CHAPTER VI

CONCERNING THE SUMMATION OF PROGRESSIONS BY INFINITE SERIES

140. In the preceding chapter we have found the general expression

$$Sz = \int zdx + \frac{1}{2}z + \frac{\mathfrak{A}dz}{1\cdot 2dx} - \frac{\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3} + \frac{\mathfrak{C}d^5z}{1\cdot 2\cdot 3\cdots 6dx^5} - \text{etc.},$$

for the summatory term of each series, of which the general term is $= z$, or corresponding to the index *x*, which will be particularly of service for the series to be summed, the general terms of which are some whole rational functions of the index *x*, because in these cases the sum finally arrives at vanishing differentials. But if *z* were not a function of *x* of this kind, then the differentials of this extend to infinity and thus the sum of an infinite series results expressing the sum of the proposed series and indeed as far as to the given term, of which the index is $= x$. On account of which if there may be put $x = \infty$, the sum of the proposed progression will be produced to infinity, and with this agreed upon the other infinite series is equal to the first infinite series.

141. But if there may be put $x = 0$, then the expression showing the sum must vanish, as we have now noted; unless which happens, a constant quantity of this kind must be added or thence taken away from the sum, in order that this condition may be satisfied. With which done if there is put $x = 1$, the sum found will give the first term of the series ; if $x = 2$, the sum of the first and second terms, if $x = 3$, there may arise the sum of the first three terms of the series, and thus so on. Therefore from these cases, because the sum of one or of two or of three terms is known, the value may become known of the infinite series, by which this sum is expressed, and from this source innumerable series will be able to be summed.

142. Because, if a constant of this kind were added to the sum, so that it may vanish on putting $x = 0$, then it is evident the sum for all the remaining cases will be satisfied, provided a constant quantity may be added to the sums found of this kind, so that in a certain case the true sum may be indicated, then in all the remaining cases the true sum must be produced. Whereby if on putting $x = 0$ it may be apparent, that the expression of the sum does not receive a value of this kind, nor therefore hence the constant requiring to be added can be found, then some other number must be put in place for *x* and with the constant to be added to be effected, so that the due sum may be indicated ; which must come about in some manner, which from the following will become more evident.

142a. First we may consider this harmonic progression

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots+\tfrac{1}{x}=s\,;$$

the general term of which since it shall be $=\frac{1}{x}$, becomes $z=\frac{1}{x}$ and the summatory term s thus may be found. In the first place there shall be $\int zdx=\int\frac{dx}{x}=lx$; then the differentials themselves thus may be had

$$\frac{dz}{dx}=-\frac{1}{x^2},\quad \frac{ddz}{2dx^2}=\frac{1}{x^3},\quad \frac{d^3z}{6dx^3}=-\frac{1}{x^4},\quad \frac{d^4z}{24dx^4}=\frac{1}{x^5},\quad \frac{d^5z}{120dx^5}=-\frac{1}{x^6}\ \text{etc.}$$

And thus hence there will be

$$s=lx+\frac{1}{2x}-\frac{\mathfrak{A}}{2x^2}+\frac{\mathfrak{B}}{4x^4}-\frac{\mathfrak{C}}{6x^6}+\frac{\mathfrak{D}}{8x^8}-\text{etc.}+\text{Constant.}$$

Therefore here the constant to be added cannot be defined from the case $x=0$. Hence there may be put $x = 1$, because then there becomes $s=1$; there will be

$$1=\frac{1}{2}-\frac{\mathfrak{A}}{2}+\frac{\mathfrak{B}}{4}-\frac{\mathfrak{C}}{6}+\frac{\mathfrak{D}}{8}-\text{etc.}+\text{Const.},$$

from which the constant itself becomes

$$=\frac{1}{2}+\frac{\mathfrak{A}}{2}-\frac{\mathfrak{B}}{4}+\frac{\mathfrak{C}}{6}-\frac{\mathfrak{D}}{8}+\text{etc.}$$

and thus the summatory term sought will be

$$\begin{aligned}s=lx+\frac{1}{2x}-\frac{\mathfrak{A}}{2x^2}+\frac{\mathfrak{B}}{4x^4}-\frac{\mathfrak{C}}{6x^6}+\frac{\mathfrak{D}}{8x^8}-\text{etc.}\\ +\frac{1}{2}\ +\frac{\mathfrak{A}}{2}\ -\frac{\mathfrak{B}}{4}\ +\frac{\mathfrak{C}}{6}\ -\frac{\mathfrak{D}}{8}\ +\text{etc.}\end{aligned}$$

143. Because the Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. constitute a divergent series, here the value of the constant cannot be known. But if in place of x there may be substituted a larger number and the sum of just as many terms actually is sought, the value of the constant will be investigated conveniently. To this end there is put $x=10$ and the sum of these first ten numbers taken together may be found [for s]

$$=2{,}928968253968253968,$$

to which the expression of the sum must be equal, if in that there may put $x=10$, which becomes

$$l10+\frac{1}{20}-\frac{\mathfrak{A}}{200}+\frac{\mathfrak{B}}{40000}-\frac{\mathfrak{C}}{6000000}+\frac{\mathfrak{D}}{800000000}-\text{etc.}+C.$$

Therefore with the hyperbolic logarithm of $l10$ taken and with the above values substituted in place of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc, that constant may be found

$$C=0,5772156649015325,$$

which number [the Euler-Mascheroni constant $\gamma=0,5772156649015328\cdots$] therefore expresses the sum of the series

$$\frac{1}{2}+\frac{\mathfrak{A}}{2}-\frac{\mathfrak{B}}{4}+\frac{\mathfrak{C}}{6}-\frac{\mathfrak{D}}{8}+\frac{\mathfrak{E}}{10}-\text{etc.}$$

144. If for x there may be substituted numbers not exceedingly large, because the sum of the series may actually be found easily, this sum of the series is obtained

$$\frac{1}{2x}-\frac{\mathfrak{A}}{2x^2}+\frac{\mathfrak{B}}{4x^4}-\frac{\mathfrak{C}}{6x^6}+\frac{\mathfrak{D}}{8x^8}-\text{etc.}=s-lx-\text{C}.$$

But if x may signify an extremely large number, because then the value of this expression on extending to infinity may be assigned easily in decimal fractions, in turn the sum of the series may be defined. And in the first place indeed it may be agreed, if the series may be continued to infinity, the sum of this becomes infinitely large; indeed on making $x=\infty$, lx also becomes infinite, although the ratio lx to x may maintain an infinitely small ratio. But so that any sum of terms of the series may be assigned conveniently, we may express the values of the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. in decimal fractions.

$$\begin{aligned}
\mathfrak{A}&=0,1666666666666\\
\mathfrak{B}&=0,0333333333333\\
\mathfrak{C}&=0,0238095238095\\
\mathfrak{D}&=0,0333333333333\\
\mathfrak{E}&=0,0757575757575\\
\mathfrak{F}&=0,2531135531135\\
\mathfrak{G}&=1,1666666666666\\
\mathfrak{H}&=7,0921568627451 \text{ etc.}
\end{aligned}$$

from which therefore there will be

$$\begin{aligned}
\frac{\mathfrak{A}}{2}&=0,0833333333333\\
\frac{\mathfrak{B}}{4}&=0,0083333333333
\end{aligned}$$

$$\frac{\mathfrak{C}}{6} = 0{,}0039682539682$$
$$\frac{\mathfrak{D}}{8} = 0{,}0041666666666$$
$$\frac{\mathfrak{E}}{10} = 0{,}0075757575757$$
$$\frac{\mathfrak{F}}{12} = 0{,}0210927960928$$
$$\frac{\mathfrak{G}}{14} = 0{,}0833333333333$$
$$\frac{\mathfrak{H}}{16} = 0{,}4432598039216 \text{ etc.}$$

EXAMPLE 1

To find the sum of one thousand terms of the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\text{etc.}$

There may be put therefore $x = 1000$, and since there shall be,

$$l10 = 2{,}3025850929940456840$$

there will be

$$\begin{array}{rl} lx = & 6{,}9077552789821 \\ \text{Const.} = & 0{,}5772156649015 \\ \frac{1}{2x} = & 0{,}0005000000000 \\ \hline & 7{,}4854709438836 \\ \text{subtr.}\frac{\mathfrak{A}}{2xx} = & 0{,}0000000833333 \\ \hline & 7{,}4854708605503 \\ \text{add } \frac{\mathfrak{B}}{4x^2} = & 0{,}0000000000000 \\ \hline & 7{,}4854708605503 \end{array}$$

Hence $7{,}4854708605503$

is the sum sought of the one thousand terms, which indeed does not make seven and a half.

EXAMPLE 2

To find the sum of a thousand times a thousand terms of the series
$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\text{etc.}$

Because there is $x = 1000000$, there will be $lx = 6\cdot l10$, hence

$$
\begin{aligned}
lx &= 13{,}8155105579642\\
\text{Const.} &= \ \ 0{,}5772156649015\\
\tfrac{1}{2x} &= \ \ 0{,}0000005000000\\
\hline
&\ \ 14{,}3927267228657 = \text{sum sought.}
\end{aligned}
$$

145. Therefore if an exceedingly large number may be put in place for *x*, the sum may be found precisely enough from the first term *lx* increased by the constant *C* ; from which outstanding corollaries can be deduced. Thus if *x* were an exceedingly large number and there may be put

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots+\tfrac{1}{x}=s$$

and

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots+\tfrac{1}{x}+\cdots+\tfrac{1}{x+y}=t,$$

because there is almost $s=lx+C$ and $t=l(x+y)+C$, there will be

$$t-s=l(x+y)-lx=l\,\tfrac{x+y}{x}$$

and thus here the logarithm may be expressed nearly by a constant harmonic series with a finite number of terms in this manner

$$l\,\tfrac{x+y}{x}=\tfrac{1}{x+1}+\tfrac{1}{x+2}+\tfrac{1}{x+3}+\cdots+\tfrac{1}{x+y}.$$

But here the logarithm will be shown more accurately, if the above sums *s* and *t* may be taken more precisely. Thus since there shall be

$$s=lx+\tfrac{1}{2x}-\tfrac{1}{12xx}\quad\text{and}\quad t=l(x+y)+C+\tfrac{1}{2(x+y)}-\tfrac{1}{12(x+y)^2},$$

there will be

$$t-s=l\,\tfrac{x+y}{x}-\tfrac{1}{2x}+\tfrac{1}{2(x+y)}+\tfrac{1}{12xx}-\tfrac{1}{12(x+y)^2}$$

and thus

$$l\,\tfrac{x+y}{x}=\tfrac{1}{x+1}+\tfrac{1}{x+2}+\tfrac{1}{x+3}+\cdots+\tfrac{1}{x+y}+\tfrac{1}{2x}-\tfrac{1}{2(x+y)}-\tfrac{1}{12xx}+\tfrac{1}{12(x+y)^2}$$

But if x shall be a number so great, that the two final terms are able to be rejected, there will be almost

$$l\frac{x+y}{x}=\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}+\cdots+\frac{1}{x+y}+\frac{1}{2}\left(\frac{1}{x}-\frac{1}{x+y}\right).$$

145a. From this too we will be able to derive from the harmonic series the sum of this series, in which odd numbers only occur,

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots+\frac{1}{2x+1}.$$

For since there shall be with all the terms taken

$$\begin{aligned}&\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2x}+\frac{1}{2x+1}\\&=l(2x+1)+\mathrm{C}+\frac{1}{2(2x+1)}-\frac{\mathfrak{A}}{2(2x+1)^2}+\frac{\mathfrak{B}}{4(2x+1)^4}-\frac{\mathfrak{C}}{6(2x+1)^6}+\text{etc.},\end{aligned}$$

truly from the order of the even terms

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2x}$$

the sum shall be of half the above, certainly

$$\frac{1}{2}\mathrm{C}+\frac{1}{2}lx+\frac{1}{4x}-\frac{\mathfrak{A}}{4x^2}+\frac{\mathfrak{B}}{8x^4}-\frac{\mathfrak{C}}{12x^6}+\frac{\mathfrak{D}}{16x^4}-\text{etc.},$$

there will be with this series taken from that

$$\begin{aligned}&\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots+\frac{1}{2x+1}\\&=\frac{1}{2}\mathrm{C}+l\frac{2x+1}{\sqrt{x}}+\frac{1}{2(2x+1)}-\frac{\mathfrak{A}}{2(2x+1)^2}+\frac{\mathfrak{B}}{4(2x+1)^4}-\text{etc.}\\&-\frac{1}{4x}+\frac{\mathfrak{A}}{4x^2}-\frac{\mathfrak{B}}{8x^4}+\text{etc.}\end{aligned}$$

146. Now the sum of any harmonic series can be found also by the same general expression; for there shall be

$$\frac{1}{m+n}+\frac{1}{2m+n}+\frac{1}{3m+n}+\frac{1}{4m+n}+\cdots+\frac{1}{mx+n}=s;$$

because the general term shall be $z=\frac{1}{mx+n}$, there will be

$$\int zdx = \frac{1}{m}l\left(mx+n\right), \quad \frac{dz}{dx} = -\frac{m}{(mx+n)^2}, \quad \frac{ddz}{2dx^2} = \frac{mm}{(mx+n)^3},$$

$$\frac{d^3z}{6dx^3} = -\frac{m^3}{(mx+n)^4}, \quad \frac{d^4z}{24dx^4} = \frac{m^4}{(mx+n)^5}, \quad \frac{d^5z}{120dx^5} = -\frac{m^5}{(mx+n)^6}, \quad \text{etc.}$$

From these therefore there will be found

$$s = D + \frac{1}{m}l\left(mx+n\right) + \frac{1}{2(mx+n)} - \frac{\mathfrak{A}m}{2(mx+n)^2} + \frac{\mathfrak{B}m^3}{4(mx+n)^3}$$
$$-\frac{\mathfrak{C}m^5}{6(mx+n)^6} + \frac{\mathfrak{D}m^7}{8(mx+n)^8} - \text{etc.}$$

Therefore on putting $x = 0$ that constant required to be added becomes

$$D = -\frac{1}{m}ln - \frac{1}{2n} + \frac{\mathfrak{A}m}{2n^2} - \frac{\mathfrak{B}m^3}{4n^3} + \frac{\mathfrak{C}m^5}{6n^4} - \text{etc.}$$

147. Now if there shall be $n = 0$, because the sum of the series

$$\frac{1}{m} + \frac{1}{2m} + \frac{1}{3m} + \frac{1}{4m} + \cdots + \frac{1}{mx}$$

is

$$= \frac{1}{m}C + \frac{1}{m}lx + \frac{1}{2mx} - \frac{\mathfrak{A}}{2mx^2} + \frac{\mathfrak{B}}{4mx^3} - \text{etc.},$$

but truly the sum of the series

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{mx}$$

is

$$= C + lmx + \frac{1}{2mx} - \frac{\mathfrak{A}}{2m^2x^2} + \frac{\mathfrak{B}}{4m^4x^4} - \text{etc.},$$

if from this series that one may be taken away *m* times, so that this series may be produced

$$1 + \frac{1}{2} + \cdots + \frac{1}{m} + \cdots + \frac{1}{2m} + \cdots + \frac{1}{3m} + \cdots + \frac{1}{mx}$$
$$-\frac{m}{m} \qquad -\frac{m}{2m} \qquad -\frac{m}{3m} \qquad -\frac{m}{mx},$$

the sum of that will be

$$= lm + \frac{1}{2mx} - \frac{\mathfrak{A}}{2m^2x^2} + \frac{\mathfrak{B}}{4m^4x^4} - \text{etc.}$$
$$-\frac{1}{2x} + \frac{\mathfrak{A}}{2xx} - \frac{\mathfrak{B}}{4x^4} + \text{etc.},$$

and if $x=\infty$ is put in place, the sum will be $=lm$. Hence on putting the numbers 2, 3, 4 etc. for *m* the numbers will be

$$l2 = 1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}-\tfrac{1}{8}+\text{etc.}$$
$$l3 = 1+\tfrac{1}{2}-\tfrac{2}{3}+\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{2}{6}+\tfrac{1}{7}+\tfrac{1}{8}-\tfrac{2}{9}+\text{etc.}$$
$$l4 = 1+\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{3}{4}+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}-\tfrac{3}{8}+\text{etc.}$$
$$l5 = 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}-\tfrac{4}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}+\tfrac{1}{9}-\tfrac{4}{10}+\text{etc.}$$
$$\text{etc.}$$

148. Moreover with the harmonic series left we may progress to the reciprocal series of the squares and there shall be

$$s=1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}+\cdots+\tfrac{1}{xx}\ ;$$

in which since the general term shall be $z=\frac{1}{xx}$, there will be $\int zdx=-\frac{1}{x}$ and the differentials of *z* thus will be had themselves

$$\frac{dz}{2dx}=-\frac{1}{x^3},\quad \frac{ddz}{2\cdot 3dx^2}=\frac{1}{x^4},\quad \frac{d^3z}{2\cdot 3\cdot 4dx^3}=-\frac{1}{x^5}\ \text{etc.},$$

from which the sum will be

$$s=C-\frac{1}{x}+\frac{1}{2xx}-\frac{\mathfrak{A}}{x^3}+\frac{\mathfrak{B}}{x^5}-\frac{\mathfrak{C}}{x^7}+\frac{\mathfrak{D}}{x^9}-\frac{\mathfrak{E}}{x^{11}}+\text{etc.},$$

in which the constant term requiring to be added *C* from one case, from which the sum may be constructed, is to be defined. Therefore we may put $x=1$; because there becomes $s=1$, there must be

$$s=1+1-\tfrac{1}{2}+\mathfrak{A}-\mathfrak{B}+\mathfrak{C}-\mathfrak{D}+\mathfrak{E}-\text{etc.},$$

but which series, since it shall be maximally diverging, has not been able to show the value of the constant *C*. But because above we have shown the sum of this series continued to infinity to be $=\frac{\pi\pi}{6}$, on making $x=\infty$, if there is put $s=\frac{\pi\pi}{6}$, there becomes $C=\frac{\pi\pi}{6}$ on account of all the remaining terms vanishing. Hence there will be

$$1+1-\tfrac{1}{2}+\mathfrak{A}-\mathfrak{B}+\mathfrak{C}-\mathfrak{D}+\mathfrak{E}-\text{etc.}=\frac{\pi\pi}{6}.$$

149. But if the sum of this series were not known, the value of that constant C must be determined from some other case, in which the sum has actually been found. This in the end we may put $x=10$ and with ten terms actually added there is found

$$s=1,5497677311665406 90;$$

then there is

$$\begin{array}{lr}
\text{add } \frac{1}{x} & =0,1 \\
\text{subtr.} \frac{1}{2xx} & =0,005 \\
\hline
 & 1,6447677311665406 90 \\
\text{add } \frac{\mathfrak{A}}{x^3} & =0,0001666666666666 66 \\
\hline
 & 1,6449343978332073 56 \\
\text{subtr.} \frac{\mathfrak{B}}{x^5} & =0,0000003333333333 33 \\
\hline
 & 1,6449340644998740 23 \\
\text{add } \frac{\mathfrak{C}}{x^7} & =0,0000000023809523 81 \\
\hline
 & 1,6449340668808264 04 \\
\text{subtr.} \frac{\mathfrak{D}}{x^9} & =0,0000000000333333 33 \\
\hline
 & 1,6449340668474930 71 \\
\text{add } \frac{\mathfrak{E}}{x^{11}} & =0,0000000000007575 75 \\
\hline
 & 1,6449340668482506 46 \\
\text{subtr.} \frac{\mathfrak{F}}{x^{13}} & =0,0000000000000253 11 \\
\hline
 & 1,6449340668482253 35 \\
\text{add } \frac{\mathfrak{G}}{x^{15}} & =0,0000000000000011 66 \\
\text{subtr.} \frac{\mathfrak{H}}{x^{17}} & =0,0000000000000000 71 \\
\hline
 & 1,6449340668482264 30=C.
\end{array}$$

And this number likewise is the value of the expression $=\frac{\pi\pi}{6}$, just as it will be apparent from the value of π known from the calculation required to be put in place. From which likewise it is understood, even if the series $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. may diverge, yet in this way the true sum is produced.

