -\ \frac{2\mathfrak{C}\beta}{4\alpha} - \frac{6\mathfrak{A}\alpha\beta}{3\cdot 4} - \frac{\mathfrak{B}\alpha^2}{3\cdot 4}\\
&\mathfrak{E} = -\ \frac{4\mathfrak{D}\beta}{5\alpha} - \frac{12\mathfrak{A}\beta^2}{4\cdot 5} - \frac{6\mathfrak{B}\alpha\beta}{4\cdot 5} - \frac{\mathfrak{C}\alpha^2}{4\cdot 5}\\
&\mathfrak{F} = -\ \frac{6\mathfrak{E}\beta}{6\alpha} - \frac{8\mathfrak{A}\beta^3}{5\cdot 6\alpha} - \frac{12\mathfrak{B}\beta\beta}{5\cdot 6} - \frac{6\mathfrak{C}\alpha\beta}{5\cdot 6} - \frac{\mathfrak{D}\alpha^2}{5\cdot 6}\\
&\mathfrak{G} = -\ \frac{8\mathfrak{F}\beta}{7\alpha} - \frac{8\mathfrak{B}\beta^3}{6\cdot 7\alpha} - \frac{12\mathfrak{C}\beta\beta}{6\cdot 7} - \frac{6\mathfrak{D}\alpha\beta}{6\cdot 7} - \frac{\mathfrak{E}\alpha^2}{6\cdot 7}\\
&\text{etc.}
\end{aligned}$$

With which values found there will be

$$\sin\left(\alpha x + \beta x^2\right) = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \text{etc.}$$

with $\mathfrak{A} = \alpha$ present.

220. In a like manner the cosine of each angle may be converted into a series ; but because the arc may be rarely expressed by a series, we may show the use of the differential of the differential equation in finding the series for the cosine of the arc x. Therefore let $s = \cos x$ and there may be constructed

$$s = 1 - \mathfrak{A}x^2 + \mathfrak{B}x^4 - \mathfrak{C}x^6 + \mathfrak{D}x^8 - \text{etc.}$$

Because there is $ds = -dx\sin x$ and $dds = -dx^2\cos x = -sdx^2$, there will be

$$dds + sdx^2 = 0\,;$$

therefore with the substitution made there becomes

$$\frac{dds}{dx^2} = -1\cdot 2\mathfrak{A} + 3\cdot 4\mathfrak{B}x^2 - 5\cdot 6\mathfrak{C}x^4 + 7\cdot 8\mathfrak{D}x^6 - \text{etc.}$$
$$s = 1 - \mathfrak{A}x^2 + \mathfrak{B}x^4 - \mathfrak{C}x^6 + \text{etc.}$$

but from the equalisation of the terms there follows

$$\mathfrak{A} = \frac{1}{1\cdot 2}$$
$$\mathfrak{B} = \frac{\mathfrak{A}}{3\cdot 4} = \frac{1}{1\cdot 2\cdot 3\cdot 4}$$
$$\mathfrak{C} = \frac{\mathfrak{B}}{5\cdot 6} = \frac{1}{1\cdot 2\cdot 3\cdots 6}$$
$$\mathfrak{D} = \frac{\mathfrak{C}}{7\cdot 8} = \frac{1}{1\cdot 2\cdot 3\cdots 8}$$
etc.

Therefore it is apparent, as now we have shown above further, to be

$$\cos x = 1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \frac{x^6}{1\cdot 2\cdot 3\cdots 6} + \frac{x^8}{1\cdot 2\cdot 3\cdots 8} - \text{etc.}$$

truly the former series for the sine on putting $\beta = 0$ and $\alpha = 1$ will give

$$\sin x = \frac{x}{1} - \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{x^7}{1\cdot 2\cdot 3\cdots 7} + \frac{x^9}{1\cdot 2\cdot 3\cdots 9} - \text{etc.}$$

221. From these most noteworthy series for the sine and cosine there are deduced the series for the tangent, cotangent, secant, and cosecant of any angle. Indeed the tangent is produced, if the sine may be divided by the cosine, the cotangent if the cosine may be divided by the sine, the secant, if the radius 1 may be divided by the cosine, and the cosecant, if the radius may be divided by the sine. But the series arising from these divisions may be seen to be greatly irregular ; truly with the series excepted showing the secant the remainder are able to be reduced by the Bernoulli numbers defined above $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. [not the ones just used above !] to an easy law of progression. Because indeed we have found above in §127 to be

$$\frac{\mathfrak{A}u^2}{1\cdot 2} + \frac{\mathfrak{B}u^4}{1\cdot 2\cdot 3\cdot 4} + \frac{\mathfrak{C}u^6}{1\cdot 2\cdot 3\cdots 6} + \frac{\mathfrak{D}u^8}{1\cdot 2\cdot 3\cdots 8} + \text{etc.} = 1 - \frac{u}{2}\cot\frac{1}{2}u\,,$$

there will be on putting $\frac{1}{2}u = x$

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1\cdot 2} - \frac{2^4\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} - \frac{2^6\mathfrak{C}x^5}{1\cdot 2\cdot 3\cdots 6} - \frac{2^8\mathfrak{D}x^7}{1\cdot 2\cdot 3\cdots 8} - \text{etc.},$$

and if there is put $\frac{1}{2}x$ for x, there will be

$$\cot\frac{1}{2}x = \frac{2}{x} - \frac{2\mathfrak{A}x}{1\cdot 2} - \frac{2\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} - \frac{2\mathfrak{C}x^5}{1\cdot 2\cdot 3\cdots 6} - \frac{2\mathfrak{D}x^7}{1\cdot 2\cdot 3\cdots 8} - \text{etc.}$$

222. Hence moreover the tangent of any arc may be expressed in the following manner by a series.
Since there shall be

$$\text{tang}\, 2x = \frac{2\,\text{tang}\, x}{1 - \text{tang}^2 x}$$

there will be

$$\cot 2x = \frac{1}{2\text{tang}\, x} - \frac{\text{tang}\, x}{2} = \frac{1}{2}\cot x - \frac{1}{2}\text{tang}\, x$$

and thus

$$\text{tang}\, x = \cot x - 2\cot 2x\,.$$

Therefore since there shall be

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1\cdot 2} - \frac{2^4\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} - \frac{2^6\mathfrak{C}x^5}{1\cdot 2\cdot 3\cdots 6} - \frac{2^8\mathfrak{D}x^7}{1\cdot 2\cdot 3\cdots 8} - \text{etc.},$$
$$2\cot 2x = \frac{1}{x} - \frac{2^4\mathfrak{A}x}{1\cdot 2} - \frac{2^8\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} - \frac{2^{12}\mathfrak{C}x^5}{1\cdot 2\cdot 3\cdots 6} - \frac{2^{16}\mathfrak{D}x^7}{1\cdot 2\cdot 3\cdots 8} - \text{etc.},$$

by subtracting that series from this there will become

$$\text{tang}\, x = \frac{2^2\left(2^2-1\right)\mathfrak{A}x}{1\cdot 2} + \frac{2^4\left(2^4-1\right)\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} + \frac{2^6\left(2^6-1\right)\mathfrak{C}x^5}{1\cdot 2\cdot 3\cdots 6} + \frac{2^8\left(2^8-1\right)\mathfrak{D}x^7}{1\cdot 2\cdot 3\cdots 8} + \text{etc.}$$

Therefore if there may be introduced here the numbers A, B, C, D etc. found in §182, there will be

$$\text{tang}\, x = \frac{2Ax}{1\cdot 2} + \frac{2^3Bx^3}{1\cdot 2\cdot 3\cdot 4} + \frac{2^5Cx^5}{1\cdot 2\cdot 3\cdots 6} + \frac{2^7Dx^7}{1\cdot 2\cdot 3\cdots 8} + \text{etc.}$$

223. But the cosecant may be found in the following way. Because there is

$$\cot x = \text{tang}\, x + 2\cot 2x = \frac{1}{\cot x} + 2\cot 2x,$$

there will be

$$\cot^2 x = 2\cot x \cdot \cot 2x + 1$$

and with the root extracted

$$\cot x = \cot 2x + \operatorname{cosec} 2x\ ,$$

from which there is made

$$\operatorname{cosec} 2x = \cot x - \cot 2x$$

and on putting *x* for *2x*

$$\operatorname{cosec} x = \cot \tfrac{1}{2} x - \cot x$$

Whereby since we may have the cotangent, evidently

$$\cot \tfrac{1}{2} x = \tfrac{2}{x} - \tfrac{2\mathfrak{A}x}{1\cdot 2} - \tfrac{2\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} - \tfrac{2\mathfrak{C}x^5}{1\cdot 2\cdot 3\cdots 6} - \text{etc.}$$

$$\cot x = \tfrac{1}{x} - \tfrac{2^2\mathfrak{A}x}{1\cdot 2} - \tfrac{2^4\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} - \tfrac{2^6\mathfrak{C}x^5}{1\cdot 2\cdot 3\cdots 6} - \text{etc.}$$

with the latter series subtracted from the former there will be

$$\operatorname{cosec} x = \tfrac{1}{x} + \frac{2(2-1)\mathfrak{A}x}{1\cdot 2} + \frac{2\left(2^3-1\right)\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} + \frac{2\left(2^5-1\right)\mathfrak{C}x^5}{1\cdot 2\cdot 3\cdots 6} + \text{etc.}$$

224. But the secant cannot be expressed by these Bernoulli numbers, but it requires other numbers, which are present in the sums of the reciprocals of the odd powers. For if there is put in place

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \text{etc.} = \alpha \cdot \tfrac{\pi}{2^2}$$

$$1 - \tfrac{1}{3^3} + \tfrac{1}{5^3} - \tfrac{1}{7^3} + \tfrac{1}{9^3} - \text{etc.} = \tfrac{\beta}{1\cdot 2} \cdot \tfrac{\pi^3}{2^4}$$

$$1 - \tfrac{1}{3^5} + \tfrac{1}{5^5} - \tfrac{1}{7^5} + \tfrac{1}{9^5} - \text{etc.} = \tfrac{\gamma}{1\cdot 2\cdot 3\cdot 4} \cdot \tfrac{\pi^5}{2^6}$$

$$1 - \tfrac{1}{3^7} + \tfrac{1}{5^7} - \tfrac{1}{7^7} + \tfrac{1}{9^7} - \text{etc.} = \tfrac{\delta}{1\cdot 2\cdots 6} \cdot \tfrac{\pi^7}{2^8}$$

$$1 - \tfrac{1}{3^9} + \tfrac{1}{5^9} - \tfrac{1}{7^9} + \tfrac{1}{9^9} - \text{etc.} = \tfrac{\varepsilon}{1\cdot 2\cdots 8} \cdot \tfrac{\pi^9}{2^{10}}$$

$$1 - \tfrac{1}{3^{11}} + \tfrac{1}{5^{11}} - \tfrac{1}{7^{11}} + \tfrac{1}{9^{11}} - \text{etc.} = \tfrac{\varepsilon}{1\cdot 2\cdots 10} \cdot \tfrac{\pi^{11}}{2^{12}}$$

etc.,

there will be

$\alpha = 1$
$\beta = 1$
$\gamma = 5$
$\delta = 61$
$\varepsilon = 1385$
$\zeta = 50521$
$\eta = 2702765$
$\theta = 199360981$
$\iota = 19391512145$
$\chi = 2404879661671$ [actually 2404879675441.]
etc.

and from these values there will be obtained

$$\sec x = \alpha + \frac{\beta}{1\cdot 2}xx + \frac{\gamma}{1\cdot 2\cdot 3\cdot 4}x^4 + \frac{\delta}{1\cdot 2\cdot 3\cdots 6}x^6 + \frac{\varepsilon}{1\cdot 2\cdot 3\cdots 8}x^8 + \text{etc.}$$

[The numbers $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$ etc. are now called the Euler Numbers.]

