CHAPTER II

CONCERNING THE INVESTIGATION OF SUMMABLE SERIES

19. If the sum of a series were known, in which the indeterminate quantity x is present, which certainly will be a function of x, then, whichever value may be attributed to x, the sum of the series will be able to be assigned always. Whereby if in place x there may be put $x+dx$, the sum of the resulting series will be equal to the sum of the former series together with the differential of this ; from which follows that the differential of the sum is equal to the differential of the series. Because indeed in this manner both the sum as well as the individual terms of the series will be multiplied by dx, if it may be divided by dx everywhere, a new series will be obtained, the sum of which will be known. In a similar manner if this series with its new sum may be differentiated again and it may be divided everywhere dx, a new series is produced with its own sum and thus from one summable series involving the indeterminate quantity x equally by continued differentiation there may be elicited innumerable new series.

20. So that these may be understood more clearly, this indeterminate geometric progression shall be proposed, obviously the sum of which is known,

$$\frac{1}{1-x}=1+x+x^2+x^3+x^4+x^5+x^6+\text{etc}.$$

If now differentiation is put in place, there shall be

$$\frac{dx}{(1-x)^2}=dx+2xdx+3x^2dx+4x^3dx+5x^4dx+\text{etc}.$$

and with division by dx carried out, there will be had

$$\frac{1}{(1-x)^2}=1+2x+3x^2+4x^3+5x^4+\text{etc}.$$

If it may be differentiated anew and divided by dx, there will be produced

$$\frac{2}{(1-x)^3}=2+2\cdot 3x+3\cdot 4x^2+4\cdot 5x^3+5\cdot 6x^4+\text{etc.}$$

or

$$\frac{1}{(1-x)^3}=1+3x+6x^2+10x^3+15x^4+21x^5+\text{etc.},$$

where the coefficients are triangular numbers. If these may be differentiated again and divided by *3dx*, there will be obtained

$$\frac{1}{(1-x)^4} = 1 + 4x + 10x^2 + 20x^3 + 35x^4 + \text{etc.},$$

the coefficients of which are the first pyramidal numbers. And thus on proceeding further the same series arise, which it is agreed to be produced from the expansion of the fraction $\frac{1}{(1-x)^n}$.

21. But this investigation of the series will extend more widely, if before whichever differentiation may be undertaken, the series itself together with the sum may be multiplied by some power or function of *x*. Thus since there shall be

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \text{etc}$$

it may be multiplied everywhere by x^m and there will be

$$\frac{x^m}{1-x} = x^m + x^{m+1} + x^{m+2} + x^{m+3} + x^{m+4} + \text{etc}$$

Now this series may be differentiated and it becomes on division by *dx*

$$\frac{mx^{m-1} - (m-1)x^m}{(1-x)^2} = mx^{m-1} + (m+1)x^m + (m+2)x^{m+1} + (m+3)x^{m+2} + \text{etc}.$$

Now it is divided by x^{m-1}; there will be obtained

$$\frac{m-(m-1)x}{(1-x)^2} = \frac{m}{1-x} + \frac{x}{(1-x)^2} = m + (m+1)x + (m+2)x^2 + \text{etc}.$$

This may be multiplied by x^n, before a new differentiation may be undertaken, so that there shall be

$$\frac{mx^n}{1-x} + \frac{x^{n+1}}{(1-x)^2} = mx^n + (m+1)x^{n+1} + (m+2)x^{n+2} + \text{etc}$$

Now the differentiation may be put in place and on division by *dx* there will be

$$\frac{mnx^{n-1}}{1-x}+\frac{(m+n+1)x^{n}}{(1-x)^{2}}+\frac{2x^{n+1}}{(1-x)^{3}}=mnx^{n-1}+(m+1)(n+1)x^{n}+(m+2)(n+2)x^{n+1}+\text{etc}.$$

But on division by x^{n-1} put in place there becomes

$$\frac{mn}{1-x}+\frac{(m+n+1)x}{(1-x)^{2}}+\frac{2xx}{(1-x)^{3}}=mn+(m+1)(n+1)x+(m+2)(n+2)x^{2}+\text{etc.}$$

and thus it will be allowed to progress further; moreover always the same series will be found, which come from expanding out of the sum of the fractions put in place.

22. Because the sum of the geometric progression first assumed can be assigned as far as some term, in this manner also constant series with a definite number of terms will be summed. Thus since there shall be

$$\frac{1-x^{n+1}}{1-x}=1+x+x^{2}+x^{3}+x^{4}+\cdots+x^{n},$$

with the differentiation put in place and with the terms divided by *dx* there will be

$$\frac{1}{(1-x)^{2}}-\frac{(n+1)x^{n}-nx^{n+1}}{(1-x)^{2}}=1+2x+3x^{2}+4x^{3}\cdots+nx^{n-1}.$$

Hence the sums of the powers of the natural numbers will be able to be found to any term. For this series may be multiplied by *x*, so that there becomes

$$\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^{2}}=x+2x^{2}+3x^{3}+\cdots+nx^{n},$$

which on differentiation again and on division by *dx* will give

$$\frac{1+x-(n+1)^{2}x^{n}+(2nn+2n-1)x^{n+1}-nnx^{n+2}}{(1-x)^{3}}=1+4x+9x^{2}+\cdots+n^{2}x^{n-1};$$

which on multiplication by *x* will give

$$\frac{x+x^{2}-(n+1)^{2}x^{n+1}+(2nn+2n-1)x^{n+2}-nnx^{n+3}}{(1-x)^{3}}=x+4x^{2}+9x^{3}+\cdots+n^{2}x^{n},$$

which differentiated, divided by *dx* and on multiplication *x* will produce this series

$$x + 8x^2 + 27x^3 + \cdots + n^3 x^n,$$

the sum of which therefore may be found. And from this in a similar manner the sum of biquadratics and of higher powers will be elicited indefinitely.

23. Therefore this method can be applied to all series containing an indeterminate quantity, of which indeed the sums are agreed on. Therefore since besides the recurring geometric series all may enjoy the same prerogative, so that the series may be summed not only to infinity, but also to whatever number of terms, also by this method from these the sums of innumerable other series will be able to be found. As because there shall be the greatest need for diversity, if we should wish that to be pursued, then we may consider a single case.

Clearly this series may be proposed

$$\frac{x}{1-x-xx} = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + \text{etc.},$$

[One may note that the coefficients of the powers of *x* are the Fibonacci numbers.]
which differentiated and divided by dx will give

$$\frac{1+xx}{(1-x-xx)^2} = 1 + 2x + 6x^2 + 12x^3 + 25x^4 + 48x^5 + 91x^6 + \text{etc.}$$

But it is readily apparent that all these series resulting in this manner are to be recurring also, the sums of which thus will be able to be found from the nature of these.

24. In general therefore, if the sum were known of a certain series contained in this form

$$ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc.},$$

which we may put $= S$, the sum can be found of the same series, if the individual terms one by one may be multiplied by the terms of an arithmetical progression. Indeed there shall be

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc.};$$

it may be multiplied by x^m; there will be

$$Sx^m = ax^{m+1} + bx^{m+2} + cx^{m+3} + dx^{m+4} + \text{etc.};$$

this equation may be differentiated and divided by dx

$$mSx^{m-1} + x^m \frac{dS}{dx} = (m+1)ax^m + (m+2)bx^{m+1} + (m+3)cx^{m+2} + \text{etc.};$$

it may be divided by x^{m-1} and there shall be

$$mS+\frac{xdS}{dx}=(m+1)ax+(m+2)bx^2+(m+3)cx^3+\text{etc.}$$

But if the sum of the following series therefore may be desired

$$\alpha ax+(\alpha+\beta)bx^2+(\alpha+2\beta)cx^3+(\alpha+3\beta)dx^4+\text{etc.},$$

the above may be multiplied by β and there is put in place $m\beta+\beta=\alpha$, so that there shall be $m=\frac{\alpha-\beta}{\beta}$, and the sum of this series will be $=(\alpha-\beta)S+\frac{\beta xdS}{dx}$.

25. Also it will be possible to find the sum of this proposed series, if the individual terms of this may be multiplied by the individual terms of the second order series, evidently the second differences shall be constant finally. Because indeed now we find

$$mS+\frac{xdS}{dx}=(m+1)ax+(m+2)bx^2+(m+3)cx^3+\text{etc.},$$

may be multiplied by x^n, so that shall be

$$mSx^n+\frac{x^{n+1}dS}{dx}=(m+1)ax^{n+1}+(m+2)bx^{n+2}+\text{etc.};$$

it may be differentiated on putting *dx* constant and divided by *dx*

$$mnSx^{n-1}+\frac{(m+n+1)x^n dS}{dx}+\frac{x^{n+1}ddS}{dx^2}=(m+1)(n+1)ax^n+(m+2)(n+2)bx^{n+1}+\text{etc.}$$

It may be divided by x^{n-1} and multiplied by *k*, so that there shall be

$$mnkS+\frac{(m+n+1)kxdS}{dx}+\frac{x^2kddS}{dx^2}=(m+1)(n+1)kax+(m+2)(n+2)kbx^2+(m+3)(n+3)kcx^3+\text{etc.}$$

Now this series may be compared with that above ; there will be

	Diff. I	Diff. II
$kmn+km+kn+k=\alpha$	$km+kn+3k=\beta$	$2k=\gamma$
$kmn+2km+2kn+4k=\alpha+\beta$	$km+kn+5k=\beta+\gamma$	
$kmn+3km+3kn+9k=\alpha+2\beta+\gamma$		

Therefore $k = \frac{1}{2}\gamma$ and $m + n = \frac{2\beta}{\gamma} - 3$ and

$$mn = \frac{\alpha}{k} - m - n - 1 = \frac{2\alpha}{\gamma} - \frac{2\beta}{\gamma} + 2 = \frac{2(\alpha-\beta+\gamma)}{\gamma}.$$

Hence the sum of the series sought will be

$$(\alpha - \beta + \gamma)S + \frac{(\beta-\gamma)xdS}{dx} + \frac{\gamma x^2 ddS}{2dx^2}$$

26. In a similar manner the sum of this series will be able to be found

$$Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \text{etc.},$$

if indeed the sum S of this series were known

$$S = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \text{etc.}$$

and A, B, C, D etc. constitute a series, which may be reduced to constant differences. Indeed the sum may be because, because the form of that is deduced from this preceding,

$$\alpha S + \frac{\beta xdS}{dx} + \frac{\gamma x^2 ddS}{dx^2} + \frac{\delta x^3 d^3S}{6dx^3} + \frac{\varepsilon x^4 d^4S}{24dx^4} + \text{etc.}$$

Now the individual series may be set out for finding the letters $\alpha,\ \beta,\ \gamma,\ \delta$ etc. and there will be

$$\begin{aligned}
\alpha S &= \alpha a + \alpha bx + \alpha cx^2 + \alpha dx^3 + \alpha ex^4 + \text{etc.}\\
\frac{\beta xdS}{dx} &= \beta bx + 2\beta cx^2 + 3\beta dx^3 + 4\beta ex^4 + \text{etc.}\\
\frac{\gamma x^2 ddS}{2dx^2} &= \gamma cx^2 + 3\gamma dx^3 + 6\gamma ex^4 + \text{etc.}\\
\frac{\delta x^3 d^3S}{6dx^3} &= \delta dx^3 + 4\delta ex^4 + \text{etc.}\\
\frac{\varepsilon x^4 d^4S}{24dx^4} &= 4\varepsilon ex^4 + \text{etc.}\\
&\text{etc.};
\end{aligned}$$

which taken at the same time may be compared with the proposed series

$$Z = Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \text{etc.}$$

and there becomes established from a comparison of the individual terms :

$$\alpha = A$$
$$\beta = B - \alpha = B - A$$
$$\gamma = C - 2\beta - \alpha = C - 2B + A$$
$$\delta = D - 3\gamma - 3\beta - \alpha = D - 3C + 3B - A$$
etc.

Therefore from these values found the sum sought will be

$$Z = AS + \frac{(B-A)xdS}{1dx} + \frac{(C-2B+A)x^2ddS}{1\cdot 2dx^2} + \frac{(D-3C+3B-A)x^3d^3S}{1\cdot 2\cdot 3dx^3} + \text{etc.},$$

or if the continued differences of the series *A*, *B*, *C*, *D*, *E* etc. may be indicated in the customary manner, there will be

$$Z = AS + \frac{\Delta A\cdot xdS}{1dx} + \frac{\Delta^2 A\cdot x^2 ddS}{1\cdot 2dx^2} + \frac{\Delta^3 A\cdot x^3 d^3S}{1\cdot 2\cdot 3dx^3} + \text{etc.}$$

If indeed there were, as we have assumed,

$$S = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \text{etc.}$$

Therefore if the series *A*, *B*, *C*, *D* etc. may have constant differences only, the sum of the series *Z* will be able to be expressed finitely.

