

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1223

CHAPTER XVIII

CONCERNING THE USE OF DIFFERENTIAL CALCULUS IN
THE RESOLUTION OF FRACTIONS

403. We have set out a method for the resolution of any fractions you please into simple fractions in the *Introduction...*[Book I, Ch. 2], and although that by itself is easy enough, yet with the help of differential calculus thus this can be perfected, so that on many occasions this [new] method can be called upon to be used with much less labour. Now especially so if the denominator of the fraction to be resolved should be of an indefinite degree, while the method set out previously generally is impeded in the substitution of values in place of unknown quantities, as must happen in setting up some factor. Especially moreover in these cases, the division by a factor now found can become exceedingly troublesome. Which operation can be avoided, if the differential calculus is called upon to help, thus as there is no need to know the other factor of the denominator which arises, if the denominator is divided by the factor now known. Moreover this method, in which we set out the use of the differential calculus in analysis, is better in determining the value of a fraction, of which the numerator as well as the denominator certainly both vanish [at some point]. In this chapter and likewise the end of this book, we establish the benefit of this method by which the resolution of fractions treated above can be returned in a way more convenient and easier to handle.

404. Therefore if some fraction $\frac{P}{Q}$ should be proposed of which the numerator and the denominator are rational integral functions of the variable x , in the first place it is to be considered, that each x in the numerator P has just as many or more dimensions than in the Q . Because if that comes about, the fraction $\frac{P}{Q}$ contains an integral function of this form $A + Bx + Cx^2 + \text{etc.}$, which thus with the aid of division can be elicited; the remaining part is a fraction having the same denominator as Q , but the numerator of which function, considered as R , contains fewer dimensions of x than the denominator Q , thus so that further resolution is to be put in place on the fraction $\frac{R}{Q}$. Yet meanwhile there is no need to know this new numerator R , but the same simple fractions, which the fraction $\frac{R}{Q}$ can supply, can at once be elicited from the proposed fraction $\frac{P}{Q}$, as we have now shown above.

405. Therefore in addition to the integral part, if the fraction $\frac{P}{Q}$ contains that, the simple fractions must be elicited, of which the denominators are either binomials of the form $f + gx$ or trinomials of the kind $ff + 2x \cos \varphi \sqrt{fg} + gxx$ or of the square or cube or of

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1224

higher powers of formulas of this kind. And these denominators are all factors of the denominator Q , thus so that the factor of the denominator of Q there is produced a simple fraction. Clearly if the denominator Q has the factor $f + gx$, from that there can arise the simple factor of the kind

$$\frac{\mathfrak{A}}{f+gx},$$

but if the factor were $(f + gx)^2$, two fractions

$$\frac{\mathfrak{A}}{(f+gx)^2} + \frac{\mathfrak{B}}{f+gx}.$$

And from the factor of the denominator Q from the cube $(f + gx)^3$ three simple fractions of this kind arise

$$\frac{\mathfrak{A}}{(f+gx)^3} + \frac{\mathfrak{B}}{(f+gx)^2} + \frac{\mathfrak{C}}{f+gx}$$

and thus henceforth. But if the denominator Q should have a trinomial factor of this kind $ff - 2xfg \cos \varphi + gxx$, from that there arises a simple fraction of such a form

$$\frac{\mathfrak{A}+ax}{ff-2xfg \cos \varphi+gxx},$$

and if two factors of this kind were equal such as $(ff - 2xfg \cos \varphi + gxx)^2$, hence two fractions are produced

$$\frac{\mathfrak{A}+ax}{(ff-2xfg \cos \varphi+gxx)^2} + \frac{\mathfrak{B}+bx}{ff-2xfg \cos \varphi+gxx}.$$

Moreover a cubic factor of this kind $(ff - 2fgx \cos \varphi + ggxx)^3$ gives three simple fractions, the biquadratic four, and thus so on.

406. Hence the resolution of any fraction of $\frac{P}{Q}$ can thus be established. First all the simple factors or binomials are sought, and then the trinomials of the denominator Q , and if which should be equal to each other, these are to be noted properly and the form of one of them is put in place. Then from these factors of the denominator the simple fractions are elicited either in the manner now shown above, or in that which here we are about to handle and which for argument's sake can be put in place of the first factor. With which done the sum of all such simple fractions together with the integral part, if the proposed fraction $\frac{P}{Q}$ should contain that, exhaust the value of this. Indeed we have assumed the finding of the factors of the denominator Q as known here, as they depend on the solution of the equation $Q = 0$, and here we describe a method for any given factor of the denominator from which hence the simple fractions arising can be defined. Because,

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1225

since the simple fractions of the denominator have now been established, it will be better if then we show how to find the numerator of each of the fractions.

407. Therefore we arrange the denominator Q of the fraction $\frac{P}{Q}$ to have a factor $f + gx$, thus so that $Q = (f + gx)S$, and indeed here, neither does S contain in addition the same factor $f + gx$. Let the simple fraction arising from this factor be equal to $\frac{\mathfrak{A}}{f + gx}$ and the complement has a form of this kind $\frac{V}{S}$, thus so that $\frac{\mathfrak{A}}{f + gx} + \frac{V}{S} = \frac{P}{Q}$.

Hence there arises : $\frac{V}{S} = \frac{P}{Q} - \frac{\mathfrak{A}}{f + gx} = \frac{P}{(f + gx)S} - \frac{\mathfrak{A}}{f + gx} = \frac{P - \mathfrak{A}S}{(f + gx)S}$, and thus $V = \frac{P - \mathfrak{A}S}{(f + gx)}$.

Therefore, as V is an integral function of x , it is necessary that $P - \mathfrak{A}S$ should be divisible by $f + gx$; and therefore, if there is put $f + gx = 0$ or $x = \frac{-f}{g}$, then the expression

$P - \mathfrak{A}S$ vanishes, [as $f + gx$ is a factor of $P - \mathfrak{A}S$ or $x = \frac{-f}{g}$ is a root of this polynomial].

Hence on making $x = \frac{-f}{g}$, since $P - \mathfrak{A}S = 0$, then $\mathfrak{A} = \frac{P}{S}$, [for this value of x] as now we

have found above [according to the original method]. But since $S = \frac{Q}{f + gx}$, then there

arises $\mathfrak{A} = \frac{(f + gx)P}{Q}$, if there is put $f + gx = 0$ or $x = \frac{-f}{g}$ everywhere. Now since in this

case both the numerator $(f + gx)P$ as well as the denominator Q vanish here, by means of that which we can establish in the investigation of fractions of this kind, then if indeed there is put $x = \frac{-f}{g}$,

$$\mathfrak{A} = \frac{(f + gx)dP + Pgdx}{dQ}.$$

But in this case on account of $(f + gx)dP = 0$ then $\mathfrak{A} = \frac{Pgdx}{dQ}$.

[One can see that Euler is thinking of the ratio becoming zero on zero when $x = \frac{-f}{g}$, and

so the ratio is to the differential of the numerator to the differential of the denominator.

One can see also that the problem is essentially an application of L'Hôpital's Rule : as is well known, in the neighbourhood of a common simple zero $x = a$ of well-behaved functions r and s of x , or a common root of polynomials, the values of the functions at a are represented by their differentials, and it is an easy matter to show that

$\lim_{x \rightarrow a} \frac{r(a)}{s(a)} = \frac{r'(a)}{s'(a)}$, where the rule can be extended to higher derivatives if necessary .]

EULER'S INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1226

408. Therefore if the denominator Q of the proposed fraction $\frac{P}{Q}$ has a simple factor $f + gx$, from that there arises the simple fraction $\frac{\mathfrak{A}}{f+gx}$ with $\mathfrak{A} = \frac{Pgdx}{dQ}$ being present, after everywhere here the value $\frac{-f}{g}$ is put in place of x arising from the equation $f + gx = 0$ must be substituted. Hence, if Q is not expressed in factors, this division can often be omitted without any bother, particularly if the denominator Q has indefinite exponents, since the value of \mathfrak{A} can be obtained from the formula $\frac{Pgdx}{dQ}$. But if the denominator now was expressed in factors, thus in order that hence the value of S should be at once apparent, then the other expression is to be preferred, in which we found $\mathfrak{A} = \frac{P}{S}$ on putting everywhere equally $x = \frac{-f}{g}$. And thus for finding the value of \mathfrak{A} itself in whatever case that formula can be used, which is seen to be the more convenient and expedient. But we illustrate the use of the new formula with some examples.