225. But so that we may show the connection of this series with the numbers $\alpha,\ \beta,\ \gamma,\ \delta$ etc., we may consider the series treated above in § 33

$$\frac{\pi}{n\sin\frac{m}{n}\pi} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \text{etc.}$$

There may be put $m = \frac{1}{2}n - k$ and there will be

$$\frac{\pi}{2n\sin\frac{k}{n}\pi} = \frac{1}{n-2k} + \frac{1}{n+2k} - \frac{1}{3n-2k} - \frac{1}{3n+2k} + \frac{1}{5n-2k} + \text{etc.}$$

Let $\frac{k\pi}{n} = x$ or $k\pi = nx$; there will be

$$\frac{\pi}{2n}\sec x = \frac{\pi}{n\pi-2nx} + \frac{\pi}{n\pi+2nx} - \frac{\pi}{3n\pi-2nx} - \frac{\pi}{3n\pi+2nx} + \frac{\pi}{5n\pi-2nx} + \text{etc.}$$

or

$$\sec x = \frac{2}{\pi-2x} + \frac{2}{\pi+2x} - \frac{2}{3\pi-2x} - \frac{2}{3\pi+2x} + \frac{2}{5\pi-2x} + \text{etc.}$$

or

$$\sec x = \frac{4\pi}{\pi^2-4x^2} - \frac{4\cdot 3\pi}{9\pi^2-4xx} + \frac{4\cdot 5\pi}{25\pi^2-4xx} - \frac{4\cdot 7\pi}{49\pi^2-4xx} + \text{etc.}$$

If now the individual terms may be converted into series, there becomes

$$\begin{aligned}\sec x = {} & \tfrac{4}{\pi}\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\text{etc.}\right)\\ & +\tfrac{2^4 x^2}{\pi^3}\left(1-\tfrac{1}{3^3}+\tfrac{1}{5^3}-\tfrac{1}{7^3}+\tfrac{1}{9^3}-\text{etc.}\right)\\ & +\tfrac{2^6 x^4}{\pi^5}\left(1-\tfrac{1}{3^5}+\tfrac{1}{5^5}-\tfrac{1}{7^5}+\tfrac{1}{9^5}-\text{etc.}\right)\\ & \text{etc.;}\end{aligned}$$

in place of which series if the values assigned above may be substituted, the same series for the secant will be produced, which we have deduced.

226. Hence a similar law may be apparent by which the numbers $\alpha, \beta, \gamma, \delta$ etc. proceed, from which the sums of the odd powers may be established. For since there shall be

$$\sec x = \frac{1}{\cos x} = \alpha + \frac{\beta}{1\cdot 2}x^2 + \frac{\gamma}{1\cdot 2\cdot 3\cdot 4}x^4 + \frac{\delta}{1\cdot 2\cdot 3\cdots 6}x^6 + \text{etc.},$$

it is necessary that this series shall be equal to the fraction

$$\frac{1}{1-\frac{xx}{1\cdot 2}+\frac{x^4}{1\cdot 2\cdot 3\cdot 4}-\frac{x^6}{1\cdot 2\cdot 3\cdots 6}+\frac{x^8}{1\cdot 2\cdot 3\cdots 8}-\text{etc.}}$$

therefore from the equality put in place there is made

$$\begin{aligned}1 = \alpha & + \frac{\beta}{1\cdot 2}x^2 + \frac{\gamma}{1\cdot 2\cdot 3\cdot 4}x^4 + \frac{\delta}{1\cdot 2\cdot 3\cdots 6}x^6 + \frac{\varepsilon}{1\cdot 2\cdot 3\cdots 8}x^8 + \text{etc.}\\ & - \frac{\alpha}{1\cdot 2} \quad - \frac{\beta}{1\cdot 2\cdot 1\cdot 2} \quad - \frac{\gamma}{1\cdot 2\cdot 1\cdots 4} \quad - \frac{\delta}{1\cdot 2\cdot 1\cdots 6} \quad - \text{etc.}\\ & \qquad + \frac{\alpha}{1\cdot 2\cdot 3\cdot 4} \quad + \frac{\beta}{1\cdots 4\cdot 1\cdot 2} \quad + \frac{\gamma}{1\cdots 4\cdot 1\cdots 4} \quad + \text{etc.}\\ & \qquad\qquad - \frac{\alpha}{1\cdot 2\cdots 6} \quad - \frac{\beta}{1\cdots 6\cdot 1\cdot 2} \quad - \text{etc.}\\ & \qquad\qquad\qquad + \frac{\alpha}{1\cdot 2\cdots 8} \quad + \text{etc.},\end{aligned}$$

from which these equations follow

$$\alpha = 1$$

$$\beta = \tfrac{2\cdot 1}{1\cdot 2}\alpha$$

$$\gamma = \tfrac{4\cdot 3}{1\cdot 2}\beta - \tfrac{4\cdot 3\cdot 2\cdot 1}{1.2.3\cdot 4}\alpha$$

$$\delta = \tfrac{6\cdot 5}{1\cdot 2}\gamma - \tfrac{6\cdot 5\cdot 4\cdot 3}{1.2.3\cdot 4}\beta + \tfrac{6\cdots 1}{1\cdots 6}\alpha$$

$$\varepsilon = \tfrac{8\cdot 7}{1\cdot 2}\delta - \tfrac{8\cdot 7\cdot 6\cdot 5}{1.2.3\cdot 4}\gamma + \tfrac{8\cdots 3}{1\cdots 6}\beta - \tfrac{8\cdots 1}{1\cdots 8}\alpha$$

etc.

And from these formulas the values of these letters have been found, which we have shown in § 224 and with the aid of which the sums of series contained in this form

$$1 - \tfrac{1}{3^n} + \tfrac{1}{5^n} - \tfrac{1}{7^n} + \tfrac{1}{9^n} - \text{etc.},$$

can be expressed, if *n* were an odd number.

CAPUT VIII

DE USU CALCULI DIFFERENTIALIS IN FORMANDIS SERIEBUS

198. Unum adhuc calculi differentialis usum in doctrina serierum commemorabimus, qui in ipsa formatione serierum consistit et ad quem iam supra provocavimus, cum quaestio esset de fractione, cuius denominator sit potestas quaecunque functionis cuiuspiam, in seriem evolvenda. Ista methodus autem similis est ei, qua iam aliquoties sumus usi, dum functio in seriem convertenda aequalis fingitur cuipiam seriei in singulis terminis coefficientes indeterminatos habenti, qui deinceps aequalitate constituta determinentur. Haec autem determinatio saepenumero mirifice adiuvatur, si, antequam ea suscipiatur, ad differentialia cum prima tum nonnunquam quoque ad secunda aequatio perducatur. Quae methodus cum in calculo integrali amplissimi sit usus, eam hic diligentius exponemus.

199. Primum igitur breviter repetamus, quae supra de evolutione fractionum in series sine calculi differentialis subsidio attulimus. Sit fractio quaecunque proposita

$$\frac{A+Bx+Cx^2+Dx^3+\text{etc.}}{\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4+\text{etc.}}=s,$$

quam in seriem secundum potestates ipsius x procedentem converti oporteat.
Fingatur pro s series indeterminata

$$s=\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3+\mathfrak{E}x^4+\mathfrak{F}x^5+\mathfrak{G}x^6+\text{etc.}$$

Cum igitur fractione per multiplicationem sublata sit

$$\begin{aligned}&A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+Gx^6+\text{etc.}\\&=s\left(\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4+\zeta x^5+\eta x^6+\text{etc.}\right),\end{aligned}$$

si pro s series ficta substituatur, prodibit sequens aequatio

$$\begin{aligned} & A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.} \\ = & \mathfrak{A}\alpha + \mathfrak{B}\alpha x + \mathfrak{C}\alpha x^2 + \mathfrak{D}\alpha x^3 + \mathfrak{E}\alpha x^4 + \mathfrak{F}\alpha x^5 + \text{etc.} \\ & \quad + \mathfrak{A}\beta + \mathfrak{B}\beta + \mathfrak{C}\beta + \mathfrak{D}\beta + \mathfrak{E}\beta + \text{etc.} \\ & \quad\quad + \mathfrak{A}\gamma + \mathfrak{B}\gamma + \mathfrak{C}\gamma + \mathfrak{D}\gamma + \text{etc.} \\ & \quad\quad\quad + \mathfrak{A}\delta + \mathfrak{B}\delta + \mathfrak{C}\delta + \text{etc.} \\ & \quad\quad\quad\quad + \mathfrak{A}\varepsilon + \mathfrak{B}\varepsilon + \text{etc.} \\ & \quad\quad\quad\quad\quad + \mathfrak{A}\zeta + \text{etc.} \end{aligned}$$

Aequalitate ergo inter singulos terminos, qui easdem ipsius x potestates continent, constituta fiet

$$\begin{aligned} & \mathfrak{A}\alpha - A = 0 \\ & \mathfrak{B}\alpha + \mathfrak{A}\beta - B = 0 \\ & \mathfrak{C}\alpha + \mathfrak{B}\beta + \mathfrak{A}\gamma - C = 0 \\ & \mathfrak{D}\alpha + \mathfrak{C}\beta + \mathfrak{B}\gamma + \mathfrak{A}\delta - D = 0 \\ & \mathfrak{E}\alpha + \mathfrak{D}\beta + \mathfrak{C}\gamma + \mathfrak{B}\delta + \mathfrak{A}\varepsilon - E = 0 \\ & \text{etc.}, \end{aligned}$$

ex quibus aequationibus coefficientes ficti $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. determinantur, sicque series infinita invenitur

$$\mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \text{etc.}$$

fractioni propositae s aequalis. Atque in hac forma, si tam numerator quam denominator fractionis s finito terminorum numero constent, omnes series recurrentes comprehenduntur, de quibus iam supra fusius est tractatum.

