CHAPTER IV

CONCERNING THE CONVERSION OF FUNCTIONS INTO SERIES

70. Now in the above chapter the use has been shown from the part, which general expressions found there have in the investigation of series from finite differences, which may show the value of any function of *x*. If indeed *y* were a given function of *x*, the value of which it adopts on putting $x = 0$ will be known ; and if this may be put $= A$, there will be as we have found,

$$y - \frac{xdy}{1dx} + \frac{x^2 ddy}{1\cdot 2dx^2} - \frac{x^3 d^3 y}{1\cdot 2\cdot 3dx^3} + \frac{x^4 d^4 y}{1\cdot 2\cdot 3\cdot 4dx^4} - \text{etc.} = A.$$

[Note: the now customary function notation is not yet being used, although *y* is a function of *x*, but instead Euler refers to the Cartesian coordinates of a point as *x, y*; by performing a backwards Taylor expansion he arrives at the point 0, *A*.]
Therefore hence not only do we have generally a series extending to infinity, the sum of which is equal to the constant *A*, even if the variable quantity in the individual terms shall be *x*, but also we will be able to express the function *y* itself by a series ; indeed there will be

$$y = A + \frac{xdy}{1dx} - \frac{x^2 ddy}{1\cdot 2dx^2} + \frac{x^3 d^3 y}{1\cdot 2\cdot 3dx^3} - \frac{x^4 d^4 y}{1\cdot 2\cdot 3\cdot 4dx^4} + \text{etc.},$$

some examples of which are to be presented.

71. But so that this investigation may extend widely, we may put the function *y* to change into *z*, if in place of *x* everywhere there is written $x + \omega$, thus so that *z* shall be such a function of $x + \omega$, just as *y* is of *x*, and we have shown [§ 48] to be

$$z = y + \frac{\omega dy}{1dx} + \frac{\omega^2 ddy}{1\cdot 2dx^2} + \frac{\omega^3 d^3 y}{1\cdot 2\cdot 3dx^3} + \frac{\omega^4 d^4 y}{1\cdot 2\cdot 3\cdot 4dx^4} + \text{etc.}$$

Therefore since the individual terms of this series to be found by the continual differentiation of *y* by putting *dx* constant and likewise the value of *z* may actually be able to be shown by the substitution of $x + \omega$ in place of *x*, in this manner always the series will be found equal to the value of *z*, which, if ω were an exceedingly small quantity, will converge maximally and indeed will give an approximate value of *z* and not with very many terms requiring to be taken. From which good use will be made of this formula in the business of approximating.

72. Therefore so that we may proceed in due manner in showing the outstanding use of this formula, in the first place we may substitute algebraic functions of x in place of y. And at first there shall be $y = x^n$ and there will be, if $x + \omega$ may be put in place of x, $z = (x + \omega)^n$. Therefore since there shall be

$$\frac{dy}{dx} = nx^{n-1}, \quad \frac{ddy}{dx^2} = n(n-1)x^{n-2}, \quad \frac{d^3y}{dx^3} = n(n-1)(\mathrm{n}-2)x^{n-3},$$

$$\frac{d^4y}{dx^4} = n(n-1)(\mathrm{n}-2)(\mathrm{n}-3)x^{n-4} \quad \text{etc.,}$$

with these values substituted there becomes

$$(x+\omega)^n = x^n + \frac{n}{1}x^{n-1}\omega + \frac{n(n-1)}{1\cdot 2}x^{n-2}\omega^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}\omega^3 + \text{etc.,}$$

which is the most noteworthy Newtonian expression, from which the powers of the binomial $(x+\omega)^n$ is converted into a series. The number of terms of this series is finite always, if n should be a positive whole number.

73. Hence we will be also to find the progression, which thus may express the value of the powers of the binomial, so that that may be broken off, as often as the exponent of the power should be a negative number. For we may put in place

$$\omega = \frac{-ux}{x+u};$$

there will be

$$z = (x+\omega)^n = \left(\frac{xx}{x+u}\right)^n$$

and thus there will be had [from §72, where $y = x^n$ with the above derivatives, etc.]

$$\frac{x^{2n}}{(x+u)^n} = x^n - \frac{nx^n u}{1(x+u)} + \frac{n(n-1)x^n u^2}{1\cdot 2(x+u)^2} - \frac{n(n-1)(n-2)x^n u^3}{1\cdot 2\cdot 3(x+u)^3} + \text{etc.}$$

It may be divided everywhere by x^{2n} and there will be

$$(x+u)^{-n} = x^{-n} - \frac{nx^{-n}u}{1(x+u)} + \frac{n(n-1)x^{-n}u^2}{1\cdot 2(x+u)^2} - \frac{n(n-1)(n-2)x^{-n}u^3}{1\cdot 2\cdot 3(x+u)^3} + \text{etc.}$$

Now there may be put $-n = m$ and there will be produced

$$(x+u)^m = x^m + \frac{mx^m u}{1(x+u)} + \frac{m(m+1)x^m u^2}{1\cdot 2(x+u)^2} + \frac{m(m+1)(m+2)x^m u^3}{1\cdot 2\cdot 3(x+u)^3} + \text{etc.,}$$

which series, as often as *m* is a negative whole number, will depend on a finite number of terms. Therefore this series is equal to the first found, if for ω and *n* there may be written *u* and *m;* indeed there will be thence

$$(x+u)^m = x^m + \frac{mx^{m-1}u}{1} + \frac{m(m-1)x^{m-2}u^2}{1\cdot 2} + \frac{m(m-1)(m-2)x^{m-3}u^3}{1\cdot 2\cdot 3} + \text{etc.}$$

74. This same series can be deduced also from the initial expression given in § 70. Since indeed, if on putting $x=0$, *y* may become *A*, there shall be

$$y - \frac{xdy}{1dx} + \frac{x^2ddy}{1\cdot 2dx^2} - \frac{x^3d^3y}{1\cdot 2\cdot 3dx^3} + \frac{x^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4} - \text{etc.} = A,$$

there may be put $y=(x+a)^n$ and there will be $A=a^n$ and on account of

$$\frac{dy}{dx} = n(x+a)^{n-1}, \quad \frac{ddy}{dx^2} = n(n-1)(x+a)^{n-2}, \quad \frac{d^3y}{dx^3} = n(n-1)(\mathrm{n}-2)(x+a)^{n-3} \text{ etc.}$$

it becomes

$$(x+a)^n - \frac{n}{1}x(x+a)^{n-1} + \frac{n(n-1)}{1\cdot 2}x^2(x+a)^{n-2} - \text{etc.} = a^n;$$

it may be divided by $a^n(x+a)^n$ and there will be produced

$$(x+a)^{-n} = a^{-n} - \frac{na^{-n}x}{1(x+a)} + \frac{n(n-1)a^{-n}x^2}{1\cdot 2(x+a)} - \text{etc.},$$

which on putting respectively *u, x* and – *m* for *x, a* and *n* the series found before may arise.

75. If fractional numbers may be put in place for *m*, both series will extend to infinity, yet meanwhile, if *u* compared with *x* were an exceedingly small quantity, they will converge very quickly to the true value. Therefore if $m=\frac{\mu}{\nu}$ and $x=a^\nu$; there will be from the series found first

$$\left(a^\nu+u\right)^{\frac{\mu}{\nu}} = a^\mu\left(1 + \frac{\mu}{\nu}\cdot\frac{u}{a^\nu} + \frac{\mu(\mu-\nu)}{\nu\cdot 2\nu}\cdot\frac{u^2}{a^{2\nu}} + \frac{\mu(\mu-\nu)(\mu-2\nu)}{\nu\cdot 2\nu\cdot 3\nu}\cdot\frac{u^3}{a^{3\nu}} + \text{etc.}\right).$$

But the latter series found will give

$$\left(a^\nu+u\right)^{\frac{\mu}{\nu}} = a^\mu\left(1 + \frac{\mu u}{\nu(a^\nu+u)} + \frac{\mu(\mu+\nu)u^2}{\nu\cdot 2\nu(a^\nu+u)^2} + \frac{\mu(\mu+\nu)(\mu+2\nu)u^3}{\nu\cdot 2\nu\cdot 3\nu(a^\nu+u)^3} + \text{etc.}\right).$$

But this latter series will converge more than the former, since the terms of this may also decrease, if there were $u > a^{\nu}$, yet in which case the first series will diverge.

If therefore there shall be $\mu = 1$, $\nu = 2$, there will be

$$\sqrt{\left(a^2+u\right)} = a\left(1+\frac{1u}{2\left(a^2+u\right)}+\frac{1\cdot 3u^2}{2\cdot 4\left(a^2+u\right)^2}+\frac{1\cdot 3\cdot 5u^3}{2\cdot 4\cdot 6\left(a^2+u\right)^3}+\text{etc.}\right).$$

In a similar manner on putting the numbers 3, 4, 5 etc. for ν with $\mu = 1$ remaining, there will be

$$\sqrt[3]{\left(a^3+u\right)} = a\left(1+\frac{1u}{3\left(a^3+u\right)}+\frac{1\cdot 4u^2}{3\cdot 6\left(a^3+u\right)^2}+\frac{1\cdot 4\cdot 7u^3}{3\cdot 6\cdot 9\left(a^3+u\right)^3}+\text{etc.}\right)$$

$$\sqrt[4]{\left(a^4+u\right)} = a\left(1+\frac{1u}{4\left(a^4+u\right)}+\frac{1\cdot 5u^2}{4\cdot 8\left(a^4+u\right)^2}+\frac{1\cdot 5\cdot 9u^3}{4\cdot 8\cdot 12\left(a^4+u\right)^3}+\text{etc.}\right)$$

$$\sqrt[5]{\left(a^5+u\right)} = a\left(1+\frac{1u}{5\left(a^5+u\right)}+\frac{1\cdot 6u^2}{5\cdot 10\left(a^5+u\right)^2}+\frac{1\cdot 6\cdot 11u^3}{5\cdot 10\cdot 15\left(a^3+u\right)^3}+\text{etc.}\right)$$

76. Therefore from these formulas the root of any power of any proposed number will be able to be found easily. For with the number c proposed the power nearest to that is sought , either larger or smaller; in the first case u becomes a negative number, in the latter a positive number. But if truly the resulting series may not seem to converge well enough, the number c may be multiplied by a certain power, for example by f^{ν}, if the root of the power ν should be extracted, and the root of the number $f^{\nu}c$ may be sought, which divided by f will give the root sought of the number c. But when a greater number f is taken, with that the series will converge more and that especially, if certain similar powers a^{ν} may not differ greatly from $f^{\nu}c$.

EXAMPLE 1

The square root of the number 2 is sought.

If there is put $a = 1$ and $u = 1$ without further preparation, there becomes

$$\sqrt{2} = 1+\frac{1}{2\cdot 2}+\frac{1\cdot 3}{2\cdot 4\cdot 2^2}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 2^3}+\text{etc.};$$

which although now it converges well enough, yet it will be better to multiply the number 2 before by a certain square, as 25, so that the number 50 produced may differ minimally from another square 49. On account of this the square root of 50 is sought, which divided by 5 will give $\sqrt{2}$. But then there will be $a = 7$ and $u = 1$, from which there becomes

$$\sqrt{50} = 5\sqrt{2} = 7\left(1 + \tfrac{1}{2\cdot 50} + \tfrac{1\cdot 3}{2\cdot 4\cdot 50^2} + \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 50^3} + \text{etc.}\right)$$

or

$$\sqrt{2} = \tfrac{7}{5}\left(1 + \tfrac{1}{100} + \tfrac{1\cdot 3}{100\cdot 200} + \tfrac{1\cdot 3\cdot 5}{100\cdot 200\cdot 300} + \text{etc.}\right),$$

which is more suitable according to the computation in decimal fractions required to be established.

Indeed there shall be

$\frac{7}{5} =$	1,4000000000000
$\frac{7}{5}\cdot\frac{1}{100} =$	140000000000
$\frac{7}{5}\cdot\frac{1}{100}\cdot\frac{3}{200} =$	2100000000
$\frac{7}{5}\cdot\frac{1}{100}\cdot\frac{3}{200}\cdot\frac{5}{300} =$	35000000
preceding by $\frac{7}{400} =$	612500
preceding by $\frac{9}{500} =$	11025
preceding by $\frac{11}{600} =$	202
preceding by $\frac{13}{700} =$	3
Hence	
$\sqrt{2} =$	1,4142135623730

EXAMPLE 2

The cube root of the number 3 is sought.

The number 3 may be multiplied by the cube 8 and the cube root of 24 is sought ; for there shall be $\sqrt[3]{24} = 2\sqrt[3]{3}$. Hence there is put $a = 3$ and $u = -3$ and there will be

$$\sqrt[3]{24} = 3\left(1 - \tfrac{1\cdot 3}{3\cdot 24} + \tfrac{1\cdot 4\cdot 3^2}{3\cdot 6\cdot 24^2} - \tfrac{1\cdot 4\cdot 7\cdot 3^3}{3\cdot 6\cdot 9\cdot 24^3} + \text{etc.}\right)$$

and

$$\sqrt[3]{3} = \tfrac{3}{2}\left(1 - \tfrac{1}{3\cdot 8} + \tfrac{1\cdot 4}{3\cdot 6\cdot 8^2} - \tfrac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9\cdot 8^3} + \text{etc.}\right)$$

or

$$\sqrt[3]{3} = \tfrac{3}{2}\left(1 - \tfrac{1}{24} + \tfrac{1}{24}\cdot\tfrac{4}{48} - \tfrac{1}{24}\cdot\tfrac{4}{48}\tfrac{7}{72} + \text{etc.}\right),$$

which series now converges strongly, since any term will be more than eight times smaller than the preceding. But if 3 may be multiplied by the cube 729, there will be made 2187 and $\sqrt[3]{2187} = \sqrt[3]{\left(13^3 - 10\right)} = 9\sqrt[3]{3}$. Therefore there will be on account of $a = 13$ and $u = -10$

$$\sqrt[3]{3} = \tfrac{13}{3}\left(1 - \tfrac{1\cdot 10}{3\cdot 2187} + \tfrac{1\cdot 4\cdot 10^2}{3\cdot 6\cdot 2187^2} - \tfrac{1\cdot 4\cdot 7\cdot 10^3}{3\cdot 6\cdot 9\cdot 2187^3} + \text{etc.}\right),$$

any term of which is more than two hundred times less than the preceding.

77. The expansion of binomial powers may be extended as widely, so that all algebraic functions are able to be dealt with in that. If indeed for the argument's sake the value of this function may be sought $\sqrt{(a+2bx+cxx)}$ expressed by a series, this will be able to come about by the preceding formulas, by considering two terms as one. Then truly this explanation can come about with the help of the expression first treated ; for if there may be put $\sqrt{(a+2bx+cxx)} = y$, because on putting $x = 0$ there becomes $y = \sqrt{a}$, there will be $A = \sqrt{a}$, and since the differentials of y may themselves be had thus :

$$\frac{dy}{dx} = \frac{b+cx}{\sqrt{(a+2bx+cxx)}}, \quad \frac{ddy}{dx^2} = \frac{ac-bb}{(a+2bx+cxx)^{\frac{3}{2}}}, \quad \frac{d^3y}{dx^3} = \frac{3(bb-ac)(b+cx)}{(a+2bx+cxx)^{\frac{5}{2}}},$$
$$\frac{d^4y}{dx^4} = \frac{3(bb-ac)(ac-5bb-8bcx-4ccxx)}{(a+2bx+cxx)^{\frac{7}{2}}} \quad \text{etc.},$$

therefore from these there will be obtained

$$\sqrt{(a+2bx+cxx)} - \frac{(b+cx)x}{\sqrt{(a+2bx+cxx)}} - \frac{(bb-ac)xx}{2(a+2bx+cxx)^{\frac{3}{2}}}$$
$$- \frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^{\frac{5}{2}}} - \frac{(bb-ac)(5bb-ac+8bcx+4ccxx)x^4}{8(a+2bx+cxx)^{\frac{7}{2}}} - \text{etc.} = \sqrt{a}.$$

But if therefore it may be multiplied everywhere by $\sqrt{(a+2bx+cxx)}$, the series becomes rational and there will be

$$\sqrt{a(a+2bx+cxx)} = a + 2bx + cxx - (b+cx)x - \frac{(bb-ac)xx}{2(a+2bx+cxx)}$$
$$- \frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^2} - \frac{(bb-ac)(5bb-ac+8bcx+4ccxx)x^4}{8(a+2bx+cxx)3} - \text{etc.}$$

or

$$\sqrt{(a+2bx+cxx)} = \sqrt{a} + \frac{bx}{\sqrt{a}} - \frac{(bb-ac)xx}{2(a+2bx+cxx)\sqrt{a}} - \frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^2\sqrt{a}} - \text{etc.}$$

78. Therefore we may cross over to transcending functions, which we may substitute in place of y. And thus there shall be the first $y = lx$ and on putting $x+\omega$ in place x there becomes $z = l(x+\omega)$. But whichever these logarithms shall be, which maintain the ratio $n:1$ to hyperbolic logarithms, there will be for hyperbolic logarithms $n = 1$ and for the logarithms from tables there will be $n = 0,4342944819032$. Hence the differentials of $y = lx$ will be

$$\frac{dy}{dx} = \frac{n}{x}, \quad \frac{ddy}{dx^2} = -\frac{n}{x^2}, \quad \frac{d^3y}{dx^3} = \frac{2n}{x^3} \text{ etc.},$$

from which there is made

$$z = l(x+\omega) = lx + \frac{n\omega}{x} - \frac{n\omega^2}{2x^2} + \frac{n\omega^3}{3x^3} - \frac{n\omega^4}{4x^4} + \text{etc.}$$

In a similar manner, if ω were put in place negative, there will be

$$z = l(x-\omega) = lx - \frac{n\omega}{x} - \frac{n\omega^2}{2x^2} - \frac{n\omega^3}{3x^3} - \frac{n\omega^4}{4x^4} - \text{ etc.}$$

But if hence this series may be subtracted from the first there comes about

$$l\,\frac{x+\omega}{x-\omega} = 2n\left(\frac{\omega}{x} + \frac{\omega^3}{3x^3} + \frac{\omega^5}{5x^5} + \frac{\omega^7}{7x^7} + \text{etc.}\right).$$

79. If in the first series found

$$l(x+\omega) = lx + \frac{n\omega}{x} - \frac{n\omega^2}{2x^2} + \frac{n\omega^3}{3x^3} - \frac{n\omega^4}{4x^4} + \text{etc.}$$

there may be put

$$\omega = \frac{xx}{u-x}$$

there will be $x+\omega = \frac{ux}{u-x}$ and

$$l(x+\omega) = lu + lx - l(u-x) = lx + \frac{nx}{u-x} - \frac{nxx}{2(u-x)^2} + \text{ etc..}$$

and

$$l(u-x) = lu - \frac{nx}{u-x} + \frac{nxx}{(u-x)^2} - \frac{nx^3}{3(u-x)^3} + \text{etc.}$$

and by taking *x* negative there is had

$$l\left(u+x\right)=lu+\frac{nx}{u+x}+\frac{nxx}{\left(u+x\right)^2}+\frac{nx^3}{3\left(u+x\right)^3}+\frac{nx^4}{4\left(u+x\right)^4}+\text{etc.}$$

Therefore logarithms will be able to be found readily with the help of these series, if indeed the series may be converging greatly. Moreover the following will be of this kind, which may be deduced easily from these found,

$$l\left(x+1\right)=lx+n\left(\frac{1}{x}-\frac{1}{2xx}+\frac{1}{3x^3}-\frac{1}{4x^4}+\text{etc.}\right)$$
$$l\left(x-1\right)=lx-n\left(\frac{1}{x}+\frac{1}{2xx}+\frac{1}{3x^3}+\frac{1}{4x^4}+\text{etc.}\right);$$

which two series with only the signs differing between themselves, if they may be recalled to the calculation, from the known logarithm of the number *x* from the same work the logarithms of both the numbers $x+1$ and $x-1$ may be found. Then from the series left there will be

$$l\left(x+1\right)=l\left(x-1\right)+2n\left(\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\frac{1}{7x^7}+\text{etc.}\right)$$
$$l\left(x-1\right)=lx-n\left(\frac{1}{x-1}-\frac{1}{2\left(x-1\right)^2}+\frac{1}{3\left(x-1\right)^3}-\frac{1}{4\left(x-1\right)^4}+\text{etc.}\right)$$
$$l\left(x+1\right)=lx+n\left(\frac{1}{x+1}+\frac{1}{2\left(x+1\right)^2}+\frac{1}{3\left(x+1\right)^3}+\frac{1}{4\left(x+1\right)^4}+\text{etc.}\right).$$

80. Therefore from the given logarithm of the number *x* the logarithms of the contiguous numbers $x+1$ and $x-1$ may be found readily ; indeed also and in turn from the logarithm of the number $x-1$ the logarithm of the number greater by two may be elicited. Because although it may be shown more fully in the *Introductione*, yet here we may add some examples.

EXAMPLE1

From the given hyperbolic logarithm of the number 10, *which is* 2,3025850929940, *to find the hyperbolic logarithms of the numbers* 11 *and* 9.