EXAMPLE 1

Let this fraction be proposed $\frac{x^9}{1+x^{17}}$, a simple fraction of which is required to be defined from the factor of the denominator $1 + x$.

Since here $Q = 1 + x^{17}$, even if $1 + x$ is agreed to be a factor of this, yet, if as the first method demands, we wish to divide by that, there is produced

$$S = 1 - x + xx - x^3 + \dots + x^{16}$$

Therefore it is more convenient to use the new formula $\mathfrak{A} = \frac{Pgdx}{dQ}$; and thus since $f = 1$,

$g = 1$ and $P = x^9$, on account of $dQ = 17x^{16}dx$ there becomes $\mathfrak{A} = \frac{x^9}{17x^{16}} = \frac{1}{17x^7}$ and on

putting $x = -1$, from which there arises $\mathfrak{A} = -\frac{1}{17}$, and the simple fraction arising from the factor of the denominator $1 + x$ is $\frac{-1}{17(1+x)}$.

EXAMPLE 2

For the proposed fraction $\frac{x^m}{1-x^{2n}}$, to find the simple fraction arising from the factor $1 - x$ of the denominator.

On account of the proposed factor $1 - x$ there is $f = 1$ and $g = -1$. Then the denominator $1 - x^{2n}$ gives $dQ = -2nx^{2n-1}dx$, from which because of $P = x^m$ there is

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1227

obtained $\mathfrak{A} = \frac{Pgdx}{dQ} = \frac{-x^m}{-2nx^{2n-1}}$. And on putting from the equation $1-x$, $x=1$ there

becomes $\mathfrak{A} = \frac{1}{2n}$, thus so that the simple fraction is $\frac{1}{2n(1-x)}$.

EXAMPLE 3

For the proposed fraction $\frac{x^m}{1-4x^k+3x^n}$, to find the simple fraction arising from the factor $1-x$ of the denominator.

Hence here there becomes $f=1$, $g=-1$, $P=x^m$, and $Q=1-4x^k+3x^n$ and $\frac{dQ}{dx} = -4kx^{k-1} + 3nx^{n-1}$; from which there arises $\mathfrak{A} = \frac{-x^m}{-4kx^{k-1} + 3nx^{n-1}}$ and on putting $x=1$ then $\mathfrak{A} = \frac{1}{4k-3n}$. The simple fraction arising from this simple factor of the denominator $1-x$ is now $\frac{1}{(4k-3n)(1-x)}$.

409. Now we put the denominator Q of the fraction $\frac{P}{Q}$ to have the square factor

$(f+gx)^2$ and the simple fractions hence arising are $\frac{\mathfrak{A}}{(f+gx)^2} + \frac{\mathfrak{B}}{f+gx}$.

Let $Q = (f+gx)^2 S$ and the complement $= \frac{V}{S}$, thus in order that

$$\frac{V}{S} = \frac{P}{Q} - \frac{\mathfrak{A}}{(f+gx)^2} - \frac{\mathfrak{B}}{f+gx} \text{ and } V = \frac{P - \mathfrak{A}S - \mathfrak{B}(f+gx)S}{(f+gx)^2}$$

Now because V is an integral function, it is necessary that $P - \mathfrak{A}S - \mathfrak{B}(f+gx)S$

is divisible by $(f+gx)^2$; and since S does not contain a further factor $f+gx$,

also this expression $\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f+gx)$ is divisible by $(f+gx)^2$, and thus with the

factor $f+gx=0$ or $x = -\frac{f}{g}$ not only this expression, but also the differential of this

vanishes, $d.\frac{P}{S} - \mathfrak{B}gdx$. Therefore on putting $x = -\frac{f}{g}$, there arises from the first

equation $\mathfrak{A} = \frac{P}{S}$, and from the second equation there now arises $\mathfrak{B} \frac{1}{gdx} = d.\frac{P}{S}$; from

which the values of the fractions sought $\frac{\mathfrak{A}}{(f+gx)^2} + \frac{\mathfrak{B}}{f+gx}$ can be found.

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1228

EXAMPLE

From the proposed fraction, $\frac{x^m}{1-4x^3+3x^4}$ the denominator of which has the factor $(1-x)^2$, to find all the simple fractions thus arising.

Since here there is $f = 1$, $g = -1$, $P = x^m$ and $Q = 1 - 4x^3 + 3x^4$ then [on factorising the denominator], $S = 1 + 2x + 3xx$, then

$$\frac{P}{S} = \frac{x^m}{1+2x+3xx} \text{ and } d.\frac{P}{S} = \frac{mx^{m-1}dx+2(m-1)x^m dx+3(m-2)x^{m+1}dx}{(1+2x+3xx)^2}.$$

Hence on putting $x = 1$ there arises

$$\mathfrak{A} = \frac{1}{6} \text{ and } \mathfrak{B} = -1 \cdot \frac{6m-8}{36} = \frac{4-3m}{18};$$

from which the fractions sought are

$$\frac{1}{6(1-x)^2} + \frac{4-3m}{18(1-x)}.$$

410. Let the denominator Q of the fraction $\frac{P}{Q}$ have three equal simple fractions, or

$Q = (f + gx)^3 S$ and let the simple fractions arising from the cube of the factor $(f + gx)^3$ be these

$$\frac{\mathfrak{A}}{(f+gx)^3} + \frac{\mathfrak{B}}{(f+gx)^2} + \frac{\mathfrak{C}}{(f+gx)};$$

now the complement of these fractions to be established for the proposed fraction $\frac{P}{Q}$ is equal to $\frac{V}{S}$ and it follows that

$$V = \frac{P - \mathfrak{A}S - \mathfrak{B}S(f+gx) - \mathfrak{C}S(f+gx)^2}{(f+gx)^3}$$

Whereby this expression $\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f + gx) - \mathfrak{C}(f + gx)^2 = 0$ is divisible by $(f + gx)^3$;

from which on putting $f + gx = 0$ or $x = \frac{-f}{g}$ not only this expression, but also the first

and second differentials are able to become equal to 0. Clearly on putting $x = \frac{-f}{g}$

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1229

$$\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f + gx) - \mathfrak{C}(f + gx)^2 = 0$$

$$d.\frac{P}{S} - \mathfrak{B}gdx - 2\mathfrak{C}gdx(f + gx) = 0$$

$$dd.\frac{P}{S} - 2\mathfrak{C}g^2dx^2 = 0.$$

Hence from the first equation there becomes

$$\mathfrak{A} = \frac{P}{S}.$$

Now from the second

$$\mathfrak{B} = \frac{1}{gdx} d.\frac{P}{S}.$$

And from the third there is defined

$$\mathfrak{C} = \frac{1}{2g^2dx^2} dd.\frac{P}{S}.$$

411. Hence generally, if the denominator Q of the fraction $\frac{P}{Q}$ has the factor $(f + gx)^n$,

thus in order that $Q = (f + gx)^n S$, with these simple fractions arising from the factor

$(f + gx)^n$:

$$\frac{\mathfrak{A}}{(f+gx)^n} + \frac{\mathfrak{B}}{(f+gx)^{n-1}} + \frac{\mathfrak{C}}{(f+gx)^{n-2}} + \frac{\mathfrak{D}}{(f+gx)^{n-3}} + \frac{\mathfrak{E}}{(f+gx)^{n-4}} + \text{etc.},$$

until finally the fraction is arrived at, of which the denominator is $f + gx$, if the method is put in place as before, then this expression is found:

$$\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f + gx) - \mathfrak{C}(f + gx)^2 - \mathfrak{D}(f + gx)^3 - \mathfrak{E}(f + gx)^4 - \text{etc.}$$

which must be divisible by $(f + gx)^n$; hence this expression as well as the individual

differentials of this as far as the degree $n - 1$ must vanish when we set $x = \frac{-f}{g}$

From which equations it can be concluded on putting $x = \frac{-f}{g}$ everywhere :

EULER'S INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1230

$$\mathfrak{A} = \frac{P}{S}$$

$$\mathfrak{B} = \frac{1}{1gdx} d \cdot \frac{P}{S}$$

$$\mathfrak{C} = \frac{1}{1 \cdot 2g^2 dx^2} dd \cdot \frac{P}{S}$$

$$\mathfrak{D} = \frac{1}{1 \cdot 2 \cdot 3g^3 dx^3} d^3 \cdot \frac{P}{S}$$

$$\mathfrak{E} = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4g^4 dx^4} d^4 \cdot \frac{P}{S}$$

etc.