27. Because on taking e for the number, of which the hyperbolic logarithm is $=1$, there is

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \text{etc.}$$

this series is taken for the prior proposition, and since there shall be $S = e^x$, there will be $\frac{dS}{dx} = e^x$, $\frac{ddS}{dx^2} = e^x$ etc.
Whereby the sum of this series, which is composed from that and here *A*, *B*, C, *D* etc.,

$$A + \frac{Bx}{1} + \frac{Cx^2}{1\cdot 2} + \frac{Dx^3}{1\cdot 2\cdot 3} + \frac{Ex^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

may be expressed in this manner

$$e^x\left(A + \frac{x\Delta A}{1} + \frac{xx\Delta^2 A}{1\cdot 2} + \frac{x^3\Delta^3 A}{1\cdot 2\cdot 3} + \text{etc.}\right).$$

Thus, if this series may be proposed

$$2+\frac{5x}{1}+\frac{10x^2}{1\cdot 2}+\frac{17x^3}{1\cdot 2\cdot 3}+\frac{26x^4}{1\cdot 2\cdot 3\cdot 4}+\frac{37x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\text{etc.},$$

on account of the series

$$\begin{array}{rccccl} & A, & B, & C, & D, & E \quad \text{etc.}\\ A= & 2, & 5, & 10, & 17, & 26 \ \text{etc.}\\ \varDelta A= & 3 & 5, & 7, & 9 & \text{etc.}\\ \varDelta^2 A= & & 2, & 2, & 2 & \text{etc.}\end{array}$$

the sum of this series will be

$$2+5x+\frac{10x^2}{2}+\frac{17}{6}x^3+\frac{26x^4}{24}+\text{etc.}$$

$$=e^x(2+3x+xx)=e^x(1+x)(2+x),$$

which indeed is apparent at once. For there is

$$\begin{aligned} 2e^x &= 2+\frac{2x}{1}+\frac{2x^2}{2}+\frac{2x^3}{6}+\frac{2x^4}{24}+\text{etc.}\\ 3xe^x &= 3x+\frac{3x^2}{1}+\frac{3x^3}{2}+\frac{3x^4}{6}+\text{etc.}\\ xxe^x &= xx+\frac{x^3}{1}+\frac{x^4}{2}+\text{etc.}\end{aligned}$$

$$e^x(2+3x+xx)=2+5x+\frac{10xx}{2}+\frac{17x^3}{6}+\frac{26x^4}{24}+\text{etc.}$$

28. So far [propositions] have been handled, which do not consider only series going off to infinity, but also the sums of certain numbers of terms ; for the coefficients *a, b, c, d* etc. are able either to progress to infinity or, wherever it may please, they may be interrupted. Truly since this does not need to be explained in more detail, at this point we may consider more carefully that which follows produced from these. Therefore with some proposed series, of which the individual terms may depend on two factors, of which the series of the other are set up to be reduced to constant differences, the sum of the first series will be possible to be assigned, provided the series were summable with these factors omitted. Evidently, if that series should be proposed

$$Z=Aa+Bbx+Ccx^2+Ddx^3+Eex^4+\text{etc.},$$

in which the quantities *A, B, C, D, E* etc. may constitute a series of this kind, which only may lead to constant differences, then the sum of this series itself will be able to be shown, provided the sum *S* of this series

$$S = a + bx + cx^2 + dx^3 + ex^4 + \text{etc}.$$

may be had. Indeed with the continued constant differences taken from the progression *A, B, C, D, E* etc., as we have shown at the start of this book,

$$\begin{array}{c} A,\quad B,\quad C,\quad D,\quad E,\quad F\quad \text{etc.} \\ \Delta A,\quad \Delta B,\quad \Delta C,\quad \Delta D,\quad \Delta E\quad \text{etc.} \\ \Delta^2 A,\ \Delta^2 B,\ \Delta^2 C,\ \Delta^2 D\ \text{etc.} \\ \Delta^3 A,\ \Delta^3 B,\ \Delta^3 C\ \text{etc.} \\ \Delta^4 A,\ \Delta^4 B\ \text{etc.} \\ \Delta^5 A\quad \text{etc.} \\ \text{etc.} \end{array}$$

the sum of the proposed series will be

$$Z = SA + \frac{xdS}{1dx}\Delta A + \frac{x^2 ddS}{1\cdot 2dx^2}\Delta^2 A + \frac{x^3 d^3 S}{1\cdot 2\cdot 3dx^3}\Delta^3 A + \text{etc.},$$

on putting in place the higher differentials of *S* with constant *dx*.
[Thus, *S* is a known power series of *x*, with known derivatives,
$S = a + bx + cx^2 + dx^3 + ex^4 + \text{etc}$ while *A, B, C, D* etc and the higher differences of *A* are present in the coefficients of another series $Z = Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \text{etc.}$ This second series may be summed in the form $Z = SA + \frac{xdS}{1dx}\Delta A + \frac{x^2 ddS}{1\cdot 2dx^2}\Delta^2 A + \frac{x^3 d^3 S}{1\cdot 2\cdot 3dx^3}\Delta^3 A + \text{etc.}$]

29. Therefore if the series *A, B, C, D* etc. under no circumstances may lead to constant differences, the sum of the series *Z* may be expressed by a new infinite series, which meanwhile will converge more than the proposed, and thus that series will be transformed into another equal to itself . This series may be proposed according to this declaration

$$Y = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \frac{y^5}{5} + \frac{y^6}{6} + \text{etc.},$$

as it may be agreed to express $l\frac{1}{1-y}$ thus so that there shall be $Y = -l(1-y)$. This series may be divided by *y* and there may be put in place $y = x$ and $Y = yZ$, so that there shall be

$$Z = l\frac{1}{1-y} = -\frac{1}{x}l(l-x);$$

there shall be

$$Z = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \text{etc.},$$

which compared with that

$$S = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \text{etc.} = \frac{1}{1-x}$$

will give these values for the series A, B, C, D, E etc.

$$\begin{array}{c} 1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5} \text{ etc.} \\ -\frac{1}{1\cdot 2}, \quad -\frac{1}{2\cdot 3}, \quad -\frac{1}{3\cdot 4}, \quad -\frac{1}{4\cdot 5} \text{ etc.} \\ \frac{1\cdot 2}{1\cdot 2\cdot 3}, \quad \frac{1\cdot 2}{2\cdot 3\cdot 4}, \quad \frac{1\cdot 2}{3\cdot 4\cdot 5} \text{ etc.} \\ -\frac{1\cdot 2\cdot 3}{1\cdot 2\cdot 3\cdot 4}, \quad -\frac{1\cdot 2\cdot 3}{2\cdot 3\cdot 4\cdot 5} \text{ etc.} \\ \text{etc.} \end{array}$$

Therefore there will be

$$A = 1, \quad \Delta A = -\frac{1}{2}, \quad \Delta^2 A = \frac{1}{3}, \quad \Delta^3 A = -\frac{1}{4} \text{ etc.}$$

Again since there shall be $S = \frac{1}{1-x}$, there will be

$$\frac{dS}{dx} = \frac{1}{(1-x)^2}, \quad \frac{ddS}{1\cdot 2dx^2} = \frac{1}{(1-x)^3}, \quad \frac{d^3S}{1\cdot 2\cdot 3dx^3} = \frac{1}{(1-x)^4} \text{ etc.}$$

With which values substituted the sum may arise

$$Z = \frac{1}{1-x} - \frac{x}{2(1-x)^2} + \frac{x^2}{3(1-x)^3} - \frac{x^4}{4(1-x)^4} + \frac{x^5}{5(1-x)^5} - \text{etc.}$$

Therefore since there shall be $x = y$ and $Y = -l(1-y) = yZ$, there will be

$$-l(1-y) = \frac{y}{1-x} - \frac{y^2}{2(1-y)^2} + \frac{y^3}{3(1-y)^3} - \frac{y^4}{4(1-y)^4} + \text{etc.},$$

which series certainly will express the series $l\left(1 + \frac{y}{1-y}\right) = l\frac{1}{1-y} = -l(1-y)$, the truth of which thus agrees with the former demonstration.

30. Now this series shall be proposed, so that the use may be apparent also, if only the odd powers occur and the signs may alternate,

$$Y = y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \frac{y^9}{9} - \frac{y^{11}}{11} + \text{etc.},$$

from which there is agreed to be $Y = \text{Atang}\, y$.
This series may be divided by y and there may be put in place $\frac{Y}{y} = Z$ and $yy = x$; there will be

$$Z = 1 - \frac{x}{3} + \frac{xx}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{y^5}{11} + \text{etc.}$$

Which if it may be compared with that series

$$S = 1 - x + xx - x^3 + x^4 - x^5 + x^6 + \text{etc.},$$

there becomes $S = \frac{1}{1+x}$ and the series of the coefficients A, B, C, D etc. become

$$\begin{array}{llllll}
A = 1, & \frac{1}{3}, & \frac{1}{5}, & \frac{1}{7}, & \frac{1}{9} & \text{etc.} \\
\varDelta A = & -\frac{2}{3}, & -\frac{2}{3\cdot 5}, & -\frac{2}{5\cdot 7}, & -\frac{2}{7\cdot 9} & \text{etc.} \\
\varDelta^2 A = & & \frac{1\cdot 2}{1\cdot 2\cdot 3}, & \frac{1\cdot 2}{2\cdot 3\cdot 4}, & \frac{1\cdot 2}{3\cdot 4\cdot 5} & \text{etc.} \\
\varDelta^3 A = & & & -\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}, & -\frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7\cdot 9} & \text{etc.} \\
\varDelta^4 A = & & & & -\frac{2\cdot 4\cdot 6\cdot 8}{3\cdot 5\cdot 7\cdot 9} & \text{etc.} \\
 & & \text{etc.} & & &
\end{array}$$

But since there shall be $S = \frac{1}{1+x}$, there will be

$$\frac{dS}{1dx} = -\frac{1}{(1+x)^2}, \quad \frac{ddS}{1\cdot 2dx^2} = \frac{1}{(1+x)^3}, \quad \frac{d^3S}{1\cdot 2\cdot 3dx^3} = -\frac{1}{(1+x)^4} \quad \text{etc.}$$

Whereby with these values substituted the form comes about

$$Z = \frac{1}{1+x} + \frac{2x}{3(1+x)^2} + \frac{2\cdot 4x^2}{3\cdot 5(1+x)^3} + \frac{2\cdot 4\cdot 6x^3}{3\cdot 5\cdot 7(1+x)^4} + \text{etc.}$$

Therefore on restoring $x = yy$ and on multiplying by y there is made

$$Y = \text{Atang}\, y = \frac{y}{1+yy} + \frac{2y^3}{3(1+yy)^2} + \frac{2\cdot 4y^5}{3\cdot 5(1+yy)^3} + \frac{2\cdot 4\cdot 6y^7}{3\cdot 5\cdot 7(1+yy)^4} + \text{etc.}$$

31. Also the above series, in which the arc of the circle is expressed by the tangent, can be transformed in another way by comparing that with the logarithmic series.

Clearly we may consider the series

$$Z = 1 - \frac{x}{3} + \frac{xx}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{y^5}{11} + \text{etc.},$$

as we may compare that with

$$S = \frac{1}{0} - \frac{x}{2} + \frac{xx}{4} - \frac{x^3}{6} + \frac{x^4}{8} - \text{etc.} = \frac{1}{0} - \frac{1}{2} l\left(1+x\right),$$

and the values of the letters *A, B, C, D* etc. will be
[i.e. $a = \frac{1}{0}, b = \frac{1}{2}$, etc; while $aA = 1, bB = \frac{1}{3}$, etc]

$$\begin{array}{lllll} A = \frac{0}{1}, & \frac{2}{3}, & \frac{4}{5}, & \frac{6}{7}, & \frac{8}{9} \quad \text{etc.} \\ \varDelta A = & \frac{2}{3}, & \frac{+2}{3\cdot5}, & \frac{+2}{5\cdot7}, & \frac{+2}{7\cdot9} \quad \text{etc.} \\ \varDelta^2 A = & & \frac{-2\cdot4}{3\cdot5}, & \frac{-2\cdot4}{3\cdot5\cdot7}, & \frac{-2\cdot4}{5\cdot7\cdot9} \quad \text{etc.} \\ \varDelta^3 A = & & & \frac{2\cdot4\cdot6}{3\cdot5\cdot7} \quad \text{etc.} & \\ & & \text{etc.} & & \end{array}$$

Then, since there shall be $S = \frac{1}{0} - \frac{1}{2} l\left(1+x\right)$, there will be

$$\frac{dS}{1dx} = -\frac{1}{2(1+x)}, \quad \frac{ddS}{1\cdot2dx^2} = \frac{1}{4(1+x)^2},$$
$$\frac{d^3S}{1\cdot2\cdot3dx^3} = -\frac{1}{6(1+x)^3}, \quad \frac{d^4S}{1\cdot2\cdot3\cdot4dx^4} = \frac{1}{8(1+x)^4} \quad \text{etc.}$$

Therefore there will be $SA = S\frac{0}{1} = 1$ and from the remaining there is made

$$Z = 1 - \frac{x}{3(1+x)} - \frac{2xx}{3\cdot5(1+x)^2} - \frac{2\cdot4x^3}{3\cdot5\cdot7(1+x)^3} - \text{etc.}$$

Now there may be put $x = yy$ and the series may be multiplied by y ; there becomes

$$Y = \text{Atang}\, y = y - \frac{y^3}{3(1+yy)} - \frac{2y^5}{3\cdot5(1+yy)^2} - \frac{2\cdot4y^7}{3\cdot5\cdot7(1+yy)^3} - \text{etc.}$$

Therefore this transmutation is not obstructed by the infinite term $\frac{1}{0}$, which may enter into the series *S*. But if doubt should remain about that, it may resolve only the first term besides the individual terms following the powers of *y* in the series and actually may cause the first series proposed to result.

