CHAPTER III

CONCERNING THE FINDING OF FINITE DIFFERENCES

44. Just as differentials may be found easily from the finite difference of functions, in the beginning we have explained this more widely and thus we have derived the principle of differentiation from this source. For if the differences, which were being assumed finite, should vanish and they may change into nothing, the differentials arise; and because in this case several and often innumerable terms, which constitute the finite differences , are rejected, the differentials can be found and expressed much more easily and succinctly than finite differences. Nor hence in turn therefore may a way be considered to be apparent to ascend from finite differentials to finite differences. Yet meanwhile in that way which we will use here, all the finite differences of these will be able to be defined from the differentials of all orders of any function.

45. Let y be some function of x; which since on putting $x+dx$ in place of x may change into $y+dy$, if again on putting $x+dx$ in place of x, the value $y+dy$ will be augmented by its own differential $dy+ddy$ and becomes $=y+2dy+ddy$, which therefore will correspond to the value $x+2dx$ of x. In a similar manner if we may put the quantity x to be continually augmented by its own differential dx, so that successively it may adopt the values $x+dx,\ x+2dx,\ x+3dx,\ x+4dx$ etc., the corresponding values of y will be those which this table shows.

Values of	Corresponding values of the function
x	y
$x+dx$	$y+dy$
$x+2dx$	$y+2dy+ddy$
$x+3dx$	$y+3dy+3ddy+d^3y$
$x+4dx$	$y+4dy+6ddy+4d^3y+d^4y$
$x+5dx$	$y+5dy+10ddy+10d^3y+5d^4y+d^5y$
$x+6dx$	$y+6dy+15ddy+20d^3y+15d^4y+6d^5y+d^6y$
etc.	etc.

46. Therefore generally if x may change into $x+ndx$, the function y may accept this form

$$y+\frac{n}{1}dy+\frac{n(n-1)}{1\cdot 2}ddy+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}d^3y+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}d^4y+\text{etc.};$$

in which expression although any term is infinitely less than the preceding, yet we have omitted nothing, so that this formula may be used conveniently in the present undertaking. For we may set up an infinitely large number for n, and because we have noted it is possible to happen, that the product from an infinitely great quantity into an infinitely small quantity may be equal to a finite quantity, in the first place the second term can become homogeneous everywhere or ndy will be able to represent a finite quantity. And on account of the same reasoning the third term $\frac{n(n-1)}{1\cdot 2}ddy$, although ddy is infinitely less than dy, yet because the other factor $\frac{n(n-1)}{1\cdot 2}$ is an infinite number of times greater than n, also the third term will be able to express a finite quantity; and thus on putting an infinite number for n no term of this expression will be permitted to be rejected.

47. Moreover on putting in place an infinite number n, however that number may be increased or diminished [finitely], the resulting number will have the ratio of equality to n and hence it will be possible to write n everywhere for the individual factors $n-1, n-2, n-3, n-4$ etc. For since there shall be $\frac{n(n-1)}{1\cdot 2}ddy=\frac{1}{2}nnddy-\frac{1}{2}nddy$, the former term $\frac{1}{2}nnddy$ will hold the ratio as n to 1 to the latter term $\frac{1}{2}nddy$ and thus here may vanish with respect to that ; therefore in place of $\frac{n(n-1)}{1\cdot 2}$ it will be possible to write $\frac{1}{2}nn$. In a similar manner the coefficient of the fourth term $\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$ can be contracted into $\frac{n^3}{6}$ and equally the numbers can be ignored in the following terms, in which n may be reduced into factors. Truly with this done the function y may receive the following value, if in place of x there may be put $x+ndx$ with an infinite number n assumed,

$$y+\frac{n}{1}dy+\frac{nn}{1\cdot 2}ddy+\frac{n^3d^3y}{1\cdot 2\cdot 3}+\frac{n^4d^4y}{1\cdot 2\cdot 3\cdot 4}+\frac{n^5d^5y}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\text{etc.}$$

48. Therefore since with an infinitely great number n taken, even if dx shall be infinitely small, the product ndx will be able to express a finite quantity, we may put $ndx=\omega$, so that there shall be $n=\frac{\omega}{dx}$; certainly n will be an infinite number, since it shall be the quotient from the division of a finite quantity ω by the resulting infinitely small dx. But with the value used in place of n, if the variable quantity x may be increased by some finite quantity ω or if in place of x there may be put $x+\omega$, then whatever the arbitrary function of y to be in this formula

$$y+\frac{\omega dy}{1dx}+\frac{\omega ddy}{1\cdot 2dx^2}+\frac{\omega^3 d^3 y}{1\cdot 2\cdot 3dx^3}+\frac{\omega^4 d^4 y}{1\cdot 2\cdot 3\cdot 4dx^4}+\text{etc.},$$

we will know that the individual terms of this expression will be able to be found by the continuous differentiation of y. For since y shall be a function of x, we have shown above all these functions $\frac{dy}{dx}$, $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$ etc . present finite quantities.

49. Therefore since, while it is assumed the quantity of the variable x to be increased by ω, any function y of this may be increased by its own first difference , as we have indicated above by Δy, with $\omega=\Delta x$ arising, the differences of y will be able to be found by continued differentiation; indeed there will be

$$\Delta y=\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}+\frac{\omega^3 d^3 y}{6dx^3}+\frac{\omega^4 d^4 y}{24dx^4}+\text{etc.}$$

or

$$\Delta y=\frac{\Delta x}{1}\cdot\frac{dy}{dx}+\frac{\Delta x^2}{2}\cdot\frac{ddy}{dx^2}+\frac{\Delta x^3}{6}\cdot\frac{d^3y}{dx^3}+\frac{\Delta x^4}{24}\cdot\frac{d^4y}{dx^4}+\text{etc.}$$

And thus the finite difference Δy is expressed by a progression, the individual terms of which proceed following the powers of Δx . And hence in turn it is apparent, if the quantity x may be increased by an infinitely small amount only, so that Δx may change into the differential of this dx, all the terms besides the first vanish and there becomes $\Delta y=dy$; for with $\Delta x=dx$ the difference Δy by definition will change into the differential dy.

50. Because, if there is put $x+\omega$ in place of x, any function y of this will adopt the following value

$$y+\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}+\frac{\omega^3 d^3 y}{6dx^3}+\frac{\omega^4 d^4 y}{24dx^4}+\text{etc.},$$

the truth of this expression will be able to be confirmed from examples of this kind, in which the higher differentials of y vanish at last; indeed from these cases the number of terms of the above expression becomes finite.

EXAMPLE 1

The value of the expression $xx-x$ is sought, if in place of x there may be put $x+1$.

There may be put $y=xx-x$, and since x may be put in place to become $x+1$, there is made $\omega=1$; now with the differentials taken there will be

$$\frac{dy}{dx}=2x-1,\ \ \frac{ddy}{dx^2}=2,\ \ \frac{d^3y}{dx^3}=0\ \ \text{etc.}$$

Hence the function $y = xx - x$ on putting $x+1$ in place of x will change into

$$xx - x + 1(2x-1) + \tfrac{1}{2}\cdot 2 = xx + x.$$

But if moreover in $xx - x$ in place of x there is put actually $x+1$, it will become

$$\begin{array}{l} xx \text{ into } xx+2x+1 \\ x \text{ into } x+1 \\ \hline \end{array}$$

Therefore

$$xx - x \text{ into } xx + x.$$

EXAMPLE 2

The value of the expression $x^3 + xx + x$ *is sought, if in place of* x *there may be put* $x+2$.

There may be put $y = x^3 + xx + x$ and there is made $\omega = 2$; now since there shall be

$$x^3 + xx + x,$$

there will be

$$\frac{dy}{dx} = 3xx + 2x + 1,\quad \frac{ddy}{dx^2} = 6x + 2,\quad \frac{d^3y}{dx^3} = 6,\quad \frac{d^4y}{dx^4} = 0 \text{ etc.},$$

From these the value of the function $y = x^3 + xx + x$, if for x there may be put in place $x+2$, will be the following

$$x^3 + xx + x + 2(3xx + 2x + 1) + \tfrac{4}{2}(6x+2) + \tfrac{8}{6}\cdot 6 = x^3 + 7xx + 17x + 14,$$

which likewise will be produced, in actually in place of x there may be substituted $x+2$.

EXAMPLE 3

The value of the expression $xx + 3x + 1$ *is sought, if in place of* x *there may be put* $x-3$.

Therefore there is made $\omega = -3$, and on putting

$$xx + 3x + 1$$

there will be

$$\frac{dy}{dx} = 2x + 3,\quad \frac{ddy}{dx^2} = 2\ ,\quad \frac{d^3y}{dx^3} = 0 \text{ etc.},$$

from which on putting $x-3$ in place of x the function $xx+3x+1$ will change into

$$x^2+3x+1-\tfrac{3}{1}(2x+3)+\tfrac{9}{2}\cdot 2 \ = x^2-3x+1 .$$

51. If a negative number is taken for ω, the value may be found, that any function of x adopts, while the quantity x is diminished by the given quantity ω. Evidently if in place of x there may be put $x-\omega$, any function y of x will take that value

$$y-\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}-\frac{\omega^3 d^3 y}{6dx^3}+\frac{\omega^4 d^4 y}{24dx^4}-\text{etc.}$$

from which all the variations, which the function y can undergo, while the quantity x may be varied on both sides, are able to be found. But if moreover y were a whole rational function of x, because finally it arrives at the vanishing differentials of this, the varied value of the expression by a finite expression ; but if y were not a function of this kind, the varied value may be expressed by an infinite series, therefore the sum of which will be able to present a finite expression, because if the substitution is actually put in place, the varied value may be assigned easily.

52. But just as the first difference had been found, thus also the following differences are able to be shown by similar expressions. For x may adopt successively the values $x+\omega,\ x+2\omega,\ x+3\omega,\ x+4\omega$ etc. and the corresponding values of y may be indicated by $y^{\text{I}}, y^{\text{II}}, y^{\text{III}}, y^{\text{IV}}$ etc ., in the same way as we have shown at the beginning of this book. Therefore because $y^{\text{I}}, y^{\text{II}}, y^{\text{III}}, y^{\text{IV}}$ etc . shall be the values, which y obtains, if in place of x there may be written respectively $x+\omega,\ x+2\omega,\ x+3\omega,\ x+4\omega$ etc ., by the manner demonstrated the values of y may be expressed thus from this :

$$y^{\text{I}}=y+\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}+\frac{\omega^3 d^3 y}{6dx^3}+\frac{\omega^4 d^4 y}{24dx^4}+\text{etc.}$$
$$y^{\text{II}}=y+\frac{2\omega dy}{dx}+\frac{4\omega^2 ddy}{2dx^2}+\frac{8\omega^3 d^3 y}{6dx^3}+\frac{16\omega^4 d^4 y}{24dx^4}+\text{etc.}$$
$$y^{\text{III}}=y+\frac{3\omega dy}{dx}+\frac{9\omega^2 ddy}{2dx^2}+\frac{27\omega^3 d^3 y}{6dx^3}+\frac{81\omega^4 d^4 y}{24dx^4}+\text{etc.}$$
$$y^{\text{IV}}=y+\frac{4\omega dy}{dx}+\frac{16\omega^2 ddy}{2dx^2}+\frac{64\omega^3 d^3 y}{6dx^3}+\frac{256\omega^4 d^4 y}{24dx^4}+\text{etc.}$$
etc.

53. Therefore since, if $\Delta y,\ \Delta^2 y,\ \Delta^3 y,\ \Delta^4 y$ etc. may denote the first, second, third, fourth, etc. differences, there shall be

$$\Delta y = y^{\mathrm{I}} - y$$
$$\Delta^2 y = y^{\mathrm{II}} - 2y^{\mathrm{I}} + y,$$
$$\Delta^3 y = y^{\mathrm{III}} - 3y^{\mathrm{II}} + 3y^{\mathrm{I}} - y,$$
$$\Delta^4 y = y^{\mathrm{IV}} - 4y^{\mathrm{III}} + 6y^{\mathrm{II}} - 4y^{\mathrm{I}} + y$$
etc.,

these differences may be expressed by the differentials in this manner :

$$\Delta y = \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^4 d^4 y}{24dx^4} + \text{etc.}$$
$$\Delta^2 y = \frac{(2^2-2\cdot 1)\omega^2 ddy}{2dx^2} + \frac{(2^3-2\cdot 1)\omega^3 d^3 y}{6dx^3} + \frac{(2^4-2\cdot 1)\omega^4 d^4 y}{24dx^4} + \text{etc.}$$
$$\Delta^3 y = \frac{(3^3-3\cdot 2^3+3\cdot 1)\omega^3 d^3 y}{6dx^3} + \frac{(3^4-3\cdot 2^4+3\cdot 1)\omega^4 d^4 y}{24dx^4} + \text{etc.}$$
$$\Delta^4 y^4 = \frac{(4^4-4\cdot 3^4+6\cdot 2^4-4\cdot 1)\omega^4 d^4 y}{24dx^4} + \frac{(4^5-4\cdot 3^5+6\cdot 2^5-4\cdot 1)\omega^5 d^5 y}{120dx^5} + \text{etc.}$$
etc.

