CHAPTER V

THE INVESTIGATION OF THE SUMS OF SERIES FROM THE GENERAL TERM

103. Let the general term $= y$ of each series correspond to the index *x*, thus in order that *y* shall be some function of *x*. Again let Sy be the sum or the summatory term of the series, expressing the aggregate of all the terms from the first or from some other fixed term as far as to *y*, inclusive. But we will compute the sum of series from the first term, from which , if there shall be $x = 1$, *y* will give the first term and *Sy* will show this first term *y*; but if there may be put *x* = 0, the term *Sy* must change into nothing, because therefore no terms are present to be summed. On which account the summatory term *Sy* will be a function of *x* of this kind, which may vanish on putting $x = 0$.

104. If the general term y may depend on several parts, so that there shall be $y = p + q + r + \text{etc.}$, then the series itself will be able to be considered as put together from several other series, the general terms of which shall be p, q, r etc. Hence if the individual sums of the series themselves should be known, likewise the sum of the proposed series will be able to be assigned; for it will be the aggregate from the sums of the individual series. On account of this if there shall be $y = p + q + r + \text{etc.}$, there will be $Sy = Sp + Sq + Sr + \text{etc.}$ Therefore since above we have shown the sums of series, the general terms of which shall be some powers of *x* having positive integral exponents, hence the general summatory term will be able to be found of each series of which the general term is $ax^{\alpha} + bx^{\beta} + cx^{\gamma} + \text{etc.}$, with α, β, γ etc. denoting positive whole numbers, or the general term of which is a rational integral function of *x*.

105. There shall be in a series, the general term of which is equal to $= y$ or corresponding to the exponent *x*, this preceding term $= v$ or corresponding to the exponent $x - 1$; because *v* arises from *y*, if in place of x there is written $x - 1$, there will be

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \frac{d^4y}{24dx^4} - \frac{d^5y}{120dx^5} + \text{etc.}$$

Therefore if y were the general term of this series

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & \cdots\cdots & x-1 & x \\ a+ & b+ & c+ & d+ & \cdots\cdots & +v+ & y \end{array}$$

and of this series the term corresponding to the index 0 were $= A$, v will be, in as much as it is a function of *x*, the general term of this series

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & \cdots\cdots & & x \\ A & a & +b & +c & +d & +\cdots\cdots & & +v, \end{array}$$

from which, if Sv may denote the sum of this series, there will be $Sv = Sy - y + A$. And thus on putting $x = 0$, because there becomes $Sy = 0$ and $y = A$, Sv will vanish also.

106. Therefore since there shall be

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \text{etc.}$$

there will be, from shown before

$$Sv = Sy - S\frac{dy}{dx} + S\frac{ddy}{2dx^2} - S\frac{d^3y}{6dx^3} + S\frac{d^4y}{24dx^4} - \text{etc.}$$

and on account of $Sv = Sy - y + A$ there will be

$$y - A = S\frac{dy}{dx} - S\frac{ddy}{2dx^2} + S\frac{d^3y}{6dx^3} - S\frac{x^4d^4y}{24dx^4} + \text{etc.}$$

and there will be had

$$S\frac{dy}{dx} = y - A + S\frac{ddy}{2dx^2} - S\frac{d^3y}{6dx^3} + S\frac{x^4d^4y}{24dx^4} - \text{etc.}$$

Therefore if the summatory terms of the series may be had, of which the general terms are $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ etc., from these the summatory term of the series will be obtained, of which the general term is $\frac{dy}{dx}$. Now the constant quantity A must be prepared thus, so that on making $x = 0$ the summatory term $S\frac{dy}{dx}$ may vanish, and it is determined more easily from this, as if we may say that it is the term corresponding to the index 0 in the series, of which the general term shall be $= y$.

107. The sums of the powers of the natural numbers are accustomed to be investigated from this source. Indeed let there be $y = x^{n+1}$; because there becomes

$$\frac{dy}{dx} = (n+1)x^n, \quad \frac{ddy}{2dx^2} = \frac{(n+1)n}{1\cdot 2}x^{n-1}, \quad \frac{d^3y}{6dx^3} = \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}x^{n-2},$$
$$\frac{x^4d^4y}{24dx^4} = \frac{(n+1)n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4}x^{n-3} \quad \text{etc.},$$

with these values substituted there will be

$$(n+1)Sx^n = x^{n+1} - A + \tfrac{(n+1)n}{1\cdot 2}Sx^{n-1} - \tfrac{(n+1)n(n-1)}{1\cdot 2\cdot 3}Sx^{n-2} + \text{etc.};$$

and if both sides may be divided by $n+1$, there will be

$$Sx^n = \tfrac{1}{n+1}x^{n+1} + \tfrac{n}{1\cdot 2}Sx^{n-1} - \tfrac{n(n-1)}{1\cdot 2\cdot 3}Sx^{n-2} + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4}Sx^{n-3} - \text{etc.} - \text{Const.},$$

which constant must be taken thus, so that on putting $x = 0$ the whole summatory term may vanish. Therefore with the aid of this formula now from known sums of lesser powers, the general terms of which are x^{n-1}, x^{n-2} etc., the sum will be able to be found of the higher powers expressed by the general term x^n.

108. If in this expression *n* may denote a positive whole number, the number of terms will be finite. And so hence the sum of boundless powers may be known completely; for there will be, if $n = 0$,

$$Sx^0 = x.$$

And with that known it will be allowed to progress to higher powers; for on putting $n = 1$ there becomes

$$Sx^1 = \tfrac{1}{2}x^2 + \tfrac{1}{2}Sx^0 = \tfrac{1}{2}x^2 + \tfrac{1}{2}x.$$

if again there may be put $n = 2$, there will be produced

$$Sx^2 = \tfrac{1}{3}x^3 + Sx - \tfrac{1}{3}Sx^0 = \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + \tfrac{1}{6}x,$$

then following

$$Sx^3 = \tfrac{1}{4}x^4 + \tfrac{3}{2}Sx^2 - Sx + \tfrac{1}{4}Sx^0 = \tfrac{1}{4}x^4 + \tfrac{1}{2}x^3 + \tfrac{1}{4}x^2,$$

$$Sx^4 = \tfrac{1}{5}x^5 + \tfrac{4}{2}Sx^3 - \tfrac{4}{2}Sx^2 + Sx - \tfrac{1}{5}Sx^0$$

or

$$Sx^4 = \tfrac{1}{5}x^5 + \tfrac{1}{2}x^4 + \tfrac{1}{3}x^3 - \tfrac{1}{30}x.$$

And thus again the sums of any higher powers may be deduced successively from the lower sums ; but this will be performed easier by the following ways.

109. Because above we have found to be

$$S\frac{dy}{dx} = y + \frac{1}{2}S\frac{ddy}{dx^2} - \frac{1}{6}S\frac{d^3y}{dx^3} + \frac{1}{24}S\frac{d^4y}{dx^4} - \frac{1}{120}S\frac{d^5y}{dx^5} + \text{etc.},$$

If we may put $\frac{dy}{dx} = z$, and $\frac{ddy}{dx^2} = \frac{dz}{dx}$, $\frac{d^3y}{dx^3} = \frac{ddz}{dx^2}$ etc. Then truly on account of $dy = zdx$ there will be the quantity *y*, the differential of which is $= zdx$, which we may indicate in this manner, so that there shall be $y = \int zdx$. But though this discovery of *y* itself depends on the calculation of the integral from a given *z*, yet now we will be able to use this form $\int zdx$ here, if indeed for *z* we may not substitute other functions of *x* unless they are of this kind, so that that function, the differential of which $= zdx$, may be able to be shown from what has preceded. Therefore with these values substituted there will be

$$Sz = \int zdx + \frac{1}{2}S\frac{dz}{dx} - \frac{1}{6}S\frac{ddz}{dx^2} + \frac{1}{24}S\frac{d^3z}{dx^3} - \text{etc.}$$

with the addition of a constant of this kind, so that on putting $x = 0$ the sum *Sz* itself may vanish.

110. Moreover on substituting the letter *z* in the above expression in place of y or, because it returns the same, on differentiating that equation there will be

$$S\frac{dz}{dx} = z + \frac{1}{2}S\frac{ddz}{dx^2} - \frac{1}{6}S\frac{d^3z}{dx^3} + \frac{1}{24}S\frac{d^4z}{dx^4} - \text{etc. };$$

but if in place of y [in §109] there may be put $\frac{dz}{dx}$ there will be

$$S\frac{ddz}{dx^2} = \frac{dz}{dx} + \frac{1}{2}S\frac{d^3z}{dx^3} - \frac{1}{6}S\frac{d^4z}{dx^4} + \frac{1}{24}S\frac{d^5z}{dx^5} - \text{etc. };$$

And in a similar manner if for *y* there may be put successively the values $\frac{ddz}{dx^2}$, $\frac{d^3z}{dx^3}$ etc., there may be found

$$S\frac{d^3z}{dx^3} = \frac{ddz}{dx^2} + \frac{1}{2}S\frac{d^4z}{dx^4} - \frac{1}{6}S\frac{d^5z}{dx^5} + \frac{1}{24}S\frac{d^6z}{dx^6} - \text{etc.},$$
$$S\frac{d^4z}{dx^4} = \frac{d^3z}{dx^3} + \frac{1}{2}S\frac{d^5z}{dx^5} - \frac{1}{6}S\frac{d^6z}{dx^6} + \frac{1}{24}S\frac{d^7z}{dx^7} - \text{etc.}$$

and thus again indefinitely.

111. If now those values for $S\frac{dz}{dx}$, $S\frac{ddz}{dx^2}$, $S\frac{d^3z}{dx^3}$ etc. may be substituted successively in the expression

$$Sz = \int zdx + \frac{1}{2}S\frac{dz}{dx} - \frac{1}{6}S\frac{ddz}{dx^2} + \frac{1}{24}S\frac{d^3z}{dx^3} - \text{etc.}$$

an expression may be found for Sz, which will be in agreement with these terms $\int zdx, z, \frac{dz}{dx}, \frac{ddz}{dx^2}, \frac{d^3z}{dx^3}$ etc., the coefficients of which are found more easily in the following manner.
There may be put

$$Sz = \int zdx + \alpha z + \frac{\beta dz}{dx} + \frac{\gamma ddz}{dx^2} + \frac{\delta d^3z}{dx^3} + \frac{\varepsilon d^4z}{dx^4} + \text{etc.}$$

and for these terms, the values for these may be substituted, which are obtained from the preceding series, and from which there is

$$\begin{aligned}
\int zdx &= Sz - \frac{1}{2}S\frac{dz}{dx} + \frac{1}{6}S\frac{ddz}{dx^2} - \frac{1}{24}S\frac{d^3z}{dx^3} + \frac{1}{120}S\frac{d^4z}{dx^4} - \text{etc.}\\
\alpha z &= \qquad + \alpha S\frac{dz}{dx} - \frac{\alpha}{2}S\frac{ddz}{dx^2} + \frac{\alpha}{6}S\frac{d^3z}{dx^3} - \frac{\alpha}{24}S\frac{d^4z}{dx^4} + \text{etc.}\\
\frac{\beta dz}{dx} &= \qquad \beta S\frac{ddz}{dx^2} - \frac{\beta}{2}S\frac{d^3z}{dx^3} + \frac{\beta}{6}S\frac{d^4z}{dx^4} - \text{etc.}\\
\frac{\gamma ddz}{dx^2} &= \qquad \gamma S\frac{d^3z}{dx^3} - \frac{\gamma}{2}S\frac{d^4z}{dx^4} + \text{etc.}\\
\frac{\delta d^3z}{dx^3} &= \qquad \delta S\frac{d^4z}{dx^4} - \text{etc.}\\
&\text{etc.}
\end{aligned}$$

Which values are required to be added since they ought to produce Sz, the coefficients $\alpha, \beta, \gamma, \delta$ etc. may be defined from the following equations

$$\begin{aligned}
&\alpha - \tfrac{1}{2} = 0, \quad \beta - \tfrac{\alpha}{2} + \tfrac{1}{6} = 0, \quad \gamma - \tfrac{\beta}{2} + \tfrac{\alpha}{6} - \tfrac{1}{24} = 0,\\
&\delta - \tfrac{\gamma}{2} + \tfrac{\beta}{6} - \tfrac{\alpha}{24} + \tfrac{1}{120} = 0, \quad \varepsilon - \tfrac{\delta}{2} + \tfrac{\gamma}{6} - \tfrac{\beta}{24} + \tfrac{\alpha}{120} - \tfrac{1}{720} = 0,\\
&\zeta - \tfrac{\varepsilon}{2} + \tfrac{\delta}{6} - \tfrac{\gamma}{24} + \tfrac{\beta}{120} - \tfrac{\alpha}{720} + \tfrac{1}{5040} = 0 \quad \text{etc.}
\end{aligned}$$

112. From these equations the values of all the letters $\alpha, \beta, \gamma, \delta$ etc. are able to be defined successively; moreover there is found

$$\alpha=\tfrac{1}{2},\quad \beta=\tfrac{\alpha}{2}-\tfrac{1}{6}=\tfrac{1}{12},\quad \gamma=\tfrac{\beta}{2}-\tfrac{\alpha}{6}+\tfrac{1}{24}=0,$$
$$\delta=\tfrac{\gamma}{2}-\tfrac{\beta}{6}+\tfrac{\alpha}{24}-\tfrac{1}{120}=-\tfrac{1}{720},\quad \varepsilon=\tfrac{\delta}{2}-\tfrac{\gamma}{6}+\tfrac{\beta}{24}-\tfrac{\alpha}{120}+\tfrac{1}{720}=0,\ \text{etc.}$$

and thus on progressing further the alternate terms vanish continually [i.e. γ, ε, etc. are zero]. Therefore the letters from the third, fifth, seventh etc. order and all the odd orders are equal to zero except the first, from which this value of the series itself may seem to strike against the law of continuity. On account of which therefore it is more necessary, so that it may be shown rigorously, that all the odd terms besides the first vanish by necessity.

113. Because the individual letters are determined constant following the law, these will constitute recurring series between themselves. Towards explaining which, this series may be considered

$$1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\zeta u^6+\text{etc.},$$

the value of which $=V$ and it is evident this recurrent series arises from the expansion of this fraction

$$V=\frac{1}{1-\frac{1}{2}u+\frac{1}{6}u^2-\frac{1}{24}u^3+\frac{1}{120}u^4-\text{etc.}}$$

And if this fraction is able to be resolved in another way into an infinite series following the progressing powers of *u*, it is necessary, that the same series

$$V=1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\text{etc.}$$

may result always; and in this manner another law may be elicited, from which these same values α, β, γ, δ etc. are determined.

114. Because, if *e* may denote the number, the hyperbolic logarithm of which is equal to unity, there will be

$$e^{-u}=1-u+\tfrac{1}{2}u^2-\tfrac{1}{6}u^3+\tfrac{1}{24}u^4-\tfrac{1}{120}u^5+\text{etc.}$$

there will be

$$\frac{1-e^{-u}}{u}=1-\tfrac{1}{2}u+\tfrac{1}{6}u^2-\tfrac{1}{24}u^3+\tfrac{1}{120}u^4-\text{etc.}$$

and thus

$$V=\frac{u}{1-e^{-u}}.$$

Now the term $\alpha u=\frac{1}{2}u$ may be removed from the second series, so that there shall be

$$V-\tfrac{1}{2}u=1+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\zeta u^6+\text{etc. };$$

there will be

$$V-\tfrac{1}{2}u=\frac{\frac{1}{2}u\left(1+e^{-u}\right)}{1-e^{-u}}$$

The numerator and denominator may be multiplied by $e^{\frac{1}{2}u}$ and there will be

$$V-\tfrac{1}{2}u=\frac{u\left(e^{\frac{1}{2}u}+e^{-\frac{1}{2}u}\right)}{2\left(e^{\frac{1}{2}u}-e^{-\frac{1}{2}u}\right)}$$

and with the quantities $e^{\frac{1}{2}u}$ and $e^{-\frac{1}{2}u}$ changed into series there becomes

$$V-\tfrac{1}{2}u=\frac{1+\frac{u^2}{2\cdot 4}+\frac{u^4}{2\cdot 4\cdot 6\cdot 8}+\frac{u^6}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot 12}+\text{etc.}}{2\left(\frac{1}{2}+\frac{u^2}{2\cdot 4\cdot 6}+\frac{u^4}{2\cdot 4\cdot 6\cdot 8\cdot 10}+\text{etc.}\right)}$$

or

$$V-\tfrac{1}{2}u=\frac{1+\frac{u^2}{2\cdot 4}+\frac{u^4}{2\cdot 4\cdot 6\cdot 8}+\frac{u^6}{2\cdot 4\cdots 12}+\frac{u^8}{2\cdot 4\cdots 16}+\text{etc.}}{1+\frac{u^2}{4\cdot 6}+\frac{u^4}{2\cdot 4\cdot 6\cdot 8\cdot 10}+\frac{u^6}{4\cdot 6\cdots 14}+\frac{u^8}{4\cdot 6\cdots 18}+\text{etc.}}$$

115. Therefore since in this fraction the odd powers shall be lacking completely, also in the expansion of this the odd powers entering are entirely nothing ; whereby since $V-\frac{1}{2}u$ may be equal to this series

$$1+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\zeta u^6+\text{etc .},$$

all the coefficients of the odd powers $\gamma,\ \varepsilon,\ \eta,\ \iota$ etc. may vanish. And thus the reason is clear, why in the series $1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+\text{etc.}$ all the terms besides the even terms and the second in order shall be $=0$ nor yet may the law of continuity undergo a test of strength. Hence there shall be

$$V=1+\beta u^2+\delta u^4+\zeta u^6+\theta u^8+\chi u^{10}+\text{etc}$$

and with the letters $\beta,\ \delta,\ \zeta,\ \theta,\ \chi$ etc. determined by the expansion of the above fraction we will obtain the term of the summation of the series *Sz*, the general term of which is $=z$ corresponding to the index *x*, expressed in this manner

$$Sz=\int zdx+\tfrac{1}{2}z+\frac{\beta dz}{dx}+\frac{\delta d^3z}{dx^3}+\frac{\zeta d^5z}{dx^5}+\frac{\theta d^7z}{dx^7}+\text{etc.}$$

116. Because the series $1+\beta u^2+\delta u^4+\zeta u^6+\theta u^8+\text{etc.}$ arises from the expansion of this fraction

$$\frac{1+\frac{u^2}{2\cdot4}+\frac{u^4}{2\cdot4\cdot6\cdot8}+\frac{u^6}{2\cdot4\cdot6\cdot8\cdot10\cdot12}+\text{etc.}}{1+\frac{u^2}{4\cdot6}+\frac{u^4}{4\cdot6\cdot8\cdot10}+\frac{u^6}{4\cdot6\cdot8\cdot10\cdot12\cdot14}+\text{etc.}}$$

the letters $\beta,\ \delta,\ \zeta,\ \theta$ etc. will maintain this law, so that there shall be

$$\begin{aligned}
\beta &= \frac{1}{2\cdot4}-\frac{1}{4\cdot6}\\
\delta &= \frac{1}{2\cdot4\cdot6\cdot8}-\frac{\beta}{4\cdot6}-\frac{1}{4\cdot6\cdot8\cdot10}\\
\zeta &= \frac{1}{2\cdot4\cdot6\cdots12}-\frac{\delta}{4\cdot6}-\frac{\beta}{4\cdot6\cdot8\cdot10}-\frac{1}{4\cdot6\cdots14}\\
\theta &= \frac{1}{2\cdot4\cdot6\cdots16}-\frac{\zeta}{4\cdot6}-\frac{\delta}{4\cdot6\cdot8\cdot10}-\frac{\beta}{4\cdot6\cdots14}-\frac{1}{4\cdot6\cdots18}\\
&\text{etc.}
\end{aligned}$$

Moreover these values alternately are made to be positive and negative .