200. Quodsi autem vel numerator vel denominator vel uterque ad dignitatem quamcunque fuerit elevatus, tum hoc modo series difficulter obtinetur, propterea quod negotium, nisi functio elevata sit binomium, perquam fit laboriosum. Calculo autem differentiali iste labor evitari potest. Adsit primum solus numerator sitque

$$s = \left(A + Bx + Cxx\right)^n,$$

unde quaeratur series secundum potestates ipsius x procedens huic trinomii dignitati aequalis; quam quidem finitam fore constat, si exponens n fuerit numerus integer affirmativus. Fingatur iterum pro s series indefinita

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \mathfrak{F}x^5 + \mathfrak{G}x^6 + \text{etc.},,$$

cuius terminum primum $\mathfrak{A}$ constat esse $= A^n$; si enim ponatur $x = 0$, ex priori forma proposita fit $s = A^n$, ex serie autem ficta $s = \mathfrak{A}$. Haec autem primi termini determinatio ex ipsa rei natura est petenda, si ad differentialia descendere velimus, quia hinc primus terminus non determinatur, uti mox patebit.

201. Cum sit $s = \left(A + Bx + Cx^2\right)^n$ erit logarithmis sumendis

$$ls = nl\left(A + Bx + Cx^2\right)$$

hincque sumtis differentialibus habebitur

$$\frac{ds}{s} = \frac{nBdx + 2nCxdx}{A + Bx + Cx^2} \quad \text{seu} \quad \left(A + Bx + Cx^2\right)\frac{ds}{dx} = ns\left(B + 2Cx\right).$$

Ex serie autem ficta est

$$\frac{ds}{dx} = \mathfrak{B} + 2\mathfrak{C}x + 3\mathfrak{D}x^2 + 4\mathfrak{E}x^3 + 5\mathfrak{F}x^4 + \text{etc.}$$

Si igitur haec series loco $\frac{ds}{dx}$ et pro *s* ipsa series ficta substituatur, prodibit sequens aequatio

$$\begin{array}{llllll} A\mathfrak{B} + & 2A\mathfrak{C}x + & 3A\mathfrak{D}x^2 + & 4A\mathfrak{E}x^3 + & 5A\mathfrak{F}x^4 & + \text{etc.} \\ & + B\mathfrak{B} & + 2B\mathfrak{C} & + 3B\mathfrak{D} & + 4B\mathfrak{E} & + \text{etc.} \\ & & + C\mathfrak{B} & + 2C\mathfrak{C} & + 3C\mathfrak{D} & + \text{etc.} \\ = & & & & & \\ nB\mathfrak{A} + & nB\mathfrak{B}x & + nB\mathfrak{C}x^2 & + nB\mathfrak{D}x^3 & + nB\mathfrak{E}x^4 & + \text{etc.} \\ & + 2nC\mathfrak{A} & + 2nC\mathfrak{B} & + 2nC\mathfrak{C} & + 2nC\mathfrak{D} & + \text{etc.} \end{array}$$

Aequalitate ergo hic inter terminos eiusdem ipsius *x* potestatis constituta erit

$$\begin{aligned} \mathfrak{B} &= \frac{nB\mathfrak{A}}{A} \\ \mathfrak{C} &= \frac{(n-1)\mathfrak{B}B + 2nC\mathfrak{A}}{2A} \\ \mathfrak{D} &= \frac{(n-2)B\mathfrak{C} + (2n-1)C\mathfrak{B}}{3A} \\ \mathfrak{E} &= \frac{(n-3)B\mathfrak{D} + (2n-2)C\mathfrak{C}}{4A} \\ \mathfrak{F} &= \frac{(n-4)B\mathfrak{E} + (2n-3)C\mathfrak{D}}{5A} \\ &\text{etc.} \end{aligned}$$

Cum igitur, ut ante vidimus, sit $\mathfrak{A} = A^n$, erit $\mathfrak{B} = nA^{n-1}B$ hincque reliqui coefficientes omnes successive determinabuntur. Lex autem, quam ipsi sequuntur, facillime ex his formulis patet, quae vehementer obscura mansisset, si trinomium actu elevare voluissemus.

202. Haec eadem methodus succedit, si polynomium quodcunque ad quampiam dignitatem elevari debeat. Sit

$$s = \left(A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}\right)^n$$

fingaturque

$$s = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}x^2 + \mathfrak{D}x^3 + \mathfrak{E}x^4 + \text{etc.};$$

erit $\mathfrak{A} = A^n$, qui valor colligitur, si ponatur $x = 0$. Sumtis iam ut ante logarithmis eorumque differentialibus reperietur

$$\frac{ds}{s} = \frac{nBdx + 2nCxdx + 3nDx^2dx + 4nEx^3dx + \text{etc.}}{A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}}$$

seu

$$\left(A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc.}\right)\frac{ds}{dx} = s\left(nB + 2nCx + 3nDx^2 + 4nEx^3 + \text{etc.}\right).$$

Cum igitur sit

$$\frac{ds}{dx} = \mathfrak{B} + 2\mathfrak{C}x + 3\mathfrak{D}x^2 + 4\mathfrak{E}x^3 + 5\mathfrak{F}x^4 + \text{etc.}$$

erit his seriebus pro s et $\frac{ds}{dx}$ substitutis

$$\begin{array}{llllll}
A\mathfrak{B} & +2A\mathfrak{C}x & +3A\mathfrak{D}x^2 & +4A\mathfrak{E}x^3 & +5A\mathfrak{F}x^4 & +\text{etc.} \\
 & +B\mathfrak{B} & +2B\mathfrak{C} & +3B\mathfrak{D} & +4B\mathfrak{E} & +\text{etc.} \\
 & & +C\mathfrak{B} & +2C\mathfrak{C} & +3C\mathfrak{D} & +\text{etc.} \\
 & & & +D\mathfrak{B} & +2C\mathfrak{C} & +\text{etc.} \\
 & & & & +E\mathfrak{B} & +\text{etc.}
\end{array}$$

$$=$$

$$\begin{array}{llllll}
nB\mathfrak{A} & +nB\mathfrak{B}x & +nB\mathfrak{C}x^2 & +nB\mathfrak{D}x^3 & +nB\mathfrak{E}x^4 & +\text{etc.} \\
 & +2nC\mathfrak{A} & +2nC\mathfrak{B} & +2nC\mathfrak{C} & +2nC\mathfrak{D} & +\text{etc.} \\
 & & +3nD\mathfrak{A} & +3nD\mathfrak{B} & +3nD\mathfrak{C} & +\text{etc.} \\
 & & & +4nE\mathfrak{A} & +4nE\mathfrak{B} & +\text{etc.} \\
 & & & & +5nF\mathfrak{A} & +\text{etc.}
\end{array}$$

Unde derivantur sequentes determinationes

$$A\mathfrak{B} = nB\mathfrak{A}$$
$$2A\mathfrak{C} = (n-1)B\mathfrak{B} + 2nC\mathfrak{A}$$
$$3A\mathfrak{D} = (n-2)B\mathfrak{C} + (2n-1)C\mathfrak{B} + 3nD\mathfrak{A}$$
$$4A\mathfrak{E} = (n-3)B\mathfrak{D} + (2n-2)C\mathfrak{C} + (3n-1)D\mathfrak{B} + 4nE\mathfrak{A}$$
$$5A\mathfrak{F} = (n-4)B\mathfrak{E} + (2n-3)C\mathfrak{D} + (3n-2)D\mathfrak{C} + (4n-1)E\mathfrak{B} + 5nF\mathfrak{A}$$
etc.,

unde, quemadmodum coefficientes ficti $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. a se invicem pendeant hincque determinentur, cum sit $\mathfrak{A} = A^n$, luculentissime apparet.

203. Quoniam, si quantitas $A + Bx + Cx^2 + Dx^3 + \text{etc.}$ ex finito terminorum numero constat numerusque *n* fuerit integer affirmativus, quaecunque potestas finito etiam terminorum numero constare debet, manifestum est hoc casu formulas modo inventas tandem evanescere debere atque, cum omnes termini adesse debeant, ut primum unus evanuerit, simul omnes sequentes evanescere debere. Ponamus formulam propositam $A + Bx + Cx^2$ esse trinomium eiusque cubum quaeri, ut sit $n = 3$; erit

$\mathfrak{A} = A^3$	ideoque	$\mathfrak{A} = A^3$
$A\mathfrak{B} = 3B\mathfrak{A}$		$\mathfrak{B} = 3A^2B$
$2A\mathfrak{C} = 2B\mathfrak{B} + 6C\mathfrak{A}$		$\mathfrak{C} = 3AB^2 + 3A^2C$
$3A\mathfrak{D} = 1B\mathfrak{C} + 5C\mathfrak{B}$		$\mathfrak{D} = B^3 + 6ABC$
$4A\mathfrak{E} = \quad 0 \quad + 4C\mathfrak{C}$		$\mathfrak{E} = 3B^2C + 3AC^2$
$5A\mathfrak{F} = -B\mathfrak{E} + 3C\mathfrak{D}$		$\mathfrak{F} = 3BC^2$
$6A\mathfrak{G} = -2B\mathfrak{F} + 2C\mathfrak{E}$		$\mathfrak{G} = C^3$
$7A\mathfrak{H} = -3B\mathfrak{G} + 1C\mathfrak{F}$		$\mathfrak{H} = 0$
$8A\mathfrak{I} = -4B\mathfrak{H} + 0$		$\mathfrak{I} = 0$.

Quoniam igitur iam bini sunt $= 0$ sequentiumque quilibet a duobus praecedentibus pendet, patet omnes sequentes pariter evanescere debere. Hancque ob causam lex, qua hi coefficientes a se invicem pendere sunt inventi, eo magis est notatu digna.

204. Si *n* fuerit numerus negativus, ita ut *s* aequale fiat fractioni, series in infinitum excurret. Sit igitur

$$s=\frac{1}{\left(\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4+\text{etc.}\right)^n}$$

fingatur pro eius valore haec series

$$s=\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3+\mathfrak{E}x^4+\mathfrak{F}x^5+\text{etc.}$$

Atque si in superioribus formulis pro litteris A, *B, C, D* etc. ponantur $\alpha,\ \beta,\ \gamma,\ \delta$ etc. simulque fiat *n* negativum, sequentes determinationes coefficientium $\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D}$ etc. prodibunt

$$\mathfrak{A}=\alpha^{-n}=\tfrac{1}{\alpha^n}$$
$$\alpha\mathfrak{B}+n\beta\mathfrak{A}=0$$
$$2\alpha\mathfrak{C}+(n+1)\beta\mathfrak{B}+2n\gamma\mathfrak{A}=0$$
$$3\alpha\mathfrak{D}+(n+2)\beta\mathfrak{C}+(2n+1)\gamma\mathfrak{B}+3n\delta\mathfrak{A}=0$$
$$4\alpha\mathfrak{E}+(n+3)\beta\mathfrak{D}+(2n+2)\gamma\mathfrak{C}+(3n+1)\delta\mathfrak{B}+4n\varepsilon\mathfrak{A}=0$$
$$5\alpha\mathfrak{F}+(n+4)\beta\mathfrak{E}+(2n+3)\gamma\mathfrak{D}+(3n+2)\delta\mathfrak{C}+(4n+1)\varepsilon\mathfrak{B}+5n\zeta\mathfrak{A}=0$$
etc.

Quae formulae eandem continent legem horum coefficientium numerorum, quam iam supra observavimus in *Introductione* cuiusque adeo veritatem nunc demum rigide demonstrare licuit.