54. How great a use these expressions of the differences may bring to the instruction of series and progressions, may both by itself be apparent, as well as we may explain more fully in the following. Yet meanwhile in this chapter we may consider carefully the use, which hence is immediately overwhelming in its application to the notion of series. Though generally the indices of the terms of any series are assumed to constitute an arithmetical progression, of which the difference is unity, yet, so that the use may appear wider and the application will be able to be made easier, we may put the difference $=\omega$, thus so that, if the general term or that, which may correspond to the index *x*, were *y*, the following may be agreed upon for the indices $x+\omega$, $x+2\omega$, $x+3\omega$ etc. But if therefore the following series of terms may correspond to these indices

$$\begin{array}{llllll} x, & x+\omega, & x+2\omega, & x+3\omega, & x+4\omega & \text{etc.} \\ y, & P, & Q, & R, & S & \text{etc.,} \end{array}$$

the individual terms may be defined in this way from *y* and from the differentials of this:

$$P = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^4 d^4 y}{24dx^4} + \text{etc.}$$
$$Q = y + \frac{2\omega dy}{dx} + \frac{4\omega^2 ddy}{2dx^2} + \frac{8\omega^3 d^3 y}{6dx^3} + \frac{16\omega^4 d^4 y}{24dx^4} + \text{etc.}$$
$$R = y + \frac{3\omega dy}{dx} + \frac{9\omega^2 ddy}{2dx^2} + \frac{27\omega^3 d^3 y}{6dx^3} + \frac{81\omega^4 d^4 y}{24dx^4} + \text{etc.}$$
$$S = y + \frac{4\omega dy}{dx} + \frac{16\omega^2 ddy}{2dx^2} + \frac{64\omega^3 d^3 y}{6dx^3} + \frac{256\omega^4 d^4 y}{24dx^4} + \text{etc.}$$
$$T = y + \frac{5\omega dy}{dx} + \frac{25\omega^2 ddy}{2dx^2} + \frac{125\omega^3 d^3 y}{6dx^3} + \frac{625\omega^4 d^4 y}{24dx^4} + \text{etc.}$$
etc.

55. If these expressions may be subtracted in turn from each other, y will no longer be present in the differences and there will be

$$P - y = \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^4 d^4 y}{24dx^4} + \text{etc.}$$
$$Q - P = \frac{\omega dy}{dx} + \frac{3\omega^2 ddy}{2dx^2} + \frac{7\omega^3 d^3 y}{6dx^3} + \frac{15\omega^4 d^4 y}{24dx^4} + \text{etc.}$$
$$R - Q = \frac{\omega dy}{dx} + \frac{5\omega^2 ddy}{2dx^2} + \frac{19\omega^3 d^3 y}{6dx^3} + \frac{65\omega^4 d^4 y}{24dx^4} + \text{etc.}$$
$$S - R = \frac{\omega dy}{dx} + \frac{7\omega^2 ddy}{2dx^2} + \frac{37\omega^3 d^3 y}{6dx^3} + \frac{175\omega^4 d^4 y}{24dx^4} + \text{etc.}$$
$$T - S = \frac{\omega dy}{dx} + \frac{9\omega^2 ddy}{2dx^2} + \frac{61\omega^3 d^3 y}{6dx^3} + \frac{369\omega^4 d^4 y}{24dx^4} + \text{etc.}$$
etc.

If these expressions may again be subtracted from each other in turn, the first differences also will cancel and there will be

$$Q - 2P + y = \frac{2\omega^2 ddy}{2dx^2} + \frac{6\omega^3 d^3 y}{6dx^3} + \frac{14\omega^4 d^4 y}{24dx^4} + \text{etc.}$$
$$R - 2Q + P = \frac{2\omega^2 ddy}{2dx^2} + \frac{12\omega^3 d^3 y}{6dx^3} + \frac{50\omega^4 d^4 y}{24dx^4} + \text{etc.}$$
$$S - 2R + Q = \frac{2\omega^2 ddy}{2dx^2} + \frac{18\omega^3 d^3 y}{6dx^3} + \frac{110\omega^4 d^4 y}{24dx^4} + \text{etc.}$$
$$T - 2S + R = \frac{2\omega^2 ddy}{2dx^2} + \frac{24\omega^3 d^3 y}{6dx^3} + \frac{194\omega^4 d^4 y}{24dx^4} + \text{etc.}$$
etc.

Moreover again from these taken from each other the second differentials also vanish from the calculation :

$$R-3Q+3P=\frac{6\omega^3d^3y}{6dx^3}+\frac{36\omega^4d^4y}{24dx^4}+\text{etc.}$$

$$S-3R+3Q-P=\frac{6\omega^3d^3y}{6dx^3}+\frac{60\omega^4d^4y}{24dx^4}+\text{etc.}$$

$$T-3S+3R-Q=\frac{6\omega^3d^3y}{6dx^3}+\frac{84\omega^4d^4y}{24dx^4}+\text{etc.}$$

etc.

Moreover by continuing the subtraction further there becomes

$$S-4R+6Q-4P+y=\frac{24\omega^4d^4y}{24dx^4}+\text{etc.}$$

$$T-4S+6R-4Q+P=\frac{24\omega^4d^4y}{24dx^4}+\text{etc.}$$

etc.

and

$$T-5S+10R-10Q+5P-y=\frac{120\omega^5d^5y}{120dx^5}+\text{etc.}$$

etc.

56. But if therefore *y* were a rational integral function of *x*, because the higher differentials of this finally vanish, by preceding in this manner vanishing expressions finally may be come upon. Therefore since these expressions shall be the differences of this *y*, we may consider the forms and coefficient of these more carefully

$$y=y$$

$$\Delta y=\frac{\omega dy}{dx}+\frac{\omega^2ddy}{2dx^2}+\frac{\omega^3d^3y}{6dx^3}+\frac{\omega^4d^4y}{24dx^4}+\frac{\omega^5d^5y}{120dx^5}+\text{etc.}$$

$$\Delta^2 y=\frac{\omega^2ddy}{dx^2}+\frac{3\omega^3d^3y}{3dx^3}+\frac{7\omega^4d^4y}{3\cdot4dx^4}+\frac{15\omega^5d^5y}{3\cdot4\cdot5dx^5}+\frac{31\omega^6d^6y}{3\cdot4\cdot5\cdot6dx^6}+\text{etc.}$$

$$\Delta^3 y=\frac{\omega^3d^3y}{dx^3}+\frac{6\omega^4d^4y}{4dx^4}+\frac{25\omega^5d^5y}{4\cdot5dx^5}+\frac{90\omega^6d^6y}{4\cdot5\cdot6dx^6}+\frac{301\omega^7d^7y}{4\cdot5\cdot6\cdot7dx^7}+\text{etc.}$$

$$\Delta^4 y=\frac{\omega^4d^4y}{dx^4}+\frac{10\omega^5d^5y}{5dx^5}+\frac{65\omega^6d^6y}{5\cdot6dx^6}+\frac{350\omega^7d^7y}{5\cdot6\cdot7dx^7}+\frac{1701\omega^8d^8y}{5\cdot6\cdot7\cdot8dx^8}+\text{etc.}$$

$$\Delta^5 y=\frac{\omega^5d^5y}{dx^5}+\frac{15\omega^6d^6y}{6dx^6}+\frac{140\omega^7d^7y}{6\cdot7dx^7}+\frac{1050\omega^8d^8y}{6\cdot7\cdot8dx^8}+\frac{6951\omega^9d^9y}{6\cdot7\cdot8\cdot9dx^9}+\text{etc.}$$

$$\Delta^6 y=\frac{\omega^6d^6y}{dx^6}+\frac{21\omega^7d^7y}{7dx^7}+\frac{266\omega^8d^8y}{7\cdot8dx^8}+\frac{2646\omega^9d^9y}{7\cdot8\cdot9dx^9}+\frac{22827\omega^{10}d^{10}y}{7\cdot8\cdot9\cdot10dx^{10}}+\text{etc.}$$

etc.

It is clear how the denominators proceed in which series ; but the coefficients of the numerators may be formed thus, so that the coefficient of any number shall be the sum from the above standing and from the preceding multiplied by the exponent of the difference. Thus in the series the difference expressed $\Delta^6 y$ is $2646=1050+6\cdot266$.

57. We may consider also the same series likewise continued backwards, which may contain the terms corresponding to the indices $x-\omega, x-2\omega, x-3\omega$ etc.,

$$\begin{array}{cccccccccc} x-4\omega, & x-3\omega, & x-2\omega, & x-\omega, & x, & x+\omega, & x+2\omega, & x+3\omega, & x+4\omega & \text{etc.} \\ s, & r, & q, & p, & y, & P, & Q, & R, & S & \text{etc.} \end{array}$$

Therefore since there shall be

$$\begin{aligned} p &= y-\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}-\frac{\omega^3 d^3y}{6dx^3}+\frac{\omega^4 d^4y}{24dx^4}-\text{etc.} \\ q &= y-\frac{2\omega dy}{dx}+\frac{4\omega^2 ddy}{2dx^2}-\frac{8\omega^3 d^3y}{6dx^3}+\frac{16\omega^4 d^4y}{24dx^4}-\text{etc.} \\ r &= y-\frac{3\omega dy}{dx}+\frac{9\omega^2 ddy}{2dx^2}-\frac{27\omega^3 d^3y}{6dx^3}+\frac{81\omega^4 d^4y}{24dx^4}-\text{etc.} \\ s &= y-\frac{4\omega dy}{dx}+\frac{16\omega^2 ddy}{2dx^2}-\frac{64\omega^3 d^3y}{6dx^3}+\frac{256\omega^4 d^4y}{24dx^4}-\text{etc.} \\ &\text{etc.} \end{aligned}$$

from these values subtracted from the above P, Q, R, S etc. there will be

$$\begin{aligned} \frac{P-p}{2} &= \frac{\omega dy}{dx}+\frac{\omega^3 d^3y}{6dx^3}+\frac{\omega^5 d^5y}{120dx^5}+\text{etc.} \\ \frac{Q-q}{2} &= \frac{2\omega dy}{dx}+\frac{8\omega^3 d^3y}{6dx^3}+\frac{32\omega^5 d^5y}{120dx^5}+\text{etc.} \\ \frac{R-r}{2} &= \frac{3\omega dy}{dx}+\frac{27\omega^3 d^3y}{6dx^3}+\frac{243\omega^5 d^5y}{120dx^5}+\text{etc.} \\ \frac{S-s}{2} &= \frac{4\omega dy}{dx}+\frac{64\omega^3 d^3y}{6dx^3}+\frac{1024\omega^5 d^5y}{120dx^5}+\text{etc.} \\ &\text{etc.} \end{aligned}$$

But if these terms may be added to these above, then, just as here the differentials of even order are present, the odd differentials depart from the calculation.
For there will be

$$\frac{P+p}{2} = y + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^4 d^4 y}{24dx^4} + \frac{\omega^6 d^6 y}{720dx^6} + \text{etc.}$$

$$\frac{Q+q}{2} = y + \frac{4\omega^2 ddy}{2dx^2} + \frac{16\omega^4 d^4 y}{24dx^4} + \frac{64\omega^6 d^6 y}{720dx^6} + \text{etc.}$$

$$\frac{R+r}{2} = y + \frac{9\omega^2 ddy}{2dx^2} + \frac{81\omega^4 d^4 y}{24dx^4} + \frac{729\omega^6 d^6 y}{720dx^6} + \text{etc.}$$

$$\frac{S+s}{2} = y + \frac{16\omega^2 ddy}{2dx^2} + \frac{256\omega^4 d^4 y}{24dx^4} + \frac{4096\omega^6 d^6 y}{720dx^6} + \text{etc.}$$

etc.

