CHAPTER VII

THE ABOVE METHOD OF SUMMATION FURTHER ENLARGED UPON

167. In order that we may make good the deficiency of the method of summation treated before, in this chapter we will consider series of such a kind, the general terms of which shall be more complex. Therefore since the expression found before from geometric progressions, even if they are able to be summed easier by other methods, may not give the true sum contained in a finite formula, here in the first place we will consider a series of this kind, the terms of which shall be produced from the terms of a geometric series and from some other [kind of series]. Therefore let this series be proposed

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & & x \\ z = ap + & bp^2 + & cp^3 + & dp^4 + & \cdots + & yp^x, \end{array}$$

which has been composed from the geometric series p, p^2, p^3 etc. and some other series $a+b+c+d+\text{etc.}$, the general term of which or corresponding to the index x shall be $= y$, and we may investigate the general expression for the value of this sum $s = S.yp^x$.

168. We may put the reasoning in place, in the same manner as we have used above, and let v be the term preceding y in the series $a+b+c+d+\text{etc.}$ and A preceding a itself or that, which corresponds to the index 0, and vp^{x-1} shall be the general term of this series

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & & x \\ A + & ap + & bp^2 + & cp^3 + & \cdots + & vp^{x-1}; \end{array}$$

the sum of which may be indicated by $S.vp^{x-1}$, there will be

$$S.vp^{x-1} = \tfrac{1}{p} S.vp^x = S.yp^x - yp^x + A.$$

But since there shall be

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \frac{d^4y}{24dx^4} - \frac{d^5y}{120dx^5} + \text{etc.},$$

there will be

$$S.yp^x - yp^x + A = \tfrac{1}{p} S.yp^x - \tfrac{1}{p} S.\frac{dy}{dx} p^x + \tfrac{1}{2p} S.\frac{ddy}{dx^2} p^x$$
$$- \tfrac{1}{6p} S.\frac{d^3y}{dx^3} p^x + \tfrac{1}{24p} S.\frac{d^4y}{dx^4} p^x - \text{etc.}$$

From which there becomes

$$S.yp^x = \frac{1}{p-1}\left(yp^{x+1} - Ap - S.\frac{dy}{dx}p^x + S.\frac{ddy}{2dx^2}p^x - S.\frac{d^3y}{6dx^3}p^x + \text{etc.}\right)$$

Therefore if the summatory terms of the series may be had, the general terms of which are $\frac{dy}{dx}p^x$, $\frac{ddy}{dx^2}p^x$, $\frac{d^3y}{dx^3}p^x$ etc., from these the term of the summation $s = S.yp^x$ will be able to defined.

169. Hence it will be possible now to find the sum of the series , the general terms of which may be maintained in this form $x^n p^x$. Indeed there shall be $y = x^n$ *;* there will be $A = 0$, unless there shall be $n = 0$, in which case there becomes $A = 1$, and because there is

$$\frac{dy}{dx} = nx^{n-1}, \quad \frac{ddy}{2dx^2} = \frac{n(n-1)}{1\cdot 2}x^{n-2}, \quad \frac{d^3y}{6dx^3} = \frac{n(n-1)(n-3)}{1\cdot 2\cdot 3}x^{n-3} \text{ etc.,}$$

there will be

$$S.x^n p^x = \frac{1}{p-1}\left\{\begin{array}{l} x^n p^{x+1} - Ap - nS.x^{n-1}p^x + \frac{n(n-1)}{1\cdot 2}S.x^{n-2}p^x - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}S.x^{n-3}p^x \\ + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}S.x^{n-4}p^x - \text{etc.} \end{array}\right\}$$

Now from this form, on substituting successively the numbers 0, 1, 2, 3 etc. for *n*, the following summations will be obtained; and at first indeed, if $n = 0$, there becomes $A = 1$, but in the remaining cases there will be $A = 0$.

$$S.x^0 p^x = S.p^x = \frac{1}{p-1}\left(p^{x+1} - p\right) = \frac{p^{x+1}-p}{p-1} = \frac{p\left(p^x-1\right)}{p-1},$$

which is the known sum of the geometric progression;

$$S.xp^x = \frac{1}{p-1}\left(xp^{x+1} - S.p^x\right) = \frac{xp^{x+1}}{p-1} - \frac{p^{x+1}-p}{(p-1)^2}$$

or

$$S.xp^x = \frac{pxp^x}{p-1} - \frac{p\left(p^x-1\right)}{(p-1)^2}\ ;$$

$$S.x^2 p^x = \frac{1}{p-1}\left(x^2 p^{x+1} - 2S.xp^x + S.p^x\right)$$

or

$$S.x^2p^x = \frac{x^2p^{x+1}}{p-1} - \frac{2xp^{x+1}}{(p-1)^2} + \frac{p(p+1)(p^x-1)}{(p-1)^3}$$

Again there shall be

$$S.x^3p^x = \frac{1}{p-1}\left(x^3p^{x+1} - 3S.x^2p^x + 3S.xp^x - S.p^x\right)$$

or

$$S.x^3p^x = \frac{x^3p^{x+1}}{p-1} - \frac{3x^2p^{x+1}}{(p-1)^2} + \frac{3(p+1)xp^{x+1}}{(p-1)^3} + \frac{p(pp+4p+1)(p^x-1)}{(p-1)^4}$$

and thus the sums will be able to be defines on progressing to higher powers x^4p^x, x^5p^x, x^6p^x etc.; truly this will be performed more conveniently with the aid of the general expression, as we will investigate now.

170. Because we have found to be

$$S.yp^x = \frac{1}{p-1}\left(yp^{x+1} - Ap - S.\frac{dy}{dx}p^x + S.\frac{ddy}{2dx^2}p^x - S.\frac{d^3y}{6dx^3}p^x + \text{etc.}\right),$$

where A is a constant of such a kind, so that the sum becomes $=0$, if there may be put $x=0$ (for in this case there becomes $y=A$ and $yp^{x+1}=Ap$), we will be able to ignore this constant, provided always we may recall always that it is required to add a constant of this kind to the sum, so that it may vanish on making $x=0$ or so that some other case may be satisfied. Therefore we may put z in place of y and there shall be

$$S.p^xz = \frac{p^{x+1}z}{p-1} - \frac{1}{p-1}S.p^x\frac{dz}{dx} + \frac{1}{2(p-1)}S.p^x\frac{ddz}{dx^2} - \frac{1}{6(p-1)}S.p^x\frac{d^3z}{dx^3}$$
$$+\frac{1}{24(p-1)}S.p^x\frac{d^4z}{dx^4} - \frac{1}{120(p-1)}S.p^x\frac{d^5z}{dx^5} + \text{etc.}$$

Then we may put in place successively $\frac{dz}{dx}$, $\frac{ddz}{dx^2}$ $\frac{d^3z}{dx^3}$ etc. in place of y and there will be

$$S.\frac{p^xdz}{dx} = \frac{p^{x+1}}{p-1}\frac{dz}{dx} - \frac{1}{p-1}S.\frac{p^xddz}{dx^2} + \frac{1}{2(p-1)}S.\frac{p^xd^3z}{dx^3} - \text{etc.}$$

$$S.\frac{p^xddz}{dx^2} = \frac{p^{x+1}}{p-1}\frac{ddz}{dx^2} - \frac{1}{p-1}S.\frac{p^xd^3z}{dx^3} + \frac{1}{2(p-1)}S.\frac{p^xd^4z}{dx^4} - \text{etc.}$$

$$S.\frac{p^xd^3z}{dx^3} = \frac{p^{x+1}}{p-1}\frac{d^3z}{dx^3} - \frac{1}{p-1}S.\frac{p^xd^4z}{dx^4} + \frac{1}{2(p-1)}S.\frac{p^xd^5z}{dx^5} + \text{etc.}$$

etc.

Therefore if these values may be substituted successively, $S.\,p^x z$ may be expressed by a form of this kind

$$S.p^x z = \frac{p^{x+1}z}{p-1} - \frac{\alpha p^{x+1}}{p-1}\cdot\frac{dz}{dx} + \frac{\beta p^{x+1}}{p-1}\cdot\frac{ddz}{dx^2} - \frac{\gamma p^{x+1}}{p-1}\cdot\frac{d^3z}{dx^3}$$
$$+\frac{\delta p^{x+1}}{p-1}\cdot\frac{d^4z}{dx^4} - \frac{\varepsilon p^{x+1}}{p-1}\cdot\frac{d^5z}{dx^5} + \text{etc.}$$

171. Towards defining the values of the letters $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$ etc. for any term the series found before may be substituted, evidently

$$\frac{p^{x+1}z}{p-1} = S.p^x z + \frac{1}{p-1}S.\frac{p^x dz}{dx} - \frac{1}{2(p-1)}S.\frac{p^x ddz}{dx^2} + \frac{1}{6(p-1)}S.\frac{p^x d^3z}{dx^3} - \text{etc.}$$
$$\frac{p^{x+1}dz}{(p-1)dx} = S.\frac{p^x dz}{dx} + \frac{1}{p-1}S.\frac{p^x ddz}{dx^2} - \frac{1}{2(p-1)}S.\frac{p^x d^3z}{dx^3} + \text{etc.}$$
$$\frac{p^{x+1}ddz}{(p-1)dx^2} = S.\frac{p^x ddz}{dx^2} + \frac{1}{p-1}S.\frac{p^x d^3z}{dx^3} - \text{etc.}$$
$$\frac{p^{x+1}d^3z}{(p-1)dx^3} = S.\frac{p^x d^3z}{dx^3} + \text{etc.}$$

Therefore we will have [note the use here of detached coefficients]

$$S.p^x z = S.p^x z$$
$$+\frac{1}{p-1}S.\frac{p^x dz}{dx} - \frac{1}{2(p-1)}S.\frac{p^x ddz}{dx^2} + \frac{1}{6(p-1)}S.\frac{p^x d^3z}{dx^3} - \frac{1}{24(p-1)}S.\frac{p^x d^4z}{dx^4} + \text{etc}$$
$$-\alpha \qquad -\frac{\alpha}{p-1} \qquad +\frac{\alpha}{2(p-1)} \qquad -\frac{\alpha}{6(p-1)}$$
$$+\beta \qquad +\frac{\beta}{p-1} \qquad -\frac{\beta}{2(p-1)}$$
$$-\gamma \qquad -\frac{\gamma}{p-1}$$
$$+\delta$$

from which the values will be obtained of the coefficients $\alpha,\ \beta,\ \gamma,\ \delta$ etc.

$$\alpha = \frac{1}{p-1},\quad \beta = \frac{1}{p-1}\left(\alpha + \frac{1}{2}\right),\quad \gamma = \frac{1}{p-1}\left(\beta + \frac{\alpha}{2} + \frac{1}{6}\right),$$
$$\delta = \frac{1}{p-1}\left(\gamma + \frac{\beta}{2} + \frac{\alpha}{6} + \frac{1}{24}\right),\quad \varepsilon = \frac{1}{p-1}\left(\delta + \frac{\gamma}{2} + \frac{\beta}{6} + \frac{\alpha}{24} + \frac{1}{120}\right)$$
etc.

172. If for the sake of brevity $\frac{1}{p-1}=q$; there will be

$$\alpha = q$$
$$\beta = \alpha q + \tfrac{1}{2}q = qq + \tfrac{1}{2}q$$
$$\gamma = \beta q + \tfrac{1}{2}\alpha q + \tfrac{1}{6}q = q^3 + qq + \tfrac{1}{6}q$$
$$\delta = \gamma q + \tfrac{1}{2}\beta q + \tfrac{1}{6}\alpha q + \tfrac{1}{24}q = q^4 + \tfrac{3}{2}q^3 + \tfrac{7}{12}q^2 + \tfrac{1}{24}q$$
$$\varepsilon = \delta q + \tfrac{1}{2}\gamma q + \tfrac{1}{6}\beta q + \tfrac{1}{24}\alpha q + \tfrac{1}{120}q = q^5 + 2q^4 + \tfrac{5}{4}q^3 + \tfrac{1}{4}q^2 + \tfrac{1}{120}q$$
$$\zeta = q^6 + \tfrac{5}{2}q^5 + \tfrac{13}{6}q^4 + \tfrac{13}{4}q^3 + \tfrac{31}{360}q^2 + \tfrac{1}{720}q$$
etc.

or expressed in this manner

$$\alpha = \frac{q}{1}$$
$$\beta = \frac{2qq+q}{1\cdot 2}$$
$$\gamma = \frac{6q^3+6q^2+q}{1\cdot 2\cdot 3}$$
$$\delta = \frac{24q^4+36q^3+14q^2+q}{1\cdot 2\cdot 3\cdot 4}$$
$$\varepsilon = \frac{120q^5+240q^4+150q^3+30q^2+q}{1\cdot 2\cdot 3\cdot 4\cdot 5}$$
$$\zeta = \frac{720q^6+1800q^5+1560q^4+540q^3+62q^2+q}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$$
$$\eta = \frac{5040q^7+15120q^6+16800q^5+8400q^4+1806q^3+126q^2+q}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$$
etc.

where any coefficient such as 16800 arises, if the sum of the above two numbers $1560+1800$ may be multiplied by the exponent of q, which is 5 here.

173. But we may restore the value $\frac{1}{p-1}$ in place of q :

$$\alpha=\frac{1}{1(p-1)}$$

$$\beta=\frac{p+1}{1\cdot 2(p-1)^2}$$

$$\gamma=\frac{pp+4p+1}{1\cdot 2\cdot 3(p-1)^3}$$

$$\delta=\frac{p^3+11p^2+11p+1}{1\cdot 2\cdot 3\cdot 4(p-1)^4}$$

$$\varepsilon=\frac{p^4+26p^3+66p^2+26p+1}{1\cdot 2\cdot 3\cdot 4\cdot 5(p-1)^5}$$

$$\zeta=\frac{p^5+57p^4+302p^3+302p^2+57p+1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6(p-1)^6}$$

$$\eta=\frac{p^6+120p^5+1191p^4+2416p^3+1191p^2+120p+1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7(p-1)^7}$$

etc.

Thus the law of these quantities itself may be had, so that if some term is put in place

$$=\frac{p^{n-2}+Ap^{n-3}+Bp^{n-4}+Cp^{n-5}+Dp^{n-6}+\text{etc.}}{1\cdot 2\cdot 3\cdots(n-1)(p-1)^{n-1}}$$

there shall become

$$A=2^{n-1}-n$$

$$B=3^{n-1}-n\cdot 2^{n-1}+\frac{n(n-1)}{1\cdot 2}$$

$$C=4^{n-1}-n\cdot 3^{n-1}+\frac{n(n-1)}{1\cdot 2}2^{n-1}-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$$

$$D=5^{n-1}-n\cdot 4^{n-1}+\frac{n(n-1)}{1\cdot 2}3^{n-1}-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}2^{n-1}+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}$$

etc.,

from which these coefficients $\alpha, \beta, \gamma, \delta$ etc. are able to be continued as long as it should please.

174. But if truly we should consider the law, by which these coefficients are related to each other, it readily becomes apparent these constitute a recurring series and to be produced, if this fraction may be expanded out

$$\frac{1}{1-\frac{u}{p-1}-\frac{u^2}{2(p-1)}-\frac{u^3}{6(p-1)}-\frac{u^4}{24(p-1)}-\text{etc.}};$$

indeed this series will appear

$$1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\zeta u^6+\text{etc.}$$

That fraction may be put $=V$, and since there shall be [simply by multiplying]

$$V=\frac{p-1}{p-1-u-\frac{u^2}{2}-\frac{u^3}{6}-\frac{u^4}{24}-\text{etc.}}$$

there will be

$$V=\frac{p-1}{p-\mathrm{e}^{-u}}$$

where *e* is the number, the hyperbolic logarithm of which is $=1$. And if the value of *V* may be expressed following the powers of *u*, there may arise

$$V=1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\zeta u^6+\text{etc.}$$

the coefficients of which $\alpha,\ \beta,\ \gamma,\ \delta$ etc. shall be these themselves, of which we have a need in the present work. Therefore from these found there will be

$$S.p^x z=\frac{p^{x+1}}{p-1}\left(z-\frac{\alpha dz}{dx}+\frac{\beta ddz}{dx^2}-\frac{\gamma d^3z}{dx^3}+\frac{\delta d^4z}{dx^4}-\text{etc.}\right)\pm\text{Const.},$$

which expression therefore is the summatory term of this series

$$ap+bp^2+cp^3+\cdots+p^xz,$$

the general term of which is $=p^xz$. [Recall that *z* was written in place of *y* near the beginning; the present development uses the exponential function to simplify the equations arising by removing the derivatives.]