32. Up to this point we have considered only series in which all powers of the variable occur. Now therefore we may progress to other series, in which the individual terms the same power of the variable may be considered, but which is of this kind

$$S = \frac{1}{a+x} + \frac{1}{b+x} + \frac{1}{c+x} + \frac{1}{d+x} + \text{etc.}$$

Indeed for a series of this kind if the sum *S* were known and it may be expressed by some function of x , there will be on differentiation and on being divided by $-dx$

$$\frac{-dS}{dx} = \frac{1}{(a+x)^2} + \frac{1}{(b+x)^2} + \frac{1}{(c+x)^2} + \frac{1}{(d+x)^2} + \text{etc.}$$

If this may be differentiated further and divided by $-2dx$, the series of cubes may be recognised

$$\frac{ddS}{2dx^2} = \frac{1}{(a+x)^3} + \frac{1}{(b+x)^3} + \frac{1}{(c+x)^3} + \frac{1}{(d+x)^3} + \text{etc.}$$

and this again differentiated and divided by $-3dx$ will give

$$\frac{-d^3S}{6dx^3} = \frac{1}{(a+x)^4} + \frac{1}{(b+x)^4} + \frac{1}{(c+x)^4} + \frac{1}{(d+x)^4} + \text{etc.}$$

And in a similar manner the sums of all the following powers may be found, provided the sum of the first series were known.

33. But above in the *Introductio* [Ch.10] we have elicited series of fractions involving an indeterminate quantity, where we have shown, if a circle of which the radius $=1$, with the semi periphery put in place $=\pi$, to be

$$\frac{\pi}{n\sin\frac{m}{n}\pi} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \text{etc.}$$

$$\frac{\pi\cos\frac{m}{n}\pi}{n\sin\frac{m}{n}\pi} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \text{etc.}$$

Therefore since it is allowed to assume any numbers for *m* and *n*, we may put in place $n=1$ and $m=x$, so that we may obtain those similar series, as we were proposing in the preceding chapter; with this done there will be

$$\frac{\pi}{\sin\pi x} = \frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} + \frac{1}{3-x} - \text{etc.}$$

$$\frac{\pi\cos\pi x}{\sin\pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \text{etc.}$$

Therefore by differentiations the sums of any powers will be able to be shown from the fractions shown.

34. We may consider the first series and for the sake of brevity let $\frac{\pi}{sin\,\pi x} = S$, of which the higher differentials may be taken with dx put constant, and there shall be

$$S = \frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} - \text{etc}$$

$$\frac{-dS}{dx} = \frac{1}{xx} - \frac{1}{(1-x)^2} - \frac{1}{(1+x)^2} + \frac{1}{(2-x)^2} + \frac{1}{(2+x)^2} - \frac{1}{(3-x)^2} - \text{etc.}$$

$$\frac{ddS}{2dx^2} = \frac{1}{x^3} + \frac{1}{(1-x)^3} - \frac{1}{(1+x)^3} - \frac{1}{(2-x)^3} + \frac{1}{(2+x)^3} + \frac{1}{(3-x)^3} - \text{etc.}$$

$$\frac{-d^3S}{6dx^3} = \frac{1}{x^4} - \frac{1}{(1-x)^4} - \frac{1}{(1+x)^4} + \frac{1}{(2-x)^4} + \frac{1}{(2+x)^4} - \frac{1}{(3-x)^4} - \text{etc.}$$

$$\frac{d^4S}{24dx^4} = \frac{1}{x^5} + \frac{1}{(1-x)^5} - \frac{1}{(1+x)^5} - \frac{1}{(2-x)^5} + \frac{1}{(2+x)^5} + \frac{1}{(3-x)^5} - \text{etc.}$$

$$\frac{-d^5S}{120dx^5} = \frac{1}{x^6} - \frac{1}{(1-x)^6} - \frac{1}{(1+x)^6} + \frac{1}{(2-x)^6} + \frac{1}{(2+x)^6} - \frac{1}{(3-x)^6} - \text{etc.}$$

etc.

where it is to be noted in even powers the signs follow the same rule and equally in odd powers the same rule of signs is to be seen. Therefore the sums of all of these series may be found from the differentials of the expression $S = \frac{\pi}{sin\,\pi x}$.

35. Towards expressing these differentials more simply, we may put

$$\sin\pi x = p \text{ and } \cos\pi x = q;$$

there will be

$$dp = \pi dx\cos\pi x = \pi q dx \quad\text{and}\quad dq = -\pi p dx.$$

Therefore since there shall be $S = \frac{\pi}{p}$, there will be

$$\frac{-dS}{dx}=\frac{\pi^2 q}{pp}$$

$$\frac{ddS}{dx^2}=\frac{\pi^3(pp+2qq)}{p^3}=\frac{\pi^3(qq+1)}{p^3}\quad \text{on account of } pp+qq=1$$

$$\frac{-d^3S}{dx^3}=\pi^4\left(\frac{5q}{pp}+\frac{6q^3}{p^4}\right)=\frac{\pi^4\left(q^3+5q\right)}{p^4}$$

$$\frac{d^4S}{dx^4}=\pi^5\left(\frac{24q^4}{p^5}+\frac{28q^2}{p^3}+\frac{5}{p}\right)=\frac{\pi^5\left(q^4+18q^2+5\right)}{p^5}$$

$$\frac{-d^5S}{dx^5}=\pi^6\left(\frac{120q^5}{p^6}+\frac{180q^3}{p^4}+\frac{61q}{pp}\right)=\frac{\pi^6\left(q^5+58q^3+61q\right)}{p^6}$$

$$\frac{d^6S}{dx^6}=\pi^7\left(\frac{720q^6}{p^7}+\frac{1320q^4}{p^5}+\frac{662q^2}{p^3}+\frac{61}{p}\right)=\frac{\pi^7\left(q^6+179q^4+479q^2+61\right)}{p^7}$$

$$\frac{-d^7S}{dx^7}=\pi^8\left(\frac{5040q^7}{p^8}+\frac{10920q^5}{p^6}+\frac{7266q^3}{p^4}+\frac{1385q}{p^2}\right)$$

or

$$=\frac{\pi^8}{p^8}\left(q^7+543q^5+3111q^3+1385q\right)$$

$$\frac{d^8S}{dx^8}=\pi^9\left(\frac{40320q^8}{p^9}+\frac{100800q^6}{p^7}+\frac{83664q^4}{p^5}+\frac{24568q^2}{p^3}+\frac{1385}{p}\right)$$

or

$$=\frac{\pi^9}{p^9}\left(q^8+1636q^6+18270q^4+19028q^2+1385\right)$$

etc.

Which expressions are able to be continued further, as far as it pleases; for if there were

$$\pm\frac{d^nS}{dx^n}=\pi^{n+1}\left(\frac{\alpha q^n}{p^{n+1}}+\frac{\beta q^{n-2}}{p^{n-1}}+\frac{\gamma q^{n-4}}{p^{n-3}}+\frac{\delta q^{n-6}}{p^{n-5}}+\text{etc.}\right),$$

there will be the following differential with the sign changed

$$\mp\frac{d^{n+1}S}{dx^{n+1}}=\pi^{n+2}\left\{\begin{array}{c}(n+1)\alpha\frac{q^{n+1}}{p^{n+2}}+\left(n\alpha+(n-1)\beta\right)\frac{q^{n-1}}{p^n}+\left((n-2)\beta+(n-3)\gamma\right)\frac{q^{n-3}}{p^{n-2}}\\+\left((n-4)\gamma+(n-5)\delta\right)\frac{q^{n-5}}{p^{n-4}}+\text{etc.}\end{array}\right\},$$

36. Therefore from these the following sums will be obtained of the above series shown in § 34

$$S = \pi \cdot \frac{1}{p}$$

$$\frac{-dS}{dx} = \frac{\pi^2}{1} \cdot \frac{q}{p^2}$$

$$\frac{ddS}{2dx^2} = \frac{\pi^3}{2}\left(\frac{2q^2}{p^3} + \frac{1}{p}\right)$$

$$\frac{-d^3S}{6dx^3} = \frac{\pi^4}{6}\left(\frac{6q^3}{p^4} + \frac{5q}{p^2}\right)$$

$$\frac{d^4S}{24dx^4} = \frac{\pi^5}{24}\left(\frac{24q^4}{p^5} + \frac{28q^2}{p^3} + \frac{5}{p}\right)$$

$$\frac{-d^5S}{120dx^5} = \frac{\pi^6}{120}\left(\frac{120q^5}{p^6} + \frac{180q^3}{p^4} + \frac{61q}{p^2}\right)$$

$$\frac{d^6S}{720dx^6} = \frac{\pi^7}{720}\left(\frac{720q^6}{p^7} + \frac{1320q^4}{p^5} + \frac{662q^2}{p^3} + \frac{61}{p}\right)$$

$$\frac{-d^7S}{5040dx^7} = \frac{\pi^8}{5040}\left(\frac{5040q^7}{p^8} + \frac{10920q^5}{p^6} + \frac{7266q^3}{p^4} + \frac{1385q}{p^2}\right)$$

$$\frac{d^8S}{40320dx^8} = \frac{\pi^9}{40320}\left(\frac{40320q^8}{p^9} + \frac{100800q^6}{p^7} + \frac{24568q^4}{p^5} + \frac{19028q^2}{p^3} + \frac{1385}{p}\right)$$

etc.

37. We may treat the other series found above [§ 33] in the same manner

$$\frac{\pi \cos \pi x}{\sin \pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \text{etc.}$$

and therefore on putting for brevity $\frac{\pi \cos \pi x}{n \sin \pi x} = T$, the following summations may arise

$$T = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \text{etc.}$$

$$\frac{-dT}{dx} = \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{(1+x)^2} + \frac{1}{(2-x)^2} + \frac{1}{(2+x)^2} + \text{etc.}$$

$$\frac{ddT}{2dx^2} = \frac{1}{x^3} - \frac{1}{(1-x)^3} + \frac{1}{(1+x)^3} - \frac{1}{(2-x)^3} + \frac{1}{(2+x)^3} - \text{etc.}$$

$$\frac{d^3T}{6dx^3} = \frac{1}{x^4} + \frac{1}{(1-x)^4} + \frac{1}{(1+x)^4} + \frac{1}{(2-x)^4} + \frac{1}{(2+x)^4} + \text{etc.}$$

$$\frac{d^4T}{24dx^4} = \frac{1}{x^5} - \frac{1}{(1-x)^5} + \frac{1}{(1+x)^5} - \frac{1}{(2-x)^5} + \frac{1}{(2+x)^5} - \text{etc.}$$

$$\frac{-d^5T}{120dx^3} = \frac{1}{x^6} + \frac{1}{(1-x)^6} + \frac{1}{(1+x)^6} + \frac{1}{(2-x)^6} + \frac{1}{(2+x)^6} + \text{etc.}$$

etc.,

where in the even powers all the terms are positive, but in the odd powers the take alternately the signs + and –.

38. So that the values of the differentials of these may be known, we may put as before

$$\sin \pi x = p \quad \text{and} \quad \cos \pi x = q ,$$

where there shall be $pp + qq = 1$; there will be

$$dp = \pi q dx \ \text{ and } \ dq = -\pi p dx .$$

With which values used there will be

$$T = \pi \cdot \frac{q}{p}$$

$$\frac{-dT}{dx} = \pi^2\left(\frac{qq}{pp}+1\right) = \frac{\pi^2}{pp}$$

$$\frac{ddT}{dx^2} = \pi^3\left(\frac{2q^3}{p^3}+\frac{2q}{p}\right) = \frac{2\pi^3 q}{p^3}$$

$$\frac{-d^3T}{dx^3} = \pi^4\left(\frac{6q^4}{p^4}+\frac{8qq}{p^2}+2\right) = \pi^4\left(\frac{6qq}{p^4}+\frac{2}{pp}\right)$$

$$\frac{d^4T}{dx^4} = \pi^5\left(\frac{24q^3}{p^5}+\frac{16q}{p^3}\right)$$

$$\frac{-d^5T}{dx^5} = \pi^6\left(\frac{120q^4}{p^6}+\frac{120qq}{p^4}+\frac{16}{pp}\right)$$

$$\frac{d^6T}{dx^6} = \pi^7\left(\frac{720q^5}{p^7}+\frac{960q^3}{p^5}+\frac{272q}{p^3}\right)$$

$$\frac{-d^7T}{dx^7} = \pi^8\left(\frac{5040q^6}{p^8}+\frac{8400q^4}{p^6}+\frac{3696q^2}{p^4}+\frac{272}{p^2}\right)$$

$$\frac{d^8T}{dx^8} = \pi^9\left(\frac{40320q^7}{p^9}+\frac{80640q^5}{p^7}+\frac{48384q^3}{p^5}+\frac{7936q}{p^3}\right)$$

etc.