58. Because all the preceding terms are able to be expressed, if these may be gathered into one sum, the summatory term of the proposed series will be produced. Clearly the first term corresponds to the first index $x - n\omega$ and the first term itself will be

$$= y - \frac{n\omega dy}{dx} + \frac{n^2\omega^2 ddy}{2dx^2} - \frac{n^3\omega^3 d^3 y}{6dx^3} + \frac{n^4\omega^4 d^4 y}{24dx^4} - \text{etc.}$$

Therefore since the term corresponding to the index x shall be $= y$ and the number of all the terms shall be $= n+1$, the sum of all from the first inclusively to the final y of the sum, or the summatory term

$$\begin{aligned} = (n+1)y &- \frac{\omega dy}{dx}(1+2+3+\cdots+n) \\ &+ \frac{\omega^2 ddy}{2dx^2}\left(1^2+2^2+3^2+\cdots+n^2\right) \\ &- \frac{\omega^3 d^3 y}{6dx^3}\left(1^3+2^3+3^3+\cdots+n^3\right) \\ &+ \frac{\omega^4 d^4 y}{24dx^4}\left(1^4+2^4+3^4+\cdots+n^4\right) \\ &- \frac{\omega^5 d^5 y}{120dx^5}\left(1^5+2^5+3^5+\cdots+n^5\right) \end{aligned}$$

etc.

59. Moreover we have shown above the sums of the individual terms of the series [see §62 of Part One] ; which if they may be substituted here, the sum of our proposed series will be

$$
\begin{aligned}
&= (n+1)y - \frac{\omega dy}{dx}\left(\tfrac{1}{2}nn + \tfrac{1}{2}n\right)\\
&\quad + \frac{\omega^2 ddy}{2dx^2}\left(\tfrac{1}{3}n^3 + \tfrac{1}{2}nn + \tfrac{1}{6}n\right)\\
&\quad - \frac{\omega^3 d^3y}{6dx^3}\left(\tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2\right)\\
&\quad + \frac{\omega^4 d^4y}{24dx^4}\left(\tfrac{1}{5}n^5 + \tfrac{1}{2}n^4 + \tfrac{1}{3}n^3 - \tfrac{1}{30}n\right)\\
&\quad - \frac{\omega^5 d^5y}{120dx^5}\left(\tfrac{1}{6}n^6 + \tfrac{1}{2}n^5 + \tfrac{5}{12}n^4 - \tfrac{1}{12}n^2\right)\\
&\qquad \text{etc.}
\end{aligned}
$$

where n will be given from the index of the first term, from which the sum be may be computed. Thus if there is put $\omega = 1$ and the index of the first term shall be $= 1$, of the second $= 2$ and of the final $= x$, thus so that this series shall be proposed

$$
\begin{matrix} 1, & 2, & 3, & 4, \cdots\cdots x \\ a, & b, & c, & d, \cdots\cdots y, \end{matrix}
$$

the sum of this series will be on account of $x - n = 1$ and $n = x - 1$

$$
\begin{aligned}
&= xy - \frac{dy}{dx}\left(\tfrac{1}{2}xx - \tfrac{1}{2}x\right)\\
&\quad + \frac{ddy}{2dx^2}\left(\tfrac{1}{3}x^3 - \tfrac{1}{2}xx + \tfrac{1}{6}x\right)\\
&\quad - \frac{d^3y}{6dx^3}\left(\tfrac{1}{4}x^4 - \tfrac{1}{2}x^3 + \tfrac{1}{4}xx\right)\\
&\quad + \frac{d^4y}{24dx^4}\left(\tfrac{1}{5}x^5 - \tfrac{1}{2}x^4 + \tfrac{1}{3}x^3 - \tfrac{1}{30}x\right)\\
&\quad - \frac{d^5y}{120dx^5}\left(\tfrac{1}{6}x^6 - \tfrac{1}{2}x^5 + \tfrac{5}{12}x^4 - \tfrac{1}{12}x^2\right)\\
&\quad + \frac{d^6y}{720dx^6}\left(\tfrac{1}{7}x^7 - \tfrac{1}{2}x^6 + \tfrac{1}{2}x^5 - \tfrac{1}{6}x^3 + \tfrac{1}{42}x\right)\\
&\qquad \text{etc.}
\end{aligned}
$$

60. From this expression of the sum, because the coefficients may be much increased, if x were a large number, it would supply little to the instruction of series; yet meanwhile it will aid other properties flowing from that to be kept in mind. Let the general term be $y = x^n$ and the summatory term may be indicated by $S.\,y$ or $S.x^n$.
[Note that in this designation of a sum, the variable is *x*, and not *n*, as we are accustomed to use nowadays with the Σ summation notation; thus *x* successively takes the values $x,\ x-1,\ x-2$, etc.] With which designation used everywhere there will be

$$\tfrac{1}{2}xx - \tfrac{1}{2}x = S.x - x$$
$$\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{6}x = S.x^2 - x^2$$
$$\tfrac{1}{4}x^4 - \tfrac{1}{2}x^3 + \tfrac{1}{4}xx = S.x^3 - x^3$$
etc.

On account of which there will be obtained from the above expression

$$S.x^n = x^{n+1} - nx^{n-1}S.x + nx^n$$
$$+\tfrac{n(n-1)}{1\cdot 2}x^{n-2}S.x^2 - \tfrac{n(n-1)}{1\cdot 2}x^n - \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}S.x^3 + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^n + \text{etc.}$$

But since there shall be

$$(1-1)^n = 0 = 1 - n + \tfrac{n(n-1)}{1\cdot 2} - \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3} + \text{etc.},$$

there will be

$$n - \tfrac{n(n-1)}{1\cdot 2} + \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3} - \text{etc.} = 1$$

and thus with the case excepted $n = 0$, from which that expression becomes $= 0$,

$$S.x^n = x^{n+1} + x^n - nx^{n-1}S.x + \tfrac{n(n-1)}{1\cdot 2}x^{n-2}S.x^2$$
$$-\tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}S.x^3 + \tfrac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^{n-4}S.x^4 - \text{etc.}$$

61. So that both the truth as well as the strength of this formula may be seen more clearly, we may set out individual cases and in the first place there shall be $S.x = x^2 + x - S.x$ and thus $S.x = \frac{xx+x}{2}$, just as is agreed upon well enough. Therefore we may put $n = 2$ and there will be

$$S.\,x^2 = x^3 + xx - 2xS.x + S.x^2,$$

which equation, since the terms on each side $S.x^2$ cancel each other, which gives the same preceding $S.x = \frac{xx+x}{2}$. If there shall be $n = 3$, there will be

$$S.\,x^3 = x^4 + x^3 - 3x^2S.x + 3xS.x^2 - S.x^3$$

and thus

$$S.x^3 = \tfrac{3}{2}xS.x^2 - \tfrac{3}{2}x^2S.x + \tfrac{1}{2}x^3(x+1);$$

if there may be put $n = 4$, there will be produced

$$S.x^4 = x^5 + x^4 - 4x^3S.x + 6x^2S.x^2 - 4xS.x^3 + S.x^4,$$

from which on account of $S.x^4$ canceling there will be

$$S.x^3 = \tfrac{3}{2}xS.x^2 - x^2S.x + \tfrac{1}{4}x^3(x+1)$$

but with the triple of which there may be subtracted twice the preceding, there will remain

$$S.x^3 = \tfrac{3}{2}xS.x^2 - \tfrac{1}{4}x^3(x+1).$$

If there is put $n = 5$, there is made

$$S.x^5 = x^6 + x^5 - 5x^4S.x + 10x^3S.x^2 - 10x^2S.x^3 + 5xS.x^4 - S.x^5$$

or

$$S.x^5 = \tfrac{5}{2}xS.x^4 - 5x^2S.x^3 + 5x^3S.x^2 - \tfrac{5}{2}x^4S.x + \tfrac{1}{2}x^5(x+1)$$

and from $n = 6$ there follows

$$S.x^6 = x^7 + x^6 - 6x^5S.x + 15x^4S.x^2 - 20x^3S.x^3 + 15x^2S.x^4 - 6xS.x^5 + S.x^6$$

or

$$S.x^5 = \tfrac{5}{2}xS.x^4 - \tfrac{10}{3}x^2S.x^3 + \tfrac{5}{2}x^3S.x^2 - x^4S.x + \tfrac{1}{6}x^5(x+1).$$

62. Therefore from these we may conclude generally, if there were $n = 2m+1$, to be

$$S.x^{2m+1} = \tfrac{2m+1}{2}xS.x^{2m} - \tfrac{(2m+1)2m}{2\cdot1\cdot2}x^2S.x^{2m-1}$$
$$+\tfrac{(2m+1)2m(2m-1)}{2\cdot1\cdot2\cdot3}x^3S.x^{2m-2} - \cdots - \tfrac{2m+1}{2}x^{2m}S.x + \tfrac{1}{2}x^{2m+1}(x+1).$$

But if there shall be $n = 2m+2$, because the terms $S.x^{2m+2}$ mutually cancel each other, there may be found

$$S.x^{2m+1} = \tfrac{2m+1}{2}xS.x^{2m} - \tfrac{(2m+1)2m}{2\cdot3}x^2S.x^{2m-1}$$
$$+\tfrac{(2m+1)2m(2m-1)}{2\cdot3\cdot4}x^3S.x^{2m-2} - \cdots - x^{2m}S.x + \tfrac{1}{2m+2}x^{2m+1}(x+1).$$

Therefore the sum of the odd powers are able to be determined in two ways from the sums of the lower powers, and from the different combinations of these two formulas boundless others are able to be formed.

63. But the sums of odd powers are able to be defined much easier from the preceding and indeed according to that it will suffice to know only the sum of the preceding even powers. Indeed from the sum of the powers shown above it is agreed that the number of terms requiring to constitute the sum to be increased only with even powers, thus so that the sum of the odd powers may depend on just as many terms as the sum of the preceding even powers. Thus if the sum of the even powers x^{2n} shall be

$$S.x^{2n} = \alpha x^{2n+1} + \beta x^{2n} + \gamma x^{2n-1} - \delta x^{2n-3} + \varepsilon x^{2n-5} - \text{etc.}$$

(we have seen indeed after the third term the alternate [even power] terms to be missing and the signs to alternate), hence the sum of the sequence of powers x^{2n+1} may be found, if the individual terms of this may be multiplied respectively by these numbers

$$\tfrac{2n+1}{2n+2}x,\quad \tfrac{2n+1}{2n+1}x,\quad \tfrac{2n+1}{2n}x,\quad \tfrac{2n+1}{2n-1}x,\quad \tfrac{2n+1}{2n-2}x,\quad \text{etc.}$$

not disregarding the missing terms; therefore there will be

$$S.x^{2n+1} = \tfrac{2n+1}{2n+2}\alpha x^{2n+2} + \tfrac{2n+1}{2n+1}\beta x^{2n+1} + \tfrac{2n+1}{2n}\gamma x^{2n} - \tfrac{2n+1}{2n-2}\delta x^{2n-2}$$
$$+ \tfrac{2n+1}{2n-4}\varepsilon x^{2n-4} - \tfrac{2n+1}{2n-6}\zeta x^{2n-6} + \text{etc.}$$

[These coefficients of the increased power sum may be thought of as arising from a term by term integration of the series on the right and of the summatory term on the left of the equation, and can be surmised from an examination of individual series.]
Therefore if the sum of the powers x^{2n} may be agreed upon, from that at once the sum of the following powers x^{2n+1} will be able to be formed.

64. This investigation of the sums of following powers is extended also to even powers ; but because the sums of these receive a new term, this is not found by that method, yet from the nature of this series, with which it agrees if there is put $x = 1$, the sum also must become $x = 1$, which will be able to be elicited always. But in turn always the sums will be able to be found from the known sum of any preceding powers. For if there were

$$S.x^{n} = \alpha x^{n+1} + \beta x^{n} + \gamma x^{n-1} - \delta x^{n-3} + \varepsilon x^{n-5} - \zeta x^{n-7} + \text{etc.},$$

there will be for the preceding power

$$S.x^{n-1} = \tfrac{n+1}{n}\alpha x^{n} + \tfrac{n}{n}\beta x^{n-1} + \tfrac{n-1}{n}\gamma x^{n-2} - \tfrac{n-3}{n}\delta x^{n-4} + \text{etc.}$$

and hence it is allowed to regress further, as far as it pleases. But it is to be noted that always there is $\alpha = \tfrac{1}{n+1}$ and $\beta = \tfrac{1}{2}$ as may be apparent from the formulas given above.