Because this question considers hyperbolic logarithms, there will be $n=1$ and thus these series will be had

$$l11 = l10 + \tfrac{1}{10} - \tfrac{1}{2\cdot 10^2} + \tfrac{1}{3\cdot 10^3} - \tfrac{1}{4\cdot 10^4} + \tfrac{1}{5\cdot 10^5} - \text{etc.}$$
$$l9 = l10 - \tfrac{1}{10} - \tfrac{1}{2\cdot 10^2} - \tfrac{1}{3\cdot 10^3} - \tfrac{1}{4\cdot 10^4} - \tfrac{1}{5\cdot 10^5} - \text{etc.}$$

According to the sums of which series requiring to be found, the even and odd terms are gathered together and there will be

$\tfrac{1}{10} = 0{,}1000000000000$	$\tfrac{1}{2\cdot 10^2} = 0{,}0050000000000$
$\tfrac{1}{3\cdot 10^3} = 0{,}0003333333333$	$\tfrac{1}{4\cdot 10^4} = 0{,}0000250000000$
$\tfrac{1}{5\cdot 10^5} = 0{,}0000020000000$	$\tfrac{1}{6\cdot 10^6} = 0{,}0000001666666$
$\tfrac{1}{7\cdot 10^7} = 0{,}0000000142857$	$\tfrac{1}{8\cdot 10^8} = 0{,}0000000012500$
$\tfrac{1}{9\cdot 10^9} = 0{,}0000000001111$	$\tfrac{1}{10\cdot 10^{10}} = 0{,}0000000000100$
$\tfrac{1}{11\cdot 10^{11}} = 0{,}0000000000009$	$\tfrac{1}{12\cdot 10^{12}} = 0{,}0000000000001$
the sum = 0,1003353477310	the sum = 0,0050251679267

The sum of both will be 0,1053605156577

The difference of both will be 0,0953101798043

Now there is $l10 = 2{,}3025850929940$

Hence there will be $l11 = 2{,}3978952727983$
and $l9 = 2{,}1972245773363$
Hence again there will be $l3 = 1{,}0986122886681$
and $l99 = 4{,}5951198501346$

EXAMPLE 2

From the hyperbolic logarithm of the number 99 *now found, to find the logarithm of the number 101.*

The series used above may be applied to this

$$l(x+1) = l(x-1) + \tfrac{2}{x} + \tfrac{2}{3x^3} + \tfrac{2}{5x^5} + \tfrac{2}{7x^7} + \text{etc.},$$

in which there may be made $x = 100$, and there will be

$$l101 = l99 + \tfrac{2}{100} + \tfrac{2}{3\cdot 100^3} + \tfrac{2}{5\cdot 100^5} + \tfrac{2}{7\cdot 100^7} + \text{etc.},$$

the sum of which series from the four terms gathered together = 0,0200006667066, which added to *l*99 will give *l*101 = 4,6151205168412.

EXAMPLE 3

From the given logarithm of 10 *of the tables, which is* = 1, *to find the logarithms of the numbers* 11 and 9.

Because here we search for the common tabulated tables, there will be

$$n = 0{,}4342944819032\,;$$

therefore on putting $x = 10$ there will be

$$l11 = l10 + \tfrac{n}{10} - \tfrac{n}{2\cdot 10^2} + \tfrac{n}{3\cdot 10^3} - \tfrac{n}{4\cdot 10^4} + \text{etc.}$$
$$l9 = l10 - \tfrac{n}{10} - \tfrac{n}{2\cdot 10^2} - \tfrac{n}{3\cdot 10^3} - \tfrac{n}{4\cdot 10^4} - \text{etc.}$$

The even and odd terms may be summed separately

$\frac{1}{10} = 0{,}0434294481903$	$\frac{1}{2\cdot 10^2} = 0{,}0021714724095$
$\frac{1}{3\cdot 10^3} = 0{,}0001447648273$	$\frac{1}{4\cdot 10^4} = 0{,}0000108573620$
$\frac{1}{5\cdot 10^5} = 0{,}0000008685889$	$\frac{1}{6\cdot 10^6} = 0{,}0000000723824$
$\frac{1}{7\cdot 10^7} = 0{,}0000000062042$	$\frac{1}{8\cdot 10^8} = 0{,}0000000005428$
$\frac{1}{9\cdot 10^9} = 0{,}0000000000482$	$\frac{1}{10\cdot 10^{10}} = 0{,}0000000000043$
$\frac{1}{11\cdot 10^{11}} = 0{,}0000000000004$	$\frac{1}{12\cdot 10^{12}} = 0{,}0000000000000$
sum = 0,0435750878593	sum = 0,0021824027010

The sum of both is

= 0,0457574905603

The difference of these is

= 0,0413926851583

Therefore since there shall be

*l*10 = 1,0000000000000

there will be

*l*11 = 1,0413926851583

and

*l*9 = 0,9542425094397

hence

$$l3 = 0,4771212547198$$

and

$$l99 = 1,9956351945980$$

EXAMPLE 4

From the table logarithm of the number 99 *found here, to find the table logarithm of the number* 10l.

Here by applying the same series, which we have used in the second example, we will have

$$l101 = l99 + 2n\left(\tfrac{1}{100} + \tfrac{1}{3\cdot 100^3} + \tfrac{1}{5\cdot 100^5} + \tfrac{1}{7\cdot 100^7} + \text{etc.}\right),,$$

the sum of which series will be found soon on putting for *n* the due value

$$= 0,0086861791849$$

with which added to

$$l99 = 1,9956351945980$$

there arises

$$l101 = 2,0043213737829$$

81. Now we may attribute in our general expression *y* the exponential value and there shall be $y = a^x$*;* on putting $x + \omega$ in place of x there will be $z = a^{x+\omega}$, the value of which on account of the differentials

$$\tfrac{dy}{dx} = a^x la, \quad \tfrac{ddy}{dx^2} = a^x \left(la\right)^2, \quad \tfrac{d^3y}{dx^3} = a^x \left(la\right)^3 \text{ etc.},$$

will be

$$a^{x+\omega} = a^x\left(1 + \tfrac{\omega la}{1} + \tfrac{\omega^2(la)^2}{1\cdot 2} + \tfrac{\omega^3(la)^3}{1\cdot 2\cdot 3} + \text{etc.}\right);$$

which if it may be divided by a^x, will produce the series expressing the values of the exponential quantity, which we have now elicited above in the *Introductione*, surely

$$a^{\omega} = 1 + \tfrac{\omega la}{1} + \tfrac{\omega^2(la)^2}{1\cdot 2} + \tfrac{\omega^3(la)^3}{1\cdot 2\cdot 3} + \tfrac{\omega^4(la)^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

In a similar manner on taking ω negative there will be

$$a^{-\omega} = 1 - \tfrac{\omega la}{1} + \tfrac{\omega^2(la)^2}{1\cdot 2} - \tfrac{\omega^3(la)^3}{1\cdot 2\cdot 3} + \tfrac{\omega^4(la)^4}{1\cdot 2\cdot 3\cdot 4} - \text{etc.}$$

from the combination of which there arises

$$\frac{a^{\omega}+a^{-\omega}}{2}=1+\frac{\omega^2(la)^2}{1\cdot 2}+\frac{\omega^4(la)^4}{1\cdot 2\cdot 3\cdot 4}+\frac{\omega^6(la)^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}+\text{etc.}$$
$$\frac{a^{\omega}-a^{-\omega}}{2}=\frac{\omega la}{1}+\frac{\omega^3(la)^3}{1\cdot 2\cdot 3}+\frac{\omega^5(la)^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\text{etc.}$$

where it is to be noted that *la* denotes the hyperbolic logarithm of the number *a*.

82. With the aid of this formula from some given logarithm, the number agreeing with that will be able to be found. For let some proposed logarithm be *u* belonging to the tables, in which the logarithm of the number $a=1$ is put in place. In the same tables there may be sought the logarithm *x* falling close to *u* and there shall be $u=x+\omega$, but the number agreeing with the logarithm *x* shall be $=y=a^x$; the number corresponding to the logarithm $u=x+\omega$ will be $=a^{x+\omega}=z$ and there arises

$$z=y\left(1+\frac{\omega la}{1}+\frac{\omega^2(la)^2}{1\cdot 2}+\frac{\omega^3(la)^3}{1\cdot 2\cdot 3}+\frac{\omega^4(la)^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.}\right),$$

which series may converge on account of the exceedingly small number ω, the use of which we may indicate in the following example.

EXAMPLE

The number is sought equal to this binary power $2^{2^{24}}$.

Since there shall be $2^{24}=16777216$, there will be $2^{2^{24}}=2^{216777216}$ and with the common logarithms taken there will be the logarithm of this number $=16777216l2$. But since there shall be
$$l2=0{,}3010299956639811952137389,$$

the logarithm of the number sought will be

$$5050445{,}259733675932039063,$$

the characteristic of which indicates the number sought to be expressed from 5050446 figures; which since all are unable to be shown, it may be sufficient to assign the initial figures, which must be investigated from the mantissa

$$,259733675932039063=u.$$

But from tables the number is deduced, the logarithm of which may fall nearest to that, to be $=1,818$, which may be put in place as *y;* the logarithm of which

$$x = 0,259593878885948644,$$

from which there will be

$$\omega = 0,000139797046090419$$

Now since there shall be

$$a = 10$$

there will be

$$la = 2,302585092994045684017991 4$$

and

$$\omega la = 0,000321894594372400$$

Then there will be

$$y = 1,818000000000000000$$

$$\frac{\omega la}{1} y = \qquad 585204372569023$$

$$\frac{\omega^2 (la)^2}{1\cdot 2} y = \qquad 94187062066$$

$$\frac{\omega^3 (la)^3}{1\cdot 2\cdot 3} y = \qquad 10106102$$

$$\frac{\omega^4 (la)^4}{1\cdot 2\cdot 3\cdot 4} y = \qquad 813$$

$$1818585298569738 00$$

and these are the initial figures of the number sought, all the figures except perhaps the last are justified.

83. We may consider transcending quantities depending on the circle and there shall be, as we put in place always, the radius of the circle $=1$ and y may denote the arc of the circle, the sine of which is $=x$, or there shall be $y = \text{Asin}\, x$. $x+\omega$ may be put in place of *x* and there will be $z = \text{Asin}(x+\omega)$; towards which value being expressed the differentials of *y* may be sought

$$\frac{dy}{dx}=\frac{1}{\sqrt{(1-xx)}},\quad \frac{ddy}{dx^2}=\frac{x}{(1-xx)^{\frac{3}{2}}},\quad \frac{d^3y}{dx^3}=\frac{1+2xx}{(1-xx)^{\frac{5}{2}}},\quad \frac{d^4y}{dx^4}=\frac{9x+6x^3}{(1-xx)^{\frac{7}{2}}},$$

$$\frac{d^5y}{dx^5}=\frac{9+72x^2+24x^4}{(1-xx)^{\frac{9}{2}}},\quad \frac{d^6y}{dx^6}=\frac{225x+600x^3+120x^5}{(1-xx)^{\frac{11}{2}}}$$

etc.

Therefore from these there may be found

$$\text{Asin}(x+\omega)=\text{Asin}\,x+\frac{\omega}{\sqrt{(1-xx)}}+\frac{\omega^2x}{2(1-xx)^{\frac{3}{2}}}+\frac{\omega^3(1+2xx)}{6(1-xx)^{\frac{5}{2}}}+$$
$$\frac{\omega^4(9x+6x^3)}{24(1-xx)^{\frac{7}{2}}}+\frac{\omega^5(9+72x^2+24x^4)}{120(1-xx)^{\frac{9}{2}}}+\text{ etc.}$$

84. Hence if the arc were known, the sine of which is $=x$, by the benefit of this formula the arc will be able to be found, the sine of which is $x+\omega$, if ω were an exceedingly small quantity. Moreover the series, the sum of which must be added, may be expressed in parts of the radius, which may be reduced easily to an arc, as may be understood from this example.

EXAMPLE

The arc of the circle is sought, the sine of which is $=\frac{1}{3}=0{,}3333333333$.

The arc is sought from the table of sines, the sine of which shall be the nearest less than $\frac{1}{3}$, which will be $19^0\ 28^{\text{I}}$, the sine of which $=0{,}3332584$. Therefore there may be put in place $19^0\ 28^{\text{I}}=\text{Asin}\,x=y$; there will be $x=0{,}3332584$ and $\omega=0{,}0000749$ and from the tables $\sqrt{(1-xx)}=\cos y=0{,}9428356$.

Therefore the arc of the circle sought *z*, the sine of which $=\frac{1}{3}$ is proposed,

$$=19^0\ 28^{\text{I}}+\frac{\omega}{\cos y}+\frac{\omega\omega\sin y}{2\cos^3 y},$$

which expression now is sufficient; therefore the sum will be by the calculation put in place through logarithms

$$l\omega=5{,}8744818$$
$$l\cos y=9{,}9744359$$
$$l\frac{\omega}{\cos y}=5{,}9000459 \qquad \frac{\omega}{\cos y}=0{,}0000794412$$

$$l\frac{\omega^2}{\cos^2 y} = 1{,}8000918$$
$$l\frac{\sin y}{\cos y} = 9{,}5483452$$
$$1{,}3484370$$
$$l2 = 0{,}3010300$$
$$l\frac{\omega^2}{2\cos^2 y} = 1{,}0474070 \qquad \frac{\omega^2 \sin y}{2\cos^3 y} = 0{,}0000000011$$
$$\text{sum} = 0{,}0000794423$$

which is the value of the arc to be added to $19^0\ 28^{\mathrm{I}}$, to expressing which in seconds of minutes we may take the logarithm of this, which is

$$5{,}9000518,$$

from which there may be taken [corresponding to $180/\pi \times 3600$, the characteristics of the logs are adapted to insure that no negative logs occur.]

$$4{,}6855749$$
$$1{,}2144769,$$

to which log. there corresponds a number

$$= 16{,}38615,$$

which is the number of seconds of minutes; now the fraction becomes the arc sought on expressing into a fraction with the third and fourth orders

$$= 19^0\ 28^{\mathrm{I}}\ 16^{\mathrm{II}}\ 23^{\mathrm{III}} 10^{\mathrm{IV}}\ 8^{\mathrm{V}}\ 24^{\mathrm{VI}}.$$

85. The expression for cosines may be elicited in a similar manner ; indeed on putting $y = \mathrm{A}\cos x$, because there is $dy = \frac{-dx}{\sqrt{(1-xx)}}$, the series found before will remain the same, as long as the signs may be interchanged. And thus there will be

$$\mathrm{A}\cos(x+\omega) = \mathrm{A}\cos x - \frac{\omega}{\sqrt{(1-xx)}} - \frac{\omega^2 x}{2(1-xx)^{\frac{3}{2}}} - \frac{\omega^3(1+2xx)}{6(1-xx)^{\frac{5}{2}}} - \frac{\omega^4(9x+6x^3)}{24(1-xx)^{\frac{7}{2}}} - \frac{\omega^5(9+72x^2+24x^4)}{120(1-xx)^{\frac{9}{2}}} - \text{ etc.}$$

which series and equally the preceding will always converge strongly, if the true angles may be selected from the table of sines, thus so that generally the single first term $\frac{\omega}{\sqrt{(1-xx)}}$ may be sufficient. Yet meanwhile, if x were equal to one itself or almost equal to the total sine, then on account of the very small denominators that series loses convergence. Therefore in these cases, in which x fails to be not much deficient from 1, because the differences become most small, we will use the customary interpolation more conveniently.

86. Also we may put the arc for y, the tangent of which is given, and let there be $y = \text{Atang}\, x$ and $z = \text{Atang}(x+\omega)$, thus so that there shall be

$$z = y + \frac{\omega dy}{1dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \text{etc.}$$

Towards finding which terms the individual differentials of y are sought

$$\frac{dy}{dx} = \frac{1}{1+xx}, \quad \frac{ddy}{dx^2} = \frac{-2x}{(1+xx)^2}, \quad \frac{d^3y}{dx^3} = \frac{-2+6xx}{(1+xx)^3}, \quad \frac{d^4y}{dx^4} = \frac{24x-24x^3}{(1+xx)^4},$$

$$\frac{d^5y}{dx^5} = \frac{24-240x^2+120x^4}{(1+xx)^5}, \quad \frac{d^6y}{dx^6} = \frac{-720x+2400x^3-720x^5}{(1+xx)^6}$$

etc.

from which there is deduced to be

$$\text{Atang}(x+\omega) = \text{Atang}x = \frac{\omega}{1+xx} - \frac{\omega^2 x}{(1+xx)^2} + \frac{\omega^3}{(1+xx)^3}\left(xx - \tfrac{1}{3}\right) - \frac{\omega^4}{(1+xx)^4}\left(x^3 - x\right)$$
$$+ \frac{\omega^5}{(1+xx)^5}\left(x^4 - 2x^2 + \tfrac{1}{5}\right) - \frac{\omega^6}{(1+xx)^6}\left(x^5 - \tfrac{10}{3}x^3 + x\right) + \text{etc.}$$

87. This series, the law of progression of which is not yet exactly clear, can be changed into another form, the progression of which springs to mind at once. To this end there is put $\text{Atang}\, x = 90^0 - u$, so that there shall be $x = \cot u = \frac{\cos u}{\sin u}$; there will be $1 + xx = \frac{1}{\sin^2 u}$, from which there becomes $\frac{dy}{dx} = \frac{1}{1+xx} = \sin^2 u$. Then since there shall be $dx = \frac{-du}{\sin^2 u}$ or $du = -dx \sin^2 u$, the becomes on taking the further differentials

$$\frac{ddy}{dx} = 2du \sin u \cdot \cos u = du \sin 2u = -dx \sin^2 u \cdot \sin 2u$$

and thus

$$\frac{ddy}{1dx^2} = -\sin^2 u \cdot \sin 2u,$$

$$\frac{d^3y}{2dx^2} = -du \sin u.\cos u \cdot \sin 2u - du \sin^2 u \cdot \cos 2u = -du \sin u \cdot \sin 3u$$
$$= dx \sin^3 u \cdot \sin 3u$$

and thus

$$\frac{d^3y}{1\cdot 2dx^3} = \sin^3 u\cdot\sin 3u\,,$$

$$\frac{d^4y}{1\cdot 2\cdot 3dx^3} = du\sin^2 u\left(\cos u\cdot\sin 3u + \sin u\cdot\cos 3u\right) = du\cdot\sin^2 u\cdot\sin 4u = -dx\sin^4 u\cdot\sin 4u$$

and thus

$$\frac{d^4y}{1\cdot 2\cdot 3dx^4} = -\sin^4 u\cdot\sin 4u$$

$$\frac{d^5y}{1\cdot 2\cdot 3\cdot 4dx^4} = -du\sin^3 u\left(\cos u\cdot\sin 4u + \sin u\cdot\cos 4u\right) = -du\cdot\sin^3 u\cdot\sin 5u = dx\sin^5 u\cdot\sin 5u$$

and thus

$$\frac{d^5y}{1\cdot 2\cdot 3\cdot 4dx^5} = +\sin^5 u\cdot\sin 5u$$

etc.

From which there is deduced to be

$$\begin{aligned}&\text{Atang}\left(x+\omega\right) = \text{Atang}\,x + \tfrac{\omega}{1}\sin u\cdot\sin u - \tfrac{\omega^2}{2}\sin^2 u\cdot\sin 2u\\ &+\tfrac{\omega^3}{3}\sin^3 u\cdot\sin 3u - \tfrac{\omega^4}{4}\sin^4 u\cdot\sin 4u + \tfrac{\omega^5}{5}\sin^5 u\cdot\sin 5u - \tfrac{\omega^6}{6}\sin^6 u\cdot\sin 6u + \text{etc.};\end{aligned}$$

where since there shall be $\text{Atang}\,x = y$ and $\text{Atang}\,x = 90^0 - u$, there will be $y = 90^0 - u$.

88. If there is put $\text{Acot}\,x = y$ and $\text{Acot}\left(x+\omega\right) = z$, there will be

$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{1\cdot 2dx^2} + \frac{\omega^3 d^3y}{1\cdot 2\cdot 3dx^3} + \frac{\omega^4 d^4y}{1\cdot 2\cdot 3\cdot 4dx^4} + \text{etc.}$$

But since there shall be $dy = \frac{-dx}{1+xx}$ the terms of this series will agree with the previous found except for the first only with the signs excepted. Whereby if there may be put as before $\text{Atang}\,x = 90^0 - u$ or $\text{Acot}\,x = u$, so that there shall be sit $u = y$, there will be

$$\begin{aligned}&\text{Acot}\left(x+\omega\right) = \text{Acot}\,x - \tfrac{\omega}{1}\sin u\cdot\sin u + \tfrac{\omega^2}{2}\sin^2 u\cdot\sin 2u\\ &-\tfrac{\omega^3}{3}\sin^3 u\cdot\sin 3u + \tfrac{\omega^4}{4}\sin^4 u\cdot\sin 4u - \tfrac{\omega^5}{5}\sin^5 u\cdot\sin 5u - \text{etc.},\end{aligned}$$

which expression follows immediately from the preceding ; because indeed there is

$$\text{Acot}(x+\omega)=90^0-\text{Atang}(x+\omega) \text{ and } \text{A cot}\, x=90^0-\text{Atang}\, x,$$

there will be

$$\text{Acot}(x+\omega)-\text{Acot}\, x=-\text{Atang}(x+\omega)+\text{Atang}\, x.$$

89. Many exceptional corollaries follow from these expressions, as given values may be substituted in place of x and ω. Therefore in the first place let $x=0$, and since there shall be $u=90^0-\text{Atang}\, x$, there becomes $u=90^0$ and
$\sin u=1,\ \sin 2u=0,\ \sin 3u=-1,\ \sin 4u=0,\ \sin 5u=1,\ \sin 6u=0,\ \sin 7u=-1$ etc., from which there becomes

$$\text{Atang}\,\omega=\frac{\omega}{1}-\frac{\omega^3}{3}+\frac{\omega^5}{5}-\frac{\omega^7}{7}+\frac{\omega^9}{9}-\frac{\omega^{11}}{11}+\text{etc.},$$

which is a most noteworthy series expressing the arc, the tangent of which is $=\omega$. [Discovered originally in 1671 by James Gregory.]