175. Because we have found to be $V=\frac{p-1}{p-\mathrm{e}^{-u}}$ there will be

$$\mathrm{e}^u=\frac{pV-p+1}{V}$$

and with the logarithms taken the equation becomes

$$u = l\left(pV - p + 1\right) - lV$$

and hence on being differentiated

$$du = \frac{(p-1)dV}{pV^2 - (p-1)V}$$

on account of which there will be

$$pV^2 = \left(p-1\right)V + \frac{(p-1)dV}{du}.$$

Therefore because there is

$$V = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \text{etc.},$$

there will be

$$\begin{aligned} pV^2 = p + 2\alpha pu + 2\beta pu^2 &+ 2\gamma pu^3 + 2\delta pu^4 + 2\varepsilon pu^5 + \text{etc.} \\ \alpha^2 pu^2 &+ 2\alpha\beta pu^3 + 2\alpha\gamma pu^4 + 2\alpha\delta pu^5 + \text{etc.} \\ &+ \beta\beta pu^4 + 2\beta\gamma pu^5 + \text{etc.} \end{aligned}$$

and

$$\begin{aligned} \left(p-1\right)V &= \left(p-1\right) + \alpha\left(p-1\right)u + \beta\left(p-1\right)u^2 \\ &+ \gamma\left(p-1\right)u^3 + \delta\left(p-1\right)u^4 + \varepsilon\left(p-1\right)u^5 + \text{etc.}, \\ \frac{(p-1)dV}{du} &= \left(p-1\right)\alpha + 2\left(p-1\right)\beta u + 3\left(p-1\right)\gamma u^2 + 4\left(p-1\right)\delta u^3 \\ &+ 5\left(p-1\right)\varepsilon u^4 + 6\left(p-1\right)\zeta u^5 + \text{etc.}, \end{aligned}$$

from which expressions equated to each other there may be found

$$\begin{aligned} &\left(p-1\right)\alpha = 1 \\ &2\left(p-1\right)\beta = \alpha\left(p+1\right) \\ &3\left(p-1\right)\gamma = \beta\left(p+1\right) + \alpha^2 p \\ &4\left(p-1\right)\delta = \gamma\left(p+1\right) + 2\alpha\beta p \\ &5\left(p-1\right)\varepsilon = \delta\left(p+1\right) + 2a\gamma p + \beta\beta p \\ &6\left(p-1\right)\zeta = \varepsilon\left(p+1\right) + 2a\delta p + 2\beta\gamma p \\ &7\left(p-1\right)\eta = \zeta\left(p+1\right) + 2\alpha\varepsilon p + 2\beta\delta p + \gamma\gamma p \end{aligned}$$

etc.,

from which formulas, if some given number may be assumed for *p*, the values of the coefficients α, β, γ, δ etc. can be determined more easily than from the law first found.

176. Before we may descend to special cases in the account of the values of p, we may put to be $z = x^n$, thus so that this series must be summed

$$s = p + 2^n p^2 + 3^n p^3 + 4^n p^4 + \cdots + x^n p^x ,$$

and there will be by the expression found before

$$s = p^x \left(\begin{array}{l} \frac{p}{p-1} \cdot x^n - \frac{p}{(p-1)^2} \cdot nx^{n-1} + \frac{pp+p}{(p-1)^3} \cdot \frac{n(n-1)}{1\cdot 2} x^{n-2} \\ - \frac{p^3+4p^2+p}{(p-1)^4} \cdot \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} x^{n-3} + \text{etc.} \end{array} \right) \pm \text{C},$$

which reduces to $s = 0$, if there may be put $x = 0$.

Hence on putting successively the numbers 0, 1, 2, 3, 4 etc. for n there will be

$$S.x^0 p^x = p^x \cdot \frac{p}{p-1} - \frac{p}{p-1}$$

$$S.x^1 p^x = p^x \left(\frac{px}{p-1} - \frac{p}{(p-1)^2} \right) + \frac{p}{(p-1)^2}$$

$$S.x^2 p^x = p^x \left(\frac{px^2}{p-1} - \frac{2px}{(p-1)^2} + \frac{p(p+1)}{(p-1)^3} \right) - \frac{p(p+1)}{(p-1)^3}$$

$$S.x^3 p^x = p^x \left(\frac{px^3}{p-1} - \frac{3px^2}{(p-1)^2} + \frac{3p(p+1)x}{(p-1)^3} - \frac{p(p^2+4p+1)}{(p-1)^4} \right) + \frac{p(p^2+4p+1)}{(p-1)^4}$$

$$S.x^4 p^x = p^x \left(\frac{px^4}{p-1} - \frac{4px^3}{(p-1)^2} + \frac{6p(p+1)x^2}{(p-1)^3} - \frac{4p(p^2+4p+1)x}{(p-1)^4} + \frac{p(p^3+11p^2+11p+1)}{(p-1)^5} \right) - \frac{p(p^3+11p^2+11p+1)}{(p-1)^5}$$

$$S.x^5 p^x = \frac{p^{x+1}x^5}{p-1} - \frac{5p^{x+1}x^4}{(p-1)^2} + \frac{10(p+1)p^{x+1}x^3}{(p-1)^3} - \frac{10(p^2+4p+1)p^{x+1}x^2}{(p-1)^4}$$
$$+ \frac{5(p^3+11p^2+11p+1)p^{x+1}x}{(p-1)^5} - \frac{(p^4+26p^3+66p^2+26p+1)(p^{x+1}-p)}{(p-1)^6}$$

$$S.x^6 p^x = \frac{p^{x+1}x^6}{p-1} - \frac{6p^{x+1}x^5}{(p-1)^2} + \frac{15(p+1)p^{x+1}x^4}{(p-1)^3} - \frac{20(p^2+4p+1)p^{x+1}x^3}{(p-1)^4}$$
$$+ \frac{15(p^3+11p^2+11p+1)p^{x+1}x^2}{(p-1)^5} - \frac{6(p^4+26p^3+66p^2+26p+1)p^{x+1}x}{(p-1)^6}$$
$$+ \frac{(p^5+57p^4+302p^3+302p^2+57p+1)(p^{x+1}-p)x}{(p-1)^7}$$

etc.

177. Hence it is understood, as often as *z* shall be a rational integral function of *x*, so is it possible often to show the sum of the series, of which the general term is $p^x z$, because on requiring therefore to take the differentials of *z* only until the vanishing point is reached. Thus if this series may be proposed

$$p+3p^2+6p^3+10p^4+\cdots+\tfrac{xx+x}{2}p^x,$$

on account of

$$z=\tfrac{xx+x}{2} \quad \text{and} \quad \tfrac{dz}{dx}=x+\tfrac{1}{2}, \quad \text{and also} \quad \tfrac{ddz}{dx^2}=1$$

the summatory term will be

$$s=\tfrac{p^{x+1}}{p-1}\left(\tfrac{1}{2}xx+\tfrac{1}{2}x-\tfrac{2x+1}{2(p-1)}+\tfrac{p+1}{2(p-1)^2}\right)-\tfrac{p}{p-1}\left(\tfrac{p+1}{2(p-1)^2}-\tfrac{1}{2(p-1)}\right)$$

or

$$s=p^{x+1}\left(\tfrac{xx}{2(p-1)}+\tfrac{(p-3)x}{2(p-1)^2}+\tfrac{1}{(p-1)^3}\right)-\tfrac{p}{(p-1)^3}.$$

But if *z* were not a rational integral function, then the expression of the summatory terms will extend to infinity. Thus if there shall be $z=\frac{1}{x}$, so that this series shall be required to be summed

$$s=p+\tfrac{1}{2}p^2+\tfrac{1}{3}p^3+\tfrac{1}{4}p^4+\cdots+\tfrac{1}{x}p^x,$$

on account of

$$\tfrac{dz}{dx}=-\tfrac{1}{xx}, \quad \tfrac{ddz}{dx^2}=\tfrac{2}{x^3}, \quad \tfrac{d^3z}{dx^3}=-\tfrac{2\cdot3}{x^4}, \quad \tfrac{d^4z}{dx^4}=-\tfrac{2\cdot3\cdot4}{x^5} \quad \text{etc.}$$

the summatory term will be produced

$$s=\tfrac{p^{x+1}}{p-1}\left(\tfrac{1}{x}+\tfrac{1}{(p-1)x^2}+\tfrac{p+1}{(p-1)^2x^3}+\tfrac{pp+4p+1}{(p-1)^3x^4}+\tfrac{p^3+11p^2+11p+1}{(p-1)^4x^5}+\text{etc.}\right)+C.$$

Therefore the constant *C* can be defined except in the case $x=0$; towards defining that therefore we may put $x=1$, and because there becomes $s=p$, there will be

$$C=p-\tfrac{pp}{p-1}\left(1+\tfrac{1}{p-1}+\tfrac{p+1}{(p-1)^2}+\tfrac{pp+4p+1}{(p-1)^3}+\text{etc.}\right).$$

178. From these it is evident, unless p may designate a determined number, hence little of use remains towards showing the approximate sums of series. [Essentially one series is replaced by another.] Moreover first it is apparent that it is not possible to write 1 for p, because all the coefficients $\alpha,\ \beta,\ \gamma,\ \delta$ etc. therefore may become infinitely great. Whereby, since the series, which we may now treat, may change into that which we have considered before, if there may be put $p=1$, it is a wonder, because that cannot be elicited from this as in the easiest case. Then truly also it is remarkable, because in the case $p=1$ the summation may require the integral $\int zdx$, since yet generally the sum is able to be shown without any integral. Thus therefore it comes about, so that while all the coefficients $\alpha,\ \beta,\ \gamma,\ \delta$ etc. increase to infinity, likewise that formula of the integral may be brought in. And so this case, in which $p=1$, is alone, towards which the general expression found here is unable to be applied. Nor truly in this case may it be agreed that the general form moves away from the truth ; for even if the individual terms become infinite, yet actually all the infinities cancel each other and there may remain a finite quantity equal to the sum and agreeing with that, which is found by the first method, which we are about to indicate further below.

179. Therefore let $p=-1$ and the signs in the series to be summed follow each other alternately

$$\begin{matrix} 1 & 2 & 3 & 4 & & x \\ -a & +b & -c & +d & -\cdots & \pm z, \end{matrix}$$

where z will be positive, if x were an even number, but negative, if x shall be an odd number. Therefore on putting

$$-a+b-c+d-\cdots\pm z=s$$

there will be

$$s=\tfrac{\pm 1}{2}\left(z-\tfrac{\alpha dz}{dx}+\tfrac{\beta ddz}{dx^2}-\tfrac{\gamma d^3z}{dx^3}+\tfrac{\delta d^4z}{dx^4}-\text{etc.}\right)+C,$$

where the upper of both signs prevail, if x shall be an even number, truly the opposite, if x shall be an odd number. Therefore with the signs changed there will be

$$a-b+c-d+\cdots\mp z=s=\mp\tfrac{1}{2}\left(z-\tfrac{\alpha dz}{dx}+\tfrac{\beta ddz}{dx^2}-\tfrac{\gamma d^3z}{dx^3}+\tfrac{\delta d^4z}{dx^4}-\text{etc.}\right)+C,$$

where the ambiguities of the signs follow the same law.

180. In this case the coefficients $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta$ etc. can be found from the values treated before on putting $p=-1$ everywhere . But they may be elicited easier from the general formulas given in § 175, from which likewise it may be seen that these alternate coefficients vanish. For on making $p=-1$ these formulas will change into

$$-2\alpha=1,\quad -4\beta=0,\quad -6\gamma=0-\alpha^2,\ -8\delta=0-2\alpha\beta,$$
$$-10\varepsilon=0-2\alpha\gamma-\beta\beta,\quad -12\zeta=0-2\alpha\delta-2\beta\gamma \text{ etc.;}$$

from which, since there shall be $\beta=0$, also there will be $\delta=0$, and again $\zeta=0,\ \theta=0$ etc. and the remaining letters thus will be determined, so that there shall be

$$\alpha=-\tfrac{1}{2},\quad \gamma=\tfrac{\alpha^2}{6},\quad \varepsilon=\tfrac{2\alpha\gamma}{10},\quad \eta=\tfrac{2\alpha\varepsilon+\gamma\gamma}{14},\quad \iota=\tfrac{2\alpha\eta+2\gamma\varepsilon}{18} \text{ etc.}$$

181. In order that this calculation will be completed more conveniently, we may introduce new letters and there shall be

$$\alpha=-\tfrac{A}{1\cdot2},\quad \gamma=\tfrac{B}{1\cdot2\cdot3\cdot4},\quad \varepsilon=-\tfrac{C}{1\cdot2\cdot3\cdot4\cdot5\cdot6},$$
$$\eta=\tfrac{D}{1\cdot2\cdot3\cdots8},\quad \iota=-\tfrac{E}{1\cdot2\cdot3\cdots10}$$
etc.

And the sum shown before will be

$$\mp\tfrac{1}{2}\left(z+\tfrac{Adz}{1\cdot2dx}-\tfrac{Bd^3z}{1\cdot2\cdot3\cdot4dx^3}+\tfrac{Cd^5z}{1\cdot2\cdot3\cdots6dx^5}-\tfrac{Dd^7z}{1\cdot2\cdot3\cdots8dx^7}+\text{ etc.}\right)+C.$$

Truly the coefficients may be defined from the following formulas

$$A=1$$
$$3B=\tfrac{4\cdot3}{1\cdot2}\cdot\tfrac{AA}{2}$$
$$5C=\tfrac{6\cdot5}{1\cdot2}\cdot AB$$
$$7D=\tfrac{8\cdot7}{1\cdot2}\cdot AC+\tfrac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4}\cdot\tfrac{BB}{2}$$
$$9E=\tfrac{10\cdot9}{1\cdot2}\cdot AD+\tfrac{10\cdot9\cdot8\cdot7}{1\cdot2\cdot3\cdot4}\cdot BC$$
$$11F=\tfrac{12\cdot11}{1\cdot2}\cdot AE+\tfrac{12\cdot11\cdot10\cdot9}{1\cdot2\cdot3\cdot4}\cdot BD+\tfrac{12\cdot11\cdot10\cdot9\cdot8\cdot7}{1\cdot2\cdot3\cdot4\cdot5\cdot6}\cdot\tfrac{CC}{2}$$
etc.,

which in this manner they are able to represent the calculation more easily and conveniently

$$A=1$$
$$B=2\cdot\tfrac{AA}{2}$$
$$C=3\cdot AB$$
$$D=4\cdot AC+4\cdot\tfrac{6\cdot5}{3\cdot4}\cdot\tfrac{BB}{2}$$
$$E=5\cdot AD+5\cdot\tfrac{8\cdot7}{3\cdot4}\cdot BC$$
$$F=6\cdot AE+6\cdot\tfrac{10\cdot9}{3\cdot4}\cdot BD+6\cdot\tfrac{10\cdot9\cdot8\cdot7}{3\cdot4\cdot5\cdot6}\cdot\tfrac{CC}{2}$$
$$G=7\cdot AE+7\cdot\tfrac{12\cdot11}{3\cdot4}\cdot BE+7\cdot\tfrac{12\cdot11\cdot10\cdot9}{3\cdot4\cdot5\cdot6}\cdot CD$$
etc.

Hence therefore with the calculation put in place there may be found

$$A=1$$
$$B=1$$
$$C=3$$
$$D=17$$
$$E=155=5\cdot31$$
$$F=2073=691\cdot3$$
$$G=38227=7\cdot5461=7\cdot\tfrac{127\cdot129}{3}$$
$$H=929569=3617\cdot257$$
$$I=28820619=43867\cdot9\cdot73$$
etc.

182. If we may consider these numbers more diligently, from the factors 691, 3617, 43867 it is permitted to conclude easily that these numbers have a connection with the Bernoulli numbers shown above and thence are able to be determined. Therefore this relation to be investigated soon will make apparent that these numbers can be formed from the Bernoulli numbers $\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D},\mathfrak{E},$etc. in the following manner :

$$A = 2\cdot 1\cdot 3\,\mathfrak{A} = 2\left(2^2 - 1\right)\mathfrak{A}$$
$$B = 2\cdot 3\cdot 5\,\mathfrak{B} = 2\left(2^4 - 1\right)\mathfrak{B}$$
$$C = 2\cdot 7\cdot 9\,\mathfrak{C} = 2\left(2^6 - 1\right)\mathfrak{C}$$
$$D = 2\cdot 15\cdot 17\,\mathfrak{D} = 2\left(2^8 - 1\right)\mathfrak{D}$$
$$E = 2\cdot 31\cdot 33\,\mathfrak{E} = 2\left(2^{10} - 1\right)\mathfrak{D}$$
$$F = 2\cdot 63\cdot 65\,\mathfrak{F} = 2\left(2^{12} - 1\right)\mathfrak{F}$$
$$G = 2\cdot 127\cdot 129\,\mathfrak{G} = 2\left(2^{14} - 1\right)\mathfrak{G}$$
$$H = 2\cdot 255\cdot 257\,\mathfrak{H} = 2\left(2^{16} - 1\right)\mathfrak{H}$$
etc.

Therefore since the Bernoulli numbers shall be fractions, truly our coefficients whole numbers, it is apparent these factors always remove the fractions and therefore there will be

$$A = 1$$
$$B = 1$$
$$C = 3$$
$$D = 17$$
$$E = 5\cdot 31 = 155$$
$$F = 3\cdot 691 = 2073$$
$$G = 7\cdot 5461 = 38227$$
$$H = 257\cdot 3617 = 929569$$
$$I = 9\cdot 73\cdot 43867 = 28820619$$
$$K = 5\cdot 31\cdot 41\cdot 174611 = 1109652905$$
$$L = 89\cdot 683\cdot 854513 = 51943281731$$
$$M = 3\cdot 4097\cdot 236364091 = 2905151042481$$
$$N = 2731\cdot 8191\cdot 8553103 = 191329672483963$$
etc.

Therefore from these whole numbers the Bernoulli numbers will be able to be found in turn.