205. Haec ita se habent, si numerator fractionis fuerit unitas vel etiam quaepiam ipsius *x* potestas, puta x^m; posteriori enim casu tantum oportebit seriem priori inventam $\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3+\text{etc.}$ multiplicare per x^m. At si numerator constet ex duobus pluribusve terminis, tum supra quidem legem progressionis non observavimus; quarnobrem eam hic per differentiationem investigemus. Sit igitur

$$s=\frac{A+Bx+Cx^2+Dx^3+\text{etc.}}{\left(\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4+\text{etc.}\right)^n}$$

fingaturque pro valore huius fractionis sequens series

$$s=\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3+\mathfrak{E}x^4+\mathfrak{F}x^5+\text{etc.}\ ;$$

cuius primus terminus $\mathfrak{A}$ ut definiatur, ponatur $x=0$ eritque ex priori expressione $s=\frac{A}{\alpha^n}$, ex ficta vera $s=\mathfrak{A}$, unde necesse est, ut sit $\mathfrak{A}=\frac{A}{\alpha^n}$. Quo termino determinato reliqui per differentiationem innotescent.

206. Sumtis logarithmis erit

$$ls = l\left(A+Bx+Cx^2+Dx^3+\text{etc.}\right)-nl\left(\alpha+\beta x+\gamma x^2+\gamma x^3+\varepsilon x^4+\text{etc.}\right)$$

hincque differentiando orietur

$$\frac{ds}{s}=\frac{Bdx+2Cxdx+3Dx^2dx+\text{etc.}}{A+Bx+Cx^2+Dx^3+\text{etc.}}$$
$$-\frac{n\beta dx+2n\gamma xdx+3n\delta x^2dx+\text{etc.}}{\alpha+\beta x+\gamma x^2+\delta x^3+\text{etc.}}$$

sublatisque per multiplicationem fractionibus erit

$$\left\{\begin{array}{l} A\alpha+A\beta x+A\gamma x^2+A\delta x^3+\text{etc.}\\ \quad +B\alpha\ +B\beta\ +B\gamma\ +\text{etc.}\\ \qquad C\alpha\ +C\beta\ +\text{etc.}\\ \qquad\quad +D\alpha\ +\text{etc.}\end{array}\right\}\frac{ds}{dx}$$
$$=\left\{\begin{array}{l} B\alpha+B\beta x+B\gamma x^2+B\delta x^3+\text{etc.}\\ \quad +2C\alpha\ +2C\beta\ +2C\gamma\ +\text{etc.}\\ \qquad 3D\alpha\ +3D\beta\ +\text{etc.}\\ \qquad\quad +4E\alpha\ +\text{etc.}\end{array}\right\}s$$
$$-\left\{\begin{array}{l} A\beta+2A\gamma x+3A\delta x^2+4A\varepsilon x^3+\text{etc.}\\ \quad +B\beta\ +2B\gamma\ +3B\delta\ +\text{etc.}\\ \qquad +C\beta\ +2C\gamma\ +\text{etc.}\\ \qquad\quad +D\beta\ +\text{etc}\end{array}\right\}ns.$$

Cum nunc sit $\frac{ds}{dx}=\mathfrak{B}+2\mathfrak{C}x+3\mathfrak{D}x^2+4\mathfrak{E}x^3+\text{etc.}$, erit factis substitutionibus

$$\left.\begin{array}{r} A\alpha\mathfrak{B}+nA\beta\mathfrak{A}\\ -\ B\alpha\mathfrak{A}\end{array}\right\}=0$$

$$\left.\begin{aligned} &2A\alpha\mathfrak{C}+(n+1)A\beta\mathfrak{B}+2nA\gamma\mathfrak{A}\\ &\quad +\ 0B\alpha\mathfrak{B}+(n-1)B\beta\mathfrak{A}\\ &\quad -\ 2C\alpha\mathfrak{A}\end{aligned}\right\}=0$$

$$\left.\begin{aligned} &3A\alpha\mathfrak{D}+(n+2)A\beta\mathfrak{C}+(2n+1)A\gamma\mathfrak{B}+3nA\delta\mathfrak{A}\\ &\quad +\ B\alpha\mathfrak{C}+nB\beta\mathfrak{B}+(2n-1)B\gamma\mathfrak{A}\\ &\quad -\ C\alpha\mathfrak{B}+(n-2)C\beta\mathfrak{A}\\ &\quad -\ 3D\alpha\mathfrak{A}\end{aligned}\right\}=0$$

$$\left.\begin{aligned} &4A\alpha\mathfrak{E}+(n+3)A\beta\mathfrak{D}+(2n+2)A\gamma\mathfrak{C}+(3n+1)A\delta\mathfrak{B}+4nA\varepsilon\mathfrak{A}\\ &\quad +\ 2B\alpha\mathfrak{D}+(n+1)B\beta\mathfrak{C}+2nB\gamma\mathfrak{B}+(3n-1)B\delta\mathfrak{A}\\ &\quad +\ 0C\alpha\mathfrak{C}+(n-1)C\beta\mathfrak{B}+(2n-2)C\gamma\mathfrak{A}\\ &\quad -\ 2D\alpha\mathfrak{B}+(n-3)D\beta\mathfrak{A}\\ &\quad -\ 4E\alpha\mathfrak{A}\end{aligned}\right\}=0.$$

etc.

Hinc lex, qua istae formulae progrediuntur, facile perspicitur; prima enim cuiusque aequationis linea eandem sequitur legem, quam § 204 habuimus. Tum vero coefficientes secundarum linearum oriuntur, si a coefficientibus superioribus subtrahatur $n+1$, similique modo ex linea secunda formatur linea tertia et sequentes a coefficientibus superioribus continuo subtrahendo $n+1$; ipsae autem litterae quemvis terminum componentes per solam inspectionem facillime formantur.

207. Sin autem quoque numerator fractionis fuerit quaepiam potestas, scilicet

$$s=\frac{(A+Bx+Cx^2+Dx^3+\text{etc.})^m}{(\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4+\text{etc.})^n},$$

fingaturque

$$s=\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}x^2+\mathfrak{D}x^3+\mathfrak{E}x^4+\text{etc.},$$

erit $\mathfrak{A}=\frac{A^m}{\alpha^n}$; reliqui vero coefficientes ex sequentibus formulis determinabuntur

$$\left.\begin{aligned} &A\alpha\mathfrak{B}+nA\beta\mathfrak{A}\\ &\quad -\ mB\alpha\mathfrak{A}\end{aligned}\right\}=0$$

$$\left.\begin{array}{r} 2A\alpha\mathfrak{C}+(n+1)A\beta\mathfrak{B}+2nA\gamma\mathfrak{A} \\ -(m-1)B\alpha\mathfrak{B}+(n-m)B\beta\mathfrak{A} \\ -2mC\alpha\mathfrak{A} \end{array}\right\}=0$$

$$\left.\begin{array}{r} 3A\alpha\mathfrak{D}+(n+2)A\beta\mathfrak{C}+(2n+1)A\gamma\mathfrak{B}+3nA\delta\mathfrak{A} \\ -(m-2)B\alpha\mathfrak{C}+(n-m+1)B\beta\mathfrak{B}+(2n-m)B\gamma\mathfrak{A} \\ -(2m-1)C\alpha\mathfrak{B}+(n-2m)C\beta\mathfrak{A} \\ -3mD\alpha\mathfrak{A} \end{array}\right\}=0$$

$$\left.\begin{array}{r} 4A\alpha\mathfrak{E}+(n+3)A\beta\mathfrak{D}+(2n+2)A\gamma\mathfrak{C}+(3n+1)A\delta\mathfrak{B}+4nA\varepsilon\mathfrak{A} \\ -(m-3)B\alpha\mathfrak{D}+(n-m+1)B\beta\mathfrak{C}+(2n-m+1)B\gamma\mathfrak{B}+(3n-m)B\delta\mathfrak{A} \\ -(2m-2)C\alpha\mathfrak{C}+(n-2m+1)C\beta\mathfrak{B}+(2n-2m)C\gamma\mathfrak{A} \\ -(3m-1)D\alpha\mathfrak{B}+(n-3m)D\beta\mathfrak{A} \\ -4mE\alpha\mathfrak{A} \end{array}\right\}=0.$$

etc.

Lex, qua istae formulae ulterius continuantur, ex inspectione facilius apparet, quam verbis describi queat. Descendendo autem coefficientes diminuuntur differentia $n+m$ *;* horizontaliter autem progrediendo augentur continuo differentia $n-1$.

208. Hoc igitur modo doctrina de seriebus recurrentibus amplificatur, dum istum defectum supplevimus atque legem coefficientium definivimus, si non solum denominator fractionis fuerit potestas quaecunque, sed etiam numerator ex quotlibet terminis constet, ad quam legem detegendam sola inductio non sufficiebat. Praeter plurimos autem usus serierum recurrentium, quos iam exposuimus, maximam quoque afferunt utilitatem ad summas quarumvis serierum proxima inveniendas; cuius specimen iam in capite primo huius sectionis exhibuimus, dum seriem substitutione $x=\frac{y}{1+ny}$ in aliam transmutavimus, quae saepenumero terminorum numero finito constet. Eaque methodus ulterius extendi potuisset, si pro *x* aliae functiones substitutae fuissent. Quoniam vero tum lex progressionis serierum, quae loco potestatum ipsius *x* poni deberent, non satis luculenter constabat, in hunc locum istam amplificationem reservare visum est, cum memorata lex iam penitus esset detecta. Interim tamen re diligentius perpensa comperimus idem negotium sine hac progressionis lege expediri posse in subsidium tantum vocando methodum, qua hic ad hanc ipsam legem investigandam, sumus usi.