Which formulas are able to be continued further easily, as far as it pleases. For if there shall be

$$\pm\frac{d^nT}{dx^n} = \pi^{n+1}\left(\frac{\alpha q^{n-1}}{p^{n+1}}+\frac{\beta q^{n-3}}{p^{n-1}}+\frac{\gamma q^{n-5}}{p^{n-3}}+\frac{\delta q^{n-7}}{p^{n-5}}+\text{etc.}\right),$$

there will be the following expression

$$\mp\frac{d^{n+1}T}{dx^{n+1}} = \pi^{n+2}\left((n+1)\frac{\alpha q^n}{p^{n+2}}+\frac{(n-1)(\alpha+\beta)q^{n-2}}{p^n}+\frac{(n-3)(\beta+\gamma)q^{n-4}}{p^{n-2}}+\text{etc.}\right).$$

39. Therefore the given series of powers § 37 will have the following sums on putting

$$\sin\pi x = p \;\text{ and }\; \cos\pi x = q$$

$$T = \pi \cdot \frac{q}{p}$$
$$\frac{-dT}{dx} = \pi^2 \frac{1}{pp}$$
$$\frac{ddT}{2dx^2} = \pi^3 \frac{q}{p^3}$$
$$\frac{-d^3T}{6dx^3} = \pi^4 \left(\frac{qq}{p^4} + \frac{1}{3pp}\right)$$
$$\frac{d^4T}{24dx^4} = \pi^5 \left(\frac{q^3}{p^5} + \frac{2q}{3p^3}\right)$$
$$\frac{-d^5T}{120dx^5} = \pi^6 \left(\frac{q^4}{p^6} + \frac{3qq}{3p^4} + \frac{2}{15pp}\right)$$
$$\frac{d^6T}{720dx^6} = \pi^7 \left(\frac{q^5}{p^7} + \frac{4q^3}{3p^5} + \frac{17q}{45p^3}\right)$$
$$\frac{-d^7T}{5040dx^7} = \pi^8 \left(\frac{q^6}{p^8} + \frac{5q^4}{3p^6} + \frac{11q^2}{15p^4} + \frac{17}{315pp}\right)$$
$$\frac{d^8T}{40320dx^8} = \pi^9 \left(\frac{q^7}{p^9} + \frac{6q^5}{3p^7} + \frac{6q^3}{5p^5} + \frac{62q}{315p^3}\right)$$
etc.

40. Besides these series we have found, there are some others in the *Introductio* [Book I, Ch. 10] from which new series are able to be elicited by differentiation.

For we have shown to be

$$\frac{1}{2x} - \frac{\pi\sqrt{x}}{2x \operatorname{tang} \pi\sqrt{x}} = \frac{1}{1-x} + \frac{1}{4-x} + \frac{1}{9-x} + \frac{1}{16-x} + \frac{1}{25-x} + \text{etc.}$$

We may put the sum of this series to be $= S$, so that there shall be

$$S = \frac{1}{2x} - \frac{\pi\sqrt{x}}{2x} \cdot \frac{\cos \pi\sqrt{x}}{\sin \pi\sqrt{x}};$$

there will be

$$\frac{dS}{dx} = -\frac{1}{2xx} + \frac{\pi}{4x\sqrt{x}} \cdot \frac{\cos \pi\sqrt{x}}{\sin \pi\sqrt{x}} + \frac{\pi\pi}{4x\left(\sin \pi\sqrt{x}\right)^2},$$

which expression therefore gives the sum of this series

$$\frac{1}{(1-x)^2} + \frac{1}{(4-x)^2} + \frac{1}{(9-x)^2} + \frac{1}{(16-x)^2} + \frac{1}{(25-x)^2} + \text{etc.}$$

Then also we have shown to be

$$\frac{\pi}{2\sqrt{x}}\cdot\frac{e^{2\pi\sqrt{x}}+1}{e^{2\pi\sqrt{x}}-1}-\frac{1}{2x}=\frac{1}{1+x}+\frac{1}{4+x}+\frac{1}{9+x}+\frac{1}{16+x}+\text{etc.}$$

But if this sum may therefore be put $= S$, there will be

$$\frac{-dS}{dx}=\frac{1}{(1+x)^2}+\frac{1}{(4+x)^2}+\frac{1}{(9+x)^2}+\frac{1}{(16+x)^2}+\text{etc.}$$

But there is

$$\frac{dS}{dx}=\frac{-\pi}{4\sqrt{x}}\cdot\frac{e^{2\pi\sqrt{x}}+1}{e^{2\pi\sqrt{x}}-1}-\frac{\pi\pi}{x}\cdot\frac{e^{2\pi\sqrt{x}}}{\left(e^{2\pi\sqrt{x}}-1\right)^2}+\frac{1}{2xx}.$$

Therefore the sum of this series will be

$$\frac{-dS}{dx}=\frac{\pi}{4x\sqrt{x}}\cdot\frac{e^{2\pi\sqrt{x}}+1}{e^{2\pi\sqrt{x}}-1}+\frac{\pi\pi}{x}\cdot\frac{e^{2\pi\sqrt{x}}}{\left(e^{2\pi\sqrt{x}}-1\right)^2}-\frac{1}{2xx}.$$

And in a similar manner the sums of the following powers may be found from further differentiations.

41. If the value of a certain product were known composed from factors involving unknown letters, innumerable summable series are able to be found from that by the same method. Indeed let the value of this product

$$(1+\alpha x)(1+\beta x)(1+\gamma x)(1+\delta x)(1+\varepsilon x)\ \text{etc}.$$

be equal to *S*, clearly of some function of *x;* with the logarithms taken there shall be

$$lS=l(1+\alpha x)+l(1+\beta x)+l(1+\gamma x)+l(1+\delta x)\ \text{etc}.$$

Now the differentials are taken ; with the division by *dx* put in place there will be

$$\frac{dS}{Sdx}=\frac{\alpha}{1+\alpha x}+\frac{\beta}{1+\beta x}+\frac{\gamma}{1+\gamma x}+\frac{\delta}{1+\delta x}+\text{etc.},$$

from the further differentiation of which the sums of any powers of these fractions may be found, plainly as we have set out further in the previous examples.

42. Moreover we have shown in the *Introductio* several expressions of this kind, to which we may apply this method. Clearly if π shall be the 180^0 arc of the circle, the radius of which is equal to 1, we have shown to be

$$\sin\frac{m\pi}{2n} = \frac{m\pi}{2n}\cdot\frac{4nn-mm}{4nn}\cdot\frac{16nn-mm}{16nn}\cdot\frac{36nn-mm}{36nn}\cdot\text{etc.}$$

$$\cos\frac{m\pi}{2n} = \frac{nn-mm}{nn}\cdot\frac{9nn-mm}{9nn}\cdot\frac{25nn-mm}{25nn}\cdot\frac{49nn-mm}{49nn}\cdot\text{etc.}$$

We may put $n = 1$ and $m = 2x$, so that there shall be

$$\sin\pi x = \pi x\cdot\frac{1-xx}{1}\cdot\frac{4-xx}{4}\cdot\frac{9-xx}{9}\cdot\frac{16-xx}{16}\cdot\text{etc.}$$

or

$$\sin\pi x = \pi x\cdot\frac{1-x}{1}\cdot\frac{1+x}{1}\cdot\frac{2-x}{2}\cdot\frac{2+x}{2}\cdot\frac{3-x}{3}\cdot\frac{3+x}{3}\cdot\frac{4-x}{4}\cdot\text{etc.}$$

and

$$\cos\pi x = \frac{1-4xx}{1}\cdot\frac{9-4xx}{9}\cdot\frac{25-4xx}{25}\cdot\frac{49-4xx}{49}\cdot\text{etc.}$$

or

$$\cos\pi x = \frac{1-2x}{1}\cdot\frac{1+2x}{1}\cdot\frac{3-2x}{3}\cdot\frac{3+2x}{3}\cdot\frac{5-2x}{5}\cdot\frac{5+2x}{5}\cdot\text{etc.}$$

Therefore from these expressions, if the logarithms may be taken, there will be

$$l\sin\pi x = l\pi x + l\,\frac{1-x}{1} + l\,\frac{1+x}{1} + l\,\frac{2-x}{2} + l\,\frac{2+x}{2} + l\,\frac{3-x}{3} + \text{etc.}$$

$$l\cos\pi x = l\,\frac{1-2x}{1} + l\,\frac{1+2x}{1} + l\,\frac{3-2x}{3} + l\,\frac{3+2x}{3} + l\,\frac{5-2x}{5} + \text{etc.}$$

43. Now we may take the differentials of the logarithms of these series and with the division made everywhere by dx the first series will give

$$\frac{\pi\cos\pi x}{\sin\pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \text{etc.},$$

which is that series itself, which we treated in §37. Now the other series will give

$$\frac{-\pi\sin\pi x}{\cos\pi x} = -\frac{2}{1-2x} + \frac{2}{1+2x} - \frac{2}{3-2x} + \frac{2}{3+2x} - \frac{2}{5-2x} + \text{etc.}$$

We may put $2x = z$, so that there shall be $x = \frac{z}{2}$, and we may divide by -2; there will be

$$\frac{\pi\sin\frac{1}{2}\pi z}{2\cos\frac{1}{2}\pi z} = \frac{1}{1-z} - \frac{1}{1+z} + \frac{1}{3-z} - \frac{1}{3+z} + \frac{1}{5-z} - \text{etc.}$$

But since there shall be

$$\sin\tfrac{1}{2}\pi z = \sqrt{\frac{1-\cos\pi z}{2}} \quad\text{and}\quad \cos\tfrac{1}{2}\pi z = \sqrt{\frac{1+\cos\pi z}{2}},$$

there will be

$$\frac{\pi\sqrt{1-\cos\pi z}}{\sqrt{1+\cos\pi z}}=\frac{2}{1-z}-\frac{2}{1+z}+\frac{2}{3-z}-\frac{2}{3+z}+\frac{2}{5-z}-\text{etc.}$$

or on writing x in place of z there will be

$$\frac{\pi\sqrt{1-\cos\pi x}}{\sqrt{1+\cos\pi x}}=\frac{2}{1-x}-\frac{2}{1+x}+\frac{2}{3-x}-\frac{2}{3+x}+\frac{2}{5-x}-\text{etc.}$$

This series may be added to the first found

$$\frac{\pi\cos\pi x}{\sin\pi x}=\frac{1}{x}-\frac{1}{1-x}+\frac{1}{1+x}-\frac{1}{2-x}+\frac{1}{2+x}-\frac{1}{3-x}+\text{etc.}$$

and the sum of this series

$$\frac{1}{x}+\frac{1}{1-x}-\frac{1}{1+x}-\frac{1}{2-x}+\frac{1}{2+x}+\frac{1}{3-x}-\frac{1}{3+x}-\text{etc.}$$

may be found $=\frac{\pi\sqrt{1-\cos\pi x}}{\sqrt{1+\cos\pi x}}+\frac{\pi\cos\pi x}{\sin\pi x}$. But this fraction $\frac{\sqrt{1-\cos\pi x}}{\sqrt{1+\cos\pi x}}$, if the numerator and the denominator may be multiplied by $\sqrt{1-\cos\pi x}$, will change into $\frac{1-\cos\pi x}{\sin\pi x}$. On account of which the sum of the series will be $=\frac{\pi}{\sin\pi x}$, which is that itself, which we treated in §34 ; from which we may not pursue that further.

CAPUT II

DE INVESTIGATIONE SERIERUM SUMMABILIUM

19. Si seriei, in cuius terminis inest quantitas indeterminata *x*, summa fuerit cognita, quae utique erit functio ipsius *x*, tum, quicunque valor ipsi *x* tribuatur, seriei summa perpetuo assignari poterit. Quare si loco *x* ponatur $x+dx$, seriei resultantis summa erit aequalis summae prioris una cum ipsius differentiali; unde sequitur fore differentiale summae = differentiali seriei. Quia vera hoc modo tam summa quam singuli seriei termini multiplicati erunt per *dx*, si ubique per *dx* dividatur, habebitur nova series, cuius summa erit cognita. Simili modo si haec series cum sua summa denuo differentietur et ubique per *dx* dividatur, nova exoritur series cum sua summa sicque ex una serie summabili quantitatem indeterminatam *x* involvente per continuam differentiationem innumerae novae series pariter summabiles elicientur.