65. But it will be apparent at once on attending to the sum of the powers x^{n-1} to be produced, if the sum of the powers x^{n} may be differentiated and the differential of this may be divided by ndx ; and thus there will be $d.S.x^{n} = ndx \cdot S.x^{n-1}$, and because there is $d.x^{n} = nx^{n-1}dx$,there will be

$$d.S.x^{n} = S.nx^{n-1}dx = S.d.x^{n}$$

from which it will be understood that the differential of the sum is equal to the sum of the differentials; thus in generation, if the general term of some series were $= y$ and $S.\,y$ the summatory term of this, there will be also $S.dy = d.S.y$, that is : the sum of the differentials of all the terms is equal to the sum of the differentials of the terms themselves. But the reasoning of this equality will be understood easily from these, which we have presented above concerning the differentiation of series. Indeed since there shall be

$$S.x^{n} = x^{n} + \left(x-1\right)^{n} + \left(x-2\right)^{n} + \left(x-3\right)^{n} + \left(x-4\right)^{n} + \text{etc.},$$

there will be

$$\tfrac{d.S.x^{n}}{ndx} = x^{n-1} + \left(x-1\right)^{n-1} + \left(x-2\right)^{n-1} + \left(x-3\right)^{n-1} + \text{etc.} = S.x^{n-1},$$

which demonstration may be extended to all other series.

66. But we may return to the differences of functions, from which we have digressed, concerning which at this point certain matters are required to be noted. Because we have seen, if *y* were some function of x and everywhere in place of x there may be put $x \pm \omega$, the function *y* will be able to adopt the following value

$$y \pm \frac{\omega dy}{1dx} + \frac{\omega^{2}ddy}{1\cdot 2dx^{2}} \pm \frac{\omega^{3}d^{3}y}{1\cdot 2\cdot 3dx^{3}} + \frac{\omega^{4}d^{4}y}{1\cdot 2\cdot 3\cdot 4dx^{4}} \pm \frac{\omega^{5}d^{5}y}{1\cdot 2\cdot 3\cdot 4\cdot 5dx^{5}} + \text{etc.},$$

this expression will have a place, either if some constant quantity may be taken for ω or also depending on the variable *x*. Indeed the variability may not be seen with the values of the fractions $\frac{dy}{dx}$, $\frac{ddy}{dx^{2}}$, $\frac{d^{3}y}{dx^{3}}$ etc. found from differentiation into the factors ω, ω^{2}, ω^{3} etc. and hence it is the same, whether ω may denote a constant quantity or a variable depending on *x*.

67. Therefore we may put $\omega = x$ and in the function *y* in place of *x* to write $x - x = 0$.
[Thus the series is evaluated for a step back to the origin, which will give the value of *y* at this point.]
On account of which in any function *y* of *x* in place of *x* there may be written 0, the value of the function will become here

$$y - \frac{xdy}{1dx} + \frac{x^2 ddy}{1 \cdot 2dx^2} - \frac{x^3 d^3 y}{1 \cdot 2 \cdot 3dx^3} + \frac{x^4 d^4 y}{1 \cdot 2 \cdot 3 \cdot 4dx^4} - \frac{x^5 d^5 y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5dx^5} + \text{etc.},$$

Therefore this expression always indicated the value, which any function *y* may adopt, if in that function there may be put $x = 0$, the truth of which will be illustrated by the following examples.

EXAMPLE 1

Let $y = xx + ax + ab$, *the value of which is sought, if there is put* $x = 0$, *which indeed may be agreed to become = ab.*

Since there shall be $y = xx + ax + ab$, there will be

$$\frac{dy}{1dx} = 2x + a, \quad \frac{ddy}{1 \cdot 2dx^2} = 1$$

and thus the value of the product sought

$$= xx + ax + ab - x(2x + a) + xx \cdot 1 = ab.$$

EXAMPLE 2

Let there be $y = x^3 - 2x + 3$, *of which the value may be sought on putting* $x = 0$, *which may be agreed to be* $= 3$.

Since there shall be $y = x^3 - 2x + 3$, there will be

$$\frac{dy}{dx} = 3xx - 2, \quad \frac{ddy}{1 \cdot 2dx^2} = 3x, \quad \frac{d^3 y}{1 \cdot 2 \cdot 3dx^3} = 1;$$

from which the value sought will be obtained

$$= x^3 - 2x + 3 - x(3xx - 2) + xx \cdot 3x - x^3 \cdot 1 = 3.$$

EXAMPLE 3

Let there be $y = \frac{x}{1-x}$ *the value of which is sought on putting* $x = 0$*, which is agreed to become* $= 0$.

Since there shall be $y = \frac{x}{1-x}$ there will be

$$\frac{dy}{dx} = \frac{1}{(1-x)^2}, \quad \frac{ddy}{1 \cdot 2dx^2} = \frac{1}{(1-x)^3}, \quad \frac{d^3y}{1 \cdot 2 \cdot 3dx^3} = \frac{1}{(1-x)^4} \text{ etc.}$$

Hence the value sought will be

$$= \frac{x}{1-x} - \frac{x}{(1-x)^2} + \frac{xx}{(1-x)^3} - \frac{x^3}{(1-x)^4} + \frac{x^4}{(1-x)^5} - \text{ etc.}$$

therefore the value of this series is $= 0$. Which also hence may be apparent, because this series with the first term truncated $\frac{x}{(1-x)^2} - \frac{xx}{(1-x)^3} + \frac{x^3}{(1-x)^4} - \text{etc.}$ will be a geometric series and the sum of this

$$= \frac{x}{(1-x)^2 + x(1-x)} = \frac{x}{(1-x)}$$

from which the value found will be

$$\frac{x}{1-x} - \frac{x}{1-x} = 0.$$

EXAMPLE 4

Let $y = e^x$ *with e denoting the number, of which the hyperbolic logarithm is one, and the value of this y is sought, if there is put* $x = 0$*, which indeed agrees to become* $= 1$.

Since there shall be $y = e^x$, there will be

$$\frac{dy}{dx} = e^x, \quad \frac{ddy}{dx^2} = e^x, \text{ etc.}$$

and thus the value sought will be

$$= e^x - \frac{e^x x}{1} + \frac{e^x xx}{1 \cdot 2} - \frac{e^x x^3}{1 \cdot 2 \cdot 3} + \frac{e^x x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \text{etc.}$$
$$= e^x \left(1 - \frac{x}{1} + \frac{xx}{1 \cdot 2} - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \text{etc.}\right)$$

But above we have seen the series

$$=1-\frac{x}{1}+\frac{xx}{1\cdot 2}-\frac{x^3}{1\cdot 2\cdot 3}+\text{ etc.}$$

to express the value e^{-x}; therefore this will be the value sought and certainly $=e^x.\,e^{-x}=\frac{e^x}{e^x}=1$.

EXAMPLE 5

Let $y=\sin x$ *and on putting* $x=0$ *it is evident to become* $y=0$, *that which also will be indicated by the formula.*

For since there shall be $y=\sin x$, there will be

$$\frac{dy}{dx}=\cos x,\quad \frac{ddy}{dx^2}=-\sin x,\quad \frac{d^3y}{dx^3}=-\cos x,\quad \text{etc.}$$

The value of y will be on putting $x=0$ here will be

$$=\sin x-\frac{x}{1}\cos x-\frac{xx}{1\cdot 2}\sin x+\frac{x^3}{1\cdot 2\cdot 3}\cos x+\frac{x^4}{1\cdot 2\cdot 3\cdot 4}\sin x-\text{etc.},$$

which is

$$=\sin x\left(1-\frac{xx}{1\cdot 2}+\frac{x^4}{1\cdot 2\cdot 3\cdot 4}-\frac{x^6}{1\cdot 2\cdot 3\cdots 6}+\text{ etc.}\right)$$
$$-\cos x\left(\frac{x}{1}-\frac{x^3}{1\cdot 2\cdot 3}+\frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\frac{x^7}{1\cdot 2\cdot 3\cdots 7}+\text{ etc.}\right).$$

But of these series the upper expresses cos *x*, moreover the lower sin *x*, from which the value sought will be

$$=\sin x.\cos x-\cos x.\sin x=0.$$

68. Hence therefore in turn we know, if y were a function of *x* of this kind, so that it may vanish on putting $x=0$, then there becomes

$$y-\frac{xdy}{1dx}+\frac{x^2ddy}{1\cdot 2dx^2}-\frac{x^3d^3y}{1\cdot 2\cdot 3dx^3}+\frac{x^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4}-\text{etc.}=0.$$

From which this is the entirely general equation of all functions of *x*, which, while there becomes $x=0$, likewise themselves vanish. And that on account of this that equation thus has been prepared, so that, whatever the function of *x*, provided that may vanish with vanishing *x*, it may be substituted in place of *y*, always to be satisfying the equation. But if truly *y* were a function of this kind of *x*, which on putting $x=0$ may take the given value $=A$, then there will be

$$y-\frac{xdy}{1dx}+\frac{x^2ddy}{1\cdot 2dx^2}-\frac{x^3d^3y}{1\cdot 2\cdot 3dx^3}+\frac{x^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4}-\frac{x^5d^5y}{1\cdot 2\cdot 3\cdot 4\cdot 5dx^5}+\text{etc.}=A,$$

in which equation there will be contained all the functions of x, which on putting $x=0$ will change into A.

69. If there may be written $2x$ or $x+x$ in place of x any function of x, which may be designated by y, may adopt this value

$$y+\frac{xdy}{1dx}+\frac{x^2ddy}{1\cdot 2dx^2}+\frac{x^3d^3y}{1\cdot 2\cdot 3dx^3}+\frac{x^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4}+\text{etc.}$$

And if in place of x we may write nx, that is $x+(n-1)x$, the function y may taker the following value

$$y+\frac{(n-1)dy}{1dx}+\frac{(n-1)^2ddy}{1\cdot 2dx^2}+\frac{(n-1)^3d^3y}{1\cdot 2\cdot 3dx^3}+\text{etc.}$$

But if generally for x we may write t, any function y of x will be changed on account of $t=x+t-x$ into the following form

$$y+\frac{(t-x)dy}{1dx}+\frac{(t-x)^2ddy}{1\cdot 2dx^2}+\frac{(t-x)^3d^3y}{1\cdot 2\cdot 3dx^3}+\text{etc.}$$

Therefore if v were such a function of t, such as y is of x, because v arises from y on putting t in place of x, there will be

$$v=y+\frac{(t-x)dy}{1dx}+\frac{(t-x)^2ddy}{1\cdot 2dx^2}+\frac{(t-x)^3d^3y}{1\cdot 2\cdot 3dx^3}+\text{etc.},$$

the truth of which we may approve by some examples.

EXAMPLE

For let there be $y=xx-x$; it is evident on putting t in place of x to become $v=tt-t$, which therefore will indicate the same expression found.

For on account of $y=xx-x$ there will be

$$\frac{dy}{dx}=2x-1 \text{ and } \frac{ddy}{2dx^2}=1\,;$$

from which there becomes

$$v=xx-x+(t-x)(2x-1)+(t-x)^2$$
$$=xx-x+2tx-2xx-t+x+tt-2tx+xx=tt-t.$$

And thus if y were a function of this kind of x, which on putting $x = a$ may change into A, on account of $t = a$ and $v = A$ becomes

$$A = y + \frac{(a-x)dy}{1dx} + \frac{(a-x)^2 ddy}{1\cdot 2dx^2} + \frac{(a-x)^3 d^3 y}{1\cdot 2\cdot 3dx^3} + \text{etc.},$$

therefore all functions of x satisfy this equation, which change into A on making $x = a$.

CAPUT III

DE INVENTIONE DIFFERENTIARUM FINITARUM

44. Quemadmodum ex functionum differentiis finitis earum differentialia facile inveniri queant, in initio fusius exposuimus atque adeo ex hoc fonte principium differentialium derivavimus. Si enim differentiae, quae assumtae erant finitae, evanescant in nihilumque abeant, oriuntur differentialia; et quia hoc casu plures et saepe innumeri termini, qui differentiam finitam constituunt, reiiciuntur, differentialia multo facilius inveniri atque commodius succinctiusque exprimi possunt quam differentiae finitae. Neque igitur hinc vicissim via patere videtur a differentialibus ad differentias finitas ascendendi. Interim tamen eo modo, quo hic utemur, ex differentialibus omnium ordinum cuiuscunque functionis eiusdem differentiae finitae omnes definiri poterunt.

45. Sit y functio quaecunque ipsius x; quae cum posito $x+dx$ loco x abeat in $y+dy$, si denuo loco x ponatur $x+dx$, valor $y+dy$ suo differentiali $dy+ddy$ augebitur fietque $=y+2dy+ddy$, qui ergo valor respondebit ipsius x valori $x+2dx$. Simili modo si ponamus quantitatem x continuo suo differentiali dx augeri, ut successive valores $x+dx,\ x+2dx,\ x+3dx,\ x+4dx$ etc. induat, valores ipsius y respondentes erunt, quos haec tabella indicat.