Where indeed it is to be observed it is required to take the differentials themselves of $\frac{P}{S}$ beforehand, as in place of x there is to be put $\frac{-f}{g}$;

412. Hence in this way these numbers $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc can be expressed more easily than in the manner treated in the *Introductio*, and the values of these can be found more readily on many occasions too by this new method. Which comparison by which we can define the values of the letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc, can be put in place more easily than in the previous manner:

Putting $x = \frac{-f}{g}$, the remainder of the variable x is put in place :

$$\mathfrak{A} = \frac{P}{S} \qquad \frac{P - \mathfrak{A}S}{f + gx} = \mathfrak{B},$$

$$\mathfrak{B} = \frac{\mathfrak{B}}{S} \qquad \frac{\mathfrak{B} - \mathfrak{B}S}{f + gx} = \mathfrak{C},$$

$$\mathfrak{C} = \frac{\mathfrak{C}}{S} \qquad \frac{\mathfrak{C} - \mathfrak{C}S}{f + gx} = \mathfrak{D},$$

$$\mathfrak{D} = \frac{\mathfrak{D}}{S} \qquad \frac{\mathfrak{D} - \mathfrak{D}S}{f + gx} = \mathfrak{E},$$

$$\mathfrak{E} = \frac{\mathfrak{E}}{S} \qquad \text{and thus henceforth.}$$

413. But if the denominator Q of the fraction $\frac{P}{Q}$ should not have all real simple factors, then two of the imaginary factors are to be taken together, the product of which is real. Hence let a factor of the denominator Q be $ff - 2fgx \cos \varphi + ggxx$, which on being put equal to zero gives this two-fold imaginary value :

$$x = \frac{f}{g} \cos \varphi \pm \frac{f}{g\sqrt{-1}} \sin \varphi;$$

from which there arises

$$x^n = \frac{f^n}{g^n} \cos n\varphi \pm \frac{f^n}{g^n\sqrt{-1}} \sin n\varphi.$$

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1231

We put the denominator $Q = (ff - 2fgx \cos \varphi + ggxx)S$ and in addition S is not to be divisible by $ff - 2fgx \cos \varphi + ggxx$. Let the fraction originating from this denominator be

$$\frac{\mathfrak{A} + \alpha x}{ff - 2fgx \cos \varphi + ggxx}$$

and the complement for the proposed fraction $\frac{P}{Q}$ is equal to $\frac{V}{S}$; then

$$V = \frac{P - (\mathfrak{A} + \alpha x)S}{ff - 2fgx \cos \varphi + ggxx},$$

from which $P - (\mathfrak{A} + \alpha x)S$ and in addition also $\frac{P}{S} - \mathfrak{A} - \alpha x$ is divisible by

$ff - 2fgx \cos \varphi + ggxx$. Hence $\frac{P}{S} - \mathfrak{A} - \alpha x$ vanishes, if there is put

$ff - 2fgx \cos \varphi + ggxx = 0$, that is, if there is put either

$$x = \frac{f}{g} \cos \varphi + \frac{f}{g\sqrt{-1}} \sin \varphi$$

or

$$x = \frac{f}{g} \cos \varphi - \frac{f}{g\sqrt{-1}} \sin \varphi$$

414. Because P and S are integral [*i. e.* polynomial] functions of x , the substitution can be made separately into each in turn; and because for any power of x , for example x^n , this binomial must be substituted

$$x^n = \frac{f^n}{g^n} \cos n\varphi \pm \frac{f^n}{g^n \sqrt{-1}} \sin n\varphi,$$

in the first place we can put $\frac{f^n}{g^n} \cos n\varphi$ for x^n everywhere, and with this done P is

changed into \mathfrak{P} and S into \mathfrak{S} . Then in place of x^n everywhere there is put $\frac{f^n}{g^n} \sin n\varphi$ and

with this done P becomes \mathfrak{p} and S becomes \mathfrak{s} ; where it is to be noted that before these substitutions each function P and S must be expanded out completely, thus in order that, if perhaps factors are to be involved, then these are to be removed. From these values

\mathfrak{P} , \mathfrak{p} , \mathfrak{S} , \mathfrak{s} found it is evident, if there is put $x = \frac{f}{g} \cos \varphi \pm \frac{f}{g\sqrt{-1}} \sin \varphi$, the function P is

changed into $\mathfrak{P} \pm \frac{\mathfrak{p}}{\sqrt{-1}}$ and the function S changed into $\mathfrak{S} \pm \frac{\mathfrak{s}}{\sqrt{-1}}$

Hence since $\frac{P}{S} - \mathfrak{A} - \alpha x$ or $P - (\mathfrak{A} + \alpha x)S$ must vanish in each case, then

$$\mathfrak{P} \pm \frac{\mathfrak{p}}{\sqrt{-1}} = \left(\mathfrak{A} + \frac{\alpha f}{g} \cos \varphi \pm \frac{\alpha f}{g\sqrt{-1}} \sin \varphi \right) \left(\mathfrak{S} \pm \frac{\mathfrak{s}}{\sqrt{-1}} \right)$$

from which on account of the sign condition these two equations arise :

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1232

$$\mathfrak{P} = \mathfrak{A}\mathfrak{S} + \frac{\mathfrak{a}f\mathfrak{S}}{g} \cos \varphi - \frac{\mathfrak{a}f\mathfrak{s}}{g} \sin \varphi$$

$$\mathfrak{p} = \mathfrak{A}\mathfrak{s} + \frac{\mathfrak{a}f\mathfrak{s}}{g} \cos \varphi + \frac{\mathfrak{a}f\mathfrak{S}}{g} \sin \varphi,$$

from which on eliminating \mathfrak{A} there arises

$$\mathfrak{S}\mathfrak{p} - \mathfrak{s}\mathfrak{P} = \frac{\mathfrak{a}f(\mathfrak{S}^2 + \mathfrak{s}^2)}{g} \sin \varphi ;$$

and thus there becomes

$$\mathfrak{a} = \frac{g(\mathfrak{S}\mathfrak{p} - \mathfrak{s}\mathfrak{P})}{f(\mathfrak{S}^2 + \mathfrak{s}^2) \sin \varphi}.$$

Then on eliminating $\sin \varphi$ there becomes

$$\mathfrak{S}\mathfrak{P} + \mathfrak{s}\mathfrak{p} = (\mathfrak{S}^2 + \mathfrak{s}^2) \left(\mathfrak{A} + \frac{\mathfrak{a}f}{g} \cos \varphi \right).$$

Hence

$$\mathfrak{A} = \frac{\mathfrak{S}\mathfrak{P} + \mathfrak{s}\mathfrak{p}}{\mathfrak{S}^2 + \mathfrak{s}^2} - \frac{(\mathfrak{S}\mathfrak{p} - \mathfrak{s}\mathfrak{P}) \cos \varphi}{(\mathfrak{S}^2 + \mathfrak{s}^2) \sin \varphi}.$$