117. Therefore if the alternates of these letters may be taken negatively, thus so that there shall be

$$Sz=\int zdx+\tfrac{1}{2}z-\frac{\beta dz}{dx}+\frac{\delta d^3z}{dx^3}-\frac{\zeta d^5z}{dx^5}+\frac{\theta d^7z}{dx^7}-\text{etc.},$$

the letters $\beta,\ \delta,\ \zeta,\ \theta$ etc. may be defined from this fraction

$$\frac{1-\frac{u^2}{2\cdot4}+\frac{u^4}{2\cdot4\cdot6\cdot8}-\frac{u^6}{2\cdot4\cdots12}+\frac{u^8}{2\cdot4\cdots16}-\text{etc.}}{1-\frac{u^2}{4\cdot6}+\frac{u^4}{4\cdot6\cdot8\cdot10}-\frac{u^6}{4\cdot6\cdots14}+\frac{u^8}{4\cdot6\cdots18}-\text{etc.}}$$

that on expanding in the series

$$1+\beta u^2+\delta u^4+\zeta u^6+\theta u^8+\text{etc.}$$

on account of which there will be

$$\beta = \frac{1}{4\cdot 6} - \frac{1}{2\cdot 4}$$
$$\delta = \frac{\beta}{4\cdot 6} - \frac{1}{4\cdot 6\cdot 8\cdot 10} + \frac{1}{2\cdot 4\cdot 6\cdot 8}$$
$$\zeta = \frac{\delta}{4\cdot 6} - \frac{\beta}{4\cdot 6\cdot 8\cdot 10} + \frac{1}{4\cdot 6\cdots 14} - \frac{1}{2\cdot 4\cdot 6\cdots 12}$$
etc. ;

now all the terms become negative.

118. Therefore we may put $\alpha = -A,\ \ \delta = -B,\ \ \zeta = -C$ etc., so that there shall be

$$Sz = \int zdx + \frac{1}{2}z + \frac{Adz}{dx} - \frac{Bd^3z}{dx^3} + \frac{Cd^5z}{dx^5} - \frac{Dd^7z}{dx^7} + \text{etc.},$$

and according to the letters *A*, *B*, *C*, *D* etc. requiring to be defined this series may be considered

$$1 - Au^2 - Bu^4 - Cu^6 - Du^8 - Eu^{10} - \text{etc.},$$

which arises from the expansion of this fraction

$$\frac{1 - \frac{u^2}{2\cdot 4} + \frac{u^4}{2\cdot 4\cdot 6\cdot 8} - \frac{u^6}{2\cdot 4\cdots 12} + \frac{u^8}{2\cdot 4\cdots 16} - \text{etc.}}{1 - \frac{u^2}{4\cdot 6} + \frac{u^4}{4\cdot 6\cdot 8\cdot 10} - \frac{u^6}{4\cdot 6\cdots 14} + \frac{u^8}{4\cdot 6\cdots 18} - \text{etc.}},$$

or that series may be considered

$$\frac{1}{u} - Au - Bu^3 - Cu^5 - Du^7 - Eu^9 - \text{etc.} = s,$$

which arises from the expansion of this fraction

$$s = \frac{1 - \frac{u^2}{2\cdot 4} + \frac{u^4}{2\cdot 4\cdot 6\cdot 8} - \frac{u^6}{2\cdot 4\cdots 12} + \text{etc.}}{u - \frac{u^3}{4\cdot 6} + \frac{u^5}{4\cdot 6\cdot 8\cdot 10} - \frac{u^7}{4\cdot 6\cdots 14} + \text{etc.}}.$$

But since there shall be

$$\cos\tfrac{1}{2}u = 1 - \frac{u^2}{2\cdot 4} + \frac{u^4}{2\cdot 4\cdot 6\cdot 8} - \frac{u^6}{2\cdot 4\cdots 12} + \text{etc.},$$
$$\sin\tfrac{1}{2}u = \frac{u}{2} - \frac{u^3}{2\cdot 4\cdot 6} + \frac{u^5}{2\cdot 4\cdot 6\cdot 8\cdot 10} - \frac{u^7}{2\cdot 4\cdots 14} + \text{etc.},$$

it follows to become

$$s = \frac{\cos\frac{1}{2}u}{2\sin\frac{1}{2}u} = \frac{1}{2}\cot\frac{1}{2}u.$$

Whereby if the cotangent of the arc $\frac{1}{2}u$ may be converted into a series, the terms of which may proceed following the powers of u, from that the values of the letters *A, B, C, D, E* etc. may become known.

119. Therefore since there shall be $s=\frac{1}{2}\cot\frac{1}{2}u$, there will be $\frac{1}{2}u=A\cot 2s$ and by differentiation there will be $\frac{1}{2}du=\frac{-2ds}{1+4ss}$ or $4ds+du+4ssdu=0$ or

$$\frac{4ds}{du}+1+4ss=0..$$

But because there is

$$s=\frac{1}{u}-Au-Bu^3-Cu^5-\text{etc.},$$

there will be

$$\begin{aligned}
\frac{4ds}{du}&=-\frac{4}{uu}-4A-3\cdot 4Bu^2-5\cdot 4Cu^4-7\cdot 4Du^6-\text{etc.}\\
1&=\quad 1\\
4ss&=\frac{4}{uu}-8A-\quad 8Bu^2-\quad Cu^4\quad-\quad 8Du^6-\text{etc.}\\
&\qquad\qquad +\ 4A^2u^2+\ 8ABu^4\ -\ 8ACu^6+\ \text{etc.}\\
&\qquad\qquad\qquad\qquad +\ 4BBu^6\ +\ \text{etc.}
\end{aligned}$$

With these homogeneous terms brought to zero there becomes

$$A=\frac{1}{12},\ B=\frac{A^2}{5},\ C=\frac{2AB}{7},\ D=\frac{2AC+BB}{9},\ E=\frac{2AD+2BC}{11},$$
$$F=\frac{2AE+2BD+CC}{13},\ G=\frac{2AF+2BE+2CD}{15},\ H=\frac{2AG+2BF+2CE+DD}{17},$$
etc.

Now from which formulas clearly it is proven that these individual values are to be positive.

120. Now because the denominators of these values become exceedingly great and they hardly impede the calculation, in place of the letters *A*, *B*, *C*, *D* etc. we may introduce these new values

$$A=\frac{\alpha}{1\cdot 2\cdot 3},\ B=\frac{\beta}{1\cdot 2\cdot 3\cdot 4\cdot 5},\ C=\frac{\gamma}{1\cdot 2\cdot 3\cdots 7},$$
$$D=\frac{\delta}{1\cdot 2\cdot 3\cdots 9},\ E=\frac{\varepsilon}{1\cdot 2\cdot 3\cdots 11}\ \text{ etc.}$$

And there may be found to become

$$\alpha=\tfrac{1}{2},\ \beta=\tfrac{2}{3}\alpha^2,\ \gamma=2\cdot\tfrac{3}{3}\alpha\beta,\ \delta=2\cdot\tfrac{4}{3}\alpha\gamma+\tfrac{8\cdot7}{4\cdot5}\beta^2,$$
$$\varepsilon=2\cdot\tfrac{5}{3}\alpha\delta+2\cdot\tfrac{10\cdot9\cdot8}{1\cdot2\cdots5}\beta\gamma,\ \zeta=2\cdot\tfrac{12}{1\cdot2\cdot3}\alpha\varepsilon+2\cdot\tfrac{12\cdot11\cdot10}{1\cdot2\cdots5}\beta\delta+\tfrac{12\cdot11\cdot10\cdot9\cdot8}{1\cdot2\cdots7}\gamma\gamma,$$
$$\eta=2\cdot\tfrac{14}{1\cdot2\cdot3}\alpha\zeta+2\cdot\tfrac{14\cdot13\cdot12}{1\cdot2\cdots5}\beta\varepsilon+2\cdot\tfrac{14\cdot13\cdot12\cdot11\cdot10}{1\cdot2\cdots7}\gamma\delta$$
etc.

121. But more conveniently we may use from these formulas

$$\alpha=\tfrac{1}{2},\ \beta=\tfrac{4}{3}\cdot\tfrac{\alpha\alpha}{2},\ \gamma=\tfrac{6}{3}\cdot\alpha\beta,\ \delta=\tfrac{8}{3}\cdot\alpha\gamma+\tfrac{8\cdot7\cdot6}{3\cdot4\cdot5}\cdot\tfrac{\beta\beta}{2},$$
$$\varepsilon=\tfrac{10}{3}\cdot\alpha\delta+\tfrac{10\cdot9\cdot8}{3\cdot4\cdot5}\cdot\beta\gamma,\ \zeta=\tfrac{12}{3}\cdot\alpha\varepsilon+\tfrac{12\cdot11\cdot10}{3\cdot4\cdot5}\cdot\beta\delta+\tfrac{12\cdot11\cdot10\cdot9\cdot8}{3\cdot4\cdot5\cdot6\cdot7}\cdot\tfrac{\gamma\gamma}{2},$$
$$\eta=\tfrac{14}{3}\cdot\alpha\zeta+\tfrac{14\cdot13\cdot12}{3\cdot4\cdot5}\cdot\beta\varepsilon+\tfrac{14\cdot13\cdot12\cdot11\cdot10}{3\cdot4\cdot5\cdot6\cdot7}\cdot\gamma\delta,$$
$$\theta=\tfrac{16}{3}\cdot\alpha\eta+\tfrac{16\cdot15\cdot14}{3\cdot4\cdot5}\cdot\beta\zeta+\tfrac{16\cdot15\cdots12}{3\cdot4\cdots7}\cdot\gamma\varepsilon+\tfrac{16\cdot15\cdots10}{3\cdot4\cdots9}\cdot\tfrac{\delta\delta}{2}$$
etc.

Therefore from this law, as the following calculation is put in place without difficulty, if the values were found of the letters $\alpha,\ \beta,\ \gamma,\ \delta$ etc., then the summatory term of any series thus may be expressed, of which the general term, or agreeing with the index x, were $=z$, so that there shall be

$$Sz=\int zdx+\tfrac{1}{2}z+\frac{\alpha dz}{1\cdot2\cdot3\cdot dx}-\frac{\beta d^3z}{1\cdot2\cdot3\cdot4\cdot5dx^3}+\frac{\gamma d^5z}{1\cdot2\cdots7dx^5}$$
$$-\frac{\delta d^7z}{1\cdot2\cdots9dx^7}+\frac{\varepsilon d^9z}{1\cdot2\cdots11dx^9}-\frac{\zeta d^{11}z}{1\cdot2\cdots13dx^9}+\text{etc.}$$

But these letters $\alpha,\ \beta,\ \gamma,\ \delta$ etc. have been found to have the following values:

$\alpha=\frac{1}{2}$	or	$1\cdot2\alpha=1$
$\beta=\frac{1}{6}$		$1\cdot2\cdot3\beta=1$
$\gamma=\frac{1}{6}$		$1\cdot2\cdot3\cdot4\gamma=4$
$\delta=\frac{3}{10}$		$1\cdot2\cdot3\cdots5\delta=36$
$\varepsilon=\frac{5}{6}$		$1\cdot2\cdot3\cdots6\varepsilon=600$
$\zeta=\frac{691}{210}$		$1\cdot2\cdot3\cdots7\zeta=24\cdot691$
$\eta=\frac{35}{2}$		$1\cdot2\cdot3\cdots8\eta=20160\cdot35$

$$\theta = \tfrac{3617}{30} \qquad 1\cdot 2\cdot 3\cdots 9\theta = 12096\cdot 3617$$
$$\iota = \tfrac{43867}{42} \qquad 1\cdot 2\cdot 3\cdots 10\iota = 86400\cdot 43867$$
$$\chi = \tfrac{1222277}{110} \qquad 1\cdot 2\cdot 3\cdots 11\chi = 362880\cdot 1222277$$
$$\lambda = \tfrac{854513}{6} \qquad 1\cdot 2\cdot 3\cdots 12\lambda = 79833600\cdot 854513$$
$$\mu = \tfrac{1181820455}{546} \qquad 1\cdot 2\cdot 3\cdots 13\mu = 11404800\cdot 1181820455$$
$$\nu = \tfrac{76977927}{2} \qquad 1\cdot 2\cdot 3\cdots 14\nu = 43589145600\cdot 76977927$$
$$\xi = \tfrac{23749461029}{30} \qquad 1\cdot 2\cdot 3\cdots 15\xi = 43589145600\cdot 23749461029$$
$$\pi = \tfrac{8615841276005}{462} \qquad 1\cdot 2\cdot 3\cdots 16\pi = 45287424000\cdot 8615841276005$$

etc.

122. These numbers have the greatest use in the general principles of series. For first from these numbers the final terms are able to be formed in the sums of even powers, which uneven powers, we have noted above that the terms remaining are able to be found from the preceding sums [in the first part of § 63]. For in the equal powers the terms of the summation are *x* multiplied by certain numbers, which numbers for the powers II, IV, VI, VIII etc. are $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, $\frac{1}{30}$ etc. with alternate signs effected. But these numbers arise, if the values of the letters $\alpha,\ \beta,\ \gamma,\ \delta$ etc. found above may be divided respectively by the odd numbers 3, 5, 7, 9 etc., from which these numbers, which from their discoverer James Bernoulli are accustomed to be called Bernoulli Numbers , are

$$\frac{\alpha}{3} = \frac{1}{6} = \mathfrak{A} \qquad \frac{\iota}{19} = \frac{43867}{798} = \mathfrak{I}$$
$$\frac{\beta}{5} = \frac{1}{30} = \mathfrak{B} \qquad \frac{\chi}{21} = \frac{174611}{330} = \mathfrak{K} = \frac{283\cdot 617}{330}$$
$$\frac{\gamma}{7} = \frac{1}{42} = \mathfrak{C} \qquad \frac{\lambda}{23} = \frac{854513}{138} = \mathfrak{L} = \frac{11\cdot 131\cdot 593}{2\cdot 3\cdot 23}$$
$$\frac{\delta}{9} = \frac{1}{30} = \mathfrak{D} \qquad \frac{\mu}{25} = \frac{236364091}{2730} = \mathfrak{M}$$
$$\frac{\varepsilon}{11} = \frac{5}{66} = \mathfrak{E} \qquad \frac{\nu}{27} = \frac{8553103}{6} = \mathfrak{N} = \frac{13\cdot 657931}{6}$$
$$\frac{\zeta}{13} = \frac{691}{2730} = \mathfrak{F} \qquad \frac{\xi}{29} = \frac{23749461029}{870} = \mathfrak{O}$$
$$\frac{\eta}{15} = \frac{7}{6} = \mathfrak{G} \qquad \frac{\pi}{31} = \frac{8615841276005}{14322} = \mathfrak{P}$$
$$\frac{\theta}{17} = \frac{3617}{510} = \mathfrak{H} \qquad \text{etc.}$$

[These numbers arise in finding the sums of powers from certain properties of the binomial coefficients, introduced by James Bernoulli in his *Ars Conjectandi.*]

123. These Bernoulli Numbers $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ etc. therefore are able to be found at once from the following equations

$$\begin{aligned}
\mathfrak{A} &= \tfrac{1}{6}\\
\mathfrak{B} &= \tfrac{4\cdot 3}{1\cdot 2}\cdot\tfrac{1}{5}\mathfrak{A}^2\\
\mathfrak{C} &= \tfrac{6\cdot 5}{1\cdot 2}\cdot\tfrac{2}{7}\mathfrak{A}\mathfrak{B}\\
\mathfrak{D} &= \tfrac{8\cdot 7}{1\cdot 2}\cdot\tfrac{2}{9}\mathfrak{A}\mathfrak{C}+\tfrac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}\cdot\tfrac{1}{9}\mathfrak{B}^2\\
\mathfrak{E} &= \tfrac{10\cdot 9}{1\cdot 2}\cdot\tfrac{2}{11}\mathfrak{A}\mathfrak{D}+\tfrac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4}\cdot\tfrac{2}{11}\mathfrak{B}\mathfrak{C}\\
\mathfrak{F} &= \tfrac{12\cdot 11}{1\cdot 2}\cdot\tfrac{2}{13}\mathfrak{A}\mathfrak{E}+\tfrac{12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4}\cdot\tfrac{2}{13}\mathfrak{B}\mathfrak{D}+\tfrac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot\tfrac{1}{13}\mathfrak{C}^2\\
\mathfrak{G} &= \tfrac{14\cdot 13}{1\cdot 2}\cdot\tfrac{2}{15}\mathfrak{A}\mathfrak{F}+\tfrac{14\cdot 13\cdot 12\cdot 11}{1\cdot 2\cdot 3\cdot 4}\cdot\tfrac{2}{15}\mathfrak{B}\mathfrak{E}+\tfrac{14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot\tfrac{2}{15}\mathfrak{C}\mathfrak{D}\\
&\text{etc.,}
\end{aligned}$$

the law of which equations is by itself evident, if only it may be noted where the square of a certain letter occurs, the coefficient of which is twice as small, as must be considered following the rule. But actually the terms, which contain the products from different letters, are agreed to occur twice ; indeed there shall be for argument's sake

$$\begin{aligned}
13\mathfrak{F} = \tfrac{12\cdot 11}{1\cdot 2}\mathfrak{A}\mathfrak{E}+\tfrac{12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4}\mathfrak{B}\mathfrak{D}+\tfrac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\mathfrak{C}\mathfrak{C}\\
+\tfrac{12\cdot 11\cdot 10\cdots 5}{1\cdot 2\cdot 3\cdots 8}\mathfrak{D}\mathfrak{B}+\tfrac{12\cdot 11\cdot 10\cdots 3}{1\cdot 2\cdot 3\cdots 10}\mathfrak{E}\mathfrak{A}.
\end{aligned}$$

124. Then truly also the same numbers $\alpha, \beta, \gamma, \delta$ etc. enter into the expressions of the sums of series of fractions contained in this general form

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\text{etc.},$$

as often as n is a positive even number. Indeed we have given these sums in the *Introductione* expressed by powers of the semi perimeter of the circle π with the radius present $=1$, and in the coefficients of these powers themselves those numbers $\alpha, \beta, \gamma, \delta$ etc. are taken to be present. But so that in case these arrangements do not come about, but by necessity it may be understood to have a place, we may investigate these same sums in a singular way, so that the law of the summation of these will be apparent. Because above we have found (§ 43) to be

$$\frac{\pi}{n}\cot\frac{m}{n}\pi=\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\text{ etc.},$$

with the two terms joined together we will have

$$\frac{\pi}{n}\cot\frac{m}{n}\pi=\frac{1}{m}-\frac{2m}{nn-m^2}-\frac{2m}{4n^2-m^2}-\frac{2m}{9n^2-m^2}-\frac{2m}{16n^2-m^2}-\text{etc.},$$

from which we may deduce to be

$$\frac{1}{n^2-m^2}+\frac{1}{4n^2-m^2}+\frac{1}{9n^2-m^2}+\frac{1}{16n^2-m^2}+\text{etc.}=\frac{1}{2mm}-\frac{\pi}{2mn}\cot\frac{m}{n}\pi.$$