Valores ipsius	Valores respondentes functionis
x	y
$x+dx$	$y+dy$
$x+2dx$	$y+2dy+ddy$
$x+3dx$	$y+3dy+3ddy+d^3y$
$x+4dx$	$y+4dy+6ddy+4d^3y+d^4y$
$x+5dx$	$y+5dy+10ddy+10d^3y+5d^4y+d^5y$
$x+6dx$	$y+6dy+15ddy+20d^3y+15d^4y+6d^5y+d^6y$
etc.	etc.

46. Generaliter ergo si x abeat in $x+ndx$, functio y recipiet hanc formam

$$y+\frac{n}{1}dy+\frac{n(n-1)}{1\cdot2}ddy+\frac{n(n-1)(n-2)}{1\cdot2\cdot3}d^3y+\frac{n(n-1)(n-2)(n-3)}{1\cdot2\cdot3\cdot4}d^4y+\text{etc.};$$

in qua expressione etsi quilibet terminus infinities minor est quam praecedens, tamen nullum praetermisimus, quo ista formula ad praesens negotium apta redderetur. Statuemus enim pro *n* numerum infinite magnum, et quoniam notavimus fieri posse, ut productum ex quantitate infinite magna in infinite parvam aequetur quantitati finitae, terminus secundus utique homogeneus fieri poterit primo seu *ndy* quantitatem finitam repraesentare poterit. Ob eandemque rationem terminus tertius $\frac{n(n-1)}{1\cdot 2}ddy$, etsi *ddy* infinities minus est quam *dy*, tamen, quia alter factor $\frac{n(n-1)}{1\cdot 2}$ infinities maior est quam *n*, terminus quoque tertius quantitatem finitam exprimere poterit; sicque posito *n* numero infinito nullum illius expressionis terminum reiicere licebit.

47. Posito autem *n* numero infinito, quocunque is numero finito sive augeatur sive diminuatur, numerus resultans ad *n* habebit rationem aequalitatis hincque pro singulis factoribus $n-1, n-2, n-3, n-4$ etc. ubique scribi poterit *n*. Cum enim sit $\frac{n(n-1)}{1\cdot 2}ddy = \frac{1}{2}nnddy - \frac{1}{2}nddy$, prior terminus $\frac{1}{2}nnddy$ ad posteriorem $\frac{1}{2}nddy$ rationem tenebit ut *n* ad 1 sicque hic respectu illius evanescet; loco $\frac{n(n-1)}{1\cdot 2}$ ergo scribi poterit $\frac{1}{2}nn$. Simili modo quarti termini coefficiens $\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$ contrahi poterit in $\frac{n^3}{6}$ pariterque in sequentibus numeri, quibus *n* in factoribus diminuitur, negligi poterunt. Hoc vero facto functio *y*, si loco *x* ponatur $x + ndx$ existente numero *n* infinito, sequentem valorem accipiet

$$y + \frac{n}{1}dy + \frac{nn}{1\cdot 2}ddy + \frac{n^3d^3y}{1\cdot 2\cdot 3} + \frac{n^4d^4y}{1\cdot 2\cdot 3\cdot 4} + \frac{n^5d^5y}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \text{etc.},$$

48. Cum igitur sumto *n* numero infinite magno, etiamsi *dx* sit infinite parvum, productum *ndx* quantitatem finitam exprimere possit, ponamus $ndx = \omega$, ut sit $n = \frac{\omega}{dx}$; erit utique *n* numerus infinitus, cum sit quotus ex divisione quantitatis finitae ω per infinite parvam *dx* resultans. Valore autem hoc loco *n* adhibito cognoscemus, si quantitas variabilis *x* augeatur quavis quantitate finita ω seu si loco *x* ponatur $x + \omega$, tum quamvis ipsius functionem *y* abituram esse in hanc formam

$$y + \frac{\omega dy}{1dx} + \frac{\omega ddy}{1\cdot 2dx^2} + \frac{\omega^3d^3y}{1\cdot 2\cdot 3dx^3} + \frac{\omega^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4} + \text{etc.}$$

cuius expressionis singuli termini per continuam ipsius *y* differentiationem inveniri poterunt. Cum enim *y* sit functio ipsius *x*, ostendimus supra has functiones omnes $\frac{dy}{dx}, \frac{ddy}{dx^2}, \frac{d^3y}{dx^3}$ etc. quantitates finitas exhibere.

49. Cum igitur, dum quantitas variabilis *x* quantitate finita ω augeri assumitur, functio eius quaecunque y augeatur sua differentia prima, quam supra per Δy indicavimus existente $\omega = \Delta x$, differentia ipsius y per continuam differentiationem reperiri poterit; erit enim

$$\Delta y = \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^4 d^4 y}{24dx^4} + \text{etc.}$$

seu

$$\Delta y = \frac{\Delta x}{1} \cdot \frac{dy}{dx} + \frac{\Delta x^2}{2} \cdot \frac{ddy}{dx^2} + \frac{\Delta x^3}{6} \cdot \frac{d^3 y}{dx^3} + \frac{\Delta x^4}{24} \cdot \frac{d^4 y}{dx^4} + \text{etc.}$$

Sicque differentia finita Δy exprimitur per progressionem, cuius singuli termini secundum potestates ipsius Δx procedunt. Atque hinc vicissim patet, si quantitas x tantum quantitate infinite parva augeatur, ut Δx abeat in eius differentiale dx, omnes terminos prae primo evanescere foreque $\Delta y = dy$; facto enim $\Delta x = dx$ differentia Δy abit per definitionem in differentiale dy.

50. Quoniam, si loco x ponatur $x + \omega$, eius functio quaecunque y induit sequentem valorem

$$y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^4 d^4 y}{24dx^4} + \text{etc.},$$

veritas huius expressionis comprobari poterit eiusmodi exemplis, quibus differentialia altiora ipsius y tandem evanescunt; his enim casibus numerus terminorum superioris expressionis fiet finitus.

EXEMPLUM 1

Quaeratur valor expressionis $xx - x$, si *loco x ponatur* $x + 1$.

Ponatur $y = xx - x$, et cum x in $x + 1$ abire statuatur, fiet $\omega = 1$; sumtis iam differentialibus erit

$$\frac{dy}{dx} = 2x - 1, \quad \frac{ddy}{dx^2} = 2, \quad \frac{d^3 y}{dx^3} = 0 \quad \text{etc.}$$

Hinc functio $y = xx - x$ posito $x + 1$ loco x abibit in

$$xx - x + 1(2x - 1) + \tfrac{1}{2} \cdot 2 = xx + x.$$

Quodsi autem in $xx - x$ loco x actu ponatur $x + 1$, abibit

$$\begin{array}{l} xx \text{ in } xx + 2x + 1 \\ x \text{ in } x + 1 \\ \hline \end{array}$$

Ergo

$$xx - x \text{ in } xx + x.$$

EXEMPLUM 2

Quaeratur valor expressionis $x^3 + xx + x$ *, si loco x ponatur* $x + 2$.
Ponatur $y = x^3 + xx + x$ fietque $\omega = 2$; nunc cum sit

$$x^3 + xx + x ,$$

erit

$$\frac{dy}{dx} = 3xx + 2x + 1, \quad \frac{ddy}{dx^2} = 6x + 2, \quad \frac{d^3y}{dx^3} = 6, \quad \frac{d^4y}{dx^4} = 0 \text{ etc.},$$

Ex his valor functionis $y = x^3 + xx + x$, si pro x statuatur $x + 2$, erit sequens

$$x^3 + xx + x + 2(3xx + 2x + 1) + \tfrac{4}{2}(6x + 2) + \tfrac{8}{6} \cdot 6 = x^3 + 7xx + 17x + 14,$$

qui idem prodit, si actu loco x substituatur $x + 2$.

EXEMPLUM 3

Quaeratur valor expressionis $xx + 3x + 1$*, si loco x ponatur* $x - 3$.

Fiet ergo $\omega = -3$, et posito

$$xx + 3x + 1$$

erit

$$\frac{dy}{dx} = 2x + 3, \quad \frac{ddy}{dx^2} = 2 , \quad \frac{d^3y}{dx^3} = 0 \text{ etc.},$$

unde posito $x - 3$ loco x functio $xx + 3x + 1$ abibit in

$$x^2 + 3x + 1 - \tfrac{3}{1}(2x + 3) + \tfrac{9}{2} \cdot 2 = x^2 - 3x + 1 .$$

51. Si pro ω sumatur numerus negativus, reperietur valor, quem functio quaecunque ipsius x induit, dum ipsa quantitas x diminuitur data quantitate ω . Scilicet si loco x ponatur $x - \omega$, functio ipsius x quaecunque y accipiet istum valorem

$$y - \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} - \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} - \text{etc.}$$

unde omnes variationes, quas functio *y* subire potest, dum quantitas *x* utrinque variatur, inveniri poterunt. Quodsi autem *y* fuerit functio rationalis integra ipsius *x*, quoniam tandem ad eius differentialia evanescentia devenitur, valor variatus per expressionem finitam exprimetur; sin autem *y* non fuerit huiusmodi functio, valor variatus per seriem infinitam exprimetur, cuius propterea summa, quoniam, si substitutio actu instituatur, valor variatus facile assignatur, expressione finita exhiberi poterit.

52. Quemadmodum autem differentia prima est inventa, ita quoque differentiae sequentes similibus expressionibus exhiberi possunt. Induat enim *x* successive valores $x+\omega,\ x+2\omega,\ x+3\omega,\ x+4\omega$ etc. atque valores ipsius *y* respondentes indicentur per $y^{\mathrm{I}}, y^{\mathrm{II}}, y^{\mathrm{III}}, y^{\mathrm{IV}}$ etc., sicut in initio huius libri posuimus. Quoniam ergo $y^{\mathrm{I}}, y^{\mathrm{II}}, y^{\mathrm{III}}, y^{\mathrm{IV}}$ etc. sunt valores, quos *y* nanciscitur, si loco *x* scribatur respective $x+\omega,\ x+2\omega,\ x+3\omega,\ x+4\omega$ etc., per modo demonstrata isti ipsius *y* valores ita exprimentur:

$$y^{\mathrm{I}} = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3y}{6dx^3} + \frac{\omega^4 d^4y}{24dx^4} + \text{etc.}$$
$$y^{\mathrm{II}} = y + \frac{2\omega dy}{dx} + \frac{4\omega^2 ddy}{2dx^2} + \frac{8\omega^3 d^3y}{6dx^3} + \frac{16\omega^4 d^4y}{24dx^4} + \text{etc.}$$
$$y^{\mathrm{III}} = y + \frac{3\omega dy}{dx} + \frac{9\omega^2 ddy}{2dx^2} + \frac{27\omega^3 d^3y}{6dx^3} + \frac{81\omega^4 d^4y}{24dx^4} + \text{etc.}$$
$$y^{\mathrm{IV}} = y + \frac{4\omega dy}{dx} + \frac{16\omega^2 ddy}{2dx^2} + \frac{64\omega^3 d^3y}{6dx^3} + \frac{256\omega^4 d^4y}{24dx^4} + \text{etc.}$$
$$\text{etc.}$$

53. Cum igitur, si $\Delta y,\ \Delta^2 y,\ \Delta^3 y,\ \Delta^4 y$ etc. denotent differentias primam, secundam, tertiam, quartam etc., sit

$$\Delta y = y^{\mathrm{I}} - y$$
$$\Delta^2 y = y^{\mathrm{II}} - 2y^{\mathrm{I}} + y,$$
$$\Delta^3 y = y^{\mathrm{III}} - 3y^{\mathrm{II}} + 3y^{\mathrm{I}} - y,$$
$$\Delta^4 y = y^{\mathrm{IV}} - 4y^{\mathrm{III}} + 6y^{\mathrm{II}} - 4y^{\mathrm{I}} + y$$
$$\text{etc.,}$$

istae differentiae per differentialia hoc modo exprimentur:

$$\Delta y = \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^4 d^4 y}{24dx^4} + \text{etc.}$$

$$\Delta^2 y = \frac{\left(2^2 - 2\cdot 1\right)\omega^2 ddy}{2dx^2} + \frac{\left(2^3 - 2\cdot 1\right)\omega^3 d^3 y}{6dx^3} + \frac{\left(2^4 - 2\cdot 1\right)\omega^4 d^4 y}{24dx^4} + \text{etc.}$$

$$\Delta^3 y = \frac{\left(3^3 - 3\cdot 2^3 + 3\cdot 1\right)\omega^3 d^3 y}{6dx^3} + \frac{\left(3^4 - 3\cdot 2^4 + 3\cdot 1\right)\omega^4 d^4 y}{24dx^4} + \text{etc.}$$

$$\Delta^4 y^4 = \frac{\left(4^4 - 4\cdot 3^4 + 6\cdot 2^4 - 4\cdot 1\right)\omega^4 d^4 y}{24dx^4} + \frac{\left(4^5 - 4\cdot 3^5 + 6\cdot 2^5 - 4\cdot 1\right)\omega^5 d^5 y}{120dx^5} + \text{etc.}$$

etc.