183. Therefore with the Bernoulli numbers of the series

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & & x \\ a & -b & +c & -d & +e & -\cdots & \mp z \end{matrix}$$

being employed there will be

$$\mp\left(\tfrac{1}{2}z+\frac{(2^2-1)\mathfrak{A}dz}{1\cdot 2dx}-\frac{(2^4-1)\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3}+\frac{(2^6-1)\mathfrak{C}d^5z}{1\cdot 2\cdots 6dx^5}-\frac{(2^8-1)\mathfrak{D}d^7z}{1\cdot 2\cdots 8dx^7}+\text{etc.}\right)+\text{Const.}$$

But hence it is evident that those numbers in no case are present in this expression ; just as it arises for the proposed series, if from that

$$a+b+c+d+\cdots+z,$$

where all the terms have the + sign, with the sum of the alternates $b+d+f+\text{etc.}$ twice taken is subtracted, thus also the expression found can be resolved into two parts, of which one is the sum of all the terms affected by the sign + , which shall be

$$\int zdx+\tfrac{1}{2}z+\frac{\mathfrak{A}dz}{1\cdot 2dx}-\frac{\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3}+\frac{\mathfrak{C}d^5z}{1\cdot 2\cdot 3\cdots 6dx^5}-\text{etc.}$$

Truly the sum of the other even part also may be found, with which we have used above [Ch.V]. For since the final term shall be *z* corresponding to the index *x*, the preceding corresponding to the index $x-2$ will be

$$z-\frac{2dz}{dx}+\frac{2^2ddz}{1\cdot 2dx^2}-\frac{2^3d^3z}{1\cdot 2\cdot 3dx^3}+\frac{2^4d^4z}{1\cdot 2\cdot 3\cdot 4dx^4}-\text{etc.},$$

which form arises from that, by which the preceding term will be expressed before, if in place of *x* there may be written $\frac{x}{2}$. Therefore the sum of the alternate signs will be had, if in the sum of the whole there may be written $\frac{x}{2}$ in place of *x* everywhere, which therefore will be

$$\tfrac{1}{2}\int zdx+\tfrac{1}{2}z+\frac{2\mathfrak{A}dz}{1\cdot 2dx}-\frac{2^3\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3}+\frac{2^5\mathfrak{C}d^5z}{1\cdot 2\cdot 3\cdots 6dx^5}-\text{etc.};$$

the double of which if subtracted from the preceding sum with *x* proving to be an even number, or if from the double of which the preceding sum is subtracted, if *x* is an odd number, then the remainder will show the sum of the series

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & & x \\ a & -b & +c & -d & +e & -\cdots & \mp z, \end{matrix}$$

which therefore will be

$$\mp\left(\tfrac{1}{2}z+\frac{(2^2-1)\mathfrak{A}dz}{1\cdot 2dx}-\frac{(2^4-1)\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3}+\text{etc.}\right)+C,$$

which is the same expression, just as we had found.

184. The powers of x may be taken for z, surely x^n, so that the sum of the series may be found

$$1-2^n+3^n-4^n+\cdots\mp x^n.$$

On account of which

$$\frac{dz}{1dx}=\frac{n}{1}x^{n-1},\quad \frac{d^3z}{1\cdot 2\cdot 3dx^3}=\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\quad \text{etc.}$$

the sum sought will be with the coefficients A, B, C, D etc. employed

$$\mp\tfrac{1}{2}\left\{\begin{matrix} x^n+\frac{A}{2}nx^{n-1}-\frac{B}{4}\cdot\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}+\frac{C}{6}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^{n-5} \\ -\frac{D}{8}\cdot\frac{n(n-1)\cdots(n-6)}{1\cdot 2\cdots 7}x^{n-7}+\text{ etc.}+\text{Const.} \end{matrix}\right\}$$

where the upper sign prevails if x shall be an even number, truly the lower if odd. But the constant must be defined thus, so that the sum may vanish, if $x=0$, in which case the upper sign prevails. Hence with the numbers 0, 1, 2, 3 etc. substituted successively for n the following summations will be produced:

$$\text{I.}\quad 1-1+1-1+\cdots\mp 1=\mp\tfrac{1}{2}(1)+\tfrac{1}{2};$$

evidently if the number of terms were even, the sum will be $=0$, but if odd, it will be $=+1$.

$$\text{II.}\quad 1-2+3-4+\cdots\mp x=\mp\tfrac{1}{2}\left(x+\tfrac{1}{2}\right)+\tfrac{1}{4}$$

evidently if the number of terms shall be even, the sum will be $=-\frac{1}{2}x$ and for an odd number of terms $=+\frac{1}{2}x+\frac{1}{2}$.

$$\text{III.}\quad 1-2^2+3^2-4^2+\cdots\mp x^2=\mp\tfrac{1}{2}\left(x^2+x\right);$$

evidently for an even number $=-\frac{1}{2}xx-\frac{1}{2}x$ and for an odd number $=+\frac{1}{2}xx+\frac{1}{2}x$.

$$\text{IV.}\quad 1-2^3+3^3-4^3+\cdots\mp x^3=\mp\ \tfrac{1}{2}\left(x^3+\tfrac{3}{2}xx-\tfrac{1}{4}\right)-\tfrac{1}{8}$$

evidently for the even $=-\frac{1}{2}x^3-\frac{3}{4}x^2$ and for the odd $=\frac{1}{2}x^3+\frac{3}{4}x^2-\frac{1}{4}$.

$$\text{V.}\quad 1-2^4+3^4-4^4+\cdots\mp x^4=\mp\ \tfrac{1}{2}\left(x^4+2x^3-x\right)$$

evidently for an even number $=-\frac{1}{2}x^4-x^3+\frac{1}{2}x$ and for an odd number
$=\frac{1}{2}x^4+x^3-\frac{1}{2}x$ etc.

185. Therefore it appears in the even powers besides $n=0$ the constant requiring to be added vanishes and in these cases the sum for the number of the terms either of even or of odd powers only differ on account of the sign. But if hence *x* were an infinite number, because this is neither even nor odd, this consideration must cease and therefore in the sum the terms of the ambiguity are to be rejected ; from which it follows the sum in the continuation to infinity of series of this kind be expressed only by a constant quantity to be added.

On this account

$$1-1\quad+1\ -1\ +\ \text{etc. to infinity}=\tfrac{1}{2}$$

$$1-2\ +3\ -4\ +\text{etc.}\ \cdot\ \cdot\ \cdot\qquad=\tfrac{A}{4}=+\frac{\left(2^2-1\right)\mathfrak{A}}{2}$$

$$1-2^2+3^2-4^2+\text{etc.}\ \cdot\ \cdot\ \cdot\qquad=\ 0$$

$$1-2^3+3^3-4^3+\text{etc.}\ \cdot\ \cdot\ \cdot\qquad=-\tfrac{B}{8}=-\frac{\left(2^4-1\right)\mathfrak{B}}{4}$$

$$1-2^4+3^4-4^4+\text{etc.}\ \cdot\ \cdot\ \cdot\qquad=\ 0$$

$$1-2^5+3^5-4^5+\text{etc.}\ \cdot\ \cdot\ \cdot\qquad=\ \tfrac{C}{12}=+\frac{\left(2^6-1\right)\mathfrak{C}}{6}$$

$$1-2^6+3^6-4^6+\text{etc.}\ \cdot\ \cdot\ \cdot\qquad=\ 0$$

$$1-2^7+3^7-4^7+\text{etc.}\ \cdot\ \cdot\ \cdot\qquad=-\tfrac{D}{16}=-\frac{\left(2^8-1\right)\mathfrak{D}}{8}$$

etc.

Which same sums are found by the above method treating series requiring to be summed, in which the signs + and – may alternate.

186. If negative numbers may be taken for *n*, the expression of the sum may extend to infinity. Let $n=-1$; the sum of the series will be

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\cdots\mp\tfrac{1}{x}$$
$$=\mp\left(\tfrac{1}{x}-\tfrac{A}{2x^2}+\tfrac{B}{4x^4}-\tfrac{C}{6x^6}+\tfrac{D}{8x^8}-\text{etc.}\right)+\text{Const.}$$

But here because the constant cannot be defined in the case $x=0$, it is required to be defined from another case. There may be put $x=1$ and on account of the sum $=1$ and the lower sign there will be

$$\text{Const.}=1-\tfrac{1}{2}\left(\tfrac{1}{1}-\tfrac{A}{2}+\tfrac{B}{4}-\tfrac{C}{6}+\text{etc.}\right)$$

or

$$\text{Const.}=\tfrac{1}{2}+\tfrac{A}{4}-\tfrac{B}{8}+\tfrac{C}{12}-\tfrac{D}{16}+\text{etc.}$$

Or there may be put $x=2$; on account of the sum$=\tfrac{1}{2}$ and the upper sign there may be found

$$\text{Const.}=\tfrac{1}{2}+\tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{A}{2\cdot 2^2}+\tfrac{B}{4\cdot 2^4}-\tfrac{C}{6\cdot 2^6}+\text{etc.}\right)$$

or

$$\text{Const.}=\tfrac{3}{4}-\tfrac{A}{4\cdot 2^2}+\tfrac{B}{8\cdot 2^4}-\tfrac{C}{12\cdot 2^6}+\tfrac{D}{16\cdot 2^8}-\text{etc.}$$

But if there is put $x=4$, there will be

$$\text{Const.}=\tfrac{17}{24}-\tfrac{A}{4\cdot 4^2}+\tfrac{B}{8\cdot 4^4}-\tfrac{C}{12\cdot 4^6}+\tfrac{D}{16\cdot 4^8}-\text{etc.}$$

But whatever constant may be defined, it will produce the same value which is$=l2$, which likewise the sum of the series continued to infinity will indicate.

187. Moreover from these new numbers *A, B, C, D, E* etc. the sums of series of reciprocals of even powers, in which only odd numbers occurs, will be able to be summed conveniently. If indeed there may be put

$$1+\tfrac{1}{2^{2n}}+\tfrac{1}{3^{2n}}+\tfrac{1}{4^{2n}}+\tfrac{1}{5^{2n}}+\tfrac{1}{6^{2n}}+\text{etc.}=s$$

there will be

$$\tfrac{1}{2^{2n}}\qquad+\tfrac{1}{4^{2n}}\qquad+\tfrac{1}{6^{2n}}+\text{etc.}=\tfrac{s}{2^{2n}},$$

which taken from that there will remain

$$1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\text{etc.}=\frac{\left(2^{2n}-1\right)s}{2^{2n}}.$$

Therefore since we have now shown above the values of s for the individual numbers n (§ 125), there will be

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\text{ etc.}=\frac{A}{1\cdot 2}\cdot\frac{\pi^2}{4}$$
$$1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\text{ etc.}=\frac{B}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{\pi^4}{4}$$
$$1+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\text{ etc.}=\frac{C}{1\cdot 2\cdot 3\cdots 6}\cdot\frac{\pi^6}{4}$$
$$1+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\text{ etc.}=\frac{D}{1\cdot 2\cdot 3\cdots 8}\cdot\frac{\pi^8}{4}$$
$$1+\frac{1}{3^{10}}+\frac{1}{5^{10}}+\frac{1}{7^{10}}+\text{ etc.}=\frac{E}{1\cdot 2\cdot 3\cdots 10}\cdot\frac{\pi^{10}}{4}$$
etc.

But if all the numbers may be present and with the signs alternating, because there will be

$$1-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\frac{1}{4^{2n}}+\text{etc.}=\frac{\left(2^{2n}-1\right)s-s}{2^{2n}},$$

there will be had

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\text{etc.}=\frac{A-2\mathfrak{A}}{1\cdot 2}\cdot\frac{\pi^2}{4}=\frac{(2-1)\mathfrak{A}}{1\cdot 2}\cdot\pi^2$$
$$1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}-\text{etc.}=\frac{B-2\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{\pi^4}{4}=\frac{\left(2^3-1\right)\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}\cdot\pi^4$$
$$1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\text{etc.}=\frac{C-2\mathfrak{C}}{1\cdot 2\cdots 6}\cdot\frac{\pi^6}{4}=\frac{\left(2^5-1\right)\mathfrak{C}}{1\cdot 2\cdots 6}\cdot\pi^6$$
$$1-\frac{1}{2^8}+\frac{1}{3^8}-\frac{1}{4^8}+\frac{1}{5^8}-\text{etc.}=\frac{D-2\mathfrak{D}}{1\cdot 2\cdots 8}\cdot\frac{\pi^8}{4}=\frac{\left(2^7-1\right)\mathfrak{D}}{1\cdot 2\cdots 8}\cdot\pi^8$$
$$1-\frac{1}{2^{10}}+\frac{1}{3^{10}}-\frac{1}{4^{10}}+\frac{1}{5^{10}}-\text{etc.}=\frac{E-2\mathfrak{E}}{1\cdot 2\cdots 10}\cdot\frac{\pi^{10}}{4}=\frac{\left(2^9-1\right)\mathfrak{E}}{1\cdot 2\cdots 10}\cdot\pi^{10}$$
etc.

188. Just as at this stage we have considered a series, the terms of which have been produced from the terms of the geometric progression p, p^2, p^3 etc. and from the terms of any series a, b, c etc., thus we will be able to pursue by similar reasoning a series, the terms of which shall be produced from the terms of any two series, of which only the one may be assumed to be known. Let the known series be

$$\begin{matrix} 1 & 2 & 3 & & x \\ A+ & B+ & C+ & \cdots+ & Z, \end{matrix}$$

truly the unknown [series *s*]

$$a+b+c+\cdots+z$$

and the sum of this series may be sought

$$Aa+Bb+Cc+\cdots+Zz,$$

which is put $=Zs$. In the known series let the penultimate term be $=Y$ and on putting $x-1$ in place of x the expression for the sum $S.Zz$ extracted will be

$$Y\left(s-\frac{ds}{dx}+\frac{dds}{2dx^2}-\frac{d^3s}{6dx^3}+\frac{d^4s}{24dx^4}-\text{etc.}\right).$$

Which since it may express the sum of the series *Zs* with the final term *Zz* extracted, there will be

$$Zs-Zz=Ys-\frac{Yds}{dx}+\frac{Ydds}{2dx^2}-\frac{Yd^3s}{6dx^3}+\text{etc.},$$

which equation will contain the relation, by which the sum *Zs* depends on *Y*, *Z* and *z*.

189. Towards the resolution of this equation at first the differential terms may be ignored, and there will be

$$s=\frac{Zz}{Z-Y};$$

this value may be put in place $\frac{Zz}{Z-Y}=P^{\mathrm{I}}$ and actually there will be $s=P^{\mathrm{I}}+p;$ with which value substituted into the equation there becomes

$$\begin{aligned}(Z-Y)p=&-\frac{YdP^{\mathrm{I}}}{dx}+\frac{YddP^{\mathrm{I}}}{2dx^2}-\text{etc.}\\ &-\frac{Ydp}{dx}+\frac{Yddp}{2dx^2}-\text{etc.};\end{aligned}$$

and to each side there may be added YP^{I}, and since $P^{\mathrm{I}}-\frac{dP^{\mathrm{I}}}{dx}+\frac{ddP^{\mathrm{I}}}{2dx^2}-\text{etc.}$ shall be the value of P^{I}, which is produced, if in place of x there may be put $x-1$, let this value $=P$ and there shall be

$$(Z-Y)p+YP^{\mathrm{I}}=YP-\tfrac{Ydp}{dx}+\tfrac{Yddp}{2dx^2}-\text{etc.},$$

from which with the differentials ignored there will be

$$p=\tfrac{Y\left(P-P^{\mathrm{I}}\right)}{Z-Y}.$$

There may be put $\tfrac{Y\left(P-P^{\mathrm{I}}\right)}{Z-Y}=Q^{\mathrm{I}}$ and there shall be $p=Q^{\mathrm{I}}+q$; there may become

$$(Z-Y)q=-\tfrac{Y\left(dQ^{\mathrm{I}}+dq\right)}{dx}+\tfrac{Y\left(ddQ^{\mathrm{I}}+ddq\right)}{2dx^2}-\text{etc.}$$

and on putting Q for the value Q^{I}, which it adopts, if in place of x there may be written $x-1$, [and adding YQ^{I} to each side as above] there will be

$$(Z-Y)q+YQ^{\mathrm{I}}=YQ-\tfrac{Ydq}{dx}+\tfrac{Yddq}{2dx^2}-\text{etc.},$$

from which with the differentials ignored there becomes

$$q=\tfrac{Y\left(Q-Q^{\mathrm{I}}\right)}{Z-Y}.$$

There may be put $\tfrac{Y\left(Q-Q^{\mathrm{I}}\right)}{Z-Y}=R^{\mathrm{I}}$ and there shall be actually $q=R^{\mathrm{I}}+r$ and in a similar manner it is found that

$$r=\tfrac{Y\left(R-R^{\mathrm{I}}\right)}{Z-Y};$$

and by proceeding thus the sum sought will be

$$Z\mathrm{s}=Z\left(P^{\mathrm{I}}+Q^{\mathrm{I}}+R^{\mathrm{I}}+\text{etc.}\right).$$

190. Therefore for any series proposed

$$Aa+Bb+Cc+\cdots+Yy+Zz$$

the sum of this may be defined in the following manner :

Putting,	and on putting $x-1$ in place of x
$\frac{Zz}{Z-Y} = P^{\mathrm{I}}$,	P^{I} is changed into P
$\frac{Y(P-P^{\mathrm{I}})}{Z-Y} = Q^{\mathrm{I}}$,	Q^{I} " " " " " Q
$\frac{Y(Q-Q^{\mathrm{I}})}{Z-Y} = R^{\mathrm{I}}$,	R^{I} " " " " " R
$\frac{Y(R-R^{\mathrm{I}})}{Z-Y} = S^{\mathrm{I}}$,	S^{I} " " " " " S

etc.