150. Now let there be $z=\frac{1}{x^3}$ and

$$s=1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\cdots+\frac{1}{x^3};$$

which is

$$\int zdx=-\frac{1}{2xx},\quad \frac{dz}{1\cdot 2\cdot 3dx}=-\frac{1}{2x^4},\quad \frac{ddz}{1\cdot 2\cdot 3\cdot 4dx^2}=\frac{1}{2x^5},$$
$$\frac{d^3z}{1\cdot 2\cdots 5dx^3}=-\frac{1}{2x^6},\quad \frac{d^4z}{1\cdot 2\cdots 6dx^4}=\frac{1}{2x^7},\quad \frac{d^5z}{1\cdot 2\cdots 7dx^5}=-\frac{1}{2x^8}\ \text{ etc.},$$

there will be

$$s=C-\frac{1}{2xx}+\frac{1}{2x^3}-\frac{3\mathfrak{A}}{2x^4}+\frac{5\mathfrak{B}}{2x^6}-\frac{7\mathfrak{C}}{2x^8}+\text{etc.}$$

and hence on putting $x=1$ on account of $s=1$ there becomes

$$C=1+\frac{1}{2}-\frac{1}{2}+\frac{3}{2}\mathfrak{A}-\frac{5}{2}\mathfrak{B}+\frac{7}{2}\mathfrak{C}-\frac{9}{2}\mathfrak{D}+\text{etc.}$$

and this value of C may likewise show the sum of the proposed series continued to infinity. Truly because the sums of the odd and even powers do not agree equally, this value of C must be defined from some known sum of terms. Therefore let there be $x=10$; there will be

$$C=s+\frac{1}{2xx}-\frac{1}{2x^3}+\frac{3\mathfrak{A}}{2x^4}-\frac{5\mathfrak{B}}{2x^6}+\frac{7\mathfrak{C}}{2x^8}-\text{ etc.}$$

Now to set up the computation more easily there is

$$\frac{3\mathfrak{A}}{2}=\ 0,2500000000000$$
$$\frac{5\mathfrak{B}}{2}=\ 0,0833333333333$$
$$\frac{7\mathfrak{C}}{2}=\ 0,0833333333333$$
$$\frac{9\mathfrak{D}}{2}=\ 0,1500000000000$$
$$\frac{11\mathfrak{E}}{2}=\ 0,4166666666666$$
$$\frac{13\mathfrak{F}}{2}=\ 1,6452380952380$$
$$\frac{15\mathfrak{G}}{2}=\ 8,7500000000000$$
$$\frac{17\mathfrak{H}}{2}=60,2833333333333$$

etc.

Hence therefore the terms become to be added to s

$$\frac{1}{2xx} = 0{,}005000000000000000$$
$$\frac{3\mathfrak{A}}{2x^4} = 0{,}000025000000000000$$
$$\frac{7\mathfrak{C}}{2x^8} = 0{,}000000000833333333$$
$$\frac{11\mathfrak{E}}{2x^{12}} = 0{,}000000000000416666$$
$$\frac{15\mathfrak{G}}{2x^{16}} = 0{,}000000000000006875$$
$$0{,}005025000833750875;$$

moreover the terms to be subtracted are

$$\frac{1}{2x^3} = 0{,}000500000000000000$$
$$\frac{5\mathfrak{B}}{2x^6} = 0{,}000000083333333333$$
$$\frac{9\mathfrak{D}}{2x^{10}} = 0{,}000000000015000000$$
$$\frac{13\mathfrak{F}}{2x^{14}} = 0{,}000000000000016452$$
$$\frac{17\mathfrak{H}}{2x^{18}} = 0{,}000000000000000060$$
$$0{,}000500083348349845$$
$$\text{from } 0{,}005025000833750875$$
$$0{,}004524917485401030$$

$$s = 1{,}197531985674193251$$
$$C = 1{,}202056903159594281$$

151. If we may progress further in this manner, we may find the sums of all the series of the reciprocals of powers expressed in decimal fractions.

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{ etc.}=1,6449340668482264=\frac{2\mathfrak{A}}{1\cdot 2}\pi^2$$

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\text{ etc.}=1,2020569031595942$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{ etc.}=1,0823232337111381=\frac{2^3\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}\pi^4$$

$$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\text{ etc.}=1,0369277551433699$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{ etc.}=1,0173430619844491=\frac{2^5\mathfrak{C}}{1\cdot 2\cdots 6}\pi^6$$

$$1+\frac{1}{2^7}+\frac{1}{3^7}+\frac{1}{4^7}+\text{ etc.}=1,0083492773819228$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\text{ etc.}=1,0040773561979443=\frac{2^7\mathfrak{D}}{1\cdot 2\cdots 8}\pi^8$$

$$1+\frac{1}{2^9}+\frac{1}{3^9}+\frac{1}{4^9}+\text{ etc.}=1,0020083928260822$$

$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\text{ etc.}=1,0009945751278180=\frac{2^9\mathfrak{E}}{1\cdot 2\cdots 10}\pi^{10}$$

$$1+\frac{1}{2^{11}}+\frac{1}{3^{11}}+\frac{1}{4^{11}}+\text{ etc.}=1,0004941886041194$$

$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\text{ etc.}=1,0002460865533080=\frac{2^{12}\mathfrak{F}}{1\cdot 2\cdots 12}\pi^{12}$$

$$1+\frac{1}{2^{13}}+\frac{1}{3^{13}}+\frac{1}{4^{13}}+\text{ etc.}=1,0001227133475784$$

$$1+\frac{1}{2^{14}}+\frac{1}{3^{14}}+\frac{1}{4^{14}}+\text{ etc.}=1,0000612481350587=\frac{2^{13}\mathfrak{G}}{1\cdot 2\cdots 14}\pi^{14}$$

$$1+\frac{1}{2^{15}}+\frac{1}{3^{15}}+\frac{1}{4^{15}}+\text{ etc.}=1,0000305882363070$$

$$1+\frac{1}{2^{16}}+\frac{1}{3^{16}}+\frac{1}{4^{16}}+\text{ etc.}=1,0000152822594086=\frac{2^{15}\mathfrak{H}}{1\cdot 2\cdots 16}\pi^{16}$$

etc.

152. Therefore from these in turn the sum of these infinite series of constants will be able to be shown from the Bernoulli numbers. For there shall be

$$1+0-\frac{1}{2}+\frac{\mathfrak{A}}{2}-\frac{\mathfrak{B}}{4}+\frac{\mathfrak{C}}{6}-\frac{\mathfrak{D}}{8}+\text{etc.}=0{,}57721\text{ etc.}$$
$$1+1-\frac{1}{2}+\mathfrak{A}-\mathfrak{B}+\mathfrak{C}-\mathfrak{D}+\text{etc.}=\frac{2\mathfrak{A}}{1\cdot 2}\pi^2$$
$$1+\frac{1}{2}-\frac{1}{2}+\frac{3\mathfrak{A}}{2}-\frac{5\mathfrak{B}}{4}+\frac{7\mathfrak{C}}{6}-\frac{9\mathfrak{D}}{8}+\text{etc.}=1{,}2020\text{ etc.}$$
$$1+\frac{1}{3}-\frac{1}{2}+\frac{3\cdot 4\mathfrak{A}}{2\cdot 3}-\frac{5\cdot 6\mathfrak{B}}{2\cdot 3}+\frac{7\cdot 8\mathfrak{C}}{2\cdot 3}-\frac{9\cdot 10\mathfrak{D}}{2\cdot 3}+\text{etc.}=\frac{2^3\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}\pi^4$$
$$1+\frac{1}{4}-\frac{1}{2}+\frac{3\cdot 4\cdot 5\mathfrak{A}}{2\cdot 3\cdot 4}-\frac{5\cdot 6\cdot 7\mathfrak{B}}{2\cdot 3\cdot 4}+\frac{7\cdot 8\cdot 9\mathfrak{C}}{2\cdot 3\cdot 4}-\frac{9\cdot 10\cdot 11\mathfrak{D}}{2\cdot 3\cdot 4}+\text{etc.}=1{,}0369\text{ etc.}$$
$$1+\frac{1}{5}-\frac{1}{2}+\frac{3\cdot 4\cdot 5\cdot 6\mathfrak{A}}{2\cdot 3\cdot 4\cdot 5}-\frac{5\cdot 6\cdot 7\cdot 8\mathfrak{B}}{2\cdot 3\cdot 4\cdot 5}+\frac{7\cdot 8\cdot 9\cdot 10\mathfrak{C}}{2\cdot 3\cdot 4\cdot 5}-\text{etc.}=\frac{2^5\mathfrak{C}}{1\cdot 2\cdots 6}\pi^6$$

Therefore with the aid of the quadrature of the circle the alternate members of these series can be summed ; truly on which transcending quantity the remainder may depend on is not clear at this point ; nor indeed are the powers of *n* having odd exponents able to be recalled, thus so that the coefficients may be rational numbers. But so that at least it may be approximately apparent, what kind of power of π the coefficients for the odd exponents may become, we have added the following table.

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\text{etc. to infinity}=\frac{\pi}{0{,}0000}=\infty$$
$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{etc. to infinity}=\frac{\pi^2}{6{,}0000}\text{ exactly}$$
$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\text{etc. to infinity}=\frac{\pi^3}{25{,}79436}\text{ approx.}$$
$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc. to infinity}=\frac{\pi^4}{90{,}00000}\text{ exactly}$$
$$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\text{etc. to infinity}=\frac{\pi^5}{295{,}1215}\text{ approx.}$$
$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc. to infinity}=\frac{\pi^6}{945{,}00000}\text{ exactly}$$
$$1+\frac{1}{2^7}+\frac{1}{3^7}+\frac{1}{4^7}+\text{etc. to infinity}=\frac{\pi^7}{2995{,}284}\text{ approx.}$$
$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\text{etc. to infinity}=\frac{\pi^8}{945{,}00000}\text{ exactly}$$
$$1+\frac{1}{2^9}+\frac{1}{3^9}+\frac{1}{4^9}+\text{etc. to infinity}=\frac{\pi^9}{29749{,}35}\text{ approx.}$$

etc.

153. From this source the series of Bernoulli numbers

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \mathfrak{A}, & \mathfrak{B}, & \mathfrak{C}, & \mathfrak{D}, & \mathfrak{E}, & \mathfrak{F}, & \mathfrak{G}, & \mathfrak{H}, & \mathfrak{I} \text{ etc.,} \end{matrix}$$

of whatever irregularity may be seen, to be interpolated or terms in the middle between any two put in place will be able to be assigned ; if indeed the middle term lying between the first $\mathfrak{A}$ and the second $\mathfrak{B}$ or with the index $1\frac{1}{2}$ corresponding were $= p$, there will certainly be

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\text{etc.}=\frac{2^2 p}{1\cdot 2\cdot 3}\pi^3$$

and thus

$$p=\frac{3}{2\pi^3}\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\text{etc.}\right)=0{,}05815227.$$

In a similar manner if the middle term lying between $\mathfrak{B}$ and $\mathfrak{C}$ or having the index $2\frac{1}{2}$ may be put $=q$, because there shall be

$$1+\frac{1}{2^5}+\frac{1}{3^5}+\text{etc.}=\frac{2^4 q}{1\cdot 2\cdot 3\cdot 4\cdot 5}\pi^5,$$

it becomes

$$q=\frac{15}{2\pi^5}\left(1+\frac{1}{2^5}+\frac{1}{3^5}+\text{etc.}\right)=0{,}02541327.$$

Therefore if the sums of these series, in which the exponents of the powers are odd numbers may be able to be shown, then also the series of the Bernoulli numbers might be able to be interpolated.

154. Now we may put $z=nn+xx$ and the sum of this series may be sought

$$s=\frac{1}{nn+1}+\frac{1}{nn+4}+\frac{1}{nn+9}+\cdots+\frac{1}{nn+xx}.$$

Because there is $\int zdx=\int\frac{dx}{nn+xx}$, there will be

$$\int zdx=\frac{1}{n}\text{Atang}\frac{x}{n}.$$

There may be put $\text{Acot}\frac{x}{n}=u$ *;* there will be $\int zdx=\frac{1}{n}\left(\frac{\pi}{2}-u\right)$

and

$$\frac{x}{n}=\cot u=\frac{\cos u}{\sin u} \quad \text{and} \quad \frac{nn+xx}{nn}=\frac{1}{\sin u^2}, \quad z=\frac{\sin^2 u}{nn}, \quad \frac{dx}{n}=-\frac{du}{\sin^2 u},$$

from which there becomes

$$du = -\frac{dx\sin^2 u}{n}.$$

Hence the differentials of z may be found in this manner

$$dz = \frac{2du\sin u\cdot\cos u}{nn} = -\frac{dx\sin^2 u\cdot\sin 2u}{n^3} \quad \text{et} \quad \frac{dz}{dx} = -\frac{\sin^2 u\cdot\sin 2u}{n^3},$$

$$\frac{ddz}{2dx} = -\frac{du\left(\sin u\cdot\cos u\cdot\sin 2u + \sin^2 u\cdot\cos 2u\right)}{n^3} = \frac{dx\sin^3 u\cdot\cos 3u}{n^4}$$

and

$$\frac{ddz}{2dx^2} = \frac{\sin^3 u\cdot\cos 3u}{n^4}.$$

In a similar manner there will be, as now we have found above for the same case (in §87),

$$\frac{d^3 z}{2\cdot 3dx^3} = -\frac{\sin^4 u\cdot\sin 4u}{n^5}, \quad \frac{d^4 z}{2\cdot 3\cdot 4dx^4} = \frac{\sin^5 u\cdot\sin 5u}{n^6} \quad \text{etc.},$$

from which the sum sought will be formed

$$s = \frac{\pi}{2n} - \frac{u}{n} + \frac{\sin u\cdot\sin u}{2nn} - \frac{\mathfrak{A}}{2}\cdot\frac{\sin^2 u\cdot\sin 2u}{n^3} + \frac{\mathfrak{B}}{4}\cdot\frac{\sin^4 u\cdot\sin 4u}{n^5}$$
$$-\frac{\mathfrak{C}}{6}\cdot\frac{\sin^6 u\cdot\sin 6u}{n^7} + \frac{\mathfrak{D}}{8}\cdot\frac{\sin^8 u\cdot\sin 8u}{n^9} - \text{etc.} + \text{Const.}$$

If here towards determining the constant there may be put $x = 0$, so that this becomes $s = 0$, there will be $\cot u = 0$ and thus the angle u becomes 90^0 and therefore $\sin u = 1$, $\sin 2u = 0$, $\sin 4u = 0$, $\sin 6u = 0$, etc.; therefore there may be seen to become

$$0 = \frac{\pi}{2n} - \frac{\pi}{2n} + \frac{1}{2nn} + C \quad \text{and hence} \quad C = -\frac{1}{2nn};$$

but truly it is to be noted, even if the remaining terms may vanish, yet, because the [number of] coefficients $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ etc. finally increase to infinity, the sum of these is able to be finite.

155. Therefore towards duly determining that constant we may put $x = \infty$; indeed now above we have defined in the *Introductione* the sum of this series extending to infinity, and we have shown that to be

$$= -\frac{1}{2nn} + \frac{\pi}{2n} + \frac{\pi}{n\left(e^{2n\pi} - 1\right)}.$$

Moreover on putting $x = \infty$ there becomes $u = 0$ and thus $\sin u = 0$ and likewise the sine of all the multiple arcs vanish. But since in this series the powers of $\sin u$ may increase, the divergence of the series cannot be impeded, unless the value of the series may vanish in this case. Therefore there becomes $s=\frac{\pi}{2n}+C$; from which there shall be

$$\frac{\pi}{2n}+C=-\frac{1}{nn}+\frac{\pi}{2n}+\frac{\pi}{n\left(e^{2n\pi}-1\right)} \quad \text{and} \quad C=-\frac{1}{2nn}+\frac{\pi}{n\left(e^{2n\pi}-1\right)}.$$

Whereby the sum of the series sought will be

$$s=\frac{\pi}{2n}-\frac{u}{n}-\frac{1}{2nn}+\frac{\sin^2 u}{2nn}-\frac{\mathfrak{A}}{2}\cdot\frac{\sin^2 u\cdot\sin 2u}{n^3}+\frac{\mathfrak{B}}{4}\cdot\frac{\sin^4 u\cdot\sin 4u}{n^5}$$
$$-\frac{\mathfrak{C}}{6}\cdot\frac{\sin^6 u\cdot\sin 6u}{n^7}+\text{etc.}+\frac{\pi}{n\left(e^{2n\pi}-1\right)}.$$

Where it is to be noted, if n were a not very large number, the final term $\frac{\pi}{n\left(e^{2n\pi}-1\right)}$ becomes so small, that it can be ignored.