415. Since now there shall be $S = \frac{Q}{ff - 2fgx \cos \varphi + ggxx}$,

because on putting $ff - 2fgx \cos \varphi + ggxx = 0$

both the numerator as well as the denominator vanish, in this case there arises

$$S = \frac{dQ:dx}{2ggx - 2fg \cos \varphi}$$

Now we may put, if $\frac{f^n}{g^n} \cos n\varphi$ is substituted everywhere, the function $\frac{dQ}{dx}$ to become \mathfrak{Q} ,

but if there is put in place everywhere $\frac{f^n}{g^n} \sin n\varphi$, that becomes \mathfrak{q} ; and it is evident, if

there is put $x = \frac{f}{g} \cos \varphi \pm \frac{f}{g\sqrt{-1}} \sin \varphi$, the function $\frac{dQ}{dx}$ becomes $\mathfrak{Q} \pm \frac{\mathfrak{q}}{\sqrt{-1}}$

From which the function S is changed into

$$\frac{\mathfrak{Q} \pm \mathfrak{q} \sqrt{-1}}{\pm 2fg \sin \varphi \sqrt{-1}}.$$

Therefore since $S = \mathfrak{S} \pm \frac{\mathfrak{s}}{\sqrt{-1}}$ with the same value for x put in place, there is obtained :

$$\mathfrak{Q} \pm \frac{\mathfrak{q}}{\sqrt{-1}} = \pm \frac{2fg\mathfrak{S}}{\sqrt{-1}} \sin \varphi - 2fg\mathfrak{s} \sin \varphi.$$

Hence there arises

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1233

$$s = \frac{-\Omega}{2fg \sin \varphi} \text{ and } \mathfrak{S} = \frac{q}{2fg \sin \varphi}$$

With these values in place

$$a = \frac{2gg(pq + \mathfrak{P}\Omega)}{\Omega^2 + q^2}$$

and

$$\mathfrak{A} = \frac{2fg(\mathfrak{P}q - p\Omega)\sin \varphi}{\Omega^2 + q^2} - \frac{2fg(pq + \mathfrak{P}\Omega)\cos \varphi}{\Omega^2 + q^2}.$$

416. Hence therefore a suitable ratio is obtained and here a simple fraction can be formed for any factor of the second power, since the denominator of the proposed denominator is retained in the computation, as we avoid a division, in which the value of the letter S must be defined and which often is more than a little troublesome. Therefore if the denominator Q of the fraction $\frac{P}{Q}$ has such a factor $ff - 2fgx \cos \varphi + ggxx = 0$, a simple fraction arising from this factor can be defined in the following manner, so that we may produce :

$$= \frac{\mathfrak{A} + ax}{ff - 2fgx \cos \varphi + ggxx}.$$

There is put $x = \frac{f}{g} \cos \varphi$ and for whatever the power of x^n there is written $\frac{f^n}{g^n} \cos n\varphi$;

with which done P is changed into \mathfrak{P} and the function $\frac{dQ}{dx}$ into Ω . Then likewise there

is put $x = \frac{f}{g} \sin \varphi$ and any power of this $x^n = \frac{f^n}{g^n} \sin n\varphi$ and P is changed into p and $\frac{dQ}{dx}$

into q . And in this manner with the values of the letters \mathfrak{P}, Ω, p and q found, \mathfrak{A} and a can thus be found, in order that there becomes

$$\mathfrak{A} = \frac{2fg(\mathfrak{P}q - p\Omega)\sin \varphi}{\Omega^2 + q^2} - \frac{2fg(pq + \mathfrak{P}\Omega)\cos \varphi}{\Omega^2 + q^2},$$

$$a = \frac{2gg(pq + \mathfrak{P}\Omega)}{\Omega^2 + q^2}.$$

Hence the fraction arising from the factor $ff - 2fgx \cos \varphi + ggxx$ of the denominator Q is given by

$$\frac{2fg(\mathfrak{P}q - p\Omega)\sin \varphi + 2g(\mathfrak{P}\Omega + pq)(gx - f \cos \varphi)}{(\Omega^2 + q^2)(ff - 2fgx \cos \varphi + ggxx)}.$$

EXAMPLE 1

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1234

If this fraction is proposed $\frac{x^m}{a+bx^n}$, the denominator of which $a+bx^n$ has this factor $ff - 2fgx \cos \varphi + ggxx$, to find the simple fraction agreeing with this factor.

Because here $P = x^m$ and $Q = a + bx^n$, then $\frac{dQ}{dx} = nbx^n$ and thus there arises

$$\begin{aligned} \mathfrak{P} &= \frac{f^m}{g^m} \cos m\varphi, & \mathfrak{p} &= \frac{f^n}{g^n} \sin m\varphi, \\ \mathfrak{Q} &= \frac{nbf^{n-1}}{g^{n-1}} \cos(n-1)\varphi, & \mathfrak{q} &= \frac{nbf^{n-1}}{g^{n-1}} \sin(n-1)\varphi. \end{aligned}$$

From these there becomes

$$\begin{aligned} \mathfrak{Q}^2 + \mathfrak{q}^2 &= \frac{n^2 b^2 f^{2(n-1)}}{g^{2(n-1)}}, \\ \mathfrak{P}\mathfrak{Q} - \mathfrak{p}\mathfrak{q} &= \frac{nbf^{m+n-1}}{g^{m+n-1}} \sin(n-m-1)\varphi, \end{aligned}$$

and

$$\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} = \frac{nbf^{m+n-1}}{g^{m+n-1}} \cos(n-m-1)\varphi.$$

On account of which the simple fraction sought is

$$\frac{2g^{n-m}(f \sin \varphi \sin(n-m-1)\varphi + gx \cos(n-m-1)\varphi - f \cos \varphi \cos(n-m-1)\varphi)}{nbf^{n-m-1}(ff - 2fgx \cos \varphi + ggxx)}$$

or

$$\frac{2g^{n-m}(gx \cos(n-m-1)\varphi - f \cos(n-m)\varphi)}{nbf^{n-m-1}(ff - 2fgx \cos \varphi + ggxx)}.$$

EXAMPLE 2

If this fraction should be proposed, $\frac{1}{x^m(a+bx^n)}$, the denominator of which has the factor $ff - 2fgx \cos \varphi + ggxx$; to find the simple fraction arising from this.

As there is $P = 1$ and $Q = ax^m + bx^{m+n}$, then $\frac{dQ}{dx} = max^{m-1} + (m+n)bx^{m+n-1}$

and thus on putting $x^n = \frac{f^n}{g^n} \cos n\varphi$, on account of $P = x^0$, $\mathfrak{P} = 1$, and

$$\mathfrak{Q} = \frac{maf^{m-1}}{g^{m-1}} \cos(m-1)\varphi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}} \cos(m+n-1)\varphi$$

and [on putting $x^n = \frac{f^n}{g^n} \sin n\varphi$] $\mathfrak{p} = 0$ and

$$\mathfrak{q} = \frac{maf^{m-1}}{g^{m-1}} \sin(m-1)\varphi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}} \sin(m+n-1)\varphi.$$

EULER'S INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1235

Hence

$$\mathfrak{A}^2 + \mathfrak{q}^2 = \frac{m^2 a^2 f^{2(m-1)}}{g^{m-1}} + \frac{2m(m+n)abf^{2m+n-2}}{g^{2m+n-2}} \cos n\varphi + \frac{(m+n)^2 b^2 f^{2(m+n-1)}}{g^{2(m+n-1)}}.$$

For if now $ff - 2fgx \cos \varphi + ggxx$ is a divisor of $a + bx^n$, then

$$a + \frac{bf^n}{g^n} \cos n\varphi = 0 \text{ and } \frac{bf^n}{g^n} \sin n\varphi = 0, \text{ on account of which } aa = \frac{bbf^{2n}}{g^{2n}}.$$

Hence there becomes

$$\mathfrak{A}^2 + \mathfrak{q}^2 = \frac{(m+n)^2 bbf^{2(m+n-1)}}{g^{2(m+n-1)}} - \frac{m(2n+m)aa f^{2(m-1)}}{g^{2(m-1)}} = \frac{nnaaf^{2(m-1)}}{g^{2(m-1)}} = \frac{nnbbf^{2(m+n-1)}}{g^{2(m+n-1)}}.$$