Now we may put in place $n=1$ and for m we may put u, so that there shall be

$$\frac{1}{1-u^2}+\frac{1}{4-u^2}+\frac{1}{9-u^2}+\frac{1}{16-u^2}+\text{etc.}=\frac{1}{2uu}-\frac{\pi}{2u}\cot\pi u.$$

These individual fractions may be resolved in series

$$\begin{aligned}
\frac{1}{1-u^2}&=1+u^2+u^4+u^6+u^8+\text{etc.}\\
\frac{1}{4-u^2}&=\frac{1}{2^2}+\frac{u^2}{2^4}+\frac{u^4}{2^6}+\frac{u^6}{2^8}+\frac{u^8}{2^{10}}+\text{etc.}\\
\frac{1}{9-u^2}&=\frac{1}{3^2}+\frac{u^2}{3^4}+\frac{u^4}{3^6}+\frac{u^6}{3^8}+\frac{u^8}{3^{10}}+\text{etc.}\\
\frac{1}{16-u^2}&=\frac{1}{4^2}+\frac{u^2}{4^4}+\frac{u^4}{4^6}+\frac{u^6}{4^8}+\frac{u^8}{4^{10}}+\text{etc.}\\
&\text{etc.}
\end{aligned}$$

125. But if therefore there is put in place

$$\begin{aligned}
&1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{etc.}=\mathfrak{a} &\quad& 1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\text{etc.}=\mathfrak{d}\\
&1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.}=\mathfrak{b} && 1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\text{etc.}=\mathfrak{e}\\
&1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc.}=\mathfrak{c} && 1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\text{etc.}=\mathfrak{f}\\
&\text{etc.,}
\end{aligned}$$

the above series will be changed into this

$$\mathfrak{a}+\mathfrak{b}u^2+\mathfrak{c}u^4+\mathfrak{d}u^6+\mathfrak{e}u^8+\mathfrak{f}u^{10}+\text{etc.}=\frac{1}{2uu}-\frac{\pi}{2u}\cot\pi u.$$

Therefore since in §118 the letters *A*, *B*, *C*, *D* etc. thus prepared may be found, so that by putting

$$s=\frac{1}{u}-Au-Bu^3-Cu^5-Du^7-Eu^9-\text{etc.}$$

there shall be $s=\frac{1}{2}\cot\frac{1}{2}u$, there will be on putting πu in place of $\frac{1}{2}u$ or $2\pi u$ in place of u

$$\tfrac{1}{2}\cot\pi u=\tfrac{1}{2\pi u}-2A\pi u-2^3B\pi^3u^3-2^5C\pi^5u^5-2^7D\pi^7u^7-\text{etc.},$$

from which on multiplying by $\frac{\pi}{u}$ there will be

$$\tfrac{\pi}{2u}\cot\pi u=\tfrac{1}{2uu}-2A\pi^2-2^3B\pi^4u^2-2^5C\pi^6u^4-2^7D\pi^8u^6-\text{etc.},$$

and hence it follows to become

$$\tfrac{1}{2uu}-\tfrac{\pi}{2u}\cot\pi u=2A\pi^2+2^3B\pi^4u^2+2^5C\pi^6u^4+2^7D\pi^8u^6+\text{etc.}$$

Therefore because in this way we have found to be

$$\tfrac{1}{2uu}-\tfrac{\pi}{2u}\cot\pi u=\mathfrak{a}+\mathfrak{b}u^2+\mathfrak{c}u^4+\mathfrak{d}u^6+\text{etc.},$$

it is necessary that there shall be

$$\begin{aligned}
\mathfrak{a}&=2A\pi^2=\tfrac{2\alpha}{1\cdot2\cdot3}\pi^2=\tfrac{2\mathfrak{A}}{1\cdot2}\pi^2\\
\mathfrak{b}&=2^3B\pi^4=\tfrac{2^3\beta}{1\cdot2\cdot3\cdot4\cdot5}\pi^4=\tfrac{2^3\mathfrak{B}}{1\cdot2\cdot3\cdot4}\pi^4\\
\mathfrak{c}&=2^5C\pi^6=\tfrac{2^5\gamma}{1\cdot2\cdot3\cdots7}\pi^6=\tfrac{2^5\mathfrak{C}}{1\cdot2\cdot3\cdots6}\pi^6\\
\mathfrak{d}&=2^7D\pi^8=\tfrac{2^7\delta}{1\cdot2\cdot3\cdots9}\pi^8=\tfrac{2^7\mathfrak{D}}{1\cdot2\cdot3\cdots8}\pi^8\\
\mathfrak{e}&=2^9E\pi^{10}=\tfrac{2^9\varepsilon}{1\cdot2\cdot3\cdots11}\pi^{10}=\tfrac{2^9\mathfrak{E}}{1\cdot2\cdot3\cdots10}\pi^{10}\\
\mathfrak{f}&=2^{11}F\pi^{12}=\tfrac{2^{11}\xi}{1\cdot2\cdot3\cdots13}\pi^{12}=\tfrac{2^{11}\mathfrak{F}}{1\cdot2\cdot3\cdots12}\pi^{12}\\
&\text{etc.}
\end{aligned}$$

126. Therefore as from this easy reasoning not only all the powers of reciprocals, which we have shown in the preceding chapter, may be summed readily, but likewise also it is evident, how these sums may be formed from the known values of the letters α, β, γ, δ, ε etc. or also from the Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. Whereby since we have defined fifteen of these numbers in §122, from these the sums of the reciprocals of all the even powers as far as to the sum of this series included will be possible to be assigned :

$$1+\tfrac{1}{2^{30}}+\tfrac{1}{3^{30}}+\tfrac{1}{4^{30}}+\text{etc.};$$

indeed the sum of this series will be

$$=\frac{2^{29}\xi}{1\cdot 2\cdot 3\cdots 31}\pi^{30}=\frac{2^{29}\mathfrak{F}}{1\cdot 2\cdot 3\cdots 30}\pi^{30}.$$

And if with which it were wished to determine these sums further, that will be most easily put in place with the numbers $\alpha,\ \beta,\ \gamma$ etc. continued, or from these $\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},$ etc.

127. Therefore the origin of these numbers $\alpha,\ \beta,\ \gamma,\ \delta$ etc. or thence of the forms $\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{D}$ etc. chiefly is owed to the expansion of the cotangent of some angle in an infinite series. For since there shall be

$$\tfrac{1}{2}\cot\tfrac{1}{2}u=\tfrac{1}{u}-Au-Bu^3-Cu^5-Du^7-Eu^9-\text{etc.},$$

there will be

$$Au^2+Bu^4+Cu^6+Du^8+Eu^9+\text{etc.}=1-\tfrac{u}{2}\cot\tfrac{1}{2}u\ ;$$

therefore if in place of the coefficients A, B, C, D etc. the values of these may be substituted, there may be found

$$\frac{\alpha u^2}{1\cdot 2\cdot 3}+\frac{\beta u^4}{1\cdot 2\cdots 5}+\frac{\gamma u^6}{1\cdot 2\cdots 7}+\frac{\delta u^8}{1\cdot 2\cdots 9}+\text{etc.}=1-\tfrac{u}{2}\cot\tfrac{1}{2}u$$

and on using the Bernoulli numbers there will be

$$\frac{\mathfrak{A}u^2}{1\cdot 2}+\frac{\mathfrak{B}u^4}{1\cdot 2\cdot 3\cdot 4}+\frac{\mathfrak{C}u^6}{1\cdot 2\cdots 6}+\frac{\mathfrak{D}u^8}{1\cdot 2\cdots 8}+\text{etc.}=1-\tfrac{u}{2}\cot\tfrac{1}{2}u\ ,$$

from which series by differentiation innumerable other series are able to be deduced and thus an infinitude of series to be summed, in which these so noteworthy numbers enter.

128. We may take the first equation, which we may multiply by u, so that there shall be

$$\frac{\alpha u^3}{1\cdot 2\cdot 3}+\frac{\beta u^5}{1\cdot 2\cdots 5}+\frac{\gamma u^7}{1\cdot 2\cdots 7}+\frac{\delta u^9}{1\cdot 2\cdots 9}+\text{etc.}=u-\tfrac{uu}{2}\cot\tfrac{1}{2}u,$$

which differentiated and divided by du gives

$$\frac{\alpha u^2}{1\cdot 2}+\frac{\beta u^4}{1\cdot 2\cdot 3\cdot 4}+\frac{\gamma u^6}{1\cdot 2\cdots 6}+\frac{\delta u^8}{1\cdot 2\cdots 8}+\text{etc.}=1-u\cot\tfrac{1}{2}u+\frac{uu}{4\left(\sin\frac{1}{2}u\right)^2};$$

and if it may be differentiated anew, there will be

$$\frac{\alpha u}{1}+\frac{\beta u^3}{1\cdot 2\cdot 3}+\frac{\gamma u^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\text{etc.}=-\cot\tfrac{1}{2}u+\frac{u}{\left(\sin\frac{1}{2}u\right)^2}-\frac{uu\cos\frac{1}{2}u}{4\left(\sin\frac{1}{2}u\right)^3}.$$

But if the other equation may be differentiated, there will be

$$\frac{\mathfrak{A}u}{1}+\frac{\mathfrak{B}u^3}{1\cdot 2\cdot 3}+\frac{\mathfrak{C}u^5}{1\cdot 2\cdots 5}+\frac{\mathfrak{D}u^7}{1\cdot 2\cdots 7}+\text{etc.}=-\tfrac{1}{2}\cot\tfrac{1}{2}u+\frac{u}{4\left(\sin\frac{1}{2}u\right)^2}.$$

Hence from these, if there may be put $u=\pi$, on account of $\cot\frac{1}{2}\pi=0$ and $\sin\frac{1}{2}\pi=1$ these summations follow

$$1=\frac{\alpha\pi^2}{1\cdot 2\cdot 3}+\frac{\beta\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\frac{\gamma\pi^6}{1\cdot 2\cdot 3\cdots 7}+\frac{\delta\pi^8}{1\cdot 2\cdot 3\cdots 9}+\text{etc.}$$
$$1+\frac{\pi^2}{4}=\frac{\alpha\pi^2}{1\cdot 2}+\frac{\beta\pi^4}{1\cdot 2\cdot 3\cdot 4}+\frac{\gamma\pi^6}{1\cdot 2\cdot 3\cdots 6}+\frac{\delta\pi^8}{1\cdot 2\cdot 3\cdots 8}+\text{etc.}$$
$$\pi=\frac{\alpha\pi}{1}+\frac{\beta\pi^3}{1\cdot 2\cdot 3}+\frac{\gamma\pi^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\frac{\delta\pi^7}{1\cdot 2\cdot 3\cdots 7}+\text{etc.}$$

or

$$1=\alpha+\frac{\beta\pi^2}{1\cdot 2\cdot 3}+\frac{\gamma\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\frac{\delta\pi^6}{1\cdot 2\cdot 3\cdots 7}+\text{etc.}$$

from which if the first may be taken away, there will remain

$$\alpha=\frac{(\alpha-\beta)\pi^2}{1\cdot 2\cdot 3}+\frac{(\beta-\gamma)\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\frac{(\gamma-\delta)\pi^6}{1\cdot 2\cdot 3\cdots 7}+\text{etc.}$$

Then truly there will be

$$1=\frac{\mathfrak{A}\pi^2}{1\cdot 2}+\frac{\mathfrak{B}\pi^4}{1\cdot 2\cdot 3\cdot 4}+\frac{\mathfrak{C}\pi^6}{1\cdot 2\cdot 3\cdots 6}+\frac{\mathfrak{D}\pi^8}{1\cdot 2\cdot 3\cdots 8}+\text{etc.}$$
$$\frac{\pi}{4}=\frac{\mathfrak{A}\pi}{1}+\frac{\mathfrak{B}\pi^3}{1\cdot 2\cdot 3}+\frac{\mathfrak{C}\pi^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\frac{\mathfrak{D}\pi^7}{1\cdot 2\cdot 3\cdots 7}+\text{etc.}$$

or

$$\frac{1}{4}=\frac{\mathfrak{A}}{1}+\frac{\mathfrak{B}\pi^2}{1\cdot 2\cdot 3}+\frac{\mathfrak{C}\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\frac{\mathfrak{D}\pi^6}{1\cdot 2\cdot 3\cdots 7}+\text{etc.}$$

129. From the table of the values of the number $\alpha,\ \beta,\ \gamma,\ \delta$ etc., which we have shown above (§ 121), it appears that these decrease at first, and then truly again indeed to increase indefinitely. Therefore it will be worthwhile to investigate, according to what method they may progress further, after they will have continued now for an exceedingly long time. Therefore let φ be some number of this series of numbers $\alpha,\ \beta,\ \gamma,\ \delta$ etc. removed at the greatest distance from the start, and let the following of the numbers be ψ. Because the sum of the reciprocals of the powers is defined by these numbers, let *2n* be the exponent of the power, in the sum of which the number φ enters; $2n+2$ will be the exponent of the power for the corresponding number ψ and the number *n* now will be exceedingly large. Hence from § 125 there will be had

$$1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\text{etc.}=\frac{2^{2n-1}\varphi}{1\cdot 2\cdot 3\cdots(2n+1)}\pi^{2n},$$
$$1+\frac{1}{2^{2n+2}}+\frac{1}{3^{2n+2}}+\frac{1}{4^{2n+2}}+\text{etc.}=\frac{2^{2n+1}\psi}{1\cdot 2\cdot 3\cdots(2n+3)}\pi^{2n+2}.$$

But if therefore the latter may be divided by the former, there will be

$$\frac{1+\frac{1}{2^{2n+2}}+\frac{1}{3^{2n+2}}+\text{etc.}}{1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\text{etc.}}=\frac{4\psi\pi^2}{(2n+2)(2n+3)\varphi}.$$

Now because *n* is an exceedingly large number, on account of the series each nearly $=1$ and there will be

$$\frac{\psi}{\varphi}=\frac{(2n+2)(2n+3)}{4\pi^2}=\frac{nn}{\pi\pi}.$$

Therefore since *n* designate, how often the number φ shall be computed from the first α, here the number φ will be itself had to its following ψ as π^2 to n^2, which ratio, if *n* were an infinite number, becomes in complete agreement to the truth. Because there is nearly $\pi\pi=10$, if there may be put $n=100$, the term will be around a thousand times smaller to the following. Therefore the numbers $\alpha, \beta, \gamma, \delta$ etc. and equally the Bernoulli numbers $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. equally constitute maximally divergent series, which also increase more than any series of geometric terms proceeding by increasing.

130. Therefore from these values of the numbers found $\alpha, \beta, \gamma, \delta$ etc. or $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc., if a series may be proposed, the general term of which *z* were some function of *x*, the term of the summation *Sz* of this series may be expressed in the following manner, so that there shall be

$$Sz=\int zdx+\frac{1}{2}z+\frac{1}{6}\cdot\frac{dz}{1\cdot 2dx}-\frac{1}{30}\cdot\frac{d^3z}{1\cdot 2\cdot 3\cdot 4dx^3}$$
$$+\frac{1}{42}\cdot\frac{d^5z}{1\cdot 2\cdot 3\cdots 6dx^5}-\frac{1}{30}\cdot\frac{d^7z}{1\cdot 2\cdot 3\cdots 8dx^7}+\frac{5}{66}\cdot\frac{d^9z}{1\cdot 2\cdot 3\cdots 10dx^9}-\frac{691}{2730}\cdot\frac{d^{11}z}{1\cdot 2\cdot 3\cdots 12dx^{11}}+$$
$$\frac{7}{6}\cdot\frac{d^{13}z}{1\cdot 2\cdot 3\cdots 14dx^{13}}-\frac{3617}{510}\cdot\frac{d^{15}z}{1\cdot 2\cdot 3\cdots 16dx^{15}}+\frac{43867}{798}\cdot\frac{d^{17}z}{1\cdot 2\cdot 3\cdots 18dx^{17}}$$
$$-\frac{174611}{330}\cdot\frac{d^{19}z}{1\cdot 2\cdot 3\cdots 20dx^{19}}+\frac{854513}{138}\cdot\frac{d^{21}z}{1\cdot 2\cdot 3\cdots 22dx^{21}}$$
$$-\frac{236364091}{2730}\cdot\frac{d^{23}z}{1\cdot 2\cdot 3\cdots 24dx^{23}}+\frac{8553103}{6}\cdot\frac{d^{25}z}{1\cdot 2\cdot 3\cdots 26dx^{25}}$$
$$-\frac{23749461029}{870}\cdot\frac{d^{27}z}{1\cdot 2\cdot 3\cdots 28dx^{27}}+\frac{8615841276005}{14322}\cdot\frac{d^{29}z}{1\cdot 2\cdot 3\cdots 30dx^{29}}-\text{etc.}$$

Therefore if the integral $\int zdx$ were known or that quantity, the differential of which shall be = zdx, the summatory term may be found with the aid of differentiation. But it is to be noted always that it is required to add a constant of this kind, so that the sum made $=0$, if the index x may be put to change into nothing.

131. Therefore if z were a rational integral function of x, because the differentials of this finally vanish, the summatory term may be expressed by a finite expression; that which we will shown in the following examples.

EXAMPLE 1

The summatory term of this series is sought.

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & & & x \\ 1+ & 9+ & 25+ & 49+ & 81+ & \cdots & & +(2x-1)^2. \end{array}$$

Because here there shall be $z=(2x-1)^2=4xx-4x+1$, there will be

$$\int zdx=\tfrac{4}{3}x^3-2x^2+x;$$

for from the differentiation of this there arises $4xxdx-4xdx+dx=zdx$. Then indeed by differentiation there will be

$$\tfrac{dz}{dx}=8x-4,\quad \tfrac{ddz}{dx^2}=8,\quad \tfrac{d^3z}{dx^3}=0 \text{ etc.}$$

Hence the summatory term sought will be

$$\tfrac{4}{3}x^3-2x^2+x+2xx-2x+\tfrac{1}{2}+\tfrac{2}{3}x-\tfrac{1}{3}\pm\text{Const.},$$

where the terms $\frac{1}{2}-\frac{1}{3}$ must be taken from the constant; from which there will be

$$S(2x-1)^2=\tfrac{4}{3}x^3-\tfrac{1}{3}x=\tfrac{x}{3}(2x-1)(2x+1).$$

Thus there will be on putting $x=4$, the sum of the first four terms

$$1+9+25+49=\tfrac{4}{3}\cdot 7\cdot 9=84\,.$$

EXEMPLUM 2

The summatory term of this series is sought.