Let $x = 1$; there will be $\text{Atang}\, x=45^0$ and thus $u=45^0$, hence
$\sin u=\frac{1}{\sqrt2},\sin 2u=1,\sin 3u=\frac{1}{\sqrt2},\sin 4u=0,\sin 5u=-\frac{1}{\sqrt2},\sin 6u=-1,\sin 7\cdot u=-\frac{1}{\sqrt2},$
$\sin 8u=0,\sin 9u=\frac{1}{\sqrt2}$ etc.
From which there becomes

$$\text{Atang}(1+\omega)=45^0+\frac{\omega}{2}-\frac{\omega^2}{2\cdot 2}+\frac{\omega^3}{3\cdot 4}-\frac{\omega^5}{5\cdot 8}+\frac{\omega^6}{6\cdot 8}-\frac{\omega^7}{7\cdot 16}+\frac{\omega^9}{9\cdot 32}-\frac{\omega^{10}}{10\cdot 32}+\frac{\omega^{11}}{11\cdot 64}$$
$$-\frac{\omega^{13}}{13\cdot 128}+\frac{\omega^{14}}{14\cdot 128}-\text{etc.}$$

If there shall be put $\omega=-1$, on account of $\text{Atang}(1+\omega)=0$ and $45^0=\frac{\pi}{4}$ there becomes

$$\frac{\pi}{4}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 2^2}-\frac{1}{5\cdot 2^3}-\frac{1}{6\cdot 2^3}-\frac{1}{7\cdot 2^4}+\frac{1}{9\cdot 2^5}+\frac{1}{10\cdot 2^5}+\frac{1}{11\cdot 2^6}-\text{etc.}$$

which value if it may be substituted in place of the arc 45^0 in that expression, there will be

$$\text{Atang}(1+\omega)=\frac{\omega+1}{1\cdot 2}-\frac{\omega^2-1}{2\cdot 2}+\frac{\omega^3+1}{3\cdot 2^2}-\frac{\omega^5+1}{5\cdot 2^3}+\frac{\omega^6-1}{6\cdot 2^3}-\frac{\omega^7+1}{7\cdot 2^4}+\text{etc.}$$

Moreover that series is most suitable for finding the approximate value of $\frac{\pi}{4}$.

90. Since there shall be

$$\frac{\pi}{4}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 2^2}-\frac{1}{5\cdot 2^3}-\frac{1}{6\cdot 2^3}-\frac{1}{7\cdot 2^4}+\text{etc.}$$

but the terms having 2, 6, 10 etc. in the denominators

$$\frac{1}{2\cdot 2}-\frac{1}{6\cdot 2^3}+\frac{1}{10\cdot 2^5}-\frac{1}{14\cdot 2^7}+\text{etc.}$$

may express $\frac{1}{2}\text{Atang}\frac{1}{2}$, [from $\text{Atang}\,\omega=\frac{\omega}{1}-\frac{\omega^3}{3}+\frac{\omega^5}{5}-\frac{\omega^7}{7}+\frac{\omega^9}{9}-\frac{\omega^{11}}{11}+\text{etc.},$]
there will be

$$\frac{\pi}{4}=\frac{1}{2}\text{Atang}\frac{1}{2}+\frac{1}{1\cdot 2}+\frac{1}{3\cdot 2^2}-\frac{1}{5\cdot 2^3}-\frac{1}{7\cdot 2^4}+\frac{1}{9\cdot 2^5}+\frac{1}{11\cdot 2^6}-\text{etc.}$$

Moreover in the other formula on putting ω negative, since there shall be

$$\begin{aligned}\text{Atang}(1-\omega)=&\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 2^2}-\frac{1}{5\cdot 2^3}-\frac{1}{6\cdot 2^3}-\frac{1}{7\cdot 2^4}+\text{etc.}\\&-\frac{\omega}{1\cdot 2}-\frac{\omega^2}{2\cdot 2}-\frac{\omega^3}{3\cdot 2^2}+\frac{\omega^5}{5\cdot 2^3}+\frac{\omega^6}{6\cdot 2^3}+\frac{\omega^7}{7\cdot 2^4}-\text{etc.},\end{aligned}$$

if there is made $\omega=\frac{1}{2}$ there will be

$$\begin{aligned}\text{Atang}\frac{1}{2}=&\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 2^2}-\frac{1}{5\cdot 2^3}-\frac{1}{6\cdot 2^3}-\frac{1}{7\cdot 2^4}+\text{etc.}\\&-\frac{1}{1\cdot 2^2}-\frac{1}{2\cdot 2^3}-\frac{1}{3\cdot 2^5}+\frac{1}{5\cdot 2^8}+\frac{1}{6\cdot 2^9}+\frac{1}{7\cdot 2^{11}}-\text{etc.},\end{aligned}$$

but with the terms divided by 2, 6, 10 etc. taken separately there will be

$$\begin{aligned}\text{Atang}\frac{1}{2}=&\frac{1}{2}\text{Atang}\frac{1}{2}+\frac{1}{1\cdot 2}+\frac{1}{3\cdot 2^2}-\frac{1}{5\cdot 2^3}-\frac{1}{7\cdot 2^4}+\frac{1}{9\cdot 2^5}\text{etc.}\\&-\frac{1}{2}\text{Atang}\frac{1}{8}-\frac{1}{1\cdot 2^2}-\frac{1}{3\cdot 2^5}+\frac{1}{5\cdot 2^8}+\frac{1}{7\cdot 2^{11}}-\frac{1}{9\cdot 2^{14}}-\text{etc.},\end{aligned}$$

and thus

$$\begin{aligned}\text{Atang}\frac{1}{2}=&\frac{1}{1\cdot 2}+\frac{1}{3\cdot 2^2}-\frac{1}{5\cdot 2^3}-\frac{1}{7\cdot 2^4}+\text{etc.}\\&-\frac{1}{2}\text{Atang}\frac{1}{8}-\frac{1}{1\cdot 2^2}-\frac{1}{3\cdot 2^5}+\frac{1}{5\cdot 2^8}+\frac{1}{7\cdot 2^{11}}-\text{etc.};\end{aligned}$$

which value if it may be substituted in the above series and $\text{Atang}\frac{1}{8}$ itself be converted into a series, there may be found

$$\frac{\pi}{4} = \begin{cases} 1 + \frac{1}{3\cdot 2^1} - \frac{1}{5\cdot 2^2} - \frac{1}{7\cdot 2^3} + \frac{1}{9\cdot 2^4} + \text{etc.} \\ -\frac{1}{1\cdot 2^2} - \frac{1}{3\cdot 2^5} + \frac{1}{5\cdot 2^8} + \frac{1}{7\cdot 2^{11}} - \frac{1}{9\cdot 2^{14}} - \text{etc.} \\ -\frac{1}{1\cdot 2^4} + \frac{1}{3\cdot 2^{10}} - \frac{1}{5\cdot 2^{16}} + \frac{1}{7\cdot 2^{22}} - \frac{1}{9\cdot 2^{28}} + \text{etc.} \end{cases}$$

90a. These any many other series follow from putting in place $x = 1$; but if we may put $x = \sqrt{3}$, so that there shall be $\text{Atang}\, x = 60^0$, there becomes $u = 30^0$ and $\sin u = \frac{1}{2}$, $\sin 2u = \frac{\sqrt{3}}{2}$, $\sin 3u = 1$, $\sin 4u = \frac{\sqrt{3}}{2}$, $\sin 5u = \frac{1}{2}$, $\sin 6u = 0$, $\sin 7u = -\frac{1}{2}$ etc., from which there will be

$$\text{Atang}\left(\sqrt{3} + \omega\right) = 60^0 + \frac{\omega}{1\cdot 2^2} - \frac{\omega^2\sqrt{3}}{2\cdot 2^3} + \frac{\omega^3}{3\cdot 2^3} - \frac{\omega^4\sqrt{3}}{4\cdot 2^5} + \frac{\omega^5}{5\cdot 2^6}$$
$$- \frac{\omega^7}{7\cdot 2^8} + \frac{\omega^8\sqrt{3}}{8\cdot 2^9} - \frac{\omega^9}{9\cdot 2^9} + \frac{\omega^{10}\sqrt{3}}{10\cdot 2^{11}} - \frac{\omega^{11}}{11\cdot 2^{11}} + \text{etc.}$$

But if there may be put $x = \frac{1}{\sqrt{3}}$ so that there shall be $\text{Atang}\, x = 30^0$, there will be $u = 60^0$ and $\sin u = \frac{\sqrt{3}}{2}$, $\sin 2u = \frac{\sqrt{3}}{2}$, $\sin 3u = 0$, $\sin 4u = -\frac{\sqrt{3}}{2}$, $\sin 5u = -\frac{\sqrt{3}}{2}$, $\sin 6u = 0$, $\sin 7u = \frac{\sqrt{3}}{2}$ etc. with which values substituted there will be

$$\text{Atang}\left(\frac{1}{\sqrt{3}} + \omega\right) = 30^0 + \frac{3\omega}{1\cdot 2^2} - \frac{3\omega^2\sqrt{3}}{2\cdot 2^3} + \frac{3^2\omega^4\sqrt{3}}{4\cdot 2^5} - \frac{3^3\omega^5}{5\cdot 2^6} + \text{etc.}$$

therefore if there shall be $\omega = -\frac{1}{\sqrt{3}}$, on account of $30^0 = \frac{\pi}{6}$ there will be

$$\frac{\pi}{6\sqrt{3}} = \frac{1}{1\cdot 2^2} + \frac{1}{2\cdot 2^3} - \frac{1}{4\cdot 2^5} - \frac{1}{5\cdot 2^6} + \frac{1}{7\cdot 2^8} + \frac{1}{8\cdot 2^9} - \text{etc.}$$

91. We may resume the general expression found

$$\text{Atang}\left(x + \omega\right) = \text{Atang}\, x + \frac{\omega}{1}\sin u \cdot \sin u - \frac{\omega^2}{2}\sin^2 u \cdot \sin 2u$$
$$+ \frac{\omega^3}{3}\sin^3 u \cdot \sin 3u - \text{etc.};$$

and we may put $\omega = -x$, so that there shall be $\text{Atang}\left(x + \omega\right) = 0$, and there will be

$$\text{Atang}\, x = \frac{x}{1}\sin u \cdot \sin u + \frac{x^2}{2}\sin^2 u \cdot \sin 2u + \frac{x^3}{3}\sin^3 u \cdot \sin 3u + \text{etc.};$$

But since there shall be $\text{Atang}\,x = 90^0 - u = \frac{\pi}{2} - u$, there will be $x = \cot u = \frac{\cos u}{\sin u}$. On account of which there will be

$$\frac{\pi}{2} = u + \cos u \cdot \sin u + \tfrac{1}{2}\cos^2 u \cdot \sin 2u + \tfrac{1}{3}\cos^3 u \cdot \sin 3u + \tfrac{1}{4}\cos^4 u \cdot \sin 4u + \text{etc.},$$

which series therefore is noteworthy, because, whichever arc may be taken in place of u, the value of the series may be produced the same always $= \frac{\pi}{2}$.

But if there shall be $\omega = -2x$, on account of $\text{Atang}(-x) = -\text{Atang}\,x$ there becomes

$$2\text{Atang}\,x = \tfrac{2x}{1}\sin u \cdot \sin u + \tfrac{4x^2}{2}\sin^2 u \cdot \sin 2u + \tfrac{8x^3}{3}\sin^3 u \cdot \sin 3u + \text{etc.}$$

But since there shall be $\text{Atang}\,x = \frac{\pi}{2} - u$ and $x = \frac{\cos u}{\sin u}$, there will be

$$\pi = 2u + \tfrac{2}{1}\cos u \cdot \sin u + \tfrac{2^2}{2}\cos^2 u \cdot \sin 2u + \tfrac{2^3}{3}\cos^3 u \cdot \sin 3u + \text{etc.}$$

Let $u = 45^0 = \frac{\pi}{4}$, there will be

$$\cos u = \tfrac{1}{\sqrt{2}},\ \sin u = \tfrac{1}{\sqrt{2}},\ \sin 2u = 1,\ \sin 3u = \tfrac{1}{\sqrt{2}},\ \sin 4u = 0,\ \sin 5u = \tfrac{-1}{\sqrt{2}},$$
$$\sin 6u = -1,\ \sin 7u = \tfrac{-1}{\sqrt{2}},\ \sin 8u = 0,\ \sin 9u = \tfrac{1}{\sqrt{2}}\ \text{etc.}$$

and there will be

$$\frac{\pi}{2} = \frac{1}{1} + \frac{2}{2} + \frac{2}{3} - \frac{2^2}{5} - \frac{2^3}{6} - \frac{2^3}{7} + \frac{2^4}{9} + \frac{2^5}{10} + \frac{2^5}{11} - \text{etc.},$$

which series, even if it diverges, is noteworthy on account of simplicity.

92. In the general expression found there is put

$$\omega = -x - \frac{1}{x} = \frac{-1}{\sin u \cdot \cos u}$$

on account of $x = \frac{\cos u}{\sin u}$; there will be

$$\text{Atang}(x+\omega) = \text{Atang}\left(-\tfrac{1}{x}\right) = -\text{Atang}\left(\tfrac{1}{x}\right) = -\tfrac{\pi}{2} + \text{Atang}\,x.$$

Hence the following expression will be obtained

$$\frac{\pi}{2}=\frac{\sin u}{1\cos u}+\frac{\sin 2u}{2\cos^2 u}+\frac{\sin 3u}{3\cos^3 u}+\frac{\sin 4u}{4\cos^4 u}+\frac{\sin 5u}{5\cos^5 u}+\text{etc.},$$

which on putting $u=45^0$ gives the same series, which we found in the final place. But if we may put $\omega=-\sqrt{(1+xx)}$, on account of $x=\frac{\cos u}{\sin u}$ there becomes

$$\omega=-\frac{1}{\sin u}$$

and

$$\text{Atang}\left(x-\sqrt{(1+xx)}\right)=-\ \text{Atang}\left(\sqrt{(1+xx)}-x\right)$$
$$=-\tfrac{1}{2}\text{Atang}\tfrac{1}{x}=-\tfrac{1}{2}\left(\tfrac{\pi}{2}-\text{Atang}\,x\right)=-\tfrac{1}{2}u$$

and

$$\text{Atang}\,x=\tfrac{\pi}{2}-\tfrac{1}{2}u.$$

Hence on this account

$$\tfrac{\pi}{2}=\tfrac{1}{2}u+\tfrac{1}{1}\sin u+\tfrac{1}{2}\sin 2u+\tfrac{1}{3}\sin 3u+\tfrac{1}{4}\sin 4u+\text{etc.}$$

But if this equation may be differentiated, there will be

$$0=\tfrac{1}{2}+\cos u+\cos 2u+\cos 3u+\cos 4u+\cos 5u+\text{etc.},$$

an account of which is understood from the nature of the recurring series.
[This series cannot be true in general !]

93. If in a similar manner the series found before may be differentiated, new summable series may be found. And indeed in the first place from

$$\text{Atang}(1+\omega)=\tfrac{\pi}{4}+\tfrac{\omega}{2}-\tfrac{\omega^2}{2\cdot 2}+\tfrac{\omega^3}{3\cdot 4}-\tfrac{\omega^5}{5\cdot 8}+\tfrac{\omega^6}{6\cdot 8}-\text{etc.}$$

it follows

$$\frac{1}{2+2\omega+\omega^2}=\tfrac{1}{2}-\tfrac{\omega}{2}+\tfrac{\omega^2}{4}-\tfrac{\omega^4}{8}+\tfrac{\omega^5}{8}-\tfrac{\omega^6}{16}+\tfrac{\omega^8}{32}-\text{etc.}$$

which arises from the expansion of the fraction $\frac{2-2\omega+\omega^2}{4+\omega^4}=\frac{1}{2+2\omega+\omega^2}$.

Then this series

$$\tfrac{\pi}{2}=u+\cos u\cdot\sin u+\tfrac{1}{2}\cos^2 u\cdot\sin 2u+\tfrac{1}{3}\cos^3 u\cdot\sin 3u+\tfrac{1}{4}\cos^4 u\cdot\sin 4u+\text{etc.}$$

by differentiation will give

$$0 = 1 + \cos 2u + \cos u \cdot \cos 3u + \cos^2 u \cdot \cos 4u + \cos^3 u \cdot \cos 5u + \text{etc.}$$

And then the series

$$\frac{\pi}{2} = \frac{\sin u}{\cos u} + \frac{\sin 2u}{2\cos^2 u} + \frac{\sin 3u}{3\cos^3 u} + \frac{\sin 4u}{4\cos^4 u} + \text{etc.}$$

gives

$$0 = \frac{1}{\cos^2 u} + \frac{\cos u}{\cos^3 u} + \frac{\cos 2u}{\cos^4 u} + \frac{\cos 3u}{\cos^5 u} + \frac{\cos 4u}{\cos^5 u} + \text{etc.}$$

or

$$0 = 1 + \frac{\cos u}{\cos u} + \frac{\cos 2u}{\cos^2 u} + \frac{\cos 3u}{\cos^3 u} + \frac{\cos 4u}{\cos^4 u} + \frac{\cos 5u}{\cos^5 u} + \text{etc.}$$

94. But the expression found initially

$$\text{Atang}\left(x + \omega\right) = \text{Atang}\, x + \frac{\omega}{1}\sin u \cdot \sin u - \frac{\omega^2}{2}\sin^2 u \cdot \sin 2u + \frac{\omega^3}{3}\sin^3 u \cdot \sin 3u - \text{etc.}$$

with

$$x = \cot u \;\text{ or }\; u = \text{Acot}\, x = 90^0 - \text{Atang}\, x \text{ present}$$

will take care of the angle or the given arc for each corresponding tangent requiring to be found. Indeed let the proposed tangent $= t$ and there is sought in tables the tangent falling closest to that $= x$, to which there may correspond the arc $= y$, and there will be $u = 90^0 - y$. Then there may be put $x + \omega = t$ or $\omega = t - x$ and the arc sought will be

$$= y + \frac{\omega}{1}\sin u \cdot \sin u - \frac{\omega^2}{2}\sin^2 u \cdot \sin 2u + \text{etc.},$$

which rule then is not only especially useful, but also if the proposed tangent were very great and therefore the arc sought might disagree little from 90^0. For in these cases on account of the tangents increasing greatly the customary method of interpolation will depart excessively from the truth. Therefore let this example be proposed.

EXAMPLE

The arc is sought, of which the tangent shall be $= 100$ *on putting the radius* $= 1$.

The nearest for the arc sought is equal to $89^0\ 25^{\mathrm{I}}$, the tangent of which is

$$x = 98{,}217943$$

which taken from

$$t = 100,000000$$

there will remain

$$\omega = 1,782057$$

Then since there shall be $89^0\ 25^{\mathrm{I}}$ there will be $u = 0^0\,35^{\mathrm{I}}$, $2u = 1^0\,10^{\mathrm{I}}$, $3u = 1^0\,45^{\mathrm{I}}$ etc. Now the individual terms may be found by logarithms. To

$$l\omega = 0,2509215$$

there may be added [note that 10.000000 is added to avoid the use of negative logarithms.]

$$l\sin u = 8,0077867$$
$$l\sin u = 8,0077867$$
$$l\omega\sin u \cdot \sin u = 6,2664949$$

there may be taken

$$4,6855749$$
$$=1,5809200$$

Hence

$$\omega\sin u \cdot \sin u = 38,09956 \text{ seconds.}$$

To	$l\omega\sin^2 u = 6,2664949$
there is added	$l\omega = 0,2509215$
	$l\sin 2u = 8,3087941$
	$4,8262105$
taken away	$l2 = 0,3010300$

$$l\tfrac{1}{2}\omega^2\sin u.\ \sin 2u = 4,5251805$$

taken away	4,6855749
remains	9,8396056

Therefore $$\tfrac{1}{2}\omega^2\sin u.\ \sin 2u = 0,69120 \text{ seconds.}$$

Again to

	$l\omega^3 = 0,7527645$
added	$l\sin^3 u = 4,0233601$
	$l\sin 3u = 8,4848479$
	$3,2609725$
taken away	$l3 = 0,4771213$
	$2,7838512$

taken away	4,6855749
	8,0982763

Therefore $\frac{1}{2}\omega^3\sin^3u.\ \sin 3u = 0{,}01254$ seconds.

Then to	$l\omega^4 = 1{,}0036860$
there is added	$l\sin^4u = 2{,}0311468$
	$l\sin 4u = 8{,}6097341$
	1,6445669
taken away	$l4 = 0{,}6020600$
	1,0425069
taken away	
	4,6855749
	6,3569320

Therefore $\frac{1}{2}\omega^4\sin^4u.\ \sin 4u = 0{,}00023$ seconds.