Then there now becomes

$$\begin{aligned} \mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} &= \frac{maf^{m-1}}{g^{m-1}} \sin(m-1)\varphi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}} \sin(m+n-1)\varphi \\ &= \frac{bf^{m+n-1}}{g^{m+n-1}} ((m+n) \sin(m+n-1)\varphi - m \cos n\varphi \cdot \sin(m-1)\varphi) \\ &= \frac{bf^{m+n-1}}{g^{m+n-1}} (n \cos n\varphi \cdot \sin(m-1)\varphi + (m+n) \sin n\varphi \cdot \cos(m-1)\varphi) \end{aligned}$$

and

$$\mathfrak{P}\mathfrak{q} + \mathfrak{p}\mathfrak{Q} = \frac{bf^{m+n-1}}{g^{m+n-1}} ((m+n) \cos(m+n-1)\varphi - m \cos n\varphi \cdot \cos(m-1)\varphi).$$

Either since $ff - 2fg \cos \varphi + ggxx$ is also a divisor of $ax^{m-1} + bx^{m+n-1}$, then

$$\frac{af^{m-1}}{g^{m-1}} \cos(m-1)\varphi + \frac{bf^{m+n-1}}{g^{m+n-1}} \cos(m+n-1)\varphi = 0$$

and

$$\frac{af^{m-1}}{g^{m-1}} \sin(m-1)\varphi + \frac{bf^{m+n-1}}{g^{m+n-1}} \sin(m+n-1)\varphi = 0,$$

from which there becomes

$$\mathfrak{Q} = \frac{nbf^{m+n-1}}{g^{m+n-1}} \cos(m+n-1)\varphi \text{ and } \mathfrak{q} = \frac{nbf^{m+n-1}}{g^{m+n-1}} \sin(m+n-1)\varphi$$

or

$$\mathfrak{Q} = \frac{-naf^{m-1}}{g^{m-1}} \cos(m-1)\varphi \text{ and } \mathfrak{q} = \frac{-naf^{m-1}}{g^{m-1}} \sin(m-1)\varphi$$

From which there results the fraction sought

$$\frac{2g^m(f \cos m\varphi - gx \cos(m-1)\varphi)}{naf^{m-1}(ff - 2fgx \cos \varphi + ggxx)}.$$

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1236

Which formula follows from the first example, if m is put negative, from which there was no need to have put in place this particular example.

EXAMPLE 3

If the denominator of this fraction $\frac{x^m}{a+bx^n+cx^{2n}}$ should have a factor of the form $ff - 2fgx \cos \varphi + ggxx$, to find the simple fraction arising from this factor.

If $ff - 2fgx \cos \varphi + ggxx$ is a factor of the denominator $a + bx^n + cx^{2n}$, then, as we have shown above,

$$a + \frac{bf^n}{g^n} \cos n\varphi + \frac{cf^{2n}}{g^{2n}} \cos 2n\varphi = 0 \text{ and } \frac{bf^n}{g^n} \sin n\varphi + \frac{cf^{2n}}{g^{2n}} \sin 2n\varphi = 0$$

Therefore since there shall be $P = x^m$ and $Q = a + bx^n + cx^{2n}$, then

$$\frac{dQ}{dx} = nbx^{n-1} + 2ncx^{2n-1},$$

from which there is produced

$$\begin{aligned} \mathfrak{P} &= \frac{f^m}{g^m} \cos m\varphi \text{ and } \mathfrak{p} = \frac{f^m}{g^m} \sin m\varphi, \\ \mathfrak{Q} &= \frac{nbf^{n-1}}{g^{n-1}} \cos(n-1)\varphi + \frac{2ncf^{2n-1}}{g^{2n-1}} \cos(2n-1)\varphi, \\ \mathfrak{q} &= \frac{nbf^{n-1}}{g^{n-1}} \sin(n-1)\varphi + \frac{2ncf^{2n-1}}{g^{2n-1}} \sin(2n-1)\varphi. \end{aligned}$$

On account of which we have

$$\mathfrak{Q}^2 + \mathfrak{q}^2 = \frac{n^2 f^{2(n-1)}}{g^{2(n-1)}} \left(bb + \frac{4bcf^n}{g^n} \cos n\varphi + \frac{4ccf^{2n}}{g^{2n}} \right)$$

But from the two former equations there is

$$\frac{f^{2n}}{g^{2n}} \left(bb + \frac{2bcf^n}{g^n} \cos n\varphi + \frac{ccf^{2n}}{g^{2n}} \right) = aa$$

and thus

$$\frac{4bcf^n}{g^n} \cos n\varphi = \frac{2g^{2n}aa}{f^{2n}} - 2bb - \frac{2ccf^{2n}}{g^{2n}};$$

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1237

from which value substituted there, there becomes

$$\mathfrak{A}^2 + \mathfrak{q}^2 = \frac{n^2 f^{2n-2}}{g^{2n-2}} \left(\frac{2aag^{2n}}{f^{2n}} - bb + \frac{2ccf^{2n}}{g^{2n}} \right)$$

or

$$\mathfrak{A}^2 + \mathfrak{q}^2 = \frac{n^2 (2aag^{4n} - bbf^{2n}g^{2n} + 2ccf^{4n})}{ffg^{4n-2}}.$$

Then there becomes

$$\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} = \frac{nbf^{m+n-1}}{g^{m+n-1}} \sin(n-m-1)\varphi + \frac{2ncf^{m+2n-1}}{g^{m+2n-1}} \sin(2n-m-1)\varphi,$$

$$\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} = \frac{nbf^{m+n-1}}{g^{m+n-1}} \cos(n-m-1)\varphi + \frac{2ncf^{m+2n-1}}{g^{m+2n-1}} \cos(2n-m-1)\varphi.$$

From which values found the simple fraction sought becomes

$$\frac{2fg(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q})\sin\varphi + 2g(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})(gx - f\cos\varphi)}{(\mathfrak{Q}^2 + \mathfrak{q}^2)(ff - 2fgx\cos\varphi + ggxx)}.$$

417. But these values are more easily expressed if we determine these factors themselves from the denominator. Therefore let the denominator of the proposed fraction be

$$a + bx^n;$$

of which if the trinomial factor is put in place $ff - 2fgx\cos\varphi + ggxx$,

there will be, as we have shown in the *Introductio*,

$$a + \frac{bf^n}{g^n} \cos n\varphi = 0 \quad \text{and} \quad \frac{bf^n}{g^n} \sin n\varphi = 0$$

therefore since $\sin n\varphi = 0$, either $n\varphi = (2k-1)\pi$ or $n\varphi = 2k\pi$; in the first case then $\cos n\varphi = -1$, in the latter $\cos n\varphi = +1$. Hence if a and b are positive quantities, in the first

case only the condition is had, in which there arises $a = \frac{bf^n}{g^n}$ and hence

$$f = a^{\frac{1}{n}} \quad \text{and} \quad g = b^{\frac{1}{n}}.$$

But we may retain in place of these irrational quantities the letters f and g or we may put rather $a = f^n$ and $b = g^n$, thus in order that the factors of this function are to be found :

$$f^n + g^n x^n.$$

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1238

Therefore since $\varphi = \frac{(2k-1)\pi}{n}$, where k can be designated to some positive number, but indeed numbers for k greater than $\frac{2k-1}{n}$ are not to be taken, than which return $\frac{2k-1}{n}$ less than unity ; hence the factors of the proposed fraction $f^n + g^n x^n$ are the following :

$$ff - 2fgxcos\frac{\pi}{n} + ggxx$$

$$ff - 2fgxcos\frac{3\pi}{n} + ggxx$$

$$ff - 2fgxcos\frac{5\pi}{n} + ggxx$$

etc,

where it is to be noted that if n is an odd number, then this one binomial factor is present,

$$f + gx,$$

but if n is an even number then no binomial factor is present.