209. Sit igitur proposita series quaecunque

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.},$$

quam in aliam transformari oporteat, cuius termini singuli sint fractiones, quarum denominatores secundum potestates formulae huiusmodi

$$\alpha + \beta x + \gamma x^2 + \delta x^3 + \text{etc.}$$

procedant. Quo igitur a simplicioribus incipiamus, ponamus esse

$$s = \frac{\mathfrak{A}}{\alpha+\beta x} + \frac{\mathfrak{B}x}{(\alpha+\beta x)^2} + \frac{\mathfrak{C}x^2}{(\alpha+\beta x)^3} + \frac{\mathfrak{D}x^3}{(\alpha+\beta x)^4} + \text{etc.};$$

aequalitate illius seriei cum hac expressione constituta multiplicetur ubique per $\alpha + \beta x$ fietque

$$\left.\begin{matrix} A\alpha + B\alpha x + C\alpha x^2 + D\alpha x^3 + \text{etc.} \\ + A\beta \quad + B\beta \quad + C\beta \quad + \text{etc.} \end{matrix}\right\} = \mathfrak{A} + \frac{\mathfrak{B}x}{\alpha+\beta x} + \frac{\mathfrak{C}x^2}{(\alpha+\beta x)^2} + \text{etc.}$$

Statuatur $\mathfrak{A} = A\alpha$ fiatque

$$\begin{aligned} A\beta + B\alpha &= A' \\ B\beta + C\alpha &= B' \\ C\beta + D\alpha &= C' \\ D\beta + E\alpha &= D' \\ &\text{etc.}; \end{aligned}$$

erit divisione per x instituta

$$A' + B'x + C'x^2 + D'x^3 + \text{etc.} = \frac{\mathfrak{B}}{\alpha+\beta x} + \frac{\mathfrak{C}x}{(\alpha+\beta x)^2} + \frac{\mathfrak{D}x^2}{(\alpha+\beta x)^3} + \text{etc.}$$

Multiplicetur denuo per $\alpha + \beta x$ positoque

$$\begin{aligned} A'\beta + B'\alpha &= A'' \\ B'\beta + C'\alpha &= B'' \\ C'\beta + D'\alpha &= C'' \\ &\text{etc.} \end{aligned}$$

fiet

$$A' + B''x + C''x^2 + D''x^3 + \text{etc.} = \mathfrak{B} + \frac{\mathfrak{C}x}{\alpha+\beta x} + \frac{\mathfrak{D}x^2}{(\alpha+\beta x)^2} + \text{etc.}$$

Sit igitur $\mathfrak{B} = A'\alpha$; atque operationem ut ante instituendo, si fiat

$$\begin{array}{ll} A''\beta + B''\alpha = A''' & A'''\beta + B'''\alpha = A'''' \\ B''\beta + C''\alpha = B''' & B'''\beta + C'''\alpha = B'''' \\ C''\beta + D''\alpha = C''' & C'''\beta + D'''\alpha = C'''' \\ \text{etc.} & \text{etc.,} \end{array}$$

erit $\mathfrak{C} = A''\alpha$, $\mathfrak{D} = A'''\alpha$, $\mathfrak{E} = A''''\alpha$ etc.; unde summa seriei propositae hoc modo exprimetur, ut sit

$$s = \frac{A\alpha}{\alpha+\beta x} + \frac{A'\alpha x}{(\alpha+\beta x)^2} + \frac{A''\alpha x^2}{(\alpha+\beta x)^3} + \frac{A'''\alpha x^3}{(\alpha+\beta x)^4} + \text{etc.}$$

Quae eadem series orta fuisset ex substitutione

$$\frac{x}{\alpha+\beta z} = y \quad \text{seu} \quad x = \frac{ay}{1-\beta y}.$$

210. Haec transformatio optimo cum successu adhibetur, si series proposita $A + Bx + Cx^2 + \text{etc}$. ita fuerit comparata, ut tandem confundatur cum serie recurrente seu potius geometrica ex fractione $\frac{P}{\alpha+\beta x}$ orta. Tum enim valores A', B', C', D' etc. tandem evanescent hincque multo magis litterae A'', A''', A'''' etc. constituent seriem maxime convergentem.

Poterimus autem simili modo denominatores trinomiales et polynomiales quoscunque adhibere, qui usum habebunt eximium, si series proposita tandem cum recurrente confundatur. Proposita ergo serie

$$s = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

statuatur

$$s = \frac{\mathfrak{A}+\mathfrak{B}x}{\alpha+\beta x+\gamma x^2} + \frac{\mathfrak{A}'x^2+\mathfrak{B}'x^3}{(\alpha+\beta x+\gamma x^2)^2} + \frac{\mathfrak{A}''x^4+\mathfrak{B}''x^5}{(\alpha+\beta x+\gamma x^2)^3} + \frac{\mathfrak{A}'''x^6+\mathfrak{B}'''x^7}{(\alpha+\beta x+\gamma x^2)^4} + \text{etc.}$$

Multiplicetur ubique per $\alpha + \beta x + \gamma x^2$ positoque

$$\begin{array}{l} A\gamma + B\beta + C\alpha = A' \\ B\gamma + C\beta + D\alpha = B' \\ C\gamma + D\beta + E\alpha = C' \\ \text{etc.} \end{array} \quad \text{et} \quad \mathfrak{A} = \alpha A, \quad \mathfrak{B} = A\beta + B\alpha$$

orietur aequatio priori similis divisione per xx instituta

$$A'+B'x+C'x^2+D'x^3+E'x^4+\text{etc.}$$
$$=\frac{\mathfrak{A}'+\mathfrak{B}'x}{\alpha+\beta x+\gamma xx}+\frac{\mathfrak{A}''x^2+\mathfrak{B}''x^3}{(\alpha+\beta x+\gamma xx)^2}+\frac{\mathfrak{A}'''x^4+\mathfrak{B}'''x^5}{(\alpha+\beta x+\gamma xx)^3}+\text{etc.}$$

Si igitur ut ante operatio instituatur faciendo

$$\begin{aligned} A'\gamma+B'\beta+C'\alpha&=A'' \\ B'\gamma+C'\beta+D'\alpha&=B'' \\ C'\gamma+D'\beta+E'\alpha&=C'' \\ &\text{etc.}\end{aligned}\qquad \text{et}\qquad \mathfrak{A}'=\alpha A',\quad \mathfrak{B}'=A'\beta+B'\alpha$$

porroque

$$\begin{aligned} A''\gamma+B''\beta+C''\alpha&=A''' \\ B''\gamma+C''\beta+D''\alpha&=B''' \\ C''\gamma+D''\beta+E''\alpha&=C''' \\ &\text{etc.}\end{aligned}\qquad \text{et}\qquad \mathfrak{A}''=\alpha A'',\quad \mathfrak{B}''=A''\beta+B''\alpha$$

sicque ulterius valores similes investigando, erit

$$s=\frac{A\alpha+(A\beta+\beta A)x}{\alpha+\beta x+\gamma xx}+\frac{(A'\alpha+(A'\beta+B'\alpha)x)x^2}{(\alpha+\beta x+\gamma xx)^2}+\frac{(A''\alpha+(A''\beta+B''\alpha)x)x^4}{(\alpha+\beta x+\gamma xx)^3}+\text{etc.}$$

211. Si ponatur $x=1$, qua positione amplitudini nihil decedit, cum α, β, γ pro lubitu accipi possint, fueritque

$$s=A+B+C+D+E+F+G+\text{etc.},$$

cum putentur successive sequentes valores

$$\begin{aligned} A\gamma+B\beta+C\alpha&=A' \\ B\gamma+C\beta+D\alpha&=B' \\ C\gamma+D\beta+E\alpha&=C' \\ &\text{etc.}\end{aligned}\qquad \begin{aligned} A'\gamma+B'\beta+C'\alpha&=A'' \\ B'\gamma+C'\beta+D'\alpha&=B'' \\ C'\gamma+D'\beta+E'\alpha&=C'' \\ &\text{etc.}\end{aligned}\qquad \text{sicque porro,}$$

insuper vero brevitatis ergo ponatur

$$\alpha+\beta+\gamma=m,$$

obtinebitur summa seriei propositae hoc modo expressa

$$s=\begin{cases}(\alpha+\beta)\left(\frac{A}{m}+\frac{A'}{m^2}+\frac{A''}{m^3}+\frac{A'''}{m^4}+\text{etc.}\right)\\ \quad+\ \alpha\left(\frac{B}{m}+\frac{B'}{m^2}+\frac{B''}{m^3}+\frac{B'''}{m^4}+\text{etc.}\right)\end{cases}$$

212. Eodem modo denominatores ex pluribus terminis constantes accipi possunt, et quoniam operatio ex praecedentibus facile perspicitur, hic tantum casum pro quadrinomio evolvamus. Sit ergo

$$s=A+B+C+D+E+F+G+\text{etc}.$$

Quaerantur valores sequentes

$$\begin{aligned}A\delta+B\gamma+C\beta+D\alpha&=A'\\ B\delta+C\gamma+D\beta+E\alpha&=B'\\ C\delta+D\gamma+E\beta+F\alpha&=C'\\ &\text{etc.}\end{aligned}$$

$$\begin{aligned}A'\delta+B'\gamma+C'\beta+D'\alpha&=A''\\ B'\delta+C'\gamma+D'\beta+E'\alpha&=B''\\ C'\delta+D'\gamma+E'\beta+F'\alpha&=C''\\ &\text{etc.}\end{aligned}$$

$$\begin{aligned}A''\delta+B''\gamma+C''\beta+D''\alpha&=A'''\\ B''\delta+C''\gamma+D''\beta+E''\alpha&=B'''\\ C''\delta+D''\gamma+E''\beta+F''\alpha&=C'''\\ &\text{etc.}\end{aligned}$$

Tum vero sit $\alpha+\beta+\gamma=m$ eritque

$$s=\begin{cases}(\alpha+\beta+\gamma)\left(\frac{A}{m}+\frac{A'}{m^2}+\frac{A''}{m^3}+\frac{A'''}{m^4}+\text{etc.}\right)\\ \quad+(\alpha+\beta)\left(\frac{B}{m}+\frac{B'}{m^2}+\frac{B''}{m^3}+\frac{B'''}{m^4}+\text{etc.}\right)\\ \quad+\qquad\alpha\left(\frac{C}{m}+\frac{C'}{m^2}+\frac{C''}{m^3}+\frac{C'''}{m^4}+\text{etc.}\right)\end{cases}$$

unde simul progressio, si adhuc plures partes denominatori *m* tribuantur, clarissime perspicitur.

213. Neque vero absolute opus est, ut denominatores fractionum, ad quas summam seriei reducimus, sint potestates eiusdem formulae

$$\alpha+\beta x+\gamma x^2+\text{etc.},$$

sed haec ipsa in singulis terminis variari potest. Quo hoc clarius pateat, sumamus primo tantum duos terminos fingaturque series

$$s=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\text{etc.}$$

in hanc seriem fractionum converti

$$s=\frac{\mathfrak{A}}{\alpha+\beta x}+\frac{\mathfrak{A}'x}{(\alpha+\beta x)(\alpha'+\beta' x)}+\frac{\mathfrak{A}''x^2}{(\alpha+\beta x)(\alpha'+\beta' x)(\alpha''+\beta'' x)}+\text{etc.}$$

Multiplicetur primum utrinque per $\alpha+\beta x$ ponaturque

$$\begin{aligned}A\beta+B\alpha&=A'\\ B\beta+C\alpha&=B'\quad \text{et}\quad \mathfrak{A}=A\alpha\\ C\beta+D\alpha&=C'\\ &\text{etc.}\end{aligned}$$

fietque per x diviso

$$A'+B'x+C'x^2+D'x^3+\text{etc.}=\frac{\mathfrak{A}'}{\alpha'+\beta' x}+\frac{\mathfrak{A}''x}{(\alpha'+\beta' x)(\alpha''+\beta'' x)}+\text{etc.}$$

Deinde simili modo multiplicando per $\alpha'+\beta' x$ tumque per $\alpha''+\beta'' x$ et ita porro, si statuatur

$$\begin{array}{lll}A'\beta'+B'\alpha'=A'' & A''\beta''+B''\alpha''=A''' & A'''\beta'''+B'''\alpha'''=A''''\\ B'\beta'+C'\alpha'=B'' & B''\beta''+C''\alpha''=B''' & B'''\beta'''+C'''\alpha'''=B''''\quad\text{etc.}\\ C'\beta'+D'\alpha'=C'' & C''\beta''+D''\alpha''=C''' & C'''\beta'''+D'''\alpha'''=C''''\\ \text{etc.} & \text{etc.} & \text{etc.}\end{array}$$

fiet

$$\mathfrak{A}'=A'\alpha',\quad \mathfrak{A}''=A''\alpha'',\quad \mathfrak{A}'''=A'''\alpha''',\quad \text{etc.}$$

atque hinc series proposita convertetur in hanc

$$s=\frac{A\alpha}{\alpha+\beta x}+\frac{A'\alpha' x}{(\alpha+\beta x)(\alpha'+\beta' x)}+\frac{A''\alpha''x^2}{(\alpha+\beta x)(\alpha'+\beta' x)(\alpha''+\beta'' x)}+\text{etc.}$$

ubi valores $\alpha, \beta, \alpha', \beta', \alpha'', \beta'', \alpha''', \beta'''$ etc. sunt arbitrarii, quovis autem casu ita accipi possunt, ut ista nova series maxime convergat.