Hence

	Terms to be added	terms to be subtracted
	38,09956	0,69120
	0,01254	0,00023
	38,11210	0,69143
subtracted	0,69143	
	37,42067=	$25^{\mathrm{I}}\ 37^{\mathrm{II}}\ 25^{\mathrm{III}}\ 14^{\mathrm{IV}}\ 24^{\mathrm{V}}\ 36^{\mathrm{V1}}$

On account of which the arc, the tangent of which exceeds the radius one hundred times, will be

$$89^0\ 25^{\mathrm{I}}\ 37^{\mathrm{II}}\ 25^{\mathrm{III}}\ 14^{\mathrm{IV}}\ 24^{\mathrm{V}}\ 36^{\mathrm{V1}}$$

nor does the error rise to the fourth order, but can be present only in the fifth, from which truly we will be able to state this angle $89^0\ 25^{\mathrm{I}}\ 37^{\mathrm{II}}\ 25^{\mathrm{III}}\ 14^{\mathrm{IV}}$. At this stage if a greater tangent may be proposed, even if perhaps ω may be produced more, yet on account of the smaller angle *u* at this stage the angle equally will be able to be defined readily.

95. Since here we have substituted for the arc *y* of the circle, now we may put the reciprocal functions in place of *y*, of which kind are the sin *x*, cos *x*, tang *x*, cot *x* etc. Therefore let there be $y = \sin x$ and on putting $x + \omega$ in place of x there becomes $z = \sin(x + \omega)$ and the equation

$$z = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \text{etc.}$$

on account of

$$\frac{dy}{dx} = \cos x, \quad \frac{ddy}{dx^2} = -\sin x, \quad \frac{d^3y}{dx^3} = -\cos x, \quad \frac{d^4y}{dx^4} = \sin x \text{ etc.},$$

will give

$$\sin(x+\omega)=\sin x+\omega\cos x-\tfrac{1}{2}\omega^2\sin x+\tfrac{1}{6}\omega^3\cos x+\tfrac{1}{24}\omega^4\sin x+\text{etc.}$$

and on taking ω negative there will be

$$\sin(x-\omega)=\sin x-\omega\cos x-\tfrac{1}{2}\omega^2\sin x+\tfrac{1}{6}\omega^3\cos x+\tfrac{1}{24}\omega^4\sin x-\text{etc.}$$

But if now there is put in place $y=\cos x$, on account of

$$\frac{dy}{dx}=-\sin x,\quad \frac{ddy}{dx^2}=-\cos x,\quad \frac{d^3y}{dx^3}=\sin x,\quad \frac{d^4y}{dx^4}=\cos x \text{ etc.}$$

there will be

$$\cos(x+\omega)=\cos x-\omega\sin x-\tfrac{1}{2}\omega^2\cos x+\tfrac{1}{6}\omega^3\sin x+\tfrac{1}{24}\omega^4\cos x-\text{etc.}$$

and from ω made negative there will be

$$\cos(x-\omega)=\cos x+\omega\sin x-\tfrac{1}{2}\omega^2\cos x-\tfrac{1}{6}\omega^3\sin x+\tfrac{1}{24}\omega^4\cos x+\text{etc.}$$

96. The uses of these formulas is excellent both in the building and interpolation of tables of sines and cosines. For if the sine and cosine of any arc x were known, from these by an easy calculation the sines and cosines of the angles $x+\omega$ and $x-\omega$ are able to be found, if indeed the difference ω were small enough ; for in this case the series found are strongly convergent. Truly for this it is necessary, that the arc ω may be expressed in parts of the radius ; which is done easily, since the arc 180^0 shall be

$$3,14159265358979323846\ ;$$

for there will be established on division by180 :

$$\text{the arc}\quad 1^0=0,01745329251994329576 9$$
$$\text{the arc}\quad 1^{\text{I}}=0,00029088820866572159 6$$
$$\text{the arc}\quad 10^{\text{II}}=0,00004848136811095359 9.$$

EXAMPLE 1

To find the sine and cosine of the angles $45^0 1^{\mathrm{I}}$ *and* $44^0 59^{\mathrm{I}}$ *from the given sine and cosine of the angle* 45^0 *, each of which is* $= \frac{1}{\sqrt{2}} = 0{,}7071067811865$.

Since there shall be

$$\sin x = \cos x = 0{,}7071067811865$$

and

$$\omega = 0{,}0002908882086$$

there will be so that the multiplications may be put in place easier

$$\begin{aligned} 2\omega &= 0{,}0005817764173 \\ 3\omega &= 0{,}0008726646259 \\ 4\omega &= 0{,}0011635528346 \\ 5\omega &= 0{,}0014544410433 \\ 6\omega &= 0{,}0017453292519 \\ 7\omega &= 0{,}0020362174606 \\ 8\omega &= 0{,}0023271056693 \\ 9\omega &= 0{,}0026179938779 \end{aligned}$$

Hence $\omega \sin x$ and $\omega \cos x$ may be found in this manner [Euler here abandons logs, and shows how perhaps he does the same calculations mentally, to a greater number of places than the log tables available] :

7	·	0,00020362174606
0	· · · · · · ·	
7	·	0,000203621746
1	· ·	2908882
0	· · · · · · ·	
6	· ·	174532
7	· ·	20362
8	· ·	2327
1	· ·	29
1	· ·	2
8	· ·	2
6	· ·	0

$$\omega\sin x = \omega\cos x = 0{,}00020568902488$$

Hence

$$\tfrac{1}{2}\omega\cos x = 0{,}00010284451244$$

per ω

1	·	0,000002908882		
0	·	· · · ·		
2	·	58177		
8	·	23271		
4	·	1163		
4	·	·	116	
5	·	·	14	

$$\tfrac{1}{2}\omega^2\cos x = 0{,}00000002991623$$

$$\tfrac{1}{6}\omega^2\cos x = 0{,}00000000997208$$

by ω

9	· 0,000000000261			
9	·	·	·	26
7	·	·	·	2

$$\tfrac{1}{6}\omega^3\cos x = 0{,}00000000000289.$$

Therefore for the sin $45^0\ 1^{\mathrm{I}}$ required to be found to

$$\sin x = 0{,}7071067811865$$

there may be added

$$\omega\cos x = \quad 2056890249$$

$$0{,}7073124702114$$

there may be subtracted

$$\tfrac{1}{2}\omega^2\sin x = \quad 299162$$

$$0{,}7073124402952$$

there may be subtracted

$$\tfrac{1}{6}\omega^3\cos x = \quad 29$$

$$\sin 45^0 1^{\mathrm{I}} = 0{,}7073124402923 = \cos 44^0 59^{\mathrm{I}}$$

But for the $\cos 45^0 1^{\mathrm{I}}$ requiring to be found from

$$\cos x = 0{,}7071067811865$$

there may be subtracted

$$\omega \sin x = \qquad 2056890249$$
$$\overline{\qquad 0{,}7069010921616}$$

there may be subtracted

$$\omega^2 \cos x = \qquad 299162$$
$$\overline{\qquad 0{,}7069010622454}$$

there may be added

$$\tfrac{1}{6}\omega^3 \sin x = \qquad 29$$
$$\overline{\cos 45^0 1^{\mathrm{I}} = 0{,}7069010622483 = \sin 44^0 59^{\mathrm{I}}}$$

EXAMPLE 2

From the given sine and cosine of the arc $67^0 30^{\mathrm{I}}$, *to find the sine and cosine of the arcs* $67^0 31^{\mathrm{I}}$ *and* $67^0 29^{\mathrm{I}}$.

We may resolve this calculation in decimal fractions only to 7 places, as the common tables are accustomed to be constructed, and thus the calculation may be easily put in place by logarithms. Since there shall be $x = 67^0 30^{\mathrm{I}}$ and $\omega = 0{,}000290888$, there will be

$$l\omega = 6{,}4637259$$

and

$$l\sin x = 9{,}9656153 \qquad l\cos x = 9{,}5828397$$
$$l\omega = 6{,}4637259 \qquad l\omega = 6{,}4637259$$
$$l\omega\sin x = 6{,}4293412 \qquad l\omega\sin x = 6{,}4293412$$

$$l\tfrac{1}{2}\omega = 6{,}1626959 \qquad l\tfrac{1}{2}\omega = 6{,}1626959$$
$$l\tfrac{1}{2}\omega^2 \sin x = 2{,}5920371 \qquad l\tfrac{1}{2}\omega^2 \cos x = 2{,}2092615$$

hence

$$\omega \sin x = 0{,}00026874 \qquad \omega \cos x = 0{,}00011132$$
$$\tfrac{1}{2}\omega^2 \sin x = 0{,}00000004 \qquad \tfrac{1}{2}\omega^2 \cos x = 0{,}00000001$$

from which there becomes

$$\sin 67^0 31^{\mathrm{I}} = 0{,}9239908 \qquad \cos 67^0 31^{\mathrm{I}} = 0{,}3824147$$
$$\sin 67^0 29^{\mathrm{I}} = 0{,}9237681 \qquad \cos 67^0 29^{\mathrm{I}} = 0{,}3829522$$

where there was no need for either of the terms $\tfrac{1}{2}\omega^2 \sin x$ et $\tfrac{1}{2}\omega^2 \cos x$.

97. From the series, which we have found above,

$$\sin(x+\omega)=\sin x+\omega\cos x-\tfrac{1}{2}\omega^2\sin x+\tfrac{1}{6}\omega^3\cos x+\tfrac{1}{24}\omega^4\sin x+\text{etc.}$$
$$\cos(x+\omega)=\cos x-\omega\sin x-\tfrac{1}{2}\omega^2\cos x+\tfrac{1}{6}\omega^3\sin x+\tfrac{1}{24}\omega^4\cos x-\text{etc.}$$
$$\sin(x-\omega)=\sin x-\omega\cos x-\tfrac{1}{2}\omega^2\sin x+\tfrac{1}{6}\omega^3\cos x+\tfrac{1}{24}\omega^4\sin x-\text{etc.}$$
$$\cos(x-\omega)=\cos x+\omega\sin x-\tfrac{1}{2}\omega^2\cos x-\tfrac{1}{6}\omega^3\sin x+\tfrac{1}{24}\omega^4\cos x+\text{etc.}$$

there follows from combination to be

$$\tfrac{\sin(x+\omega)+\sin(x-\omega)}{2}$$
$$=\sin x-\tfrac{1}{2}\omega^2\sin x+\tfrac{1}{24}\omega^4\sin x-\tfrac{1}{720}\omega^6\sin x+\text{etc.}=\sin x\cdot\cos\omega$$

and

$$\tfrac{\sin(x+\omega)-\sin(x-\omega)}{2}$$
$$=\omega\cos x-\tfrac{1}{6}\omega^3\cos x+\tfrac{1}{120}\omega^5\cos x-\tfrac{1}{5040}\omega^7\cos x+\text{etc.}=\cos x\cdot\sin\omega,$$

from which the series for sines and cosine emerges now found previously

$$\cos\omega=1-\tfrac{1}{2}\omega^2+\tfrac{1}{24}\omega^4-\tfrac{1}{720}+\text{etc.}$$
$$\sin\omega=\omega-\tfrac{1}{6}\omega^3+\tfrac{1}{120}\omega^5-\tfrac{1}{5040}\omega^7+\text{etc.},$$

which same series follow from the first on putting $x=0$; since indeed there shall be $\cos x=1$ and $\sin x=0$, the first series will show $\sin\omega$, truly the second $\cos\omega$.

98. Now we may put also $y=\operatorname{tang}x$, so that there shall be $z=\operatorname{tang}(x+\omega)$*;* there will be on account of $y=\frac{\sin x}{\cos x}$

$$\frac{dy}{dx}=\frac{1}{\cos^2 x},\quad \frac{ddy}{2dx^2}=\frac{\sin x}{\cos^3 x},\quad \frac{d^3y}{2dx^3}=\frac{1}{\cos^2 x}+\frac{3\sin^2 x}{\cos^4 x}=\frac{1}{\cos^4 x}-\frac{2}{\cos^2 x},$$
$$\frac{d^4y}{2\cdot 4dx^4}=\frac{3\sin x}{\cos^5 x}-\frac{\sin x}{\cos^3 x},\quad \frac{d^5y}{2\cdot 4dx^5}=\frac{15}{\cos^6 x}-\frac{15}{\cos^4 x}+\frac{2}{\cos^2 x},$$

from which there follows to be

$$z = \operatorname{tang}(x+\omega) = \operatorname{tang} x + \begin{cases} \frac{\omega}{\cos^2 x} + \frac{\omega^2 \sin x}{\cos^3 x} + \frac{\omega^3}{\cos^4 x} + \frac{\omega^4 \sin x}{\cos^5 x} + \text{etc.} \\ \qquad - \frac{2\omega^3}{3\cos^2 x} - \frac{\omega^4 \sin x}{3\cos^3 x} - \text{etc.} \end{cases}$$

with the aid of which formulas the tangents of nearby angles will be able to be found. Now because the above series is geometric, these may be gathered into one sum and there will be

$$z = \operatorname{tang}(x+\omega) = \operatorname{tang} x + \frac{\omega + \omega^2 \operatorname{tang} x}{\cos^2 x - \omega^2} - \frac{2\omega^3}{3\cos^2 x} - \frac{\omega^4 \sin x}{3\cos^3 x} - \text{etc.}$$

or

$$z = \operatorname{tang}(x+\omega) = \operatorname{tang} x + \frac{\sin x \cdot \cos x + \omega}{\cos^2 x - \omega^2} - \frac{2\omega^3}{3\cos^2 x} - \frac{\omega^4}{3\cos^3 x} - \text{etc.},$$

which formula may be used in this final form.

99. Similar expressions for the logarithms of the sines, cosines, and tangents are able to be found also. Indeed let y = logarithm of the of the angle x, because we may thus express

$y = l \sin x$, and $z = l \sin(x+\omega)$ on account of

$\frac{dy}{dx} = \frac{n\cos x}{\sin x}$ there will be $\frac{ddy}{dx^2} = \frac{-n}{\sin^2 x}$, $\frac{d^3 y}{2dx^3} = \frac{2n\cos x}{\sin^3 x}$ etc. from which there becomes

$$z = l\sin(x+\omega) = l\sin x + \frac{n\omega \cos x}{\sin x} - \frac{n\omega^2}{2\sin^2 x} + \frac{n\omega^3 \cos x}{3\sin^3 x} - \text{etc.},$$

where n denotes the number, by which the hyperbolic logarithms must be multiplied, so that the proposed logarithms may be produced. But if there shall be $y = l \operatorname{tang} x$ and $z = l \operatorname{tang}(x+\omega)$, there becomes

$$\frac{dy}{dx} = \frac{n}{\cos x \cdot \sin x} = \frac{2n}{\sin 2x}, \quad \frac{ddy}{2dx^2} = \frac{-2n\cos 2x}{(\sin 2x)^2} \text{ etc.}$$

and thus

$$z = l \operatorname{tang}(x+\omega) = l \operatorname{tang} x \frac{dy}{dx} + \frac{2n\omega}{\sin 2x} - \frac{2n\omega^2 \cos 2x}{(\sin 2x)^2} + \text{etc.},$$

with the help of which formulas the logarithms of the sines and tangents can be interpolated.

100. We may put y to denote the arc, of which the logarithm of the sine shall be $= x$, or so that there shall be $y = \mathrm{A}.l\sin x$, [we regard this notation as representing the inverse of the logsin function] and z to be the arc, of which the logarithm of the sine shall be $= x+\omega$ or $z = \mathrm{A}.l\sin(x+\omega)$; there will be $x = l\sin y$ and

$$\frac{dx}{dy} = \frac{n\cos y}{\sin y}, \text{ from which } \frac{dy}{dx} = \frac{\sin y}{n\cos y};$$

there will be

$$\frac{ddy}{dx} = \frac{dy}{n\cos^2 y} = \frac{dx\sin y}{n^2\cos^3 y}, \quad \text{hence} \quad \frac{ddy}{dx^2} = \frac{\sin y}{n^2\cos^3 y}.$$

Consequently

$$z = y + \frac{\omega\sin y}{n\cos y} + \frac{\omega^2\sin y}{2n^2\cos^3 y} + \text{etc.}$$

In a similar manner, if the logarithm of the cosine were given, the expression may be found.

But if there shall be $y = \mathrm{A}.l\,\mathrm{tang}\,x$ and $z = \mathrm{A}.l\,\mathrm{tang}(x+\omega)$, since there shall be $x = l\,\mathrm{tang}\,y$, there becomes

$$\frac{dx}{dy} = \frac{n}{\sin y\cdot\cos y} \text{ and } \frac{dy}{dx} = \frac{\sin y\cdot\cos y}{n} = \frac{\sin 2y}{2n},$$

and

$$\frac{ddy}{dx^2} = \frac{\sin 2y\cdot\cos 2y}{2nn} = \frac{\sin 4y}{4nn}, \quad \frac{d^3y}{dx^3} = \frac{\sin 2y\cdot\cos 4y}{2n^3} \text{ etc. };$$

hence

$$z = y + \frac{\omega\sin 2y}{2n} + \frac{\omega^2\sin 2y\cdot\cos 2y}{4nn} + \frac{\omega^3\sin 2y\cdot\cos 4y}{12n^3} + \text{etc.}$$

101. Because the use of these expressions in the construction from tables of the logarithms of sines and tangents may be easily seen from the preceding, we will not spend a long time with these. Hence we will consider at this point the value of this kind $y = e^x\sin nx$ and there shall be $z = e^{x+\omega}\sin n(x+\omega)$; because there is

$$\frac{dy}{dx} = e^x(\sin nx + n\sin nx)$$
$$\frac{ddy}{dx^2} = e^x((1-nn)\sin nx + 2n\cos nx)$$
$$\frac{d^3y}{dx^3} = e^x((1-3nn)\sin nx + n(3-2nn)\cos nx)$$
$$\frac{d^4y}{dx^4} = e^x((1-6nn+n^4)\sin nx + n(4-4nn)\cos nx)$$
$$\frac{d^5y}{dx^5} = e^x((1-10nn+5n^4)\sin nx + n(5-10nn+n^4)\cos nx),$$

etc.,

with these substituted and with division by e^{x} put in place there will be

$$\begin{aligned}&e^{\omega}\sin n(x+\omega)=\sin nx\\&+\omega\sin nx+\tfrac{1-nn}{2}\omega^{2}\sin nx+\tfrac{1-3nn}{6}\omega^{3}\sin nx+\tfrac{1-6nn+n^{4}}{24}\omega^{4}\sin nx+\text{etc.}\\&+n\omega\cos nx+\tfrac{2n}{2}\omega^{2}\cos nx+\frac{n(3-2nn)}{6}\omega^{3}\cos nx+\frac{n(4-4nn)}{24}\omega^{4}\cos nx+\text{ etc.}\end{aligned}$$

102. Hence many outstanding corollaries are able to be deduced ; but it may suffice for us to have noted these. If there were $x=0$, there will be

$$e^{\omega}\sin n\omega=n\omega+\tfrac{2n}{2}\omega^{2}+\frac{n(3-2nn)}{6}\omega^{3}+\frac{n(4-4nn)}{24}\omega^{4}+\frac{n\left(5-10n^{2}+n^{4}\right)}{120}\omega^{5}+\text{ etc.}$$

If there shall be $\omega=-x$, on account of $\sin n(x+\omega)=0$, there will be

$$\operatorname{tang} nx=\frac{nx-\frac{2n}{2}x^{2}-\frac{n(3-2nn)}{6}x^{3}-\frac{n(4-4nn)}{24}x^{4}-\frac{n\left(5-10n^{2}+n^{4}\right)}{120}x^{5}-\text{ etc.}}{1-x+\frac{1-nn}{2}x^{2}-\frac{1-3nn}{6}x^{3}+\frac{1-6nn+n^{4}}{24}x^{4}-\text{etc.}}$$

Generally indeed, if there shall be $n=1$, there will be had

$$\begin{aligned}&e^{\omega}\sin(x+\omega)=\sin x\left(1+\omega-\tfrac{1}{3}\omega^{3}-\tfrac{1}{6}\omega^{4}-\tfrac{1}{30}\omega^{5}+\tfrac{1}{630}\omega^{7}+\text{etc.}\right)\\&+\omega\cos x\left(1+\omega+\tfrac{1}{3}\omega^{3}-\tfrac{1}{30}\omega^{4}-\tfrac{1}{90}\omega^{5}-\tfrac{1}{630}\omega^{7}+\text{etc}\right).\end{aligned}$$

But if there shall be $n=0$, on account of $\sin n(x+\omega)=n(x+\omega)$ and $\sin nx=nx$ and $\cos nx=1$, if everywhere it may be divided by *n*, there will be produced

$$\begin{aligned}&e^{\omega}(x+\omega)=x+\omega x+\tfrac{1}{2}\omega^{2}+\tfrac{1}{6}\omega^{3}x+\tfrac{1}{24}\omega^{4}x+\text{etc.}\\&+\omega+\omega^{2}+\tfrac{1}{2}\omega^{3}+\tfrac{1}{6}\omega^{4}+\tfrac{1}{24}\omega^{5}+\text{etc.},\end{aligned}$$

the account of which series is clear.