20. Quo haec clarius perspiciantur, proposita sit progressio geometrica indeterminata, quippe cuius summa est cognita, haec

$$\frac{1}{1-x}=1+x+x^2+x^3+x^4+x^5+x^6+\text{etc}.$$

Si nunc differentiatio instituatur, erit

$$\frac{dx}{(1-x)^2}=dx+2xdx+3x^2dx+4x^3dx+5x^4dx+\text{etc}.$$

atque divisione per *dx* instituta habebitur

$$\frac{1}{(1-x)^2}=1+2x+3x^2+4x^3+5x^4+\text{etc}.$$

Si denuo differentietur atque per *dx* dividatur, prodibit

$$\frac{2}{(1-x)^3}=2+2\cdot 3x+3\cdot 4x^2+4\cdot 5x^3+5\cdot 6x^4+\text{etc.}$$

seu

$$\frac{1}{(1-x)^3}=1+3x+6x^2+10x^3+15x^4+21x^5+\text{etc.},$$

ubi coefficientes sunt numeri trigonales. Si haec porro differentietur atque per *3dx* dividatur, obtinebitur

$$\frac{1}{(1-x)^4}=1+4x+10x^2+20x^3+35x^4+\text{etc.},$$

cuius coefficientes sunt numeri pyramidales primi. Sicque ulterius procedendo oriuntur eaedem series, quas ex evolutione fractionis $\frac{1}{(1-x)^n}$ nasci constat.

21. Latius autem patebit haec serierum investigatio, si, antequam quaevis differentiatio suscipiatur, ipsa series una cum summa per quamvis ipsius *x* potestatem seu functionem multiplicetur. Sic cum sit

$$\frac{1}{1-x}=1+x+x^2+x^3+x^4+x^5+x^6+\text{etc}$$

multiplicetur ubique per x^m eritque

$$\frac{x^m}{1-x}=x^m+x^{m+1}+x^{m+2}+x^{m+3}+x^{m+4}+\text{etc}$$

Nunc differentietur haec series fietque per *dx* diviso

$$\frac{mx^{m-1}-(m-1)x^m}{(1-x)^2}=mx^{m-1}+(m+1)x^m+(m+2)x^{m+1}+(m+3)x^{m+2}+\text{etc}.$$

Dividatur nunc per x^{m-1}; habebitur

$$\frac{m-(m-1)x}{(1-x)^2}=\frac{m}{1-x}+\frac{x}{(1-x)^2}=m+(m+1)x+(m+2)x^2+\text{etc}.$$

Multiplicetur haec, antequam nova differentiatio suscipiatur, per x^n ut sit

$$\frac{mx^n}{1-x}+\frac{x^{n+1}}{(1-x)^2}=mx^n+(m+1)x^{n+1}+(m+2)x^{n+2}+\text{etc}$$

Nunc instituatur differentiatio et diviso per *dx* erit

$$\frac{mnx^{n-1}}{1-x}+\frac{(m+n+1)x^n}{(1-x)^2}+\frac{2x^{n+1}}{(1-x)^3}=mnx^{n-1}+(m+1)(n+1)x^n+(m+2)(n+2)x^{n+1}+\text{etc}.$$

Divisione autem per x^{n-1} instituta fiet

$$\frac{mn}{1-x}+\frac{(m+n+1)x}{(1-x)^2}+\frac{2xx}{(1-x)^3}=mn+(m+1)(n+1)x+(m+2)(n+2)x^2+\text{etc.}$$

sicque ulterius progredi licebit; invenientur autem perpetuo eaedem series, quae ex evolutione fractionum summam constituentium nascuntur.

22. Quoniam progressionis geometricae primum assumtae summa ad quemvis terminum usque assignari potest, hoc modo quoque series definito terminorum numero constantes summabuntur. Ita cum sit

$$\frac{1-x^{n+1}}{1-x}=1+x+x^2+x^3+x^4+\cdots+x^n,$$

erit differentiatione instituta et terminis per dx divisis

$$\frac{1}{(1-x)^2}-\frac{(n+1)x^n-nx^{n+1}}{(1-x)^2}=1+2x+3x^2+4x^3\cdots+nx^{n-1}.$$

Hinc summae potestatum numerorum naturalium ad quemvis terminum inveniri poterunt. Multiplicetur enim haec series per x, ut fiat

$$\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}=x+2x^2+3x^3+\cdots+nx^n,$$

quae denuo differentiata ac per dx divisa dabit

$$\frac{1+x-(n+1)^2x^n+(2nn+2n-1)x^{n+1}-nnx^{n+2}}{(1-x)^3}=1+4x+9x^2+\cdots+n^2x^{n-1};$$

quae per x multiplicata dabit

$$\frac{x+x^2-(n+1)^2x^{n+1}+(2nn+2n-1)x^{n+2}-nnx^{n+3}}{(1-x)^3}=x+4x^2+9x^3+\cdots+n^2x^n,$$

quae differentiata, per dx divisa ac per x multiplicata producet seriem hanc

$$x+8x^2+27x^3+\cdots+n^3x^n,$$

cuius summa propterea invenietur. Ex hacque simili modo summa biquadratorum altiorumque potestatum, indefinita eruetur.

23. Methodus igitur haec ad omnes series quantitatem indeterminatam continentes accommodari potest, quarum quidem summae constant. Cum igitur praeter geometricas series recurrentes omnes eadem praerogativa gaudeant, ut non solum in infinitum, sed etiam ad quemvis terminum summari queant, ex iis quoque hac methodo innumerae aliae series summabiles inveniri poterunt. Quod cum opus foret maxime diffusum, si id persequi vellemus, unicum casum perpendamus.

Sit scilicet proposita haec series

$$\frac{x}{1-x-xx} = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + \text{etc.},$$

quae differentiata ac per dx divisa dabit

$$\frac{1+xx}{(1-x-xx)^2} = 1 + 2x + 6x^2 + 12x^3 + 25x^4 + 48x^5 + 91x^6 + \text{etc.}$$

Facile autem patet omnes has series hoc modo resultantes fore quoque recurrentes, quarum adeo summae ex ipsarum natura inveniri poterunt.

24. In genere igitur, si seriei cuiuspiam in hac forma contentae

$$ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc.}$$

summa fuerit cognita, quam ponamus $= S$, eiusdem seriei, si singuli termini singulatim per terminos progressionis arithmeticae multiplicentur, summa inveniri poterit. Sit enim

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc.};$$

multiplicetur per x^m; erit

$$Sx^m = ax^{m+1} + bx^{m+2} + cx^{m+3} + dx^{m+4} + \text{etc.};$$

differentietur haec aequatio et dividatur per dx

$$mSx^{m-1} + x^m \frac{dS}{dx} = (m+1)ax^m + (m+2)bx^{m+1} + (m+3)cx^{m+2} + \text{etc.};$$

dividatur per x^{m-1} eritque

$$mS + \frac{xdS}{dx} = (m+1)ax + (m+2)bx^2 + (m+3)cx^3 + \text{etc.}$$

Quodsi ergo huius sequentis seriei summa desideretur

$$\alpha ax + (\alpha+\beta)bx^2 + (\alpha+2\beta)cx^3 + (\alpha+3\beta)dx^4 + \text{etc.},$$

multiplicetur superior per β ac statuatur $m\beta+\beta=\alpha$, ut sit $m=\frac{\alpha-\beta}{\beta}$, eritque huius seriei summa

$$=(\alpha-\beta)S+\frac{\beta xdS}{dx}.$$

25. Poterit etiam huius seriei propositae summa inveniri, si singuli eius termini multiplicentur per terminos seriei secundi ordinis singulatim, cuius scilicet differentiae demum secundae sint constantes. Quoniam enim iam invenimus

$$mS+\frac{xdS}{dx}=(m+1)ax+(m+2)bx^2+(m+3)cx^3+\text{etc.},$$

multiplicetur per x^n, ut sit

$$mSx^n+\frac{x^{n+1}dS}{dx}=(m+1)ax^{n+1}+(m+2)bx^{n+2}+\text{etc.};$$

differentietur posito dx constante et per dx dividatur

$$mnSx^{n-1}+\frac{(m+n+1)x^n dS}{dx}+\frac{x^{n+1}ddS}{dx^2}=(m+1)(n+1)ax^n+(m+2)(n+2)bx^{n+1}+\text{etc.}$$

Dividatur per x^{n-1} ac multiplicetur per k, ut sit

$$mnkS+\frac{(m+n+1)kxdS}{dx}+\frac{x^2kddS}{dx^2}=(m+1)(n+1)kax+(m+2)(n+2)kbx^2+(m+3)(n+3)kcx^3+\text{etc.}$$

Comparetur nunc haec series cum ista; erit

	Diff. I	Diff. II
$kmn+km+kn+k=\alpha$	$km+kn+3k=\beta$	$2k=\gamma$
$kmn+2km+2kn+4k=\alpha+\beta$	$km+kn+5k=\beta+\gamma$	
$kmn+3km+3kn+9k=\alpha+2\beta+\gamma$		

Ergo $k=\frac{1}{2}\gamma$ et $m+n=\frac{2\beta}{\gamma}-3$ atque

$$mn = \frac{\alpha}{k} - m - n - 1 = \frac{2\alpha}{\gamma} - \frac{2\beta}{\gamma} + 2 = \frac{2(\alpha - \beta + \gamma)}{\gamma}.$$

Hinc summa seriei quaesita erit

$$(\alpha - \beta + \gamma)S + \frac{(\beta - \gamma)xdS}{dx} + \frac{\gamma x^2 ddS}{2dx^2}$$

26. Simili modo summa reperiri poterit seriei huius

$$Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \text{etc.},$$

si quidem cognita fuerit summa S seriei huius

$$S = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \text{etc.}$$

atque *A, B, C, D* etc. constituant seriem, quae ad differentias constantes deducitur. Fingatur enim summa, quoniam eius forma ex antecedentibus colligitur,
haec

$$\alpha S + \frac{\beta xdS}{dx} + \frac{\gamma x^2 ddS}{dx^2} + \frac{\delta x^3 d^3 S}{6dx^3} + \frac{\varepsilon x^4 d^4 S}{24dx^4} + \text{etc.}$$

Nunc ad litteras α, β, γ, δ etc. inveniendas evolvantur singulae series eritque

$$\begin{aligned}
\alpha S &= \alpha a + \alpha bx + \alpha cx^2 + \alpha dx^3 + \alpha ex^4 + \text{etc.}\\
\frac{\beta xdS}{dx} &= \beta bx + 2\beta cx^2 + 3\beta dx^3 + 4\beta ex^4 + \text{etc.}\\
\frac{\gamma x^2 ddS}{2dx^2} &= \gamma cx^2 + 3\gamma dx^3 + 6\gamma ex^4 + \text{etc.}\\
\frac{\delta x^3 d^3 S}{6dx^3} &= \delta dx^3 + 4\delta ex^4 + \text{etc.}\\
\frac{\varepsilon x^4 d^4 S}{24dx^4} &= 4\varepsilon ex^4 + \text{etc.}\\
&\text{etc.};
\end{aligned}$$

quae simul sumtae comparentur cum proposita

$$Z = Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \text{etc.}$$

fietque comparatione singulorurn terminorum instituta

$$\alpha = A$$
$$\beta = B - \alpha = B - A$$
$$\gamma = C - 2\beta - \alpha = C - 2B + A$$
$$\delta = D - 3\gamma - 3\beta - \alpha = D - 3C + 3B - A$$
etc.

His igitur valoribus inventis erit summa quaesita

$$Z = AS + \frac{(B-A)xdS}{1dx} + \frac{(C-2B+A)x^2ddS}{1\cdot 2dx^2} + \frac{(D-3C+3B-A)x^3d^3S}{1\cdot 2\cdot 3dx^3} + \text{etc.},$$

seu si seriei *A, B, C, D, E* etc. differentiae continuae more consueto indicentur, erit

$$Z = AS + \frac{\Delta A\cdot xdS}{1dx} + \frac{\Delta^2 A\cdot x^2 ddS}{1\cdot 2dx^2} + \frac{\Delta^3 A\cdot x^3 d^3S}{1\cdot 2\cdot 3dx^3} + \text{etc.}$$

Siquidem fuerit, uti assumsimus,

$$S = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \text{etc.}$$

Si ergo series *A, B, C, D* etc. tandem habeat differentias constantes, summa seriei *Z* finite exprimi poterit.