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & & & x \\ 1+ & 27+ & 125+ & 343 & +\cdots+ & & (2x-1)^3. \end{array}$$

Because there is $z=(2x-1)^3=8x^3-12x^2+6x-1$, there will be

$$\int zdx = 2x^4-4x^3+3x^2-x,$$
$$\tfrac{dz}{dx}=24x^2-24x+6,\quad \tfrac{ddz}{dx^2}=48x-24,\quad \tfrac{d^3z}{dx^3}= 48\ ;$$

the following vanish. Whereby there shall be

$$\begin{aligned} S(2x-1)^3 = 2x^4-4x^3+3x^2-x \\ +4x^3-6x^2+3x-\tfrac{1}{2} \\ +2x^2-2x+\tfrac{1}{2} \\ -\tfrac{1}{15}\pm \text{ Const.}, \end{aligned}$$

that is

$$S(2x-1)^3=2x^4-x^2=x^2(2xx-1).$$

Thus there will be on putting $x=4$

$$1+27+125+343=16\cdot 31=496\,.$$

132. From this general expression for the summatory term that summatory term follows unaided, which in the above part we have dedicated to the sums of powers of natural numbers and which there it was not possible to report. For if indeed we may put $z=x^n$, certainly there will be $\int zdx=\frac{1}{n+1}x^{n+1}$; now the differentials thus will be had themselves

$$\tfrac{dz}{dx}=nx^{n-1},\quad \tfrac{ddz}{dx^2}=(n-1)x^{n-2},\quad \tfrac{d^3z}{dx^3}=(n-1)(n-2)x^{n-3},$$
$$\tfrac{d^5z}{dx^5}=(n-1)(n-2)(n-3)(n-4)x^{n-5},\quad \tfrac{d^7z}{dx^7}=n(n-1)\cdots(n-6)x^{n-7}\ \text{ etc.}$$

Therefore from these the following summatory term may be deduced corresponding to the general term x^n, clearly

$$Sx^n = \frac{1}{n+1}x^{n+1} + \frac{1}{2}x^n + \frac{1}{6}\cdot\frac{n}{2}x^{n-1} - \frac{1}{30}\cdot\frac{n(n-1)(n-2)}{2\cdot3\cdot4}x^{n-3}$$
$$+\frac{1}{42}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)}{2\cdot3\cdot4\cdot5\cdot6}x^{n-5} - \frac{1}{30}\cdot\frac{n(n-1)\cdots(n-6)}{2\cdot3\cdots8}x^{n-7} + \frac{5}{66}\cdot\frac{n(n-1)\cdots(n-8)}{1\cdot2\cdot3\cdots10}x^{n-9}$$
$$-\frac{691}{2730}\cdot\frac{n(n-1)\cdots(n-10)}{1\cdot2\cdot3\cdots12}x^{n-11} + \frac{7}{6}\cdot\frac{n(n-1)\cdots(n-12)}{1\cdot2\cdot3\cdots14dx^{13}}x^{n-13}$$
$$-\frac{3617}{510}\cdot\frac{n(n-1)\cdots(n-14)}{1\cdot2\cdot3\cdots16}x^{n-15} + \frac{43867}{798}\cdot\frac{n(n-1)\cdots(n-16)}{1\cdot2\cdot3\cdots18}x^{n-17}$$
$$-\frac{174611}{330}\cdot\frac{n(n-1)\cdots(n-18)}{1\cdot2\cdot3\cdots20}x^{n-19} + \frac{854513}{138}\cdot\frac{n(n-1)\cdots(n-20)}{1\cdot2\cdot3\cdots22}x^{n-21}$$
$$-\frac{236364091}{2730}\cdot\frac{n(n-1)\cdots(n-22)}{1\cdot2\cdot3\cdots24}x^{n-23} + \frac{8553103}{6}\cdot\frac{n(n-1)\cdots(n-24)}{1\cdot2\cdot3\cdots26}x^{n-25}$$
$$-\frac{23749461029}{870}\cdot\frac{n(n-1)\cdots(n-26)}{1\cdot2\cdot3\cdots28}x^{n-27} + \frac{8615841276005}{14322}\cdot\frac{n(n-1)\cdots(n-28)}{1\cdot2\cdot3\cdots30}x^{n-29} - \text{etc.}$$

which expression does not differ from that, which we have given above, only here we have introduced the Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc., since above we have used the numbers $\alpha, \beta, \gamma, \delta$, etc.; yet meanwhile it elicits agreement at once. Here therefore the summatory term of all powers as far as to the thirtieth term inclusive will be allowed to be shown; which investigation, if another way were undertaken, would not be possible to be resolved without the longest and most tedious calculations.

133. Now above (§ 59) we have given an almost similar expression for the summatory term from the general term being defined. Indeed equally from that the following differentials of the general term will proceed; but it will be different from that chiefly in that, because that integral $\int zdx$ will not be required, now the individual differentials of the general term will have multiples of these functions of x. Therefore we may elicit anew the same expression in the following manner more convenient for the nature of the series, from which likewise the law will be shown more clearly, following which the coefficients of the differential for that are proceeding. Therefore let the general term of the series be *z*, some function of the index *x;* now the summatory term sought shall be *s;* which because, as we have seen, it will be a function of this kind of *x*, so that it may vanish on putting $x = 0$, there will be by that, which we have shown above concerning the nature of functions of this kind,

$$s - \frac{xds}{1dx} + \frac{x^2dds}{1\cdot2dx^2} - \frac{x^3d^3s}{1\cdot2\cdot3dx^3} + \frac{x^4d^4s}{1\cdot2\cdot3\cdot4dx^4} - \text{etc.} = 0.$$

134. Because *s* specifies the sum of all the terms of the series from the first as far as to the final *z*, it is evident, if in *s* in place of x there may be put $x-1$, then the first sum to the final term *z* pays a price; evidently there will be [for the negative unit step]

$$s - z = s - \frac{ds}{dx} + \frac{dds}{2dx^2} - \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} - \text{etc.}$$

and thus

$$z = \frac{ds}{dx} - \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} - \frac{d^4s}{24dx^4} + \text{etc.},$$

which equation supplies the manner of defining the general term from the given summatorial term s, which is indeed the most easy by itself. But with a suitable combination of this equation with that, which we have found in the previous paragraph, the value of s will be able to be determined through x and z. To this end we may put in place to be

$$s - Az + \frac{Bdz}{dx} - \frac{Cddz}{dx^2} + \frac{Dd^3z}{dx^3} - \frac{Ed^4z}{dx^4} + \text{etc.} = 0,$$

where A, B, C, D etc. may signify necessary coefficients, either constants or variables; for since there shall be

$$z = \frac{ds}{dx} - \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} - \frac{d^4s}{24dx^4} + \frac{d^5s}{120dx^5} - \text{etc.},$$

if hence the values for z, $\frac{dz}{dx}$, $\frac{ddz}{dx^2}$, $\frac{d^3z}{dx^3}$ etc. may be substituted in the above equation, there will be produced

$$\begin{aligned}
s &= s \\
-Az &= -\frac{Ads}{dx} + \frac{Adds}{2dx^2} - \frac{Ad^3s}{6dx^3} + \frac{Ad^4s}{24dx^4} - \frac{Ad^5s}{120dx^5} + \text{etc.} \\
+\frac{Bdz}{dx} &= \qquad\quad + \frac{Bdds}{dx^2} - \frac{Bd^3s}{2dx^3} + \frac{Bd^4s}{6dx^4} - \frac{Bd^5s}{24dx^5} + \text{etc.} \\
-\frac{Cddz}{dx^2} &= \qquad\qquad\qquad - \frac{Cd^3s}{dx^3} + \frac{Cd^4s}{2dx^4} - \frac{Cd^5s}{6dx^5} + \text{etc.} \\
+\frac{Dd^3z}{dx^3} &= \qquad\qquad\qquad\qquad\quad + \frac{Dd^4s}{dx^4} - \frac{Dd^5s}{2dx^5} + \text{etc.} \\
-\frac{Ed^4z}{dx^4} &= \qquad\qquad\qquad\qquad\qquad\qquad - \frac{Ed^5s}{dx^5} + \text{etc.} \\
& \text{etc.,}
\end{aligned}$$

which series therefore taken together will be equal to nothing.

135. Therefore since before we have found to be

$$0 = s - \frac{xds}{dx} + \frac{x^2dds}{2dx^2} - \frac{x^3d^3s}{6dx^3} + \frac{x^4d^4s}{24dx^4} - \frac{x^5d^5s}{120dx^5} + \text{etc.},$$

if the above equation may be put in place equal to this equation, the following denominations of the letters A, B, C, D etc. will be produced

$$A = x, \quad B = \frac{x^2}{2} - \frac{A}{2}, \quad C = \frac{x^3}{6} - \frac{B}{2} - \frac{A}{6},$$
$$D = \frac{x^4}{24} - \frac{C}{2} - \frac{B}{6} - \frac{A}{24}, \quad E = \frac{x^5}{120} - \frac{D}{2} - \frac{C}{6} - \frac{B}{24} - \frac{A}{120}, \quad \text{etc.}$$

Therefore from the values found of these letters A, B, C, D etc. from the general term z the summatory term $s = Sz$ thus will be determined, so that there shall be

$$Sz = Az - \frac{Bdz}{dx} + \frac{Cddz}{dx^2} - \frac{Dd^3z}{dx^3} + \frac{Ed^4z}{dx^4} - \frac{Fd^5z}{dx^5} + \text{etc.}$$

136. But since there becomes

$$A = x, \quad B = \tfrac{1}{2}x^2 - \tfrac{1}{2}x, \quad C = \tfrac{1}{6}x^3 - \tfrac{1}{4}x^2 + \tfrac{1}{12}x,$$
$$D = \tfrac{1}{24}x^4 - \tfrac{1}{12}x^3 + \tfrac{1}{24}xx, \quad \text{etc.},$$

it is apparent these coefficients to be the same as we have found above (§ 59); from which this expression of the summatory term is the same, as we have found there, and therefore there will be

$$A = Sx^0 = S1, \quad B = \tfrac{1}{1}Sx^1 - \tfrac{1}{1}x, \quad C = \tfrac{1}{2}Sx^2 - \tfrac{1}{2}x^2,$$
$$D = \tfrac{1}{6}Sx^3 - \tfrac{1}{6}x^3, \quad E = \tfrac{1}{24}Sx^4 - \tfrac{1}{24}x^4 \quad \text{etc.}$$

Hence therefore there will be

$$Sz = xz - \frac{dz}{dx}Sx + \frac{ddz}{2dx^2}Sx^2 - \frac{d^3z}{6dx^3}Sx^3 + \frac{d^4z}{24dx^4}Sx^4 - \text{etc.}$$
$$+ \frac{xdz}{dx} - \frac{x^2ddz}{2dx^2} + \frac{x^3d^3z}{6dx^3} - \frac{x^4d^4z}{24dx^4} + \text{etc.}$$

But if in the general term z there is put $x = 0$, there will be produced the term with the corresponding index $= 0$; which if it is put $= a$, there will be

$$a = z - \frac{xdz}{dx} + \frac{x^2ddz}{2dx^2} - \frac{x^3d^3z}{6dx^3} + \text{etc.}$$

and thus

$$\frac{xdz}{dx} - \frac{x^2ddz}{2dx^2} + \frac{x^3d^3z}{6dx^3} - \frac{x^4d^4z}{24dx^4} + \text{etc.} = z - a,$$

with which value substituted there will be had

$$Sz = (x+1)z - a - \frac{dz}{dx}Sx + \frac{ddz}{2dx^2}Sx^2 - \frac{d^3z}{6dx^3}Sx^3 + \frac{d^4z}{24dx^4}Sx^4 - \text{etc.}$$

Therefore with the sum of the powers known for this for whatever general term agreeing with that the summatory term will be able to be shown.

137. Therefore because we have found a twin expression of the summatory term *Sz* for the general term *z* and the other of these may contain the integral formula $\int zdx$, if these two expressions may be put equal to each other, the value of $\int zdx$ itself will be obtained expressed by a series. For since there shall be

$$\int zdx + \tfrac{1}{2}z + \frac{\mathfrak{A}dz}{1\cdot 2dx} - \frac{\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3} + \frac{\mathfrak{C}d^5z}{1\cdot 2\cdots 6dx^5} - \text{etc.}$$
$$= (x+1)z - a - \frac{dz}{1dx}Sx + \frac{ddz}{1\cdot 2dx^2}Sx^2 - \frac{d^3z}{1\cdot 2\cdot 3dx^3}Sx^3 + \text{etc.},$$

there will be

$$\int zdx = \left(x+\tfrac{1}{2}\right)z - a - \frac{dz}{dx}\left(Sx + \tfrac{1}{2}\mathfrak{A}\right) + \frac{ddz}{2dx^2}Sx^2 - \frac{d^3z}{6dx^3}\left(Sx^3 - \frac{\mathfrak{B}}{4}\right)$$
$$+ \frac{d^4z}{24dx^4}Sx^4 - \frac{d^5z}{120dx^5}\left(Sx^5 + \tfrac{1}{6}\mathfrak{C}\right) + \frac{d^6z}{720dx^6}Sx^6 - \frac{d^7z}{5040dx^7}\left(Sx^7 - \tfrac{1}{8}\mathfrak{D}\right) + \text{etc.},$$
$$+\text{etc.},$$

where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. signify the Bernoulli numbers shown above (§ 122).
For argument's sake let there be $z = xx$; there becomes $a = 0$, $\frac{dz}{dx} = 2x$ and $\frac{ddz}{2dx^2} = 1$; hence there will be

$$\int xxdx = \left(x+\tfrac{1}{2}\right)xx - 2x\left(\tfrac{1}{2}xx + \tfrac{1}{2}x + \tfrac{1}{12}\right) + 2x\left(\tfrac{1}{2}xx + \tfrac{1}{2}x + \tfrac{1}{12}\right) + 1\left(\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + \tfrac{1}{6}x\right)$$

or $\int xxdx = \tfrac{1}{3}x^3$; but $\tfrac{1}{3}x^3$ certainly gives the differential *xxdx*.

138. Therefore a new way is apparent for finding the summatory term of a series of powers; because indeed from the coefficients assumed before *A*, *B*, *C*, *D* etc. these summatory terms may be formed easily, but of which any coefficients may be constructed from the preceding: if in the formulas given in § 135 in place of the values of the letters the values reported in § 136 may be substituted, there will be

$$Sx^1 - x = \tfrac{1}{2}xx - \tfrac{1}{2}x$$
$$Sx^2 - x^2 = \tfrac{1}{3}x^3 - \tfrac{1}{3}x - \tfrac{2}{2}\left(Sx - x\right)$$
$$Sx^3 - x^3 = \tfrac{1}{4}x^4 - \tfrac{1}{4}x - \tfrac{3}{2}\left(Sx^2 - x^2\right) - \tfrac{3\cdot 2}{2\cdot 3}\left(Sx - x\right)$$
$$Sx^4 - x^4 = \tfrac{1}{5}x^5 - \tfrac{1}{5}x - \tfrac{4}{2}\left(Sx^3 - x^3\right) - \tfrac{4\cdot 3}{2\cdot 3}\left(Sx^2 - x\right) - \tfrac{4\cdot 3\cdot 2}{2\cdot 3\cdot 4}\left(Sx - x\right)$$
etc.

Hence therefore the sums of the higher powers will be able to be formed from the sums of the lower powers.

139. But if truly the law, by which the coefficients *A*, *B*, *C*, *D* etc. above (§ 135) have been found to be progressing, we may examine more carefully, we may come upon the recurring series to be put in place. For if we may expand this fraction

$$\frac{x + \frac{1}{2}xxu + \frac{1}{6}x^3u^2 + \frac{1}{24}x^4u^3 + \frac{1}{120}x^5u^4 + \text{etc.}}{x + \frac{1}{2}u + \frac{1}{6}u^2 + \frac{1}{24}u^3 + \frac{1}{120}u^4 + \text{etc.}}$$

we may assume the following powers of u and this series to result

$$A + Bu + Cu^2 + Du^3 + Eu^4 + \text{etc.},$$

there will be, as we have found before,

$$A = x, \quad B = \tfrac{1}{2}xx - \tfrac{1}{2}A \quad \text{etc.}$$

and thus the summatory terms of the series of powers will be found from this series.
But that fraction, from the expansion of which that series arises, will change into this form $\frac{e^{xu}-1}{e^u-1}$, which, if x were a positive whole number, will change into

$$1 + e^u + e^{2u} + e^{3u} + \cdots + e^{(x-1)u};$$

since there shall be

$$1 = 1$$
$$e^{u} = 1 + \frac{u}{1} + \frac{u^2}{1\cdot 2} + \frac{u^3}{1\cdot 2\cdot 3} + \frac{u^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$
$$e^{2u} = 1 + \frac{2u}{1} + \frac{4u^2}{1\cdot 2} + \frac{8u^3}{1\cdot 2\cdot 3} + \frac{16u^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$
$$e^{3u} = 1 + \frac{3u}{1} + \frac{9u^2}{1\cdot 2} + \frac{27u^3}{1\cdot 2\cdot 3} + \frac{81u^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$
$$e^{(x-1)u} = 1 + \frac{(x-1)u}{1} + \frac{(x-1)^2u^2}{1\cdot 2} + \frac{(x-1)^3u^3}{1\cdot 2\cdot 3} + \frac{(x-1)^4u^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.},$$

and thus there will be

$$A = x$$
$$B = S(x-1) = Sx - x$$
$$C = \tfrac{1}{2}S(x-1)^2 = \tfrac{1}{2}Sx^2 - \tfrac{1}{2}x^2$$
$$D = \tfrac{1}{6}S(x-1)^3 = \tfrac{1}{6}Sx^3 - \tfrac{1}{6}x^3$$
$$\text{etc.}$$

From which the intertwining of these coefficients with the sums of the powers now observed before is completely confirmed and demonstrated.

CAPUT V

INVESTIGATIO SUMMAE SERIERUM EX TERMINO GENERALI

103. Sit seriei cuiusque terminus generalis $=y$ respondens indici *x*, ita ut *y* sit functio quaecunque ipsius *x*. Sit porro *Sy* summa seu terminus summatorius seriei exprimens aggregatum omnium terminorum a primo seu alio termino fixo usque ad *y* inclusive. Computabimus autem summas serierum a termino primo, unde , si sit $x=1$, dabit *y* terminum primum atque *Sy* hunc *y* terminum primum exhibebit; sin autem ponatur *x* = 0, terminus summatorius *Sy* in nihilum abire debet, propterea quod nulli termini summandi adsunt. Quocirca terminus summatorius *Sy* eiusmodi erit functio ipsius *x*, quae evanescat posito $x=0$.

104. Si terminus generalis *y* ex pluribus partibus constet, ut sit $y=p+q+r+\text{etc.}$, tum ipsa series considerari poterit tanquam conflata ex pluribus aliis seriebus, quarum termini generales sint *p, q, r* etc. Hinc si singularum istarum serierum summae fuerint cognitae, simul seriei propositae summa poterit assignari; erit enim aggregatum ex summis singularum serierum. Hanc ob rem si sit $y=p+q+r+\text{etc.}$, erit $Sy=Sp+Sq+Sr+\text{etc.}$ Cum igitur supra exhibuerimus summas serierum, quarum termini generales sint quaecunque potestates ipsius *x* habentes exponentes integros affirmativos, hinc euiusque seriei, cuius terminus generalis est $ax^{\alpha}+bx^{\beta}+cx^{\gamma}+\text{etc.}$ denotantibus α,β,γ etc . numeros integros affirmativos seu cuius terminus generalis est functio rationalis integra ipsius *x*, terminus summatorius inveniri poterit.