214. Applicemus hoc quoque ad factores trinomiales sitque proposita serie quacunque $s = A + B + C + D + E + F + G + \text{etc.}$

$$\begin{array}{ll} A\gamma + B\beta + C\alpha = A' & A'\gamma' + B'\beta' + C'\alpha' = A'' \\ B\gamma + C\beta + D\alpha = B' & B'\gamma' + C'\beta' + D'\alpha' = B'' \\ C\gamma + D\beta + E\alpha = C' & C'\gamma' + D'\beta' + E'\alpha' = C'' \\ \text{etc.} & \text{etc.} \end{array}$$

$$\begin{array}{ll} A''\gamma'' + B''\beta'' + C''\alpha'' = A''' & A'''\gamma''' + B'''\beta''' + C'''\alpha''' = A'''' \\ B''\gamma'' + C''\beta'' + D''\alpha'' = B''' & B'''\gamma''' + C'''\beta''' + D'''\alpha''' = B'''' \\ C''\gamma'' + D''\beta'' + E''\alpha'' = C' & C'''\gamma' + D'''\beta''' + E'''\alpha' = C'''' \\ \text{etc.} & \text{etc.} \end{array}$$

Deinde statuatur brevitatis gratia

$$\begin{aligned} \alpha + \beta + \gamma &= m \\ \alpha' + \beta' + \gamma' &= m' \\ \alpha'' + \beta'' + \gamma'' &= m'' \\ \alpha''' + \beta''' + \gamma''' &= m''' \\ &\text{etc.} \end{aligned}$$

eritque seriei propositae summa

$$\begin{aligned} s = {} & \frac{\alpha(A+B)}{m} + \frac{\alpha'(A'+B')}{mm'} + \frac{\alpha''(A''+B'')}{mm'm''} + \frac{\alpha'''(A'''+B''')}{mm'm''m'''} + \text{etc.} \\ & + \frac{\beta A}{m} + \frac{\beta' A'}{mm'} + \frac{\beta'' A''}{mm'm''} + \frac{\beta''' A'''}{mm'm''m'''} + \text{etc.} \end{aligned}$$

215. Quoniam haec tam late patent, ut usus minus clare percipi possit, restringamus transformationem § 213 traditam sitque $x = -1$, ut habeatur haec series

$$s = A - B + C - D + E - F + G - \text{etc.},$$

statuaturque

$$\begin{array}{llll}
B-A=A' & B'-2A'=A'' & B''-3A''=A''' & B'''-4A'''=A'''' \\
C-B=B' & C'-2B'=B'' & C''-3B''=B''' & C'''-4B'''=B'''' \\
D-C=C' & D'-2C'=C'' & D''-3C''=C''' & D'''-4C'''=C'''' \\
E-D=D' & E'-2D'=D'' & E''-3D''=D''' & E'''-4D'''=D'''' \\
\text{etc.} & \text{etc.} & \text{etc.} & \text{etc.}
\end{array}$$

Quibus valoribus inventis erit summa seriei propositae aequalis sequenti seriei

$$s=\frac{A}{2}-\frac{A'}{2\cdot3}+\frac{A''}{2\cdot3\cdot4}-\frac{A'''}{2\cdot3\cdot4\cdot5}+\frac{A''''}{2\cdot3\cdot4\cdot5\cdot6}-\text{etc.}$$

Simili igitur modo series quaecunque proposita in innumerabiles alias sibi aequales transmutari potest, inter quas sine dubio series maxime convergentes reperientur, quarum ope summa proposita vero proxime indagari queat.

216. Revertamur autem ad inventionem serierum, quarum progressionis legem calculus differentialis declarat. Cum igitur hoc in quantitatibus algebraicis iam sit praestitum, progrediamur ad transcendentes quaeraturque series huic logarithmo aequalis

$$s=l\left(1+\alpha x+\beta x^2+\gamma x^3+\delta x^4+\varepsilon x^5+\text{etc.}\right);$$

fingatur quaesito satisfacere haec series

$$s=\mathfrak{A}x+\mathfrak{B}x^2+\mathfrak{C}x^3+\mathfrak{D}x^4+\mathfrak{E}x^5+\mathfrak{F}x^6+\text{etc.}$$

Cum igitur ex illius aequationis differentiatione sequatur

$$\frac{ds}{dx}=\frac{\alpha+2\beta x+3\gamma x^2+4\delta x^3+5\varepsilon x^4+\text{etc.}}{1+\alpha x+\beta x^2+\gamma x^3+\delta x^4+\varepsilon x^5+\text{etc.}},$$

erit

$$\left(1+\alpha x+\beta x^2+\gamma x^3+\delta x^4+\varepsilon x^5+\text{etc.}\right)\frac{ds}{dx}=\alpha+2\beta x+3\gamma x^2+4\delta x^3+5\varepsilon x^4+\text{etc.}$$

Quia vero ex ficta aequatione est

$$\frac{ds}{dx}=\mathfrak{A}+2\mathfrak{B}x+3\mathfrak{C}x^2+4\mathfrak{D}x^3+5\mathfrak{E}x^4+\text{etc.},$$

facta hac substitutione oritur haec aequatio

$$\begin{array}{llllll}
\mathfrak{A}+2\mathfrak{B}x+ & 3\mathfrak{C}x^2+ & 4\mathfrak{D}x^3+ & 5\mathfrak{E}x^4 & +\text{etc.} \\
\quad +\mathfrak{A}\alpha & +2\mathfrak{B}\alpha & +3\mathfrak{C}\alpha & +4\mathfrak{D}\alpha & +\text{etc.} \\
 & +\mathfrak{A}\beta & +2\mathfrak{B}\beta & +3\mathfrak{C}\beta & +\text{etc.} \\
 & & +\mathfrak{A}\gamma & +2\mathfrak{B}\gamma & +\text{etc.} \\
 & & & +\mathfrak{A}\delta & +\text{etc.} \\
=\alpha+2\beta x+ & 3\gamma x^2 & +4\delta x^3 & +5\varepsilon x^4 & +\text{etc.}
\end{array}$$

Ex qua sequentes determinationes obtinentur

$$\begin{aligned}
&\mathfrak{A}=\alpha \\
&\mathfrak{B}=-\tfrac{1}{2}\mathfrak{A}\alpha+\beta \\
&\mathfrak{C}=-\tfrac{2}{3}\mathfrak{B}\alpha-\tfrac{1}{3}\mathfrak{A}\beta+\gamma \\
&\mathfrak{D}=-\tfrac{3}{4}\mathfrak{C}\alpha-\tfrac{2}{4}\mathfrak{B}\beta-\tfrac{1}{4}\mathfrak{A}\gamma+\delta \\
&\mathfrak{E}=-\tfrac{4}{5}\mathfrak{D}\alpha-\tfrac{3}{5}\mathfrak{C}\beta-\tfrac{2}{5}\mathfrak{B}\gamma-\tfrac{1}{5}\mathfrak{A}\delta+\varepsilon \\
&\qquad\qquad \text{etc.}
\end{aligned}$$

217. Proposita nunc sit quantitas exponentialis

$$s=e^{\alpha x+\beta x^2+\gamma x^3+\delta x^4+\varepsilon x^5+\text{etc.}}$$

in qua *e* denotet numerum, cuius logarithmus hyperbolicus est = 1, atque fingatur series quaesita

$$s=1+\mathfrak{A}x+\mathfrak{B}x^2+\mathfrak{C}x^3+\mathfrak{D}x^4+\mathfrak{E}x^5+\text{etc.}$$

Iam enim ex casu $x=0$ patet primum terminum esse debere unitatem. Cum igitur sumendis logarithmis sit

$$ls=\alpha x+\beta x^2+\gamma x^3+\delta x^4+\varepsilon x^5+\zeta x^6+\text{etc.}$$

erit differentialibus sumtis

$$\tfrac{ds}{dx}=s\left(\alpha+2\beta x+3\gamma x^2+4\delta x^3+5\varepsilon x^4+\text{etc.}\right)$$

At vero ex aequatione ficta erit

$$\frac{ds}{dx} = \mathfrak{A} + 2\mathfrak{B}x + 3\mathfrak{C}x^2 + 4\mathfrak{D}x^3 + 5\mathfrak{E}x^4 + \text{etc.}$$

$$\begin{aligned} = \alpha &+ \mathfrak{A}\alpha x + \mathfrak{B}\alpha x^2 + \mathfrak{C}\alpha x^3 + \mathfrak{D}\alpha x^4 + \text{etc.} \\ &+ 2\beta \quad + 2\mathfrak{A}\beta \quad + 2\mathfrak{B}\beta \quad + 2\mathfrak{C}\beta \quad + \text{etc.} \\ &\qquad\quad + 3\gamma \quad + 3\mathfrak{A}\gamma \quad + 3\mathfrak{B}\gamma \quad + \text{etc.} \\ &\qquad\qquad\qquad + 4\delta \quad + 4\mathfrak{A}\delta \quad + \text{etc.} \\ &\qquad\qquad\qquad\qquad + 5\varepsilon \quad + \text{etc.}, \end{aligned}$$

ex quibus sequentes prodeunt litterarum $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. determinationes

$$\begin{aligned} \mathfrak{A} &= \alpha \\ \mathfrak{B} &= \beta + \tfrac{1}{2}\mathfrak{A}\alpha \\ \mathfrak{C} &= \gamma + \tfrac{2}{3}\mathfrak{A}\beta + \tfrac{1}{3}\mathfrak{B}\alpha \\ \mathfrak{D} &= \delta + \tfrac{3}{4}\mathfrak{A}\gamma + \tfrac{2}{4}\mathfrak{B}\beta + \tfrac{1}{4}\mathfrak{C}\alpha \\ \mathfrak{E} &= \varepsilon + \tfrac{4}{5}\mathfrak{A}\delta + \tfrac{3}{5}\mathfrak{B}\gamma + \tfrac{2}{5}\mathfrak{C}\beta + \tfrac{1}{5}\mathfrak{D}\alpha \\ &\text{etc.} \end{aligned}$$