27. Quia sumto *e* pro numero, cuius logarithmus hyperbolicus est $=1$, est

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \text{etc.}$$

sumatur haec series pro priori, et cum sit $S = e^x$, erit $\frac{dS}{dx} = e^x$, $\frac{ddS}{dx^2} = e^x$ etc.
Quare huius seriei, quae ex illa et hac *A, B*, C, *D* etc. componitur,

$$A + \frac{Bx}{1} + \frac{Cx^2}{1\cdot 2} + \frac{Dx^3}{1\cdot 2\cdot 3} + \frac{Ex^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

summa hoc modo exprimetur

$$e^x\left(A + \frac{x\Delta A}{1} + \frac{xx\Delta^2 A}{1\cdot 2} + \frac{x^3\Delta^3 A}{1\cdot 2\cdot 3} + \text{etc.}\right).$$

Sic, si proponatur haec series

$$2 + \frac{5x}{1} + \frac{10x^2}{1\cdot 2} + \frac{17x^3}{1\cdot 2\cdot 3} + \frac{26x^4}{1\cdot 2\cdot 3\cdot 4} + \frac{37x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \text{etc.},$$

ob seriem

$$\begin{array}{rccccc} & A, & B, & C, & D, & E \text{ etc.} \\ A = & 2, & 5, & 10, & 17, & 26 \text{ etc.} \\ \varDelta A = & & 3 & 5, & 7, & 9 \text{ etc.} \\ \varDelta^2 A = & & & 2, & 2, & 2 \text{ etc.} \end{array}$$

erit huius seriei

$$2 + 5x + \tfrac{10x^2}{2} + \tfrac{17}{6}x^3 + \tfrac{26x^4}{24} + \text{etc.}$$

summa

$$= e^x\left(2 + 3x + xx\right) = e^x\left(1 + x\right)\left(2 + x\right),$$

quod quidem sponte patet. Est enim

$$\begin{aligned} 2e^x &= 2 + \tfrac{2x}{1} + \tfrac{2x^2}{2} + \tfrac{2x^3}{6} + \tfrac{2x^4}{24} + \text{etc.} \\ 3xe^x &= \qquad 3x + \tfrac{3x^2}{1} + \tfrac{3x^3}{2} + \tfrac{3x^4}{6} + \text{etc.} \\ xxe^x &= \qquad\qquad xx + \tfrac{x^3}{1} + \tfrac{x^4}{2} + \text{etc.} \end{aligned}$$

$$e^x\left(2 + 3x + xx\right) = 2 + 5x + \tfrac{10xx}{2} + \tfrac{17x^3}{6} + \tfrac{26x^4}{24} + \text{etc.}$$

28. Quae hactenus sunt tradita, non solum ad series in infinitum excurrentes spectant, sed etiam ad summas quotcunque terminorum; coefficientes enim *a, b, c, d* etc. vel in infinitum progredi vel, ubicunque libuerit, abrumpi possunt. Verum cum hoc non egeat uberiori explicatione, quae ex hactenus allatis sequuntur, accuratius perpendamus. Proposita ergo quacunque serie, cuius singuli termini duobus constent factoribus, quorum alteri seriem ad differentias constantes deducentem constituant, huius seriei summa poterit assignari, dummodo omissis istis factoribus series fuerit summabilis. Scilicet si proposita sit ista series

$$Z = Aa + Bbx + Ccx^2 + Ddx^3 + Eex^4 + \text{etc.},$$

in qua quantitates *A, B, C, D, E* etc. eiusmodi seriem constituant, quae tandem ad differentias constantes perducatur, tum istius seriei summa exhiberi poterit, dummodo habeatur summa *S* huius seriei

$$S = a + bx + cx^2 + dx^3 + ex^4 + \text{etc}.$$

Sumtis enim ex progressione *A, B, C, D, E* etc. differentiis continuis, uti initio huius libri ostendimus,

$$\begin{array}{c} A,\quad B,\quad C,\quad D,\quad E,\quad F\quad \text{etc.}\\ \Delta A,\quad \Delta B,\quad \Delta C,\quad \Delta D,\quad \Delta E\quad \text{etc.}\\ \Delta^2 A,\ \Delta^2 B,\ \Delta^2 C,\ \Delta^2 D\ \text{etc.}\\ \Delta^3 A,\ \Delta^3 B,\ \Delta^3 C\ \text{etc.}\\ \Delta^4 A,\ \Delta^4 B\ \text{etc.}\\ \Delta^5 A\quad \text{etc.}\\ \text{etc.}\end{array}$$

erit seriei propositae summa

$$Z = SA + \frac{xdS}{1dx}\Delta A + \frac{x^2 ddS}{1\cdot 2dx^2}\Delta^2 A + \frac{x^3 d^3 S}{1\cdot 2\cdot 3dx^3} + \text{etc.}$$

posito in altioribus ipsius *S* differentialibus *dx* constante.

29. Si igitur series *A, B, C, D* etc. nunquam ad differentias constantes deducat, summa seriei *Z* per novam seriem infinitam exprimetur, quae interdum magis converget quam proposita, sicque ista series in aliam sibi aequalem transformabitur. Sit ad hoc declarandum proposita haec series

$$Y = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \frac{y^5}{5} + \frac{y^6}{6} + \text{etc.},$$

quam constat exprimere $l\frac{1}{1-y}$ ita ut sit $Y = -l(1-y)$. Dividatur haec series per *y* et statuatur $y = x$ et $Y = yZ$, ut sit

$$Z = l\frac{1}{1-y} = -\frac{1}{x}l(l-x);$$

erit

$$Z = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \frac{x^5}{6} + \text{etc.},$$

quae comparata cum ista

$$S = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \text{etc.} = \frac{1}{1-x}$$

dabit pro serie *A, B,* C, *D, E* etc. hos valores

$$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \frac{1}{5} \text{ etc.}$$

$$-\frac{1}{1\cdot 2}, \quad -\frac{1}{2\cdot 3}, \quad -\frac{1}{3\cdot 4}, \quad -\frac{1}{4\cdot 5} \text{ etc.}$$

$$\frac{1\cdot 2}{1\cdot 2\cdot 3}, \quad \frac{1\cdot 2}{2\cdot 3\cdot 4}, \quad \frac{1\cdot 2}{3\cdot 4\cdot 5} \text{ etc.}$$

$$-\frac{1\cdot 2\cdot 3}{1\cdot 2\cdot 3\cdot 4}, \quad -\frac{1\cdot 2\cdot 3}{2\cdot 3\cdot 4\cdot 5} \text{ etc.}$$

etc.

Erit ergo

$$A=1, \quad \Delta A=-\frac{1}{2}, \quad \Delta^2 A=\frac{1}{3}, \quad \Delta^3 A=-\frac{1}{4} \text{ etc.}$$

Porro cum sit $S=\frac{1}{1-x}$, erit

$$\frac{dS}{dx}=\frac{1}{(1-x)^2}, \quad \frac{ddS}{1\cdot 2dx^2}=\frac{1}{(1-x)^3}, \quad \frac{d^3S}{1\cdot 2\cdot 3dx^3}=\frac{1}{(1-x)^4} \text{ etc.}$$

Quibus valoribus substitutis orietur summa

$$Z=\frac{1}{1-x}-\frac{x}{2(1-x)^2}+\frac{x^2}{3(1-x)^3}-\frac{x^4}{4(1-x)^4}+\frac{x^5}{5(1-x)^5}-\text{etc.}$$

Cum ergo sit $x=y$ et $Y=-l(1-y)=yZ$, erit

$$-l(1-y)=\frac{y}{1-x}-\frac{y^2}{2(1-y)^2}+\frac{y^3}{3(1-y)^3}-\frac{y^4}{4(1-y)^4}+\text{etc.},$$

quae series utique exprimit $l\left(1+\frac{y}{1-y}\right)=l\frac{1}{1-y}=-l(1-y)$, cuius adeo veritas per ante demonstrata constat.

30. Proposita nunc sit ista series, ut etiam usus pateat, si potestates tantum impares occurrant et signa alternentur,

$$Y=y-\frac{y^3}{3}+\frac{y^5}{5}-\frac{y^7}{7}+\frac{y^9}{9}-\frac{y^{11}}{11}+\text{etc.},$$

ex qua constat esse $Y=\text{Atang}\, y$.
Dividatur haec series per y et ponatur $\frac{Y}{y}=Z$ et $yy=x$; erit

$$Z=1-\frac{x}{3}+\frac{xx}{5}-\frac{x^3}{7}+\frac{x^4}{9}-\frac{y^5}{11}+\text{ etc.}$$

Quae si comparetur cum ista

$$S=1-x+xx-x^3+x^4-x^5+x^6+\text{etc.},$$

fiet $S = \frac{1}{1+x}$ et series coefficientium *A*, *B*, *C*, *D* etc. fiet

$$\begin{array}{llllll} A = 1, & \frac{1}{3}, & \frac{1}{5}, & \frac{1}{7}, & \frac{1}{9} & \text{etc.} \\ \varDelta A = & -\frac{2}{3}, & -\frac{2}{3\cdot5}, & -\frac{2}{5\cdot7}, & -\frac{2}{7\cdot9} & \text{etc.} \\ \varDelta^2 A = & & \frac{1\cdot2}{1\cdot2\cdot3}, & \frac{1\cdot2}{2\cdot3\cdot4}, & \frac{1\cdot2}{3\cdot4\cdot5} & \text{etc.} \\ \varDelta^3 A = & & & -\frac{2\cdot4\cdot6}{3\cdot5\cdot7}, & -\frac{2\cdot4\cdot6}{3\cdot5\cdot7\cdot9} & \text{etc.} \\ \varDelta^4 A = & & & & -\frac{2\cdot4\cdot6\cdot8}{3\cdot5\cdot7\cdot9} & \text{etc.} \\ & & \text{etc.} & & & \end{array}$$

At cum sit $S = \frac{1}{1+x}$, erit

$$\frac{dS}{1dx} = -\frac{1}{(1+x)^2}, \quad \frac{ddS}{1\cdot2dx^2} = \frac{1}{(1+x)^3}, \quad \frac{d^3S}{1\cdot2\cdot3dx^3} = -\frac{1}{(1+x)^4} \quad \text{etc.}$$

Quare substitutis his valoribus fiet forma

$$Z = \frac{1}{1+x} + \frac{2x}{3(1+x)^2} + \frac{2\cdot4x^2}{3\cdot5(1+x)^3} + \frac{2\cdot4\cdot6x^3}{3\cdot5\cdot7(1+x)^4} + \text{etc.}$$

Restituto ergo $x = yy$ et per y multiplicato fiet

$$Y = \text{Atang}\, y = \frac{y}{1+yy} + \frac{2y^3}{3(1+yy)^2} + \frac{2\cdot4y^5}{3\cdot5(1+yy)^3} + \frac{2\cdot4\cdot6y^7}{3\cdot5\cdot7(1+yy)^4} + \text{etc.}$$

31. Potest quoque superior series, qua arcus circuli per tangentem exprimitur, alio modo transmutari eam comparando cum serie logarithmica.

Consideremus nempe seriem

$$Z = 1 - \frac{x}{3} + \frac{xx}{5} - \frac{x^3}{7} + \frac{x^4}{9} - \frac{y^5}{11} + \text{etc.},$$

quam comparemus cum hac

$$S = \frac{1}{0} - \frac{x}{2} + \frac{xx}{4} - \frac{x^3}{6} + \frac{x^4}{8} - \text{etc.} = \frac{1}{0} - \frac{1}{2} l(1+x),$$

atque valores litterarum *A*, *B*, *C*, *D* etc. erunt

$$\begin{array}{llllll} A = \frac{0}{1}, & \frac{2}{3}, & \frac{4}{5}, & \frac{6}{7}, & \frac{8}{9} & \text{etc.} \\ \Delta A = & \frac{2}{3}, & \frac{+2}{3\cdot5}, & \frac{+2}{5\cdot7}, & \frac{+2}{7\cdot9} & \text{etc.} \\ \Delta^2 A = & \frac{-2\cdot4}{3\cdot5}, & \frac{-2\cdot4}{3\cdot5\cdot7}, & \frac{-2\cdot4}{5\cdot7\cdot9} & \text{etc.} & \\ \Delta^3 A = & \frac{2\cdot4\cdot6}{3\cdot5\cdot7} & \text{etc.} & & & \\ & \text{etc.} & & & & \end{array}$$

Deinde, cum sit $S = \frac{1}{0} - \frac{1}{2} l(1+x)$, erit

$$\frac{dS}{1dx} = -\frac{1}{2(1+x)}, \quad \frac{ddS}{1\cdot2dx^2} = \frac{1}{4(1+x)^2},$$
$$\frac{d^3S}{1\cdot2\cdot3dx^3} = -\frac{1}{6(1+x)^3}, \quad \frac{d^4S}{1\cdot2\cdot3\cdot4dx^4} = \frac{1}{8(1+x)^4} \quad \text{etc.}$$

Erit igitur $SA = S\frac{0}{1} = 1$ et ex reliquis fiet

$$Z = 1 - \frac{x}{3(1+x)} - \frac{2xx}{3\cdot5(1+x)^2} - \frac{2\cdot4x^3}{3\cdot5\cdot7(1+x)^3} - \text{etc.}$$

Ponatur nunc $x = yy$ et multiplicetur per y ; fiet

$$Y = \text{Atang}\,y = y - \frac{y^3}{3(1+yy)} - \frac{2y^5}{3\cdot5(1+yy)^2} - \frac{2\cdot4y^7}{3\cdot5\cdot7(1+yy)^3} - \text{etc.}$$

Haec ergo transmutatio non impediebatur termino infinito $\frac{1}{0}$, qui in seriem S ingrediebatur. Sin autem cui supersit dubium, is tantum singulos terminos praeter primum secundum potestates ipsius y in series resolvat atque deprehendet actu seriem primum propositam resultare.