156. We may put in place $x=n$ thus so that it may denote

$$s=\frac{1}{nn+1}+\frac{1}{nn+4}+\frac{1}{nn+9}+\cdots+\frac{1}{nn+xx}.$$

Then indeed there will be $\cot u=1$ and $u=45^0=\frac{\pi}{4}\cdot$ On account of which there will be had

$$\sin u=\frac{1}{\sqrt{2}},\sin 2u=1,\ \sin 4u=0,\ \sin 6u=-1,\ \sin 8u=0,\ \sin 10u=1 \text{ etc.}$$

On account of this,

$$s=\frac{\pi}{4n}-\frac{1}{2nn}+\frac{1}{4nn}-\frac{\mathfrak{A}}{2\cdot 2n^3}+\frac{\mathfrak{C}}{6\cdot 8n^7}-\frac{\mathfrak{E}}{10\cdot 2^5 n^{11}}+\frac{\mathfrak{G}}{14\cdot 2^7 n^{15}}-\text{etc.}+\frac{\pi}{n\left(e^{2n\pi}-1\right)},$$

in which expression only the alternate from the Bernoulli numbers occur. Therefore if the value of *s* by actual computation put in place were now found, hence the quantity *n* will be able to be defined; then there will be

$$\pi=4ns+\frac{1}{n}+\frac{\mathfrak{A}}{1\cdot n^2}-\frac{\mathfrak{C}}{3\cdot 2n^6}+\frac{\mathfrak{E}}{5\cdot 2^4 n^{10}}-\frac{\mathfrak{G}}{7\cdot 2^6 n^{14}}+\text{etc.}-\frac{4\pi}{e^{2n\pi}-1}.$$

For even if π is not present in the final term, yet, because it is so small, it provides an approximate value of π to be known.

EXAMPLE

Let $n = 5$; there will be

$$s = \tfrac{1}{26} + \tfrac{1}{29} + \tfrac{1}{34} + \tfrac{1}{41} + \tfrac{1}{50};$$

which terms added will give

$$s = 0{,}146746305690549494,$$

from which there will be these terms

$$4ns = 2{,}93492611381098988$$
$$\frac{1}{n} = 0{,}2$$
$$\frac{\mathfrak{A}}{1\cdot n^2} = \underline{0{,}00666666666666666}$$
$$3{,}14159278047765654$$
$$\frac{\mathfrak{C}}{3\cdot 2n^6} = \underline{0{,}00000012698412698}$$
$$3{,}14159265349352956$$

$$\frac{\mathfrak{E}}{5\cdot 2^4 n^{10}} = \underline{0{,}00000000009696969}$$
$$3{,}14159265359049925$$
$$\frac{\mathfrak{G}}{7\cdot 2^6 n^{14}} = \underline{0{,}00000000000042666}$$
$$3{,}14159265359007259$$
$$\frac{\mathfrak{I}}{9\cdot 2^8\cdot n^{18}} = \underline{\qquad\qquad 625}$$
$$3{,}14159265359007884.$$

This value now falls so closely to the true value, that it is to be wondered at that it is able to arrive at that point with so light a calculation. Now this correct expression is a small amount larger; for at this point there must thence be subtracted $\frac{4\pi}{e^{2n\pi}-1}$, the value of which, provided π shall be almost known, is able to be shown; which can be arranged by logarithms.

Because there is $\pi le = 1{,}3643763538$, there will be

$$le^{2n\pi} = 10\pi le = 13{,}6437635.$$

Now since there shall be

$$\frac{4\pi}{e^{2n\pi}-1} = \frac{4\pi}{e^{2n\pi}} + \frac{4\pi}{e^{4n\pi}} + \text{etc.},$$

it is easily understood to be sufficient for our calculation to have selected the first of the terms. Hence we may increase the characteristic to the number 17, because we may have as many decimal places; there will be

$$\begin{array}{rr} l\pi = & 17,4971498 \\ l4 = & \underline{0,6020600} \\ & 18,0992098 \\ \text{subtr.}\quad le^{2n\pi} = & \underline{13,6437635} \\ & 4,4554463 \end{array}$$

Hence

$$\frac{4\pi}{e^{2n\pi}} = \qquad 28539$$

it is taken from

$$\underline{3,14159265359007884}$$

and there will be

$$\pi = 3,14159265358979345$$

which expression departs from the true value in the second last figure; which is not to be wondered at, since at this point we ought to have subtracted the term $\frac{\mathfrak{L}}{11\cdot 2^{10}\cdot n^{22}}$ which gives 22, and thus indeed it may not have erred in the final figures. Moreover it is understood, if in place of *n* we may have assumed a greater number as 10, then it will be possible to find the periphery π to 25 and more places in an easy calculation.

157. Now we may put also transcending functions of *x* for *z* and let there be $z = lx$ by taking the hyperbolic logarithms, because the common logarithms may be easily recalled from that, and let there be

$$\mathrm{s} = l1 + l2 + l3 + l4 + \cdots + lx\,.$$

Therefore because there is $z = lx$, there will be

$$\int zdx = xlx - x \;;$$

for the differential of this gives *dxlx.* Then there is

$$\frac{dz}{dx} = \frac{1}{x}\,,\quad \frac{ddz}{dx^2} = -\frac{1}{x^2}\,,\quad \frac{d^3z}{1\cdot 2dx^3} = \frac{1}{x^3}\,,\quad \frac{d^4z}{1\cdot 2\cdot 3dx^4} = -\frac{1}{x^4}\,,\quad \frac{d^5z}{1\cdot 2\cdot 3\cdot 4dx^5} = \frac{1}{x^5}\quad \text{etc.}$$

Hence therefore it is concluded to become

$$\mathrm{s} = xlx - x + \frac{1}{2}lx + \frac{\mathfrak{A}}{1\cdot 2x} - \frac{\mathfrak{B}}{3\cdot 4x^3} + \frac{\mathfrak{C}}{5\cdot 6x^5} - \frac{\mathfrak{D}}{7\cdot 8x^7} + \text{etc.} + \text{Const.}$$

But this constant thus may be defined on putting $x=1$, because there shall be $s=l1=0$, thus it may be defined, so that there shall be

$$C=1-\frac{\mathfrak{A}}{1\cdot 2}+\frac{\mathfrak{B}}{3\cdot 4}-\frac{\mathfrak{C}}{5\cdot 6}+\frac{\mathfrak{D}}{7\cdot 8}-\text{etc.},$$

which series on account of excessive divergence is unsuited at any rate for eliciting the approximate value of *C*.

158. But not only may we find a close value, but also the true value of the true value of C itself, if we may consider the expression of Wallis for the value of π found and shown in the *Introductione*, which was

$$\frac{\pi}{2}=\frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10\cdot 12\cdot\text{etc.}}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot 11\cdot 11\cdot\text{etc.}}.$$

Hence indeed with the logarithms taken there will be

$$\begin{aligned} l\pi-l2&=2l2+2l4+2l6+2l8+2l10+l12+\text{etc.}\\ &-l1-\ 2l3-2l5-2l7-2l9-2l11-\text{etc.}\end{aligned}$$

Therefore we may put $x=\infty$ in the series assumed, and since there shall be

$$l1+l2+l3+l4+\cdots+lx=C+\left(x+\tfrac{1}{2}\right)lx-x,$$

there will be

$$l1+l2+l3+l4+\cdots+l2x=C+\left(2x+\tfrac{1}{2}\right)l2x-2x$$

and

$$l2+l4+l6+l8+\cdots+l2x=C+\left(x+\tfrac{1}{2}\right)lx+xl2-x,$$

hence

$$l1+l3+l5+l7+\cdots+l\left(2x-1\right)=xlx+\left(x+\tfrac{1}{2}\right)l2-x\,.$$

Therefore since there shall be

$$\begin{aligned} l\tfrac{\pi}{2}&=2l2+2l4+2l6+\cdots+2l2x-l2x\\ &-2l1-2l3-2l5-\cdots-2l\left(2x-1\right),\end{aligned}$$

on putting $x=\infty$ there will be

$$l\tfrac{\pi}{2}=2C+\left(2x+1\right)lx+2xl2-2x-l2-lx-2xlx-\left(2x+1\right)l2+2x$$

and thus

$$l\tfrac{\pi}{2}=2C-2l2,\quad\text{therefore}\quad 2C=l2\pi\quad\text{and}\quad C=\tfrac{1}{2}l2\pi,$$

from which in decimal fractions there is found

$$C = 0{,}91893853320467274178032927,$$

and likewise the following series may be summed

$$1 - \frac{\mathfrak{A}}{1\cdot 2} + \frac{\mathfrak{B}}{3\cdot 4} - \frac{\mathfrak{C}}{5\cdot 6} + \frac{\mathfrak{D}}{7\cdot 8} - \frac{\mathfrak{E}}{9\cdot 10} + \text{etc.} = \tfrac{1}{2} l2\pi .$$

159. Now with this constant known $C = \frac{1}{2} l2\pi$ the sum of any number of logarithms can be shown from this series $l1 + l2 + l3 + \text{etc}$. For if there is put

$$s = l1 + l2 + l3 + l4 + \cdots + lx,$$

there will be

$$s = \tfrac{1}{2} l2\pi + \left(x + \tfrac{1}{2}\right) xlx - x + \frac{\mathfrak{A}}{1\cdot 2x} - \frac{\mathfrak{B}}{3\cdot 4x^3} + \frac{\mathfrak{C}}{5\cdot 6x^5} - \frac{\mathfrak{D}}{7\cdot 8x^7} + \text{etc.},$$

if indeed the proposed logarithms were hyperbolic ; but if common logarithms may be proposed, then in the terms $\frac{1}{2} l2\pi + \left(x + \frac{1}{2}\right) xlx$ for $l2\pi$ and lx common logarithms will be assumed, but the remainder of the terms of the series

$$-x + \frac{\mathfrak{A}}{1\cdot 2x} - \frac{\mathfrak{B}}{3\cdot 4x^3} + \text{etc.}$$

must be multiplied by $0{,}434294481903251827 = n$. Therefore in this case for the common logarithms

$$l\pi = 0{,}497149872694133854351268$$
$$l2 = 0{,}301029995663981195213738$$

$$l2\pi = 0{,}798179868358115049565006$$
$$\tfrac{1}{2} l2\pi = 0{,}399089934179057524782503.$$

EXAMPLE

The sum of the thousand tables of logarithms is sought
$s = l1 + l2 + l3 + \cdots + l1000$.

Therefore there will be $x = 1000$ and

$$lx = 3{,}0000000000000,$$

from which there is made

$$xlx = 3000,0000000000000$$
$$\tfrac{1}{2}lx = \quad 1,5000000000000$$
$$\tfrac{1}{2}l2\pi = \quad \underline{0,3990899341790}$$
$$3001,8990899341790$$
$$\text{subtr.} \quad nx = \underline{\;434,2944819032518}$$
$$2567,6046080309272$$

Then there is

$$\frac{n\mathfrak{A}}{1\cdot 2x} = \quad 0,0000361912068$$
$$\text{subtr.} \frac{n\mathfrak{B}}{3\cdot 4x^3} = \quad \underline{0,0000000000012}$$
$$0,0000361912056$$
$$\text{there is added} \quad \underline{2567,6046080309272}$$
$$\text{the sum sought} \quad s = 2567,6046442221328.$$

Therefore since s shall be the logarithm for the product of numbers

$$1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdots 1000,$$

it is apparent this product, if actually it may be multiplied, to consist of 2568 figures, and the initial figures noted from the left to be 4023872, which in addition 2561 figures follow.

160. Therefore with the aid of this summation of logarithms the products from some number of factors, which proceed following the natural numbers, will be able to be assigned approximately. This can refer to problems chiefly, in which the middle increased part or maximum
[*i.e.* the largest binomial coefficient, which is a little larger than neighbouring coefficients, which Euler designates as the ' uncia media' : the 'middle inch' or small extra part ; there was occasionally a problem finding an appropriate word to describe precisely what was required, in Latin.]
is sought in some binomial power $(a+b)^m$, where indeed it is to be noted, if m shall be an odd number, two middle values are to be given equal to each other, which taken together may give a single middle value to the following even power. Whereby since the small maxima in some even powers shall be twice as great as the maximum may be necessary for even middle powers in the preceding power odd power, the middle maximum may be determined. Therefore let $m = 2n$, and the middle maximum may be expressed thus, so that there shall be

$$\frac{2n(2n-1)(2n-2)(2n-3)\cdots(n+1)}{1\cdot 2\cdot 3\cdot 4\cdots n}$$

This small middle maximum which is sought may be called $= u$ and that may be represented in this way, so that there shall be

$$\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots 2n}{(1\cdot 2\cdot 3\cdot 4\cdot 5\cdots n)^2},$$

and with the logarithms taken there will be

$$\begin{aligned} lu &= lI + l2 + l3 + l4 + l5 + \cdots + l2n \\ &-2l1 - 2l2 - 2l3 - 2l4 - 2l5 - \cdots - 2ln. \end{aligned}$$

161. Now truly with these hyperbolic logarithms taken there will be

$$\begin{aligned} l1 + l2 + l3 + l4 + \cdots + l2n &= \tfrac{1}{2} l2\pi + \left(2n + \tfrac{1}{2}\right) ln + \left(2n + \tfrac{1}{2}\right) l\,2 - 2n \\ &+ \frac{\mathfrak{A}}{1\cdot 2\cdot 2n} - \frac{\mathfrak{B}}{3\cdot 4\cdot 2^3 n^3} + \frac{\mathfrak{C}}{5\cdot 6\cdot 2^5 n^5} - \text{etc.} \end{aligned}$$

and

$$\begin{aligned} 2l1 + 2l2 + 2l3 + 2l4 + \cdots + 2ln &= l2\pi + (2n+1) ln - 2n \\ &+ \frac{2\mathfrak{A}}{1\cdot 2n} - \frac{2\mathfrak{B}}{3\cdot 4\cdot n^3} + \frac{2\mathfrak{C}}{5\cdot 6n^5} - \text{etc.}, \end{aligned}$$

with which expression taken from that there will be remaining

$$\begin{aligned} lu = -\tfrac{1}{2} l\pi - \tfrac{1}{2} ln + 2nl2 &+ \frac{\mathfrak{A}}{1\cdot 2\cdot 2n} - \frac{\mathfrak{B}}{3\cdot 4\cdot 2^3 n^3} + \frac{\mathfrak{C}}{5\cdot 6\cdot 2^5 n^5} - \text{etc.} \\ &- \frac{2\mathfrak{A}}{1\cdot 2n} + \frac{2\mathfrak{B}}{3\cdot 4\cdot n^3} - \frac{2\mathfrak{C}}{5\cdot 6n^5} + \text{etc.}; \end{aligned}$$

truly from these two terms taken together there will be

$$lu = l\frac{2^{2n}}{\sqrt{n\pi}} - \frac{3\mathfrak{A}}{1\cdot 2\cdot 2n} + \frac{15\mathfrak{B}}{3\cdot 4\cdot 2^3 n^3} - \frac{63\mathfrak{C}}{5\cdot 6\cdot 2^5 n^5} + \frac{255\mathfrak{D}}{7\cdot 8\cdot 2^7 n^7} - \text{etc.}$$

Let there be

$$\begin{aligned} &\frac{3\mathfrak{A}}{1\cdot 2\cdot 2^2 n^2} - \frac{15\mathfrak{B}}{3\cdot 4\cdot 2^4 n^4} + \frac{63\mathfrak{C}}{5\cdot 6\cdot 2^6 n^6} - \frac{255\mathfrak{D}}{7\cdot 8\cdot 2^8 n^8} + \text{etc.} \\ &= l\left(1 + \frac{A}{2^2 n^2} + \frac{B}{2^4 n^4} + \frac{C}{2^6 n^6} + \frac{D}{2^8 n^8} + \text{etc.}\right); \end{aligned}$$

there will be

$$lu = l\frac{2^{2n}}{\sqrt{n\pi}} - 2nl\left(1 + \frac{A}{2^2 n^2} + \frac{B}{2^4 n^4} + \frac{C}{2^6 n^6} + \frac{D}{2^8 n^8} + \text{etc.}\right)$$

and thus

$$u=\frac{2^{2n}}{\left(1+\frac{A}{2^2n^2}+\frac{B}{2^4n^4}+\frac{C}{2^6n^6}+\frac{D}{2^8n^8}+\text{etc.}\right)^{2n}\sqrt{n\pi}}.$$

Now on putting $2n=m$ there will be

$$\begin{aligned}
&l\left(1+\frac{A}{2^2n^2}+\frac{B}{2^4n^4}+\frac{C}{2^6n^6}+\frac{D}{2^8n^8}+\text{etc.}\right)=\\
&\frac{A}{m^2}+\frac{B}{m^4}+\frac{C}{m^6}+\frac{D}{m^8}+\frac{E}{m^{10}}+\text{etc.}\\
&-\frac{A^2}{2m^4}-\frac{AB}{m^6}-\frac{AC}{m^8}-\frac{AD}{m^{10}}-\text{etc.}\\
&-\frac{BB}{2m^8}-\frac{BC}{m^{10}}-\text{etc.}\\
&+\frac{A^3}{3m^6}+\frac{A^2B}{m^8}+\frac{A^2C}{m^{10}}+\text{etc.}\\
&+\frac{AB^2}{m^{10}}+\text{etc.}\\
&-\frac{A^4}{4m^8}-\frac{A^3B}{m^{10}}-\text{etc.}\\
&+\frac{A^5}{5m^{10}}+\text{etc.};
\end{aligned}$$

which expression since it must be equal to this

$$\frac{3\mathfrak{A}}{1\cdot2\cdot m^2}-\frac{15\mathfrak{B}}{3\cdot4m^4}+\frac{63\mathfrak{C}}{5\cdot6m^6}-\frac{255\mathfrak{D}}{7\cdot8m^8}+\text{etc.},$$

becomes

$$\begin{aligned}
A&=\frac{3\mathfrak{A}}{1\cdot2}\\
B&=\frac{A^2}{2}-\frac{15\mathfrak{B}}{3\cdot4}\\
C&=AB-\frac{1}{3}A^3+\frac{63\mathfrak{C}}{5\cdot6}\\
D&=AC+\frac{1}{2}B^2-A^2B+\frac{1}{4}A^4-\frac{255\mathfrak{D}}{7\cdot8}\\
E&=AD+BC-A^2C-AB^2+A^3B-\frac{1}{5}A^5+\frac{1023\mathfrak{E}}{9\cdot10}
\end{aligned}$$

etc.