From these values found, the sum of the series will be

$$= ZP^{\mathrm{I}} + ZQ^{\mathrm{I}} + ZR^{\mathrm{I}} + ZS^{\mathrm{I}} + \text{etc.} + \text{Const.},$$

which may return the sum $=0$, if there may be put $x=0$, or which returns with the same, which may be effected, so that a certain case may be satisfied.

191. This formula, because it is not involved with any differentials, may be employed readily in most cases and also the true sum may be shown repeatedly.
Thus if this series is proposed

$$p + 4p^2 + 9p^3 + 16p^3 + \cdots + x^2 p^x,$$

there may be made $Z = p^x$ and $z = x^2$; there will be $Y = p^{x-1}$ and

$$\frac{Z}{Z-Y} = \frac{p}{p-1} \text{ and } \frac{Y}{Z-Y} = \frac{1}{p-1}.$$

Hence there arises

$$P^{\mathrm{I}} = \frac{px^2}{p-1} \qquad P = \frac{pxx-2px+p}{p-1}$$

$$Q^{\mathrm{I}} = \frac{-2px+p}{(p-1)^2} \qquad Q = \frac{-2px+3p}{(p-1)^2}$$

$$R^{\mathrm{I}} = \frac{2p}{(p-1)^3} \qquad R = \frac{2p}{(p-1)^3}$$

$$S^{\mathrm{I}} = 0$$

and all the remaining terms vanish ; from which the sum will be

$$= p^x\left(\frac{px^2}{p-1}-\frac{2px-p}{(p-1)^2}+\frac{2p}{(p-1)^3}\right)-\frac{p}{(p-1)^2}-\frac{2p}{(p-1)^3}$$
$$= p^{x+1}\left(\frac{x^2}{p-1}-\frac{2x}{(p-1)^2}+\frac{p+1}{(p-1)^3}\right)-\frac{p(p+1)}{(p-1)^3},$$

just as we have now found above.

192. In a similar manner, so that we may arrive at this expression of the sum, we may be able to arrive at another expression, if the proposed series shall not be composed from two others; which chiefly will be able to be called into use in these cases, when there are vanishing denominators come upon in the preceding expression. Therefore let this series be proposed

$$\begin{matrix} & 1 & & 2 & & 3 & & 4 & & & & x \\ s = & a & + & b & + & c & + & d & + & \cdots & + & z; \end{matrix}$$

because on putting $x-1$ in place of x the sum is cut off at the final term, there will be

$$s-z=s-\frac{ds}{dx}+\frac{dds}{2dx^2}-\frac{d^3s}{6dx^3}+\frac{d^4s}{24dx^4}-\text{etc.}$$

or

$$z=\frac{ds}{dx}-\frac{dds}{2dx^2}+\frac{d^3s}{6dx^3}-\frac{d^4s}{24dx^4}+\text{etc.}$$

[*i.e.* the backwards Taylor expansion generates the same unit displacement, and corresponding change in the sum.]
Because here the sum s has not occurred, the higher differentials may be ignored and there becomes $s=\int zdx$; there may be put $\int zdx = P^{\mathrm{I}}$, the value of which may change into P, if there may be written $x-1$ for x, and in fact there shall be $s = P^{\mathrm{I}} + p$; there will be

$$z=\frac{dP^{\mathrm{I}}}{dx}-\frac{ddP^{\mathrm{I}}}{2dx^2}+\text{etc.}+\frac{dp}{dx}-\frac{ddp}{2dx^2}+\text{ etc.};$$

because there is

$$P=P^{\mathrm{I}}-\frac{dP^{\mathrm{I}}}{dx}+\frac{ddP^{\mathrm{I}}}{2dx^2}-\text{etc.},$$

there will be

$$z-P^{\mathrm{I}}+P=\frac{dp}{dx}-\frac{ddp}{2dx^2}+\text{etc.},$$

from which there becomes

$$p=\int\left(z-P^{\mathrm{I}}+P\right)dx.$$

If again there may be put $\int\left(z-P^{\mathrm{I}}+P\right)dx=Q^{\mathrm{I}}$ and this value will change into Q on putting $x-1$ in place of *x*, there shall be

$$\int\left(z-P^{\mathrm{I}}+P-Q^{\mathrm{I}}+Q\right)dx=R^{\mathrm{I}}=Q^{\mathrm{I}}-\int\left(Q^{\mathrm{I}}-Q\right)dx,$$

again

$$R^{\mathrm{I}}-\int\left(R^{\mathrm{I}}-R\right)dx=S^{\mathrm{I}}$$

etc.; the sum sought will be

$$s=P^{\mathrm{I}}+Q^{\mathrm{I}}+R^{\mathrm{I}}+S^{\mathrm{I}}+\text{etc.}+\text{Const.},$$

which will be satisfied by a single case.

193. With the letters changed a little this summation here is returned. For the proposed series required to be summed

$$\begin{matrix} & 1 & & 2 & & 3 & & 4 & & & & x\\ s= & a & + & b & + & c & + & d & + & \cdots & + & z\end{matrix}$$

there is put	$x-1$ in place of *x*,
$\int zdx=P$	P is changed into p
$P-\int\left(P-p\right)dx=Q$	Q is changed into q
$Q-\int\left(Q-q\right)dx=R$	R is changed into r

etc.;

with which values found the sum sought will be

$$s=P+Q+R+S+etc.$$

and this expression will show the sum expediently, if the formulas are able to be shown to be integrated. Let there be $z=xx+x$, so that we may illustrate the use of this with an example, and there will be

$$P=\tfrac{1}{3}x^3+\tfrac{1}{2}xx,\quad p=\tfrac{1}{3}x^3-\tfrac{1}{2}xx+\tfrac{1}{6}$$

and

$$\int (P-p)dx = \tfrac{1}{3}x^3 - \tfrac{1}{6}x \ ;$$

$$Q = \tfrac{1}{2}xx + \tfrac{1}{6}x, \quad q = \tfrac{1}{2}xx - \tfrac{5}{6}x + \tfrac{1}{3}, \quad Q - q = x - \tfrac{1}{3}$$

and

$$\int (Q-q)dx = \tfrac{1}{2}xx - \tfrac{1}{3}x \ ;$$

$$R = \tfrac{1}{2}x, \quad r = \tfrac{1}{2}x - \tfrac{1}{2}, \quad R - r = \tfrac{1}{2}$$

and

$$\int (R-r)dx = \tfrac{1}{2}x \ ;$$

$S = 0$ and the remaining values vanish. Whereby the sum sought will be

$$\begin{aligned} P &= \tfrac{1}{3}x^3 + \tfrac{1}{2}xx + \tfrac{1}{2}xx + \tfrac{1}{6}x + \tfrac{1}{2}x \\ &= \tfrac{1}{3}x^3 + xx + \tfrac{2}{3}x = \tfrac{1}{3}x(x+1)(x+2). \end{aligned}$$

And therefore in this manner the sums of all the series, of which the general terms are rational integral functions of *x*, are able to be found with the help of continued integrations. From which it is seen easily that several volumes suffice to be taken, as a great space may be occupied by the science concerning the summation of series and nor by all the methods to be had at one time, but also which are able to be devised at some other.

194. Until now we have investigated the sums of series from the first term as far as to that, of which the index is *x,* with which known on putting $x = \infty$ the sum of the series continued to infinity may become known. But many times this can be performed better, if instead of the sum of the terms from the first as that, of which the index is *x* may be sought, but rather the sum of all the terms from that, of which the index is *x,* as far as to infinity may be sought, and in this case in the first place the final expressions become manageable. Therefore let the series be proposed, the general term of which or corresponding to the index *x* shall be $= z$, the following corresponding to the index $x+1$ shall be $= z^{\mathrm{I}}$ and the following beyond this shall be $z^{\mathrm{II}}, z^{\mathrm{III}}$, etc. and the sum of this infinite series may be sought

$$\begin{array}{llllll} & x & x+1 & x+2 & x+3 & \text{etc.} \\ s = & z & + z^{\mathrm{I}} & + z^{\mathrm{II}} & + z^{\mathrm{III}} & + \text{etc.} \quad \text{to infinity.} \end{array}$$

Therefore this sum *s* will be a function of *x;* in which if there may be put $x+1$ in place of *x*, the first sum may arise truncated by the term *z*. Therefore since by this change *s* may be changed into

$$s+\frac{ds}{dx}+\frac{dds}{2dx^2}+\text{etc.},$$

there will be

$$s-z=s+\frac{ds}{dx}+\frac{dds}{2dx^2}+\frac{d^3s}{6dx^3}+\frac{d^4s}{24dx^4}+\frac{d^5s}{120dx^5}+\text{etc.}$$

or

$$0=z+\frac{ds}{dx}+\frac{dds}{2dx^2}+\frac{d^3s}{6dx^3}+\frac{d^4s}{24dx^4}+\frac{d^5s}{120dx^5}+\text{etc.}$$

[Thus, *s* and its derivatives are evaluated now at the point *x* + 1.]

195. If now so that we may put in place the previous reasoning, with the higher derivatives ignored [so that $0=z+\frac{ds}{dx}$], there may become $s=C-\int zdx$. Therefore there may be put $\int zdx=P$ and in fact there becomes $s=C-P+p$*;* there will be

$$\begin{aligned}0=z-\frac{dP}{dx}-\frac{ddP}{2dx^2}-\frac{d^3P}{6dx^3}-\text{etc.}\\+\frac{dp}{dx}+\frac{ddp}{2dx^2}+\frac{d^3p}{6dx^3}+\text{etc.}\end{aligned}$$

P may change into P^{I}, if in place of x there may be put $x+1$, and there will be

$$0=z+P-P^{\text{I}}+\frac{dp}{dx}+\frac{ddp}{2dx^2}+\frac{d^3p}{6dx^3}+\text{etc.}$$

Hence with the higher differentials ignored there becomes $p=\int\left(P^{\text{I}}-P\right)dx-P$. There may be put in place $\int\left(P^{\text{I}}-P\right)dx-P=-Q$ and there shall be $p=-Q+q$*;* there will be

$$0=z+P-P^{\text{I}}-\frac{dQ}{dx}-\frac{ddQ}{2dx^2}-\text{etc.}+\frac{dq}{dx}+\frac{ddq}{2dx^2}+\text{etc.}$$

Q may change into Q^{I}, if in place of x there may be put $x+1$, and there will be

$$0 = z + P - P^{\mathrm{I}} + Q - Q^{\mathrm{I}} + \frac{dq}{dx} + \frac{ddq}{2dx^2} + \text{etc.},$$

from which it follows $q = \int \left(Q^{\mathrm{I}} - Q\right) dx - Q$. On account of which if the division for each fixed quantity may denote the value of this , which it adopts on putting $x+1$ in place of x, and there may be put

$$\int z dx = P$$

$$P - \int \left(P^{\mathrm{I}} - P\right) dx = Q$$

$$Q - \int \left(Q^{\mathrm{I}} - Q\right) dx = R$$

$$R - \int \left(R^{\mathrm{I}} - R\right) dx = S$$

etc.,

the sum of the proposed series will be $z + z^{\mathrm{I}} + z^{\mathrm{II}} + z^{\mathrm{III}} + z^{\mathrm{IV}} +$ etc.

$$= C - P - Q - R - S - \text{ etc.},$$

where the constant C must be defined thus, so that on putting $x = \infty$ the whole sum may vanish. But because the application of this expression will require integration, it may not be possible to indicate the use of that in this place.

196. But in order that we may avoid integral formulas, we may establish the sum of a series $= ys$ with any known function y of x present, the values of which $y^{\mathrm{I}}, y^{\mathrm{II}}$ etc. will be known, which may be produced on putting $x+1, x+2$ etc. in place of x. If now there may be put $x+1$ in place of x, the above series will be produced diminished by the first term, the sum of which therefore will be

$$y^{\mathrm{I}} \left(s + \frac{ds}{dx} + \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} + \text{etc.}\right) = ys - z$$

or

$$z + \frac{y^{\mathrm{I}} ds}{dx} + \frac{y^{\mathrm{I}} dds}{2dx^2} + \frac{y^{\mathrm{I}} d^3 s}{6dx^3} + \text{etc.} = \left(y - y^{\mathrm{I}}\right) s$$

from which with the differentials ignored there arises $s = \frac{z}{y - y^{\mathrm{I}}}$. There may be put in place $\frac{z}{y - y^{\mathrm{I}}} = P$ and there becomes actually $s = -P + p$; there will be

$$\left.\begin{array}{l} -\frac{y^{\mathrm{I}}dP}{dx}-\frac{y^{\mathrm{I}}ddP}{2dx^2}-\frac{y^{\mathrm{I}}d^3P}{6dx^3}-\text{etc.} \\ +\frac{y^{\mathrm{I}}dp}{dx}+\frac{y^{\mathrm{I}}ddp}{2dx^2}+\frac{y^{\mathrm{I}}d^3p}{6dx^3}+\text{etc.} \end{array}\right\} = \left(y-y^{\mathrm{I}}\right)p$$

and thus

$$\frac{y^{\mathrm{I}}dp}{dx}+\frac{y^{\mathrm{I}}ddp}{2dx^2}+\frac{y^{\mathrm{I}}d^3p}{6dx^3}+\text{etc.}=y^{\mathrm{I}}\left(P^{\mathrm{I}}-P\right)-\left(y^{\mathrm{I}}-y\right)p.$$

There may be put in place $Q=\frac{y^{\mathrm{I}}\left(P^{\mathrm{I}}-P\right)}{y^{\mathrm{I}}-y}$ and there shall be $p=Q+q$; there will be

$$y^{\mathrm{I}}\left(Q^{\mathrm{I}}-Q\right)+y^{\mathrm{I}}\left(\frac{dq}{dx}+\frac{ddq}{2dx^2}+\text{etc.}\right)=-\left(y^{\mathrm{I}}-y\right)q.$$

There may be put $R=\frac{y^{\mathrm{I}}\left(Q^{\mathrm{I}}-Q\right)}{y^{\mathrm{I}}-y}$ and there becomes $q=-R+r$.

In this manner, if we may progress further, the sum of the proposed series

$$z+z^{\mathrm{I}}+z^{\mathrm{II}}+z^{\mathrm{III}}+z^{\mathrm{IV}}+\text{etc.}$$

may be found in the following manner. On taking as it pleases some function of x, which shall be $=y$, there may be established

$$P=\frac{z}{y^{\mathrm{I}}-y}=\frac{z}{\Delta y}$$

$$Q=\frac{y^{\mathrm{I}}\left(P^{\mathrm{I}}-P\right)}{y^{\mathrm{I}}-y}=\frac{y\Delta P}{\Delta y}+\Delta P$$

$$R=\frac{y^{\mathrm{I}}\left(Q^{\mathrm{I}}-Q\right)}{y^{\mathrm{I}}-y}=\frac{y\Delta Q}{\Delta y}+\Delta Q$$

$$S=\frac{y^{\mathrm{I}}\left(R^{\mathrm{I}}-R\right)}{y^{\mathrm{I}}-y}=\frac{y\Delta R}{\Delta y}+\Delta R$$

etc.

And hence the sum sought will be

$$=C-Py+Qy-Ry+Sy-\text{etc.}$$

on taking for C a constant of such a kind, so that on putting $x=\infty$ the sum may vanish.

197. There may be taken $y = a^x$; on account of $y^{\mathrm{I}} = a^{x+1}$ there will be $y^{\mathrm{I}} - y = a^x(a-1)$, from which there becomes

$$P = \frac{z}{a^x(a-1)} \qquad P^{\mathrm{I}} = \frac{z^{\mathrm{I}}}{a^{x+1}(a-1)}$$

$$Q = \frac{a\left(P^{\mathrm{I}} - P\right)}{(a-1)} = \frac{z^{\mathrm{I}} - az}{a^x(a-1)^2} \qquad Q^{\mathrm{I}} = \frac{z^{\mathrm{II}} - az^{\mathrm{I}}}{a^{x+1}(a-1)^2}$$

$$R = \frac{a\left(Q^{\mathrm{I}} - Q\right)}{(a-1)} = \frac{z^{\mathrm{II}} - 2az^{\mathrm{I}} + aaz}{a^x(a-1)^3} \qquad R^{\mathrm{I}} = \frac{z^{\mathrm{III}} - 2az^{\mathrm{II}} + aaz^{\mathrm{I}}}{a^{x+1}(a-1)^3}$$

$$S = \frac{a\left(R^{\mathrm{I}} - R\right)}{(a-1)} = \frac{z^{\mathrm{III}} - 3az^{\mathrm{II}} + 3a^2z^{\mathrm{I}} - a^3z}{a^x(a-1)^4}$$

etc.

On account of which the sum of the proposed series will be

$$C - \frac{z}{a-1} + \frac{z^{\mathrm{I}} - az}{(a-1)^2} - \frac{z^{\mathrm{II}} - 2az^{\mathrm{I}} + a^2z}{(a-1)^3} + \frac{z^{\mathrm{III}} - 3az^{\mathrm{II}} + 3a^2z^{\mathrm{I}} - a^3z}{(a-1)^4} - \text{etc.}$$

Truly this same expression for the sum has been found above in the first chapter. But hence with other values to be taken for *y*, infinitely many other expressions will be able to be elicited from this, and which can be selected to be suited especially for each case.