CAPUT IV

DE CONVERSIONE FUNCTIONUM IN SERIES

70. In capite superiori iam ex parte ostensus est usus, quem expressiones generales ibi pro differentiis finitis inventae habent in investigatione serierum, quae valorem cuiusque functionis ipsius *x* exhibeant. Si enim *y* fuerit functio data ipsius *x*, eius valor, quem induit posito $x=0$, erit cognitus; hicque si ponatur $=A$, erit, uti invenimus,

$$y-\frac{xdy}{1dx}+\frac{x^2ddy}{1\cdot 2dx^2}-\frac{x^3d^3y}{1\cdot 2\cdot 3dx^3}+\frac{x^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4}-\text{etc.}=A.$$

Hinc ergo non solum habemus seriem plerumque in infinitum excurrentem, cuius summa aequetur quantitati constanti *A*, etiamsi in singulis terminis sit quantitas variabilis *x*, sed etiam ipsam functionem *y* per seriem exprimere poterimus; erit enim

$$y=A+\frac{xdy}{1dx}-\frac{x^2ddy}{1\cdot 2dx^2}+\frac{x^3d^3y}{1\cdot 2\cdot 3dx^3}-\frac{x^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4}+\text{etc.},$$

cuius exempla iam aliquot sunt allata.

71. Quo autem haec investigatio latius pateat, ponamus functionem *y* abire in *z*, si loco *x* ubique scribatur $x+\omega$, ita ut *z* talis sit functio ipsius $x+\omega$, qualis *y* est ipsius *x*, atque ostendimus [§ 48] fore

$$z=y+\frac{\omega dy}{1dx}+\frac{\omega^2ddy}{1\cdot 2dx^2}+\frac{\omega^3d^3y}{1\cdot 2\cdot 3dx^3}+\frac{\omega^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4}+\text{etc.}$$

Cum igitur huius seriei singuli termini per continuam ipsius *y* differentiationem ponendo *dx* constans inveniri simulque valor ipsius *z* per substitutionem $x+\omega$ in locum ipsius *x* actu exhiberi queat, hoc modo perpetuo obtinebitur series valori ipsius *z* aequalis, quae, si ω fuerit quantitas vehementer parva, maxime convergit atque non admodum multis terminis capiendis valorem ipsius *z* proxime verum praebebit. Ex quo huius formulae in negotio approximationum uberrimus erit usus.

72. Ut igitur in insigni huius formulae usu ostendendo ordine procedamus, substituamus primo in locum ipsius *y* functiones ipsius *x* algebraicas. Ac primo quidem sit $y=x^n$ eritque, si $x+\omega$ loco *x* ponatur, $z=(x+\omega)^n$.
Cum igitur sit

$$\frac{dy}{dx} = nx^{n-1}, \quad \frac{ddy}{dx^2} = n(n-1)x^{n-2}, \quad \frac{d^3y}{dx^3} = n(n-1)(\mathrm{n}-2)x^{n-3},$$

$$\frac{d^4y}{dx^4} = n(n-1)(\mathrm{n}-2)(\mathrm{n}-3)x^{n-4} \quad \text{etc.},$$

his valoribus substitutis fiet

$$(x+\omega)^n = x^n + \frac{n}{1}x^{n-1}\omega + \frac{n(n-1)}{1\cdot 2}x^{n-2}\omega^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}\omega^3 + \text{etc.},$$

quae est notissima expressio NEUTONIANA, qua potestas binomii $(x+\omega)^n$ in seriem convertitur. Huiusque seriei terminorum numerus semper est finitus, si *n* fuerit numerus integer affirmativus.

73. Poterimus hinc quoque progressionem invenire, quae valorem potestatis binomii ita exprimat, ut ea abrumpatur, quoties exponens potestatis fuerit numerus negativus. Statuamus enim

$$\omega = \frac{-ux}{x+u};$$

erit

$$z = (x+\omega)^n = \left(\frac{xx}{x+u}\right)^n$$

ideoque habebitur

$$\frac{x^{2n}}{(x+u)^n} = x^n - \frac{nx^nu}{1(x+u)} + \frac{n(n-1)x^nu^2}{1\cdot 2(x+u)^2} - \frac{n(n-1)(n-2)x^nu^3}{1\cdot 2\cdot 3(x+u)^3} + \text{etc.}$$

Dividatur ubique per x^{2n} eritque

$$(x+u)^{-n} = x^{-n} - \frac{nx^{-n}u}{1(x+u)} + \frac{n(n-1)x^{-n}u^2}{1\cdot 2(x+u)^2} - \frac{n(n-1)(n-2)x^{-n}u^3}{1\cdot 2\cdot 3(x+u)^3} + \text{etc.}$$

Ponatur nunc $-n = m$ prodibitque

$$(x+u)^m = x^m + \frac{mx^mu}{1(x+u)} + \frac{m(m+1)x^mu^2}{1\cdot 2(x+u)^2} + \frac{m(m+1)(m+2)x^mu^3}{1\cdot 2\cdot 3(x+u)^3} + \text{etc.},$$

quae series, quoties *m* est numerus integer negativus, finito terminorum numero constabit. Haec igitur series aequalis est primum inventae, si pro ω et *n* scribantur *u* et *m;* erit enim inde

$$(x+u)^m = x^m + \frac{mx^{m-1}u}{1} + \frac{m(m-1)x^{m-2}u^2}{1\cdot 2} + \frac{m(m-1)(m-2)x^{m-3}u^3}{1\cdot 2\cdot 3} + \text{etc.}$$

74. Haec eadem series quoque deduci potest ex expressione initio § 70 data. Cum enim, si posito $x=0$ abeat *y* in *A*, sit

$$y-\frac{xdy}{1dx}+\frac{x^2ddy}{1\cdot 2dx^2}-\frac{x^3d^3y}{1\cdot 2\cdot 3dx^3}+\frac{x^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4}-\text{etc.}=A,$$

ponatur $y=(x+a)^n$ eritque $A=a^n$ et ob

$$\frac{dy}{dx}=n(x+a)^{n-1},\quad \frac{ddy}{dx^2}=n(n-1)(x+a)^{n-2},\quad \frac{d^3y}{dx^3}=n(n-1)(\mathrm{n}-2)(x+a)^{n-3}\text{etc.}$$

fiet

$$(x+a)^n-\tfrac{n}{1}x(x+a)^{n-1}+\tfrac{n(n-1)}{1\cdot 2}x^2(x+a)^{n-2}-\text{etc.}=a^n;$$

dividatur per $a^n(x+a)^n$ atque prodibit

$$(x+a)^{-n}=a^{-n}-\frac{na^{-n}x}{1(x+a)}+\frac{n(n-1)a^{-n}x^2}{1\cdot 2(x+a)}-\text{etc.},$$

quae positis respective *u*, *x* et – *m* pro *x*, *a* et *n* orietur series ante inventa.

75. Si pro *m* statuantur numeri fracti, ambae series in infinitum excurrent, interim tamen, si *u* prae *x* fuerit quantitas valde parva, vehementer ad verum valorem convergent. Sit igitur $m=\frac{\mu}{\nu}$ et $x=a^\nu$; erit ex serie primum inventa

$$\left(a^\nu+u\right)^{\frac{\mu}{\nu}}=a^\mu\left(1+\frac{\mu}{\nu}\cdot\frac{u}{a^\nu}+\frac{\mu(\mu-\nu)}{\nu\cdot 2\nu}\cdot\frac{u^2}{a^{2\nu}}+\frac{\mu(\mu-\nu)(\mu-2\nu)}{\nu\cdot 2\nu\cdot 3\nu}\cdot\frac{u^3}{a^{3\nu}}+\text{etc.}\right).$$

Series autem posterius inventa dabit

$$\left(a^\nu+u\right)^{\frac{\mu}{\nu}}=a^\mu\left(1+\frac{\mu u}{\nu\left(a^\nu+u\right)}+\frac{\mu(\mu+\nu)u^2}{\nu\cdot 2\nu\left(a^\nu+u\right)^2}+\frac{\mu(\mu+\nu)(\mu+2\nu)u^3}{\nu\cdot 2\nu\cdot 3\nu\left(a^\nu+u\right)^3}+\text{etc.}\right).$$

Haec autem posterior series magis convergit quam prior, cum eius termini etiam decrescant, si fuerit $u>a^\nu$, quo casu tamen prior series divergit.

Si igitur sit $\mu = 1$, $v = 2$, erit

$$\sqrt{(a^2+u)} = a\left(1 + \frac{1u}{2(a^2+u)} + \frac{1\cdot3u^2}{2\cdot4(a^2+u)^2} + \frac{1\cdot3\cdot5u^3}{2\cdot4\cdot6(a^2+u)^3} + \text{etc.}\right).$$

Simili modo pro *v* ponendo numeros 3, 4, 5 etc., manente $\mu = 1$, erit

$$\sqrt[3]{(a^3+u)} = a\left(1 + \frac{1u}{3(a^3+u)} + \frac{1\cdot4u^2}{3\cdot6(a^3+u)^2} + \frac{1\cdot4\cdot7u^3}{3\cdot6\cdot9(a^3+u)^3} + \text{etc.}\right)$$

$$\sqrt[4]{(a^4+u)} = a\left(1 + \frac{1u}{4(a^4+u)} + \frac{1\cdot5u^2}{4\cdot8(a^4+u)^2} + \frac{1\cdot5\cdot9u^3}{4\cdot8\cdot12(a^4+u)^3} + \text{etc.}\right)$$

$$\sqrt[5]{(a^5+u)} = a\left(1 + \frac{1u}{5(a^5+u)} + \frac{1\cdot6u^2}{5\cdot10(a^5+u)^2} + \frac{1\cdot6\cdot11u^3}{5\cdot10\cdot15(a^3+u)^3} + \text{etc.}\right)$$

76. Ex his ergo formulis facile cuiusque numeri propositi radix cuiusvis potestatis inveniri poterit. Proposito enim numero *c* quaeratur potestas ei proxima, sive maior sive minor; priori casu *u* fiet numerus negativus, posteriori affirmativus. Quodsi vero series resultans non satis convergere videatur, multiplicetur numerus *c* per quampiam potestatem, puta per f^v, si radix dignitatis *v* extrahi debeat, et quaeratur numeri $f^v c$ radix, quae per *f* divisa dabit radicem numeri *c* quaesitam. Quo maior autem accipitur numerus *f*, eo magis series converget idque imprimis, si quaepiam similis potestas a^v non multum ab $f^v c$ discrepet.

EXEMPLUM 1

Quaeratur radix quadrata ex numero 2.

Si sine ulteriori praeparatione ponatur $a = 1$ et $u = 1$, fiet

$$\sqrt{2} = 1 + \frac{1}{2\cdot2} + \frac{1\cdot3}{2\cdot4\cdot2^2} + \frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot2^3} + \text{etc.};$$

quae etsi iam satis convergit, tamen praestabit numerum 2 ante per quadratum quodpiam, uti 25, multiplicare, ut productum 50 ab alio quadrato 49 minime discrepet. Hanc ob rem quaeratur radix quadrata ex 50, quae per 5 divisa dabit $\sqrt{2}$. Erit autem tum *a* = 7 et *u* = 1, unde fiet

$$\sqrt{50} = 5\sqrt{2} = 7\left(1 + \tfrac{1}{2\cdot 50} + \tfrac{1\cdot 3}{2\cdot 4\cdot 50^2} + \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 50^3} + \text{etc.}\right)$$

seu

$$\sqrt{2} = \tfrac{7}{5}\left(1 + \tfrac{1}{100} + \tfrac{1\cdot 3}{100\cdot 200} + \tfrac{1\cdot 3\cdot 5}{100\cdot 200\cdot 300} + \text{etc.}\right),$$

quae ad computum in fractionibus decimalibus instituendum est aptissima.
Erit enim

$\tfrac{7}{5} =$	1,4000000000000
$\tfrac{7}{5}\cdot\tfrac{1}{100} =$	140000000000
$\tfrac{7}{5}\cdot\tfrac{1}{100}\cdot\tfrac{3}{200} =$	2100000000
$\tfrac{7}{5}\cdot\tfrac{1}{100}\cdot\tfrac{3}{200}\cdot\tfrac{5}{300} =$	35000000
praec. in $\tfrac{7}{400} =$	612500
praec. in $\tfrac{9}{500} =$	11025
praec. in $\tfrac{11}{600} =$	202
praec. in $\tfrac{13}{700} =$	3
Ergo	
$\sqrt{2} =$	1,4142135623730

EXEMPLUM 2

Quaeratur radix cubica ex 3.

Multiplicetur 3 per cubum 8 et quaeratur radix cubica ex 24; erit enim $\sqrt[3]{24} = 2\sqrt[3]{3}$. Ponatur ergo $a = 3$ et $u = -3$ eritque

$$\sqrt[3]{24} = 3\left(1 - \tfrac{1\cdot 3}{3\cdot 24} + \tfrac{1\cdot 4\cdot 3^2}{3\cdot 6\cdot 24^2} - \tfrac{1\cdot 4\cdot 7\cdot 3^3}{3\cdot 6\cdot 9\cdot 24^3} + \text{etc.}\right)$$

et

$$\sqrt[3]{3} = \tfrac{3}{2}\left(1 - \tfrac{1}{3\cdot 8} + \tfrac{1\cdot 4}{3\cdot 6\cdot 8^2} - \tfrac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9\cdot 8^3} + \text{etc.}\right)$$

seu

$$\sqrt[3]{3} = \tfrac{3}{2}\left(1 - \tfrac{1}{24} + \tfrac{1}{24}\cdot\tfrac{4}{48} - \tfrac{1}{24}\cdot\tfrac{4}{48}\tfrac{7}{72} + \text{etc.}\right),$$

quae series iam vehementer convergit, cum quilibet terminus plus quam octies minor sit praecedente. Sin autem 3 multiplicetur per cubum 729, fiet 2187 et $\sqrt[3]{2187} = \sqrt[3]{\left(13^3 - 10\right)} = 9\sqrt[3]{3}$. Erit ergo ob $a = 13$ et $u = -10$

$$\sqrt[3]{3}=\tfrac{13}{3}\left(1-\tfrac{1\cdot 10}{3\cdot 2187}+\tfrac{1\cdot 4\cdot 10^2}{3\cdot 6\cdot 2187^2}-\tfrac{1\cdot 4\cdot 7\cdot 10^3}{3\cdot 6\cdot 9\cdot 2187^3}+\text{etc.}\right),$$

cuius quivis terminus plus quam ducenties minor est quam praecedens.

77. Evolutio binomii potestatis tam late patet, ut omnes functiones algebraicae in ea comprehendi queant. Si enim verbi gratia quaeratur valor huius functionis $\sqrt{(a+2bx+cxx)}$ per seriem expressus, hoc per praecedentes formulas, duos terminos tanquam unum considerando, fieri poterit. Deinde vero haec explicatio fieri poterit ope expressionis primum traditae; nam si ponatur $\sqrt{(a+2bx+cxx)}=y$, quia posito $x=0$ fit $y=\sqrt{a}$, erit $A=\sqrt{a}$, et cum differentialia ipsius y ita se habeant

$$\frac{dy}{dx}=\frac{b+cx}{\sqrt{(a+2bx+cxx)}},\quad \frac{ddy}{dx^2}=\frac{ac-bb}{(a+2bx+cxx)^{\frac{3}{2}}},\quad \frac{d^3y}{dx^3}=\frac{3(bb-ac)(b+cx)}{(a+2bx+cxx)^{\frac{5}{2}}},$$
$$\frac{d^4y}{dx^4}=\frac{3(bb-ac)(ac-5bb-8bcx-4ccxx)}{(a+2bx+cxx)^{\frac{7}{2}}}\quad \text{etc.},$$

ex his ergo obtinebitur

$$\sqrt{(a+2bx+cxx)}-\frac{(b+cx)x}{\sqrt{(a+2bx+cxx)}}-\frac{(bb-ac)xx}{2(a+2bx+cxx)^{\frac{3}{2}}}$$
$$-\frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^{\frac{5}{2}}}-\frac{(bb-ac)(5bb-ac+8bcx+4ccxx)x^4}{8(a+2bx+cxx)^{\frac{7}{2}}}-\text{etc.}=\sqrt{a}.$$

Quodsi ergo ubique per $\sqrt{(a+2bx+cxx)}$ multiplicetur, series fiet rationalis eritque

$$\sqrt{a(a+2bx+cxx)}=a+2bx+cxx-(b+cx)x-\frac{(bb-ac)xx}{2(a+2bx+cxx)}$$
$$-\frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^2}-\frac{(bb-ac)(5bb-ac+8bcx+4ccxx)x^4}{8(a+2bx+cxx)3}-\text{etc.}$$

sive

$$\sqrt{(a+2bx+cxx)}=\sqrt{a}+\frac{bx}{\sqrt{a}}-\frac{(bb-ac)xx}{2(a+2bx+cxx)\sqrt{a}}-\frac{(bb-ac)(b+cx)x^3}{2(a+2bx+cxx)^2\sqrt{a}}-\text{etc.}$$

78. Transeamus ergo ad functiones transcendentes, quas loco y substituamus. Sit itaque primum $y = lx$ ac posito $x+\omega$ loco x fiet $z = l(x+\omega)$. Sint autem hi logarithmi quicunque, qui ad hyperbolicos rationem teneant $n:1$, eritque pro logarithmis hyperbolicis $n = 1$ et pro tabularibus erit $n = 0,4342944819032$. Hinc differentialia ipsius $y = lx$ erunt

$$\frac{dy}{dx} = \frac{n}{x}, \quad \frac{ddy}{dx^2} = -\frac{n}{x^2}, \quad \frac{d^3y}{dx^3} = \frac{2n}{x^3} \text{ etc.},$$

ex quibus conficitur

$$z = l(x+\omega) = lx + \frac{n\omega}{x} - \frac{n\omega^2}{2x^2} + \frac{n\omega^3}{3x^3} - \frac{n\omega^4}{4x^4} + \text{ etc.}$$

Simili modo, si ω statuatur negativum, erit

$$z = l(x-\omega) = lx - \frac{n\omega}{x} - \frac{n\omega^2}{2x^2} - \frac{n\omega^3}{3x^3} - \frac{n\omega^4}{4x^4} - \text{ etc.}$$

Quodsi ergo haec series a priori subtrahatur, fiet

$$l\,\frac{x+\omega}{x-\omega} = 2n\left(\frac{\omega}{x} + \frac{\omega^3}{3x^3} + \frac{\omega^5}{5x^5} + \frac{\omega^7}{7x^7} + \text{ etc.}\right).$$

79. Si in serie primum inventa

$$l(x+\omega) = lx + \frac{n\omega}{x} - \frac{n\omega^2}{2x^2} + \frac{n\omega^3}{3x^3} - \frac{n\omega^4}{4x^4} + \text{ etc.}$$

ponatur

$$\omega = \frac{xx}{u-x}$$

erit $x+\omega = \frac{ux}{u-x}$ et

$$l(x+\omega) = lu + lx - l(u-x) = lx + \frac{nx}{u-x} - \frac{nxx}{2(u-x)^2} + \text{ etc.}.$$

atque

$$l(u-x) = lu - \frac{nx}{u-x} + \frac{nxx}{(u-x)^2} - \frac{nx^3}{3(u-x)^3} + \text{etc.}$$

sumtoque x negativo habebitur

$$l(u+x) = lu + \frac{nx}{u+x} + \frac{nxx}{(u+x)^2} + \frac{nx^3}{3(u+x)^3} + \frac{nx^4}{4(u+x)^4} + \text{etc.}$$

Harum ergo serierum ope logarithmi expedite inveniri poterunt, si quidem series valde convergant. Huiusmodi autem erunt sequentes, quae ex inventis facile deducuntur,

$$l(x+1) = lx + n\left(\frac{1}{x} - \frac{1}{2xx} + \frac{1}{3x^3} - \frac{1}{4x^4} + \text{etc.}\right)$$
$$l(x-1) = lx - n\left(\frac{1}{x} + \frac{1}{2xx} + \frac{1}{3x^3} + \frac{1}{4x^4} + \text{etc.}\right);$$

quae duae series cum tantum signis a se invicem discrepent, si ad calculum revocentur, ex logarithmo numeri x cognito eadem opera logarithmi amborum numerorum $x+1$ et $x-1$ reperientur. Deinde ex reliquis seriebus erit

$$l(x+1) = l(x-1) + 2n\left(\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \frac{1}{7x^7} + \text{etc.}\right)$$
$$l(x-1) = lx - n\left(\frac{1}{x-1} - \frac{1}{2(x-1)^2} + \frac{1}{3(x-1)^3} - \frac{1}{4(x-1)^4} + \text{etc.}\right)$$
$$l(x+1) = lx + n\left(\frac{1}{x+1} + \frac{1}{2(x+1)^2} + \frac{1}{3(x+1)^3} + \frac{1}{4(x+1)^4} + \text{etc.}\right).$$

80. Ex dato ergo logarithmo numeri x logarithmi numerorum contiguorum $x+1$ et $x-1$ facile inveniri poterunt; quin etiam ex logarithmo numeri $x-1$ logarithmus numeri binario maioris et vicissim eruetur. Quod quamvis in *Introductione* uberius sit ostensum, tamen hic quaedam exempla adiungemus.