162. Now since there shall be $\mathfrak{A}=\frac{1}{6}$, $\mathfrak{B}=\frac{1}{30}$, $\mathfrak{C}=\frac{1}{42}$, $\mathfrak{D}=\frac{1}{30}$, $\mathfrak{E}=\frac{5}{66}$, there will be

$$A=\frac{1}{4},\quad B=-\frac{1}{96},\quad C=\frac{27}{640},\quad D=-\frac{90031}{2^{11}\cdot3^2\cdot5\cdot7}\quad \text{etc.}$$

Hence there is brought about

$$u=\frac{2^{2n}}{\left(1+\frac{1}{2^4n^2}-\frac{1}{2^9.3n^4}+\frac{27}{2^{13}.5n^6}-\frac{90031}{2^{19}.3^2.5\cdot7n^8}+\text{etc.}\right)^{2n}\sqrt{n\pi}}$$

or

$$u=\frac{2^{2n}\left(1-\frac{1}{2^4n^2}+\frac{1}{2^9.3n^4}-\frac{27}{2^{13}.5n^6}+\frac{90031}{2^{19}.3^2.5\cdot7n^8}-\text{etc.}\right)^{2n}}{\sqrt{n\pi}},$$

or if that raising of the series may be actually put in place, there will be approximately

$$u=\frac{2^{2n}}{\sqrt{n\pi}\left(1+\frac{1}{4n}+\frac{1}{32n^2}-\frac{1}{128n^3}-\frac{5}{16\cdot5128n^4}+\text{etc.}\right)};$$

hence the middle term in $(1+1)^{2n}$ will be to the sum of all the terms 2^{2n}

$$\text{as 1 to }\sqrt{n\pi}\left(1+\tfrac{1}{4n}+\tfrac{1}{32n^2}-\tfrac{1}{128n^3}-\tfrac{5}{16\cdot5128n^4}+\text{etc.}\right);$$

or on putting for the sake of brevity $4n=v$ there will be that ratio

$$\text{as 1 to }\sqrt{n\pi}\left(1+\tfrac{1}{v}+\tfrac{1}{2v^2}-\tfrac{1}{2v^3}-\tfrac{5}{8v^4}+\tfrac{23}{8v^5}+\tfrac{53}{16v^6}-\text{etc.}\right).$$

EXAMPLE 1

The maximum middle term is sought in the expanded binomial $(a+b)^{10}$ *, which it is agreed to be*

$$\tfrac{10\cdot9\cdot8\cdot7\cdot6}{1\cdot2\cdot3\cdot4\cdot5}=252.$$

By using the final formula for u found there will be $n=5$

$$\tfrac{1}{4n}=0{,}0500000$$
$$\tfrac{1}{32n^2}=0{,}0012500$$
$$0{,}0512500$$

$$\text{subtr. }\tfrac{1}{128n^3}=\quad 625$$
$$0{,}0511875$$
$$\text{subtr. }\tfrac{1}{16\cdot128n^4}=\quad 39$$

Hence $\quad 1+\frac{1}{4n}+\text{etc.}=\underline{1,0511836}$

The log. of which

$$
\begin{aligned}
&= 0,0216784\\
ln &= 0,6989700\\
l\pi &= \underline{0,4971498}\\
&\ 1,2177982\\
l\sqrt{n\pi}\left(1+\text{etc.}\right) &= 0,6088991\\
l2^{2n} &= \underline{3,0102999}\\
lu &= 2,4014008
\end{aligned}
$$

from which there becomes $\quad u=252$.

EXAMPLE 2

The ratio may be investigated, which the middle term may hold to the sum of all 2^{100} *in the hundredth power of the binomial* $1+1$.

According to this with the formula first found, we may use

$$lu=l\frac{2^{2n}}{\sqrt{\pi n}}-\frac{3\mathfrak{A}}{1\cdot 2\cdot 2n}+\frac{15\mathfrak{B}}{3\cdot 4\cdot 2^3 n^3}-\frac{63\mathfrak{C}}{5\cdot 6\cdot 2^5 n^5}+\frac{255\mathfrak{D}}{7\cdot 8\cdot 2^7 n^7}-\text{etc.}$$

in which on putting $2n=m$, so that these powers $(1+1)^m$ may be had, and in place of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. with the values substituted, the equation becomes

$$lu=l\frac{2^m}{\sqrt{\frac{1}{2}m\pi}}-\frac{1}{4m}+\frac{1}{24m^3}-\frac{1}{20m^5}+\frac{17}{112m^7}-\frac{31}{36m^9}+\frac{691}{88m^{11}}-\text{etc.;}$$

which logarithms since they shall be hyperbolic, may be multiplied by

$$k=0,434294481903251,$$

so that they may be changed into tabulated ones, and there will be

$$lu=l\frac{2^m}{\sqrt{\frac{1}{2}m\pi}}-\frac{k}{4m}+\frac{k}{24m^3}-\frac{k}{20m^5}+\frac{17k}{112m^7}-\frac{31k}{36m^9}+\text{etc.;}$$

from which, since the small central maximum shall be u, the ratio sought will be $2m:u$, and therefore

$$l\frac{2^m}{u}=l\sqrt{\frac{1}{2}m\pi}+\frac{k}{4m}-\frac{k}{24m^3}+\frac{k}{20m^5}-\frac{17k}{112m^7}+\frac{31k}{36m^9}-\frac{691k}{88m^{11}}+\text{etc.}$$

Whereby, since there shall be on account of the exponent $m = 100$

$$\frac{k}{m} = 0{,}0043429448, \quad \frac{k}{m^3} = 0{,}0000004343, \quad \frac{k}{m^5} = 0{,}0000000000,$$

there will be

$$\frac{k}{4m} = 0{,}0010857362$$
$$\frac{k}{24m^3} = \underline{0{,}0000000181}$$
$$\underline{0{,}0010857181}$$

Then there is

$$l\pi = 0{,}4971498726$$
$$l\tfrac{1}{2}m = \underline{1{,}6989700043}$$
$$l\tfrac{1}{2}m\pi = 2{,}1961198769$$
$$l\sqrt{\tfrac{1}{2}m\pi} = 1{,}0980599384$$
$$\frac{k}{4m} - \frac{k}{24m^3} + \text{etc.} = \underline{0{,}0010857181}$$
$$1{,}0991456565 = l\,\frac{2^{100}}{u}.$$

Hence there will be $\frac{2^{100}}{u} = 12{,}56451$ and so in the power $(1+1)^m$ evaluated the middle term will itself be had in the ratio to the sum of all 2^{100} as 1 to 12,5645l.

163. Now the general term *z* may denote the exponential function a^x, thus so that this geometric series ought to be summed

$$s = a + a^2 + a^3 + a^4 + \cdots + a^x \;;$$

which since it shall be geometric, the sum of this now may be agreed upon ; indeed it shall be $s = \frac{(a^x-1)a}{a-1}$. But we may investigate this sum in the manner set out here. Because there is $z = a^x$, there will be $\int z dz = \frac{a^x}{la}$; for the differential of this is $a^x dx$; then truly there will be

$$\frac{dz}{dx} = a^x la, \quad \frac{ddz}{dx^2} = a^x (la)^2, \quad \frac{d^3z}{dx^3} = a^x (la)^3 \quad \text{etc.},$$

from which it follows that the sum becomes

$$s = a^x\left(\frac{1}{la} + \frac{1}{2} + \frac{\mathfrak{A}}{1\cdot 2} la - \frac{\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}(la)^3 + \frac{\mathfrak{C}}{1\cdot 2\cdot 3\cdots 6}(la)^5 - \text{etc.}\right) + C.$$

Towards defining the constant C there may be put $x=0$ and on account of $s=0$ there will be

$$C=-\frac{1}{la}-\frac{1}{2}-\frac{\mathfrak{A}}{1\cdot 2}la+\frac{\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}(la)^3-\text{etc.}$$

and thus there arises

$$s=\left(a^x-1\right)\left(\frac{1}{la}+\frac{1}{2}+\frac{\mathfrak{A}}{1\cdot 2}la-\frac{\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}(la)^3+\frac{\mathfrak{C}}{1\cdot 2\cdot 3\cdots 6}(la)^5-\text{etc.}\right).$$

Therefore since the sum shall be $s=\frac{(a^x-1)a}{a-1}$, there will be

$$\frac{a}{a-1}=\frac{1}{la}+\frac{1}{2}+\frac{\mathfrak{A}}{1\cdot 2}la-\frac{\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}(la)^3+\frac{\mathfrak{C}}{1\cdot 2\cdot 3\cdots 6}(la)^5-\text{etc.},$$

where la denotes the hyperbolic logarithm of a; hence there becomes

$$\frac{(a+1)la}{2(a-1)}=1+\frac{\mathfrak{A}(la)^2}{1\cdot 2}-\frac{\mathfrak{B}(la)^4}{1\cdot 2\cdot 3\cdot 4}+\frac{\mathfrak{C}(la)^6}{1\cdot 2\cdot 3\cdots 6}-\text{etc.},$$

and thus the sum of this series will be able to be shown.

164. Let the general term be $z=\sin ax$ and

$$s=\sin a+\sin 2a+\sin 3a+\cdots+\sin ax;$$

which series, since it shall be recurring, also is able to be summed; indeed there shall be

$$s=\frac{\sin a+\sin ax-\sin(ax+a)}{1-2\cos a+1}=\frac{\sin a+(1-\cos a)\sin ax-\sin a\cdot\cos ax}{2(1-\cos a)}.$$

Now there will be

$$\int zdx=\int dx\sin ax=-\frac{1}{a}\cos ax$$

and

$$\frac{dz}{dx}=a\cos ax,\quad \frac{d^3z}{dx^3}=-a^3\cos ax,\quad \frac{d^5z}{dx^5}=a^5\cos ax \text{ etc.}$$

Hence

$$s=C-\frac{1}{a}\cos ax+\frac{1}{2}\sin ax+\frac{\mathfrak{A}a\cos ax}{1\cdot 2}+\frac{\mathfrak{B}a^3\cos ax}{1\cdot 2\cdot 3\cdot 4}$$
$$+\frac{\mathfrak{C}a^5\cos ax}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}+\frac{\mathfrak{D}a^7\cos ax}{1\cdot 2\cdots 8}+\text{ etc.}$$

There may be put $x = 0$, so that there becomes $s = 0$, and there will be

$$C = \frac{1}{a} - \frac{\mathfrak{A}a}{1\cdot 2} - \frac{\mathfrak{B}a^3}{1\cdot 2\cdot 3\cdot 4} - \frac{\mathfrak{C}a^5}{1\cdot 2\cdots 6} - \text{etc.},$$

hence

$$s = \frac{1}{2}\sin ax + (1-\cos ax)\left(\frac{1}{a} - \frac{\mathfrak{A}a}{1\cdot 2} - \frac{\mathfrak{B}a^3}{1\cdot 2\cdot 3\cdot 4} - \frac{\mathfrak{C}a^5}{1\cdot 2\cdots 6} - \text{etc.}\right).$$

But since there shall be

$$s = \frac{1}{2}\sin ax + \frac{(1-\cos ax)\sin a}{2(1-\cos a)},$$

the equation becomes

$$\frac{\sin a}{2(1-\cos a)} = \frac{1}{2}\cot\frac{1}{2}a = \frac{1}{a} - \frac{\mathfrak{A}a}{1\cdot 2} - \frac{\mathfrak{B}a^3}{1\cdot 2\cdot 3\cdot 4} - \frac{\mathfrak{C}a^5}{1\cdot 2\cdots 6} - \text{etc.},$$

as now we had the same series above (§ 127).

165. Now let there be $z = \cos ax$ and the series required to be summed

$$s = \cos a + \cos 2a + \cos 3a + \cdots + \cos ax;$$

the sum of which series, because it is recurring, will be

$$s = \frac{\cos a - 1 + \cos ax - \cos(ax+a)}{1 - 2\cos a + 1} = -\frac{1}{2} + \frac{1}{2}\cos ax + \frac{1}{2}\cot\frac{1}{2}a\cdot\sin ax.$$

But truly towards expressing the sum by our method there will be

$$\int z dx = \int dx \cos ax = \frac{1}{a}\sin ax$$

and

$$\frac{dz}{dx} = -a\sin ax, \quad \frac{d^3z}{dx^3} = a^3\sin ax, \quad \frac{d^5z}{dx^5} = -a^5\sin ax \text{ etc.}$$

Therefore

$$s = C + \frac{1}{a}\sin ax + \frac{1}{2}\cos ax - \frac{\mathfrak{A}a\sin ax}{1\cdot 2} - \frac{\mathfrak{B}a^3\sin ax}{1\cdot 2\cdot 3\cdot 4} - \text{etc.}$$

Let $x = 0$; there will be $s = 0$ and $C = -\frac{1}{2}$ and hence there will be

$$s = -\tfrac{1}{2} + \tfrac{1}{2}\cos ax + \tfrac{1}{a}\sin ax - \frac{\mathfrak{A} a \sin ax}{1 \cdot 2} - \frac{\mathfrak{B} a^3 \sin ax}{1 \cdot 2 \cdot 3 \cdot 4} - \text{etc.}$$

Whereby, since there shall be

$$s = -\tfrac{1}{2} + \tfrac{1}{2}\cos ax + \tfrac{1}{2}\cot \tfrac{1}{2} a \cdot \sin ax,$$

there will be, as now we have just shown (in §164),

$$\tfrac{1}{2}\cot \tfrac{1}{2} a = \tfrac{1}{a} - \frac{\mathfrak{A} a}{1 \cdot 2} - \frac{\mathfrak{B} a^3}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{\mathfrak{C} a^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \text{etc.}$$

166. Since we have found above (in §92), if *a* may denote some arc, there shall be

$$\tfrac{\pi}{2} = \tfrac{a}{2} + \sin a + \tfrac{1}{2}\sin 2a + \tfrac{1}{3}\sin 3a + \tfrac{1}{4}\sin 4a + \text{etc.},$$

we may consider this series and there shall be $z = \tfrac{1}{x}\sin ax$, so that there shall be

$$s = \sin a + \tfrac{1}{2}\sin 2a + \tfrac{1}{3}\sin 3a + \tfrac{1}{4}\sin 4a + \cdots + \tfrac{1}{x}\sin ax.$$

But in this case there becomes $\int z dx = \int \frac{dx}{x}\sin ax$, which integral cannot be shown. Truly there shall be

$$\frac{dz}{dx} = \frac{a}{x}\cos ax - \frac{1}{xx}\sin ax, \quad \frac{ddz}{dx^2} = -\frac{a^2}{x}\sin ax - \frac{2a}{xx}\cos ax + \frac{2}{x^3}\sin ax,$$
$$\frac{d^3 z}{dx^3} = -\frac{a^3}{x}\cos ax + \frac{3a^2}{x^2}\sin ax + \frac{6a}{x^3}\cos ax - \frac{6}{x^4}\sin ax,$$
$$\frac{d^4 z}{dx^4} = \frac{a^4}{x}\sin ax + \frac{4a^3}{xx}\cos ax - \frac{12a^2}{x^3}\sin ax - \frac{24a}{x^4}\cos ax + \frac{24}{x^5}\sin ax.$$

Therefore because neither the integral formula $\int z dx$ can be shown nor are these differentials permitted to be expressed conveniently enough, we are not able to define the sum of this series by this method, thus so that anything thence may be concluded. The same inconvenience occurs in many other series, as often as the general term is not simple enough, so that the differentials of this may be able to be expressed by a suitable law. On account of which in the following chapter we will elicit other general expressions for the sum of series, the general terms of which either are exceedingly well arranged or may be completely unable to be given ; which with a more fruitful approach will be able to be called into use. But here especially the insufficiency of the method used may become apparent, if the signs of the proposed series may alternate; for then, however much the general terms shall be simple, yet the summatory terms cannot be conveniently shown by this method.

CAPUT VI

DE SUMMATIONE PROGRESSIONUM PER SERIES INFINITAS

140. Expressio generalis, quam in capite praecedente pro termino summatorio cuiusque seriei, cuius terminus generalis seu indici *x* respondens est $=z$, invenimus,

$$Sz=\int zdx+\tfrac{1}{2}z+\frac{\mathfrak{A}dz}{1\cdot 2dx}-\frac{\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3}+\frac{\mathfrak{C}d^5z}{1\cdot 2\cdot 3\cdot\cdot 6dx^5}-\text{etc.}$$

proprie inservit seriebus summandis, quarum termini generales sunt functiones quaecunque rationales integrae indicis *x*, quoniam his casibus ad differentialia tandem evanescentia pervenitur. Sin autem *z* non fuerit eiusmodi functio ipsius *x*, tum eius differentialia in infinitum progrediuntur sicque resultat series infinita summam seriei propositae exprimens et quidem ad datum usque terminum, cuius index est $=x$. Quocirca progressionis propositae in infinitum continuatae summa prodibit, si ponatur $x=\infty$; hocque pacto alia invenitur series infinita priori aequalis.