EXAMPLE 1

To resolve this fraction $\frac{x^m}{f^n + g^n x^n}$ into its simple fractions.

Since a factor of each trinomial denominator is contained in this form

$$ff - 2fgxcos\frac{(2k-1)\pi}{n} + ggxx,$$

in example 1 in the preceding paragraph, $a = f^n$, $b = g^n$, and $\varphi = \frac{(2k-1)\pi}{n}$,

from which there shall be

$$\sin(n - m - 1)\varphi = \sin(m + 1)\varphi = \sin\frac{(m+1)(2k-1)\pi}{n},$$

and

$$\cos(n - m - 1)\varphi = -\cos(m + 1)\varphi = -\cos\frac{(m+1)(2k-1)\pi}{n}.$$

Hence from this factor this simple fraction arises

$$\frac{2f \sin.\frac{(2k-1)\pi}{n} \sin.\frac{(m+1)(2k-1)\pi}{n} - 2\cos.\frac{(m+1)(2k-1)\pi}{n} \left(gx - f \cos.\frac{(2k-1)\pi}{n} \right)}{nf^{n-m-1} g^m \left(ff - 2fgxcos.\frac{(2k-1)\pi}{n} + ggxx \right)}$$

On account of which the proposed fraction can be resolved into these simple parts :

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1239

$$\frac{2f \sin \frac{\pi}{n} \sin \frac{(m+1)\pi}{n} - 2 \cos \frac{(m+1)\pi}{n} (gx - f \cos \frac{\pi}{n})}{nf^{n-m-1} g^m (ff - 2fgx \cos \frac{\pi}{n} + ggxx)}$$

$$+ \frac{2f \sin \frac{3\pi}{n} \sin \frac{3(m+1)\pi}{n} - 2 \cos \frac{3(m+1)\pi}{n} (gx - f \cos \frac{3\pi}{n})}{nf^{n-m-1} g^m (ff - 2fgx \cos \frac{3\pi}{n} + ggxx)}$$

$$+ \frac{2f \sin \frac{5\pi}{n} \sin \frac{5(m+1)\pi}{n} - 2 \cos \frac{5(m+1)\pi}{n} (gx - f \cos \frac{5\pi}{n})}{nf^{n-m-1} g^m (ff - 2fgx \cos \frac{5\pi}{n} + ggxx)}$$

etc.

Therefore if n were an even number, in this way all the simple fractions arise; but if n should be an odd number, on account of the binomial factor $f + gx$ this must be added to the resulting fractions above

$$\frac{\pm 1}{nf^{n-m-1} g^m (f + gx)}$$

where the sign $+$ prevails, if m is an even number, and conversely for the $-$ sign. If m should be a number greater than n , then to these fractions the above integral parts of this kind must be added

$$Ax^{m-n} + Bx^{m-2n} + Cx^{m-3n} + Dx^{m-4n} + \text{etc.},$$

as long as the exponents remain positive, and there becomes

$$Ag^n = 1 \quad \text{hence} \quad A = \frac{1}{g^n}$$

$$Af^n + Bg^n = 0 \quad B = -\frac{f^n}{g^{2n}}$$

$$Bf^n + Cg^n = 0 \quad C = +\frac{f^{2n}}{g^{3n}}$$

$$Cf^n + Dg^n = 0 \quad D = -\frac{f^{3n}}{g^{4n}}$$

etc.

etc.

EXEMPLUM 2

To resolve this fraction $\frac{1}{x^m (f^n + g^n x^n)}$ into its simple fractions.

Because it retains the factors of $f^n + g^n x^n$, from these the same fractions arise that we elicited in the preceding example, provided here m is taken negative; therefore there only remains, in order that we define the simple fractions of the denominator from the other factor x^n , which is most conveniently done in this manner. The proposed fraction is put in place, equal to

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1240

$$\frac{\mathfrak{A}}{x^m} + \frac{\mathfrak{N}x^{n-m}}{(f^n + g^n x^n)}$$

and there becomes

$$\mathfrak{A}f^n = 1 \quad \text{hence} \quad \mathfrak{A} = \frac{1}{f^n}$$

$$\mathfrak{A}g^n + \mathfrak{N} = 0 \quad \mathfrak{N} = -\frac{g^n}{f^n}.$$

If at this stage $n - m$ should be a negative number, in a like manner it is to be worked out thus in order that, if m should be however large a number, there results simple fractions of this kind

$$\frac{\mathfrak{A}}{x^m} + \frac{\mathfrak{B}}{x^{m-n}} + \frac{\mathfrak{C}}{x^{m-2n}} + \frac{\mathfrak{D}}{x^{m-3n}} + \text{etc.},$$

the whole series of terms of which are to be taken, as long as there is a positive exponent of x in the denominator. And there comes about :

$$\mathfrak{A}f^n = 1 \quad \text{hence} \quad \mathfrak{A} = \frac{1}{f^n}$$

$$\mathfrak{A}g^n + \mathfrak{B}f^n = 0 \quad \mathfrak{B} = -\frac{g^n}{f^{2n}}$$

$$\mathfrak{B}g^n + \mathfrak{C}f^n = 0 \quad \mathfrak{C} = +\frac{g^{2n}}{f^{3n}}$$

$$\mathfrak{C}g^n + \mathfrak{D}f^n = 0 \quad \mathfrak{D} = -\frac{g^{3n}}{f^{4n}}$$

etc.

etc.

Hence the proposed fraction can be resolved into these simple fractions :

$$\begin{aligned} & \frac{1}{f^n x^m} - \frac{g^n}{f^{2n} x^{m-n}} + \frac{g^{2n}}{f^{3n} x^{m-2n}} - \frac{g^{3n}}{f^{4n} x^{m-3n}} + \text{etc.} \\ & \frac{2fg^m \sin \frac{\pi}{n} \sin \frac{(m-1)\pi}{n} + 2g^m \cos \frac{(m-1)\pi}{n} (gx - f \cos \frac{\pi}{n})}{nf^{n+m-1} (ff - 2fgx \cos \frac{\pi}{n} + ggxx)} \\ & \frac{2fg^m \sin \frac{3\pi}{n} \sin \frac{3(m-1)\pi}{n} + 2g^m \cos \frac{3(m-1)\pi}{n} (gx - f \cos \frac{3\pi}{n})}{nf^{n+m-1} (ff - 2fgx \cos \frac{3\pi}{n} + ggxx)} \\ & \frac{2fg^m \sin \frac{5\pi}{n} \sin \frac{5(m-1)\pi}{n} + 2g^m \cos \frac{5(m-1)\pi}{n} (gx - f \cos \frac{5\pi}{n})}{nf^{n+m-1} (ff - 2fgx \cos \frac{5\pi}{n} + ggxx)} \\ & \text{etc.} \end{aligned}$$

From which formulas, if n should be an odd number, on account of $f + gx$, the factor of the above denominator must be added

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1241

$$\frac{\pm g^m}{nf^{m+m-1}(f+gx)},$$

where the upper of the ambiguous signs prevails, if m should be an even number, and indeed the lower if m should be odd.

418. Now we can consider the formula $a + bx^n$ too, if b should be a negative number, and let this function be proposed,

$$f^n - g^n x^n,$$

and in the first place $f - gx$ is always a factor; and if n is an even number, also $f + gx$ is a factor of this. Now the remainder are indeed trinomials; of which the general form, if there is put

$$ff - 2fgx \cos \varphi + ggxx$$

will be

$$f^n - f^n \cos n\varphi = 0 \quad \text{and} \quad f^n \sin \varphi = 0$$

or

$$\sin n\varphi = 0 \quad \text{and} \quad \cos n\varphi = 1.$$

With which in order that it is satisfied, it is required that $n\varphi = 2k\pi$ with some integer number k present and therefore there becomes $\varphi = \frac{2k\pi}{n}$. Hence the general factor is

$$ff - 2fgx \cos \frac{2k\pi}{n} + ggxx;$$

hence by taking for $2k$ all the even powers less than the exponent n they produce all the trinomial factors

$$ff - 2fgx \cos \frac{2\pi}{n} + ggxx$$

$$ff - 2fgx \cos \frac{4\pi}{n} + ggxx$$

$$ff - 2fgx \cos \frac{6\pi}{n} + ggxx$$

etc.