EXEMPLUM 1

Ex dato numeri 10 *logarithmo hyperbolico, qui est* 2,3025850929940, *logarithmos hyperbolicos numerorum* 11 *et* 9 *invenire.*

Quoniam haec quaestio logarithmos hyperbolicos spectat, erit $n=1$ ideoque habebuntur hae series

$$l11 = l10 + \frac{1}{10} - \frac{1}{2\cdot 10^2} + \frac{1}{3\cdot 10^3} - \frac{1}{4\cdot 10^4} + \frac{1}{5\cdot 10^5} - \text{etc.}$$
$$l9 = l10 - \frac{1}{10} - \frac{1}{2\cdot 10^2} - \frac{1}{3\cdot 10^3} - \frac{1}{4\cdot 10^4} - \frac{1}{5\cdot 10^5} - \text{etc.}$$

Ad quarum serierum summas inveniendas colligantur termini pares et impares seorsim eritque

$\frac{1}{10} = 0{,}1000000000000$	$\frac{1}{2\cdot 10^2} = 0{,}0050000000000$
$\frac{1}{3\cdot 10^3} = 0{,}0003333333333$	$\frac{1}{4\cdot 10^4} = 0{,}0000250000000$
$\frac{1}{5\cdot 10^5} = 0{,}0000020000000$	$\frac{1}{6\cdot 10^6} = 0{,}0000001666666$
$\frac{1}{7\cdot 10^7} = 0{,}0000000142857$	$\frac{1}{8\cdot 10^8} = 0{,}0000000012500$
$\frac{1}{9\cdot 10^9} = 0{,}0000000001111$	$\frac{1}{10\cdot 10^{10}} = 0{,}0000000000100$
$\frac{1}{11\cdot 10^{11}} = 0{,}0000000000009$	$\frac{1}{12\cdot 10^{12}} = 0{,}0000000000001$
summa = 0,1003353477310	summa = 0,0050251679267

Summa utriusque erit 0,1053605156577

Differentia ambarum erit 0,0953101798043

Iam est $l10 = 2{,}3025850929940$

Ergo erit $l11 = 2{,}3978952727983$
et $l9 = 2{,}1972245773363$
Hinc porro erit $l3 = 1{,}0986122886681$
et $l99 = 4{,}5951198501346$

EXEMPLUM 2

Ex logarithmo hyperbolico numeri 99 *nunc invento invenire logarithmum numeri 101*.

Adhibeatur ad hoc series supra inventa

$$l(x+1) = l(x-1) + \frac{2}{x} + \frac{2}{3x^3} + \frac{2}{5x^5} + \frac{2}{7x^7} + \text{etc.},$$

in qua fiat $x = 100$, eritque

$$l101 = l99 + \frac{2}{100} + \frac{2}{3\cdot 100^3} + \frac{2}{5\cdot 100^5} + \frac{2}{7\cdot 100^7} + \text{etc.},$$

cuius seriei summa ex his quatuor terminis colligitur = 0,0200006667066, quae ad $l99$ addita dabit $l101$ = 4,6151205168412.

EXEMPLUM 3

Ex dato logarithmo tabulari numeri 10, *qui est* = 1, *invenire logarithmos numerorum* 11 et 9.

Quoniam hic logarithmos communes tabulares quaerimus, erit

$$n = 0{,}4342944819032\,;$$

posito ergo $x = 10$ erit

$$l11 = l10 + \frac{n}{10} - \frac{n}{2\cdot 10^2} + \frac{n}{3\cdot 10^3} - \frac{n}{4\cdot 10^4} + \text{etc.}$$

$$l9 \;= l10 - \frac{n}{10} - \frac{n}{2\cdot 10^2} - \frac{n}{3\cdot 10^3} - \frac{n}{4\cdot 10^4} - \text{etc.}$$

Colligantur termini pares et impares seorsim

$\frac{1}{10} = 0{,}0434294481903$	$\frac{1}{2\cdot 10^2} = 0{,}0021714724095$
$\frac{1}{3\cdot 10^3} = 0{,}0001447648273$	$\frac{1}{4\cdot 10^4} = 0{,}0000108573620$
$\frac{1}{5\cdot 10^5} = 0{,}0000008685889$	$\frac{1}{6\cdot 10^6} = 0{,}0000000723824$
$\frac{1}{7\cdot 10^7} = 0{,}0000000062042$	$\frac{1}{8\cdot 10^8} = 0{,}0000000005428$
$\frac{1}{9\cdot 10^9} = 0{,}0000000000482$	$\frac{1}{10\cdot 10^{10}} = 0{,}0000000000043$
$\frac{1}{11\cdot 10^{11}} = 0{,}0000000000004$	$\frac{1}{12\cdot 10^{12}} = 0{,}0000000000000$
summa = 0,0435750878593	summa = 0,0021824027010

Aggregatum ambarum est

$$= 0{,}0457574905603$$

Differentia earum est

$$= 0{,}0413926851583$$

Cum ergo sit

$$l10 = 1{,}0000000000000$$

erit

$$l11 = 1{,}0413926851583$$

et

$$l9 = 0{,}9542425094397$$

hinc

$$l3 = 0{,}4771212547198$$

et

$$l99 = 1{,}9956351945980$$

EXEMPLUM 4

Ex logarithmo tabulari numeri 99 *hic invento invenire logarithmum tabularem numeri* 10l.

Adhibendo hic eandem seriem, qua in exemplo secundo usi sumus, habebimus

$$l101 = l99 + 2n\left(\frac{1}{100} + \frac{1}{3\cdot 100^3} + \frac{1}{5\cdot 100^5} + \frac{1}{7\cdot 100^7} + \text{etc.}\right),,$$

cuius seriei posito pro n valore debito summa mox reperietur

$$= 0{,}0086861791849$$

qua addita ad

$$l99 = 1{,}9956351945980$$

oritur

$$l101 = 2{,}0043213737829$$

81. Tribuamus nunc in expressione nostra generali y valorem exponentialem sitque $y = a^x$ *;* posito $x + \omega$ loco x erit $z = a^{x+\omega}$, cuius valor ob differentialia

$$\frac{dy}{dx} = a^x la, \quad \frac{ddy}{dx^2} = a^x (la)^2, \quad \frac{d^3y}{dx^3} = a^x (la)^3 \text{ etc.},$$

erit

$$a^{x+\omega} = a^x\left(1 + \frac{\omega la}{1} + \frac{\omega^2 (la)^2}{1\cdot 2} + \frac{\omega^3 (la)^3}{1\cdot 2\cdot 3} + \text{etc.}\right);$$

quae si dividatur per a^x, prodibit series valores quantitatis exponentialis exprimens, quam supra in *Introductione* iam elicuimus, nempe

$$a^{\omega} = 1 + \frac{\omega la}{1} + \frac{\omega^2 (la)^2}{1\cdot 2} + \frac{\omega^3 (la)^3}{1\cdot 2\cdot 3} + \frac{\omega^4 (la)^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

Simili modo sumto ω negativo erit

$$a^{-\omega} = 1 - \frac{\omega la}{1} + \frac{\omega^2 (la)^2}{1\cdot 2} - \frac{\omega^3 (la)^3}{1\cdot 2\cdot 3} + \frac{\omega^4 (la)^4}{1\cdot 2\cdot 3\cdot 4} - \text{etc.}$$

ex quarum combinatione oritur

$$\frac{a^{\omega}+a^{-\omega}}{2}=1+\frac{\omega^2(la)^2}{1\cdot 2}+\frac{\omega^4(la)^4}{1\cdot 2\cdot 3\cdot 4}+\frac{\omega^6(la)^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}+\text{etc.}$$
$$\frac{a^{\omega}-a^{-\omega}}{2}=\frac{\omega la}{1}+\frac{\omega^3(la)^3}{1\cdot 2\cdot 3}+\frac{\omega^5(la)^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\text{etc.}$$

ubi notandum est *la* denotare logarithmum hyperbolicum numeri *a*.

82. Huius formulae ope ex dato quovis logarithmo numerus ei conveniens reperiri poterit. Sit enim propositus logarithmus quicunque *u* ad canonem, in quo numeri *a* logarithmus $=1$ statuitur, pertinens. Quaeratur in eodem canone logarithmus *x* proxime ad *u* accedens sitque $u=x+\omega$, numerus autem logarithmo *x* conveniens sit $=y=a^x$; erit numerus logarithmo $u=x+\omega$ respondens $=a^{x+\omega}=z$ fietque

$$z=y\left(1+\frac{\omega la}{1}+\frac{\omega^2(la)^2}{1\cdot 2}+\frac{\omega^3(la)^3}{1\cdot 2\cdot 3}+\frac{\omega^4(la)^4}{1\cdot 2\cdot 3\cdot 4}+\text{etc.}\right),$$

quae series ob ω numerum valde parvum vehementer converget, cuius usum sequenti exemplo declaremus.

EXEMPLUM

Quaeratur numerus isti binarii potestati $2^{2^{24}}$ *aequalis.*

Cum sit $2^{24}=16777216$, erit $2^{2^{24}}=2^{216777216}$ sumendisque logarithmis vulgaribus erit huius numeri logarithmus $=16777216l2$. Cum autem sit

$$l2=0{,}301029995663981195213738 89,$$

numeri quaesiti logarithmus erit

$$5050445{,}259733675932039063,$$

cuius characteristica indicat numerum quaesitum exprimi 5050446 figuris; quae cum omnes exhiberi nequeant, sufficiet figuras initiales assignasse, quae ex mantissa

$$,259733675932039063=u$$

investigari debent. Ex tabulis autem colligitur numerum, cuius logarithmus proxime ad hunc accedat, fore $=1{,}818$, qui ponatur *y;* cuius logarithmus

$$x=0{,}259593878885948644$$

unde erit

$$\omega = 0,000139797046090419$$

Cum iam sit

$$a = 10$$

erit

$$\underline{la = 2,302585092994045684017991 4}$$

et

$$\underline{\omega la = 0,000321894594372400}$$

Deinde erit

$$y = 1,818000000000000000$$

$$\frac{\omega la}{1} y = 585204372569023$$

$$\frac{\omega^2 (la)^2}{1 \cdot 2} y = \qquad 94187062066$$

$$\frac{\omega^3 (la)^3}{1 \cdot 2 \cdot 3} y = \qquad 10106102$$

$$\frac{\omega^4 (la)^4}{1 \cdot 2 \cdot 3 \cdot 4} y = \qquad 813$$

$$181858529856973800$$

haeque sunt figurae initiales numeri quaesiti, cuius omnes figurae excepta forte ultima sunt iustae.

83. Consideremus quantitates transcendentes a circulo pendentes sitque, uti perpetuo ponimus, radius circuli $=1$ atque y denotet arcum circuli, cuius sinus $= x$, seu sit $y = \mathrm{Asin}\, x$. Ponatur $x + \omega$ loco x eritque $z = \mathrm{Asin}(x + \omega)$; ad quem valorem exprimendum quaerantur differentialia ipsius y

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1-xx)}}, \quad \frac{ddy}{dx^2} = \frac{x}{(1-xx)^{\frac{3}{2}}}, \quad \frac{d^3y}{dx^3} = \frac{1+2xx}{(1-xx)^{\frac{5}{2}}}, \quad \frac{d^4y}{dx^4} = \frac{9x+6x^3}{(1-xx)^{\frac{7}{2}}},$$

$$\frac{d^5y}{dx^5} = \frac{9+72x^2+24x^4}{(1-xx)^{\frac{9}{2}}}, \quad \frac{d^6y}{dx^6} = \frac{225x+600x^3+120x^5}{(1-xx)^{\frac{11}{2}}}$$

etc.

Ex his ergo invenitur

$$\text{Asin}(x+\omega) = \text{Asin}\, x + \frac{\omega}{\sqrt{(1-xx)}} + \frac{\omega^2 x}{2(1-xx)^{\frac{3}{2}}} + \frac{\omega^3(1+2xx)}{6(1-xx)^{\frac{5}{2}}} +$$
$$\frac{\omega^4(9x+6x^3)}{24(1-xx)^{\frac{7}{2}}} + \frac{\omega^5(9+72x^2+24x^4)}{120(1-xx)^{\frac{9}{2}}} + \text{etc.}$$

84. Si ergo cognitus fuerit arcus, cuius sinus est $= x$, huius formulae beneficio inveniri poterit arcus, cuius sinus est $x+\omega$, si fuerit ω quantitas valde parva. Series autem, cuius summa addi debet, exprimetur in partibus radii, quae ad arcum facile reducentur, uti ex hoc exemplo intelligetur.

EXEMPLUM

Quaeratur arcus circuli, cuius sinus est $= \frac{1}{3} = 0{,}3333333333$.

Quaeratur ex tabulis sinuum arcus, cuius sinus sit proxime minor quam $\frac{1}{3}$, qui erit $19^0\ 28^{\text{I}}$, cuius sinus est $= 0{,}3332584$. Statuatur ergo $19^0\ 28^{\text{I}} = \text{Asin}\, x = y$;
erit $x = 0{,}3332584$ et $\omega = 0{,}0000749$ atque ex tabulis $\sqrt{(1-xx)} = \cos y = 0{,}9428356$.
Erit ergo arcus quaesitus *z*, cuius sinus $= \frac{1}{3}$ proponitur,

$$= 19^0\ 28^{\text{I}} + \frac{\omega}{\cos y} + \frac{\omega\omega \sin y}{2\cos^3 y},$$

quae expressio iam sufficit; erit ergo per logarithmos calculum instituendo

$$l\omega = 5{,}8744818$$
$$l\cos y = 9{,}9744359$$
$$l\frac{\omega}{\cos y} = 5{,}9000459 \qquad \frac{\omega}{\cos y} = 0{,}0000794412$$
$$l\frac{\omega^2}{\cos^2 y} = 1{,}8000918$$
$$l\frac{\sin y}{\cos y} = 9{,}5483452$$
$$1{,}3484370$$
$$l2 = 0{,}3010300$$
$$l\frac{\omega^2}{2\cos^2 y} = 1{,}0474070 \qquad \frac{\omega^2 \sin y}{2\cos^3 y} = 0{,}0000000011$$
$$\text{summa} = 0{,}0000794423$$

qui est valor arcus ad $19^0\ 28^I$ addendi, ad quem in minutis secundis exprimendum sumamus eius logarithmum, qui est

$$5,9000518,$$

a quo subtrahatur

$$\begin{array}{r} 4,6855749 \\ \hline 1,2144769, \end{array}$$

cui log. respondet num.

$$= 16,38615,$$

qui est numerus minutorum secundorum; fractionem vero in tertiis et quartis exprimendo fiet arcus quaesitus

$$=19^0\ 28^I\ 16^{II}\ 23^{III}10^{IV}\ 8^V\ 24^{VI}.$$

85. Simili modo expressio pro cosinibus eruetur; posito enim $y = \mathrm{Acos}\, x$, quia est $dy = \frac{-dx}{\sqrt{(1-xx)}}$, series ante inventa invariata manebit, dummodo eius signa permutentur. Erit itaque

$$\mathrm{Acos}(x+\omega) = \mathrm{Acos}\, x - \frac{\omega}{\sqrt{(1-xx)}} - \frac{\omega^2 x}{2(1-xx)^{\frac{3}{2}}} - \frac{\omega^3(1+2xx)}{6(1-xx)^{\frac{5}{2}}} - \frac{\omega^4(9x+6x^3)}{24(1-xx)^{\frac{7}{2}}} - \frac{\omega^5(9+72x^2+24x^4)}{120(1-xx)^{\frac{9}{2}}} - \text{ etc.}$$

quae series pariter ac praecedens vehementer semper converget, si ex tabulis sinuum proxime veri anguli excerpantur, ita ut plerumque unicus terminus primus $\frac{\omega}{\sqrt{(1-xx)}}$ sufficiat. Interim tamen, si x fuerit ipsi 1 seu sinui toti proxime aequalis, tum ob denominatores admodum parvos illa series convergentiam amittit. His igitur casibus, quibus x non multum ab 1 deficit, quoniam tum differentiae fiunt minimae, commodius utemur solita interpolatione.

86. Ponamus quoque pro y arcum, cuius tangens datur, sitque $y = \mathrm{Atang}\, x$ et $z = \mathrm{Atang}(x+\omega)$, ita ut sit

$$z = y + \frac{\omega dy}{1dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3 y}{6dx^3} + \text{etc.}$$

Ad quos terminos indagandos quaerantur ipsius y singula differentialia

$$\frac{dy}{dx}=\frac{1}{1+xx},\quad \frac{ddy}{dx^2}=\frac{-2x}{(1+xx)^2},\quad \frac{d^3y}{dx^3}=\frac{-2+6xx}{(1+xx)^3},\quad \frac{d^4y}{dx^4}=\frac{24x-24x^3}{(1+xx)^4},$$

$$\frac{d^5y}{dx^5}=\frac{24-240x^2+120x^4}{(1+xx)^5},\quad \frac{d^6y}{dx^6}=\frac{-720x+2400x^3-720x^5}{(1+xx)^6}$$

etc.

unde colligitur fore

$$\text{Atang}(x+\omega)=\text{Atang}x=\frac{\omega}{1+xx}-\frac{\omega^2x}{(1+xx)^2}+\frac{\omega^3}{(1+xx)^3}\left(xx-\tfrac{1}{3}\right)-\frac{\omega^4}{(1+xx)^4}\left(x^3-x\right)$$
$$+\frac{\omega^5}{(1+xx)^5}\left(x^4-2x^2+\tfrac{1}{5}\right)-\frac{\omega^6}{(1+xx)^6}\left(x^5-\tfrac{10}{3}x^3+x\right)+\text{etc.}$$

87. Haec series, cuius lex progressionis non adeo manifesta est, transmutari potest in aliam formam, cuius progressio statim in oculos incurrit. Ponatur in hunc finem $\text{Atang}\,x=90^0-u$, ut sit $x=\cot u=\frac{\cos u}{\sin u}$; erit $1+xx=\frac{1}{\sin^2 u}$, unde fit $\frac{dy}{dx}=\frac{1}{1+xx}=\sin^2 u$.
Cum deinde sit $dx=\frac{-du}{\sin^2 u}$ seu $du=-dx\sin^2 u$, fiet ulteriora differentialia sumendo
$\frac{ddy}{dx}=2du\sin u\cdot\cos u=du\sin 2u=-dx\sin^2 u\cdot\sin 2u$
ideoque

$$\frac{ddy}{1dx^2}=-\sin^2 u\cdot\sin 2u,$$

$$\frac{d^3y}{2dx^2}=-du\sin u.\cos u\cdot\sin 2u-du\sin^2 u\cdot\cos 2u=-du\sin u\cdot\sin 3u$$
$$=dx\sin^3 u\cdot\sin 3u$$

ideoque

$$\frac{d^3y}{1\cdot 2dx^3}=\sin^3 u\cdot\sin 3u,$$

$$\frac{d^4y}{1\cdot 2\cdot 3dx^3}=du\sin^2 u\left(\cos u\cdot\sin 3u+\sin u\cdot\cos 3u\right)=du\cdot\sin^2 u\cdot\sin 4u=-dx\sin^4 u\cdot\sin 4u$$

ideoque

$$\frac{d^4y}{1\cdot 2\cdot 3dx^4}=-\sin^4 u\cdot\sin 4u$$

$$\frac{d^5y}{1\cdot 2\cdot 3\cdot 4dx^4}=-du\sin^3 u\left(\cos u\cdot\sin 4u+\sin u\cdot\cos 4u\right)=-du\cdot\sin^3 u\cdot\sin 5u=dx\sin^5 u\cdot\sin 5u$$

ideoque

$$\frac{d^5y}{1\cdot 2\cdot 3\cdot 4dx^5}=+\sin^5 u\cdot\sin 5u$$

etc.

Ex quibus colligitur fore

$$\begin{aligned}&\mathrm{Atang}(x+\omega)=\mathrm{Atang}\,x+\tfrac{\omega}{1}\sin u\cdot\sin u-\tfrac{\omega^2}{2}\sin^2 u\cdot\sin 2u\\&+\tfrac{\omega^3}{3}\sin^3 u\cdot\sin 3u-\tfrac{\omega^4}{4}\sin^4 u\cdot\sin 4u+\tfrac{\omega^5}{5}\sin^5 u\cdot\sin 5u-\tfrac{\omega^6}{6}\sin^6 u\cdot\sin 6u+\text{etc.};\end{aligned}$$

ubi cum sit $\mathrm{Atang}\,x=y$ et $\mathrm{Atang}\,x=90^0-u$, erit $y=90^0-u$.

88. Si ponatur $\mathrm{Acot}\,x=y$ et $\mathrm{Acot}(x+\omega)=z$, erit

$$z=y+\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{1\cdot 2dx^2}+\frac{\omega^3 d^3y}{1\cdot 2\cdot 3dx^3}+\frac{\omega^4 d^4y}{1\cdot 2\cdot 3\cdot 4dx^4}+\text{etc.}$$

Cum autem sit $dy=\frac{-dx}{1+xx}$ termini huius seriei congruent praeter primum cum ante inventis exceptis tantum signis. Quare si ponatur ut ante
$\mathrm{Atang}\,x=90^0-u$ seu $\mathrm{Acot}\,x=u$, ut sit $u=y$, erit

$$\begin{aligned}&\mathrm{Acot}(x+\omega)=\mathrm{Acot}\,x-\tfrac{\omega}{1}\sin u\cdot\sin u+\tfrac{\omega^2}{2}\sin^2 u\cdot\sin 2u\\&-\tfrac{\omega^3}{3}\sin^3 u\cdot\sin 3u+\tfrac{\omega^4}{4}\sin^4 u\cdot\sin 4u-\tfrac{\omega^5}{5}\sin^5 u\cdot\sin 5u-\text{etc.},\end{aligned}$$

quae expressio immediate ex praecedente sequitur; quia enim est

$$\mathrm{Acot}(x+\omega)=90^0-\mathrm{Atang}(x+\omega)\quad\text{et}\quad \mathrm{A\,cot}\,x=90^0-\mathrm{Atang}\,x,$$

erit

$$\mathrm{Acot}(x+\omega)-\mathrm{Acot}\,x=-\mathrm{Atang}(x+\omega)+\mathrm{Atang}\,x.$$

89. Ex his expressionibus multa egregia corollaria consequuntur, prout loco x et ω dati valores substituantur. Sit igitur primum $x=0$, et cum sit $u=90^0-\mathrm{Atang}\,x$, fiet $u=90^0$ atque $\sin u=1$, $\sin 2u=0$, $\sin 3u=-1$, $\sin 4u=0$, $\sin 5u=1$, $\sin 6u=0$, $\sin 7u=-1$ etc., unde fiet

$$\mathrm{Atang}\,\omega=\frac{\omega}{1}-\frac{\omega^3}{3}+\frac{\omega^5}{5}-\frac{\omega^7}{7}+\frac{\omega^9}{9}-\frac{\omega^{11}}{11}+\text{etc.},$$

quae est notissima series exprimens arcum, cuius tangens est $=\omega$.