141. Sin, autem ponatur $x=0$, tum expressio summam exhibens debet evanescere, uti iam annotavimus; quod nisi fiat, eiusmodi quantitas constans ad summam addi vel inde auferri debet, ut huic conditioni satisfiat. Quo facto si ponatur $x=1$, summa inventa praebebit terminum primum seriei; sin $x=2$, aggregatum primi et secundi, sin $x=3$, orietur aggregatum trium terminorum initialium seriei, et ita porro. His igitur casibus, quia summa unius vel duorum vel trium etc. terminorum est cognita, seriei infinitae, qua ista summa exprimitur, valor innotescet, ex hocque fonte innumerabiles series summari poterunt.

142. Quoniam, si eiusmodi constans summae fuerit adiecta, ut ea evanescat posito $x=0$, tum summa omnibus reliquis casibus, quicunque numeri pro *x* substituantur, satisfacit, manifestum est, dummodo summae inventae eiusmodi quantitas constans adiiciatur, ut uno quodam casu vera summa indicetur, tum omnibus reliquis casibus veram summam prodire debere. Quare si ponendo $x=0$ non pateat, cuiusmodi valorem expressio summae recipiat, neque igitur constans adiicienda hinc inveniri queat, tum alius quicunque numerus pro *x* statui poterit adiiciendaque constante effici, ut debita summa indicetur; quod quomodo fieri debeat, ex sequentibus magis fiet perspicuum.

142a. Consideremus primum hanc progressionem harmonicam

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots+\tfrac{1}{x}=s\,;$$

cuius terminus generalis cum sit $=\frac{1}{x}$, fiet $z=\frac{1}{x}$ et terminus summatorius s ita invenietur. Primo erit $\int zdx=\int\frac{dx}{x}=lx$; deinde differentialia ita se habebunt

$$\frac{dz}{dx}=-\frac{1}{x^2},\quad \frac{ddz}{2dx^2}=\frac{1}{x^3},\quad \frac{d^3z}{6dx^3}=-\frac{1}{x^4},\quad \frac{d^4z}{24dx^4}=\frac{1}{x^5},\quad \frac{d^5z}{120dx^5}=-\frac{1}{x^6}\text{ etc.}$$

Hinc itaque erit

$$s=lx+\frac{1}{2x}-\frac{\mathfrak{A}}{2x^2}+\frac{\mathfrak{B}}{4x^4}-\frac{\mathfrak{C}}{6x^6}+\frac{\mathfrak{D}}{8x^8}-\text{etc.}+\text{Constante.}$$

Constans igitur hic addenda ex casu $x=0$ non potest definiri. Ponatur ergo $x = 1$, quia tum fit $s=1$; erit

$$1=\frac{1}{2}-\frac{\mathfrak{A}}{2}+\frac{\mathfrak{B}}{4}-\frac{\mathfrak{C}}{6}+\frac{\mathfrak{D}}{8}-\text{etc.}+\text{Const.},$$

unde fit ista constans

$$=\frac{1}{2}+\frac{\mathfrak{A}}{2}-\frac{\mathfrak{B}}{4}+\frac{\mathfrak{C}}{6}-\frac{\mathfrak{D}}{8}+\text{etc.}$$

eritque ideo terminus summatorius quaesitus

$$\begin{aligned}s=lx&+\frac{1}{2x}-\frac{\mathfrak{A}}{2x^2}+\frac{\mathfrak{B}}{4x^4}-\frac{\mathfrak{C}}{6x^6}+\frac{\mathfrak{D}}{8x^8}-\text{etc.}\\&+\frac{1}{2}+\frac{\mathfrak{A}}{2}-\frac{\mathfrak{B}}{4}+\frac{\mathfrak{C}}{6}-\frac{\mathfrak{D}}{8}+\text{etc.}\end{aligned}$$

143. Quoniam numeri BERNOULLIANI $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. constituunt seriem divergentem, hic valor constantis cognosci nequit. Sin autem loco x substituatur numerus maior atque summa totidem terminorum actu quaeratur, valor constantis commode investigabitur. Ponatur in hunc finem $x=10$ decemque primis terminis colligendis reperietur eorum summa

$$=2{,}928968253968253968,$$

cui aequalis esse debet expressio summae, si in ea ponatur $x=10$, quae fit

$$l10+\frac{1}{20}-\frac{\mathfrak{A}}{200}+\frac{\mathfrak{B}}{40000}-\frac{\mathfrak{C}}{6000000}+\frac{\mathfrak{D}}{800000000}-\text{etc.}+C.$$

Sumto ergo pro $l10$ logarithmo hyperbolico denarii et loco $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. substitutis valoribus supra inventis reperietur constans illa

$$C=0{,}5772156649015325,$$

qui numerus ergo exprimit summam seriei

$$\frac{1}{2}+\frac{\mathfrak{A}}{2}-\frac{\mathfrak{B}}{4}+\frac{\mathfrak{C}}{6}-\frac{\mathfrak{D}}{8}+\frac{\mathfrak{E}}{10}-\text{etc.}$$

144. Si pro *x* numeri non nimis magni substituantur, quia summa seriei facile actu invenitur, obtinebitur summa seriei huius

$$\frac{1}{2x}-\frac{\mathfrak{A}}{2x^2}+\frac{\mathfrak{B}}{4x^4}-\frac{\mathfrak{C}}{6x^6}+\frac{\mathfrak{D}}{8x^8}-\text{etc.}=s-lx-\text{C}.$$

Sin autem *x* significet numerum valde magnum, quia tum valor huius expressionis in infinitum excurrentis facile in fractionibus decimalibus assignatur, vicissim summa seriei definietur. Ac primo quidem constat, si series in infinitum continuetur, eius summam futuram esse infinite magnam; facto enim $x=\infty$ fit *lx* quoque infinitus, etsi *lx* ad *x* rationem infinite parvam teneat. Quo autem commodius summa quotcunque terminorum seriei assignari queat, valores litterarum $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. in fractionibus decimalibus exprimamus.

$$\begin{aligned}
\mathfrak{A}&=0,16666666666666\\
\mathfrak{B}&=0,0333333333333\\
\mathfrak{C}&=0,0238095238095\\
\mathfrak{D}&=0,0333333333333\\
\mathfrak{E}&=0,0757575757575\\
\mathfrak{F}&=0,2531135531135\\
\mathfrak{G}&=1,1666666666666\\
\mathfrak{H}&=7,0921568627451 \text{ etc.}
\end{aligned}$$

unde ergo erit

$$\begin{aligned}
\frac{\mathfrak{A}}{2}&=0,0833333333333\\
\frac{\mathfrak{B}}{4}&=0,0083333333333\\
\frac{\mathfrak{C}}{6}&=0,0039682539682\\
\frac{\mathfrak{D}}{8}&=0,0041666666666\\
\frac{\mathfrak{E}}{10}&=0,0075757575757\\
\frac{\mathfrak{F}}{12}&=0,0210927960928\\
\frac{\mathfrak{G}}{14}&=0,0833333333333\\
\frac{\mathfrak{H}}{16}&=0,4432598039216 \text{ etc.}
\end{aligned}$$

EXEMPLUM 1

Invenire summam mille terminorum seriei $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\text{etc.}$

Ponatur ergo $x = 1000$, et cum sit,

$$l10 = 2{,}3025850929940456840$$

erit

$$\begin{array}{rl} lx = & 6{,}9077552789821 \\ \text{Const.} = & 0{,}5772156649015 \\ \frac{1}{2x} = & 0{,}0005000000000 \\ \hline & 7{,}4854709438836 \\ \text{subtr.}\frac{\mathfrak{A}}{2xx} = & 0{,}0000000833333 \\ \hline & 7{,}4854708605503 \\ \text{add.}\frac{\mathfrak{B}}{4x^2} = & 0{,}0000000000000 \\ \hline & 7{,}4854708605503 \end{array}$$

Ergo 7,4854708605503

est summa quaesita mille terminorum, qui nequidem septem unitates cum semisse conficiunt.

EXEMPLUM 2

Invenire summam millies mille terminorum seriei $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\text{etc.}$

Quia est $x = 1000000$, erit $lx = 6 \cdot l10$, ergo

$$\begin{array}{rl} lx = & 13{,}8155105579642 \\ \text{Const.} = & 0{,}5772156649015 \\ \frac{1}{2x} = & 0{,}0000005000000 \\ \hline & 14{,}3927267228657 = \text{summae quaesitae} \end{array}$$

145. Si ergo pro x statuatur numerus vehementer magnus, summa satis exacte invenitur ex solo primo termino lx constante C aucto; unde egregia corollaria deduci possunt. Sic si x fuerit numerus vehementer magnus ponaturque

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{x} = s$$

et

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots+\tfrac{1}{x}+\cdots+\tfrac{1}{x+y}=t,$$

quia est proxime $s=lx+C$ et $t=l(x+y)+C$, erit

$$t-s=l(x+y)-lx=l\,\tfrac{x+y}{x}$$

ideoque hic logarithmus proxime per seriem harmonicam finito terminorum numero constantem exprimetur hoc modo

$$l\,\tfrac{x+y}{x}=\tfrac{1}{x+1}+\tfrac{1}{x+2}+\tfrac{1}{x+3}+\cdots+\tfrac{1}{x+y}.$$

Accuratius autem hic logarithmus exhibebitur, si superiores summae *s* et *t* exactius capiantur. Sic cum sit

$$s=lx+\tfrac{1}{2x}-\tfrac{1}{12xx}\quad\text{et}\quad t=l(x+y)+C+\tfrac{1}{2(x+y)}-\tfrac{1}{12(x+y)^2},$$

erit

$$t-s=l\,\tfrac{x+y}{x}-\tfrac{1}{2x}+\tfrac{1}{2(x+y)}+\tfrac{1}{12xx}-\tfrac{1}{12(x+y)^2}$$

ideoque

$$l\,\tfrac{x+y}{x}=\tfrac{1}{x+1}+\tfrac{1}{x+2}+\tfrac{1}{x+3}+\cdots+\tfrac{1}{x+y}+\tfrac{1}{2x}-\tfrac{1}{2(x+y)}-\tfrac{1}{12xx}+\tfrac{1}{12(x+y)^2}$$

Sin autem *x* sit numerus tam magnus, ut bini termini ultimi reiici queant, erit proxime

$$l\,\tfrac{x+y}{x}=\tfrac{1}{x+1}+\tfrac{1}{x+2}+\tfrac{1}{x+3}+\cdots+\tfrac{1}{x+y}+\tfrac{1}{2}\left(\tfrac{1}{x}-\tfrac{1}{x+y}\right).$$

145a. Ex hac quoque serie harmonica derivare poterimus summam huius seriei, in qua tantum numeri impares occurrunt,

$$\tfrac{1}{1}+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{9}+\cdots+\tfrac{1}{2x+1}.$$

Cum enim omnibus terminis capiendis sit

$$\begin{aligned}&\tfrac{1}{1}+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots+\tfrac{1}{2x}+\tfrac{1}{2x+1}\\&=l(2x+1)+\mathrm{C}+\tfrac{1}{2(2x+1)}-\tfrac{\mathfrak{A}}{2(2x+1)^2}+\tfrac{\mathfrak{B}}{4(2x+1)^4}-\tfrac{\mathfrak{C}}{6(2x+1)^6}+\text{etc.},\end{aligned}$$

terminorum vero ordine parium

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2x}$$

summa sit semissis superioris, nempe

$$\frac{1}{2}\mathrm{C}+\frac{1}{2}lx+\frac{1}{4x}-\frac{\mathfrak{A}}{4x^2}+\frac{\mathfrak{B}}{8x^4}-\frac{\mathfrak{C}}{12x^6}+\frac{\mathfrak{D}}{16x^4}-\text{etc.},$$

erit hac serie ab illa ablata

$$\begin{aligned}&\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots+\frac{1}{2x+1}\\&=\frac{1}{2}\mathrm{C}+l\frac{2x+1}{\sqrt{x}}+\frac{1}{2(2x+1)}-\frac{\mathfrak{A}}{2(2x+1)^2}+\frac{\mathfrak{B}}{4(2x+1)^4}-\text{etc.}\\&-\frac{1}{4x}+\frac{\mathfrak{A}}{4x^2}-\frac{\mathfrak{B}}{8x^4}+\text{etc.}\end{aligned}$$

146. Potest vero etiam per eandem expressionem generalem summa cuiusque seriei harmonicae inveniri; sit enim

$$\frac{1}{m+n}+\frac{1}{2m+n}+\frac{1}{3m+n}+\frac{1}{4m+n}+\cdots+\frac{1}{mx+n}=s\,;$$

quia est terminus generalis $z=\frac{1}{mx+n}$, erit

$$\int zdx=\frac{1}{m}l\left(mx+n\right),\quad \frac{dz}{dx}=-\frac{m}{(mx+n)^2},\quad \frac{ddz}{2dx^2}=\frac{mm}{(mx+n)^3},$$
$$\frac{d^3z}{6dx^3}=-\frac{m^3}{(mx+n)^4},\quad \frac{d^4z}{24dx^4}=\frac{m^4}{(mx+n)^5},\quad \frac{d^5z}{120dx^5}=-\frac{m^5}{(mx+n)^6},\quad \text{etc.}$$

Ex his ergo reperitur

$$\begin{aligned}s=D+\frac{1}{m}l\left(mx+n\right)+\frac{1}{2(mx+n)}-\frac{\mathfrak{A}m}{2(mx+n)^2}+\frac{\mathfrak{B}m^3}{4(mx+n)^3}\\-\frac{\mathfrak{C}m^5}{6(mx+n)^6}+\frac{\mathfrak{D}m^7}{8(mx+n)^8}-\text{etc.}\end{aligned}$$

Posito ergo $x=0$ fiet constans illa addenda

$$D=-\frac{1}{m}ln-\frac{1}{2n}+\frac{\mathfrak{A}m}{2n^2}-\frac{\mathfrak{B}m^3}{4n^3}+\frac{\mathfrak{C}m^5}{6n^4}-\text{etc.}$$

147. Si vero sit $n = 0$, quoniam seriei

$$\frac{1}{m}+\frac{1}{2m}+\frac{1}{3m}+\frac{1}{4m}+\cdots+\frac{1}{mx}$$

summa est

$$=\frac{1}{m}C+\frac{1}{m}lx+\frac{1}{2mx}-\frac{\mathfrak{A}}{2mx^2}+\frac{\mathfrak{B}}{4mx^3}-\text{etc.},$$

at vero huius seriei

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{mx}$$

summa est

$$=C+lmx+\frac{1}{2mx}-\frac{\mathfrak{A}}{2m^2x^2}+\frac{\mathfrak{B}}{4m^4x^4}-\text{etc.},$$

si ab hac serie ilIa *m* vicibus sumta subtrahatur, ut prodeat haec series

$$\begin{aligned}1+\frac{1}{2}+\cdots+\frac{1}{m}&+\cdots+\frac{1}{2m}+\cdots+\frac{1}{3m}+\cdots+\frac{1}{mx}\\ -\frac{m}{m}&\qquad -\frac{m}{2m}\qquad -\frac{m}{3m}\qquad -\frac{m}{mx},\end{aligned}$$

eius summa erit

$$\begin{aligned}&=lm+\frac{1}{2mx}-\frac{\mathfrak{A}}{2m^2x^2}+\frac{\mathfrak{B}}{4m^4x^4}-\text{etc.}\\ &\quad-\frac{1}{2x}+\frac{\mathfrak{A}}{2xx}-\frac{\mathfrak{B}}{4x^4}+\text{etc.},\end{aligned}$$

atque si statuatur $x=\infty$, summa erit $=lm$. Hinc pro *m* ponendo numeros 2, 3, 4 etc. erit

$$\begin{aligned}l2&=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\text{etc.}\\ l3&=1+\frac{1}{2}-\frac{2}{3}+\frac{1}{4}+\frac{1}{5}-\frac{2}{6}+\frac{1}{7}+\frac{1}{8}-\frac{2}{9}+\text{etc.}\\ l4&=1+\frac{1}{2}+\frac{1}{3}-\frac{3}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}-\frac{3}{8}+\text{etc.}\\ l5&=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{4}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\frac{4}{10}+\text{etc.}\\ &\text{etc.}\end{aligned}$$

148. Relicta autem serie harmonica progrediamur ad seriem quadratorum reciprocam sitque

$$s=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots+\frac{1}{xx};$$

in qua cum sit terminus generalis $z=\frac{1}{xx}$, erit $\int zdx=-\frac{1}{x}$ et differentialia ipsius z ita se habebunt

$$\frac{dz}{2dx}=-\frac{1}{x^3},\quad \frac{ddz}{2\cdot 3dx^2}=\frac{1}{x^4},\quad \frac{d^3z}{2\cdot 3\cdot 4dx^3}=-\frac{1}{x^5} \text{ etc.},$$

unde erit summa

$$s=C-\frac{1}{x}+\frac{1}{2xx}-\frac{\mathfrak{A}}{x^3}+\frac{\mathfrak{B}}{x^5}-\frac{\mathfrak{C}}{x^7}+\frac{\mathfrak{D}}{x^9}-\frac{\mathfrak{E}}{x^7}+\text{ etc.},$$

in qua constans addenda C ex uno casu, quo summa constat, est definienda. Ponamus ergo $x=1$; quia fit $s=1$, debet esse

$$s=1+1-\tfrac{1}{2}+\mathfrak{A}-\mathfrak{B}+\mathfrak{C}-\mathfrak{D}+\mathfrak{E}-\text{etc.},$$

quae series autem, cum sit maxime divergens, valorem constantis C non ostendit. Quia autem supra demonstravimus summam huius seriei in infinitum continuatae esse $=\frac{\pi\pi}{6}$, facto $x=\infty$, si ponatur $s=\frac{\pi\pi}{6}$, fiet $C=\frac{\pi\pi}{6}$ ob reliquos terminos omnes evanescentes. Edt ergo

$$1+1-\tfrac{1}{2}+\mathfrak{A}-\mathfrak{B}+\mathfrak{C}-\mathfrak{D}+\mathfrak{E}-\text{etc.}=\frac{\pi\pi}{6}.$$

149. Sin autem summa huius seriei cognita non fuisset, valor constantis illius C ex alio quopiam casu, quo summa actu est inventa, determinari deberet. Hunc in finem ponamus $x=10$ atque decem terminis actu addendis reperietur

$$s=1{,}549767731166540690;$$

tum est

add.	$\frac{1}{x}$	$=0{,}1$
subtr.	$\frac{1}{2xx}$	$=0{,}005$
		1,644767731166540690
add.	$\frac{\mathfrak{A}}{x^3}$	$=0{,}000166666666666666$
		1,644934397833207356
subtr.	$\frac{\mathfrak{B}}{x^5}$	$=0{,}000000333333333333$
		1,644934064499874023
add.	$\frac{\mathfrak{C}}{x^7}$	$=0{,}000000002380952381$
		1,644934066880826404
subtr.	$\frac{\mathfrak{D}}{x^9}$	$=0{,}000000000033333333$
		1,644934066847493071
add.	$\frac{\mathfrak{E}}{x^{11}}$	$=0{,}000000000000757575$

$$\begin{array}{r} 1,644934066848250646 \\ \text{subtr.}\frac{\mathfrak{F}}{x^{13}} = 0,000000000000025311 \\ \hline 1,644934066848225335 \\ \text{add. } \frac{\mathfrak{G}}{x^{15}} = 0,000000000000001166 \\ \text{subtr.}\frac{\mathfrak{H}}{x^{17}} = 0,000000000000000071 \\ \hline 1,644934066848226430 = C. \end{array}$$

Hicque numerus simul est valor expressionis $= \frac{\pi\pi}{6}$, quemadmodum ex valore ipsius π cognito calculum instituenti patebit. Unde simul intelligitur, etiamsi series $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. divergat, tamen hoc modo veram prodire summam.