EXAMPLE 1

To resolve this fraction $\frac{x^m}{f^n - g^n x^n}$ into its simple fractions.

Because $f - gx$ is a factor of the denominator, thus a fraction of this kind arises $\frac{\mathfrak{A}}{f - gx}$;

to find the numerator there is put in place $x^m = P$ and $f^n - g^n x^n = Q$;

then there arises

$$dQ = -ng^n x^{n-1} dx$$

and there becomes

$$\mathfrak{A} = \frac{-gx^m}{-ng^n x^{n-1}} = \frac{x^m}{ng^{n-1} x^{n-1}}$$

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1242

on putting $x = \frac{f}{g}$. Hence there becomes $\mathfrak{A} = \frac{1}{nf^{n-m-1}g^m}$ and the simple fraction arising from the factor $f - gx$ is

$$\frac{1}{nf^{n-m-1}g^m(f-gx)}$$

If n is an even number, because also $f + gx$ is a factor of the denominator, hence a simple fraction equal to $\frac{\mathfrak{A}}{f+gx}$ is put in place ; then

$$\mathfrak{A} = \frac{-gx^m}{ng^n x^{n-1}} = \frac{-x^m}{ng^{n-1}x^{n-1}},$$

on putting $x = \frac{-f}{g}$. Hence there becomes on account of $n-1$ being an odd number

$ng^{n-1}x^{n-1} = -f^{n-1}$; but then $x^m = \frac{\pm f^m}{g^m}$, where the upper sign prevails, if m is an even

number, and the lower, if m is odd. Whereby since $\mathfrak{A} = \frac{\mp 1}{nf^{n-m-1}g^m}$ then this simple

fraction arises from the factor $f + gx$:

$$\mathfrak{A} = \frac{\mp 1}{nf^{n-m-1}g^m(f+gx)}.$$

Then since the general form of trinomial factors is

$$ff - 2fgx \cos \frac{2k\pi}{n} + ggxx,$$

if a comparison with example 1 § 416 is put in place, then

$a = f^n$, $b = -g^n$ and $\varphi = \frac{2k\pi}{n}$; from which

$$\sin n\varphi = 0 \text{ and } \cos n\varphi = 1$$

and

$$\sin(n-m-1)\varphi = -\sin(m+1)\varphi = -\sin \frac{2k(m+1)\pi}{n}$$

and

$$\cos(n-m-1)\varphi = \cos(m+1)\varphi = \cos \frac{2k(m+1)\pi}{n}$$

From which hence this simple factor arises :

$$\frac{2f \sin \frac{2k\pi}{n} \cdot \sin \frac{2k(m+1)\pi}{n} - 2 \cos \frac{2k(m+1)\pi}{n} (gx - f \cos \frac{2k\pi}{n})}{nf^{n-m-1}g^m(ff - 2fgx \cos \frac{2k\pi}{n} + ggxx)}$$

On account of the above, these are the simple fractions sought arising :

EULER'S INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1243

$$\begin{aligned} & \frac{1}{nf^{n-m-1}g^m(f-gx)} \\ & + \frac{2f \sin \frac{2\pi}{n} \cdot \sin \frac{2(m+1)\pi}{n} - 2 \cos \frac{2(m+1)\pi}{n} (gx - f \cos \frac{2\pi}{n})}{nf^{n-m-1}g^m(ff - 2fgx \cos \frac{2\pi}{n} + ggxx)} \\ & + \frac{2f \sin \frac{4\pi}{n} \cdot \sin \frac{4(m+1)\pi}{n} - 2 \cos \frac{4(m+1)\pi}{n} (gx - f \cos \frac{4\pi}{n})}{nf^{n-m-1}g^m(ff - 2fgx \cos \frac{4\pi}{n} + ggxx)} \\ & + \frac{2f \sin \frac{6\pi}{n} \cdot \sin \frac{6(m+1)\pi}{n} - 2 \cos \frac{6(m+1)\pi}{n} (gx - f \cos \frac{6\pi}{n})}{nf^{n-m-1}g^m(ff - 2fgx \cos \frac{6\pi}{n} + ggxx)} \\ & \text{etc.,} \end{aligned}$$

from which, if n should be an even number, this fraction must be added above

$$\mathfrak{A} = \frac{\mp 1}{nf^{n-m-1}g^m(f+gx)},$$

of which the upper sign $-$ is to be taken, if m is a positive number, and the lower if odd. Now in addition, if m is a number not less than n , the integral parts are to be added

$$Ax^{m-n} + Bx^{m-2n} + Cx^{m-3n} + Dx^{m-4n} + \text{etc.},$$

as long as the exponents do not become negative, and then

$$\begin{aligned} -Ag^n &= 1 \quad \text{or} \quad A = \frac{-1}{g^n} \\ Af^n - Bg^n &= 0 \quad B = -\frac{f^n}{g^{2n}} \\ Bf^n - Cg^n &= 0 \quad C = -\frac{f^{2n}}{g^{3n}} \\ Cf^n - Dg^n &= 0 \quad D = -\frac{f^{3n}}{g^{4n}} \\ \text{etc.} & \qquad \qquad \text{etc.} \end{aligned}$$

EXAMPLE 2

To resolve this fraction $\frac{1}{x^m(f^n - g^n x^n)}$ into its simple fractions.

Fractions, which arise from the factor of the denominator $f^n - g^n x^n$, are the same as before, provided m is taken negative in these formulas. Whereby consideration is to be given to the other factor x^m ; from which if we put these factors to be returned

EULER'S INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1244

$$\frac{\mathfrak{A}}{x^m} + \frac{\mathfrak{B}}{x^{m-n}} + \frac{\mathfrak{C}}{x^{m-2n}} + \frac{\mathfrak{D}}{x^{m-3n}} + \text{etc.},$$

which series from these is to be continued until the exponents of x become negative, then

$$\mathfrak{A}f^n = 1 \quad \text{hence} \quad \mathfrak{A} = \frac{1}{f^n}$$

$$\mathfrak{B}f^n - \mathfrak{A}g^n = 0 \quad \mathfrak{B} = \frac{g^n}{f^{2n}}$$

$$\mathfrak{C}f^n - \mathfrak{B}g^n = 0 \quad \mathfrak{C} = \frac{g^{2n}}{f^{3n}}$$

$$\mathfrak{D}f^n - \mathfrak{C}g^n = 0 \quad \mathfrak{C} = \frac{g^{3n}}{f^{4n}}$$

etc. etc.

Hence the fraction proposed can be resolved into these simple fractions:

$$\begin{aligned} & \frac{1}{f^n x^m} + \frac{g^n}{f^{2n} x^{m-n}} + \frac{g^{2n}}{f^{3n} x^{m-2n}} + \frac{g^{3n}}{f^{4n} x^{m-3n}} + \text{etc.} \\ & + \frac{g^m}{nf^{n+m-1}(f-gx)} \\ & - \frac{2fg^m \sin \frac{2\pi}{n} \sin \frac{2(m-1)\pi}{n} + 2g^m \cos \frac{2(m-1)\pi}{n} (gx - f \cos \frac{2\pi}{n})}{nf^{n+m-1} (ff - 2fgx \cos \frac{2\pi}{n} + ggxx)} \\ & - \frac{2fg^m \sin \frac{4\pi}{n} \sin \frac{4(m-1)\pi}{n} + 2g^m \cos \frac{4(m-1)\pi}{n} (gx - f \cos \frac{4\pi}{n})}{nf^{n+m-1} (ff - 2fgx \cos \frac{4\pi}{n} + ggxx)} \\ & - \frac{2fg^m \sin \frac{6\pi}{n} \sin \frac{6(m-1)\pi}{n} + 2g^m \cos \frac{6(m-1)\pi}{n} (gx - f \cos \frac{6\pi}{n})}{nf^{n+m-1} (ff - 2fgx \cos \frac{6\pi}{n} + ggxx)} \\ & \text{etc.} \end{aligned}$$

for which, if n should be even, this fraction must be added above :

$$\frac{\mp g^m}{nf^{n+m-1}(f-gx)}$$

but which can be omitted if n should be an odd number. Now the upper – of the two signs prevails if m is an even number, and the lower +, if m is an odd number.