105. Sit in serie, cuius terminus generalis seu exponenti *x* respondens est $=y$, terminus hunc praecedens seu exponenti $x-1$ respondens $=v$; quoniam *v* oritur ex *y*, si loco *x* scribatur $x-1$, erit

$$v=y-\tfrac{dy}{dx}+\tfrac{ddy}{2dx^2}-\tfrac{d^3y}{6dx^3}+\tfrac{d^4y}{24dx^4}-\tfrac{d^5y}{120dx^5}+\text{etc.}$$

Si igitur *y* fuerit terminus generalis huius seriei

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & \cdots\cdots & x-1 & x \\ a+ & b+ & c+ & d+ & \cdots\cdots & +v+ & y \end{array}$$

huiusque seriei terminus indici 0 respondens fuerit $=A$, erit *v*, quatenus est functio ipsius *x*, terminus generalis huius seriei

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & \cdots\cdots & x \\ A & a+ & b+ & c+ & d+ & \cdots\cdots & +v, \end{array}$$

unde, si Sv denotet summam huius seriei, erit $Sv = Sy - y + A$. Sicque posito $x = 0$, quia fit $Sy = 0$ et $y = A$, quoque Sv evanescet.

106. Cum igitur sit

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \text{etc.}$$

erit per ante ostensa

$$Sv = Sy - S\frac{dy}{dx} + S\frac{ddy}{2dx^2} - S\frac{d^3y}{6dx^3} + S\frac{d^4y}{24dx^4} - \text{etc.}$$

atque ob $Sv = Sy - y + A$ erit

$$y - A = S\frac{dy}{dx} - S\frac{ddy}{2dx^2} + S\frac{d^3y}{6dx^3} - S\frac{x^4d^4y}{24dx^4} + \text{etc.}$$

ideoque habebitur

$$S\frac{dy}{dx} = y - A + S\frac{ddy}{2dx^2} - S\frac{d^3y}{6dx^3} + S\frac{x^4d^4y}{24dx^4} - \text{etc.}$$

Si ergo habeantur termini summatorii serierum, quarum termini generales sunt $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ etc., ex iis obtinebitur terminus summatonus seriei, cuius terminus generalis est $\frac{dy}{dx}$. Quantitas vero constans A ita debet esse comparata, ut facto $x = 0$ terminus summatorius $S\frac{dy}{dx}$ evanescat, hacque conditione facilius determinatur, quam si diceremus eam esse terminum indici 0 respondentem in serie, cuius terminus generalis sit $= y$.

107. Ex hoc fonte summae potestatum numerorum naturalium investigari solent. Sit enim $y = x^{n+1}$; quoniam fit

$$\frac{dy}{dx} = (n+1)x^n, \quad \frac{ddy}{2dx^2} = \frac{(n+1)n}{1\cdot 2}x^{n-1}, \quad \frac{d^3y}{6dx^3} = \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}x^{n-2},$$
$$\frac{x^4d^4y}{24dx^4} = \frac{(n+1)n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4}x^{n-3} \quad \text{etc.,}$$

erit his valoribus substitutis

$$(n+1)Sx^n = x^{n+1} - A + \tfrac{(n+1)n}{1\cdot 2} Sx^{n-1} - \tfrac{(n+1)n(n-1)}{1\cdot 2\cdot 3} Sx^{n-2} + \text{etc.};$$

atque si utrinque per $n+1$ dividatur, erit

$$Sx^n = \tfrac{1}{n+1} x^{n+1} + \tfrac{n}{1\cdot 2} Sx^{n-1} - \tfrac{n(n-1)}{1\cdot 2\cdot 3} Sx^{n-2} + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3\cdot 4} Sx^{n-3} - \text{etc.} - \text{Const.},$$

quae constans ita accipi debet, ut posito $x=0$ totus terminus summatorius evanescat. Ope huius ergo formulae ex iam cognitis summis potestatum inferiorum, quarum termini generales sunt x^{n-1}, x^{n-2} etc., inveniri poterit summa potestatum superiorum termino generali x^n expressarum.

108. Si in hac expressione *n* denotet numerum integrum affirmativum, numerus terminorum erit finitus. Atque adeo hinc summa infinitarum potestatum absolute cognoscetur; erit enim, si $n=0$,

$$Sx^0 = x.$$

Hacque cognita ad superiores progredi licebit; posito enim $n=1$ fiet

$$Sx^1 = \tfrac{1}{2} x^2 + \tfrac{1}{2} Sx^0 = \tfrac{1}{2} x^2 + \tfrac{1}{2} x.$$

si porro ponatur $n = 2$, prodibit

$$Sx^2 = \tfrac{1}{3} x^3 + Sx - \tfrac{1}{3} Sx^0 = \tfrac{1}{3} x^3 + \tfrac{1}{2} x^2 + \tfrac{1}{6} x,$$

deinde

$$Sx^3 = \tfrac{1}{4} x^4 + \tfrac{3}{2} Sx^2 - Sx + \tfrac{1}{4} Sx^0 = \tfrac{1}{4} x^4 + \tfrac{1}{2} x^3 + \tfrac{1}{4} x^2,$$

$$Sx^4 = \tfrac{1}{5} x^5 + \tfrac{4}{2} Sx^3 - \tfrac{4}{2} Sx^2 + Sx - \tfrac{1}{5} Sx^0$$

sive

$$Sx^4 = \tfrac{1}{5} x^5 + \tfrac{1}{2} x^4 + \tfrac{1}{3} x^3 - \tfrac{1}{30} x.$$

Sicque porro quarumvis potestatum superiorum summae successivae ex inferioribus colligentur; hoc autem facilius per sequentes modos praestabitur.

109. Quoniam supra invenimus esse

$$S\tfrac{dy}{dx} = y + \tfrac{1}{2} S\tfrac{ddy}{dx^2} - \tfrac{1}{6} S\tfrac{d^3y}{dx^3} + \tfrac{1}{24} S\tfrac{d^4y}{dx^4} - \tfrac{1}{120} S\tfrac{d^5y}{dx^5} + \text{etc.},$$

Si ponamus $\frac{dy}{dx} = z$, et $\frac{ddy}{dx^2} = \frac{dz}{dx}$, $\frac{d^3y}{dx^3} = \frac{ddz}{dx^2}$ etc. Tum vero ob $dy = zdx$ erit *y* quantitas, cuius differentiale est $= zdx$, quod hoc modo indicamus, ut sit $y = \int zdx$. Quanquam autem haec inventio ipsius *y* ex dato *z* a calculo integrali pendet, tamen hic iam ista forma $\int zdx$ uti poterimus, si quidem pro *z* alias ipsius *x* functiones non substituamus nisi eiusmodi, ut functio illa, cuius differentiale est $= zdx$, ex praecedentibus exhiberi queat. His igitur valoribus substitutis erit

$$Sz = \int zdx + \tfrac{1}{2} S \tfrac{dz}{dx} - \tfrac{1}{6} S \tfrac{ddz}{dx^2} + \tfrac{1}{24} S \tfrac{d^3z}{dx^3} - \text{etc.}$$

adiiciendo eiusmodi constantem, ut posito $x = 0$ ipsa summa *Sz* evanescat.

110. Substituendo autem loco *y* in superiori expressione litteram *z* vel, quod eodem redit, differentiando istam aequationem erit

$$S \tfrac{dz}{dx} = z + \tfrac{1}{2} S \tfrac{ddz}{dx^2} - \tfrac{1}{6} S \tfrac{d^3z}{dx^3} + \tfrac{1}{24} S \tfrac{d^4z}{dx^4} - \text{etc.};$$

sin autem loco *y* ponatur $\frac{dz}{dx}$ erit

$$S \tfrac{ddz}{dx^2} = \tfrac{dz}{dx} + \tfrac{1}{2} S \tfrac{d^3z}{dx^3} - \tfrac{1}{6} S \tfrac{d^4z}{dx^4} + \tfrac{1}{24} S \tfrac{d^5z}{dx^5} - \text{etc.};$$

Similique modo si pro *y* successive ponantur valores $\frac{ddz}{dx^2}$, $\frac{d^3z}{dx^3}$ etc., reperitur

$$S \tfrac{d^3z}{dx^3} = \tfrac{ddz}{dx^2} + \tfrac{1}{2} S \tfrac{d^4z}{dx^4} - \tfrac{1}{6} S \tfrac{d^5z}{dx^5} + \tfrac{1}{24} S \tfrac{d^6z}{dx^6} - \text{etc.},$$
$$S \tfrac{d^4z}{dx^4} = \tfrac{d^3z}{dx^3} + \tfrac{1}{2} S \tfrac{d^5z}{dx^5} - \tfrac{1}{6} S \tfrac{d^6z}{dx^6} + \tfrac{1}{24} S \tfrac{d^7z}{dx^7} - \text{etc.}$$

sicque porro in infinitum.

111. Si nunc isti valores pro $S \frac{dz}{dx}$, $S \frac{ddz}{dx^2}$, $S \frac{d^3z}{dx^3}$ etc. successive substituantur in expressione

$$Sz = \int zdx + \tfrac{1}{2} S \tfrac{dz}{dx} - \tfrac{1}{6} S \tfrac{ddz}{dx^2} + \tfrac{1}{24} S \tfrac{d^3z}{dx^3} - \text{etc.}$$

invenietur expressio pro *Sz,* quae constabit ex his terminis $\int zdx, z, \frac{dz}{dx}, \frac{ddz}{dx^2}, \frac{d^3z}{dx^3}$ etc., quorum coefficientes facilius sequenti modo investigabuntur.

Ponatur

$$Sz = \int zdx + \alpha z + \frac{\beta dz}{dx} + \frac{\gamma ddz}{dx^2} + \frac{\delta d^3z}{dx^3} + \frac{\varepsilon d^4z}{dx^4} + \text{etc.}$$

atque pro his terminis sui valores substituantur, quos obtinent ex praecedentibus seriebus, ex quibus est

$$\begin{aligned}
\int zdx &= Sz - \tfrac{1}{2}S\tfrac{dz}{dx} + \tfrac{1}{6}S\tfrac{ddz}{dx^2} - \tfrac{1}{24}S\tfrac{d^3z}{dx^3} + \tfrac{1}{120}S\tfrac{d^4z}{dx^4} - \text{etc.}\\
\alpha z &= \quad + \alpha S\tfrac{dz}{dx} - \tfrac{\alpha}{2}S\tfrac{ddz}{dx^2} + \tfrac{\alpha}{6}S\tfrac{d^3z}{dx^3} - \tfrac{1}{24}S\tfrac{d^4z}{dx^4} + \text{etc.}\\
\frac{\beta dz}{dx} &= \quad \beta S\tfrac{ddz}{dx^2} - \tfrac{\beta}{2}S\tfrac{d^3z}{dx^3} + \tfrac{\beta}{6}S\tfrac{d^4z}{dx^4} - \text{etc.}\\
\frac{\gamma ddz}{dx^2} &= \quad \gamma S\tfrac{d^3z}{dx^3} - \tfrac{\gamma}{2}S\tfrac{d^4z}{dx^4} + \text{etc.}\\
\frac{\delta d^3z}{dx^3} &= \quad \delta S\tfrac{d^4z}{dx^4} - \text{etc.}\\
&\text{etc.}
\end{aligned}$$

Qui valores additi cum producere debeant *Sz*, coefficientes $\alpha, \beta, \gamma, \delta$ etc. ex sequentibus aequationibus definientur

$$\alpha - \tfrac{1}{2} = 0, \quad \beta - \tfrac{\alpha}{2} + \tfrac{1}{6} = 0, \quad \gamma - \tfrac{\beta}{2} + \tfrac{\alpha}{6} - \tfrac{1}{24} = 0,$$
$$\delta - \tfrac{\gamma}{2} + \tfrac{\beta}{6} - \tfrac{\alpha}{24} + \tfrac{1}{120} = 0, \quad \varepsilon - \tfrac{\delta}{2} + \tfrac{\gamma}{6} - \tfrac{\beta}{24} + \tfrac{\alpha}{120} - \tfrac{1}{720} = 0,$$
$$\zeta - \tfrac{\varepsilon}{2} + \tfrac{\delta}{6} - \tfrac{\gamma}{24} + \tfrac{\beta}{120} - \tfrac{\alpha}{720} + \tfrac{1}{5040} = 0 \quad \text{etc.}$$

112. Ex his ergo aequationibus successive valores omnium litterarum $\alpha, \beta, \gamma, \delta$ etc. definiri poterunt; reperietur autem

$$\alpha = \tfrac{1}{2}, \quad \beta = \tfrac{\alpha}{2} - \tfrac{1}{6} = \tfrac{1}{12}, \quad \gamma = \tfrac{\beta}{2} - \tfrac{\alpha}{6} + \tfrac{1}{24} = 0,$$
$$\delta = \tfrac{\gamma}{2} - \tfrac{\beta}{6} + \tfrac{\alpha}{24} - \tfrac{1}{120} = -\tfrac{1}{720}, \quad \varepsilon = \tfrac{\delta}{2} - \tfrac{\gamma}{6} + \tfrac{\beta}{24} - \tfrac{\alpha}{120} + \tfrac{1}{720} = 0, \text{ etc.}$$

sicque ulterius progrediendo reperientur continuo termini alterni evanescentes. Litterae ergo ordine tertia, quinta, septima etc. omnesque impares erunt $= 0$ excepta prima, quo ipso haec valorum series contra legem continuitatis impingere videtur. Quamobrem eo magis necesse est, ut rigide demonstretur omnes terminos impares praeter primum necessario evanescere.

113. Quoniam singulae litterae secundum legem constantem ex praecedentibus determinantur, eae seriem recurrentem inter se constituent. Ad quam explicandam concipiatur ista series

$$1+\alpha u+\beta u^{2}+\gamma u^{3}+\delta u^{4}+\varepsilon u^{5}+\zeta u^{6}+\text{etc.},$$

cuius valor sit $=V$ atque manifestum est hanc seriem recurrentem oriri ex evolutione huius fractionis

$$V=\frac{1}{1-\frac{1}{2}u+\frac{1}{6}u^{2}-\frac{1}{24}u^{3}+\frac{1}{120}u^{4}-\text{etc.}}$$

Atque si ista fractio alio modo in seriem infinitam secundum potestates ipsius u progredientem resolvi queat, necesse est, ut semper eadem series

$$V=1+\alpha u+\beta u^{2}+\gamma u^{3}+\delta u^{4}+\varepsilon u^{5}+\text{etc.}$$

resultet; hocque modo alia lex, qua isti iidem valores $\alpha,\ \beta,\ \gamma,\ \delta$ etc. determinantur, eruetur.

114. Quoniam, si e denotet numerum, cuius logarithmus hyperbolicus unitati aequatur, erit

$$e^{-u}=1-u+\tfrac{1}{2}u^{2}-\tfrac{1}{6}u^{3}+\tfrac{1}{24}u^{4}-\tfrac{1}{120}u^{5}+\text{etc.}$$

erit

$$\frac{1-e^{-u}}{u}=1-\tfrac{1}{2}u+\tfrac{1}{6}u^{2}-\tfrac{1}{24}u^{3}+\tfrac{1}{120}u^{4}-\text{etc.}$$

ideoque

$$V=\frac{u}{1-e^{-u}}\,.$$

Nunc extinguatur ex serie secundus terminus $\alpha u=\frac{1}{2}u$, ut sit

$$V-\tfrac{1}{2}u=1+\beta u^{2}+\gamma u^{3}+\delta u^{4}+\varepsilon u^{5}+\zeta u^{6}+\text{etc.}\ ;$$

erit

$$V-\tfrac{1}{2}u=\frac{\frac{1}{2}u\left(1+e^{-u}\right)}{1-e^{-u}}$$

Multiplicentur numerator ac denominator per $e^{\frac{1}{2}u}$ et eritque

$$V-\tfrac{1}{2}u=\frac{u\left(e^{\frac{1}{2}u}+e^{-\frac{1}{2}u}\right)}{2\left(e^{\frac{1}{2}u}-e^{-\frac{1}{2}u}\right)}$$

et quantitatibus $e^{\frac{1}{2}u}$ et $e^{-\frac{1}{2}u}$ in series conversis fiet

$$V-\frac{1}{2}u=\frac{1+\frac{u^2}{2\cdot4}+\frac{u^4}{2\cdot4\cdot6\cdot8}+\frac{u^6}{2\cdot4\cdot6\cdot8\cdot10\cdot12}+\text{etc.}}{2\left(\frac{1}{2}+\frac{u^2}{2\cdot4\cdot6}+\frac{u^4}{2\cdot4\cdot6\cdot8\cdot10}+\text{etc.}\right)}$$

sive

$$V-\frac{1}{2}u=\frac{1+\frac{u^2}{2\cdot4}+\frac{u^4}{2\cdot4\cdot6\cdot8}+\frac{u^6}{2\cdot4\cdots12}+\frac{u^8}{2\cdot4\cdots16}+\text{etc.}}{1+\frac{u^2}{4\cdot6}+\frac{u^4}{2\cdot4\cdot6\cdot8\cdot10}+\frac{u^6}{4\cdot6\cdots14}+\frac{u^8}{4\cdot6\cdots18}+\text{etc.}}$$

115. Cum igitur in hac fractione potestates impares prorsus desint, in eius quoque evolutione potestates impares omnino nullae ingredientur; quare cum $V-\frac{1}{2}u$ aequetur isti seriei

$$1+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\zeta u^6+\text{etc .},$$

coefficientes imparium potestatum $\gamma,\ \varepsilon,\ \eta,\ \iota$ etc. omnes evanescent. Sicque ratio manifesta est, cur in serie $1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+\text{etc.}$ termini ordine pares omnes praeter secundum sint $=0$ neque tamen lex continuitatis vim patiatur. Erit ergo

$$V=1+\beta u^2+\delta u^4+\zeta u^6+\theta u^8+\chi u^{10}+\text{etc}$$

litterisque $\beta,\ \delta,\ \zeta,\ \theta,\ \chi$ etc. per evolutionem superioris fractionis determinatis obtinebimus terminum summatorium Sz seriei, cuius terminus generalis est $=z$ indici x respondens, hoc modo expressum

$$Sz=\int zdx+\frac{1}{2}z+\frac{\beta dz}{dx}+\frac{\delta d^3z}{dx^3}+\frac{\zeta d^5z}{dx^5}+\frac{\theta d^7z}{dx^7}+\text{etc.}$$

116. Quia series $1+\beta u^2+\delta u^4+\zeta u^6+\theta u^8+\text{etc.}$ oritur ex evolutione huius fractionis

$$\frac{1+\frac{u^2}{2\cdot4}+\frac{u^4}{2\cdot4\cdot6\cdot8}+\frac{u^6}{2\cdot4\cdot6\cdot8\cdot10\cdot12}+\text{etc.}}{1+\frac{u^2}{4\cdot6}+\frac{u^4}{4\cdot6\cdot8\cdot10}+\frac{u^6}{4\cdot6\cdot8\cdot10\cdot12\cdot14}+\text{etc.}}$$

litterae $\beta,\ \delta,\ \zeta,\ \theta$ etc. hanc legem tenebunt, ut sit

$$\beta = \frac{1}{2\cdot 4} - \frac{1}{4\cdot 6}$$
$$\delta = \frac{1}{2\cdot 4\cdot 6\cdot 8} - \frac{\beta}{4\cdot 6} - \frac{1}{4\cdot 6\cdot 8\cdot 10}$$
$$\zeta = \frac{1}{2\cdot 4\cdot 6\cdots 12} - \frac{\delta}{4\cdot 6} - \frac{\beta}{4\cdot 6\cdot 8\cdot 10} - \frac{1}{4\cdot 6\cdots 14}$$
$$\theta = \frac{1}{2\cdot 4\cdot 6\cdots 16} - \frac{\zeta}{4\cdot 6} - \frac{\delta}{4\cdot 6\cdot 8\cdot 10} - \frac{\beta}{4\cdot 6\cdots 14} - \frac{1}{4\cdot 6\cdots 18}$$
etc.