54. Quantam utilitatem afferant istae differentiarum expressiones in doctrina serierum et progressionum, cum sponte patet, tum in sequentibus uberius exponemus. Interim tamen in hoc capite usum, qui hinc ad serierum notitiam immediate redundat, perpendamus. Quanquam vulgo indices terminorum seriei cuiuscunque progressionem arithmeticam, cuius differentia est unitas, constituere assumuntur, tamen, quo usus latius pateat atque applicatio facilius fieri possit, differentiam statuamus $= \omega$, ita ut, si terminus generalis seu is, qui indici x respondet, fuerit y, sequentes conveniant indicibus $x+\omega$, $x+2\omega$, $x+3\omega$ etc. Quodsi ergo his indicibus respondeant sequentes seriei termini

$$\begin{array}{llllll} x, & x+\omega, & x+2\omega, & x+3\omega, & x+4\omega & \text{etc.} \\ y, & P, & Q, & R, & S & \text{etc.,} \end{array}$$

singuli ex y eiusque differentialibus definientur hoc modo:

$$P = y + \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^4 d^4 y}{24dx^4} + \text{etc.}$$

$$Q = y + \frac{2\omega dy}{dx} + \frac{4\omega^2 ddy}{2dx^2} + \frac{8\omega^3 d^3 y}{6dx^3} + \frac{16\omega^4 d^4 y}{24dx^4} + \text{etc.}$$

$$R = y + \frac{3\omega dy}{dx} + \frac{9\omega^2 ddy}{2dx^2} + \frac{27\omega^3 d^3 y}{6dx^3} + \frac{81\omega^4 d^4 y}{24dx^4} + \text{etc.}$$

$$S = y + \frac{4\omega dy}{dx} + \frac{16\omega^2 ddy}{2dx^2} + \frac{64\omega^3 d^3 y}{6dx^3} + \frac{256\omega^4 d^4 y}{24dx^4} + \text{etc.}$$

$$T = y + \frac{5\omega dy}{dx} + \frac{25\omega^2 ddy}{2dx^2} + \frac{125\omega^3 d^3 y}{6dx^3} + \frac{625\omega^4 d^4 y}{24dx^4} + \text{etc.}$$

etc.

55. Si hae expressiones a se invicem subtrahantur, in differentias non amplius ingredietur y eritque

$$P-y=\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}+\frac{\omega^3 d^3y}{6dx^3}+\frac{\omega^4 d^4y}{24dx^4}+\text{etc.}$$
$$Q-P=\frac{\omega dy}{dx}+\frac{3\omega^2 ddy}{2dx^2}+\frac{7\omega^3 d^3y}{6dx^3}+\frac{15\omega^4 d^4y}{24dx^4}+\text{etc.}$$
$$R-Q=\frac{\omega dy}{dx}+\frac{5\omega^2 ddy}{2dx^2}+\frac{19\omega^3 d^3y}{6dx^3}+\frac{65\omega^4 d^4y}{24dx^4}+\text{etc.}$$
$$S-R=\frac{\omega dy}{dx}+\frac{7\omega^2 ddy}{2dx^2}+\frac{37\omega^3 d^3y}{6dx^3}+\frac{175\omega^4 d^4y}{24dx^4}+\text{etc.}$$
$$T-S=\frac{\omega dy}{dx}+\frac{9\omega^2 ddy}{2dx^2}+\frac{61\omega^3 d^3y}{6dx^3}+\frac{369\omega^4 d^4y}{24dx^4}+\text{etc.}$$
etc.

Si hae expressiones denuo a se invicem subtrahantur, etiam differentialia prima se destruent eritque

$$Q-2P+y=\frac{2\omega^2 ddy}{2dx^2}+\frac{6\omega^3 d^3y}{6dx^3}+\frac{14\omega^4 d^4y}{24dx^4}+\text{etc.}$$
$$R-2Q+P=\frac{2\omega^2 ddy}{2dx^2}+\frac{12\omega^3 d^3y}{6dx^3}+\frac{50\omega^4 d^4y}{24dx^4}+\text{etc.}$$
$$S-2R+Q=\frac{2\omega^2 ddy}{2dx^2}+\frac{18\omega^3 d^3y}{6dx^3}+\frac{110\omega^4 d^4y}{24dx^4}+\text{etc.}$$
$$T-2S+R=\frac{2\omega^2 ddy}{2dx^2}+\frac{24\omega^3 d^3y}{6dx^3}+\frac{194\omega^4 d^4y}{24dx^4}+\text{etc.}$$
etc.

His autem denuo a se invicem subtractis differentialia quoque secunda ex computo egredientur:

$$R-3Q+3P=\frac{6\omega^3 d^3y}{6dx^3}+\frac{36\omega^4 d^4y}{24dx^4}+\text{etc.}$$
$$S-3R+3Q-P=\frac{6\omega^3 d^3y}{6dx^3}+\frac{60\omega^4 d^4y}{24dx^4}+\text{etc.}$$
$$T-3S+3R-Q=\frac{6\omega^3 d^3y}{6dx^3}+\frac{84\omega^4 d^4y}{24dx^4}+\text{etc.}$$
etc.

Subtractionem autem ulterius continuando fiet

$$S-4R+6Q-4P+y=\frac{24\omega^4 d^4y}{24dx^4}+\text{etc.}$$
$$T-4S+6R-4Q+P=\frac{24\omega^4 d^4y}{24dx^4}+\text{etc.}$$
etc.

atque

$$T - 5S + 10R - 10Q + 5P - y = \frac{120\omega^5 d^5 y}{120 dx^5} + \text{etc.}$$

etc.

56. Quodsi ergo *y* fuerit functio rationalis integra ipsius *x*, quia eius differentialia altiora tandem evanescent, hoc modo procedendo tandem ad expressiones evanescentes pervenietur. Cum igitur istae expressiones sint differentiae ipsius *y*, earum formas et coefficientes diligentius perpendamus.

$$y = y$$

$$\Delta y = \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^4 d^4 y}{24dx^4} + \frac{\omega^5 d^5 y}{120dx^5} + \text{etc.}$$

$$\Delta^2 y = \frac{\omega^2 ddy}{dx^2} + \frac{3\omega^3 d^3 y}{3dx^3} + \frac{7\omega^4 d^4 y}{3\cdot 4dx^4} + \frac{15\omega^5 d^5 y}{3\cdot 4\cdot 5dx^5} + \frac{31\omega^6 d^6 y}{3\cdot 4\cdot 5\cdot 6dx^6} + \text{etc.}$$

$$\Delta^3 y = \frac{\omega^3 d^3 y}{dx^3} + \frac{6\omega^4 d^4 y}{4dx^4} + \frac{25\omega^5 d^5 y}{4\cdot 5dx^5} + \frac{90\omega^6 d^6 y}{4\cdot 5\cdot 6dx^6} + \frac{301\omega^7 d^7 y}{4\cdot 5\cdot 6\cdot 7dx^7} + \text{etc.}$$

$$\Delta^4 y = \frac{\omega^4 d^4 y}{dx^4} + \frac{10\omega^5 d^5 y}{5dx^5} + \frac{65\omega^6 d^6 y}{5\cdot 6dx^6} + \frac{350\omega^7 d^7 y}{5\cdot 6\cdot 7dx^7} + \frac{1701\omega^8 d^8 y}{5\cdot 6\cdot 7\cdot 8dx^8} + \text{etc.}$$

$$\Delta^5 y = \frac{\omega^5 d^5 y}{dx^5} + \frac{15\omega^6 d^6 y}{6dx^6} + \frac{140\omega^7 d^7 y}{6\cdot 7dx^7} + \frac{1050\omega^8 d^8 y}{6\cdot 7\cdot 8dx^8} + \frac{6951\omega^9 d^9 y}{6\cdot 7\cdot 8\cdot 9dx^9} + \text{etc.}$$

$$\Delta^6 y = \frac{\omega^6 d^6 y}{dx^6} + \frac{21\omega^7 d^7 y}{7dx^7} + \frac{266\omega^8 d^8 y}{7\cdot 8dx^8} + \frac{2646\omega^9 d^9 y}{7\cdot 8\cdot 9dx^9} + \frac{22827\omega^{10} d^{10} y}{7\cdot 8\cdot 9\cdot 10dx^{10}} + \text{etc.}$$

etc.

In quibus seriebus quemadmodum denominatores procedant, clarum est; numeratorum autem coefficientes ita formantur, ut quivis coefficiens numeratoris sit aggregatum ex supra stante et praecedente per exponentem differentiae multiplicato. Sic in serie differentiam $\Delta^6 y$ exprimente est $2646 = 1050 + 6\cdot 266$.

57. Consideremus quoque seriem eandem simul retro continuatam, quae continet terminos indicibus $x-\omega,\ x-2\omega,\ x-3\omega$ etc. respondentes,

$$\begin{array}{cccccccccc} x-4\omega, & x-3\omega, & x-2\omega, & x-\omega, & x, & x+\omega, & x+2\omega, & x+3\omega, & x+4\omega & \text{etc.} \\ s, & r, & q, & p, & y, & P, & Q, & R, & S & \text{etc.} \end{array}$$

Cum igitur sit

$$p = y - \frac{\omega dy}{dx} + \frac{\omega^2 ddy}{2dx^2} - \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^4 d^4 y}{24dx^4} - \text{etc.}$$
$$q = y - \frac{2\omega dy}{dx} + \frac{4\omega^2 ddy}{2dx^2} - \frac{8\omega^3 d^3 y}{6dx^3} + \frac{16\omega^4 d^4 y}{24dx^4} - \text{etc.}$$
$$r = y - \frac{3\omega dy}{dx} + \frac{9\omega^2 ddy}{2dx^2} - \frac{27\omega^3 d^3 y}{6dx^3} + \frac{81\omega^4 d^4 y}{24dx^4} - \text{etc.}$$
$$s = y - \frac{4\omega dy}{dx} + \frac{16\omega^2 ddy}{2dx^2} - \frac{64\omega^3 d^3 y}{6dx^3} + \frac{256\omega^4 d^4 y}{24dx^4} - \text{etc.}$$
$$\text{etc.}$$

erit his valoribus a superioribus *P*, *Q*, *R*, *S* etc. subtrahendis

$$\frac{P-p}{2} = \frac{\omega dy}{dx} + \frac{\omega^3 d^3 y}{6dx^3} + \frac{\omega^5 d^5 y}{120dx^5} + \text{etc.}$$
$$\frac{Q-q}{2} = \frac{2\omega dy}{dx} + \frac{8\omega^3 d^3 y}{6dx^3} + \frac{32\omega^5 d^5 y}{120dx^5} + \text{etc.}$$
$$\frac{R-r}{2} = \frac{3\omega dy}{dx} + \frac{27\omega^3 d^3 y}{6dx^3} + \frac{243\omega^5 d^5 y}{120dx^5} + \text{etc.}$$
$$\frac{S-s}{2} = \frac{4\omega dy}{dx} + \frac{64\omega^3 d^3 y}{6dx^3} + \frac{1024\omega^5 d^5 y}{120dx^5} + \text{etc.}$$
$$\text{etc.}$$

Sin autem termini hi ad superiores addantur, tum, quemadmodum hic differentialia parium ordinum deerant, differentialia imparia ex computo egredientur
Erit enim

$$\frac{P+p}{2} = y + \frac{\omega^2 ddy}{2dx^2} + \frac{\omega^4 d^4 y}{24dx^4} + \frac{\omega^6 d^6 y}{720dx^6} + \text{etc.}$$
$$\frac{Q+q}{2} = y + \frac{4\omega^2 ddy}{2dx^2} + \frac{16\omega^4 d^4 y}{24dx^4} + \frac{64\omega^6 d^6 y}{720dx^6} + \text{etc.}$$
$$\frac{R+r}{2} = y + \frac{9\omega^2 ddy}{2dx^2} + \frac{81\omega^4 d^4 y}{24dx^4} + \frac{729\omega^6 d^6 y}{720dx^6} + \text{etc.}$$
$$\frac{S+s}{2} = y + \frac{16\omega^2 ddy}{2dx^2} + \frac{256\omega^4 d^4 y}{24dx^4} + \frac{4096\omega^6 d^6 y}{720dx^6} + \text{etc.}$$
$$\text{etc.}$$