Sit $x = 1$; erit $\text{Atang}\, x = 45^0$ ideoque $u = 45^0$, hinc

$\sin u = \frac{1}{\sqrt{2}}, \sin 2u = 1, \sin 3u = \frac{1}{\sqrt{2}}, \sin 4u = 0, \sin 5u = -\frac{1}{\sqrt{2}}, \sin 6u = -1, \sin 7 \cdot u = -\frac{1}{\sqrt{2}},$
$\sin 8u = 0, \sin 9u = \frac{1}{\sqrt{2}}$ etc.

Ex quibus fit

$$\text{Atang}(1+\omega) = 45^0 + \frac{\omega}{2} - \frac{\omega^2}{2\cdot 2} + \frac{\omega^3}{3\cdot 4} - \frac{\omega^5}{5\cdot 8} + \frac{\omega^6}{6\cdot 8} - \frac{\omega^7}{7\cdot 16} + \frac{\omega^9}{9\cdot 32} - \frac{\omega^{10}}{10\cdot 32} + \frac{\omega^{11}}{11\cdot 64}$$
$$- \frac{\omega^{13}}{13\cdot 128} + \frac{\omega^{14}}{14\cdot 128} - \text{etc.}$$

Si igitur sit $\omega = -1$, ob $\text{Atang}(1+\omega) = 0$ et $45^0 = \frac{\pi}{4}$ fiet

$$\frac{\pi}{4} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 2^2} - \frac{1}{5\cdot 2^3} - \frac{1}{6\cdot 2^3} - \frac{1}{7\cdot 2^4} + \frac{1}{9\cdot 2^5} + \frac{1}{10\cdot 2^5} + \frac{1}{11\cdot 2^6} - \text{etc.}$$

qui valor si loco arcus 45^0 substituatur in illa expressione, erit

$$\text{Atang}(1+\omega) = \frac{\omega+1}{1\cdot 2} - \frac{\omega^2-1}{2\cdot 2} + \frac{\omega^3+1}{3\cdot 2^2} - \frac{\omega^5+1}{5\cdot 2^3} + \frac{\omega^6-1}{6\cdot 2^3} - \frac{\omega^7+1}{7\cdot 2^4} + \text{etc.}$$

Illa autem series maxime est idonea ad valorem ipsius $\frac{\pi}{4}$ proxime inveniendum.

90. Cum sit

$$\frac{\pi}{4} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 2^2} - \frac{1}{5\cdot 2^3} - \frac{1}{6\cdot 2^3} - \frac{1}{7\cdot 2^4} + \text{etc.}$$

termini autem in denominatoribus habentes 2, 6, 10 etc.

$$\frac{1}{2\cdot 2} - \frac{1}{6\cdot 2^3} + \frac{1}{10\cdot 2^5} - \frac{1}{14\cdot 2^7} + \text{etc.}$$

exprimant $\frac{1}{2}\text{Atang}\frac{1}{2}$, erit

$$\frac{\pi}{4} = \frac{1}{2}\text{Atang}\frac{1}{2} + \frac{1}{1\cdot 2} + \frac{1}{3\cdot 2^2} - \frac{1}{5\cdot 2^3} - \frac{1}{7\cdot 2^4} + \frac{1}{9\cdot 2^5} + \frac{1}{11\cdot 2^6} - \text{etc.}$$

In altera autem formula posito ω negativo cum sit

$$\text{Atang}(1-\omega)=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 2^2}-\frac{1}{5\cdot 2^3}-\frac{1}{6\cdot 2^3}-\frac{1}{7\cdot 2^4}+\text{etc.}$$
$$-\frac{\omega}{1\cdot 2}-\frac{\omega^2}{2\cdot 2}-\frac{\omega^3}{3\cdot 2^2}+\frac{\omega^5}{5\cdot 2^3}+\frac{\omega^6}{6\cdot 2^3}+\frac{\omega^7}{7\cdot 2^4}-\text{etc.},$$

si fiat $\omega=\frac{1}{2}$ erit

$$\text{Atang}\frac{1}{2}=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2}+\frac{1}{3\cdot 2^2}-\frac{1}{5\cdot 2^3}-\frac{1}{6\cdot 2^3}-\frac{1}{7\cdot 2^4}+\text{etc.}$$
$$-\frac{1}{1\cdot 2^2}-\frac{1}{2\cdot 2^3}-\frac{1}{3\cdot 2^5}+\frac{1}{5\cdot 2^8}+\frac{1}{6\cdot 2^9}+\frac{1}{7\cdot 2^{11}}-\text{etc.},$$

at terminis per 2, 6, 10 etc. divisis seorsim sumtis erit

$$\text{Atang}\frac{1}{2}=\frac{1}{2}\text{Atang}\frac{1}{2}+\frac{1}{1\cdot 2}+\frac{1}{3\cdot 2^2}-\frac{1}{5\cdot 2^3}-\frac{1}{7\cdot 2^4}+\frac{1}{9\cdot 2^5}\text{etc.}$$
$$-\frac{1}{2}\text{Atang}\frac{1}{8}-\frac{1}{1\cdot 2^2}-\frac{1}{3\cdot 2^5}+\frac{1}{5\cdot 2^8}+\frac{1}{7\cdot 2^{11}}-\frac{1}{9\cdot 2^{14}}-\text{etc.},$$

ideoque

$$\text{Atang}\frac{1}{2}=\frac{1}{1\cdot 2}+\frac{1}{3\cdot 2^2}-\frac{1}{5\cdot 2^3}-\frac{1}{7\cdot 2^4}+\text{etc.}$$
$$-\frac{1}{2}\text{Atang}\frac{1}{8}-\frac{1}{1\cdot 2^2}-\frac{1}{3\cdot 2^5}+\frac{1}{5\cdot 2^8}+\frac{1}{7\cdot 2^{11}}-\text{etc.};$$

qui valor si in superiore serie substituatur atque $\text{Atang}\frac{1}{8}$ ipse in seriem convertatur, reperietur

$$\frac{\pi}{4}=\begin{cases}1+\frac{1}{3\cdot 2^1}-\frac{1}{5\cdot 2^2}-\frac{1}{7\cdot 2^3}+\frac{1}{9\cdot 2^4}+\text{etc.}\\ -\frac{1}{1\cdot 2^2}-\frac{1}{3\cdot 2^5}+\frac{1}{5\cdot 2^8}+\frac{1}{7\cdot 2^{11}}-\frac{1}{9\cdot 2^{14}}-\text{etc.}\\ -\frac{1}{1\cdot 2^4}+\frac{1}{3\cdot 2^{10}}-\frac{1}{5\cdot 2^{16}}+\frac{1}{7\cdot 2^{22}}-\frac{1}{9\cdot 2^{28}}+\text{etc.}\end{cases}$$

90a. Sequuntur hae multaeque aliae series ex positione $x=1$; sin autem ponamus $x=\sqrt{3}$, ut sit $\text{Atang}\,x=60^0$, fiet $u=30^0$ et
$\sin u=\frac{1}{2}$, $\sin 2u=\frac{\sqrt{3}}{2}$, $\sin 3u=1$, $\sin 4u=\frac{\sqrt{3}}{2}$, $\sin 5u=\frac{1}{2}$, $\sin 6u=0$, $\sin 7u=-\frac{1}{2}$ etc., unde erit

$$\text{Atang}\left(\sqrt{3}+\omega\right)=60^0+\frac{\omega}{1\cdot 2^2}-\frac{\omega^2\sqrt{3}}{2\cdot 2^3}+\frac{\omega^3}{3\cdot 2^3}-\frac{\omega^4\sqrt{3}}{4\cdot 2^5}+\frac{\omega^5}{5\cdot 2^6}$$
$$-\frac{\omega^7}{7\cdot 2^8}+\frac{\omega^8\sqrt{3}}{8\cdot 2^9}-\frac{\omega^9}{9\cdot 2^9}+\frac{\omega^{10}\sqrt{3}}{10\cdot 2^{11}}-\frac{\omega^{11}}{11\cdot 2^{11}}+\text{etc.}$$

Sin autem ponatur $x=\frac{1}{\sqrt{3}}$ ut sit $\text{Atang}\,x=30^0$, erit $u=60^0$ atque $\sin u=\frac{\sqrt{3}}{2}$,

$\sin 2u = \frac{\sqrt{3}}{2}$, $\sin 3u = 0$, $\sin 4u = -\frac{\sqrt{3}}{2}$, $\sin 5u = -\frac{\sqrt{3}}{2}$, $\sin 6u = 0$, $\sin 7u = \frac{\sqrt{3}}{2}$ etc.,
quibus valoribus substitutis erit

$$\text{Atang}\left(\frac{1}{\sqrt{3}} + \omega\right) = 30^0 + \frac{3\omega}{1\cdot 2^2} - \frac{3\omega^2\sqrt{3}}{2\cdot 2^3} + \frac{3^2\omega^4\sqrt{3}}{4\cdot 2^5} - \frac{3^3\omega^5}{5\cdot 2^6} + \text{etc.}$$

si igitur sit $\omega = -\frac{1}{\sqrt{3}}$, ob $30^0 = \frac{\pi}{6}$ erit

$$\frac{\pi}{6\sqrt{3}} = \frac{1}{1\cdot 2^2} + \frac{1}{2\cdot 2^3} - \frac{1}{4\cdot 2^5} - \frac{1}{5\cdot 2^6} + \frac{1}{7\cdot 2^8} + \frac{1}{8\cdot 2^9} - \text{etc.}$$

91. Resumamus expressionem generalem inventam

$$\text{Atang}(x+\omega) = \text{Atang}\, x + \frac{\omega}{1}\sin u \cdot \sin u - \frac{\omega^2}{2}\sin^2 u \cdot \sin 2u$$
$$+ \frac{\omega^3}{3}\sin^3 u \cdot \sin 3u - \text{etc.};$$

ac ponamus $\omega = -x$, ut sit $\text{Atang}(x+\omega) = 0$, eritque

$$\text{Atang}\, x = \frac{x}{1}\sin u \cdot \sin u + \frac{x^2}{2}\sin^2 u \cdot \sin 2u + \frac{x^3}{3}\sin^3 u \cdot \sin 3u + \text{etc.};$$

Cum autem sit $\text{Atang}\, x = 90^0 - u = \frac{\pi}{2} - u$, erit $x = \cot u = \frac{\cos u}{\sin u}$. Quamobrem erit

$$\frac{\pi}{2} = u + \cos u \cdot \sin u + \frac{1}{2}\cos^2 u \cdot \sin 2u + \frac{1}{3}\cos^3 u \cdot \sin 3u + \frac{1}{4}\cos^4 u \cdot \sin 4u + \text{etc.},$$

quae series eo magis est notatu digna, quod, quicunque arcus loco u accipiatur, valor seriei semper prodeat idem $= \frac{\pi}{2}$.

Sin autem sit $\omega = -2x$, ob $\text{Atang}(-x) = -\text{Atang}\, x$ fiet

$$2\text{Atang}\, x = \frac{2x}{1}\sin u \cdot \sin u + \frac{4x^2}{2}\sin^2 u \cdot \sin 2u + \frac{8x^3}{3}\sin^3 u \cdot \sin 3u + \text{etc.}$$

Cum autem sit $\text{Atang}\, x = \frac{\pi}{2} - u$ et $x = \frac{\cos u}{\sin u}$, erit

$$\pi = 2u + \frac{2}{1}\cos u \cdot \sin u + \frac{2^2}{2}\cos^2 u \cdot \sin 2u + \frac{2^3}{3}\cos^3 u \cdot \sin 3u + \text{etc.}$$

Sit $u = 45^0 = \frac{\pi}{4}$ erit

$$\cos u = \frac{1}{\sqrt{2}},\ \sin u = \frac{1}{\sqrt{2}},\ \sin 2u = 1,\ \sin 3u = \frac{1}{\sqrt{2}},\ \sin 4u = 0,\ \sin 5u = \frac{-1}{\sqrt{2}},$$
$$\sin 6u = -1,\ \sin 7u = \frac{-1}{\sqrt{2}},\ \sin 8u = 0,\ \sin 9u = \frac{1}{\sqrt{2}} \text{ etc.}$$

eritque

$$\frac{\pi}{2} = \frac{1}{1} + \frac{2}{2} + \frac{2}{3} - \frac{2^2}{5} - \frac{2^3}{6} - \frac{2^3}{7} + \frac{2^4}{9} + \frac{2^5}{10} + \frac{2^5}{11} - \text{etc.},$$

quae series, etsi divergit, tamen ob simplicitatem est notatu digna.

92. Ponatur in expressione generali inventa

$$\omega = -x - \frac{1}{x} = \frac{-1}{\sin u \cdot \cos u}$$

ob $x = \frac{\cos u}{\sin u}$; erit

$$\text{Atang}(x+\omega) = \text{Atang}\left(-\frac{1}{x}\right) = -\text{Atang}\left(\frac{1}{x}\right) = -\frac{\pi}{2} + \text{Atang}\, x.$$

Hinc ergo obtinebitur sequens expressio

$$\frac{\pi}{2} = \frac{\sin u}{1\cos u} + \frac{\sin 2u}{2\cos^2 u} + \frac{\sin 3u}{3\cos^3 u} + \frac{\sin 4u}{4\cos^4 u} + \frac{\sin 5u}{5\cos^5 u} + \text{etc.},$$

quae posito $u = 45^0$ dat eandem seriem, quam ultimo loco invenimus.
Sin autem ponamus $\omega = -\sqrt{(1+xx)}$, ob $x = \frac{\cos u}{\sin u}$ fiet

$$\omega = -\frac{1}{\sin u}$$

et

$$\text{Atang}\left(x - \sqrt{(1+xx)}\right) = -\text{Atang}\left(\sqrt{(1+xx)} - x\right)$$
$$= -\frac{1}{2}\text{Atang}\frac{1}{x} = -\frac{1}{2}\left(\frac{\pi}{2} - \text{Atang}\, x\right) = -\frac{1}{2}u$$

et

$$\text{Atang}\, x = \frac{\pi}{2} - \frac{1}{2}u.$$

Hanc ob rem erit

$$\frac{\pi}{2} = \frac{1}{2}u + \frac{1}{1}\sin u + \frac{1}{2}\sin 2u + \frac{1}{3}\sin 3u + \frac{1}{4}\sin 4u + \text{etc.}$$

Quodsi haec aequatio differentietur, erit

$$0=\tfrac{1}{2}+\cos u+\cos 2u+\cos 3u+\cos 4u+\cos 5u+\text{etc.},$$

cuius ratio ex natura serierum recurrentium intelligitur.

93. Si simili modo series ante inventae differentientur, novae series summabiles reperientur. Ac primo quidem ex serie

$$\text{Atang}(1+\omega)=\tfrac{\pi}{4}+\tfrac{\omega}{2}-\tfrac{\omega^2}{2\cdot 2}+\tfrac{\omega^3}{3\cdot 4}-\tfrac{\omega^5}{5\cdot 8}+\tfrac{\omega^6}{6\cdot 8}-\text{etc.}$$

sequitur

$$\frac{1}{2+2\omega+\omega^2}=\frac{1}{2}-\frac{\omega}{2}+\frac{\omega^2}{4}-\frac{\omega^4}{8}+\frac{\omega^5}{8}-\frac{\omega^6}{16}+\frac{\omega^8}{32}-\text{etc.}$$

quae oritur ex evolutione fractionis $\frac{2-2\omega+\omega^2}{4+\omega^4}=\frac{1}{2+2\omega+\omega^2}$.

Deinde ista series

$$\tfrac{\pi}{2}=u+\cos u\cdot\sin u+\tfrac{1}{2}\cos^2 u\cdot\sin 2u+\tfrac{1}{3}\cos^3 u\cdot\sin 3u+\tfrac{1}{4}\cos^4 u\cdot\sin 4u+\text{etc.}$$

per differentiationem dabit

$$0=1+\cos 2u+\cos u\cdot\cos 3u+\cos^2 u\cdot\cos 4u+\cos^3 u\cdot\cos 5u+\text{etc.}$$

Denique series

$$\frac{\pi}{2}=\frac{\sin u}{\cos u}+\frac{\sin 2u}{2\cos^2 u}+\frac{\sin 3u}{3\cos^3 u}+\frac{\sin 4u}{4\cos^4 u}+\text{etc.}$$

dat

$$0=\frac{1}{\cos^2 u}+\frac{\cos u}{\cos^3 u}+\frac{\cos 2u}{\cos^4 u}+\frac{\cos 3u}{\cos^5 u}+\frac{\cos 4u}{\cos^5 u}+\text{etc.}$$

seu

$$0=1+\frac{\cos u}{\cos u}+\frac{\cos 2u}{\cos^2 u}+\frac{\cos 3u}{\cos^3 u}+\frac{\cos 4u}{\cos^4 u}+\frac{\cos 5u}{\cos^5 u}+\text{etc.}$$

94. Imprimis autem expressio inventa

$$\text{Atang}(x+\omega)=\text{Atang}\,x+\tfrac{\omega}{1}\sin u\cdot\sin u-\tfrac{\omega^2}{2}\sin^2 u\cdot\sin 2u+\tfrac{\omega^3}{3}\sin^3 u\cdot\sin 3u-\text{etc.}$$

existente

$$x=\cot u\quad\text{seu}\quad u=\text{Acot}\,x=90^0-\text{Atang}\,x$$

inserviet ad angulum seu arcum datae cuique tangenti respondentem inveniendum.

Sit enim proposita tangens $=t$ quaeraturque in tabulis tangens ad hanc proxime accedens $=x$, cui respondeat arcus $=y$, eritque $u=90^0-y$. Tum ponatur $x+\omega=t$ seu $\omega=t-x$ eritque arcus quaesitus

$$=y+\tfrac{\omega}{1}\sin u\cdot\sin u-\tfrac{\omega^2}{2}\sin^2 u\cdot\sin 2u+\text{etc.},$$

quae regula tum praecipue est utilis, cum tangens proposita fuerit admodum magna ac propterea arcus quaesitus parum a 90^0 discrepet. His enim casibus ob tangentes vehementer increscentes solita methodus interpolationum nimium a veritate abducit. Sit ergo propositum hoc exemplum.

EXEMPLUM

Quaeratur arcus, cuius tangens sit $=100$ *posito radio* $=1$.

Arcus proxime quaesito aequalis est , $89^0\ 25^{\text{I}}$ cuius tangens est

$$x=98{,}217943$$

quae subtrahatur a

$$t=100{,}000000$$

remanebit

$$\omega=1{,}782057$$

Deinde cum sit $89^0\ 25^{\text{I}}$ erit $u=0^0\,35^{\text{I}}$, $2u=1^0\,10^{\text{I}}$, $3u=1^0\,45^{\text{I}}$ etc. Iam singuli termini per logarithmos investigentur. Ad

$$l\omega=0{,}2509215$$

addatur

$$l\sin u=8{,}0077867$$
$$l\sin u=8{,}0077867$$
$$l\omega\sin u\cdot\sin u=6{,}2664949$$

subtrahatur

$$4{,}6855749$$
$$=1{,}5809200$$

Ergo

$$l\omega\sin u\cdot\sin u=38{,}09956 \text{ secund.}$$

Ad $\quad l\omega\sin^2 u=6{,}2664949$

addatur $\quad l\omega=0{,}2509215$

$$l\sin 2u=8{,}3087941$$
$$4{,}8262105$$

subtrahatur	$l2 = 0{,}3010300$
	$l\frac{1}{2}\omega^2 \sin u.\ \sin 2u = 4{,}5251805$
subtrahatur	$4{,}6855749$
remanet	$9{,}8396056$

Ergo $\frac{1}{2}\omega^2 \sin u.\ \sin 2u = 0{,}69120$ secund.

Porro ad

	$l\omega^3 = 0{,}7527645$
addatur	$l\sin^3 u = 4{,}0233601$
	$l\sin 3u = 8{,}4848479$
	$3{,}2609725$
subtrahatur	$l3 = 0{,}4771213$
	$2{,}7838512$
subtrahatur	$4{,}6855749$
	$8{,}0982763$

Ergo $\frac{1}{2}\omega^3 \sin^3 u.\ \sin 3u = 0{,}01254$ secund.