CAPUT VII

METHODUS SUMMANDI SUPERIOR ULTERIUS PROMOTA

167. Ut defectum methodi summandi ante traditae suppleamus, in hoc capite eiusmodi series considerabimus, quarum termini generales magis sint complexi. Cum igitur expressio ante inventa in progressionibus geometricis, etsi aliis methodis facillime summari possunt, veram summa finita formula contentam non praebeat, hic primum eiusmodi series contemplabimur, quarum termini sint producta ex terminis seriei geometricae et alius cuiuscunque. Sit igitur proposita haec series

$$\begin{matrix} & 1 & 2 & 3 & 4 & & x \\ z = & ap + & bp^2 + & cp^3 + & dp^4 + & \cdots + & yp^x, \end{matrix}$$

quae est composita ex geometrica p, p^2, p^3 etc. et alia quacunque serie $a+b+c+d+\text{etc.}$, cuius terminus generalis seu indici x respondens sit $=y$, atque expressionem generalem investigemus pro valore eius summae $s = S.yp^x$.

168. Instituamus ratiocinium, eodem modo, quo supra usi sumus, sitque v terminus antecedens ipsi y in serie $a+b+c+d+\text{etc.}$ atque A praecedens ipsi a seu is, qui indici 0 respondet, eritque vp^{x-1} terminus generalis huius seriei

$$\begin{matrix} & 1 & 2 & 3 & 4 & x \\ A + & ap + & bp^2 + & cp^3 + & \cdots + & vp^{x-1}; \end{matrix}$$

cuius summa si indicetur per $S.vp^{x-1}$, erit

$$S.vp^{x-1} = \tfrac{1}{p} S.vp^x = S.yp^x - yp^x + A.$$

Cum autem sit

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \frac{d^4y}{24dx^4} - \frac{d^5y}{120dx^5} + \text{etc.},$$

erit

$$S.yp^x - yp^x + A = \tfrac{1}{p} S.yp^x - \tfrac{1}{p} S.\frac{dy}{dx} p^x + \tfrac{1}{2p} S.\frac{ddy}{dx^2} p^x$$
$$- \tfrac{1}{6p} S.\frac{d^3y}{dx^3} p^x + \tfrac{1}{24p} S.\frac{d^4y}{dx^4} p^x - \text{etc.}$$

Ex qua fit

$$S.yp^x = \tfrac{1}{p-1}\left(yp^{x+1} - Ap - S.\tfrac{dy}{dx}p^x + S.\tfrac{ddy}{2dx^2}p^x - S.\tfrac{d^3y}{6dx^3}p^x + \text{etc.}\right)$$

Si ergo habeantur termini summatorii serierum, quarum termini generales sunt $\frac{dy}{dx}p^x$, $\frac{ddy}{dx^2}p^x$, $\frac{d^3y}{dx^3}p^x$ etc., ex iis definiri poterit terminus summatorius $s = S.yp^x$.

169. Hinc iam summae inveniri poterunt serierum, quarum termini generales in hac forma $x^n p^x$ continentur. Sit enim $y = x^n$; erit $A = 0$, nisi sit $n = 0$, quo casu foret $A = 1$, et quia est

$$\tfrac{dy}{dx} = nx^{n-1}, \qquad \tfrac{ddy}{2dx^2} = \tfrac{n(n-1)}{1\cdot 2}x^{n-2}, \qquad \tfrac{d^3y}{6dx^3} = \tfrac{n(n-1)(n-3)}{1\cdot 2\cdot 3}x^{n-3} \quad \text{etc.},$$

erit

$$S.x^n p^x = \tfrac{1}{p-1}\left\{\begin{array}{l} x^n p^{x+1} - Ap - nS.x^{n-1}p^x + \tfrac{n(n-1)}{1\cdot 2}S.x^{n-2}p^x - \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}S.x^{n-3}p^x \\ + \tfrac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}S.x^{n-4}p^x - \text{etc.} \end{array}\right\}$$

Ex hac forma nunc successive pro *n* substituendo numeros 0, 1, 2, 3 etc. obtinebuntur sequentes summationes; ac primo quidem, si $n = 0$, fit $A = 1$, in reliquis autem casibus erit $A = 0$.

$$S.x^0 p^x = S.p^x = \tfrac{1}{p-1}\left(p^{x+1} - p\right) = \tfrac{p^{x+1}-p}{p-1} = \tfrac{p\left(p^x-1\right)}{p-1},$$

quae est summa progressionis geometricae cognita;

$$S.xp^x = \tfrac{1}{p-1}\left(xp^{x+1} - S.p^x\right) = \tfrac{xp^{x+1}}{p-1} - \tfrac{p^{x+1}-p}{(p-1)^2}$$

seu

$$S.xp^x = \tfrac{pxp^x}{p-1} - \tfrac{p\left(p^x-1\right)}{(p-1)^2} \;;$$

$$S.x^2 p^x = \tfrac{1}{p-1}\left(x^2 p^{x+1} - 2S.xp^x + S.p^x\right)$$

seu

$$S.x^2 p^x = \tfrac{x^2 p^{x+1}}{p-1} - \tfrac{2xp^{x+1}}{(p-1)^2} + \tfrac{p(p+1)\left(p^x-1\right)}{(p-1)^3}$$

Porro est

$$S.x^3p^x = \tfrac{1}{p-1}\left(x^3p^{x+1} - 3S.x^2p^x + 3S.xp^x - S.p^x\right)$$

seu

$$S.x^3p^x = \frac{x^3p^{x+1}}{p-1} - \frac{3x^2p^{x+1}}{(p-1)^2} + \frac{3(p+1)xp^{x+1}}{(p-1)^3} + \frac{p(pp+4p+1)(p^x-1)}{(p-1)^4}$$

sicque ulterius progrediendo superiorum potestatum x^4p^x, x^5p^x, x^6p^x etc. summae definiri poterunt; hoc vero commodius praestabitur ope expressionis generalis, quam nunc investigabimus.

170. Quoniam invenimus esse

$$S.yp^x = \tfrac{1}{p-1}\left(yp^{x+1} - Ap - S.\tfrac{dy}{dx}p^x + S.\tfrac{ddy}{2dx^2}p^x - S.\tfrac{d^3y}{6dx^3}p^x + \text{etc.}\right),$$

ubi A est eiusmodi constans, ut summa fiat $=0$, si ponatur $x=0$ (namque hoc casu fit $y=A$ et $yp^{x+1}=Ap$), hanc constantem omittere poterimus, dummodo perpetuo meminerimus ad summam quamque semper eiusmodi constantem adiici oportere, ut facto $x=0$ evanescat seu ut alii cuipiam casui satisfiat. Statuamus ergo z loco y eritque

$$S.p^xz = \frac{p^{x+1}z}{p-1} - \frac{1}{p-1}S.p^x\frac{dz}{dx} + \frac{1}{2(p-1)}S.p^x\frac{ddz}{dx^2} - \frac{1}{6(p-1)}S.p^x\frac{d^3z}{dx^3}$$
$$+\frac{1}{24(p-1)}S.p^x\frac{d^4z}{dx^4} - \frac{1}{120(p-1)}S.p^x\frac{d^5z}{dx^5} + \text{etc.}$$

Deinde statuamus successive $\frac{dz}{dx}$, $\frac{ddz}{dx^2}$ $\frac{d^3z}{dx^3}$ etc. in locum y eritque

$$S.\frac{p^xdz}{dx} = \frac{p^{x+1}}{p-1}\frac{dz}{dx} - \frac{1}{p-1}S.\frac{p^xddz}{dx^2} + \frac{1}{2(p-1)}S.\frac{p^xd^3z}{dx^3} - \text{etc.}$$
$$S.\frac{p^xddz}{dx^2} = \frac{p^{x+1}}{p-1}\frac{ddz}{dx^2} - \frac{1}{p-1}S.\frac{p^xd^3z}{dx^3} + \frac{1}{2(p-1)}S.\frac{p^xd^4z}{dx^4} - \text{etc.}$$
$$S.\frac{p^xd^3z}{dx^3} = \frac{p^{x+1}}{p-1}\frac{d^3z}{dx^3} - \frac{1}{p-1}S.\frac{p^xd^4z}{dx^4} + \frac{1}{2(p-1)}S.\frac{p^xd^5z}{dx^5} + \text{etc.}$$
etc.

Si igitur hi valores successive substituantur, $S.p^xz$ huiusmodi forma exprimetur

$$S.p^x z = \frac{p^{x+1}z}{p-1} - \frac{\alpha p^{x+1}}{p-1}\cdot\frac{dz}{dx} + \frac{\beta p^{x+1}}{p-1}\cdot\frac{ddz}{dx^2} - \frac{\gamma p^{x+1}}{p-1}\cdot\frac{d^3z}{dx^3}$$
$$+\frac{\delta p^{x+1}}{p-1}\cdot\frac{d^4z}{dx^4} - \frac{\varepsilon p^{x+1}}{p-1}\cdot\frac{d^5z}{dx^5} + \text{etc.}$$

171. Ad valores litterarum $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$ etc. definiendos substituantur pro quovis termino series ante inventae, nempe

$$\frac{p^{x+1}z}{p-1} = S.p^x z + \frac{1}{p-1}S.\frac{p^x dz}{dx} - \frac{1}{2(p-1)}S.\frac{p^x ddz}{dx^2} + \frac{1}{6(p-1)}S.\frac{p^x d^3z}{dx^3} - \text{etc.}$$
$$\frac{p^{x+1}dz}{(p-1)dx} = S.\frac{p^x dz}{dx} + \frac{1}{p-1}S.\frac{p^x ddz}{dx^2} - \frac{1}{2(p-1)}S.\frac{p^x d^3z}{dx^3} + \text{etc.}$$
$$\frac{p^{x+1}ddz}{(p-1)dx^2} = S.\frac{p^x ddz}{dx^2} + \frac{1}{p-1}S.\frac{p^x d^3z}{dx^3} - \text{etc.}$$
$$\frac{p^{x+1}d^3z}{(p-1)dx^3} = S.\frac{p^x d^3z}{dx^3} + \text{etc.}$$

Habebimus ergo

$$S.p^x z = S.p^x z$$
$$\begin{array}{lllll}
\frac{1}{p-1}S.\frac{p^x dz}{dx} & -\frac{1}{2(p-1)}S.\frac{p^x ddz}{dx^2} & +\frac{1}{6(p-1)}S.\frac{p^x d^3z}{dx^3} & -\frac{1}{24(p-1)}S.\frac{p^x d^4z}{dx^4} & +\text{etc} \\
-\alpha & -\frac{\alpha}{p-1} & +\frac{\alpha}{2(p-1)} & -\frac{\alpha}{6(p-1)} & \\
 & +\beta & +\frac{\beta}{p-1} & -\frac{\beta}{2(p-1)} & \\
 & & -\gamma & -\frac{\gamma}{p-1} & \\
 & & & +\delta &
\end{array}$$

unde coefficientium $\alpha,\ \beta,\ \gamma,\ \delta$ etc. valores sequentes obtinebuntur

$$\alpha = \frac{1}{p-1},\quad \beta = \frac{1}{p-1}\left(\alpha + \frac{1}{2}\right),\quad \gamma = \frac{1}{p-1}\left(\beta + \frac{\alpha}{2} + \frac{1}{6}\right),$$
$$\delta = \frac{1}{p-1}\left(\gamma + \frac{\beta}{2} + \frac{\alpha}{6} + \frac{1}{24}\right),\quad \varepsilon = \frac{1}{p-1}\left(\delta + \frac{\gamma}{2} + \frac{\beta}{6} + \frac{\alpha}{24} + \frac{1}{120}\right)$$
etc.

172. Sit brevitatis gratia $\frac{1}{p-1}=q$; erit

$$\alpha = q$$
$$\beta = \alpha q + \tfrac{1}{2}q = qq + \tfrac{1}{2}q$$
$$\gamma = \beta q + \tfrac{1}{2}\alpha q + \tfrac{1}{6}q = q^3 + qq + \tfrac{1}{6}q$$
$$\delta = \gamma q + \tfrac{1}{2}\beta q + \tfrac{1}{6}\alpha q + \tfrac{1}{24}q = q^4 + \tfrac{3}{2}q^3 + \tfrac{7}{12}q^2 + \tfrac{1}{24}q$$
$$\varepsilon = \delta q + \tfrac{1}{2}\gamma q + \tfrac{1}{6}\beta q + \tfrac{1}{24}\alpha q + \tfrac{1}{120}q = q^5 + 2q^4 + \tfrac{5}{4}q^3 + \tfrac{1}{4}q^2 + \tfrac{1}{120}q$$
$$\zeta = q^6 + \tfrac{5}{2}q^5 + \tfrac{13}{6}q^4 + \tfrac{13}{4}q^3 + \tfrac{31}{360}q^2 + \tfrac{1}{720}q$$
etc.

seu hoc modo exprimantur

$$\alpha = \frac{q}{1}$$
$$\beta = \frac{2qq+q}{1\cdot 2}$$
$$\gamma = \frac{6q^3+6q^2+q}{1\cdot 2\cdot 3}$$
$$\delta = \frac{24q^4+36q^3+14q^2+q}{1\cdot 2\cdot 3\cdot 4}$$
$$\varepsilon = \frac{120q^5+240q^4+150q^3+30q^2+q}{1\cdot 2\cdot 3\cdot 4\cdot 5}$$
$$\zeta = \frac{720q^6+1800q^5+1560q^4+540q^3+62q^2+q}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$$
$$\eta = \frac{5040q^7+15120q^6+16800q^5+8400q^4+1806q^3+126q^2+q}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$$
etc.

ubi quilibet coefficiens 16800 oritur, si summa binorum superiorum $1560+1800$ per exponentem ipsius q, qui hic est 5, multiplicetur.

173. Restituamus autem loco q valorem $\frac{1}{p-1}$:

$$\alpha = \frac{1}{1(p-1)}$$

$$\beta = \frac{p+1}{1\cdot 2(p-1)^2}$$

$$\gamma = \frac{pp+4p+1}{1\cdot 2\cdot 3(p-1)^3}$$

$$\delta = \frac{p^3+11p^2+11p+1}{1\cdot 2\cdot 3\cdot 4(p-1)^4}$$

$$\varepsilon = \frac{p^4+26p^3+66p^2+26p+1}{1\cdot 2\cdot 3\cdot 4\cdot 5(p-1)^5}$$

$$\zeta = \frac{p^5+57p^4+302p^3+302p^2+57p+1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6(p-1)^6}$$

$$\eta = \frac{p^6+120p^5+1191p^4+2416p^3+1191p^2+120p+1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7(p-1)^7}$$

etc.

Lex harum quantitatum ita se habet, ut, si ponatur terminus quicunque

$$= \frac{p^{n-2}+Ap^{n-3}+Bp^{n-4}+Cp^{n-5}+Dp^{n-6}+\text{etc.}}{1\cdot 2\cdot 3\cdots(n-1)(p-1)^{n-1}}$$

futurum sit

$$A = 2^{n-1} - n$$

$$B = 3^{n-1} - n\cdot 2^{n-1} + \frac{n(n-1)}{1\cdot 2}$$

$$C = 4^{n-1} - n\cdot 3^{n-1} + \frac{n(n-1)}{1\cdot 2}2^{n-1} - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$$

$$D = 5^{n-1} - n\cdot 4^{n-1} + \frac{n(n-1)}{1\cdot 2}3^{n-1} - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}2^{n-1} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}$$

etc.,

unde isti coefficientes α, β, γ, δ etc., quousque libuerit, continuari possunt.

174. Quodsi vero legem, qua hi coefficientes inter se cohaerent, consideremus, facile patet eos seriem recurrentem constituere atque prodire, si haec fractio evolvatur

$$\frac{1}{1-\frac{u}{p-1}-\frac{u^2}{2(p-1)}-\frac{u^3}{6(p-1)}-\frac{u^4}{24(p-1)}-\text{etc.}};$$

prodibit enim haec series

$$1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\zeta u^6+\text{etc.}$$

Ponatur illa fractio $=V$, et cum sit

$$V=\frac{p-1}{p-1-u-\frac{u^2}{2}-\frac{u^3}{6}-\frac{u^4}{24}-\text{etc.}}$$

erit

$$V=\frac{p-1}{p-\mathrm{e}^{-u}}$$

ubi *e* est numerus, cuius logarithmus hyperbolicus est $=1$. Atque si valor ipsius *V* per seriem exprimatur secundum potestates ipsius *u*, orietur

$$V=1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\zeta u^6+\text{etc.}$$

cuius coefficientes α, β, γ, δ etc. erunt ii ipsi, quorum in praesenti negotio opus habemus. Iis igitur inventis erit

$$S.p^x z=\frac{p^{x+1}}{p-1}\left(z-\frac{\alpha dz}{dx}+\frac{\beta ddz}{dx^2}-\frac{\gamma d^3z}{dx^3}+\frac{\delta d^4z}{dx^4}-\text{etc.}\right)\pm\text{Const.},$$

quae ergo expressio est terminus summatorius seriei huius

$$ap+bp^2+cp^3+\cdots+p^x z,$$

cuius terminus generalis est $=p^x z$.