150. Sit nunc $z = \frac{1}{x^3}$ atque

$$s = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{x^3};$$

quia est

$$\int zdx = -\frac{1}{2xx}, \quad \frac{dz}{1\cdot 2\cdot 3dx} = -\frac{1}{2x^4}, \quad \frac{ddz}{1\cdot 2\cdot 3\cdot 4dx^2} = \frac{1}{2x^5},$$
$$\frac{d^3z}{1\cdot 2\cdots 5dx^3} = -\frac{1}{2x^6}, \quad \frac{d^4z}{1\cdot 2\cdots 6dx^4} = \frac{1}{2x^7}, \quad \frac{d^5z}{1\cdot 2\cdots 7dx^5} = -\frac{1}{2x^8} \quad \text{etc.},$$

erit

$$s = C - \frac{1}{2xx} + \frac{1}{2x^3} - \frac{3\mathfrak{A}}{2x^4} + \frac{5\mathfrak{B}}{2x^6} - \frac{7\mathfrak{C}}{2x^8} + \text{etc.}$$

hincque posito $x = 1$ ob $s = 1$ fiet

$$C = 1 + \frac{1}{2} - \frac{1}{2} + \frac{3}{2}\mathfrak{A} - \frac{5}{2}\mathfrak{B} + \frac{7}{2}\mathfrak{C} - \frac{9}{2}\mathfrak{D} + \text{etc.}$$

atque iste valor ipsius C simul ostendet summam seriei propositae in infinitum continuatae. Quoniam vero summae potestatum imparium non aeque ac parium constant, iste ipsius C valor ex cognita summa aliquot terminorum definiri debet. Sit ergo $x = 10$; erit

$$C = s + \frac{1}{2xx} - \frac{1}{2x^3} + \frac{3\mathfrak{A}}{2x^4} - \frac{5\mathfrak{B}}{2x^6} + \frac{7\mathfrak{C}}{2x^8} - \text{etc.}$$

Est vero ad computum facilius instituendum

$$\frac{3\mathfrak{A}}{2} = 0{,}2500000000000$$
$$\frac{5\mathfrak{B}}{2} = 0{,}0833333333333$$
$$\frac{7\mathfrak{C}}{2} = 0{,}0833333333333$$
$$\frac{9\mathfrak{D}}{2} = 0{,}1500000000000$$
$$\frac{11\mathfrak{E}}{2} = 0{,}4166666666666$$
$$\frac{13\mathfrak{F}}{2} = 1{,}6452380952380$$
$$\frac{15\mathfrak{G}}{2} = 8{,}7500000000000$$
$$\frac{17\mathfrak{H}}{2} = 60{,}2833333333333$$

etc.

Hinc ergo fient termini ad s addendi

$$\frac{1}{2xx} = 0{,}005000000000000000$$
$$\frac{3\mathfrak{A}}{2x^4} = 0{,}000025000000000000$$
$$\frac{7\mathfrak{C}}{2x^8} = 0{,}000000000833333333$$
$$\frac{11\mathfrak{E}}{2x^{12}} = 0{,}000000000000416666$$
$$\frac{15\mathfrak{G}}{2x^{16}} = 0{,}000000000000006875$$
$$\overline{0{,}005025000833750875;}$$

termini autem subtrahendi sunt

$$\frac{1}{2x^3} = 0{,}000500000000000000$$
$$\frac{5\mathfrak{B}}{2x^6} = 0{,}000000083333333333$$
$$\frac{9\mathfrak{D}}{2x^{10}} = 0{,}000000000015000000$$
$$\frac{13\mathfrak{F}}{2x^{14}} = 0{,}000000000000016452$$
$$\frac{17\mathfrak{H}}{2x^{18}} = 0{,}000000000000000060$$
$$\overline{0{,}000500083348349845}$$
$$\text{ab } \underline{0{,}005025000833750875}$$
$$0{,}004524917485401030$$
$$s = \underline{1{,}197531985674193251}$$
$$C = 1{,}202056903159594281$$

151. Si hoc modo ulterius progrediamur, inveniemus summas omnium serierum potestatum reciprocarum in fractionibus decimalibus expressas.

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{ etc.}=1,6449340668482264=\frac{2\mathfrak{A}}{1\cdot 2}\pi^2$$
$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\text{ etc.}=1,2020569031595942$$
$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{ etc.}=1,0823232337111381=\frac{2^3\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}\pi^4$$
$$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\text{ etc.}=1,0369277551433699$$
$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{ etc.}=1,0173430619844491=\frac{2^5\mathfrak{C}}{1\cdot 2\cdots 6}\pi^6$$
$$1+\frac{1}{2^7}+\frac{1}{3^7}+\frac{1}{4^7}+\text{ etc.}=1,0083492773819228$$
$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\text{ etc.}=1,0040773561979443=\frac{2^7\mathfrak{D}}{1\cdot 2\cdots 8}\pi^8$$
$$1+\frac{1}{2^9}+\frac{1}{3^9}+\frac{1}{4^9}+\text{ etc.}=1,0020083928260822$$
$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\text{ etc.}=1,0009945751278180=\frac{2^9\mathfrak{E}}{1\cdot 2\cdots 10}\pi^{10}$$
$$1+\frac{1}{2^{11}}+\frac{1}{3^{11}}+\frac{1}{4^{11}}+\text{ etc.}=1,0004941886041194$$
$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\text{ etc.}=1,0002460865533080=\frac{2^{12}\mathfrak{F}}{1\cdot 2\cdots 12}\pi^{12}$$
$$1+\frac{1}{2^{13}}+\frac{1}{3^{13}}+\frac{1}{4^{13}}+\text{ etc.}=1,0001227133475784$$
$$1+\frac{1}{2^{14}}+\frac{1}{3^{14}}+\frac{1}{4^{14}}+\text{ etc.}=1,0000612481350587=\frac{2^{13}\mathfrak{G}}{1\cdot 2\cdots 14}\pi^{14}$$
$$1+\frac{1}{2^{15}}+\frac{1}{3^{15}}+\frac{1}{4^{15}}+\text{ etc.}=1,0000305882363070$$
$$1+\frac{1}{2^{16}}+\frac{1}{3^{16}}+\frac{1}{4^{16}}+\text{ etc.}=1,0000152822594086=\frac{2^{15}\mathfrak{H}}{1\cdot 2\cdots 16}\pi^{16}$$
etc.

152. Ex his ergo vicissim summae illarum serierum infinitarum numeris BERNOULLIANIS constantium exhiberi poterunt. Erit enim

$$1+0-\tfrac{1}{2}+\tfrac{\mathfrak{A}}{2}-\tfrac{\mathfrak{B}}{4}+\tfrac{\mathfrak{C}}{6}-\tfrac{\mathfrak{D}}{8}+\text{etc.}=0{,}57721\text{ etc.}$$

$$1+1-\tfrac{1}{2}+\mathfrak{A}-\mathfrak{B}+\mathfrak{C}-\mathfrak{D}+\text{etc.}=\tfrac{2\mathfrak{A}}{1\cdot 2}\pi^2$$

$$1+\tfrac{1}{2}-\tfrac{1}{2}+\tfrac{3\mathfrak{A}}{2}-\tfrac{5\mathfrak{B}}{4}+\tfrac{7\mathfrak{C}}{6}-\tfrac{9\mathfrak{D}}{8}+\text{etc.}=1{,}2020\text{ etc.}$$

$$1+\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{3\cdot 4\mathfrak{A}}{2\cdot 3}-\tfrac{5\cdot 6\mathfrak{B}}{2\cdot 3}+\tfrac{7\cdot 8\mathfrak{C}}{2\cdot 3}-\tfrac{9\cdot 10\mathfrak{D}}{2\cdot 3}+\text{etc.}=\tfrac{2^3\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}\pi^4$$

$$1+\tfrac{1}{4}-\tfrac{1}{2}+\tfrac{3\cdot 4\cdot 5\mathfrak{A}}{2\cdot 3\cdot 4}-\tfrac{5\cdot 6\cdot 7\mathfrak{B}}{2\cdot 3\cdot 4}+\tfrac{7\cdot 8\cdot 9\mathfrak{C}}{2\cdot 3\cdot 4}-\tfrac{9\cdot 10\cdot 11\mathfrak{D}}{2\cdot 3\cdot 4}+\text{etc.}=1{,}0369\text{ etc.}$$

$$1+\tfrac{1}{5}-\tfrac{1}{2}+\tfrac{3\cdot 4\cdot 5\cdot 6\mathfrak{A}}{2\cdot 3\cdot 4\cdot 5}-\tfrac{5\cdot 6\cdot 7\cdot 8\mathfrak{B}}{2\cdot 3\cdot 4\cdot 5}+\tfrac{7\cdot 8\cdot 9\cdot 10\mathfrak{C}}{2\cdot 3\cdot 4\cdot 5}-\text{etc.}=\tfrac{2^5\mathfrak{C}}{1\cdot 2\cdots 6}\pi^6$$

Harum ergo serierum alternae ope quadraturae circuli summari possunt; a quanam vero quantitate transcendente reliquae pendeant, adhuc non constat; neque enim ad potestates ipsius n exponentes impares habentes revocari possunt, ita ut coefficientes essent numeri rationales. Quo autem saltem proxime appareat, quales futuri sint coefficientes potestatum ipsius π pro exponentibus imparibus, tabellam sequentem adiunximus.

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\text{etc. in infinin.}=\tfrac{\pi}{0{,}0000}=\infty$$

$$1+\tfrac{1}{2^2}+\tfrac{1}{3^2}+\tfrac{1}{4^2}+\text{etc. in infin.}=\tfrac{\pi^2}{6{,}0000}\text{ vere}$$

$$1+\tfrac{1}{2^3}+\tfrac{1}{3^3}+\tfrac{1}{4^3}+\text{etc. in infin.}=\tfrac{\pi^3}{25{,}79436}\text{ prox.}$$

$$1+\tfrac{1}{2^4}+\tfrac{1}{3^4}+\tfrac{1}{4^4}+\text{etc. in infin.}=\tfrac{\pi^4}{90{,}00000}\text{ vere}$$

$$1+\tfrac{1}{2^5}+\tfrac{1}{3^5}+\tfrac{1}{4^5}+\text{etc. in infin.}=\tfrac{\pi^5}{295{,}1215}\text{ prox.}$$

$$1+\tfrac{1}{2^6}+\tfrac{1}{3^6}+\tfrac{1}{4^6}+\text{etc. in infin.}=\tfrac{\pi^6}{945{,}00000}\text{ vere}$$

$$1+\tfrac{1}{2^7}+\tfrac{1}{3^7}+\tfrac{1}{4^7}+\text{etc. in infin.}=\tfrac{\pi^7}{2995{,}284}\text{ prox.}$$

$$1+\tfrac{1}{2^8}+\tfrac{1}{3^8}+\tfrac{1}{4^8}+\text{etc. in infin.}=\tfrac{\pi^8}{945{,}00000}\text{ vere}$$

$$1+\tfrac{1}{2^9}+\tfrac{1}{3^9}+\tfrac{1}{4^9}+\text{etc. in infin.}=\tfrac{\pi^9}{29749{,}35}\text{ prox.}$$

etc.

153. Ex hoc fonte series numerorum BERNOULLIANORUM

1	2	3	4	5	6	7	8	9
$\mathfrak{A}$,	$\mathfrak{B}$,	$\mathfrak{C}$,	$\mathfrak{D}$,	$\mathfrak{E}$,	$\mathfrak{F}$,	$\mathfrak{G}$,	$\mathfrak{H}$,	$\mathfrak{I}$ etc.,

quantumvis irregularis videatur, interpolari seu termini in medlo binorum quorumcunque constituti assignari poterunt; si enim terminus medium interiacens inter primum $\mathfrak{A}$ et secundum $\mathfrak{B}$ seu indici $1\frac{1}{2}$ respondens fuerit $= p$,
erit utique

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\text{etc.}=\frac{2^2 p}{1\cdot 2\cdot 3}\pi^3$$

ideoque

$$p=\frac{3}{2\pi^3}\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\text{etc.}\right)=0{,}05815227.$$

Simili modo si terminus inter $\mathfrak{B}$ et $\mathfrak{C}$ medium interiacens seu indicem habens $2\frac{1}{2}$ ponatur $= q$, quia erit

$$1+\frac{1}{2^5}+\frac{1}{3^5}+\text{etc.}=\frac{2^4 q}{1\cdot 2\cdot 3\cdot 4\cdot 5}\pi^5,$$

fiet

$$q=\frac{15}{2\pi^5}\left(1+\frac{1}{2^5}+\frac{1}{3^5}+\text{etc.}\right)=0{,}02541327.$$

Si ergo istarum serierum, in quibus exponentes potestatum sunt numeri impares, summae exhiberi possent, tum quoque series numerorum BERNOULLIANORUM interpolari posset.

154. Ponamus nunc $z = nn + xx$ et quaeratur summa huius seriei

$$s=\frac{1}{nn+1}+\frac{1}{nn+4}+\frac{1}{nn+9}+\cdots+\frac{1}{nn+xx}.$$

Quia est $\int zdx=\int\frac{dx}{nn+xx}$, erit

$$\int zdx=\frac{1}{n}\text{Atang}\frac{x}{n}.$$

Ponatur $\text{Acot}\frac{x}{n}=u$; erit $\int zdx=\frac{1}{n}\left(\frac{\pi}{2}-u\right)$
et

$$\frac{x}{n}=\cot u=\frac{\cos u}{\sin u}\quad\text{and}\quad\frac{nn+xx}{nn}=\frac{1}{\sin u^2},\quad z=\frac{\sin^2 u}{nn},\quad\frac{dx}{n}=-\frac{du}{\sin^2 u},$$

unde fit

$$du=-\frac{dx\sin^2 u}{n}.$$

Hinc differentialia ipsius z invenientur hoc modo

$$dz=\frac{2du\sin u\cdot\cos u}{nn}=-\frac{dx\sin^2 u\cdot\sin 2u}{n^3} \quad \text{et} \quad \frac{dz}{dx}=-\frac{\sin^2 u\cdot\sin 2u}{n^3},$$

$$\frac{ddz}{2dx}=-\frac{du\left(\sin u\cdot\cos u\cdot\sin 2u+\sin^2 u\cdot\cos 2u\right)}{n^3}=\frac{dx\sin^3 u\cdot\cos 3u}{n^4}$$

et

$$\frac{ddz}{2dx^2}=\frac{\sin^3 u\cdot\cos 3u}{n^4}.$$

Simili modo erit, uti iam supra pro eodem casu invenimus,

$$\frac{d^3 z}{2\cdot 3dx^3}=-\frac{\sin^4 u\cdot\sin 4u}{n^5}, \quad \frac{d^4 z}{2\cdot 3\cdot 4dx^4}=\frac{\sin^5 u\cdot\sin 5u}{n^6} \quad \text{etc.},$$

ex quibus formabitur summa quaesita

$$s=\frac{\pi}{2n}-\frac{u}{n}+\frac{\sin u\cdot\sin u}{2nn}-\frac{\mathfrak{A}}{2}\cdot\frac{\sin^2 u\cdot\sin 2u}{n^3}+\frac{\mathfrak{B}}{4}\cdot\frac{\sin^4 u\cdot\sin 4u}{n^5}$$
$$-\frac{\mathfrak{C}}{6}\cdot\frac{\sin^6 u\cdot\sin 6u}{n^7}+\frac{\mathfrak{D}}{8}\cdot\frac{\sin^8 u\cdot\sin 8u}{n^9}-\text{etc.}+\text{Const.}$$

Si hic ad constantem determinandam ponatur $x=0$, quo fiat $s=0$, erit $\cot u=0$ ideoque u angulus 90^0 ac propterea $\sin u=1$, $\sin 2u=0$, $\sin 4u=0$, $\sin 6u=0$, etc.; videtur ergo fore

$$0=\frac{\pi}{2n}-\frac{\pi}{2n}+\frac{1}{2nn}+C \quad \text{hincque} \quad C=-\frac{1}{2nn};$$

at vero notandum est, etiamsi reliqui termini evanescant, tamen, quia coefficientes $\mathfrak{A},\mathfrak{B},\mathfrak{C}$ etc. tandem in infinitum excrescunt, eorum summam posse esse finitam.