Hi autem valores alternative fiunt affirmativi et negativi.

117. Si igitur harum litterarum alternae capiantur negative, ita ut sit

$$Sz = \int zdx + \frac{1}{2}z - \frac{\beta dz}{dx} + \frac{\delta d^3 z}{dx^3} - \frac{\zeta d^5 z}{dx^5} + \frac{\theta d^7 z}{dx^7} - \text{etc.},$$

litterae β, δ, ζ, θ etc. definientur ex hac fractione

$$\frac{1 - \frac{u^2}{2\cdot 4} + \frac{u^4}{2\cdot 4\cdot 6\cdot 8} - \frac{u^6}{2\cdot 4\cdots 12} + \frac{u^8}{2\cdot 4\cdots 16} - \text{etc.}}{1 - \frac{u^2}{4\cdot 6} + \frac{u^4}{4\cdot 6\cdot 8\cdot 10} - \frac{u^6}{4\cdot 6\cdots 14} + \frac{u^8}{4\cdot 6\cdots 18} - \text{etc.}}$$

eam evolvendo in seriem
$$1 + \beta u^2 + \delta u^4 + \zeta u^6 + \theta u^8 + \text{etc.}$$

quocirca erit

$$\beta = \frac{1}{4\cdot 6} - \frac{1}{2\cdot 4}$$
$$\delta = \frac{\beta}{4\cdot 6} - \frac{1}{4\cdot 6\cdot 8\cdot 10} + \frac{1}{2\cdot 4\cdot 6\cdot 8}$$
$$\zeta = \frac{\delta}{4\cdot 6} - \frac{\beta}{4\cdot 6\cdot 8\cdot 10} + \frac{1}{4\cdot 6\cdots 14} - \frac{1}{2\cdot 4\cdot 6\cdots 12}$$
etc.

nunc autem omnes termini fient negativi.

118. Ponamus ergo $\alpha = -A$, $\delta = -B$, $\zeta = -C$ etc., ut sit

$$Sz = \int zdx + \frac{1}{2}z + \frac{Adz}{dx} - \frac{Bd^3 z}{dx^3} + \frac{Cd^5 z}{dx^5} - \frac{Dd^7 z}{dx^7} + \text{etc.},$$

atque ad litteras A, B, C, D etc. definiendas consideretur haec series

$$1-Au^2-Bu^4-Cu^6-Du^8-Eu^{10}-\text{etc.},$$

quae oritur ex evolutione huius fractionis

$$\frac{1-\frac{u^2}{2\cdot 4}+\frac{u^4}{2\cdot 4\cdot 6\cdot 8}-\frac{u^6}{2\cdot 4\cdots\cdots 12}+\frac{u^8}{2\cdot 4\cdots\cdots 16}-\text{etc.}}{1-\frac{u^2}{4\cdot 6}+\frac{u^4}{4\cdot 6\cdot 8\cdot 10}-\frac{u^6}{4\cdot 6\cdots\cdot 14}+\frac{u^8}{4\cdot 6\cdots\cdots 18}-\text{etc.}},$$

vel consideretur ista series

$$\frac{1}{u}-Au-Bu^3-Cu^5-Du^7-Eu^9-\text{etc.}=s,$$

quae oritur ex evolutione huius fractionis

$$s=\frac{1-\frac{u^2}{2\cdot 4}+\frac{u^4}{2\cdot 4\cdot 6\cdot 8}-\frac{u^6}{2\cdot 4\cdots\cdots 12}+\text{etc.}}{u-\frac{u^3}{4\cdot 6}+\frac{u^5}{4\cdot 6\cdot 8\cdot 10}-\frac{u^7}{4\cdot 6\cdots\cdot 14}+\text{etc.}}.$$

Cum autem sit

$$\cos\tfrac{1}{2}u=1-\frac{u^2}{2\cdot 4}+\frac{u^4}{2\cdot 4\cdot 6\cdot 8}-\frac{u^6}{2\cdot 4\cdots\cdots 12}+\text{etc.},$$

$$\sin\tfrac{1}{2}u=\frac{u}{2}-\frac{u^3}{2\cdot 4\cdot 6}+\frac{u^5}{2\cdot 4\cdot 6\cdot 8\cdot 10}-\frac{u^7}{2\cdot 4\cdots\cdots 14}+\text{etc.},$$

sequitur fore

$$s=\frac{\cos\frac{1}{2}u}{2\sin\frac{1}{2}u}=\frac{1}{2}\cot\frac{1}{2}u.$$

Quare si cotangens arcus $\frac{1}{2}u$ in seriem convertatur, cuius termini secundum potestates ipsius u procedant, ex ea cognoscentur valores litterarum *A, B, C, D, E* etc.

119. Cum igitur sit $s=\frac{1}{2}\cot\frac{1}{2}u$, erit $\frac{1}{2}u=A\cot 2s$ et differentiando erit $\frac{1}{2}du=\frac{-2ds}{1+4ss}$ seu $4ds+du+4ssdu=0$ sive

$$\frac{4ds}{du}+1+4ss=0..$$

Quia autem est

$$s=\frac{1}{u}-Au-Bu^3-Cu^5-\text{etc.},$$

erit

$$\begin{aligned}
\frac{4ds}{du} &= -\frac{4}{uu} - 4A - 3\cdot 4Bu^2 - 5\cdot 4Cu^4 - 7\cdot 4Du^6 - \text{etc.}\\
1 &= \quad 1\\
4ss &= \frac{4}{uu} - 8A - \quad 8Bu^2 - \quad Cu^4 - \quad 8Du^6 - \text{etc.}\\
&\qquad + \quad 4A^2u^2 + 8ABu^4 - 8ACu^6 + \text{etc.}\\
&\qquad\qquad 4BBu^6 + \text{etc.}
\end{aligned}$$

Perductis his terminis homogeneis ad cyphram fiet

$$A = \tfrac{1}{12},\ B = \tfrac{A^2}{5},\ C = \tfrac{2AB}{7},\ D = \tfrac{2AC+BB}{9},\ E = \tfrac{2AD+2BC}{11},$$
$$F = \tfrac{2AE+2BD+CC}{13},\ G = \tfrac{2AF+2BE+2CD}{15},\ H = \tfrac{2AG+2BF+2CE+DD}{17},$$
$$\text{etc.}$$

Ex quibus formulis iam manifesto liquet singulos hos valores esse affirmativos.

120. Quoniam vero denominatores horum valorum fiunt vehementer magni calculumque non mediocriter impediunt, loco litterarum *A, B, C, D* etc. has novas introducamus

$$A = \tfrac{\alpha}{1\cdot 2\cdot 3},\ B = \tfrac{\beta}{1\cdot 2\cdot 3\cdot 4\cdot 5},\ C = \tfrac{\gamma}{1\cdot 2\cdot 3\cdots 7},$$
$$D = \tfrac{\delta}{1\cdot 2\cdot 3\cdots 9},\ E = \tfrac{\varepsilon}{1\cdot 2\cdot 3\cdots 11}\ \text{ etc.}$$

Atque reperietur fore

$$\alpha = \tfrac{1}{2},\ \beta = \tfrac{2}{3}\alpha^2,\ \gamma = 2\cdot\tfrac{3}{3}\alpha\beta,\ \delta = 2\cdot\tfrac{4}{3}\alpha\gamma + \tfrac{8\cdot 7}{4\cdot 5}\beta^2,$$
$$\varepsilon = 2\cdot\tfrac{5}{3}\alpha\delta + 2\cdot\tfrac{10\cdot 9\cdot 8}{1\cdot 2\cdots 5}\beta\gamma,\ \zeta = 2\cdot\tfrac{12}{1\cdot 2\cdot 3}\alpha\varepsilon + 2\cdot\tfrac{12\cdot 11\cdot 10}{1\cdot 2\cdots 5}\beta\delta + \tfrac{12\cdot 11\cdot 10\cdot 9\cdot 8}{1\cdot 2\cdots 7}\gamma\gamma,$$
$$\eta = 2\cdot\tfrac{14}{1\cdot 2\cdot 3}\alpha\zeta + 2\cdot\tfrac{14\cdot 13\cdot 12}{1\cdot 2\cdots 5}\beta\varepsilon + 2\cdot\tfrac{14\cdot 13\cdot 12\cdot 11\cdot 10}{1\cdot 2\cdots 7}\gamma\delta$$
$$\text{etc.}$$

121. Commodius autem utemur his formulis

$$\alpha = \tfrac{1}{2},\ \beta = \tfrac{4}{3}\cdot\tfrac{\alpha\alpha}{2},\ \gamma = \tfrac{6}{3}\cdot\alpha\beta,\ \delta = \tfrac{8}{3}\cdot\alpha\gamma + \tfrac{8\cdot 7\cdot 6}{3\cdot 4\cdot 5}\cdot\tfrac{\beta\beta}{2},$$
$$\varepsilon = \tfrac{10}{3}\cdot\alpha\delta + \tfrac{10\cdot 9\cdot 8}{3\cdot 4\cdot 5}\cdot\beta\gamma,\ \zeta = \tfrac{12}{3}\cdot\alpha\varepsilon + \tfrac{12\cdot 11\cdot 10}{3\cdot 4\cdot 5}\cdot\beta\delta + \tfrac{12\cdot 11\cdot 10\cdot 9\cdot 8}{3\cdot 4\cdot 5\cdot 6\cdot 7}\cdot\tfrac{\gamma\gamma}{2},$$
$$\eta = \tfrac{14}{3}\cdot\alpha\zeta + \tfrac{14\cdot 13\cdot 12}{3\cdot 4\cdot 5}\cdot\beta\varepsilon + \tfrac{14\cdot 13\cdot 12\cdot 11\cdot 10}{3\cdot 4\cdot 5\cdot 6\cdot 7}\cdot\gamma\delta,$$
$$\theta = \tfrac{16}{3}\cdot\alpha\eta + \tfrac{16\cdot 15\cdot 14}{3\cdot 4\cdot 5}\cdot\beta\zeta + \tfrac{16\cdot 15\cdots 12}{3\cdot 4\cdots 7}\cdot\gamma\varepsilon + \tfrac{16\cdot 15\cdots 10}{3\cdot 4\cdots 9}\cdot\tfrac{\delta\delta}{2}$$
$$\text{etc.}$$

Ex hac igitur lege, secundum quam calculus non difficulter instituitur, si inventi fuerint valores litterarum $\alpha,\ \beta,\ \gamma,\ \delta$ etc., tum seriei cuiuscunque, cuius terminus generalis seu indici x conveniens fuerit = z, terminus summatorius ita exprimetur, ut sit

$$Sz=\int zdx+\tfrac{1}{2}z+\frac{\alpha dz}{1\cdot 2\cdot 3\cdot dx}-\frac{\beta d^3z}{1\cdot 2\cdot 3\cdot 4\cdot 5dx^3}+\frac{\gamma d^5z}{1\cdot 2\cdots 7dx^5}$$
$$-\frac{\delta d^7z}{1\cdot 2\cdots 9dx^7}+\frac{\varepsilon d^9z}{1\cdot 2\cdots 11dx^9}-\frac{\zeta d^{11}z}{1\cdot 2\cdots 13dx^9}+\text{etc.}$$

Istae autem litterae $\alpha,\ \beta,\ \gamma,\ \delta$ etc. sequentes valores habere inventae sunt:

$\alpha=\frac{1}{2}$	sive	$1\cdot 2\alpha=1$
$\beta=\frac{1}{6}$		$1\cdot 2\cdot 3\beta=1$
$\gamma=\frac{1}{6}$		$1\cdot 2\cdot 3\cdot 4\gamma=4$
$\delta=\frac{3}{10}$		$1\cdot 2\cdot 3\cdots 5\delta=36$
$\varepsilon=\frac{5}{6}$		$1\cdot 2\cdot 3\cdots 6\varepsilon=600$
$\zeta=\frac{691}{210}$		$1\cdot 2\cdot 3\cdots 7\zeta=24\cdot 691$
$\eta=\frac{35}{2}$		$1\cdot 2\cdot 3\cdots 8\eta=20160\cdot 35$
$\theta=\frac{3617}{30}$		$1\cdot 2\cdot 3\cdots 9\theta=12096\cdot 3617$
$\iota=\frac{43867}{42}$		$1\cdot 2\cdot 3\cdots 10\iota=86400\cdot 43867$
$\chi=\frac{1222277}{110}$		$1\cdot 2\cdot 3\cdots 11\chi=362880\cdot 1222277$
$\lambda=\frac{854513}{6}$		$1\cdot 2\cdot 3\cdots 12\lambda=79833600\cdot 854513$
$\mu=\frac{1181820455}{546}$		$1\cdot 2\cdot 3\cdots 13\mu=11404800\cdot 1181820455$
$\nu=\frac{76977927}{2}$		$1\cdot 2\cdot 3\cdots 14\nu=43589145600\cdot 76977927$
$\xi=\frac{23749461029}{30}$		$1\cdot 2\cdot 3\cdots 15\xi=43589145600\cdot 23749461029$
$\pi=\frac{8615841276005}{462}$		$1\cdot 2\cdot 3\cdots 16\pi=45287424000\cdot 8615841276005$

etc.

122. Numeri isti per universam serierum doctrinam amplissimum habent usum. Primum enim ex his numeris formari possunt ultimi termini in summis potestatum parium, quos non aeque ac reliquos terminos ex summis praecedentium reperiri posse supra annotavimus. In potestatibus enim paribus postremi summarum termini sunt x per certos numeros multiplicati, qui numeri pro potestatibus II, IV, VI, VIII etc. sunt $\frac{1}{6},\ \frac{1}{30},\ \frac{1}{42},\ \frac{1}{30}$ etc.

signis alternantibus affecti. Oriuntur autem hi numeri, si valores litterarum $\alpha, \beta, \gamma, \delta$ etc. supra inventi respective dividantur per numeros impares 3, 5, 7, 9 etc., unde isti numeri, qui ab inventore IACOBO BERNOULLIO vocari solent BERNOULLIANI, erunt

$$\frac{\alpha}{3}=\frac{1}{6}=\mathfrak{A} \qquad \frac{\iota}{19}=\frac{43867}{798}=\mathfrak{I}$$
$$\frac{\beta}{5}=\frac{1}{30}=\mathfrak{B} \qquad \frac{\chi}{21}=\frac{174611}{330}=\mathfrak{K}=\frac{283\cdot 617}{330}$$
$$\frac{\gamma}{7}=\frac{1}{42}=\mathfrak{C} \qquad \frac{\lambda}{23}=\frac{854513}{138}=\mathfrak{L}=\frac{11\cdot 131\cdot 593}{2\cdot 3\cdot 23}$$
$$\frac{\delta}{9}=\frac{1}{30}=\mathfrak{D} \qquad \frac{\mu}{25}=\frac{236364091}{2730}=\mathfrak{M}$$
$$\frac{\varepsilon}{11}=\frac{5}{66}=\mathfrak{E} \qquad \frac{\nu}{27}=\frac{8553103}{6}=\mathfrak{N}=\frac{13\cdot 657931}{6}$$
$$\frac{\zeta}{13}=\frac{691}{2730}=\mathfrak{F} \qquad \frac{\xi}{29}=\frac{23749461029}{870}=\mathfrak{O}$$
$$\frac{\eta}{15}=\frac{7}{6}=\mathfrak{G} \qquad \frac{\pi}{31}=\frac{8615841276005}{14322}=\mathfrak{P}$$
$$\frac{\theta}{17}=\frac{3617}{510}=\mathfrak{H} \qquad \text{etc.}$$

123. Isti igitur numeri BERNOULLIANI $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ etc. immediate ex sequentibus aequationibus inveniri poterunt

$$\mathfrak{A} = \frac{1}{6}$$
$$\mathfrak{B} = \frac{4\cdot 3}{1\cdot 2}\cdot\frac{1}{5}\mathfrak{A}^2$$
$$\mathfrak{C} = \frac{6\cdot 5}{1\cdot 2}\cdot\frac{2}{7}\mathfrak{A}\mathfrak{B}$$
$$\mathfrak{D} = \frac{8\cdot 7}{1\cdot 2}\cdot\frac{2}{9}\mathfrak{A}\mathfrak{C}+\frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1}{9}\mathfrak{B}^2$$
$$\mathfrak{E} = \frac{10\cdot 9}{1\cdot 2}\cdot\frac{2}{11}\mathfrak{A}\mathfrak{D}+\frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{2}{11}\mathfrak{B}\mathfrak{C}$$
$$\mathfrak{F} = \frac{12\cdot 11}{1\cdot 2}\cdot\frac{2}{13}\mathfrak{A}\mathfrak{E}+\frac{12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{2}{13}\mathfrak{B}\mathfrak{D}+\frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot\frac{1}{13}\mathfrak{C}^2$$
$$\mathfrak{G} = \frac{14\cdot 13}{1\cdot 2}\cdot\frac{2}{15}\mathfrak{A}\mathfrak{F}+\frac{14\cdot 13\cdot 12\cdot 11}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{2}{15}\mathfrak{B}\mathfrak{E}+\frac{14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot\frac{2}{15}\mathfrak{C}\mathfrak{D}$$
$$\text{etc.,}$$

quarum aequationum lex per se est manifesta, si tantum notetur, ubi quadratum cuiuspiam litterae occurrit, eius coefficientem duplo esse minorem, quam secundum regulam esse debere videatur. Revera autem termini, qui continent producta ex disparibus litteris, bis occurrere censendi sunt; erit enim verbi gratia

$$13\mathfrak{F}=\frac{12\cdot 11}{1\cdot 2}\mathfrak{A}\mathfrak{E}+\frac{12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4}\mathfrak{B}\mathfrak{D}+\frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\mathfrak{C}\mathfrak{C}$$
$$+\frac{12\cdot 11\cdot 10\cdots 5}{1\cdot 2\cdot 3\cdots 8}\mathfrak{D}\mathfrak{B}+\frac{12\cdot 11\cdot 10\cdots 3}{1\cdot 2\cdot 3\cdots 10}\mathfrak{E}\mathfrak{A}.$$

124. Deinde vero etiam iidem numeri $\alpha,\ \beta,\ \gamma,\ \delta$ etc. ingrediuntur in expressiones summarum serierum fractionum in hac forma generali

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\text{etc.},$$

quoties *n* est numerus par affirmativus, contentarum. Has enim summas in *Introductione* per potestates semiperipheriae circuli π radio exisente $=1$ expressas dedimus atque in harum potestatum coefficientibus isti ipsi numeri $\alpha,\ \beta,\ \gamma,\ \delta$ etc. ingredi deprehenduntur. Quo autem haec convenientia non casu evenire, sed necessario locum habere intelligatur, has easdem summas singulari modo investigemus, quo lex summarum illarum facilius patebit. Quoniam supra (§ 43) invenimus esse

$$\frac{\pi}{n}\cot\frac{m}{n}\pi=\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\text{ etc.},$$

binis terminis coniungendis habebimus

$$\frac{\pi}{n}\cot\frac{m}{n}\pi=\frac{1}{m}-\frac{2m}{nn-m^2}-\frac{2m}{4n^2-m^2}-\frac{2m}{9n^2-m^2}-\frac{2m}{16n^2-m^2}-\text{etc.},$$

unde colligimus fore

$$\frac{1}{n^2-m^2}+\frac{1}{4n^2-m^2}+\frac{1}{9n^2-m^2}+\frac{1}{16n^2-m^2}+\text{etc.}=\frac{1}{2mm}-\frac{\pi}{2mn}\cot\frac{m}{n}\pi.$$

Statuamus nunc $n=1$ et pro *m* ponamus *u*, ut sit

$$\frac{1}{1-u^2}+\frac{1}{4-u^2}+\frac{1}{9-u^2}+\frac{1}{16-u^2}+\text{etc.}=\frac{1}{2uu}-\frac{\pi}{2u}\cot\pi u.$$

Resolvantur singulae istae fractiones in series

$$\frac{1}{1-u^2}=1+u^2+u^4+u^6+u^8+\text{etc.}$$
$$\frac{1}{4-u^2}=\frac{1}{2^2}+\frac{u^2}{2^4}+\frac{u^4}{2^6}+\frac{u^6}{2^8}+\frac{u^8}{2^{10}}+\text{etc.}$$
$$\frac{1}{9-u^2}=\frac{1}{3^2}+\frac{u^2}{3^4}+\frac{u^4}{3^6}+\frac{u^6}{3^8}+\frac{u^8}{3^{10}}+\text{etc.}$$
$$\frac{1}{16-u^2}=\frac{1}{4^2}+\frac{u^2}{4^4}+\frac{u^4}{4^6}+\frac{u^6}{4^8}+\frac{u^8}{4^{10}}+\text{etc.}$$
etc.