58. Quoniam termini antecedentes omnes exprimi possunt, si ii in unam summam colligantur, prodibit seriei propositae terminus summatorius. Respondeat scilicet terminus primus indici $x - n\omega$ eritque ipse terminus primus

$$= y - \frac{n\omega dy}{dx} + \frac{n^2\omega^2 ddy}{2dx^2} - \frac{n^3\omega^3 d^3 y}{6dx^3} + \frac{n^4\omega^4 d^4 y}{24dx^4} - \text{etc.}$$

Cum igitur terminus indici x respondens sit $= y$ terminorumque omnium numerus sit $= n+1$, erit summa omnium a primo ad ultimum y inclusive sumtorum seu terminus summatorius

$$\begin{aligned} &= (n+1)y - \frac{\omega dy}{dx}(1+2+3+\cdots+n) \\ &+ \frac{\omega^2 ddy}{2dx^2}\left(1^2+2^2+3^2+\cdots+n^2\right) \\ &- \frac{\omega^3 d^3y}{6dx^3}\left(1^3+2^3+3^3+\cdots+n^3\right) \\ &+ \frac{\omega^4 d^4y}{24dx^4}\left(1^4+2^4+3^4+\cdots+n^4\right) \\ &- \frac{\omega^5 d^5y}{120dx^5}\left(1^5+2^5+3^5+\cdots+n^5\right) \\ &\text{etc.} \end{aligned}$$

59. Supra autem singularum harum serierum summas exhibuimus ; quae si hic substituantur, erit summa seriei nostrae propositae

$$\begin{aligned} &= (n+1)y - \frac{\omega dy}{dx}\left(\tfrac{1}{2}nn+\tfrac{1}{2}n\right) \\ &+ \frac{\omega^2 ddy}{2dx^2}\left(\tfrac{1}{3}n^3+\tfrac{1}{2}nn+\tfrac{1}{6}n\right) \\ &- \frac{\omega^3 d^3y}{6dx^3}\left(\tfrac{1}{4}n^4+\tfrac{1}{2}n^3+\tfrac{1}{4}n^2\right) \\ &+ \frac{\omega^4 d^4y}{24dx^4}\left(\tfrac{1}{5}n^5+\tfrac{1}{2}n^4+\tfrac{1}{3}n^3-\tfrac{1}{30}n\right) \\ &- \frac{\omega^5 d^5y}{120dx^5}\left(\tfrac{1}{6}n^6+\tfrac{1}{2}n^5+\tfrac{5}{12}n^4-\tfrac{1}{12}n^2\right) \\ &\text{etc.} \end{aligned}$$

ubi n dabitur ex indice termini primi, a qua summa computatur. Ita si ponatur $\omega = 1$ et index termini primi sit $= 1$, secundi $= 2$ et ultimi $= x$, ita ut haec series sit proposita

$$\begin{array}{l} 1,\ 2,\ 3,\ 4,\cdots\cdots x \\ a,\ b,\ c,\ d,\cdots\cdots y, \end{array}$$

erit huius seriei summa ob $x - n = 1$ et $n = x - 1$

$$\begin{aligned}= xy - \frac{dy}{dx}\left(\tfrac{1}{2}xx - \tfrac{1}{2}x\right) \\ + \frac{ddy}{2dx^2}\left(\tfrac{1}{3}x^3 - \tfrac{1}{2}xx + \tfrac{1}{6}x\right) \\ - \frac{d^3y}{6dx^3}\left(\tfrac{1}{4}x^4 - \tfrac{1}{2}x^3 + \tfrac{1}{4}xx\right) \\ + \frac{d^4y}{24dx^4}\left(\tfrac{1}{5}x^5 - \tfrac{1}{2}x^4 + \tfrac{1}{3}x^3 - \tfrac{1}{30}x\right) \\ - \frac{d^5y}{120dx^5}\left(\tfrac{1}{6}x^6 - \tfrac{1}{2}x^5 + \tfrac{5}{12}x^4 - \tfrac{1}{12}x^2\right) \\ + \frac{d^6y}{720dx^6}\left(\tfrac{1}{7}x^7 - \tfrac{1}{2}x^6 + \tfrac{1}{2}x^5 - \tfrac{1}{6}x^3 + \tfrac{1}{42}x\right) \\ \text{etc.}\end{aligned}$$

60. Ex hac summae expressione, quia coefficientes vehementer augentur, si *x* fuerit numerus magnus, parum utilitatis ad doctrinam serierum redundat; interim tamen iuvabit aliquas proprietates inde fluentes commemorasse. Sit terminus generalis $y = x^n$ atque terminus summatorius per *S. y* seu $S.x^n$ indicetur. Qua designatione ubique adhibita erit

$$\begin{aligned}\tfrac{1}{2}xx - \tfrac{1}{2}x = S.x - x \\ \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{6}x = S.x^2 - x^2 \\ \tfrac{1}{4}x^4 - \tfrac{1}{2}x^3 + \tfrac{1}{4}xx = S.x^3 - x^3 \\ \text{etc.}\end{aligned}$$

Quamobrem ex superiori expressione obtinebitur

$$S.x^n = x^{n+1} - nx^{n-1}S.x + nx^n$$
$$+\frac{n(n-1)}{1\cdot 2}x^{n-2}S.x^2 - \frac{n(n-1)}{1\cdot 2}x^n - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}S.x^3 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^n + \text{etc.}$$

At cum sit

$$(1-1)^n = 0 = 1 - n + \frac{n(n-1)}{1\cdot 2} - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} + \text{etc.},$$

erit

$$n - \frac{n(n-1)}{1\cdot 2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} - \text{etc.} = 1$$

ideoque excepto casu $n = 0$, quo ista expressio fit $= 0$,

$$S.x^n = x^{n+1} + x^n - nx^{n-1}S.x + \frac{n(n-1)}{1\cdot 2}x^{n-2}S.x^2$$
$$-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}S.x^3 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^{n-4}S.x^4 - \text{etc.}$$

61. Quo tam veritas quam vis huius formulae clarius perspiciatur, evolvamus singulos casus sitque primo eritque $S.x = x^2 + x - S.x$ ideoque $S.x = \frac{xx+x}{2}$, quemadmodum satis constat. Ponamus ergo $n = 2$ et erit

$$S.\,x^2 = x^3 + xx - 2xS.x + S.x^2\,,$$

quae aequatio, cum utrinque termini $S.x^2$ se tollant, idem dat quod praecedens $S.x = \frac{xx+x}{2}$. Si sit $n = 3$, erit

$$S.\,x^3 = x^4 + x^3 - 3x^2S.x + 3xS.x^2 - S.x^3$$

ideoque

$$S.x^3 = \tfrac{3}{2}xS.x^2 - \tfrac{3}{2}x^2S.x + \tfrac{1}{2}x^3(x+1)$$

si ponatur $n = 4$, prodibit

$$S.x^4 = x^5 + x^4 - 4x^3S.x + 6x^2S.x^2 - 4xS.x^3 + S.x^4,$$

unde ob $S.x^4$ destructum erit

$$S.x^3 = \tfrac{3}{2}xS.x^2 - x^2S.x + \tfrac{1}{4}x^3(x+1)$$

a cuius triplo si praecedentis duplum subtrahatur, remanebit

$$S.x^3 = \tfrac{3}{2}xS.x^2 - \tfrac{1}{4}x^3(x+1).$$

Si ponatur $n = 5$, fiet

$$S.x^5 = x^6 + x^5 - 5x^4S.x + 10x^3S.x^2 - 10x^2S.x^3 + 5xS.x^4 - S.x^5$$

seu

$$S.x^5 = \tfrac{5}{2}xS.x^4 - 5x^2S.x^3 + 5x^3S.x^2 - \tfrac{5}{2}x^4S.x + \tfrac{1}{2}x^5(x+1)$$

atque ex $n = 6$ sequitur

$$S.x^6 = x^7 + x^6 - 6x^5 S.x + 15x^4 S.x^2 - 20x^3 S.x^3 + 15x^2 S.x^4 - 6xS.x^5 + S.x^6$$

seu

$$S.x^5 = \tfrac{5}{2} xS.x^4 - \tfrac{10}{3} x^2 S.x^3 + \tfrac{5}{2} x^3 S.x^2 - x^4 S.x + \tfrac{1}{6} x^5 (x+1).$$

62. Ex his ergo generaliter concludimus, si fuerit $n = 2m+1$, fore

$$S.x^{2m+1} = \tfrac{2m+1}{2} xS.x^{2m} - \tfrac{(2m+1)2m}{2\cdot 1\cdot 2} x^2 S.x^{2m-1}$$
$$+ \tfrac{(2m+1)2m(2m-1)}{2\cdot 1\cdot 2\cdot 3} x^3 S.x^{2m-2} - \cdots - \tfrac{2m+1}{2} x^{2m} S.x + \tfrac{1}{2} x^{2m+1} (x+1).$$

Sin autem sit $n = 2m+2$, quia termini $S.x^{2m+2}$ se mutuo destruunt, reperietur

$$S.x^{2m+1} = \tfrac{2m+1}{2} xS.x^{2m} - \tfrac{(2m+1)2m}{2\cdot 3} x^2 S.x^{2m-1}$$
$$+ \tfrac{(2m+1)2m(2m-1)}{2\cdot 3\cdot 4} x^3 S.x^{2m-2} - \cdots - x^{2m} S.x + \tfrac{1}{2m+2} x^{2m+1} (x+1).$$

Duplici ergo modo summae potestatum imparium ex summis potestatum inferiorum determinari possunt atque ex varia combinatione harum duarum formularum infinitae aliae formari possunt.

63. Multo facilius autem summae potestatum imparium ex antecedentibus definiri possunt atque ad hoc quidem sufficit solam summam potestatis paris antecedentis novisse. Ex summis enim potestatum supra exhibitis constat numerum terminorum summas constituentium in paribus tantum potestatibus augeri, ita ut summa potestatis imparis totidem constet terminis quot summa potestatis paris praecedentis. Sic si potestatis paris x^{2n} summa sit

$$S.x^{2n} = \alpha x^{2n+1} + \beta x^{2n} + \gamma x^{2n-1} - \delta x^{2n-3} + \varepsilon x^{2n-5} - \text{etc.}$$

(vidimus enim post terminum tertium alternos terminos deficere simulque signa alternari), hinc summa sequentis potestatis x^{2n+1} invenietur, si singuli huius termini respective multiplicentur per hos numeros

$$\tfrac{2n+1}{2n+2} x, \quad \tfrac{2n+1}{2n+1} x, \quad \tfrac{2n+1}{2n} x, \quad \tfrac{2n+1}{2n-1} x, \quad \tfrac{2n+1}{2n-2} x, \quad \text{etc.}$$

non omittendo terminos deficientes; eritque ergo

$$S.x^{2n+1} = \tfrac{2n+1}{2n+2}\alpha x^{2n+2} + \tfrac{2n+1}{2n+1}\beta x^{2n+1} + \tfrac{2n+1}{2n}\gamma x^{2n} - \tfrac{2n+1}{2n-2}\delta x^{2n-2}$$
$$+ \tfrac{2n+1}{2n-4}\varepsilon x^{2n-4} - \tfrac{2n+1}{2n-6}\zeta x^{2n-6} + \text{etc.}$$

Quodsi ergo constet summa potestatis x^{2n}, ex ea expedite summa sequentis potestatis x^{2n+1} formari poterit.

64. Haec sequentium summarum investigatio etiam ad potestates pares extenditur; quoniam autem harum summae novum terminum recipiunt, hic per istam methodum non invenitur, ex natura tamen ipsius seriei, qua constat, si ponatur $x = 1$, summam quoque fieri debere $x = 1$, semper erui poterit. Vicissim autem semper ex summa cuiusvis potestatis cognita praecedentium potestatum summae inveniri poterunt. Si enim fuerit

$$S.x^n = \alpha x^{n+1} + \beta x^n + \gamma x^{n-1} - \delta x^{n-3} + \varepsilon x^{n-5} - \zeta x^{n-7} + \text{etc.},$$

erit pro potestate praecedente

$$S.x^{n-1} = \tfrac{n+1}{n}\alpha x^n + \tfrac{n}{n}\beta x^{n-1} + \tfrac{n-1}{n}\gamma x^{n-2} - \tfrac{n-3}{n}\delta x^{n-4} + \text{etc.}$$

hincque ulterius regredi licet, quousque libuerit. Notandum autem est esse perpetuo $\alpha = \frac{1}{n+1}$ et $\beta = \frac{1}{2}$ uti ex formulis iam supra datis apparet.