175. Quoniam invenimus esse $V=\frac{p-1}{p-\mathrm{e}^{-u}}$ erit

$$\mathrm{e}^u=\frac{pV-p+1}{V}$$

et logarithmis sumendis fiet

$$u=l\left(pV-p+1\right)-lV$$

hincque differentiando

$$du=\frac{(p-1)dV}{pV^2-(p-1)V}$$

quocirca erit

$$pV^2=\left(p-1\right)V+\frac{(p-1)dV}{du}.$$

Quoniam ergo est

$$V = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \zeta u^6 + \text{etc.}$$

erit

$$\begin{aligned} pV^2 = p + 2\alpha pu + 2\beta pu^2 + \quad & 2\gamma pu^3 + \quad 2\delta pu^4 + \quad 2\varepsilon pu^5 + \text{etc.} \\ \alpha^2 pu^2 \ & + 2\alpha\beta pu^3 + 2\alpha\gamma pu^4 + 2\alpha\delta pu^5 + \text{etc.} \\ & + \beta\beta pu^4 + 2\beta\gamma pu^5 + \text{etc.} \end{aligned}$$

$$\begin{aligned} (p-1)V &= (p-1) + \alpha(p-1)u + \beta(p-1)u^2 \\ &\quad + \gamma(p-1)u^3 + \delta(p-1)u^4 + \varepsilon(p-1)u^5 + \text{etc.} \\ \frac{(p-1)dV}{du} &= (p-1)\alpha + 2(p-1)\beta u + 3(p-1)\gamma u^2 + 4(p-1)\delta u^3 \\ &\quad + 5(p-1)\varepsilon u^4 + 6(p-1)\zeta u^5 + \text{etc.}, \end{aligned}$$

quibus expressionibus inter se coaequatis reperietur

$$\begin{aligned} &(p-1)\alpha = 1 \\ &2(p-1)\beta = \alpha(p+1) \\ &3(p-1)\gamma = \beta(p+1) + \alpha^2 p \\ &4(p-1)\delta = \gamma(p+1) + 2\alpha\beta p \\ &5(p-1)\varepsilon = \delta(p+1) + 2a\gamma p + \beta\beta p \\ &6(p-1)\zeta = \varepsilon(p+1) + 2a\delta p + 2\beta\gamma p \\ &7(p-1)\eta = \zeta(p+1) + 2\alpha\varepsilon p + 2\beta\delta p + \gamma\gamma p \end{aligned}$$

etc.,

ex quibus formulis, si pro p datus numerus assumatur, valores coefficientium α, β, γ, δ etc. facilius determinari possunt quam ex lege primum inventa.

176. Antequam ad casus speciales ratione valoris ipsius p descendamus, ponamus esse $z = x^n$, ita ut haec series summari debeat

$$s = p + 2^n p^2 + 3^n p^3 + 4^n p^4 + \cdots + x^n p^x,$$

eritque per expressionem ante inventam

$$s = p^x \left(\begin{array}{l} \frac{p}{p-1} \cdot x^n - \frac{p}{(p-1)^2} \cdot nx^{n-1} + \frac{pp+p}{(p-1)^3} \cdot \frac{n(n-1)}{1 \cdot 2} x^{n-2} \\ - \frac{p^3+4p^2+p}{(p-1)^4} \cdot \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^{n-3} + \text{etc.} \end{array} \right) \pm \text{C},$$

quae reddat $s = 0$, si ponatur $x = 0$.

Hinc ponendo pro *n* successive numeros 0, 1, 2, 3, 4 etc. erit

$$S.x^0 p^x = p^x \cdot \frac{p}{p-1} - \frac{p}{p-1}$$

$$S.x^1 p^x = p^x \left(\frac{px}{p-1} - \frac{p}{(p-1)^2} \right) + \frac{p}{(p-1)^2}$$

$$S.x^2 p^x = p^x \left(\frac{px^2}{p-1} - \frac{2px}{(p-1)^2} + \frac{p(p+1)}{(p-1)^3} \right) - \frac{p(p+1)}{(p-1)^3}$$

$$S.x^3 p^x = p^x \left(\frac{px^3}{p-1} - \frac{3px^2}{(p-1)^2} + \frac{3p(p+1)x}{(p-1)^3} - \frac{p(p^2+4p+1)}{(p-1)^4} \right) + \frac{p(p^2+4p+1)}{(p-1)^4}$$

$$S.x^4 p^x = p^x \left(\frac{px^4}{p-1} - \frac{4px^3}{(p-1)^2} + \frac{6p(p+1)x^2}{(p-1)^3} - \frac{4p(p^2+4p+1)x}{(p-1)^4} + \frac{p(p^3+11p^2+11p+1)}{(p-1)^5} \right) - \frac{p(p^3+11p^2+11p+1)}{(p-1)^5}$$

$$S.x^5 p^x = \frac{p^{x+1}x^5}{p-1} - \frac{5p^{x+1}x^4}{(p-1)^2} + \frac{10(p+1)p^{x+1}x^3}{(p-1)^3} - \frac{10(p^2+4p+1)p^{x+1}x^2}{(p-1)^4}$$
$$+ \frac{5(p^3+11p^2+11p+1)p^{x+1}x}{(p-1)^5} - \frac{(p^4+26p^3+66p^2+26p+1)(p^{x+1}-p)}{(p-1)^6}$$

$$S.x^6 p^x = \frac{p^{x+1}x^6}{p-1} - \frac{6p^{x+1}x^5}{(p-1)^2} + \frac{15(p+1)p^{x+1}x^4}{(p-1)^3} - \frac{20(p^2+4p+1)p^{x+1}x^3}{(p-1)^4}$$
$$+ \frac{15(p^3+11p^2+11p+1)p^{x+1}x^2}{(p-1)^5} - \frac{6(p^4+26p^3+66p^2+26p+1)p^{x+1}x}{(p-1)^6}$$
$$+ \frac{(p^5+57p^4+302p^3+302p^2+57p+1)(p^{x+1}-p)x}{(p-1)^7}$$

etc.

177. Hinc intelligitur, quoties *z* fuerit functio rationalis integra ipsius *x*, toties seriei, cuius terminus generalis est $p^x z$, summam exhiberi posse, propterea quod differentialia ipsius *z* sumendo tandem ad evanescentia perveniatur. Ita si proponatur haec series

$$p + 3p^2 + 6p^3 + 10p^4 + \cdots + \frac{xx+x}{2} p^x,$$

ob

$$z = \frac{xx+x}{2} \quad \text{et} \quad \frac{dz}{dx} = x + \frac{1}{2} \quad \text{atque} \quad \frac{ddz}{dx^2} = 1$$

erit terminus summatorius

$$s = \frac{p^{x+1}}{p-1}\left(\frac{1}{2}xx + \frac{1}{2}x - \frac{2x+1}{2(p-1)} + \frac{p+1}{2(p-1)^2}\right) - \frac{p}{p-1}\left(\frac{p+1}{2(p-1)^2} - \frac{1}{2(p-1)}\right)$$

seu

$$s = p^{x+1}\left(\frac{xx}{2(p-1)} + \frac{(p-3)x}{2(p-1)^2} + \frac{1}{(p-1)^3}\right) - \frac{p}{(p-1)^3}.$$

Sin autem z fuerit functio non rationalis integra, tum ista termini summatorii expressio in infinitum excurret. Ita si sit $z = \frac{1}{x}$, ut summanda sit haec series

$$s = p + \frac{1}{2}p^2 + \frac{1}{3}p^3 + \frac{1}{4}p^4 + \cdots + \frac{1}{x}p^x,$$

ob

$$\frac{dz}{dx} = -\frac{1}{xx}, \quad \frac{ddz}{dx^2} = \frac{2}{x^3}, \quad \frac{d^3z}{dx^3} = -\frac{2\cdot 3}{x^4}, \quad \frac{d^4z}{dx^4} = -\frac{2\cdot 3\cdot 4}{x^5} \quad \text{etc.}$$

prodibit terminus summatorius

$$s = \frac{p^{x+1}}{p-1}\left(\frac{1}{x} + \frac{1}{(p-1)x^2} + \frac{p+1}{(p-1)^2x^3} + \frac{pp+4p+1}{(p-1)^3x^4} + \frac{p^3+11p^2+11p+1}{(p-1)^4x^5} + \text{etc.}\right) + C.$$

Hoc ergo casu constans C non ex casu $x = 0$ definiri potest; ad eam igitur definiendam ponatur $x = 1$, et quia fit $s = p$, erit

$$C = p - \frac{pp}{p-1}\left(1 + \frac{1}{p-1} + \frac{p+1}{(p-1)^2} + \frac{pp+4p+1}{(p-1)^3} + \text{etc.}\right).$$

178. It is apparent from these, unless p determinatum numerum significet, parum utilitatis hinc ad summas serierum proxime exhibendas redundare. Primum autem patet pro p non posse scribi 1, propterea quod omnes coefficientes $\alpha, \beta, \gamma, \delta$ etc. fierent infinite magni. Quare, cum series, quam nunc tractamus, abeat in eam, quam ante iam sumus contemplati, si ponatur $p = 1$, mirum est, quod ille casus tanquam facillimus ex hoc erui nequeat. Tum vero quoque notabile est, quod casu $p = 1$ summatio requirat integrale $\int zdx$, cum tamen generaliter summa sine ullo integrali exhiberi queat. Sic igitur fit, ut, dum omnes coefficientes $\alpha, \beta, \gamma, \delta$ etc. in infinitum excrescunt, simul formula illa integralis invehatur. Hicque adeo casus, quo $p=1$, est solus, ad quem generalis expressio hic inventa applicari nequeat. Neque vero hoc casu generalis forma a vero recedere censenda est; nam etsi singuli termini fiunt infiniti, tamen revera omnia infinita se destruunt restatque quantitas finita summae aequalis et congruens cum ea, quae per priorem methodum invenitur, quod infra fusius sumus declaraturi.

179. Sit igitur $p=-1$ atque signa in serie summanda alternatim se excipient

$$\overset{1}{-a}+\overset{2}{b}-\overset{3}{c}+\overset{4}{d}-\cdots\pm\overset{x}{z},$$

ubi *z* erit affirmativum, si *x* fuerit numerus par, negativum autem, si *x* sit numerus impar. Posito ergo

$$-a+b-c+d-\cdots\pm z=s$$

erit

$$s=\tfrac{\pm 1}{2}\left(z-\tfrac{\alpha dz}{dx}+\tfrac{\beta ddz}{dx^2}-\tfrac{\gamma d^3z}{dx^3}+\tfrac{\delta d^4z}{dx^4}-\text{etc.}\right)+C,$$

ubi signorum ambiguorum superius valet, si *x* sit numerus par, contra vero, si *x* sit numerus impar. Mutandis ergo signis erit

$$a-b+c-d+\cdots\mp z=s=\mp\tfrac{1}{2}\left(z-\tfrac{\alpha dz}{dx}+\tfrac{\beta ddz}{dx^2}-\tfrac{\gamma d^3z}{dx^3}+\tfrac{\delta d^4z}{dx^4}-\text{etc.}\right)+C,$$

ubi signorum ambiguitas eandem sequitur legem.

180. Hoc casu coefficientes $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ etc. inveniri possunt ex valoribus ante traditis ponendo ubique $p=-1$. Facilius autem eruentur ex formulis generalibus § 175 datis, ex quibus simul perspicietur alternos istos coefficientes evanescere. Facto enim $p=-1$ istae formulae abibunt in

$$-2\alpha=1,\quad -4\beta=0,\quad -6\gamma=0-\alpha^2,\ -8\delta=0-2\alpha\beta,$$
$$-10\varepsilon=0-2\alpha\gamma-\beta\beta,\quad -12\zeta=0-2\alpha\delta-2\beta\gamma\ \text{etc.};$$

unde, cum sit $\beta=0$, erit quoque $\delta=0$ porroque $\zeta=0,\ \theta=0$ etc. et reliquae litterae ita determinabuntur, ut sit

$$\alpha=-\tfrac{1}{2},\quad \gamma=\tfrac{\alpha^2}{6},\quad \varepsilon=\tfrac{2\alpha\gamma}{10},\quad \eta=\tfrac{2\alpha\varepsilon+\gamma\gamma}{14},\quad \iota=\tfrac{2\alpha\eta+2\gamma\varepsilon}{18}\ \text{etc.}$$

181. Quo iste calculus commodius absolvi possit, introducamus novas litteras sitque

$$\alpha = -\frac{A}{1\cdot 2}, \quad \gamma = \frac{B}{1\cdot 2\cdot 3\cdot 4}, \quad \varepsilon = -\frac{C}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6},$$
$$\eta = \frac{D}{1\cdot 2\cdot 3\cdots 8}, \quad \iota = -\frac{E}{1\cdot 2\cdot 3\cdots 10}$$
$$\text{etc.}$$

Eritque summa ante exhibita

$$\mp\frac{1}{2}\left(z + \frac{Adz}{1\cdot 2dx} - \frac{Bd^3z}{1\cdot 2\cdot 3\cdot 4dx^3} + \frac{Cd^5z}{1\cdot 2\cdot 3\cdots 6dx^5} - \frac{Dd^7z}{1\cdot 2\cdot 3\cdots 8dx^7} + \text{etc.}\right) + C.$$

Coefficientes vero ex sequentibus formulis definientur

$$A = 1$$
$$3B = \frac{4\cdot 3}{1\cdot 2}\cdot\frac{AA}{2}$$
$$5C = \frac{6\cdot 5}{1\cdot 2}\cdot AB$$
$$7D = \frac{8\cdot 7}{1\cdot 2}\cdot AC + \frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{BB}{2}$$
$$9E = \frac{10\cdot 9}{1\cdot 2}\cdot AD + \frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4}\cdot BC$$
$$11F = \frac{12\cdot 11}{1\cdot 2}\cdot AE + \frac{12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4}\cdot BD + \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot\frac{CC}{2}$$
$$\text{etc.},$$

quae hoc modo facilius atque ad calculum accommodatius repraesentari possunt

$$A = 1$$
$$B = 2\cdot\frac{AA}{2}$$
$$C = 3\cdot AB$$
$$D = 4\cdot AC + 4\cdot\frac{6\cdot 5}{3\cdot 4}\cdot\frac{BB}{2}$$
$$E = 5\cdot AD + 5\cdot\frac{8\cdot 7}{3\cdot 4}\cdot BC$$
$$F = 6\cdot AE + 6\cdot\frac{10\cdot 9}{3\cdot 4}\cdot BD + 6\cdot\frac{10\cdot 9\cdot 8\cdot 7}{3\cdot 4\cdot 5\cdot 6}\cdot\frac{CC}{2}$$
$$G = 7\cdot AE + 7\cdot\frac{12\cdot 11}{3\cdot 4}\cdot BE + 7\cdot\frac{12\cdot 11\cdot 10\cdot 9}{3\cdot 4\cdot 5\cdot 6}\cdot CD$$
$$\text{etc.}$$

Hinc igitur calculo instituto reperietur

$$A = 1$$
$$B = 1$$
$$C = 3$$
$$D = 17$$
$$E = 155 = 5 \cdot 31$$
$$F = 2073 = 691 \cdot 3$$
$$G = 38227 = 7 \cdot 5461 = 7 \cdot \tfrac{127 \cdot 129}{3}$$
$$H = 929569 = 3617 \cdot 257$$
$$I = 28820619 = 43867 \cdot 9 \cdot 73$$
etc.

182. Si hos numeros attentius perpendamus, ex factoribus 691, 3617, 43867 facile concludere licet hos numeros cum supra exhibitis BERNOULLIANIS nexum habere indeque determinari posse. Hanc igitur relationem investiganti mox patebit hos numeros ex BERNOULLIANIS $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E},$ etc. sequenti modo formari posse:

$$A = 2 \cdot 1 \cdot 3\, \mathfrak{A} = 2\left(2^2 - 1\right)\mathfrak{A}$$
$$B = 2 \cdot 3 \cdot 5\, \mathfrak{B} = 2\left(2^4 - 1\right)\mathfrak{B}$$
$$C = 2 \cdot 7 \cdot 9\, \mathfrak{C} = 2\left(2^6 - 1\right)\mathfrak{C}$$
$$D = 2 \cdot 15 \cdot 17\, \mathfrak{D} = 2\left(2^8 - 1\right)\mathfrak{D}$$
$$E = 2 \cdot 31 \cdot 33\, \mathfrak{E} = 2\left(2^{10} - 1\right)\mathfrak{D}$$
$$F = 2 \cdot 63 \cdot 65\, \mathfrak{F} = 2\left(2^{12} - 1\right)\mathfrak{F}$$
$$G = 2 \cdot 127 \cdot 129\, \mathfrak{G} = 2\left(2^{14} - 1\right)\mathfrak{G}$$
$$H = 2 \cdot 255 \cdot 257\, \mathfrak{H} = 2\left(2^{16} - 1\right)\mathfrak{H}$$
etc.

Cum igitur numeri BERNOULLIANI sint fracti, coefficientes vero nostri integri, patet hos factores semper tollere fractiones eruntque ergo

$$A = 1$$
$$B = 1$$
$$C = 3$$
$$D = 17$$
$$E = 5 \cdot 31 = 155$$
$$F = 3 \cdot 691 = 2073$$
$$G = 7 \cdot 5461 = 38227$$
$$H = 257 \cdot 3617 = 929569$$
$$I = 9 \cdot 73 \cdot 43867 = 28820619$$
$$K = 5 \cdot 31 \cdot 41 \cdot 174611 = 1109652905$$
$$L = 89 \cdot 683 \cdot 854513 = 51943281731$$
$$M = 3 \cdot 4097 \cdot 236364091 = 2905151042481$$
$$N = 2731 \cdot 8191 \cdot 8553103 = 191329672483963$$
etc.

Ex his ergo numeris integris vicissim Numeri BERNOULLIANI inveniri poterunt.