125. Quodsi ergo ponatur

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{etc.}=\mathfrak{a} \quad 1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\text{etc.}=\mathfrak{d}$$
$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.}=\mathfrak{b} \quad 1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\text{etc.}=\mathfrak{e}$$
$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc.}=\mathfrak{c} \quad 1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\text{etc.}=\mathfrak{f}$$
$$\text{etc.},$$

superior series transmutabitur in hanc

$$\mathfrak{a}+\mathfrak{b}u^2+\mathfrak{c}u^4+\mathfrak{d}u^6+\mathfrak{e}u^8+\mathfrak{f}u^{10}+\text{etc.}=\frac{1}{2uu}-\frac{\pi}{2u}\cot\pi u.$$

Cum igitur in § 118 litterae *A, B, C, D* etc. ita comparatae sint inventae, ut posito

$$s=\frac{1}{u}-Au-Bu^3-Cu^5-Du^7-Eu^9-\text{etc.}$$

sit $s=\frac{1}{2}\cot\frac{1}{2}u$, erit posito πu loco $\frac{1}{2}u$ seu $2\pi u$ loco u

$$\tfrac{1}{2}\cot\pi u=\frac{1}{2\pi u}-2A\pi u-2^3B\pi^3u^3-2^5C\pi^5u^5-2^7D\pi^7u^7-\text{etc.},$$

unde per $\frac{\pi}{u}$ multiplicando erit

$$\frac{\pi}{2u}\cot\pi u=\frac{1}{2uu}-2A\pi^2-2^3B\pi^4u^2-2^5C\pi^6u^4-2^7D\pi^8u^6-\text{etc.},$$

hincque sequitur fore

$$\frac{1}{2uu}-\frac{\pi}{2u}\cot\pi u=2A\pi^2+2^3B\pi^4u^2+2^5C\pi^6u^4+2^7D\pi^8u^6+\text{etc.}$$

Quia igitur modo invenimus esse

$$\frac{1}{2uu}-\frac{\pi}{2u}\cot\pi u=\mathfrak{a}+\mathfrak{b}u^2+\mathfrak{c}u^4+\mathfrak{d}u^6+\text{etc.},$$

necesse est, ut sit

$$\mathfrak{a} = 2A\pi^2 = \tfrac{2\alpha}{1\cdot 2\cdot 3}\pi^2 = \tfrac{2\mathfrak{A}}{1\cdot 2}\pi^2$$
$$\mathfrak{b} = 2^3 B\pi^4 = \tfrac{2^3\beta}{1\cdot 2\cdot 3\cdot 4\cdot 5}\pi^4 = \tfrac{2^3\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}\pi^4$$
$$\mathfrak{c} = 2^5 C\pi^6 = \tfrac{2^5\gamma}{1\cdot 2\cdot 3\cdots 7}\pi^6 = \tfrac{2^5\mathfrak{C}}{1\cdot 2\cdot 3\cdots 6}\pi^6$$
$$\mathfrak{d} = 2^7 D\pi^8 = \tfrac{2^7\delta}{1\cdot 2\cdot 3\cdots 9}\pi^8 = \tfrac{2^7\mathfrak{D}}{1\cdot 2\cdot 3\cdots 8}\pi^8$$
$$\mathfrak{e} = 2^9 E\pi^{10} = \tfrac{2^9\varepsilon}{1\cdot 2\cdot 3\cdots 11}\pi^{10} = \tfrac{2^9\mathfrak{E}}{1\cdot 2\cdot 3\cdots 10}\pi^{10}$$
$$\mathfrak{f} = 2^{11} F\pi^{12} = \tfrac{2^{11}\xi}{1\cdot 2\cdot 3\cdots 13}\pi^{12} = \tfrac{2^{11}\mathfrak{F}}{1\cdot 2\cdot 3\cdots 12}\pi^{12}$$
etc.

126. Ex hoc ergo tam facili ratiocinio non solum omnes series potestatum reciprocarum, quas paragrapho praecedenti exhibuimus, expedite summantur, sed simul quoque perspicitur, quemadmodum istae summae ex cognitis valoribus litterarum $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. vel etiam ex numeris BERNOULLIANIS $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. formentur. Quare cum istorum numerorum quindecim §122 definiverimus, ex iis summae omnium potestatum parium usque ad sunimam huius seriei inclusive assignari poterunt:

$$1 + \tfrac{1}{2^{30}} + \tfrac{1}{3^{30}} + \tfrac{1}{4^{30}} + \text{etc.};$$

erit enim huius seriei summa

$$= \tfrac{2^{29}\xi}{1\cdot 2\cdot 3\cdots 31}\pi^{30} = \tfrac{2^{29}\mathfrak{F}}{1\cdot 2\cdot 3\cdots 30}\pi^{30}.$$

Atque si quis voluerit has summas ulterius determinare, id continuandis numeris α, β, γ etc. vel his $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, etc. facillime praestabitur.

127. Origo ergo horum numerorum $\alpha, \beta, \gamma, \delta$ etc. vel inde formatorum $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. potissimum debetur evolutioni cotangentis cuiusvis anguli in seriem infinitam. Cum enim sit

$$\tfrac{1}{2}\cot\tfrac{1}{2}u = \tfrac{1}{u} - Au - Bu^3 - Cu^5 - Du^7 - Eu^9 - \text{etc.},$$

erit

$$Au^2 + Bu^4 + Cu^6 + Du^8 + Eu^9 + \text{etc.} = 1 - \tfrac{u}{2}\cot\tfrac{1}{2}u\ ;$$

si igitur loco coefficientium A, B, C, D etc. valores ipsorum substituantur, reperietur

$$\tfrac{\alpha u^2}{1\cdot 2\cdot 3} + \tfrac{\beta u^4}{1\cdot 2\cdots 5} + \tfrac{\gamma u^6}{1\cdot 2\cdots 7} + \tfrac{\delta u^8}{1\cdot 2\cdots 9} + \text{etc.} = 1 - \tfrac{u}{2}\cot\tfrac{1}{2}u$$

atque numeros BERNOULLIANOS adhibendo erit

$$\frac{\mathfrak{A}u^2}{1\cdot 2}+\frac{\mathfrak{B}u^4}{1\cdot 2\cdot 3\cdot 4}+\frac{\mathfrak{C}u^6}{1\cdot 2\cdots 6}+\frac{\mathfrak{D}u^8}{1\cdot 2\cdots 8}+\text{etc.}=1-\tfrac{u}{2}\cot\tfrac{1}{2}u\ ,$$

ex quibus seriebus per differentiationem innumerabiles aliae deduci possunt sicque infinitae series summari, in quas isti numeri notatu tantopere digni ingrediuntur.

128. Sumamus aequationem priorem, quam per *u* multiplicemus, ut sit

$$\frac{\alpha u^3}{1\cdot 2\cdot 3}+\frac{\beta u^5}{1\cdot 2\cdots 5}+\frac{\gamma u^7}{1\cdot 2\cdots 7}+\frac{\delta u^9}{1\cdot 2\cdots 9}+\text{etc.}=u-\tfrac{uu}{2}\cot\tfrac{1}{2}u,,$$

quae differentiata ac per *du* divisa dat

$$\frac{\alpha u^2}{1\cdot 2}+\frac{\beta u^4}{1\cdot 2\cdot 3\cdot 4}+\frac{\gamma u^6}{1\cdot 2\cdots 6}+\frac{\delta u^8}{1\cdot 2\cdots 8}+\text{etc.}=1-u\cot\tfrac{1}{2}u+\frac{uu}{4\left(\sin\frac{1}{2}u\right)^2};$$

et si denuo differentietur, erit

$$\frac{\alpha u}{1}+\frac{\beta u^3}{1\cdot 2\cdot 3}+\frac{\gamma u^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\text{etc.}=-\cot\tfrac{1}{2}u+\frac{u}{\left(\sin\frac{1}{2}u\right)^2}-\frac{uu\cos\frac{1}{2}u}{4\left(\sin\frac{1}{2}u\right)^3}.$$

Sin autem altera aequatio differentietur, erit

$$\frac{\mathfrak{A}u}{1}+\frac{\mathfrak{B}u^3}{1\cdot 2\cdot 3}+\frac{\mathfrak{C}u^5}{1\cdot 2\cdots 5}+\frac{\mathfrak{D}u^7}{1\cdot 2\cdots 7}+\text{etc.}=-\tfrac{1}{2}\cot\tfrac{1}{2}u+\frac{u}{4\left(\sin\frac{1}{2}u\right)^2}.$$

Ex his ergo, si ponatur $u=\pi$, ob $\cot\frac{1}{2}\pi=0$ et $\sin\frac{1}{2}\pi=1$ sequuntur istae summationes

$$1=\frac{\alpha\pi^2}{1\cdot 2\cdot 3}+\frac{\beta\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\frac{\gamma\pi^6}{1\cdot 2\cdot 3\cdots 7}+\frac{\delta\pi^8}{1\cdot 2\cdot 3\cdots 9}+\text{etc.}$$

$$1+\frac{\pi^2}{4}=\frac{\alpha\pi^2}{1\cdot 2}+\frac{\beta\pi^4}{1\cdot 2\cdot 3\cdot 4}+\frac{\gamma\pi^6}{1\cdot 2\cdot 3\cdots 6}+\frac{\delta\pi^8}{1\cdot 2\cdot 3\cdots 8}+\text{etc.}$$

$$\pi=\frac{\alpha\pi}{1}+\frac{\beta\pi^3}{1\cdot 2\cdot 3}+\frac{\gamma\pi^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\frac{\delta\pi^7}{1\cdot 2\cdot 3\cdots 7}+\text{etc.}$$

seu

$$1=\alpha+\frac{\beta\pi^2}{1\cdot 2\cdot 3}+\frac{\gamma\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\frac{\delta\pi^6}{1\cdot 2\cdot 3\cdots 7}+\text{etc.}$$

a qua si prima subtrahatur, remanebit

$$\alpha=\frac{(\alpha-\beta)\pi^2}{1\cdot 2\cdot 3}+\frac{(\beta-\gamma)\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\frac{(\gamma-\delta)\pi^6}{1\cdot 2\cdot 3\cdots 7}+\text{etc.}$$

Tum vero erit

$$1=\frac{\mathfrak{A}\pi^2}{1\cdot 2}+\frac{\mathfrak{B}\pi^4}{1\cdot 2\cdot 3\cdot 4}+\frac{\mathfrak{C}\pi^6}{1\cdot 2\cdot 3\cdots 6}+\frac{\mathfrak{D}\pi^8}{1\cdot 2\cdot 3\cdots 8}+\text{etc.}$$

$$\frac{\pi}{4}=\frac{\mathfrak{A}\pi}{1}+\frac{\mathfrak{B}\pi^3}{1\cdot 2\cdot 3}+\frac{\mathfrak{C}\pi^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\frac{\mathfrak{D}\pi^7}{1\cdot 2\cdot 3\cdots 7}+\text{etc.}$$

seu

$$\frac{1}{4}=\frac{\mathfrak{A}}{1}+\frac{\mathfrak{B}\pi^2}{1\cdot 2\cdot 3}+\frac{\mathfrak{C}\pi^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\frac{\mathfrak{D}\pi^6}{1\cdot 2\cdot 3\cdots 7}+\text{etc.}$$

129. Ex tabula valorum numerorum α, β, γ, δ etc., quam supra (§ 121) exhibuimus, patet eos primum decrescere, tum vero iterum crescere et, quidem in infinitum. Operae igitur pretium erit investigare, in quanam ratione hi numeri, postquam iam vehementer longe fuerint continuati, ulterius progredi pergant. Sit igitur φ numerus quicunque huius seriei numerorum α, β, γ, δ etc. longissime ab initio remotus et sit ψ numerorum sequens. Quoniam per hos numeros summae potestatum reciprocarum definiuntur, sit $2n$ exponens potestatis, in cuius summam numerus φ ingreditur; erit $2n+2$ exponens potestatis numero ψ respondens atque numerus n iam erit vehementer magnus. Hinc ex § 125 habebitur

$$1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\text{etc.}=\frac{2^{2n-1}\varphi}{1\cdot 2\cdot 3\cdots(2n+1)}\pi^{2n},$$

$$1+\frac{1}{2^{2n+2}}+\frac{1}{3^{2n+2}}+\frac{1}{4^{2n+2}}+\text{etc.}=\frac{2^{2n+1}\psi}{1\cdot 2\cdot 3\cdots(2n+3)}\pi^{2n+2}.$$

Quodsi ergo haec per istam dividatur, erit

$$\frac{1+\frac{1}{2^{2n+2}}+\frac{1}{3^{2n+2}}+\text{etc.}}{1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\text{etc.}}=\frac{4\psi\pi^2}{(2n+2)(2n+3)\varphi}.$$

Quia vera n est numerus vehementer magnus, ob seriem utramque proxime $=1$ erit

$$\frac{\psi}{\varphi}=\frac{(2n+2)(2n+3)}{4\pi^2}=\frac{nn}{\pi\pi}.$$

Cum igitur n designet, quotus sit numerus φ a primo α computatus, se habebit hic numerus φ ad suum sequentem ψ ut π^2 ad n^2, quae ratio, si n fuerit numerus infinitus, veritati penitus fit consentanea. Quoniam est fere $\pi\pi=10$, si ponatur $n=100$, erit terminus centesimus circiter millies minor suo sequente. Constituunt ergo numeri α, β, γ, δ etc. pariter ac BERNOULLIANI $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. seriem maxime divergentem, quae etiam magis increscat quam ulla series geometrica terminis crescentibus procedens.

130. Inventis ergo his valoribus numerorum α, β, γ, δ etc. seu $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc., si proponatur series, cuius terminus generalis *z* fuerit functio quaecunque ipsius *x*, terminus summatorius *Sz* huius seriei sequenti modo exprimetur, ut sit

$$\begin{aligned} Sz = &\int zdx + \tfrac{1}{2}z + \tfrac{1}{6}\cdot\frac{dz}{1\cdot 2dx} - \frac{1}{30}\cdot\frac{d^3z}{1\cdot 2\cdot 3\cdot 4dx^3} \\ &+ \frac{1}{42}\cdot\frac{d^5z}{1\cdot 2\cdot 3\cdots 6dx^5} - \frac{1}{30}\cdot\frac{d^7z}{1\cdot 2\cdot 3\cdots 8dx^7} + \frac{5}{66}\cdot\frac{d^9z}{1\cdot 2\cdot 3\cdots 10dx^9} - \frac{691}{2730}\cdot\frac{d^{11}z}{1\cdot 2\cdot 3\cdots 12dx^{11}} + \\ &\frac{7}{6}\cdot\frac{d^{13}z}{1\cdot 2\cdot 3\cdots 14dx^{13}} - \frac{3617}{510}\cdot\frac{d^{15}z}{1\cdot 2\cdot 3\cdots 16dx^{15}} + \frac{43867}{798}\cdot\frac{d^{17}z}{1\cdot 2\cdot 3\cdots 18dx^{17}} \\ &- \frac{174611}{330}\cdot\frac{d^{19}z}{1\cdot 2\cdot 3\cdots 20dx^{19}} + \frac{854513}{138}\cdot\frac{d^{21}z}{1\cdot 2\cdot 3\cdots 22dx^{21}} \\ &- \frac{236364091}{2730}\cdot\frac{d^{23}z}{1\cdot 2\cdot 3\cdots 24dx^{23}} + \frac{8553103}{6}\cdot\frac{d^{25}z}{1\cdot 2\cdot 3\cdots 26dx^{25}} \\ &- \frac{23749461029}{870}\cdot\frac{d^{27}z}{1\cdot 2\cdot 3\cdots 28dx^{27}} + \frac{8615841276005}{14322}\cdot\frac{d^{29}z}{1\cdot 2\cdot 3\cdots 30dx^{29}} - \text{etc.} \end{aligned}$$

Si igitur innotuerit integrale $\int zdx$ seu quantitas illa, cuius differentiale sit = *zdx*, terminus summatorius ope continuae differentiationis invenietur. Perpetuo autem notandum est ad hanc expressionem semper eiusmodi constantem addi oportere, ut summa fiat $=0$, si index *x* ponatur in nihilum abire.

131. Si igitur *z* fuerit functio rationalis integra ipsius *x*, quia eius differentialia tandem evanescunt, terminus summatorius per expressionem finitam exprimetur; id quod sequentibus exemplis illustrabimus.