218. Si quoque arcus, cuius sinus vel cosinus quaeritur, exprimatur binomio vel polynomio vel etiam serie infinita, hoc modo quoque eius sinus et cosinus per seriem infinitam exprimi possunt. At vero quo hoc commodissime fiat, non sufficit ad differentialia prima processisse, sed opus est, ut differentialia secundi gradus in subsidium vocemus. Sit igitur

$$s = \sin\left(\alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \text{etc.}\right)$$

fingaturque series, quae quaeritur,

$$s = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \text{etc.}$$

Primum enim terminum constat evanescere; quia vero ad differentialia secunda descendendum est, coefficientem $\mathfrak{A}$ quoque aliunde definiri oportet, quod fiet, si x ponamus infinite parvum. Tum enim ob arcum $= \alpha x$ sinus ipsi fiet aequalis eritque ergo $\mathfrak{A} = \alpha$. Ponamus nunc brevitatis gratia

$$z = \alpha x + \beta x^2 + \gamma x^3 + \text{etc.},$$

ut sit $s = \sin z$; erit differentiando $ds = dz\cos z$ denuoque differentiando $dds = ddz\cos z - dz^2 \sin z$. Quia igitur est $\sin z = s$ et $\cos z = \frac{ds}{dz}$, erit

$$dds = \frac{dsddz}{dz} - sdz^2 \quad \text{or} \quad dzdds + sdz^3 = dsddz.$$

219. Ponamus arcum z tantum binomio exprimi esseque

$$z = \alpha x + \beta x^2;$$

erit

$$dz = (\alpha + 2\beta x)dx$$

et posito dx constante

$$ddz = 2\beta dx^2$$

atque

$$dz^3 = \left(\alpha^3 + 6\alpha^2\beta x + 12\alpha\beta^2 x^2 + 8\beta^3 x^3\right)dx^3$$

Deinde ob $s = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \text{etc.}$ erit

$$\frac{ds}{dx} = \mathfrak{A} + 2\mathfrak{B}x + 3\mathfrak{C}x^2 + 4\mathfrak{D}x^3 + \text{etc.}$$

et

$$\frac{dds}{dx^2} = 2\mathfrak{B} + 6\mathfrak{C}x + 12\mathfrak{D}x^2 + \text{etc.}$$

Quibus valoribus in aequatione differentio-differentiali substitutis fiet

$$\begin{array}{llllll}
\frac{dzdds}{dx^3} = 1\cdot 2\mathfrak{B}\alpha & + \; 2\cdot 3\mathfrak{C}\alpha x & + \; 3\cdot 4\mathfrak{D}\alpha x^2 & + \; 4\cdot 5\mathfrak{E}\alpha x^3 & + \; 5\cdot 6\mathfrak{F}\alpha x^4 & + \text{etc.} \\
 & + 2\cdot 1\cdot 2\mathfrak{B}\beta & + 2\cdot 2\cdot 3\mathfrak{C}\beta & + 2\cdot 3\cdot 4\mathfrak{D}\beta & + 2\cdot 4\cdot 5\mathfrak{E}\beta & + \text{etc.} \\
\frac{sdz^3}{dx^3} = & + \; \mathfrak{A}\alpha^3 & + \; \mathfrak{B}\alpha^3 & + \; \mathfrak{C}\alpha^3 & + \; \mathfrak{C}\alpha^3 & + \text{etc.} \\
 & & + \; 6\mathfrak{A}\alpha^2\beta & + \; 6\mathfrak{B}\alpha^2\beta & + \; 6\mathfrak{C}\alpha^2\beta & + \text{etc.} \\
 & & & + \; 12\mathfrak{A}\alpha\beta^2 & + 12\mathfrak{B}\alpha\beta^2 & + \text{etc.} \\
 & & & & + \; 8\mathfrak{A}\beta^3 & + \text{etc.} \\
\frac{dsddz}{dx^3} = 2\mathfrak{A}\beta & + \; 4\mathfrak{B}\beta & + \; 6\mathfrak{C}\beta & + \; 8\mathfrak{D}\beta & + \; 10\mathfrak{E}\beta & + \text{etc}
\end{array}$$

Unde coefficientes sequenti modo definientur:

$$\mathfrak{B} = \frac{2\mathfrak{A}\beta}{2\alpha}$$
$$\mathfrak{C} = 0 - \frac{\mathfrak{A}\alpha^2}{2\cdot 3}$$
$$\mathfrak{D} = -\frac{2\mathfrak{C}\beta}{4\alpha} - \frac{6\mathfrak{A}\alpha\beta}{3\cdot 4} - \frac{\mathfrak{B}\alpha^2}{3\cdot 4}$$
$$\mathfrak{E} = -\frac{4\mathfrak{D}\beta}{5\alpha} - \frac{12\mathfrak{A}\beta^2}{4\cdot 5} - \frac{6\mathfrak{B}\alpha\beta}{4\cdot 5} - \frac{\mathfrak{C}\alpha^2}{4\cdot 5}$$
$$\mathfrak{F} = -\frac{6\mathfrak{E}\beta}{6\alpha} - \frac{8\mathfrak{A}\beta^3}{5\cdot 6\alpha} - \frac{12\mathfrak{B}\beta\beta}{5\cdot 6} - \frac{6\mathfrak{C}\alpha\beta}{5\cdot 6} - \frac{\mathfrak{D}\alpha^2}{5\cdot 6}$$
$$\mathfrak{G} = -\frac{8\mathfrak{F}\beta}{7\alpha} - \frac{8\mathfrak{B}\beta^3}{6\cdot 7\alpha} - \frac{12\mathfrak{C}\beta\beta}{6\cdot 7} - \frac{6\mathfrak{D}\alpha\beta}{6\cdot 7} - \frac{\mathfrak{E}\alpha^2}{6\cdot 7}$$
etc.

Quibus valoribus inventis erit

$$\sin\left(\alpha x + \beta x^2\right) = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \mathfrak{E}x^5 + \text{etc.}$$

existente $\mathfrak{A} = \alpha$.

220. Pari modo cosinus cuiusque anguli in seriem convertitur; quia autem arcus rarissime per polynomium exprimitur, ostendamus usum differentio-differentialium in invenienda serie pro cosinu arcus *x*. Sit ergo
$s = \cos x$ et fingatur

$$s = 1 - \mathfrak{A}x^2 + \mathfrak{B}x^4 - \mathfrak{C}x^6 + \mathfrak{D}x^8 - \text{etc.}$$

Quia est $ds = -dx\sin x$ et $dds = -dx^2\cos x = -sdx^2$, erit

$$dds + sdx^2 = 0\,;$$

substitutione ergo facta fiet

$$\frac{dds}{dx^2} = -1\cdot 2\mathfrak{A} + 3\cdot 4\mathfrak{B}x^2 - 5\cdot 6\mathfrak{C}x^4 + 7\cdot 8\mathfrak{D}x^6 - \text{etc.}$$
$$s = \quad 1 - \quad \mathfrak{A}x^2 + \quad \mathfrak{B}x^4 - \quad \mathfrak{C}x^6 + \text{etc.}$$

at ex coaequatione terminorum sequitur

$$\mathfrak{A} = \frac{1}{1\cdot 2}$$
$$\mathfrak{B} = \frac{\mathfrak{A}}{3\cdot 4} = \frac{1}{1\cdot 2\cdot 3\cdot 4}$$
$$\mathfrak{C} = \frac{\mathfrak{B}}{5\cdot 6} = \frac{1}{1\cdot 2\cdot 3\cdots 6}$$
$$\mathfrak{D} = \frac{\mathfrak{C}}{7\cdot 8} = \frac{1}{1\cdot 2\cdot 3\cdots 8}$$
etc.

Patet ergo, quod iam supra fusius demonstravimus, esse

$$\cos x = 1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \frac{x^6}{1\cdot 2\cdot 3\cdots 6} + \frac{x^8}{1\cdot 2\cdot 3\cdots 8} - \text{etc.}$$

prior vero series pro sinu posito $\beta = 0$ et $\alpha = 1$ dabit

$$\sin x = \frac{x}{1} - \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{x^7}{1\cdot 2\cdot 3\cdots 7} + \frac{x^9}{1\cdot 2\cdot 3\cdots 9} - \text{etc.}$$

221. Ex his seriebus pro sinu et cosinu notissimis deducuntur series pro tangente, cotangente, secanta et cosecante cuiusvis anguli. Tangens enim prodit, si sinus per cosinum, cotangens, si cosinus per sinum, secans, si radius 1 per cosinum, et cosecans, si radius per sinum dividatur. Series autem ex his divisionibus ortae maxime videntur irregulares; verum excepta serie secantem exhibente reliquae per numeros BERNOULLIANOS supra definitos $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. ad facilem progressionis legem reduci possunt. Quoniam enim supra (§ 127) invenimus esse

$$\frac{\mathfrak{A}u^2}{1\cdot 2} + \frac{\mathfrak{B}u^4}{1\cdot 2\cdot 3\cdot 4} + \frac{\mathfrak{C}u^6}{1\cdot 2\cdot 3\cdots 6} + \frac{\mathfrak{D}u^8}{1\cdot 2\cdot 3\cdots 8} + \text{etc.} = 1 - \frac{u}{2}\cot\frac{1}{2}u,$$

erit posito $\frac{1}{2}u = x$

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1\cdot 2} - \frac{2^4\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} - \frac{2^6\mathfrak{C}x^5}{1\cdot 2\cdot 3\cdots 6} - \frac{2^8\mathfrak{D}x^7}{1\cdot 2\cdot 3\cdots 8} - \text{etc.},$$

atque si ponatur $\frac{1}{2}x$ pro x, erit

$$\cot\frac{1}{2}x = \frac{2}{x} - \frac{2\mathfrak{A}x}{1\cdot 2} - \frac{2\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} - \frac{2\mathfrak{C}x^5}{1\cdot 2\cdot 3\cdots 6} - \frac{2\mathfrak{D}x^7}{1\cdot 2\cdot 3\cdots 8} - \text{etc.}$$