183. Adhibendo igitur numeros BERNOULLIANOS seriei propositae

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & & x \\ a & -b & +c & -d & +e & -\cdots & \mp z \end{matrix}$$

summa erit

$$\mp\left(\frac{1}{2}z + \frac{(2^2-1)\mathfrak{A}dz}{1\cdot 2dx} - \frac{(2^4-1)\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3} + \frac{(2^6-1)\mathfrak{C}d^5z}{1\cdot 2\cdots 6dx^5} - \frac{(2^8-1)\mathfrak{D}d^7z}{1\cdot 2\cdots 8dx^7} + \text{etc.}\right) + \text{Const.}$$

Hinc autem perspicitur istos numeros non casu in hanc expressionem ingredi; quemadmodum enim series proposita oritur, si ab ista

$$a + b + c + d + \cdots + z\,,$$

ubi omnes termini signum habent +, subtrahatur summa alternorum $b + d + f + \text{etc.}$ bis sumta, ita quoque expressio inventa in duas resolvi potest partes, quarum altera est summa omnium terminorum signo + affectorum, quae erit

$$\int zdx+\frac{1}{2}z+\frac{\mathfrak{A}dz}{1\cdot 2dx}-\frac{\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3}+\frac{\mathfrak{C}d^5z}{1\cdot 2\cdot 3\cdots 6dx^5}-\text{etc.}$$

Summa vero alternorum pari modo invenietur, quo supra usi sumus. Cum enim ultimus terminus sit z indici x respondens, antecedens indici $x-2$ respondens erit

$$z-\frac{2dz}{dx}+\frac{2^2ddz}{1\cdot 2dx^2}-\frac{2^3d^3z}{1\cdot 2\cdot 3dx^3}+\frac{2^4d^4z}{1\cdot 2\cdot 3\cdot 4dx^4}-\text{etc.},$$

quae forma ex illa, qua ante terminus antecedens exprimebatur, oritur, si loco x scribatur $\frac{x}{2}$. Habebitur ergo summa alternorum, si in summa omnium ubique loco x scribatur $\frac{x}{2}$, quae propterea erit

$$\frac{1}{2}\int zdx+\frac{1}{2}z+\frac{2\mathfrak{A}dz}{1\cdot 2dx}-\frac{2^3\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3}+\frac{2^5\mathfrak{C}d^5z}{1\cdot 2\cdot 3\cdots 6dx^5}-\text{etc.};$$

cuius duplum si a summa praecedente subtrahatur existente x numero pari, vel si praecedens summa a duplo huius, si x est numerus impar, subtrahatur, residuum ostendet summam seriei

$$\begin{array}{l} 1 \quad 2 \quad 3 \quad 4 \quad 5 \qquad\quad x \\ a-b+c-d+e-\cdots\mp z, \end{array}$$

quae ergo erit

$$\mp\left(\frac{1}{2}z+\frac{(2^2-1)\mathfrak{A}dz}{1\cdot 2dx}-\frac{(2^4-1)\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3}+\text{etc.}\right)+C,$$

quae est eadem expressio, quam modo inveneramus.

184. Sumatur pro z potestas ipsius x, nempe x^n, ut reperiatur summa seriei

$$1-2^n+3^n-4^n+\cdots\mp x^n.$$

Ob

$$\frac{dz}{1dx}=\frac{n}{1}x^{n-1},\quad \frac{d^3z}{1\cdot 2\cdot 3dx^3}=\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\quad \text{etc.}$$

erit adhibendis coefficientibus A, B, C, D etc. summa quaesita

$$\mp\frac{1}{2}\left\{\begin{array}{l} x^n+\frac{A}{2}nx^{n-1}-\frac{B}{4}\cdot\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}+\frac{C}{6}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^{n-5} \\ -\frac{D}{8}\cdot\frac{n(n-1)\cdots(n-6)}{1\cdot 2\cdots 7}x^{n-7}+\text{etc.}+\text{Const.} \end{array}\right\}$$

ubi signum superius valet, si sit x numerus par, inferius vero, si impar. Constans autem ita definiri debet, ut summa evanescat, si $x = 0$, quo casu signum superius valet. Pro n ergo successive numeros 0, 1, 2, 3 etc. substituendo sequentes prodibunt summationes:

$$\text{I.}\quad 1-1+1-1+\cdots\mp 1 = \mp\ \tfrac{1}{2}(1)+\tfrac{1}{2};$$

scilicet si numerus terminorum fuerit par, summa erit $= 0$, sin impar, erit $= +1$.

$$\text{II.}\quad 1-2+3-4+\cdots\mp x = \mp\ \tfrac{1}{2}\left(x+\tfrac{1}{2}\right)+\tfrac{1}{4}$$

scilicet si numerus terminorum sit par, summa erit $= -\frac{1}{2}x$ et pro numero terminorum impari $= +\frac{1}{2}x+\frac{1}{2}$.

$$\text{III.}\quad 1-2^2+3^2-4^2+\cdots\mp x^2 = \mp\ \tfrac{1}{2}\left(x^2+x\right);$$

scilicet pro pari numero $= -\frac{1}{2}xx-\frac{1}{2}x$ et pro impari numero $= +\frac{1}{2}xx+\frac{1}{2}x$.

$$\text{IV.}\quad 1-2^3+3^3-4^3+\cdots\mp x^3 = \mp\ \tfrac{1}{2}\left(x^3+\tfrac{3}{2}xx-\tfrac{1}{4}\right)-\tfrac{1}{8}$$

scilicet pari $= -\frac{1}{2}x^3-\frac{3}{4}x^2$ et pro impari $= \frac{1}{2}x^3+\frac{3}{4}x^2-\frac{1}{4}$.

$$\text{V.}\quad 1-2^4+3^4-4^4+\cdots\mp x^4 = \mp\ \tfrac{1}{2}\left(x^4+2x^3-x\right)$$

scilicet pro numero pari $= -\frac{1}{2}x^4-x^3+\frac{1}{2}x$ et pro numero impari $= \frac{1}{2}x^4+x^3-\frac{1}{2}x$ etc.

185. Apparet ergo in potestatibus paribus praeter $n = 0$ constantem adiiciendam evanescere hisque casibus summam terminorum numero sive parium sive imparium tantum ratione signi discrepare. Quodsi ergo x fuerit numerus infinitus, quoniam is est neque par neque impar, haec consideratio cessare debet ac propterea in summa termini ambigui sunt reiiciendi; unde sequitur huiusmodi serierum in infinitum continuatarum summam exprimi per solam quantitatem constantem adiiciendam.

Hanc ob rem erit

$$1-1\quad+1\;-1\;\;+\;\text{etc. infinitum in} = \tfrac{1}{2}$$
$$1-2\;\;+3\;-4\;\;+\text{etc.}\;\cdot\;\cdot\;\cdot\qquad = \tfrac{A}{4} = +\frac{(2^2-1)\mathfrak{A}}{2}$$
$$1-2^2+3^2-4^2+\text{etc.}\;\cdot\;\cdot\;\cdot\qquad = \;0$$
$$1-2^3+3^3-4^3+\text{etc.}\;\cdot\;\cdot\;\cdot\qquad = -\tfrac{B}{8} = -\frac{(2^4-1)\mathfrak{B}}{4}$$
$$1-2^4+3^4-4^4+\text{etc.}\;\cdot\;\cdot\;\cdot\qquad = \;0$$
$$1-2^5+3^5-4^5+\text{etc.}\;\cdot\;\cdot\;\cdot\qquad = \;\tfrac{C}{12} = +\frac{(2^6-1)\mathfrak{C}}{6}$$
$$1-2^6+3^6-4^6+\text{etc.}\;\cdot\;\cdot\;\cdot\qquad = \;0$$
$$1-2^7+3^7-4^7+\text{etc.}\;\cdot\;\cdot\;\cdot\qquad = -\tfrac{D}{16} = -\frac{(2^8-1)\mathfrak{D}}{8}$$
etc.

Quae eaedem summae per methodum supra traditam series, in quibus signa + et – alternantur, summandi inveniuntur.

186. Si pro n statuantur numeri negativi, expressio summae in infinitum excurret. Sit $n=-1$; erit summa seriei

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\cdots\mp\tfrac{1}{x}$$
$$=\mp\left(\tfrac{1}{x}-\tfrac{A}{2x^2}+\tfrac{B}{4x^4}-\tfrac{C}{6x^6}+\tfrac{D}{8x^8}-\text{etc.}\right)+\text{Const.}$$

Hic autem quia constans non ex casu $x=0$ definiri potest, ex alia casu erit definienda. Ponatur $x=1$ atque ob summam $=1$ et signum inferius erit

$$\text{Const.}=1-\tfrac{1}{2}\left(\tfrac{1}{1}-\tfrac{A}{2}+\tfrac{B}{4}-\tfrac{C}{6}+\text{etc.}\right)$$

seu

$$\text{Const.}=\tfrac{1}{2}+\tfrac{A}{4}-\tfrac{B}{8}+\tfrac{C}{12}-\tfrac{D}{16}+\text{ etc.}$$

Vel ponatur $x=2$; ob summam $=\tfrac{1}{2}$ et signum superius reperietur

$$\text{Const.}=\tfrac{1}{2}+\tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{A}{2\cdot 2^2}+\tfrac{B}{4\cdot 2^4}-\tfrac{C}{6\cdot 2^6}+\text{etc.}\right)$$

seu

$$\text{Const.} = \frac{3}{4} - \frac{A}{4\cdot 2^2} + \frac{B}{8\cdot 2^4} - \frac{C}{12\cdot 2^6} + \frac{D}{16\cdot 2^8} - \text{etc.}$$

Sin autem ponatur $x = 4$, erit

$$\text{Const.} = \frac{17}{24} - \frac{A}{4\cdot 4^2} + \frac{B}{8\cdot 4^4} - \frac{C}{12\cdot 4^6} + \frac{D}{16\cdot 4^8} - \text{etc.}$$

Utcunque autem constans definiatur, idem prodibit valor, qui simul summam seriei in infinitum continuatae, quae est $= l2$, indicabit.

187. Ceterum ex his novis numeris *A, B, C, D, E* etc. summae serierum potestatum reciprocarum parium, in quibus tantum numeri impares occurrunt, commode summari poterunt. Si enim ponatur

$$1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \text{etc.} = s$$

erit

$$\frac{1}{2^{2n}} + \frac{1}{4^{2n}} + \frac{1}{6^{2n}} + \text{etc.} = \frac{s}{2^{2n}},$$

quae ab illa subtracta relinquet

$$1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \text{etc.} = \frac{\left(2^{2n}-1\right)s}{2^{2n}}.$$

Cum igitur valores ipsius *s* pro singulis numeris *n* iam supra exhibuerimus (§ 125), erit

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.} = \frac{A}{1\cdot 2} \cdot \frac{\pi^2}{4}$$

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.} = \frac{B}{1\cdot 2\cdot 3\cdot 4} \cdot \frac{\pi^4}{4}$$

$$1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \text{etc.} = \frac{C}{1\cdot 2\cdot 3\cdots 6} \cdot \frac{\pi^6}{4}$$

$$1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \text{etc.} = \frac{D}{1\cdot 2\cdot 3\cdots 8} \cdot \frac{\pi^8}{4}$$

$$1 + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \text{etc.} = \frac{E}{1\cdot 2\cdot 3\cdots 10} \cdot \frac{\pi^{10}}{4}$$

etc.

Sin autem omnes numeri ingrediantur signaque alternentur, quia erit

$$1 - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \frac{1}{4^{2n}} + \text{etc.} = \frac{\left(2^{2n}-1\right)s - s}{2^{2n}},$$

habebitur

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\text{etc.}=\frac{A-2\mathfrak{A}}{1\cdot 2}\cdot\frac{\pi^2}{4}=\frac{(2-1)\mathfrak{A}}{1\cdot 2}\cdot\pi^2$$

$$1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}-\text{etc.}=\frac{B-2\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{\pi^4}{4}=\frac{(2^3-1)\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}\cdot\pi^4$$

$$1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\text{etc.}=\frac{C-2\mathfrak{C}}{1\cdot 2\cdots 6}\cdot\frac{\pi^6}{4}=\frac{(2^5-1)\mathfrak{C}}{1\cdot 2\cdots 6}\cdot\pi^6$$

$$1-\frac{1}{2^8}+\frac{1}{3^8}-\frac{1}{4^8}+\frac{1}{5^8}-\text{etc.}=\frac{D-2\mathfrak{D}}{1\cdot 2\cdots 8}\cdot\frac{\pi^8}{4}=\frac{(2^7-1)\mathfrak{D}}{1\cdot 2\cdots 8}\cdot\pi^8$$

$$1-\frac{1}{2^{10}}+\frac{1}{3^{10}}-\frac{1}{4^{10}}+\frac{1}{5^{10}}-\text{etc.}=\frac{E-2\mathfrak{E}}{1\cdot 2\cdots 10}\cdot\frac{\pi^{10}}{4}=\frac{(2^9-1)\mathfrak{E}}{1\cdot 2\cdots 10}\cdot\pi^{10}$$

etc.

188. Quemadmodum hactenus seriem sumus contemplati, cuius termini erant producta ex terminis progressionis geometricae p, p^2, p^3 etc. et ex terminis seriei cuiuscunque *a, b, c* etc., ita poterimus simili ratione prosequi seriem, cuius termini sint producta ex terminis duarum quarumcunque serierum, quarum altera tanquam cognita assumatur. Sit series cognita

$$\begin{matrix} 1 & 2 & 3 & & x \\ A+ & B+ & C+ & \cdots+ & Z, \end{matrix}$$

altera vero incognita

$$a+b+c+\cdots+z$$

atque quaeratur summa huius seriei

$$Aa+Bb+Cc+\cdots+Zz,$$

quae ponatur $=Zs$. Sit in serie cognita terminus penultimus $=Y$ atque posito $x-1$ loco x expressio summae $S.Zz$ abibit in

$$Y\left(s-\frac{ds}{dx}+\frac{dds}{2dx^2}-\frac{d^3s}{6dx^3}+\frac{d^4s}{24dx^4}-\text{etc.}\right).$$

Quae cum exprimat summam seriei *Zs* termino ultimo *Zz* mulctatae, erit

$$Zs-Zz=Ys-\frac{Yds}{dx}+\frac{Ydds}{2dx^2}-\frac{Yd^3s}{6dx^3}+\text{etc.},$$

quae aequatio continet relationem, qua summa *Zs* pendet ab *Y*, *Z* et *z*.

189. Ad hanc aequationem resolvendam negligantur primum termini differentiales eritque

$$s = \frac{Zz}{Z-Y};$$

ponatur iste valor $\frac{Zz}{Z-Y} = P^{\mathrm{I}}$ sitque revera $s = P^{\mathrm{I}} + p;$ quo valore in aequatione substituto fiet

$$\left(Z-Y\right)p = -\frac{YdP^{\mathrm{I}}}{dx} + \frac{YddP^{\mathrm{I}}}{2dx^2} - \text{etc.}$$
$$-\frac{Ydp}{dx} + \frac{Yddp}{2dx^2} - \text{etc.};$$

addatur utrinque YP^{I}, et cum $P^{\mathrm{I}} - \frac{dP^{\mathrm{I}}}{dx} + \frac{ddP^{\mathrm{I}}}{2dx^2} - \text{etc.}$ sit valor ipsius P^{I}, qui prodit, si loco x ponatur $x-1$, sit iste valor = P eritque

$$\left(Z-Y\right)p + YP^{\mathrm{I}} = YP - \frac{Ydp}{dx} + \frac{Yddp}{2dx^2} - \text{etc.},$$

unde neglectis differentialibus erit

$$p = \frac{Y\left(P-P^{\mathrm{I}}\right)}{Z-Y}.$$

Ponatur $\frac{Y\left(P-P^{\mathrm{I}}\right)}{Z-Y} = Q^{\mathrm{I}}$ sitque $p = Q^{\mathrm{I}} + q;$ fiet

$$\left(Z-Y\right)q = -\frac{Y\left(dQ^{\mathrm{I}}+dq\right)}{dx} + \frac{Y\left(ddQ^{\mathrm{I}}+ddq\right)}{2dx^2} - \text{etc.}$$

positoque Q pro valore ipsius Q^{I}, quem induit, si loco x scribatur $x-1$, erit

$$\left(Z-Y\right)q + YQ^{\mathrm{I}} = YQ - \frac{Ydq}{dx} + \frac{Yddq}{2dx^2} - \text{etc.},$$

unde neglectis differentialibus fit

$$q = \frac{Y\left(Q-Q^{\mathrm{I}}\right)}{Z-Y}.$$

Ponatur $\frac{Y\left(Q-Q^{\mathrm{I}}\right)}{Z-Y} = R^{\mathrm{I}}$ sitque revera $q = R^{\mathrm{I}} + r$ ac simili modo reperitur

$$r = \frac{Y\left(R-R^{\mathrm{I}}\right)}{Z-Y};$$

sicque procedendo erit summa quaesita

$$Zs = Z\left(P^{\mathrm{I}} + Q^{\mathrm{I}} + R^{\mathrm{I}} + \text{etc.}\right).$$

190. Proposita ergo serie quacunque

$$Aa + Bb + Cc + \cdots + Yy + Zz$$

eius summa sequenti modo definietur:

Ponatur	posito $x-1$ loco x
$\frac{Zz}{Z-Y} = P^{\mathrm{I}}$	abeatque P^{I} in P
$\frac{Y\left(P-P^{\mathrm{I}}\right)}{Z-Y} = Q^{\mathrm{I}}$	abeatque Q^{I} in Q
$\frac{Y\left(Q-Q^{\mathrm{I}}\right)}{Z-Y} = R^{\mathrm{I}}$	abeatque R^{I} in R
$\frac{Y\left(R-R^{\mathrm{I}}\right)}{Z-Y} = S^{\mathrm{I}}$	abeatque S^{I} in S
etc.	