32. Hactenus eiusmodi tantum series sumus contemplati, in quibus omnes potestates variabilis occurrunt. Nunc igitur ad alias series progrediamur, quae in singulis terminis eandem potestatem ipsius variabilis complectantur, cuiusmodi est haec

$$S = \frac{1}{a+x} + \frac{1}{b+x} + \frac{1}{c+x} + \frac{1}{d+x} + \text{etc.}$$

Huius enim seriei si summa S fuerit cognita ac per functionem quampiam ipsius x exprimatur, erit differentiando ac per $-dx$ dividendo

$$\frac{-dS}{dx}=\frac{1}{(a+x)^2}+\frac{1}{(b+x)^2}+\frac{1}{(c+x)^2}+\frac{1}{(d+x)^2}+\text{etc.}$$

Si haec ulterius differentietur atque per $-2dx$ dividatur, cognoscetur series cuborum

$$\frac{ddS}{2dx^2}=\frac{1}{(a+x)^3}+\frac{1}{(b+x)^3}+\frac{1}{(c+x)^3}+\frac{1}{(d+x)^3}+\text{etc.}$$

haecque denuo differentiata atque per $-3dx$ divisa dabit

$$\frac{-d^3S}{6dx^3}=\frac{1}{(a+x)^4}+\frac{1}{(b+x)^4}+\frac{1}{(c+x)^4}+\frac{1}{(d+x)^4}+\text{etc.}$$

Similique modo omnium sequentium potestatum summae reperientur, dummodo summa seriei primae fuerit cognita.

33. Huiusmodi autem series fractionum quantitatem indeterminatam involventes supra in *Introductione* elicuimus, ubi ostendimus, si circuli, cuius radius $=1$, semiperipheria statuatur $=\pi$, fore

$$\frac{\pi}{n\sin\frac{m}{n}\pi}=\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\frac{1}{3n-m}-\text{etc.}$$

$$\frac{\pi\cos\frac{m}{n}\pi}{n\sin\frac{m}{n}\pi}=\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\text{etc.}$$

Cum igitur pro m et n numeros quoscunque assumere liceat, statuamus $n=1$ et $m=x$, ut obtineamus series illi, quam in paragrapho praecedenti proposueramus, similes; hoc facto erit

$$\frac{\pi}{\sin\pi x}=\frac{1}{x}+\frac{1}{1-x}-\frac{1}{1+x}-\frac{1}{2-x}+\frac{1}{2+x}+\frac{1}{3-x}-\text{etc.}$$

$$\frac{\pi\cos\pi x}{\sin\pi x}=\frac{1}{x}-\frac{1}{1-x}+\frac{1}{1+x}-\frac{1}{2-x}+\frac{1}{2+x}-\frac{1}{3-x}+\text{etc.}$$

Per differentiationes ergo summae quarumvis potestatum ex his fractionibus oriundarum exhiberi poterunt.

34. Consideremus seriem priorem sitque brevitatis gratia $\frac{\pi}{\sin\pi x}=S$, cuius differentialia altiora capiantur posito dx constante, eritque

$$S = \frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} - \text{etc}$$

$$\frac{-dS}{dx} = \frac{1}{xx} - \frac{1}{(1-x)^2} - \frac{1}{(1+x)^2} + \frac{1}{(2-x)^2} + \frac{1}{(2+x)^2} - \frac{1}{(3-x)^2} - \text{etc.}$$

$$\frac{ddS}{2dx^2} = \frac{1}{x^3} + \frac{1}{(1-x)^3} - \frac{1}{(1+x)^3} - \frac{1}{(2-x)^3} + \frac{1}{(2+x)^3} + \frac{1}{(3-x)^3} - \text{etc.}$$

$$\frac{-d^3S}{6dx^3} = \frac{1}{x^4} - \frac{1}{(1-x)^4} - \frac{1}{(1+x)^4} + \frac{1}{(2-x)^4} + \frac{1}{(2+x)^4} - \frac{1}{(3-x)^4} - \text{etc.}$$

$$\frac{d^4S}{24dx^4} = \frac{1}{x^5} + \frac{1}{(1-x)^5} - \frac{1}{(1+x)^5} - \frac{1}{(2-x)^5} + \frac{1}{(2+x)^5} + \frac{1}{(3-x)^5} - \text{etc.}$$

$$\frac{-d^5S}{120dx^5} = \frac{1}{x^6} - \frac{1}{(1-x)^6} - \frac{1}{(1+x)^6} + \frac{1}{(2-x)^6} + \frac{1}{(2+x)^6} - \frac{1}{(3-x)^6} - \text{etc.}$$

etc.

ubi notandum est in potestatibus paribus signa eandem sequi legem pariterque in imparibus eandem legem signorum observari. Omnium ergo istarum serierum summae invenientur ex differentialibus expressionis $S = \frac{\pi}{\sin \pi x}$.

35. Ad differentialia haec simplicius exprimenda ponamus
$\sin \pi x = p$ et $\cos \pi x = q$;
erit

$$dp = \pi dx \cos \pi x = \pi q dx \quad \text{et} \quad dq = -\pi p dx .$$

Cum ergo sit $S = \frac{\pi}{p}$ erit

$$\frac{-dS}{dx} = \frac{\pi^2 q}{pp}$$

$$\frac{ddS}{dx^2} = \frac{\pi^3 (pp+2qq)}{p^3} = \frac{\pi^3 (qq+1)}{p^3} \quad \text{ob} \quad pp + qq = 1$$

$$\frac{-d^3S}{dx^3} = \pi^4 \left(\frac{5q}{pp} + \frac{6q^3}{p^4} \right) = \frac{\pi^4 (q^3+5q)}{p^4}$$

$$\frac{d^4S}{dx^4} = \pi^5 \left(\frac{24q^4}{p^5} + \frac{28q^2}{p^3} + \frac{5}{p} \right) = \frac{\pi^5 (q^4+18q^2+5)}{p^5}$$

$$\frac{-d^5S}{dx^5} = \pi^6 \left(\frac{120q^5}{p^6} + \frac{180q^3}{p^4} + \frac{61q}{pp} \right) = \frac{\pi^6 (q^5+58q^3+61q)}{p^6}$$

$$\frac{d^6S}{dx^6} = \pi^7 \left(\frac{720q^6}{p^7} + \frac{1320q^4}{p^5} + \frac{662q^2}{p^3} + \frac{61}{p} \right) = \frac{\pi^7 (q^6+179q^4+479q^2+61)}{p^7}$$

$$\frac{-d^7S}{dx^7} = \pi^8 \left(\frac{5040q^7}{p^8} + \frac{10920q^5}{p^6} + \frac{7266q^3}{p^4} + \frac{1385q}{p^2} \right)$$

vel

$$=\frac{\pi^8}{p^8}\left(q^7+543q^5+3111q^3+1385q\right)$$

$$\frac{d^8S}{dx^8}=\pi^9\left(\frac{40320q^8}{p^9}+\frac{100800q^6}{p^7}+\frac{83664q^4}{p^5}+\frac{24568q^2}{p^3}+\frac{1385}{p}\right)$$

vel

$$=\frac{\pi^9}{p^9}\left(q^8+1636q^6+18270q^4+19028q^2+1385\right)$$

etc.

Quae expressiones facile ulterius, quousque libuerit, continuari possunt; si enim fuerit

$$\pm\frac{d^nS}{dx^n}=\pi^{n+1}\left(\frac{\alpha q^n}{p^{n+1}}+\frac{\beta q^{n-2}}{p^{n-1}}+\frac{\gamma q^{n-4}}{p^{n-3}}+\frac{\delta q^{n-6}}{p^{n-5}}+\text{etc.}\right),$$

erit differentiale sequens signis mutatis

$$\mp\frac{d^{n+1}S}{dx^{n+1}}=\pi^{n+2}\left\{\begin{array}{c}(n+1)\alpha\frac{q^{n+1}}{p^{n+2}}+\left(n\alpha+(n-1)\beta\right)\frac{q^{n-1}}{p^n}+\left((n-2)\beta+(n-3)\gamma\right)\frac{q^{n-3}}{p^{n-2}}\\+\left((n-4)\gamma+(n-5)\delta\right)\frac{q^{n-5}}{p^{n-4}}+\text{etc.}\end{array}\right\},$$

36. Ex his ergo obtinebuntur summae serierum superiorum § 34 exhibitarum sequentes

$$S = \pi \cdot \frac{1}{p}$$

$$\frac{-dS}{dx} = \frac{\pi^2}{1} \cdot \frac{q}{p^2}$$

$$\frac{ddS}{2dx^2} = \frac{\pi^3}{2}\left(\frac{2q^2}{p^3} + \frac{1}{p}\right)$$

$$\frac{-d^3S}{6dx^3} = \frac{\pi^4}{6}\left(\frac{6q^3}{p^4} + \frac{5q}{p^2}\right)$$

$$\frac{d^4S}{24dx^4} = \frac{\pi^5}{24}\left(\frac{24q^4}{p^5} + \frac{28q^2}{p^3} + \frac{5}{p}\right)$$

$$\frac{-d^5S}{120dx^5} = \frac{\pi^6}{120}\left(\frac{120q^5}{p^6} + \frac{180q^3}{p^4} + \frac{61q}{p^2}\right)$$

$$\frac{d^6S}{720dx^6} = \frac{\pi^7}{720}\left(\frac{720q^6}{p^7} + \frac{1320q^4}{p^5} + \frac{662q^2}{p^3} + \frac{61}{p}\right)$$

$$\frac{-d^7S}{5040dx^7} = \frac{\pi^8}{5040}\left(\frac{5040q^7}{p^8} + \frac{10920q^5}{p^6} + \frac{7266q^3}{p^4} + \frac{1385q}{p^2}\right)$$

$$\frac{d^8S}{40320dx^8} = \frac{\pi^9}{40320}\left(\frac{40320q^8}{p^9} + \frac{100800q^6}{p^7} + \frac{24568q^4}{p^5} + \frac{19028q^2}{p^3} + \frac{1385}{p}\right)$$

etc.

37. Tractemus simili modo alteram seriem supra [§ 33] inventam

$$\frac{\pi\cos\pi x}{\sin\pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \text{etc.}$$

atque posito brevitatis ergo $\frac{\pi\cos\pi x}{n\sin\pi x} = T$ orientur sequentes summationes

$$T = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \text{etc.}$$

$$\frac{-dT}{dx} = \frac{1}{x^2} + \frac{1}{(1-x)^2} + \frac{1}{(1+x)^2} + \frac{1}{(2-x)^2} + \frac{1}{(2+x)^2} + \text{etc.}$$

$$\frac{ddT}{2dx^2} = \frac{1}{x^3} - \frac{1}{(1-x)^3} + \frac{1}{(1+x)^3} - \frac{1}{(2-x)^3} + \frac{1}{(2+x)^3} - \text{etc.}$$

$$\frac{d^3T}{6dx^3} = \frac{1}{x^4} + \frac{1}{(1-x)^4} + \frac{1}{(1+x)^4} + \frac{1}{(2-x)^4} + \frac{1}{(2+x)^4} + \text{etc.}$$

$$\frac{d^4T}{24dx^4} = \frac{1}{x^5} - \frac{1}{(1-x)^5} + \frac{1}{(1+x)^5} - \frac{1}{(2-x)^5} + \frac{1}{(2+x)^5} - \text{etc.}$$

$$\frac{-d^5T}{120dx^3} = \frac{1}{x^6} + \frac{1}{(1-x)^6} + \frac{1}{(1+x)^6} + \frac{1}{(2-x)^6} + \frac{1}{(2+x)^6} + \text{etc.}$$

etc.,

ubi in potestatibus paribus omnes termini sunt affirmativi, in imparibus autem signa + et – alternatim se excipiunt.