155. Ad hanc ergo constantem rite determinandum ponamus $x=\infty$; summam enim huius seriei in infinitum excurrentis supra iam in *Introductione* definivimus ostendimusque esse eam

$$=-\frac{1}{2nn}+\frac{\pi}{2n}+\frac{\pi}{n\left(e^{2n\pi}-1\right)}.$$

Posito autem $x=\infty$ fiet $u=0$ ideoque $\sin u=0$ simulque sinus omnium arcuum multiplorum evanescent. Cum autem in hac serie potestates ipsius $\sin u$ crescant, divergentia seriei impedire nequit, quominus valor seriei hoc casu evanescat. Fiet ergo $s=\frac{\pi}{2n}+C$; unde erit

$$\frac{\pi}{2n}+C=-\frac{1}{nn}+\frac{\pi}{2n}+\frac{\pi}{n\left(e^{2n\pi}-1\right)} \quad \text{et} \quad C=-\frac{1}{2nn}+\frac{\pi}{n\left(e^{2n\pi}-1\right)}.$$

Quare summa seriei quaesita erit

$$s=\frac{\pi}{2n}-\frac{u}{n}-\frac{1}{2nn}+\frac{\sin^2 u}{2nn}-\frac{\mathfrak{A}}{2}\cdot\frac{\sin^2 u\cdot\sin 2u}{n^3}+\frac{\mathfrak{B}}{4}\cdot\frac{\sin^4 u\cdot\sin 4u}{n^5}$$
$$-\frac{\mathfrak{C}}{6}\cdot\frac{\sin^6 u\cdot\sin 6u}{n^7}+\text{etc.}+\frac{\pi}{n\left(e^{2n\pi}-1\right)}.$$

Ubi notandum est, si n fuerit numerus mediocriter magnus, ultimum terminum $\frac{\pi}{n\left(e^{2n\pi}-1\right)}$ tantopere fieri exiguum, ut negligi queat.

156. Ponamus esse $x=n$ ita ut denotet

$$s=\frac{1}{nn+1}+\frac{1}{nn+4}+\frac{1}{nn+9}+\cdots+\frac{1}{nn+xx}.$$

Tum vero erit $\cot u=1$ et $u=45^0=\frac{\pi}{4}\cdot$ Quamobrem habebitur

$$\sin u=\frac{1}{\sqrt{2}},\sin 2u=1,\ \sin 4u=0,\ \sin 6u=-1,\ \sin 8u=0,\ \sin 10u=1\ \text{etc.}$$

Hanc ob rem erit

$$s=\frac{\pi}{4n}-\frac{1}{2nn}+\frac{1}{4nn}-\frac{\mathfrak{A}}{2\cdot 2n^3}+\frac{\mathfrak{C}}{6\cdot 8n^7}-\frac{\mathfrak{E}}{10\cdot 2^5 n^{11}}+\frac{\mathfrak{G}}{14\cdot 2^7 n^{15}}-\text{etc.}+\frac{\pi}{n\left(e^{2n\pi}-1\right)},$$

in qua expressione tantum numeri alterni ex BERNOULLIANIS occurrunt. Si igitur valor ipsius s per computum actu institutum iam fuerit inventus, hinc quantitas n definiri poterit; erit enim

$$\pi=4ns+\frac{1}{n}+\frac{\mathfrak{A}}{1\cdot n^2}-\frac{\mathfrak{C}}{3\cdot 2n^6}+\frac{\mathfrak{E}}{5\cdot 2^4 n^{10}}-\frac{\mathfrak{G}}{7\cdot 2^6 n^{14}}+\text{etc.}-\frac{4\pi}{e^{2n\pi}-1}.$$

Etsi enim in termino ultimo inest π, tamen, quia is tantopere est parvus, sufficit valorem ipsius π proxime nosse.

EXEMPLUM

Sit $n=5$; erit

$$s=\frac{1}{26}+\frac{1}{29}+\frac{1}{34}+\frac{1}{41}+\frac{1}{50};$$

qui termini actu additi dabunt

$$s = 0{,}146746305690549494,$$

unde erunt termini illi

$$4ns = 2{,}93492611381098988$$

$$\frac{1}{n} = 0{,}2$$

$$\frac{\mathfrak{A}}{1\cdot n^2} = \underline{0{,}00666666666666666}$$

$$3{,}14159278047765654$$

$$\frac{\mathfrak{C}}{3\cdot 2n^6} = \underline{0{,}00000012698412698}$$

$$3{,}14159265349352956$$

$$\frac{\mathfrak{E}}{5\cdot 2^4 n^{10}} = \underline{0{,}00000000009696969}$$

$$3{,}14159265359049925$$

$$\frac{\mathfrak{G}}{7\cdot 2^6 n^{14}} = \underline{0{,}00000000000042666}$$

$$3{,}14159265359007259$$

$$\frac{\mathfrak{I}}{9\cdot 2^8\cdot n^{18}} = \underline{\qquad\qquad 625}$$

$$3{,}14159265359007884.$$

Hic valor iam tam prope ad veritatem accedit, ut mirandum sit tam levi calculo eousque perveniri posse. Est vero haec expressio aliquantillum iusto maior; subtrahi enim adhuc iude debet $\frac{4\pi}{e^{2n\pi}-1}$, cuius valor, dummodo π prope sit cognitum, exhiberi potest; quod per logarithmos ita expedietur.

Quia est $\pi le = 1{,}3643763538$, erit

$$le^{2n\pi} = 10\pi le = 13{,}6437635.$$

Cum iam sit

$$\frac{4\pi}{e^{2n\pi}-1} = \frac{4\pi}{e^{2n\pi}} + \frac{4\pi}{e^{4n\pi}} + \text{etc.},$$

facile intelligitur ad nostrum calculum sufficere primum terminorum sumsisse. Augeamus ergo characteristicam numero 17, quia habemus totidem figuras decimales; erit

$$l\pi = 17{,}4971498$$

$$l4 = \underline{0{,}6020600}$$

$$18{,}0992098$$

$$\text{subtr.} \quad le^{2n\pi} = \underline{13{,}6437635}$$

$$4{,}4554463$$

Ergo

$$\frac{4\pi}{e^{2n\pi}} = \qquad 28539$$

subtrahatur ab

$$3{,}14159265359007884$$

erit

$$\pi = 3,14159265358979345$$

quae expressio in figura demum penultima a veritate recedit; quod mirandum non est, cum adhuc terminum $\frac{\mathfrak{L}}{11\cdot 2^{10}\cdot n^{22}}$ qui dat 22, subtrahere debuissemus sicque ne ultima quidem figura aberrasset. Ceterum intelligitur, si pro n maiorem numerum uti 10 assumsissemus, tum facili negotio peripheriam π ad 25 pluresque figuras, inveniri potuisse.

157. Ponamus nunc quoque pro z functiones transcendentes ipsius x sitque $z = lx$ sumendo logarithmos hyperbolicos, quoniam vulgares facile eo revocantur, sitque

$$s = l1 + l2 + l3 + l4 + \cdots + lx\,.$$

Quia igitur est $z = lx$, erit

$$\int z dx = xlx - x \;\;;$$

huius enim differentiale dat *dxlx*. Deinde est

$$\frac{dz}{dx} = \frac{1}{x}\,,\quad \frac{ddz}{dx^2} = -\frac{1}{x^2}\,,\quad \frac{d^3z}{1\cdot 2dx^3} = \frac{1}{x^3}\,,\quad \frac{d^4z}{1\cdot 2\cdot 3dx^4} = -\frac{1}{x^4}\,,\quad \frac{d^5z}{1\cdot 2\cdot 3\cdot 4dx^5} = \frac{1}{x^5}\quad \text{etc.}$$

Hinc itaque concluditur fore

$$s = xlx - x + \frac{\mathfrak{A}}{1\cdot 2x} - \frac{\mathfrak{B}}{3\cdot 4x^3} + \frac{\mathfrak{C}}{5\cdot 6x^5} - \frac{\mathfrak{D}}{7\cdot 8x^7} + \text{etc.} + \text{Const.}$$

Haec autem constans ponendo $x = 1$, quia fit $s = l1 = 0$, ita definietur, ut sit

$$C = 1 - \frac{\mathfrak{A}}{1\cdot 2} + \frac{\mathfrak{B}}{3\cdot 4} - \frac{\mathfrak{C}}{5\cdot 6} + \frac{\mathfrak{D}}{7\cdot 8} - \text{etc.},$$

quae series ob nimiam divergentiam est inepta ad valorem ipsius C saltem proxime eruendum.

158. Non solum autem proximum, sed etiam ipsum verum valorem ipsius C inveniemus, si consideremus expressionem WALLISIANAM pro valore ipsius π inventam atque in *Introductione* demonstratam, quae erat

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10\cdot 12\cdot \text{etc.}}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot 11\cdot 11\cdot \text{etc.}}\,.$$

Hinc enim logarithmis sumendis erit

$$l\pi - l2 = 2l2 + 2l4 + 2l6 + 2l8 + 2l10 + l12 + \text{etc.}$$
$$- l1 - 2l3 - 2l5 - 2l7 - 2l9 - 2l11 - \text{etc.}$$

Ponamus ergo in serie assumta $x = \infty$, et cum sit

$$l1 + l2 + l3 + l4 + \cdots + lx = C + \left(x + \tfrac{1}{2}\right)lx - x,$$

erit

$$l1 + l2 + l3 + l4 + \cdots + l2x = C + \left(2x + \tfrac{1}{2}\right)l2x - 2x$$

et

$$l2 + l4 + l6 + l8 + \cdots + l2x = C + \left(x + \tfrac{1}{2}\right)lx + xl2 - x,$$

hinc

$$l1 + l3 + l5 + l7 + \cdots + l\left(2x - 1\right) = xlx + \left(x + \tfrac{1}{2}\right)l2 - x\,.$$

Cum igitur sit

$$l\tfrac{\pi}{2} = 2l2 + 2l4 + 2l6 + \cdots + 2l2x - l2x$$
$$- 2l1 - 2l3 - 2l5 - \cdots - 2l\left(2x - 1\right),$$

posito $x = \infty$ erit

$$l\tfrac{\pi}{2} = 2C + \left(2x + 1\right)lx + 2xl2 - 2x - l2 - lx - 2xlx - \left(2x + 1\right)l2 + 2x$$

ideoque

$$l\tfrac{\pi}{2} = 2C - 2l2, \quad \text{ergo} \quad 2C = l2\pi \quad \text{et} \quad C = \tfrac{1}{2}l2\pi,$$

unde in fractionibus decimalibus reperitur

$$C = 0{,}9189385332046727417803297\,,$$

atque simul sequens series summatur

$$1 - \frac{\mathfrak{A}}{1\cdot 2} + \frac{\mathfrak{B}}{3\cdot 4} - \frac{\mathfrak{C}}{5\cdot 6} + \frac{\mathfrak{D}}{7\cdot 8} - \frac{\mathfrak{E}}{9\cdot 10} + \text{etc.} = \tfrac{1}{2}l2\pi.$$

159. Cognita nunc ista constante $C = \frac{1}{2}l2\pi$ summa quotcunque logarithmorum ex hac serie $l1 + l2 + l3 + \text{etc}$. exhiberi potest. Si enim ponatur

$$s = l1 + l2 + l3 + l4 + \cdots + lx\,,$$

erit

$$s = \tfrac{1}{2}l2\pi + \left(x + \tfrac{1}{2}\right)xlx - x + \frac{\mathfrak{A}}{1\cdot 2x} - \frac{\mathfrak{B}}{3\cdot 4x^3} + \frac{\mathfrak{C}}{5\cdot 6x^5} - \frac{\mathfrak{D}}{7\cdot 8x^7} + \text{etc.},$$

si quidem logarithmi propositi fuerint hyperbolici; sin autem proponantur logarithmi vulgares, tum in terminis $\frac{1}{2}l2\pi+\left(x+\frac{1}{2}\right)xlx$ pro $l2\pi$ et lx sumi debebunt logarithmi vulgares, reliqui autem seriei termini

$$-x+\frac{\mathfrak{A}}{1\cdot 2x}-\frac{\mathfrak{B}}{3\cdot 4x^3}+\text{etc.}$$

multiplicari debent per $0,434294481903251827=n$. Erit igitur hoc casu pro logarithmis vulgaribus

$$l\pi=0,497149872694133854351268$$
$$l2=0,301029995663981195213738$$
$$l2\pi=0,798179868358115049565006$$
$$\frac{1}{2}l2\pi=0,399089934179057524782503.$$

EXEMPLUM

Quaeratur aggregatum mille logarithmorum tabularium
$s=l1+l2+l3+\cdots+l1000$.

Erit ergo x = 1000 et

$$lx=3,0000000000000,$$

unde fit

$$xlx=3000,0000000000000$$
$$\frac{1}{2}lx=\quad 1,5000000000000$$
$$\frac{1}{2}l2\pi=\quad 0,3990899341790$$
$$3001,8990899341790$$
$$\text{subtr.}\quad nx=\ 434,2944819032518$$
$$2567,6046080309272$$

Deinde est

$$\frac{n\mathfrak{A}}{1\cdot 2x}=\quad 0,0000361912068$$
$$\text{subtr.}\ \frac{n\mathfrak{B}}{3\cdot 4x^3}=\quad 0,0000000000012$$
$$0,0000361912056$$
$$\text{addatur}\ 2567,6046080309272$$

summa quaesita s = 2567,6046442221328.

Cum igitur s sit logarithmus producti numerorum

$$1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdots 1000,$$

patet hoc productum, si actu multiplicetur, constare ex 2568 figuris atque notas a laeva initiales fore 4023872, quas insuper 2561 figurae sequentur.

160. Ope ergo huius logarithmorum summationis producta ex quotcunque factoribus, qui secundum numeros naturales procedunt, proxime assignari poterunt. Huc potissimum referri potest problema, quo quaeritur uncia media seu maxima in potestate binomii quacunque $(a+b)^m$, ubi quidem notandum est, si m sit numerus impar, binas dari medias inter se aequales, quae iunctim sumtae praebeant unciam mediam in potestate sequente pari. Quare cum uncia maxima in quaque potestate pari sit duplo maior quam uncia sufficiet pro potestatibus paribus media in potestate praecedente impari, unciam mediam maximam determinasse. Sit igitur $m=2n$ et uncia media ita exprimetur, ut sit

$$\frac{2n(2n-1)(2n-2)(2n-3)\cdots(n+1)}{1\cdot 2\cdot 3\cdot 4\cdots n}$$

Vocetur ista uncia media, quae quaeritur, $=u$ atque ea hoc modo repraesentari poterit, ut sit

$$\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots 2n}{(1\cdot 2\cdot 3\cdot 4\cdot 5\cdots n)^2},$$

sumtisque logarithmis erit

$$\begin{aligned} lu &= lI + l2 + l3 + l4 + l5 + \cdots + l2n \\ &-2l1 - 2l2 - 2l3 - 2l4 - 2l5 - \cdots - 2ln. \end{aligned}$$

161. Iam vero sumendis his logarithmis hyperbolicis erit

$$\begin{aligned} &l1 + l2 + l3 + l4 + \cdots + l2n = \tfrac{1}{2}l2\pi + \left(2n+\tfrac{1}{2}\right)ln + \left(2n+\tfrac{1}{2}\right)l2 - 2n \\ &+ \frac{\mathfrak{A}}{1\cdot 2\cdot 2n} - \frac{\mathfrak{B}}{3\cdot 4\cdot 2^3 n^3} + \frac{\mathfrak{C}}{5\cdot 6\cdot 2^5 n^5} - \text{etc.} \end{aligned}$$

et

$$\begin{aligned} &2l1 + 2l2 + 2l3 + 2l4 + \cdots + 2ln = l2\pi + (2n+1)ln - 2n \\ &+ \frac{2\mathfrak{A}}{1\cdot 2n} - \frac{2\mathfrak{B}}{3\cdot 4\cdot n^3} + \frac{2\mathfrak{C}}{5\cdot 6n^5} - \text{etc.}, \end{aligned}$$

qua expressione ab illa sublata relinquetur

$$\begin{aligned} lu = &-\tfrac{1}{2}l\pi - \tfrac{1}{2}ln + 2nl2 + \frac{\mathfrak{A}}{1\cdot 2\cdot 2n} - \frac{\mathfrak{B}}{3\cdot 4\cdot 2^3 n^3} + \frac{\mathfrak{C}}{5\cdot 6\cdot 2^5 n^5} - \text{etc.} \\ &- \frac{2\mathfrak{A}}{1\cdot 2n} + \frac{2\mathfrak{B}}{3\cdot 4\cdot n^3} - \frac{2\mathfrak{C}}{5\cdot 6n^5} + \text{etc.}; \end{aligned}$$

his vero binis terminis colligendis erit

$$lu = l\,\frac{2^{2n}}{\sqrt{\pi n}} - \frac{3\mathfrak{A}}{1\cdot 2\cdot 2n} + \frac{15\mathfrak{B}}{3\cdot 4\cdot 2^3 n^3} - \frac{63\mathfrak{C}}{5\cdot 6\cdot 2^5 n^5} + \frac{255\mathfrak{D}}{7\cdot 8\cdot 2^7 n^7} - \text{etc.}$$

Sit

$$\frac{3\mathfrak{A}}{1\cdot 2\cdot 2^2 n^2} - \frac{15\mathfrak{B}}{3\cdot 4\cdot 2^4 n^4} + \frac{63\mathfrak{C}}{5\cdot 6\cdot 2^6 n^6} - \frac{255\mathfrak{D}}{7\cdot 8\cdot 2^8 n^8} + \text{etc.}$$
$$= l\left(1 + \frac{A}{2^2 n^2} + \frac{B}{2^4 n^4} + \frac{C}{2^6 n^6} + \frac{D}{2^8 n^8} + \text{etc.}\right)\ ;$$

erit

$$lu = l\,\frac{2^{2n}}{\sqrt{\pi n}} - 2nl\left(1 + \frac{A}{2^2 n^2} + \frac{B}{2^4 n^4} + \frac{C}{2^6 n^6} + \frac{D}{2^8 n^8} + \text{etc.}\right)$$

ideoque

$$u = \frac{2^{2n}}{\left(1 + \frac{A}{2^2 n^2} + \frac{B}{2^4 n^4} + \frac{C}{2^6 n^6} + \frac{D}{2^8 n^8} + \text{etc.}\right)^{2n}\sqrt{n\pi}}.$$

Erit vero posito $2n = m$

$$\begin{aligned} l\left(1 + \frac{A}{2^2 n^2} + \frac{B}{2^4 n^4} + \frac{C}{2^6 n^6} + \frac{D}{2^8 n^8} + \text{etc.}\right) = \\ \frac{A}{m^2} + \frac{B}{m^4} + \frac{C}{m^6} + \frac{D}{m^8} + \frac{E}{m^{10}} + \text{etc.} \\ - \frac{A^2}{2m^4} - \frac{AB}{m^6} - \frac{AC}{m^8} - \frac{AD}{m^{10}} - \text{etc.} \\ - \frac{BB}{2m^8} - \frac{BC}{m^{10}} - \text{etc.} \\ + \frac{A^3}{3m^6} + \frac{A^2B}{m^8} + \frac{A^2C}{m^{10}} + \text{etc.} \\ + \frac{AB^2}{m^{10}} + \text{etc.} \\ - \frac{A^4}{4m^8} - \frac{A^3B}{m^{10}} - \text{etc.} \\ + \frac{A^5}{5m^{10}} + \text{etc.}; \end{aligned}$$

quae expressio cum aequalis esse debeat huic

$$\frac{3\mathfrak{A}}{1\cdot 2\cdot m^2} - \frac{15\mathfrak{B}}{3\cdot 4m^4} + \frac{63\mathfrak{C}}{5\cdot 6m^6} - \frac{255\mathfrak{D}}{7\cdot 8m^8} + \text{etc.},$$

fiet

$$A = \frac{3\mathfrak{A}}{1\cdot 2}$$
$$B = \frac{A^2}{2} - \frac{15\mathfrak{B}}{3\cdot 4}$$
$$C = AB - \frac{1}{3}A^3 + \frac{63\mathfrak{C}}{5\cdot 6}$$
$$D = AC + \frac{1}{2}B^2 - A^2B + \frac{1}{4}A^4 - \frac{255\mathfrak{D}}{7\cdot 8}$$
$$E = AD + BC - A^2C - AB^2 + A^3B - \frac{1}{5}A^5 + \frac{1023\mathfrak{E}}{9\cdot 10}$$
etc.