His valoribus inventis erit summa seriei

$$= ZP^{\mathrm{I}} + ZQ^{\mathrm{I}} + ZR^{\mathrm{I}} + ZS^{\mathrm{I}} + \text{etc.}$$

+ Constante, quae reddat summam $=0$, si ponatur $x=0$, seu, quod eodem redit, quae efficiat, ut cuipiam casui satisfiat.

191. Formula haec, quia nullis differentialibus est implicata, in plurimis casibus facillime adhibetur atque etiam veram summam saepenumero exhibet.
Sic si proponatur haec series

$$p + 4p^2 + 9p^3 + 16p^3 + \cdots + x^2 p^x,$$

fiat $Z = p^x$ et $z = x^2$; erit $Y = p^{x-1}$ atque $\frac{Z}{Z-Y} = \frac{p}{p-1}$ et $\frac{Y}{Z-Y} = \frac{1}{p-1}$.
Hinc fiet

$$P^{\mathrm{I}} = \frac{px^2}{p-1} \qquad P = \frac{pxx-2px+p}{p-1}$$

$$Q^{\mathrm{I}} = \frac{-2px+p}{(p-1)^2} \qquad Q = \frac{-2px+3p}{(p-1)^2}$$

$$R^{\mathrm{I}} = \frac{2p}{(p-1)^3} \qquad R = \frac{2p}{(p-1)^3}$$

$$S^{\mathrm{I}} = 0$$

et reliqui evanescunt omnes; unde erit summa

$$= p^x\left(\frac{px^2}{p-1} - \frac{2px-p}{(p-1)^2} + \frac{2p}{(p-1)^3}\right) - \frac{p}{(p-1)^2} - \frac{2p}{(p-1)^3}$$
$$= p^{x+1}\left(\frac{x^2}{p-1} - \frac{2x}{(p-1)^2} + \frac{p+1}{(p-1)^3}\right) - \frac{p(p+1)}{(p-1)^3},$$

quemadmodum iam supra invenimus.

192. Simili modo, quo ad hanc summae expressionem pervenimus, aliam invenire poterimus expressionem, si series proposita non ex duabus aliis sit composita; quae illis potissimum casibus in usum vocari poterit, cum in praecedente expressione ad denominatores evanescentes pervenitur. Sit igitur proposita haec series

$$\begin{matrix} & 1 & & 2 & & 3 & & 4 & & & & x \\ s = & a & + & b & + & c & + & d & + & \cdots & + & z; \end{matrix}$$

quoniam posito $x-1$ loco x summa ultimo termino truncatur, erit

$$s - z = s - \frac{ds}{dx} + \frac{dds}{2dx^2} - \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} - \text{etc.}$$

seu

$$z = \frac{ds}{dx} - \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} - \frac{d^4s}{24dx^4} + \text{etc.}$$

Quia hic ipsa summa s non occurrit, negligantur differentialia altiora fietque $s = \int zdx$; ponatur $\int zdx = P^{\mathrm{I}}$, cuius valor abeat in P, si pro x scribatur $x-1$, sitque revera $s = P^{\mathrm{I}} + p$; erit

$$z = \frac{dP^{\mathrm{I}}}{dx} - \frac{ddP^{\mathrm{I}}}{2dx^2} + \text{etc.} + \frac{dp}{dx} - \frac{ddp}{2dx^2} + \text{ etc.};$$

quia est

$$P = P^{\mathrm{I}} - \frac{dP^{\mathrm{I}}}{dx} + \frac{ddP^{\mathrm{I}}}{2dx^2} - \text{etc.},$$

erit

$$z - P^{\mathrm{I}} + P = \frac{dp}{dx} - \frac{ddp}{2dx^2} + \text{etc.},$$

unde fit

$$p = \int \left(z - P^{\mathrm{I}} + P \right) dx.$$

Si porro ponatur $\int \left(z - P^{\mathrm{I}} + P \right) dx = Q^{\mathrm{I}}$ hicque valor abeat in Q posito $x-1$ loco x, sit

$$\int \left(z - P^{\mathrm{I}} + P - Q^{\mathrm{I}} + Q \right) dx = R^{\mathrm{I}} = Q^{\mathrm{I}} - \int \left(Q^{\mathrm{I}} - Q \right) dx,$$

porro

$$R^{\mathrm{I}} - \int \left(R^{\mathrm{I}} - R \right) dx = S^{\mathrm{I}}$$

etc.; erit summa quaesita

$$s = P^{\mathrm{I}} + Q^{\mathrm{I}} + R^{\mathrm{I}} + S^{\mathrm{I}} + \text{etc.} + \text{Const.},$$

qua uni casui satisfiat.

193. Mutatis aliquantum litteris ista summatio huc redit. Proposita serie summanda

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & & x \\ s = a + & b + & c + & d + & \cdots + & z \end{array}$$

ponatur	posito $x-1$ loco x
$\int z dx = P$	abeatque P in p
$P - \int (P - p) dx = Q$	abeatque Q in q
$Q - \int (Q - q) dx = R$	abeatque R in r

etc.;

quibus valoribus inventis erit summa quaesita

$$s = P + Q + R + S + etc.$$

haecque expressio expedite ostendit summam, si formulae istae integrales exhiberi queant. Sit, ut usum eius exemplo illustremus, $z = xx + x$ eritque

$$P = \tfrac{1}{3}x^3 + \tfrac{1}{2}xx, \quad p = \tfrac{1}{3}x^3 - \tfrac{1}{2}xx + \tfrac{1}{6}$$

et

$$\int (P - p)dx = \tfrac{1}{3}x^3 - \tfrac{1}{6}x \ ;$$

$$Q = \tfrac{1}{2}xx + \tfrac{1}{6}x, \quad q = \tfrac{1}{2}xx - \tfrac{5}{6}x + \tfrac{1}{3}, \quad Q - q = x - \tfrac{1}{3}$$

et

$$\int (Q - q)dx = \tfrac{1}{2}x^2 - \tfrac{1}{3}x \ ;$$

$$R = \tfrac{1}{2}x, \quad r = \tfrac{1}{2}x - \tfrac{1}{2}, \quad R - r = \tfrac{1}{2}$$

et

$$\int (R - r)dx = \tfrac{1}{2}x \,;$$

$S = 0$ reliquique valores evanescunt. Quare summa quaesita erit

$$P = \tfrac{1}{3}x^3 + \tfrac{1}{2}xx + \tfrac{1}{2}xx + \tfrac{1}{6}x + \tfrac{1}{2}x$$
$$= \tfrac{1}{3}x^3 + xx + \tfrac{2}{3}x = \tfrac{1}{3}x(x+1)(x+2).$$

Hocque ergo modo omnium serierum, quarum termini generales sunt functiones rationales integrae ipsius *x*, summae ope integrationum continuarum inveniri possunt. Ex quibus facile perspicitur, quam amplum occupet campum doctrina de summatione serierum neque omnibus methodis, quae tum habentur tum adhuc excogitari possunt, capiendis plura volumina sufficere.

194. Hactenus summas serierum investigavimus a termino primo usque ad eum, cuius index est *x*, quibus cognitis ponendo $x = \infty$ ipsius seriei in infinitum continuatae summa innotescet. Saepenumero autem hoc expeditius praestatur, si non summa terminorum a primo usque ad eum, cuius index est *x*, sed summa omnium terminorum ab isto, cuius index est *x*, in infinitum usque quaeratur, hocque casu imprimis expressiones ultimae fiunt tractabiliores. Sit igitur proposita series, cuius terminus generalis seu indici *x* respondens sit

$= z$, sequens indici $x+1$ respondens sit $= z^{\mathrm{I}}$ huncque ultra sequentes sint $z^{\mathrm{II}}, z^{\mathrm{III}}$, etc. quaeraturque summa huius seriei infinitae

$$\begin{array}{llllll} & x & x+1 & x+2 & x+3 & \text{etc.} \\ s = & z & + z^{\mathrm{I}} & + z^{\mathrm{II}} & + z^{\mathrm{III}} & + \text{etc. in infinitum.} \end{array}$$

Haec igitur summa *s* erit functio ipsius *x;* in qua si ponatur $x+1$ loco *x*, orietur summa prior termino *z* truncata. Cum ergo hac mutatione *s* abeat in

$$s + \frac{ds}{dx} + \frac{dds}{2dx^2} + \text{etc.},$$

erit

$$s - z = s + \frac{ds}{dx} + \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} + \frac{d^5s}{120dx^5} + \text{etc.}$$

seu

$$0 = z + \frac{ds}{dx} + \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} + \frac{d^5s}{120dx^5} + \text{etc.}$$

195. Si nunc ut ante ratiocinium instituamus, fiet neglectis differentialibus superioribus $s = C - \int zdx$. Ponatur ergo $\int zdx = P$ sitque revera
$s = C - P + p$; erit

$$\begin{aligned} 0 = z &- \frac{dP}{dx} - \frac{ddP}{2dx^2} - \frac{d^3P}{6dx^3} - \text{etc.} \\ &+ \frac{dp}{dx} + \frac{ddp}{2dx^2} + \frac{d^3p}{6dx^3} + \text{etc.} \end{aligned}$$

Abeat *P* in P^{I}, si loco *x* ponatur $x+1$, eritque

$$0 = z + P - P^{\mathrm{I}} + \frac{dp}{dx} + \frac{ddp}{2dx^2} + \frac{d^3p}{6dx^3} + \text{etc.}$$

Hinc neglectis differentialibus altioribus fiet $p = \int \left(P^{\mathrm{I}} - P\right)dx - P$. Statuatur $\int \left(P^{\mathrm{I}} - P\right)dx - P = -Q$ sitque $p = -Q + q$; erit

$$0 = z + P - P^{\mathrm{I}} - \frac{dQ}{dx} - \frac{ddQ}{2dx^2} - \text{etc.} + \frac{dq}{dx} + \frac{ddq}{2dx^2} + \text{etc.}$$

Abeat *Q* in Q^{I}, si loco *x* ponatur $x+1$, eritque

$$0 = z + P - P^{\mathrm{I}} + Q - Q^{\mathrm{I}} + \frac{dq}{dx} + \frac{ddq}{2dx^2} + \text{etc.},$$

unde sequitur $q=\int\left(Q^{\mathrm{I}}-Q\right)dx-Q$. Quamobrem si comma cuique quantitati infixum denotet eius valorem, quem induit posito $x+1$ loco *x*, ponaturque

$$\begin{gathered}\int zdx=P\\ P-\int\left(P^{\mathrm{I}}-P\right)dx=Q\\ Q-\int\left(Q^{\mathrm{I}}-Q\right)dx=R\\ R-\int\left(R^{\mathrm{I}}-R\right)dx=S\\ \text{etc.,}\end{gathered}$$

erit seriei propositae $z+z^{\mathrm{I}}+z^{\mathrm{II}}+z^{\mathrm{III}}+z^{\mathrm{IV}}+$ etc. summa

$$=C-P-Q-R-S-\text{ etc.},$$

ubi constans *C* ita debet definiri, ut posito $x=\infty$ tota summa evanescat. Quia autem applicatio huius expressionis integrationes requirit, hoc loco eius usum declarare non licet.

196. Ut autem formulas integrales evitemus, statuamus summam seriei $=ys$ existente *y* functione ipsius *x* quacunque cognita, cuius valores $y^{\mathrm{I}}, y^{\mathrm{II}}$ etc., qui prodeunt ponendo $x+1, x+2$ etc. loco *x*, erunt noti. Si iam ponatur $x+1$ loco *x*, prodibit superior series termino primo mulctata, cuius summa propterea erit

$$y^{\mathrm{I}}\left(s+\frac{ds}{dx}+\frac{dds}{2dx^2}+\frac{d^3s}{6dx^3}+\text{etc.}\right)=ys-z$$

seu

$$z+\frac{y^{\mathrm{I}}ds}{dx}+\frac{y^{\mathrm{I}}dds}{2dx^2}+\frac{y^{\mathrm{I}}d^3s}{6dx^3}+\text{etc.}=\left(y-y^{\mathrm{I}}\right)s$$

unde neglectis differentialibus oritur $s=\frac{z}{y-y^{\mathrm{I}}}$. Statuatur $\frac{z}{y-y^{\mathrm{I}}}=P$ sitque revera $s=-P+p$; erit

$$\left.\begin{aligned}&-\frac{y^{\mathrm{I}}dP}{dx}-\frac{y^{\mathrm{I}}ddP}{2dx^2}-\frac{y^{\mathrm{I}}d^3P}{6dx^3}-\text{etc.}\\&+\frac{y^{\mathrm{I}}dp}{dx}+\frac{y^{\mathrm{I}}ddp}{2dx^2}+\frac{y^{\mathrm{I}}d^3p}{6dx^3}+\text{etc.}\end{aligned}\right\}=\left(y-y^{\mathrm{I}}\right)p$$

ideoque

$$\frac{y^{\mathrm{I}}dp}{dx}+\frac{y^{\mathrm{I}}ddp}{2dx^2}+\frac{y^{\mathrm{I}}d^3p}{6dx^3}+\text{etc.}=y^{\mathrm{I}}\left(P^{\mathrm{I}}-P\right)-\left(y^{\mathrm{I}}-y\right)p.$$

Statuatur $Q = \frac{y^{\mathrm{I}}\left(P^{\mathrm{I}} - P\right)}{y^{\mathrm{I}} - y}$ sitque $p = Q + q$; erit

$$y^{\mathrm{I}}\left(Q^{\mathrm{I}} - Q\right) + y^{\mathrm{I}}\left(\frac{dq}{dx} + \frac{ddq}{2dx^2} + \text{etc.}\right) = -\left(y^{\mathrm{I}} - y\right)q.$$

Statuatur $R = \frac{y^{\mathrm{I}}\left(Q^{\mathrm{I}} - Q\right)}{y^{\mathrm{I}} - y}$ sitque $q = -R + r$.

Hocque modo si ulterius progrediamur, seriei propositae

$$z + z^{\mathrm{I}} + z^{\mathrm{II}} + z^{\mathrm{III}} + z^{\mathrm{IV}} + \text{etc.}$$

summa sequenti modo invenietur. Sumta pro lubitu functione ipsius *x*, quae sit $= y$, statuatur

$$P = \frac{z}{y^{\mathrm{I}} - y} = \frac{z}{\Delta y}$$

$$Q = \frac{y^{\mathrm{I}}\left(P^{\mathrm{I}} - P\right)}{y^{\mathrm{I}} - y} = \frac{y\Delta P}{\Delta y} + \Delta P$$

$$R = \frac{y^{\mathrm{I}}\left(Q^{\mathrm{I}} - Q\right)}{y^{\mathrm{I}} - y} = \frac{y\Delta Q}{\Delta y} + \Delta Q$$

$$S = \frac{y^{\mathrm{I}}\left(R^{\mathrm{I}} - R\right)}{y^{\mathrm{I}} - y} = \frac{y\Delta R}{\Delta y} + \Delta R$$

etc.

Hincque erit summa quaesita

$$= C - Py + Qy - Ry + Sy - \text{etc.}$$

sumta pro *C* eiusmodi constante, ut posito $x = \infty$ summa evanescat.

197. Sumatur $y = a^x$; ob $y^{\mathrm{I}} = a^{x+1}$ erit $y^{\mathrm{I}} - y = a^x(a-1)$, unde fiet

$$P = \frac{z}{a^x(a-1)} \qquad P^{\mathrm{I}} = \frac{z^{\mathrm{I}}}{a^{x+1}(a-1)}$$

$$Q = \frac{a(P^{\mathrm{I}} - P)}{(a-1)} = \frac{z^{\mathrm{I}} - az}{a^x(a-1)^2} \qquad Q^{\mathrm{I}} = \frac{z^{\mathrm{II}} - az^{\mathrm{I}}}{a^{x+1}(a-1)^2}$$

$$R = \frac{a(Q^{\mathrm{I}} - Q)}{(a-1)} = \frac{z^{\mathrm{II}} - 2az^{\mathrm{I}} + aaz}{a^x(a-1)^3} \qquad R^{\mathrm{I}} = \frac{z^{\mathrm{III}} - 2az^{\mathrm{II}} + aaz^{\mathrm{I}}}{a^{x+1}(a-1)^3}$$

$$S = \frac{a(R^{\mathrm{I}} - R)}{(a-1)} = \frac{z^{\mathrm{III}} - 3az^{\mathrm{II}} + 3a^2 z^{\mathrm{I}} - a^3 z}{a^x(a-1)^4}$$

etc.

Quocirca summa seriei propositae erit

$$C - \frac{z}{a-1} + \frac{z^{\mathrm{I}} - az}{(a-1)^2} - \frac{z^{\mathrm{II}} - 2az^{\mathrm{I}} + a^2 z}{(a-1)^3} + \frac{z^{\mathrm{III}} - 3az^{\mathrm{II}} + 3a^2 z^{\mathrm{I}} - a^3 z}{(a-1)^4} - \text{etc.}$$

Haec vero eadem summae expressio iam supra est inventa capite primo. Hinc autem aliis pro y valoribus accipiendis infinitae aliae expressiones erui poterunt, unde ea, quae cuique casui maxime sit accommodata, eligi potest.