EXEMPLUM 1

Quaeratur terminus summatorius huius seriei

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \qquad x \\ 1+9 & +25 & +49 & +81 & +\cdots & +(2x-1)^2. \end{array}$$

Quia hic est $z=(2x-1)^2=4xx-4x+1$, erit

$$\int zdx = \tfrac{4}{3}x^3 - 2x^2 + x;$$

ex huius enim differentiatione oritur $4xxdx - 4xdx + dx = zdx$. Deinde vero per differentiationem erit

$$\frac{dz}{dx} = 8x - 4, \quad \frac{ddz}{dx^2} = 8, \quad \frac{d^3z}{dx^3} = 0 \text{ etc.}$$

Hinc erit terminus summatorius quaesitus

$$\frac{4}{3}x^3 - 2x^2 + x + 2xx - 2x + \frac{1}{2} + \frac{2}{3}x - \frac{1}{3} \pm \text{Const.},$$

qua constante tolli debent termini $\frac{1}{2} - \frac{1}{3}$; unde erit

$$S(2x-1)^2 = \frac{4}{3}x^3 - \frac{1}{3}x = \frac{x}{3}(2x-1)(2x+1).$$

Sic erit posito $x = 4$ summa quatuor primorum terminorum

$$1 + 9 + 25 + 49 = \frac{4}{3} \cdot 7 \cdot 9 = 84 .$$

EXEMPLUM 2

Quaeratur terminus summatorius huius seriei

$$\begin{array}{lllll} 1 & 2 & 3 & 4 & \quad\quad x \\ 1 + & 27 + & 125 + & 343 & + \cdots + (2x-1)^3. \end{array}$$

Quia est $z = (2x-1)^3 = 8x^3 - 12x^2 + 6x - 1$, erit

$$\int z dx = 2x^4 - 4x^3 + 3x^2 - x,$$

$$\frac{dz}{dx} = 24x^2 - 24x + 6, \quad \frac{ddz}{dx^2} = 48x - 24, \quad \frac{d^3z}{dx^3} = 48 ;$$

sequentia evanescunt. Quare erit

$$\begin{aligned} S(2x-1)^3 = 2x^4 - 4x^3 + 3x^2 - x& \\ + 4x^3 - 6x^2 + 3x - \tfrac{1}{2}& \\ + 2x^2 - 2x + \tfrac{1}{2}& \\ - \tfrac{1}{15} \pm \text{ Const.},& \end{aligned}$$

hoc est

$$S(2x-1)^3 = 2x^4 - x^2 = x^2(2xx-1).$$

Sic erit posito $x = 4$

$$1 + 27 + 125 + 343 = 16 \cdot 31 = 496 .$$

132. Ex hac inventa generali expressione pro termino summatorio sponte sequitur ille terminus summatorius, quem superiori parte pro potestatibus numerorum naturalium dedimus cuiusque demonstrationem ibi tradere non licuerat. Quodsi enim ponamus $z = x^n$, erit utique $\int zdx = \frac{1}{n+1}x^{n+1}$; differentialia vero ita se habebunt

$$\frac{dz}{dx} = nx^{n-1},\quad \frac{ddz}{dx^2} = (n-1)x^{n-2},\quad \frac{d^3z}{dx^3} = (n-1)(n-2)x^{n-3},$$
$$\frac{d^5z}{dx^5} = (n-1)(n-2)(n-3)(n-4)x^{n-5},\quad \frac{d^7z}{dx^7} = n(n-1)\cdots(n-6)x^{n-7}\ \text{ etc.}$$

Ex his ergo deducetur sequens terminus summatorius respondens termino generali x^n, scilicet

$$\begin{aligned}
Sx^n &= \frac{1}{n+1}x^{n+1} + \frac{1}{2}x^n + \frac{1}{6}\cdot\frac{n}{2}x^{n-1} - \frac{1}{30}\cdot\frac{n(n-1)(n-2)}{2\cdot3\cdot4}x^{n-3}\\
&+\frac{1}{42}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)}{2\cdot3\cdot4\cdot5\cdot6}x^{n-5} - \frac{1}{30}\cdot\frac{n(n-1)\cdots(n-6)}{2\cdot3\cdots8}x^{n-7} + \frac{5}{66}\cdot\frac{n(n-1)\cdots(n-8)}{1\cdot2\cdot3\cdots10}x^{n-9}\\
&-\frac{691}{2730}\cdot\frac{n(n-1)\cdots(n-10)}{1\cdot2\cdot3\cdots12}x^{n-11} + \frac{7}{6}\cdot\frac{n(n-1)\cdots(n-12)}{1\cdot2\cdot3\cdots14dx^{13}}x^{n-13}\\
&-\frac{3617}{510}\cdot\frac{n(n-1)\cdots(n-14)}{1\cdot2\cdot3\cdots16}x^{n-15} + \frac{43867}{798}\cdot\frac{n(n-1)\cdots(n-16)}{1\cdot2\cdot3\cdots18}x^{n-17}\\
&-\frac{174611}{330}\cdot\frac{n(n-1)\cdots(n-18)}{1\cdot2\cdot3\cdots20}x^{n-19} + \frac{854513}{138}\cdot\frac{n(n-1)\cdots(n-20)}{1\cdot2\cdot3\cdots22}x^{n-21}\\
&-\frac{236364091}{2730}\cdot\frac{n(n-1)\cdots(n-22)}{1\cdot2\cdot3\cdots24}x^{n-23} + \frac{8553103}{6}\cdot\frac{n(n-1)\cdots(n-24)}{1\cdot2\cdot3\cdots26}x^{n-25}\\
&-\frac{23749461029}{870}\cdot\frac{n(n-1)\cdots(n-26)}{1\cdot2\cdot3\cdots28}x^{n-27} + \frac{8615841276005}{14322}\cdot\frac{n(n-1)\cdots(n-28)}{1\cdot2\cdot3\cdots30}x^{n-29} - \text{etc.}
\end{aligned}$$

quae expressio non differt ab ea, quam supra dedimus, nisi quod hic numeros BERNOULLIANOS $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. introduximus, cum supra usi essemus numeris $\alpha, \beta, \gamma, \delta$, etc.; interim tamen consensus sponte elucet. Hinc ergo terminos summatorios omnium potestatum usque ad potestatem trigesimam inclusive exhibere licuit; quae investigatio, si alia via fuisset suscepta, sine longissimis et taediosissimis calculis absolvi non potuisset.

133. Iam supra (§ 59) similem fere expressionem pro termino summatorio dedimus ex termino generali definiendo. Ea enim pariter secundum differentialia termini generalis procedebat; ab ista autem in hoc potissimum erat diversa, quod illa non integrale $\int zdx$ requirebat, singula vero termini generalis differentialia per certas ipsius x functiones habebat multiplicata. Eandem igitur expressionem sequenti modo ad naturam serierum magis accommodato denuo eliciamus, ex quo simul lex clarius patebit, secundum quam coefficientes illi differentialium progrediuntur. Sit igitur seriei terminus generalis z, functio quaecunque ipsius indicis *x;* terminus vero summatorius quaesitus sit *s;* qui quoniam, uti vidimus, eiusmodi erit functio ipsius *x*, ut evanescat posito $x = 0$, erit per ea, quae supra de natura huiusmodi functionum demonstravimus,

$$s - \frac{xds}{1dx} + \frac{x^2dds}{1\cdot 2dx^2} - \frac{x^3d^3s}{1\cdot 2\cdot 3dx^3} + \frac{x^4d^4s}{1\cdot 2\cdot 3\cdot 4dx^4} - \text{etc.} = 0.$$

134. Quia *s* denotat summam omnium terminorum serei a primo usque ad ultimum *z*, perspicuum est, si in *s* loco *x* ponatur $x-1$, tum priorem summam ultimo termino *z* mulctari; erit scilicet

$$s - z = s - \frac{ds}{dx} + \frac{dds}{2dx^2} - \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} - \text{etc.}$$

ideoque

$$z = \frac{ds}{dx} - \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} - \frac{d^4s}{24dx^4} + \text{etc.},$$

quae aequatio modum suppeditat ex dato termino summatorio *s* definiendi terminum generalem, quod quidem per se est facillimum. Ex idonea autem combinatione huius aequationis cum ea, quam paragrapho praecedenti invenimus, valor ipsius *s* per *x* et *z* determinari poterit. Ponamus in hunc finem esse

$$s - Az + \frac{Bdz}{dx} - \frac{Cddz}{dx^2} + \frac{Dd^3z}{dx^3} - \frac{Ed^4z}{dx^4} + \text{etc.} = 0,$$

ubi *A, B, C, D* etc. denotent coefficientes necessanos, sive constantes sive variabiles; nam cum sit

$$z = \frac{ds}{dx} - \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} - \frac{d^4s}{24dx^4} + \frac{d^5s}{120dx^5} - \text{etc.},$$

si hinc valores pro z, $\frac{dz}{dx}$, $\frac{ddz}{dx^2}$, $\frac{d^3z}{dx^3}$ etc. in superiori aequatione substituantur, prodibit

$$\begin{aligned}
s &= s\\
-Az &= -\frac{Ads}{dx} + \frac{Adds}{2dx^2} - \frac{Ad^3s}{6dx^3} + \frac{Ad^4s}{24dx^4} - \frac{Ad^5s}{120dx^5} + \text{etc.}\\
+\frac{Bdz}{dx} &= \quad + \frac{Bdds}{dx^2} - \frac{Bd^3s}{2dx^3} + \frac{Bd^4s}{6dx^4} - \frac{Bd^5s}{24dx^5} + \text{etc.}\\
-\frac{Cddz}{dx^2} &= \quad - \frac{Cd^3s}{dx^3} + \frac{Cd^4s}{2dx^4} - \frac{Cd^5s}{6dx^5} + \text{etc.}\\
+\frac{Dd^3z}{dx^3} &= \quad + \frac{Dd^4s}{dx^4} - \frac{Dd^5s}{2dx^5} + \text{etc.}\\
-\frac{Ed^4z}{dx^4} &= \quad - \frac{Ed^5s}{dx^5} + \text{etc.}\\
&\text{etc.,}
\end{aligned}$$

quae igitur series iunctim sumtae aequales erunt nihilo.

135. Cum ergo ante invenimus esse

$$0 = s - \frac{xds}{dx} + \frac{x^2dds}{2dx^2} - \frac{x^3d^3s}{6dx^3} + \frac{x^4d^4s}{24dx^4} - \frac{x^5d^5s}{120dx^5} + \text{etc.},$$

si superior aequatio huic aequalis statuatur, prodibunt sequentes litterarum *A*, *B*, C, *D* etc. denominationes

$$A = x, \quad B = \frac{x^2}{2} - \frac{A}{2}, \quad C = \frac{x^3}{6} - \frac{B}{2} - \frac{A}{6},$$
$$D = \frac{x^4}{24} - \frac{C}{2} - \frac{B}{6} - \frac{A}{24}, \quad E = \frac{x^5}{120} - \frac{D}{2} - \frac{C}{6} - \frac{B}{24} - \frac{A}{120}, \quad \text{etc.}$$

His igitur litterarum *A*, *B*, *C*, *D* etc. valoribus inventis ex termino generali *z* terminus summatorius $s = Sz$ ita determinabitur, ut sit

$$Sz = Az - \frac{Bdz}{dx} + \frac{Cddz}{dx^2} - \frac{Dd^3z}{dx^3} + \frac{Ed^4z}{dx^4} - \frac{Fd^5z}{dx^5} + \text{etc.}$$

136. Cum autem fiat

$$A = x, \quad B = \tfrac{1}{2}x^2 - \tfrac{1}{2}x, \quad C = \tfrac{1}{6}x^3 - \tfrac{1}{4}x^2 + \tfrac{1}{12}x,$$
$$D = \tfrac{1}{24}x^4 - \tfrac{1}{12}x^3 + \tfrac{1}{24}xx, \quad \text{etc.},$$

patet hos coefficientes esse eosdem, quos supra (§ 59) habuimus; unde ista termini summatorii expressio eadem est, quam ibi invenimus, eritque propterea

$$A = Sx^0 = S1, \quad B = \tfrac{1}{1}Sx^1 - \tfrac{1}{1}x, \quad C = \tfrac{1}{2}Sx^2 - \tfrac{1}{2}x^2,$$
$$D = \tfrac{1}{6}Sx^3 - \tfrac{1}{6}x^3, \quad E = \tfrac{1}{24}Sx^4 - \tfrac{1}{24}x^4 \quad \text{etc.}$$

Hinc ergo erit

$$Sz = xz - \frac{dz}{dx}Sx + \frac{ddz}{2dx^2}Sx^2 - \frac{d^3z}{6dx^3}Sx^3 + \frac{d^4z}{24dx^4}Sx^4 - \text{etc.}$$
$$+ \frac{xdz}{dx} - \frac{x^2ddz}{2dx^2} + \frac{x^3d^3z}{6dx^3} - \frac{x^4d^4z}{24dx^4} + \text{etc.}$$

Quodsi autem in termino generali *z* ponatur $x = 0$, prodibit terminus indici $= 0$ respondens; qui si ponatur $= a$, erit

$$a = z - \frac{xdz}{dx} + \frac{x^2ddz}{2dx^2} - \frac{x^3d^3z}{6dx^3} + \text{etc.}$$

ideoque

$$\frac{xdz}{dx} - \frac{x^2ddz}{2dx^2} + \frac{x^3d^3z}{6dx^3} - \frac{x^4d^4z}{24dx^4} + \text{etc.} = z - a,$$

quo valore substituto habebitur

$$Sz = (x+1)z - a - \frac{dz}{dx}Sx + \frac{ddz}{2dx^2}Sx^2 - \frac{d^3z}{6dx^3}Sx^3 + \frac{d^4z}{24dx^4}Sx^4 - \text{etc.}$$

Cognitis ergo summis potestatum hiuc pro quovis termino generali ei conveniens terminus summatorius exhiberi potest.

137. Quoniam ergo geminam invenimus expressionem termini summatorii Sz pro termino generali z earumque altera formulam integralem $\int zdx$ continet, si istae duae expressiones sibi aequales ponantur, obtinebitur valor ipsius $\int zdx$ per seriem expressus. Cum enim sit

$$\begin{aligned}&\int zdx + \tfrac{1}{2}z + \frac{\mathfrak{A}dz}{1\cdot 2dx} - \frac{\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3} + \frac{\mathfrak{C}d^5z}{1\cdot 2\cdots 6dx^5} - \text{etc.}\\ &= (x+1)z - a - \frac{dz}{1dx}Sx + \frac{ddz}{1\cdot 2dx^2}Sx^2 - \frac{d^3z}{1\cdot 2\cdot 3dx^3}Sx^3 + \text{etc.},\end{aligned}$$

erit

$$\begin{aligned}&\int zdx = \left(x+\tfrac{1}{2}\right)z - a - \frac{dz}{dx}\left(Sx + \tfrac{1}{2}\mathfrak{A}\right) + \frac{ddz}{2dx^2}Sx^2 - \frac{d^3z}{6dx^3}\left(Sx^3 - \frac{\mathfrak{B}}{4}\right)\\ &+ \frac{d^4z}{24dx^4}Sx^4 - \frac{d^5z}{120dx^5}\left(Sx^5 + \tfrac{1}{6}\mathfrak{C}\right) + \frac{d^6z}{720dx^6}Sx^6 - \frac{d^7z}{5040dx^7}\left(Sx^7 - \tfrac{1}{8}\mathfrak{D}\right) + \text{etc.},\\ &+\text{etc.},\end{aligned}$$

ubi $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. denotant numeros BERNOULLIANOS supra (§ 122) exhibitos.
Sit verbi gratia $z = xx$; fiet $a = 0$, $\frac{dz}{dx} = 2x$ et $\frac{ddz}{2dx^2} = 1$; hinc erit

$$\int xxdx = \left(x+\tfrac{1}{2}\right)xx - 2x\left(\tfrac{1}{2}xx + \tfrac{1}{2}x + \tfrac{1}{12}\right) + 2x\left(\tfrac{1}{2}xx + \tfrac{1}{2}x + \tfrac{1}{12}\right) + 1\left(\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + \tfrac{1}{6}x\right)$$

seu $\int xxdx = \frac{1}{3}x^3$; dat autem $\frac{1}{3}x^3$ differentiatum utique $xxdx$.

138. Nova ergo hinc patet via ad terminos summatorios serierum potestatum inveniendos; quoniam enim ex coefficientibus ante assumtis A, B, C, D etc. hi termini summatorii facillime formantur, horum autem coefficientium quilibet ex praecedentibus conflatur: si in formulis § 135 datis loco istarum litterarum valores in § 136 traditi substituantur, erit

$$Sx^1 - x = \tfrac{1}{2}xx - \tfrac{1}{2}x$$
$$Sx^2 - x^2 = \tfrac{1}{3}x^3 - \tfrac{1}{3}x - \tfrac{2}{2}\left(Sx - x\right)$$
$$Sx^3 - x^3 = \tfrac{1}{4}x^4 - \tfrac{1}{4}x - \tfrac{3}{2}\left(Sx^2 - x^2\right) - \tfrac{3\cdot 2}{2\cdot 3}\left(Sx - x\right)$$
$$Sx^4 - x^4 = \tfrac{1}{5}x^5 - \tfrac{1}{5}x - \tfrac{4}{2}\left(Sx^3 - x^3\right) - \tfrac{4\cdot 3}{2\cdot 3}\left(Sx^2 - x\right) - \tfrac{4\cdot 3\cdot 2}{2\cdot 3\cdot 4}\left(Sx - x\right)$$
$$\text{etc.}$$

Hinc ergo summae potestatum superiorum ex summis inferiorum formari poterunt.

139. Quodsi vero legem, qua coefficientes A, B, C, D etc. supra (§ 135) progredi inventi sunt, attentius intueamur, eos seriem recurrentem constituere deprehendemus. Si enim evolvamus hanc fractionem

$$\frac{x+\frac{1}{2}xxu+\frac{1}{6}x^3u^2+\frac{1}{24}x^4u^3+\frac{1}{120}x^5u^4+\text{etc.}}{x+\frac{1}{2}u+\frac{1}{6}u^2+\frac{1}{24}u^3+\frac{1}{120}u^4+\text{etc.}}$$

secundum potestates ipsius u hancque seriem resultare sumamus

$$A + Bu + Cu^2 + Du^3 + Eu^4 + \text{etc.},$$

erit, uti ante invenimus,

$$A = x,\quad B = \tfrac{1}{2}xx - \tfrac{1}{2}A\quad \text{etc.}$$

sicque inventa hac serie obtinebuntur termini summatorii serierum potestatum.
Illa autem fractio, ex cuius evolutione ista series nascitur, transit in hanc formam $\frac{e^{xu}-1}{e^{u}-1}$, quae, si x fuerit numerus integer affirmativus, abit in

$$1 + e^u + e^{2u} + e^{3u} + \cdots + e^{(x-1)u};$$

cum ergo sit

$$\begin{aligned}
1 &= 1\\
e^u &= 1 + \frac{u}{1} + \frac{u^2}{1\cdot 2} + \frac{u^3}{1\cdot 2\cdot 3} + \frac{u^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}\\
e^{2u} &= 1 + \frac{2u}{1} + \frac{4u^2}{1\cdot 2} + \frac{8u^3}{1\cdot 2\cdot 3} + \frac{16u^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}\\
e^{3u} &= 1 + \frac{3u}{1} + \frac{9u^2}{1\cdot 2} + \frac{27u^3}{1\cdot 2\cdot 3} + \frac{81u^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}\\
e^{(x-1)u} &= 1 + \frac{(x-1)u}{1} + \frac{(x-1)^2u^2}{1\cdot 2} + \frac{(x-1)^3u^3}{1\cdot 2\cdot 3} + \frac{(x-1)^4u^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.},
\end{aligned}$$

ideoque erit

$$\begin{aligned}
&A = x \\
&B = S(x-1) = Sx - x \\
&C = \tfrac{1}{2}S(x-1)^2 = \tfrac{1}{2}Sx^2 - \tfrac{1}{2}x^2 \\
&D = \tfrac{1}{6}S(x-1)^3 = \tfrac{1}{6}Sx^3 - \tfrac{1}{6}x^3 \\
&\qquad\qquad \text{etc.}
\end{aligned}$$

Unde nexus horum coefficientium cum summis potestatum ante iam observatus penitus confirmatur ac demonstratur.