162. Cum iam sit $\mathfrak{A} = \frac{1}{6}$, $\mathfrak{B} = \frac{1}{30}$, $\mathfrak{C} = \frac{1}{42}$, $\mathfrak{D} = \frac{1}{30}$, $\mathfrak{E} = \frac{5}{66}$, erit

$$A = \frac{1}{4},\quad B = -\frac{1}{96},\quad C = \frac{27}{640},\quad D = -\frac{90031}{2^{11}\cdot 3^2\cdot 5\cdot 7}\quad \text{etc.}$$

Hinc efficitur

$$u = \frac{2^{2n}}{\left(1+\frac{1}{2^4n^2}-\frac{1}{2^9.3n^4}+\frac{27}{2^{13}.5n^6}-\frac{90031}{2^{19}.3^2.5.7n^8}+\text{etc.}\right)^{2n}\sqrt{n\pi}}$$

seu

$$u = \frac{2^{2n}\left(1-\frac{1}{2^4n^2}+\frac{1}{2^9.3n^4}-\frac{27}{2^{13}.5n^6}+\frac{90031}{2^{19}.3^2.5.7n^8}-\text{etc.}\right)^{2n}}{\sqrt{n\pi}},$$

vel si ista seriei elevatio actu instituatur, erit proxime

$$u = \frac{2^{2n}}{\sqrt{n\pi}\left(1+\frac{1}{4n}+\frac{1}{32n^2}-\frac{1}{128n^3}-\frac{5}{16\cdot 5128n^4}+\text{etc.}\right)};$$

hinc terminus medius in $(1+1)^{2n}$ erit ad summam omnium terminorum 2^{2n}

$$\text{uti 1 ad } \sqrt{n\pi}\left(1+\frac{1}{4n}+\frac{1}{32n^2}-\frac{1}{128n^3}-\frac{5}{16\cdot 5128n^4}+\text{etc.}\right);$$

vel posito brevitatis gratia $4n = v$ erit ista ratio

$$\text{uti 1 ad } \sqrt{n\pi}\left(1+\frac{1}{v}+\frac{1}{2v^2}-\frac{1}{2v^3}-\frac{5}{8v^4}+\frac{23}{8v^5}+\frac{53}{16v^6}-\text{etc.}\right).$$

EXEMPLUM 1

Quaeratur uncia media in binomio $(a+b)^{10}$ *evoluto, quam constat esse*

$$\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 252.$$

Adhibendo ultimam formulam pro u inventam erit $n = 5$

$$\frac{1}{4n} = 0{,}0500000$$
$$\frac{1}{32n^2} = 0{,}0012500$$
$$0{,}0512500$$

$$\text{subtr.} \quad \frac{1}{128n^3} = \qquad 625$$
$$0{,}0511875$$
$$\text{subtr.} \quad \frac{1}{16 \cdot 128n^4} = \qquad 39$$

Ergo $\qquad 1 + \frac{1}{4n} + \text{etc.} = 1{,}0511836$

Huius log.

$$= 0{,}0216784$$
$$ln = 0{,}6989700$$
$$l\pi = 0{,}4971498$$
$$1{,}2177982$$
$$l\sqrt{n\pi}\left(1 + \text{etc.}\right) = 0{,}6088991$$
$$l2^{2n} = 3{,}0102999$$
$$lu = 2{,}4014008$$

unde fit $\qquad u = 252$.

EXEMPLUM 2

Investigetur ratio, quam in potestate centesima binomii $1+1$ *terminus medius ad summam omnium* 2^{100} *tenet.*

Utamur ad hoc formula primum inventa

$$lu = l\frac{2^{2n}}{\sqrt{\pi n}} - \frac{3\mathfrak{A}}{1 \cdot 2 \cdot 2n} + \frac{15\mathfrak{B}}{3 \cdot 4 \cdot 2^3 n^3} - \frac{63\mathfrak{C}}{5 \cdot 6 \cdot 2^5 n^5} + \frac{255\mathfrak{D}}{7 \cdot 8 \cdot 2^7 n^7} - \text{etc.}$$

in qua posito $2n = m$, ut habeatur ista potestas $\left(1+1\right)^m$, et loco $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. substitutis valoribus fiet

$$lu = l\frac{2^m}{\sqrt{\frac{1}{2}m\pi}} - \frac{1}{4m} + \frac{1}{24m^3} - \frac{1}{20m^5} + \frac{17}{112m^7} - \frac{31}{36m^9} + \frac{691}{88m^{11}} - \text{etc.};$$

qui logarithmi cum sint hyperbolici, multiplicentur ii per

$$k = 0{,}434294481903251,$$

ut transmutentur in tabulares, eritque

$$lu = l\frac{2^m}{\sqrt{\frac{1}{2}m\pi}} - \frac{k}{4m} + \frac{k}{24m^3} - \frac{k}{20m^5} + \frac{17k}{112m^7} - \frac{31k}{36m^9} + \text{etc.};$$

unde, cum uncia media sit u, erit $2m : u$ ratio quaesita ideoque

$$l\frac{2^m}{u} = l\sqrt{\frac{1}{2}m\pi} + \frac{k}{4m} - \frac{k}{24m^3} + \frac{k}{20m^5} - \frac{17k}{112m^7} + \frac{31k}{36m^9} - \frac{691k}{88m^{11}} + \text{etc.}$$

Quare, cum sit ob exponentem $m = 100$

$$\frac{k}{m} = 0{,}0043429448, \quad \frac{k}{m^3} = 0{,}0000004343, \quad \frac{k}{m^5} = 0{,}0000000000,$$

erit

$$\frac{k}{4m} = 0{,}0010857362$$
$$\frac{k}{24m^3} = 0{,}0000000181$$
$$\underline{0{,}0010857181}$$

Tum est

$$l\pi = 0{,}4971498726$$
$$l\frac{1}{2}m = 1{,}6989700043$$
$$l\frac{1}{2}m\pi = 2{,}1961198769$$
$$l\sqrt{\frac{1}{2}m\pi} = 1{,}0980599384$$
$$\frac{k}{4m} - \frac{k}{24m^3} + \text{etc.} = 0{,}0010857181$$
$$1{,}0991456565 = l\frac{2^{100}}{u}.$$

Erit ergo $\frac{2^{100}}{u} = 12{,}56451$ atque adeo in potestate $(1+1)^m$ evoluta terminus medius se habebit ad summam omnium 2^{100} uti 1 ad 12,5645l.

163. Denotet nunc terminus generalis z functionem exponentialem a^x, ita ut summari debeat haec series geometrica

$$s = a + a^2 + a^3 + a^4 + \cdots + a^x \;;$$

quae cum sit geomerica, eius summa iam constat ; erit enim $s = \frac{(a^x - 1)a}{a-1}$.
Modo autem hic exposito hane summam investigemus. Quia est $z = a^x$, erit $\int zdz = \frac{a^x}{la}$; huius enim differentiale est $azdx$; tum vero erit

$$\frac{dz}{dx} = a^x la, \quad \frac{ddz}{dx^2} = a^x (la)^2, \quad \frac{d^3z}{dx^3} = a^x (la)^3 \quad \text{etc.,}$$

unde sequitur fore

$$s = a^x \left(\frac{1}{la} + \frac{1}{2} + \frac{\mathfrak{A}}{1\cdot 2} la - \frac{\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}(la)^3 + \frac{\mathfrak{C}}{1\cdot 2\cdot 3\cdots 6}(la)^5 - \text{etc.} \right) + C.$$

Ad constantem C definiendam ponatur $x = 0$ et ob $s = 0$ erit

$$C = -\frac{1}{la} - \frac{1}{2} - \frac{\mathfrak{A}}{1\cdot 2} la + \frac{\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}(la)^3 - \text{etc.}$$

ideoque fiet

$$s = \left(a^x - 1\right)\left(\frac{1}{la} + \frac{1}{2} + \frac{\mathfrak{A}}{1\cdot 2} la - \frac{\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}(la)^3 + \frac{\mathfrak{C}}{1\cdot 2\cdot 3\cdots 6}(la)^5 - \text{etc.} \right).$$

Cum igitur summa sit $s = \frac{(a^x - 1)a}{a-1}$, erit

$$\frac{a}{a-1} = \frac{1}{la} + \frac{1}{2} + \frac{\mathfrak{A}}{1\cdot 2} la - \frac{\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}(la)^3 + \frac{\mathfrak{C}}{1\cdot 2\cdot 3\cdots 6}(la)^5 - \text{etc.,}$$

ubi *la* denotat logarithmum hyperbolicum ipsius *a;* hinc fit

$$\frac{(a+1)la}{2(a-1)} = 1 + \frac{\mathfrak{A}(la)^2}{1\cdot 2} - \frac{\mathfrak{B}(la)^4}{1\cdot 2\cdot 3\cdot 4} + \frac{\mathfrak{C}(la)^6}{1\cdot 2\cdot 3\cdots 6} - \text{etc.,}$$

sicque istius seriei summa exhiberi poterit.

164. Sit terminus generalis $z = \sin ax$ et

$$s = \sin a + \sin 2a + \sin 3a + \cdots + \sin ax;$$

quae series, cum sit recurrens, quoque summari potest; erit enim

$$s=\frac{\sin a+\sin ax-\sin(ax+a)}{1-2\cos a+1}=\frac{\sin a+(1-\cos a)\sin ax-\sin a\cdot\cos ax}{2(1-\cos a)}.$$

Erit vero

$$\int zdx=\int dx\sin ax=-\tfrac{1}{a}\cos ax$$

et

$$\tfrac{dz}{dx}=a\cos ax,\quad \tfrac{d^3z}{dx^3}=-a^3\cos ax,\quad \tfrac{d^5z}{dx^5}=a^5\cos ax \text{ etc.}$$

Ergo

$$s=C-\tfrac{1}{a}\cos ax+\tfrac{1}{2}\sin ax+\frac{\mathfrak{A}a\cos ax}{1\cdot 2}+\frac{\mathfrak{B}a^3\cos ax}{1\cdot 2\cdot 3\cdot 4}$$
$$+\frac{\mathfrak{C}a^5\cos ax}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}+\frac{\mathfrak{D}a^7\cos ax}{1\cdot 2\cdots 8}+\text{ etc.}$$

Ponatur $x=0$, ut fiat $s=0$, eritque

$$C=\tfrac{1}{a}-\frac{\mathfrak{A}a}{1\cdot 2}-\frac{\mathfrak{B}a^3}{1\cdot 2\cdot 3\cdot 4}-\frac{\mathfrak{C}a^5}{1\cdot 2\cdots 6}-\text{etc.},$$

ergo

$$s=\tfrac{1}{2}\sin ax+(1-\cos ax)\left(\tfrac{1}{a}-\frac{\mathfrak{A}a}{1\cdot 2}-\frac{\mathfrak{B}a^3}{1\cdot 2\cdot 3\cdot 4}-\frac{\mathfrak{C}a^5}{1\cdot 2\cdots 6}-\text{etc.}\right).$$

At cum sit

$$s=\tfrac{1}{2}\sin ax+\frac{(1-\cos ax)\sin a}{2(1-\cos a)},$$

fiet

$$\frac{\sin a}{2(1-\cos a)}=\tfrac{1}{2}\cot\tfrac{1}{2}a=\tfrac{1}{a}-\frac{\mathfrak{A}a}{1\cdot 2}-\frac{\mathfrak{B}a^3}{1\cdot 2\cdot 3\cdot 4}-\frac{\mathfrak{C}a^5}{1\cdot 2\cdots 6}-\text{etc.},$$

quam eandem seriem iam supra (§ 127) habuimus.

165. Sit nunc $z=\cos ax$ ac series summanda

$$s=\cos a+\cos 2a+\cos 3a+\cdots+\cos ax;$$

cuius seriei, quia est recurrens, erit summa

$$s=\frac{\cos a-1+\cos ax-\cos(ax+a)}{1-2\cos a+1}=-\tfrac{1}{2}+\tfrac{1}{2}\cos ax+\tfrac{1}{2}\cot\tfrac{1}{2}a\cdot\sin ax.$$

At vero ad summam nostra methodo exprimendam erit

$$\int zdx=\int dx\cos ax=\tfrac{1}{a}\sin ax$$

et

$$\frac{dz}{dx} = -a\sin ax,\quad \frac{d^3z}{dx^3} = a^3\sin ax,\quad \frac{d^5z}{dx^5} = -a^5\sin ax \text{ etc.}$$

Ergo

$$s = C + \frac{1}{a}\sin ax + \frac{1}{2}\cos ax - \frac{\mathfrak{A}a\sin ax}{1\cdot 2} - \frac{\mathfrak{B}a^3\sin ax}{1\cdot 2\cdot 3\cdot 4} - \text{etc.}$$

Sit $x = 0$; erit $s = 0$ et $C = -\frac{1}{2}$ hincque erit

$$s = -\frac{1}{2} + \frac{1}{2}\cos ax + \frac{1}{a}\sin ax - \frac{\mathfrak{A}a\sin ax}{1\cdot 2} - \frac{\mathfrak{B}a^3\sin ax}{1\cdot 2\cdot 3\cdot 4} - \text{etc.}$$

Quare, cum sit

$$s = -\frac{1}{2} + \frac{1}{2}\cos ax + \frac{1}{2}\cot\frac{1}{2}a\cdot\sin ax,$$

erit, uti iam modo invenimus,

$$\frac{1}{2}\cot\frac{1}{2}a = \frac{1}{a} - \frac{\mathfrak{A}a}{1\cdot 2} - \frac{\mathfrak{B}a^3}{1\cdot 2\cdot 3\cdot 4} - \frac{\mathfrak{C}a^5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} - \text{etc.}$$

166. Quoniam supra invenimus, si a denotet arcum quemcunque, esse

$$\frac{\pi}{2} = \frac{a}{2} + \sin a + \frac{1}{2}\sin 2a + \frac{1}{3}\sin 3a + \frac{1}{4}\sin 4a + \text{etc.},$$

consideremus hanc seriem sitque $z = \frac{1}{x}\sin ax$, ut sit

$$s = \sin a + \frac{1}{2}\sin 2a + \frac{1}{3}\sin 3a + \frac{1}{4}\sin 4a + \cdots + \frac{1}{x}\sin ax.$$

Hoc autem casu fit $\int zdx = \int\frac{dx}{x}\sin ax$, quod integrale exhiberi nequit. Erit vero

$$\begin{aligned}
&\frac{dz}{dx} = \frac{a}{x}\cos ax - \frac{1}{xx}\sin ax,\quad \frac{ddz}{dx^2} = -\frac{a^2}{x}\sin ax - \frac{2a}{xx}\cos ax + \frac{2}{x^3}\sin ax,\\
&\frac{d^3z}{dx^3} = -\frac{a^3}{x}\cos ax + \frac{3a^2}{x^2}\sin ax + \frac{6a}{x^3}\cos ax - \frac{6}{x^4}\sin ax,\\
&\frac{d^4z}{dx^4} = \frac{a^4}{x}\sin ax + \frac{4a^3}{xx}\cos ax - \frac{12a^2}{x^3}\sin ax - \frac{24a}{x^4}\cos ax + \frac{24}{x^5}\sin ax.
\end{aligned}$$

Quia igitur neque formulam integralem $\int zdx$ exhibere neque haec differentialia satis commode exprimere licet, summam huius seriei per hanc methodum definire non possumus, ita ut quicquam inde concludi posset. Idem incommodum in multis allis seriebus occurrit, quoties terminus generalis non satis est simplex, ut eius differentialia ad commodam legem exprimi queant. Quamobrem in sequenti capite alias expressiones generales pro summis serierum, quarum termini generales vel nimis sunt compositi vel prorsus dari nequeunt, eliciemus; quae feliciori successu in usum vocari poterunt. Imprimis autem insufficientia methodi hic traditae elucet, si signa terminorum seriei propositae alternentur; tum enim, quantumvis termini generales sint simplices, tamen termini summatorii hac methodo exhiberi commode nequeunt.