38. Quo differentialium horum valores innotescant, ponamus ut ante

$$\sin \pi x = p \quad \text{et} \quad \cos \pi x = q\,,$$

ut sit $pp + qq = 1$; erit

$$dp = \pi q dx \ \text{ et } \ dq = -\pi p dx\,.$$

Quibus valoribus adhibitis erit

$$T = \pi \cdot \frac{q}{p}$$

$$\frac{-dT}{dx} = \pi^2 \left(\frac{qq}{pp} + 1 \right) = \frac{\pi^2}{pp}$$

$$\frac{ddT}{dx^2} = \pi^3 \left(\frac{2q^3}{p^3} + \frac{2q}{p} \right) = \frac{2\pi^3 q}{p^3}$$

$$\frac{-d^3T}{dx^3} = \pi^4 \left(\frac{6q^4}{p^4} + \frac{8qq}{p^2} + 2 \right) = \pi^4 \left(\frac{6qq}{p^4} + \frac{2}{pp} \right)$$

$$\frac{d^4T}{dx^4} = \pi^5 \left(\frac{24q^3}{p^5} + \frac{16q}{p^3} \right)$$

$$\frac{-d^5T}{dx^5} = \pi^6 \left(\frac{120q^4}{p^6} + \frac{120qq}{p^4} + \frac{16}{pp} \right)$$

$$\frac{d^6T}{dx^6} = \pi^7 \left(\frac{720q^5}{p^7} + \frac{960q^3}{p^5} + \frac{272q}{p^3} \right)$$

$$\frac{-d^7T}{dx^7} = \pi^8 \left(\frac{5040q^6}{p^8} + \frac{8400q^4}{p^6} + \frac{3696q^2}{p^4} + \frac{272}{p^2} \right)$$

$$\frac{d^8T}{dx^8} = \pi^9 \left(\frac{40320q^7}{p^9} + \frac{80640q^5}{p^7} + \frac{48384q^3}{p^5} + \frac{7936q}{p^3} \right)$$

etc.

Quae formulae facile ulterius, quousque libuerit, continuari possunt. Si enim sit

$$\pm \frac{d^nT}{dx^n} = \pi^{n+1} \left(\frac{\alpha q^{n-1}}{p^{n+1}} + \frac{\beta q^{n-3}}{p^{n-1}} + \frac{\gamma q^{n-5}}{p^{n-3}} + \frac{\delta q^{n-7}}{p^{n-5}} + \text{etc.} \right),$$

erit expressio sequens

$$\mp \frac{d^{n+1}T}{dx^{n+1}} = \pi^{n+2} \left((n+1) \frac{\alpha q^n}{p^{n+2}} + \frac{(n-1)(\alpha+\beta) q^{n-2}}{p^n} + \frac{(n-3)(\beta+\gamma) q^{n-4}}{p^{n-2}} + \text{etc.} \right).$$

39. Series ergo potestatum § 37 datae sequentes habebunt summas posito

$$\sin\pi x = p \text{ et } \cos\pi x = q$$

$$T = \pi\cdot\frac{q}{p}$$

$$\frac{-dT}{dx} = \pi^2\frac{1}{pp}$$

$$\frac{ddT}{2dx^2} = \pi^3\frac{q}{p^3}$$

$$\frac{-d^3T}{6dx^3} = \pi^4\left(\frac{qq}{p^4}+\frac{1}{3pp}\right)$$

$$\frac{d^4T}{24dx^4} = \pi^5\left(\frac{q^3}{p^5}+\frac{2q}{3p^3}\right)$$

$$\frac{-d^5T}{120dx^5} = \pi^6\left(\frac{q^4}{p^6}+\frac{3qq}{3p^4}+\frac{2}{15pp}\right)$$

$$\frac{d^6T}{720dx^6} = \pi^7\left(\frac{q^5}{p^7}+\frac{4q^3}{3p^5}+\frac{17q}{45p^3}\right)$$

$$\frac{-d^7T}{5040dx^7} = \pi^8\left(\frac{q^6}{p^8}+\frac{5q^4}{3p^6}+\frac{11q^2}{15p^4}+\frac{17}{315pp}\right)$$

$$\frac{d^8T}{40320dx^8} = \pi^9\left(\frac{q^7}{p^9}+\frac{6q^5}{3p^7}+\frac{6q^3}{5p^5}+\frac{62q}{315p^3}\right)$$

etc.

40. Praeter has series invenimus in *Introductione* nonnullas alias, ex quibus simili modo per differentiationes novae elici possunt.

Ostendimus enim esse

$$\frac{1}{2x}-\frac{\pi\sqrt{x}}{2x\operatorname{tang}\pi\sqrt{x}} = \frac{1}{1-x}+\frac{1}{4-x}+\frac{1}{9-x}+\frac{1}{16-x}+\frac{1}{25-x}+\text{etc.},$$

Ponamus summam huius seriei esse $=S$, ut sit

$$S = \frac{1}{2x}-\frac{\pi\sqrt{x}}{2x}\cdot\frac{\cos\pi\sqrt{x}}{\sin\pi\sqrt{x}};$$

erit

$$\frac{dS}{dx} = -\frac{1}{2xx}+\frac{\pi}{4x\sqrt{x}}\cdot\frac{\cos\pi\sqrt{x}}{\sin\pi\sqrt{x}}+\frac{\pi\pi}{4x\left(\sin\pi\sqrt{x}\right)^2}$$

quae ergo expressio praebet summam huius seriei

$$\frac{1}{(1-x)^2}+\frac{1}{(4-x)^2}+\frac{1}{(9-x)^2}+\frac{1}{(16-x)^2}+\frac{1}{(25-x)^2}+\text{etc.}$$

Deinde quoque ostendimus esse

$$\frac{\pi}{2\sqrt{x}}\cdot\frac{e^{2\pi\sqrt{x}}+1}{e^{2\pi\sqrt{x}}-1}-\frac{1}{2x}=\frac{1}{1+x}+\frac{1}{4+x}+\frac{1}{9+x}+\frac{1}{16+x}+\text{etc.}$$

Quodsi ergo haec summa ponatur = *S*, erit

$$\frac{-dS}{dx}=\frac{1}{(1+x)^2}+\frac{1}{(4+x)^2}+\frac{1}{(9+x)^2}+\frac{1}{(16+x)^2}+\text{etc.}$$

At est

$$\frac{dS}{dx}=\frac{-\pi}{4\sqrt{x}}\cdot\frac{e^{2\pi\sqrt{x}}+1}{e^{2\pi\sqrt{x}}-1}-\frac{\pi\pi}{x}\cdot\frac{e^{2\pi\sqrt{x}}}{\left(e^{2\pi\sqrt{x}}-1\right)^2}+\frac{1}{2xx}.$$

Ergo summa huius seriei erit

$$\frac{-dS}{dx}=\frac{\pi}{4x\sqrt{x}}\cdot\frac{e^{2\pi\sqrt{x}}+1}{e^{2\pi\sqrt{x}}-1}+\frac{\pi\pi}{x}\cdot\frac{e^{2\pi\sqrt{x}}}{\left(e^{2\pi\sqrt{x}}-1\right)^2}-\frac{1}{2xx}.$$

Similique modo ulterioribus differentiationibus summae sequentium potestatum invenientur.

41. Si cognitus fuerit valor producti cuiuspiam ex factoribus indeterminatam litteram involventibus compositi, ex eo per eandem methodum innumerabiles series summabiles inveniri poterunt. Sit enim huius producti

$$(1+\alpha x)(1+\beta x)(1+\gamma x)(1+\delta x)(1+\varepsilon x)\ \text{etc}.$$

valor $=S$, functioni scilicet cuipiam ipsius *x;* erit logarithmis sum endis

$$lS=l(1+\alpha x)+l(1+\beta x)+l(1+\gamma x)+l(1+\delta x)\ \text{etc}.$$

Sumantur iam differentialia; erit divisione per *dx* instituta

$$\frac{dS}{Sdx}=\frac{\alpha}{1+\alpha x}+\frac{\beta}{1+\beta x}+\frac{\gamma}{1+\gamma x}+\frac{\delta}{1+\delta x}+\text{etc.},$$

ex cuius ulteriori differentiatione summae potestatum quarumvis istarum fractionum reperientur, plane uti in exemplis praecedentibus fusius exposuimus

42. Exhibuimus autem in *Introductione* nonnullas istiusmodi expressiones, ad quas hanc methodum accommodemus. Scilicet si sit π arcus 180^0 circuli, cuius radius $=1$, ostendimus esse

$$\sin\frac{m\pi}{2n}=\frac{m\pi}{2n}\cdot\frac{4nn-mm}{4nn}\cdot\frac{16nn-mm}{16nn}\cdot\frac{36nn-mm}{36nn}\cdot\text{etc.}$$
$$\cos\frac{m\pi}{2n}=\frac{nn-mm}{nn}\cdot\frac{9nn-mm}{9nn}\cdot\frac{25nn-mm}{25nn}\cdot\frac{49nn-mm}{49nn}\cdot\text{etc.}$$

Ponamus $n = 1$ et $m = 2x$, ut sit

$$\sin\pi x=\pi x\cdot\frac{1-xx}{1}\cdot\frac{4-xx}{4}\cdot\frac{9-xx}{9}\cdot\frac{16-xx}{16}\cdot\text{etc.}$$

vel

$$\sin\pi x=\pi x\cdot\frac{1-x}{1}\cdot\frac{1+x}{1}\cdot\frac{2-x}{2}\cdot\frac{2+x}{2}\cdot\frac{3-x}{3}\cdot\frac{3+x}{3}\cdot\frac{4-x}{4}\cdot\text{etc.}$$

et

$$\cos\pi x=\frac{1-4xx}{1}\cdot\frac{9-4xx}{9}\cdot\frac{25-4xx}{25}\cdot\frac{49-4xx}{49}\cdot\text{etc.}$$

seu

$$\cos\pi x=\frac{1-2x}{1}\cdot\frac{1+2x}{1}\cdot\frac{3-2x}{3}\cdot\frac{3+2x}{3}\cdot\frac{5-2x}{5}\cdot\frac{5+2x}{5}\cdot\text{etc.}$$

Ex his ergo expressionibus, si logarithmi sumantur, erit

$$l\sin\pi x=l\pi x+l\frac{1-x}{1}+l\frac{1+x}{1}+l\frac{2-x}{2}+l\frac{2+x}{2}+l\frac{3-x}{3}+\text{etc.}$$
$$l\cos\pi x=l\frac{1-2x}{1}+l\frac{1+2x}{1}+l\frac{3-2x}{3}+l\frac{3+2x}{3}+l\frac{5-2x}{5}+\text{etc.}$$

43. Sumamus nunc harum serierum logarithmicarum differentialia et divisione ubique per dx facta prior series dabit

$$\frac{\pi\cos\pi x}{\sin\pi x}=\frac{1}{x}-\frac{1}{1-x}+\frac{1}{1+x}-\frac{1}{2-x}+\frac{1}{2+x}-\frac{1}{3-x}+\text{etc.},$$

quae est ea ipsa series, quam § 37 tractavimus. Altera vero series dabit

$$\frac{-\pi\sin\pi x}{\cos\pi x}=-\frac{2}{1-2x}+\frac{2}{1+2x}-\frac{2}{3-2x}+\frac{2}{3+2x}-\frac{2}{5-2x}+\text{etc.}$$

Ponamus $2x=z$, ut sit $x=\frac{z}{2}$, et dividamus per – 2; erit

$$\frac{\pi\sin\frac{1}{2}\pi z}{2\cos\frac{1}{2}\pi z}=\frac{1}{1-z}-\frac{1}{1+z}+\frac{1}{3-z}-\frac{1}{3+z}+\frac{1}{5-z}-\text{etc.}$$

Cum autem sit

$$\sin\tfrac{1}{2}\pi z = \sqrt{\tfrac{1-\cos\pi z}{2}} \quad \text{et} \quad \cos\tfrac{1}{2}\pi z = \sqrt{\tfrac{1+\cos\pi z}{2}},$$

erit

$$\frac{\pi\sqrt{1-\cos\pi z}}{\sqrt{1+\cos\pi z}} = \frac{2}{1-z} - \frac{2}{1+z} + \frac{2}{3-z} - \frac{2}{3+z} + \frac{2}{5-z} - \text{etc.}$$

seu loco *z* scribendo *x* erit

$$\frac{\pi\sqrt{1-\cos\pi x}}{\sqrt{1+\cos\pi x}} = \frac{2}{1-x} - \frac{2}{1+x} + \frac{2}{3-x} - \frac{2}{3+x} + \frac{2}{5-x} - \text{etc.}$$

Addatur haec series ad primum inventam

$$\frac{\pi\cos\pi x}{\sin\pi x} = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \frac{1}{3-x} + \text{etc.}$$

atque reperietur huius seriei

$$\frac{1}{x} + \frac{1}{1-x} - \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} + \frac{1}{3-x} - \frac{1}{3+x} - \text{etc.}$$

summa $= \frac{\pi\sqrt{1-\cos\pi x}}{\sqrt{1+\cos\pi x}} + \frac{\pi\cos\pi x}{\sin\pi x}$. At fractio haec $\frac{\sqrt{1-\cos\pi x}}{\sqrt{1+\cos\pi x}}$, si numerator et denominator multiplicetur per $\sqrt{1-\cos\pi x}$, abit in $\frac{1-\cos\pi x}{\sin\pi x}$. Quocirca summa seriei erit $= \frac{\pi}{\sin\pi x}$, quae est ea ipsa, quam § 34 habuimus; unde eam ulterius non persequemur.