Denique ad	$l\omega^4 = 1{,}0036860$
addatur	$l\sin^4 u = 2{,}0311468$
	$l\sin 4u = 8{,}6097341$
	$1{,}6445669$
subtrahatur	$l4 = 0{,}6020600$
	$1{,}0425069$
subtrahatur	
	$4{,}6855749$
	$6{,}3569320$

Ergo $\frac{1}{2}\omega^4 \sin^4 u.\ \sin 4u = 0{,}00023$ secund.

Hinc

	termini addendi	termini subtrahendi
	38,09956	0,69120
	0,01254	0,00023
	38,11210	0,69143
subtrahatur	0,69143	
	37,42067=	$25^{I}\ 37^{II}\ 25^{III}\ 14^{IV}\ 24^{V}\ 36^{VI}$

Quocirca arcus, cuius tangens centies superat radium, erit

$$89^0\ 25^{I}\ 37^{II}\ 25^{III}\ 14^{IV}\ 24^{V}\ 36^{VI}$$

neque error ad minuta quarta ascendit, sed in minutis tantum quintis inesse potest, ex quo vere hunc angulum pronunciare poterimus $89^0\ 25^{I}\ 37^{II}\ 25^{III}\ 14^{IV}$. Si tangens adhuc maior proponatur, etiamsi fortasse ω maius prodeat, tamen ob *u* angulum adhuc minorem aeque expedite arcus definiri poterit.

95. Cum hic pro *y* arcum circuli substituerimus, nunc functiones reciprocas in locum *y* ponamus, cuiusmodi sunt sin *x*, cos *x*, tang *x*, cot *x* etc. Sit igitur $y=\sin x$ positoque $x+\omega$ loco *x* fiet $z=\sin(x+\omega)$ atque aequatio

$$z=y+\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}+\frac{\omega^3 d^3y}{6dx^3}+\frac{\omega^4 d^4y}{24dx^4}+\text{etc.}$$

ob

$$\frac{dy}{dx}=\cos x,\quad \frac{ddy}{dx^2}=-\sin x,\quad \frac{d^3y}{dx^3}=-\cos x,\quad \frac{d^4y}{dx^4}=\sin x\ \text{etc.},$$

dabit

$$\sin(x+\omega)=\sin x+\omega\cos x-\tfrac{1}{2}\omega^2\sin x+\tfrac{1}{6}\omega^3\cos x+\tfrac{1}{24}\omega^4\sin x+\text{etc.}$$

et sumto ω negativo erit

$$\sin(x-\omega)=\sin x-\omega\cos x-\tfrac{1}{2}\omega^2\sin x+\tfrac{1}{6}\omega^3\cos x+\tfrac{1}{24}\omega^4\sin x-\text{etc.}$$

Quodsi vero statuatur $y=\cos x$, ob

$$\frac{dy}{dx}=-\sin x,\quad \frac{ddy}{dx^2}=-\cos x,\quad \frac{d^3y}{dx^3}=\sin x,\quad \frac{d^4y}{dx^4}=\cos x\ \text{etc.}$$

erit

$$\cos(x+\omega)=\cos x-\omega\sin x-\tfrac{1}{2}\omega^2\cos x+\tfrac{1}{6}\omega^3\sin x+\tfrac{1}{24}\omega^4\cos x-\text{etc.}$$

et facto ω negativo erit

$$\cos(x-\omega)=\cos x+\omega\sin x-\tfrac{1}{2}\omega^2\cos x-\tfrac{1}{6}\omega^3\sin x+\tfrac{1}{24}\omega^4\cos x+\text{etc.}$$

96. Usus harum formularum eximius est cum in condendis tum interpolandis tabulis sinuum et cosinuum. Si enim cogniti fuerint sinus et cosinus cuiuspiam arcus *x*, ex iis facili negotio sinus et cosinus angulorum $x+\omega$ et $x-\omega$ inveniri possunt, siquidem differentia ω fuerit satis exigua; hoc enim casu series inventae vehementer convergunt. Ad hoc vero necesse est, ut arcus ω in partibus radii exprimatur; quod, cum arcus 180^0 sit

3,14159265358979323846,

facile fiet; erit enim divisione per 180 instituta

$$\text{arcus}\quad 1^0 = 0{,}01745329251994329576 9$$
$$\text{arcus}\quad 1^{\mathrm{I}} = 0{,}00029088820866572159 6$$
$$\text{arcus}\ l0^{\mathrm{II}} = 0{,}00000484813681109535 99.$$

EXEMPLUM 1

Invenire sinus et cosinus angulorum $45^0 1^{\mathrm{I}}$ *et* $44^0 59^{\mathrm{I}}$ *ex datis sinu et cosinu anguli* 45^0, *quorum uterque est* $= \frac{1}{\sqrt{2}} = 0{,}7071067811865$.

Cum igitur sit

$$\sin x = \cos x = 0{,}7071067811865$$

atque

$$\omega = 0{,}0002908882086$$

erit ad multiplicationes facilius instituendas

$$2\omega = 0{,}0005817764173$$
$$3\omega = 0{,}0008726646259$$
$$4\omega = 0{,}0011635528346$$
$$5\omega = 0{,}0014544410433$$
$$6\omega = 0{,}0017453292519$$
$$7\omega = 0{,}0020362174606$$
$$8\omega = 0{,}0023271056693$$
$$9\omega = 0{,}0026179938779$$

Ergo $\omega \sin x$ et $\omega \cos x$ hoc modo invenietur:

7	·	0,00020362174606
0	·	· · · · · ·
7	·	0,000203621746
1	· ·	2908882
0	·	· · · · · ·
6	· ·	174532
7	· ·	20362
8	· ·	2327
1	· ·	29
1	· ·	2
8	· ·	2
6	· ·	0

$$\omega\sin x = \omega\cos x = 0{,}00020568902488$$

Ergo

$$\tfrac{1}{2}\omega\cos x = 0{,}00010284451244$$

per ω

1	·	0,000002908882
0	·	· · · ·
2	·	58177
8	·	23271
4	·	1163
4	· ·	116
5	· ·	14

$$\tfrac{1}{2}\omega^2\cos x = 0{,}00000002991623$$

$$\tfrac{1}{6}\omega^2\cos x = 0{,}00000000997208$$

per ω

9	·	0,000000000261
9	· · ·	26
7	· · ·	2

$$\tfrac{1}{6}\omega^3\cos x = 0{,}00000000000289.$$

Ergo ad sin $45^0\ 1^I$ inveniendum ad

$$\sin x = 0{,}7071067811865$$

addatur

$$\omega\cos x = \qquad 2056890249$$

$$0{,}7073124702114$$

subtrahatur

$$\tfrac{1}{2}\omega^2\sin x = \qquad 299162$$

$$0{,}7073124402952$$

subtrahatur

$$\tfrac{1}{6}\omega^3\cos x = \qquad 29$$

$$\sin 45^0 1^{\mathrm{I}} = 0{,}7073124402923 = \cos 44^0 59^{\mathrm{I}}$$

At ad $\cos 45^0 1^{\mathrm{I}}$ inveniendum a

$$\cos x = 0{,}7071067811865$$

subtrahatur

$$\omega\sin x = \qquad 2056890249$$

$$0{,}7069010921616$$

subtrahatur

$$\omega^2\cos x = \qquad 299162$$

$$0{,}7069010622454$$

addatur

$$\tfrac{1}{6}\omega^3\sin x = \qquad 29$$

$$\cos 45^0 1^{\mathrm{I}} = 0{,}7069010622483 = \sin 44^0 59^{\mathrm{I}}$$

.

EXEMPLUM 2

Ex datis sinu et cosinu arcus $67^0 30^{\mathrm{I}}$ *invenire sinus et cosinus arcuum* $67^0 31^{\mathrm{I}}$ *et* $67^0 29^{\mathrm{I}}$.

Absolvamus hunc calculum in fractionibus decimalibus tantum ad 7 notas, uti tabulae vulgares construi solent, sicque negotium facile per logarithmos conficietur. Cum sit $x = 67^0 30^{\mathrm{I}}$ et $\omega = 0{,}000290888$, erit

$$l\omega = 6{,}4637259$$

et

$$\begin{array}{ll} l\sin x = 9{,}9656153 & l\cos x = 9{,}5828397 \\ \hline l\omega = 6{,}4637259 & l\omega = 6{,}4637259 \\ \hline l\omega\sin x = 6{,}4293412 & l\omega\sin x = 6{,}4293412 \end{array}$$

$$\begin{array}{ll} l\tfrac{1}{2}\omega = 6{,}1626959 & l\tfrac{1}{2}\omega = 6{,}1626959 \\ \hline l\tfrac{1}{2}\omega^2\sin x = 2{,}5920371 & l\tfrac{1}{2}\omega^2\cos x = 2{,}2092615 \end{array}$$

ergo

$$\begin{array}{ll} \omega\sin x = 0{,}00026874 & \omega\cos x = 0{,}00011132 \\ \tfrac{1}{2}\omega^2\sin x = 0{,}00000004 & \tfrac{1}{2}\omega^2\cos x = 0{,}00000001 \end{array}$$

unde fit

$$\begin{array}{ll} \sin 67^0 31^{\mathrm{I}} = 0{,}9239908 & \cos 67^0 31^{\mathrm{I}} = 0{,}3824147 \\ \sin 67^0 29^{\mathrm{I}} = 0{,}9237681 & \cos 67^0 29^{\mathrm{I}} = 0{,}3829522 \end{array}$$

ubi nequidem terminis $\frac{1}{2}\omega^2\sin x$ et $\frac{1}{2}\omega^2\cos x$ erat opus.

97. Ex seriebus, quas supra invenimus,

$$\sin(x+\omega) = \sin x + \omega\cos x - \tfrac{1}{2}\omega^2\sin x + \tfrac{1}{6}\omega^3\cos x + \tfrac{1}{24}\omega^4\sin x + \text{etc.}$$
$$\cos(x+\omega) = \cos x - \omega\sin x - \tfrac{1}{2}\omega^2\cos x + \tfrac{1}{6}\omega^3\sin x + \tfrac{1}{24}\omega^4\cos x - \text{etc.}$$
$$\sin(x-\omega) = \sin x - \omega\cos x - \tfrac{1}{2}\omega^2\sin x + \tfrac{1}{6}\omega^3\cos x + \tfrac{1}{24}\omega^4\sin x - \text{etc.}$$
$$\cos(x-\omega) = \cos x + \omega\sin x - \tfrac{1}{2}\omega^2\cos x - \tfrac{1}{6}\omega^3\sin x + \tfrac{1}{24}\omega^4\cos x + \text{etc.}$$

sequitur per combinationem fore

$$\frac{\sin(x+\omega)+\sin(x-\omega)}{2}$$
$$= \sin x - \tfrac{1}{2}\omega^2\sin x + \tfrac{1}{24}\omega^4\sin x - \tfrac{1}{720}\omega^6\sin x + \text{etc.} = \sin x\cdot\cos\omega$$

et

$$\frac{\sin(x+\omega)-\sin(x-\omega)}{2}$$
$$=\omega\cos x-\tfrac{1}{6}\omega^3\cos x+\tfrac{1}{120}\omega^5\cos x-\tfrac{1}{5040}\omega^7\cos x+\text{etc.}=\cos x\cdot\sin\omega,$$

unde prodeunt series pro sinibus et cosinibus iam supra inventae

$$\cos\omega=1-\tfrac{1}{2}\omega^2+\tfrac{1}{24}\omega^4-\tfrac{1}{720}+\text{etc.}$$
$$\sin\omega=\omega-\tfrac{1}{6}\omega^3+\tfrac{1}{120}\omega^5-\tfrac{1}{5040}\omega^7+\text{etc.},$$

quae eaedem series ex primis ponendo $x=0$ consequuntur; cum enim sit $\cos x=1$ et $\sin x=0$, prima series $\sin\omega$, secunda vero $\cos\omega$ exhibebit.

98. Ponamus nunc quoque $y=\operatorname{tang}x$, ut sit $z=\operatorname{tang}(x+\omega)$*;* erit ob $y=\frac{\sin x}{\cos x}$

$$\frac{dy}{dx}=\frac{1}{\cos^2 x},\quad \frac{ddy}{2dx^2}=\frac{\sin x}{\cos^3 x},\quad \frac{d^3y}{2dx^3}=\frac{1}{\cos^2 x}+\frac{3\sin^2 x}{\cos^4 x}=\frac{1}{\cos^4 x}-\frac{2}{\cos^2 x},$$
$$\frac{d^4y}{2\cdot 4dx^4}=\frac{3\sin x}{\cos^5 x}-\frac{\sin x}{\cos^3 x},\quad \frac{d^5y}{2\cdot 4dx^5}=\frac{15}{\cos^6 x}-\frac{15}{\cos^4 x}+\frac{2}{\cos^2 x},$$

unde sequitur fore

$$z=\operatorname{tang}(x+\omega)=\operatorname{tang}x+\left\{\begin{aligned}&\frac{\omega}{\cos^2 x}+\frac{\omega^2\sin x}{\cos^3 x}+\frac{\omega^3}{\cos^4 x}+\frac{\omega^4\sin x}{\cos^5 x}+\text{etc.}\\&\qquad\qquad\qquad-\frac{2\omega^3}{3\cos^2 x}-\frac{\omega^4\sin x}{3\cos^3 x}-\text{etc.}\end{aligned}\right.$$

cuius formulae ope ex data cuiusvis anguli tangente inveniri possunt tangentes angulorum proximorum. Quia vero superior series est geometrica, ea in unam summam collecta erit

$$z=\operatorname{tang}(x+\omega)=\operatorname{tang}x+\frac{\omega+\omega^2\operatorname{tang}x}{\cos^2 x-\omega^2}-\frac{2\omega^3}{3\cos^2 x}-\frac{\omega^4\sin x}{3\cos^3 x}-\text{etc.}$$

seu

$$z=\operatorname{tang}(x+\omega)=\operatorname{tang}x+\frac{\sin x\cdot\cos x+\omega}{\cos^2 x-\omega^2}-\frac{2\omega^3}{3\cos^2 x}-\frac{\omega^4}{3\cos^3 x}-\text{etc.},$$

quae formula in hunc finem commodius adhibetur.

99. Similes expressiones quoque pro logarithmis sinuum, cosinuum et tangentium inveniri possunt. Sit enim y = logarithmo sinus anguli x, quod ita

exprimamus $y = l\sin x$, et $z = l\sin(x+\omega)$ ob

$\frac{dy}{dx} = \frac{n\cos x}{\sin x}$ erit $\frac{ddy}{dx^2} = \frac{-n}{\sin^2 x}$, $\frac{d^3y}{2dx^3} = \frac{2n\cos x}{\sin^3 x}$ etc.

unde fit

$$z = l\sin(x+\omega) = l\sin x + \frac{n\omega\cos x}{\sin x} - \frac{n\omega^2}{2\sin^2 x} + \frac{n\omega^3\cos x}{3\sin^3 x} - \text{etc.},$$

ubi n denotat numerum, per quem logarithmi hyperbolici multiplicari debent, ut prodeant logarithmi propositi. Sin autem sit $y = l\,\text{tang}\,x$ et $z = l\,\text{tang}(x+\omega)$, fiet

$$\frac{dy}{dx} = \frac{n}{\cos x\cdot\sin x} = \frac{2n}{\sin 2x}, \quad \frac{ddy}{2dx^2} = \frac{-2n\cos 2x}{(\sin 2x)^2} \text{ etc.}$$

ideoque

$$z = l\,\text{tang}(x+\omega) = l\,\text{tang}\,x\frac{dy}{dx} + \frac{2n\omega}{\sin 2x} - \frac{2n\omega^2\cos 2x}{(\sin 2x)^2} + \text{etc.},$$

quarum formularum ope logarithmi sinuum et tangentium interpolari possunt.

100. Ponamus denotare y arcum, cuius sinus logarithmus sit $= x$, seu ut sit $y = \text{A}.l\sin x$, et z esse arcum, cuius sinus logarithmus sit $= x+\omega$ seu $z = \text{A}.l\sin(x+\omega)$; erit $x = l\sin y$ et

$$\frac{dx}{dy} = \frac{n\cos y}{\sin y}, \text{ unde } \frac{dy}{dx} = \frac{\sin y}{n\cos y} ;$$

erit

$$\frac{ddy}{dx} = \frac{dy}{n\cos^2 y} = \frac{dx\sin y}{n^2\cos^3 y}, \quad \text{ergo} \quad \frac{ddy}{dx^2} = \frac{\sin y}{n^2\cos^3 y}.$$

Consequenter

$$z = y + \frac{\omega\sin y}{n\cos y} + \frac{\omega^2\sin y}{2n^2\cos^3 y} + \text{etc.}$$

Simili modo, si logarithmus cosinus detur, expressio reperietur.

Sin autem sit $y = \text{A}.l\,\text{tang}\,x$ et $z = \text{A}.l\,\text{tang}(x+\omega)$, cum sit $x = l\text{tang}\,y$, fiet

$$\frac{dx}{dy} = \frac{n}{\sin y\cdot\cos y} \quad \text{et} \quad \frac{dy}{dx} = \frac{\sin y\cdot\cos y}{n} = \frac{\sin 2y}{2n},$$

et

$$\frac{ddy}{dx^2} = \frac{\sin 2y\cdot\cos 2y}{2nn} = \frac{\sin 4y}{4nn}, \quad \frac{d^3y}{dx^3} = \frac{\sin 2y\cdot\cos 4y}{2n^3} \text{ etc. ;}$$

hinc

$$z = y + \frac{\omega \sin 2y}{2n} + \frac{\omega^2 \sin 2y \cdot \cos 2y}{4nn} + \frac{\omega^3 \sin 2y \cdot \cos 4y}{12n^3} + \text{ etc.}$$

101. Quoniam usus harum expressionum in condendis tabulis logarithmorum sinuum et tangentium ex antecedentibus facile perspici potest, his diutius non immorabimur. Consideremus ergo adhuc huiusmodi valorem
$y = e^x \sin nx$ sitque $z = e^{x+\omega} \sin n(x+\omega)$; quia est

$$\frac{dy}{dx} = e^x(\sin nx + n\sin nx)$$
$$\frac{ddy}{dx^2} = e^x((1-nn)\sin nx + 2n\cos nx)$$
$$\frac{d^3y}{dx^3} = e^x((1-3nn)\sin nx + n(3-2nn)\cos nx)$$
$$\frac{d^4y}{dx^4} = e^x\left(\left(1-6nn+n^4\right)\sin nx + n(4-4nn)\cos nx\right)$$
$$\frac{d^5y}{dx^5} = e^x\left(\left(1-10nn+5n^4\right)\sin nx + n\left(5-10nn+n^4\right)\cos nx\right),$$
etc.,

his substitutis et divisione per e^x instituta erit

$$\begin{aligned} e^{\omega}\sin n(x+\omega) = \sin nx \\ +\omega \sin nx + \tfrac{1-nn}{2}\omega^2 \sin nx + \tfrac{1-3nn}{6}\omega^3 \sin nx + \tfrac{1-6nn+n^4}{24}\omega^4 \sin nx + \text{etc.} \\ +n\omega\cos nx + \tfrac{2n}{2}\omega^2\cos nx + \tfrac{n(3-2nn)}{6}\omega^3 \cos nx + \tfrac{n(4-4nn)}{24}\omega^4\cos nx + \text{ etc.}\end{aligned}$$

102. Hinc plurima egregia corollaria deduci possunt; sufficiat autem nobis haec annotasse. Si fuerit $x = 0$, erit

$$e^{\omega}\sin n\omega = n\omega + \tfrac{2n}{2}\omega^2 + \frac{n(3-2nn)}{6}\omega^3 + \frac{n(4-4nn)}{24}\omega^4 + \frac{n\left(5-10n^2+n^4\right)}{120}\omega^5 + \text{ etc.}$$

Si sit $\omega = -x$, ob $\sin n(x+\omega) = 0$ erit

$$\operatorname{tang} nx = \frac{nx - \tfrac{2n}{2}x^2 - \tfrac{n(3-2nn)}{6}x^3 - \tfrac{n(4-4nn)}{24}x^4 - \tfrac{n\left(5-10n^2+n^4\right)}{120}x^5 - \text{ etc.}}{1 - x + \tfrac{1-nn}{2}x^2 - \tfrac{1-3nn}{6}x^3 + \tfrac{1-6nn+n^4}{24}x^4 - \text{etc.}}$$

Generaliter vero, si sit $n = 1$, habebitur

$$e^{\omega}\sin(x+\omega) = \sin x\left(1+\omega-\tfrac{1}{3}\omega^3-\tfrac{1}{6}\omega^4-\tfrac{1}{30}\omega^5+\tfrac{1}{630}\omega^7+\text{etc.}\right)$$
$$+\omega\cos x\left(1+\omega+\tfrac{1}{3}\omega^3-\tfrac{1}{30}\omega^4-\tfrac{1}{90}\omega^5-\tfrac{1}{630}\omega^7+\text{etc}\right).$$

Sin autem sit $n=0$, ob $\sin n(x+\omega) = n(x+\omega)$ et $\sin nx = nx$ atque $\cos nx = 1$, si ubique per n dividatur, prodibit

$$e^{\omega}(x+\omega) = x+\omega x+\tfrac{1}{2}\omega^2+\tfrac{1}{6}\omega^3 x+\tfrac{1}{24}\omega^4 x+\text{etc.}$$
$$+\omega+\omega^2+\tfrac{1}{2}\omega^3+\tfrac{1}{6}\omega^4+\tfrac{1}{24}\omega^5+\text{etc.},$$

cuius seriei ratio est manifesta.