65. Attendenti statim patebit summam potestatum x^{n-1} prodire, si summa potestatum x^n differentietur eiusque differentiale per *ndx* dividatur ; eritque adeo $d.S.x^n = ndx \cdot S.x^{n-1}$, et quia est $d.x^n = nx^{n-1}dx$, erit

$$d.S.x^n = S.nx^{n-1}dx = S.d.x^n;$$

ex quo intelligitur differentiale summae aequari summae differentialis; ita in genere, si seriei cuiuspiarn terminus generalis fuerit = *y* et *S. y* eius terminus summatorius, erit quoque $S.dy = d.S.y$, hoc est: summa differentialium omnium terminorum aequatur differentiali summae ipsorum terminorum. Ratio autem huius aequalitatis facile perspicitur ex iis, quae supra de serierum differentiatione attulimus. Cum enim sit

$$S.x^n = x^n + (x-1)^n + (x-2)^n + (x-3)^n + (x-4)^n + \text{etc.},$$

erit

$$\frac{d.S.x^n}{ndx} = x^{n-1} + (x-1)^{n-1} + (x-2)^{n-1} + (x-3)^{n-1} + \text{etc.} = S.x^{n-1},$$

quae demonstratio ad omnes alias series patet.

66. Revertamur autem, unde digressi sumus, ad differentias functionum, circa quas adhuc quaedam annotanda sunt. Quoniam vidimus, si y fuerit functio quaecunque ipsius x atque loco x ubique ponatur $x \pm \omega$, functionem y adepturam esse sequentem valorem

$$y \pm \frac{\omega dy}{1dx} + \frac{\omega^2 ddy}{1\cdot 2dx^2} \pm \frac{\omega^3 d^3 y}{1\cdot 2\cdot 3dx^3} + \frac{\omega^4 d^4 y}{1\cdot 2\cdot 3\cdot 4dx^4} \pm \frac{\omega^5 d^5 y}{1\cdot 2\cdot 3\cdot 4\cdot 5dx^5} + \text{etc.},$$

haec expressio locum habebit, sive pro ω quantitas quaecunque constans accipiatur sive etiam variabilis ab ipsa x pendens. Inventis enim per differentiationem valoribus fractionum $\frac{dy}{dx}$, $\frac{ddy}{dx^2}$, $\frac{d^3y}{dx^3}$ etc. in factoribus ω, ω^2, ω^3 etc. variabilitas non spectatur hincque perinde est, sive ω denotet quantitatem constantem sive variabilem ab x pendentem.

67. Ponamus ergo esse $\omega = x$ atque in functione y loco x scribi $x - x = 0$. Quamobrem si in functione ipsius x quacunque y loco x ubique scribatur 0, valor functionis erit hic

$$y - \frac{xdy}{1dx} + \frac{x^2 ddy}{1\cdot 2dx^2} - \frac{x^3 d^3 y}{1\cdot 2\cdot 3dx^3} + \frac{x^4 d^4 y}{1\cdot 2\cdot 3\cdot 4dx^4} - \frac{x^5 d^5 y}{1\cdot 2\cdot 3\cdot 4\cdot 5dx^5} + \text{etc.},$$

Haec ergo expressio semper indicat valorem, quem functio quaecunque y induit, si in ea ponatur $x = 0$, cuius veritatem sequentia exempla illustrabunt.

EXEMPLUM 1

Sit $y = xx + ax + ab$, *cuius valor,* si *ponatur* $x = 0$, *quaeratur, quem quidem constat fore* $= ab$.

Cum sit $y = xx + ax + ab$, erit

$$\frac{dy}{1dx} = 2x + a, \quad \frac{ddy}{1\cdot 2dx^2} = 1$$

ideoque prodibit valor quaesitus

$$= xx + ax + ab - x(2x + a) + xx\cdot 1 = ab.$$

EXEMPLUM 2

Sit $y = x^3 - 2x + 3$, *cuius valor posito* $x = 0$ *quaeratur, quem constat fore* $= 3$.

Cum sit $y = x^3 - 2x + 3$, erit

$$\frac{dy}{dx} = 3xx - 2, \quad \frac{ddy}{1\cdot 2dx^2} = 3x, \quad \frac{d^3y}{1\cdot 2\cdot 3dx^3} = 1;$$

obtinebitur valor quaesitus

$$= x^3 - 2x + 3 - x(3xx - 2) + xx \cdot 3x - x^3 \cdot 1 = 3.$$

EXEMPLUM 3

Sit $y = \frac{x}{1-x}$ *cuius valor posito* $x = 0$ *quaeratur, quem constat fore* = 0.

Cum sit $y = \frac{x}{1-x}$ erit

$$\frac{dy}{dx} = \frac{1}{(1-x)^2}, \quad \frac{ddy}{1\cdot 2dx^2} = \frac{1}{(1-x)^3}, \quad \frac{d^3y}{1\cdot 2\cdot 3dx^3} = \frac{1}{(1-x)^4} \text{ etc.}$$

Hinc erit valor quaesitus

$$= \frac{x}{1-x} - \frac{x}{(1-x)^2} + \frac{xx}{(1-x)^3} - \frac{x^3}{(1-x)^4} + \frac{x^4}{(1-x)^5} - \text{ etc.}$$

huiusque ergo seriei valor est $= 0$. Quod etiam hinc patet, quod haec series primo termino truncata $\frac{x}{(1-x)^2} - \frac{xx}{(1-x)^3} + \frac{x^3}{(1-x)^4} - \text{etc.}$ sit series geometrica eiusque summa

$$= \frac{x}{(1-x)^2 + x(1-x)} = \frac{x}{(1-x)}$$

unde valor inventus erit

$$\frac{x}{1-x} - \frac{x}{1-x} = 0.$$

EXEMPLUM 4

Sit $y = e^x$ *denotante e numerum, cuius logarithmus hyperbolicus est unitas, et quaeratur valor ipsius y, si ponatur* $x = 0$, *quem quidem constat fore* $= 1$.

Cum sit $y = e^x$, erit

$$\frac{dy}{dx} = e^x, \quad \frac{ddy}{dx^2} = e^x, \text{ etc.}$$

ideoque valor quaesitus erit

$$= e^x - \frac{e^x x}{1} + \frac{e^x xx}{1\cdot 2} - \frac{e^x x^3}{1\cdot 2\cdot 3} + \frac{e^x x^4}{1\cdot 2\cdot 3\cdot 4} - \text{etc.}$$
$$= e^x \left(1 - \frac{x}{1} + \frac{xx}{1\cdot 2} - \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \text{etc.}\right)$$

At supra vidimus seriem

$$= 1 - \frac{x}{1} + \frac{xx}{1\cdot 2} - \frac{x^3}{1\cdot 2\cdot 3} + \text{ etc.}$$

exprimere valorem e^{-x}; erit ergo valor quaesitus utique $= e^{x}.\, e^{-x} = \frac{e^x}{e^x} = 1$.

EXEMPLUM 5

Sit $y = \sin x$ *atque posito* $x = 0$ *manifestum est fore* $y = 0$, *id quod etiam formula generalis indicabit.*

Cum enim sit $y = \sin x$, erit

$$\frac{dy}{dx} = \cos x, \quad \frac{ddy}{dx^2} = -\sin x, \quad \frac{d^3y}{dx^3} = -\cos x, \quad \text{etc.}$$

Erit posito $x = 0$ valor ipsius y hic

$$= \sin x - \frac{x}{1}\cos x - \frac{xx}{1\cdot 2}\sin x + \frac{x^3}{1\cdot 2\cdot 3}\cos x + \frac{x^4}{1\cdot 2\cdot 3\cdot 4}\sin x - \text{etc.},$$

qui est

$$= \sin x\left(1 - \frac{xx}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \frac{x^6}{1\cdot 2\cdot 3\cdots 6} + \text{ etc.}\right)$$
$$-\cos x\left(\frac{x}{1} - \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{x^7}{1\cdot 2\cdot 3\cdots 7} + \text{ etc.}\right).$$

Harum autem serierum superior exprimit cos *x*, inferior autem sin *x*, unde valor quaesitus erit

$$= \sin x.\cos x - \cos x.\sin x = 0.$$

68. Hinc igitur vicissim cognoscimus, si *y* eiusmodi fuerit functio ipsius *x*, ut ipsa evanescat posito $x = 0$, tum fore

$$y - \frac{xdy}{1dx} + \frac{x^2ddy}{1\cdot 2dx^2} - \frac{x^3d^3y}{1\cdot 2\cdot 3dx^3} + \frac{x^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4} - \text{etc.} = 0.$$

Unde haec est aequatio generalis omnium omnino functionum ipsius *x*, quae, dum fit $x = 0$, simul ipsae evanescunt. Et hanc ob rem ista aequatio ita est comparata, ut, quaecunque functio ipsius *x*, dummodo ea evanescat evanescente *x*, loco *y* substituatur, aequationi perpetuo satisfiat. Quodsi vero *y* eiusmodi fuerit functio ipsius *x*, quae posito $x = 0$ recipiat valorem datum $= A$, tum erit

$$y - \frac{xdy}{1dx} + \frac{x^2ddy}{1\cdot 2dx^2} - \frac{x^3d^3y}{1\cdot 2\cdot 3dx^3} + \frac{x^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4} - \frac{x^5d^5y}{1\cdot 2\cdot 3\cdot 4\cdot 5dx^5} + \text{etc.} = A,$$

in qua aequatione omnes continentur functiones ipsius *x*, quae posito $x = 0$ abeunt in *A*.

69. Si loco *x* scribatur *2x* seu $x + x$, functio quaecunque ipsius *x*, quae designetur per *y*, hunc induet valorem

$$y + \frac{xdy}{1dx} + \frac{x^2ddy}{1\cdot 2dx^2} + \frac{x^3d^3y}{1\cdot 2\cdot 3dx^3} + \frac{x^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4} + \text{etc.}$$

Atque si loco *x* scribamus *nx*, hoc est $x + (n-1)x$, functio *y* accipiet valorem sequentem

$$y + \frac{(n-1)dy}{1dx} + \frac{(n-1)^2 ddy}{1\cdot 2dx^2} + \frac{(n-1)^3 d^3y}{1\cdot 2\cdot 3dx^3} + \text{etc.}$$

Sin autem generaliter pro *x* scribamus *t*, functio quaecunque *y* ipsius *x* transmutabitur ob $t = x + t - x$ in formam sequentem

$$y + \frac{(t-x)dy}{1dx} + \frac{(t-x)^2 ddy}{1\cdot 2dx^2} + \frac{(t-x)^3 d^3y}{1\cdot 2\cdot 3dx^3} + \text{etc.}$$

Si igitur *v* fuerit talis functio ipsius *t*, qualis *y* est ipsius *x*, quia *v* ex *y* nascitur ponendo *t* loco *x*, erit

$$v = y + \frac{(t-x)dy}{1dx} + \frac{(t-x)^2 ddy}{1\cdot 2dx^2} + \frac{(t-x)^3 d^3y}{1\cdot 2\cdot 3dx^3} + \text{etc.},$$

cuius veritas quibuscunque exemplis comprobari potest.

EXEMPLUM

Sit enim $y = xx - x$ *; manifestum est posito t loco x fore* $v = tt - t$ *, quod idem expressio inventa declarabit.*

Nam ob $y = xx - x$ erit

$$\frac{dy}{dx} = 2x - 1 \text{ et } \frac{ddy}{2dx^2} = 1 ;$$

unde fiet

$$\begin{aligned} v &= xx - x + (t-x)(2x-1) + (t-x)^2 \\ &= xx - x + 2tx - 2xx - t + x + tt - 2tx + xx = tt - t. \end{aligned}$$

Si itaque *y* fuerit eiusmodi functio ipsius *x*, quae posito $x = a$ abeat in *A*, ob $t = a$ et $v = A$ fiet

$$A = y + \frac{(a-x)dy}{1dx} + \frac{(a-x)^2 ddy}{1\cdot 2dx^2} + \frac{(a-x)^3 d^3y}{1\cdot 2\cdot 3dx^3} + \text{etc.},$$

huicque ergo aequationi omnes functiones ipsius *x*, quae facto $x = a$ abeunt in *A*, satisfaciunt.