222. Hinc autem tangens cuiusvis arcus sequenti modo per seriem exprimetur.
Cum sit

$$\text{tang}\, 2x = \frac{2\,\text{tang}\, x}{1 - \text{tang}^2 x}$$

erit

$$\cot 2x = \frac{1}{2\operatorname{tang} x} - \frac{\operatorname{tang} x}{2} = \frac{1}{2}\cot x - \frac{1}{2}\operatorname{tang} x$$

ideoque

$$\operatorname{tang} x = \cot x - 2\cot 2x .$$

Cum igitur sit

$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1\cdot 2} - \frac{2^4\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} - \frac{2^6\mathfrak{C}x^5}{1\cdot 2\cdot 3\cdots 6} - \frac{2^8\mathfrak{D}x^7}{1\cdot 2\cdot 3\cdots 8} - \text{etc.},$$
$$2\cot 2x = \frac{1}{x} - \frac{2^4\mathfrak{A}x}{1\cdot 2} - \frac{2^8\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} - \frac{2^{12}\mathfrak{C}x^5}{1\cdot 2\cdot 3\cdots 6} - \frac{2^{16}\mathfrak{D}x^7}{1\cdot 2\cdot 3\cdots 8} - \text{etc.},$$

erit hanc seriem ab illa subtrahendo

$$\operatorname{tang} x = \frac{2^2\left(2^2-1\right)\mathfrak{A}x}{1\cdot 2} + \frac{2^4\left(2^4-1\right)\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} + \frac{2^6\left(2^6-1\right)\mathfrak{C}x^5}{1\cdot 2\cdot 3\cdots 6} + \frac{2^8\left(2^8-1\right)\mathfrak{D}x^7}{1\cdot 2\cdot 3\cdots 8} + \text{etc.}$$

Si ergo hic introducantur numeri A, B, C, D etc. §182 inventi, erit

$$\operatorname{tang} x = \frac{2Ax}{1\cdot 2} + \frac{2^3Bx^3}{1\cdot 2\cdot 3\cdot 4} + \frac{2^5Cx^5}{1\cdot 2\cdot 3\cdots 6} + \frac{2^7Dx^7}{1\cdot 2\cdot 3\cdots 8} + \text{etc.}$$

223. Cosecans autem sequenti modo invenietur. Quia est

$$\cot x = \operatorname{tang} x + 2\cot 2x = \frac{1}{\cot x} + 2\cot 2x,$$

erit

$$\cot^2 x = 2\cot x \cdot \cot 2x + 1$$

et radice extracta

$$\cot x = \cot 2x + \operatorname{cosec} 2x ,$$

unde fit

$$\operatorname{cosec} 2x = \cot x - \cot 2x$$

et x pro $2x$ posito

$$\operatorname{cosec} x = \cot \tfrac{1}{2} x - \cot x$$

Quare cum cotangentes habeamus, scilicet

$$\cot \tfrac{1}{2} x = \frac{2}{x} - \frac{2\mathfrak{A}x}{1\cdot 2} - \frac{2\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} - \frac{2\mathfrak{C}x^5}{1\cdot 2\cdot 3\cdots 6} - \text{etc.}$$
$$\cot x = \frac{1}{x} - \frac{2^2\mathfrak{A}x}{1\cdot 2} - \frac{2^4\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} - \frac{2^6\mathfrak{C}x^5}{1\cdot 2\cdot 3\cdots 6} - \text{etc.}$$

erit hac serie ab illa subtracta

$$\operatorname{cosec} x = \frac{1}{x} + \frac{2(2-1)\mathfrak{A}x}{1\cdot 2} + \frac{2\left(2^3-1\right)\mathfrak{B}x^3}{1\cdot 2\cdot 3\cdot 4} + \frac{2\left(2^5-1\right)\mathfrak{C}x^5}{1\cdot 2\cdot 3\cdots 6} + \text{etc.}$$

224. Per hos autem numeros BERNOULLIANOS secans exprimi non potest, sed requirit alios numeros, qui in summas potestatum reciprocarum imparium ingrediuntur. Si enim ponatur

$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\text{etc.}=\alpha\cdot\tfrac{\pi}{2^2}$$
$$1-\tfrac{1}{3^3}+\tfrac{1}{5^3}-\tfrac{1}{7^3}+\tfrac{1}{9^3}-\text{etc.}=\tfrac{\beta}{1\cdot2}\cdot\tfrac{\pi^3}{2^4}$$
$$1-\tfrac{1}{3^5}+\tfrac{1}{5^5}-\tfrac{1}{7^5}+\tfrac{1}{9^5}-\text{etc.}=\tfrac{\gamma}{1\cdot2\cdot3\cdot4}\cdot\tfrac{\pi^5}{2^6}$$
$$1-\tfrac{1}{3^7}+\tfrac{1}{5^7}-\tfrac{1}{7^7}+\tfrac{1}{9^7}-\text{etc.}=\tfrac{\delta}{1\cdot2\cdots6}\cdot\tfrac{\pi^7}{2^8}$$
$$1-\tfrac{1}{3^9}+\tfrac{1}{5^9}-\tfrac{1}{7^9}+\tfrac{1}{9^9}-\text{etc.}=\tfrac{\varepsilon}{1\cdot2\cdots8}\cdot\tfrac{\pi^9}{2^{10}}$$
$$1-\tfrac{1}{3^{11}}+\tfrac{1}{5^{11}}-\tfrac{1}{7^{11}}+\tfrac{1}{9^{11}}-\text{etc.}=\tfrac{\varepsilon}{1\cdot2\cdots10}\cdot\tfrac{\pi^{11}}{2^{12}}$$
etc.,

erit

$$\alpha=1$$
$$\beta=1$$
$$\gamma=5$$
$$\delta=61$$
$$\varepsilon=1385$$
$$\zeta=50521$$
$$\eta=2702765$$
$$\theta=199360981$$
$$\iota=19391512145$$
$$\chi=2404879661671$$
etc.

ex hisque valoribus obtinebitur

$$\sec x=\alpha+\tfrac{\beta}{1\cdot2}xx+\tfrac{\gamma}{1\cdot2\cdot3\cdot4}x^4+\tfrac{\delta}{1\cdot2\cdot3\cdots6}x^6+\tfrac{\varepsilon}{1\cdot2\cdot3\cdots8}x^8+\text{etc.}$$

225. Ut autem nexum huius seriei cum numeris $\alpha, \beta, \gamma, \delta$ etc. ostendamus, consideremus seriem supra [§ 33] tractatam

$$\frac{\pi}{n\sin\frac{m}{n}\pi}=\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\frac{1}{3n-m}-\text{etc.}$$

Ponatur $m=\frac{1}{2}n-k$ eritque

$$\frac{\pi}{2n\sin\frac{k}{n}\pi} = \frac{1}{n-2k} + \frac{1}{n+2k} - \frac{1}{3n-2k} - \frac{1}{3n+2k} + \frac{1}{5n-2k} + \text{etc.}$$

Sit $\frac{k\pi}{n} = x$ seu $k\pi = nx$ erit

$$\frac{\pi}{2n}\sec x = \frac{\pi}{n\pi-2nx} + \frac{\pi}{n\pi+2nx} - \frac{\pi}{3n\pi-2nx} - \frac{\pi}{3n\pi+2nx} + \frac{\pi}{5n\pi-2nx} + \text{etc.}$$

seu

$$\sec x = \frac{2}{\pi-2x} + \frac{2}{\pi+2x} - \frac{2}{3\pi-2x} - \frac{2}{3\pi+2x} + \frac{2}{5\pi-2x} + \text{etc.}$$

seu

$$\sec x = \frac{4\pi}{\pi^2-4x^2} - \frac{4\cdot 3\pi}{9\pi^2-4xx} + \frac{4\cdot 5\pi}{25\pi^2-4xx} - \frac{4\cdot 7\pi}{49\pi^2-4xx} + \text{etc.}$$

Si nunc singuli termini in series convertantur, fiet

$$\begin{aligned}\sec x = &\tfrac{4}{\pi}\left(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\text{etc.}\right)\\ &+\tfrac{2^4x^2}{\pi^3}\left(1-\tfrac{1}{3^3}+\tfrac{1}{5^3}-\tfrac{1}{7^3}+\tfrac{1}{9^3}-\text{etc.}\right)\\ &+\tfrac{2^6x^4}{\pi^5}\left(1-\tfrac{1}{3^5}+\tfrac{1}{5^5}-\tfrac{1}{7^5}+\tfrac{1}{9^5}-\text{etc.}\right)\\ &\text{etc.};\end{aligned}$$

quarum serierum loco si valores supra assignati substituantur, prodibit eadem series pro secante, quam dedimus.

226. Hinc simul patet lex, qua numeri α, β, γ, δ etc. quibus summae potestatum imparium constituuntur, procedunt. Cum enim sit

$$\sec x = \frac{1}{\cos x} = \alpha + \frac{\beta}{1\cdot 2}x^2 + \frac{\gamma}{1\cdot 2\cdot 3\cdot 4}x^4 + \frac{\delta}{1\cdot 2\cdot 3\cdots 6}x^6 + \text{etc.},$$

necesse est, ut haec series aequalis sit fractioni

$$\frac{1}{1-\frac{xx}{1\cdot 2}+\frac{x^4}{1\cdot 2\cdot 3\cdot 4}-\frac{x^6}{1\cdot 2\cdot 3\cdots 6}+\frac{x^8}{1\cdot 2\cdot 3\cdots 8}-\text{etc.}}$$

aequalitate ergo constituta fiet

$$\begin{aligned}1=\alpha+\tfrac{\beta}{1\cdot2}x^2&+\tfrac{\gamma}{1\cdot2\cdot3\cdot4}x^4+\tfrac{\delta}{1\cdot2\cdot3\cdots6}x^6+\tfrac{\varepsilon}{1\cdot2\cdot3\cdots8}x^8+\text{etc.}\\ -\tfrac{\alpha}{1\cdot2}&-\tfrac{\beta}{1\cdot2\cdot1\cdot2}-\tfrac{\gamma}{1\cdot2\cdot1\cdots4}-\tfrac{\delta}{1\cdot2\cdot1\cdots6}-\text{etc.}\\ &+\tfrac{\alpha}{1\cdot2\cdot3\cdot4}+\tfrac{\beta}{1\cdots4\cdot1\cdot2}+\tfrac{\gamma}{1\cdots4\cdot1\cdots4}+\text{etc.}\\ &-\tfrac{\alpha}{1\cdot2\cdots6}-\tfrac{\beta}{1\cdots6\cdot1\cdot2}-\text{etc.}\\ &+\tfrac{\alpha}{1\cdot2\cdots8}+\text{etc.,}\end{aligned}$$

unde sequuntur hae aequationes

$$\begin{aligned}\alpha&=1\\ \beta&=\tfrac{2\cdot1}{1\cdot2}\alpha\\ \gamma&=\tfrac{4\cdot3}{1\cdot2}\beta-\tfrac{4\cdot3\cdot2\cdot1}{1.2.3\cdot4}\alpha\\ \delta&=\tfrac{6\cdot5}{1\cdot2}\gamma-\tfrac{6\cdot5\cdot4\cdot3}{1.2.3\cdot4}\beta+\tfrac{6\cdots1}{1\cdots6}\alpha\\ \varepsilon&=\tfrac{8\cdot7}{1\cdot2}\delta-\tfrac{8\cdot7\cdot6\cdot5}{1.2.3\cdot4}\gamma+\tfrac{8\cdots3}{1\cdots6}\beta-\tfrac{8\cdots1}{1\cdots8}\alpha\\ &\text{etc.}\end{aligned}$$

Ex hisque formulis inventi sunt istarum litterarum valores, quos in § 224 exhibuimus et quorum ope summae serierum in hac forma contentarum

$$1-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\frac{1}{9^n}-\text{etc.,}$$

si n fuerit numerus impar, exprimi possunt.