419. Hence in this manner all the fractions, of which the denominator is composed from two members of the kind $a + bx^n$, can be resolved into simple fractions. But if the denominator should be constructed from three members of the kind $a + bx^n + cx^{2n}$, then it is to be observed initially, that each can be resolved in the first place into two real

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1245

factors. For if this comes about, the resolution into simple fractions can be put in place in the way described before. For if a fraction of this kind is put in place

$$\frac{x^m}{(f^n + g^n x^n)(f^n + h^n x^n)},$$

that initially is transformed into two factors of this kind

$$\frac{\alpha x^m}{f^n + g^n x^n} + \frac{\beta x^m}{f^n + h^n x^n},$$

and then

$$\alpha f^n + \beta f^n = 1 \quad \text{and} \quad \alpha h^n + \beta g^n = 0,$$

from which

$$\alpha = \frac{1}{f^n} - \beta = -\frac{\beta g^n}{h^n},$$

and thus there is found

$$\beta = \frac{h^n}{f^n(h^n - g^n)} \quad \text{and} \quad \alpha = \frac{g^n}{f^n(g^n - h^n)}$$

If the exponent m is greater than n , a change into the following fractions is more convenient

$$\frac{\alpha x^{m-n}}{f^n + g^n x^n} + \frac{\beta x^{m-n}}{f^n + h^n x^n},$$

from which there becomes

$$\alpha + \beta = 0 \quad \text{and} \quad \alpha h^n + \beta g^n = 1$$

and thus

$$\alpha = \frac{1}{h^n - g^n} \quad \text{and} \quad \beta = \frac{1}{g^n - h^n}.$$

But whichever transformation is used, each fraction arising is resolved into its simple fractions in the manner established previously by the method, which together are equal to the proposed fractions.

420. In a similar manner the method treated up to this stage is sufficient, if the denominator is constructed from a number of terms of this kind

$$a + bx^n + cx^{2n} + dx^{3n} + ex^{4n} + \text{etc.},$$

provided this can be resolved into factors of the form $f^n \pm g^n x^n$. For we put this fraction to be resolved into its simple factors:

$$\frac{x^m}{(a-x^n)(b-x^n)(c-x^n)(d-x^n) \text{ etc.}}.$$

This in the first place is resolved into these :

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1246

$$\frac{Ax^m}{a-x^n} + \frac{Bx^m}{b-x^n} + \frac{Cx^m}{c-x^n} + \frac{Dx^m}{d-x^n} + \text{etc.},$$

the numerators of which are determined in following manner, in order that there becomes :

$$A = \frac{1}{(b-a)(c-a)(d-a) \text{ etc.}}$$

$$B = \frac{1}{(a-b)(c-b)(d-b) \text{ etc.}}$$

$$C = \frac{1}{(a-c)(b-c)(d-c) \text{ etc.}}$$

etc.

Hence with these preparations made these individual fractions themselves can be resolved into their simple fractions by the method shown previously; which are all collected together into one sum.

421. But if now a denominator of this kind

$$a + bx^n + cx^{2n} + dx^{3n} + ex^{4n} + \text{etc.},$$

does not have all the real factors of the form $f^n + g^n x^n$, two imaginary [factors] are to be taken together. Hence we put the product of the two factors to be of the form :

$$f^{2n} - 2f^n g^n x^n \cos \omega + g^{2n} x^{2n},$$

and since this expression does not have simple real factors, we put the trinomial factors to be contained in this general form :

$$ff - 2fgx \cos \varphi + ggxx,$$

the number of which is equal to n . Hence on putting $x^n = \frac{f^n}{g^n} \cos n\varphi$ this equation arises :

$$1 - 2 \cos \omega \cdot \cos n\varphi + \cos 2n\varphi = 0.$$

Then on putting $x^n = \frac{f^n}{g^n} \sin n\varphi$ this equation also :

$$-2 \cos \omega \cdot \sin n\varphi + \sin 2n\varphi = 0,$$

which divided by $\sin n\varphi$ gives $\cos n\varphi = \cos \omega$ and thus likewise the first equation is satisfied. Hence there becomes $n\varphi = 2k\pi \pm \omega$ with k denoting some integer and thus there becomes $\varphi = \frac{2k\pi \pm \omega}{n}$ and all the factors are contained in this form :

$$ff - 2fgx \cos \frac{2k\pi \pm \omega}{n} + ggxx,$$

from which the following factors are obtained :

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1247

$$ff - 2fgx \cos \frac{\omega}{n} + ggxx$$

$$ff - 2fgx \cos \frac{2\pi - \omega}{n} + ggxx$$

$$ff - 2fgx \cos \frac{2\pi + \omega}{n} + ggxx$$

$$ff - 2fgx \cos \frac{4\pi - \omega}{n} + ggxx$$

$$ff - 2fgx \cos \frac{4\pi + \omega}{n} + ggxx$$

etc.,

of which as many are to be taken until the number becomes equal to n .

422. Therefore if there is put in place this fraction to be resolved into its simplest fractions :

$$\frac{x^{m-1}}{f^{2n} - 2f^n g^n x^n \cos \omega + g^{2n} x^{2n}},$$

because the factor of the denominator of any trinomial is contained in this form

$$ff - 2fgx \cos \varphi + ggxx$$

with $\varphi = \frac{2k\pi \pm \omega}{n}$ arising, this fraction is considered :

$$\frac{x^m}{f^{2n} x - 2f^n g^n x^{n+1} \cos \omega + g^{2n} x^{2n+1}}$$

and the numerator for that is put $x^m = P$ and the denominator

$$f^{2n} x - 2f^n g^n x^{n+1} \cos \omega + g^{2n} x^{2n+1} = Q;$$

then

$$\frac{dQ}{dx} = f^{2n} - 2f^n (n+1) g^n x^n \cos \omega + (2n+1) g^{2n} x^{2n}.$$

Hence there is put in place

$$x^n = \frac{f^n}{g^n} \cos n\varphi$$

then

$$\mathfrak{P} = \frac{f^m}{g^m} \cos m\varphi \quad \text{or} \quad \mathfrak{P} = \frac{f^m}{g^m} \cos \frac{m(2k\pi \pm \omega)}{n}$$

and

$$\mathfrak{Q} = f^{2n} (1 - 2(n+1) \cos \omega \cdot \cos n\varphi + (2n+1) \cos 2n\varphi).$$

But since there is the relation $\cos n\varphi = \cos \omega$, then

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1248

$$\cos 2n\varphi = 2 \cos^2 \omega - 1$$

and thus

$$\Omega = f^{2n} \left(-2n + 2n \cos^2 \omega \right) = -2nf^{2n} \sin^2 \omega.$$

Then on putting

$$x^n = \frac{f^n}{g^n} \sin n\varphi$$

there arises

$$p = \frac{f^m}{g^m} \sin m\varphi = \frac{f^m}{g^m} \sin \frac{m(2k\pi \pm \omega)}{n}$$

and

$$q = -f^{2n} \left(2(n+1) \cos \omega \cdot \sin n\varphi - (2n+1) \sin 2n\varphi \right);$$

and on account of

$$\sin 2n\varphi = 2 \sin n\varphi \cdot \cos n\varphi = 2 \cos \omega \cdot \sin n\varphi$$

then

$$q = 2nf^{2n} \cos \omega \cdot \sin n\varphi.$$

But since there is $n\varphi = 2k\pi \pm \omega$, then $\sin n\varphi = \pm \sin \omega$ and

$$q = \pm 2nf^{2n} \sin \omega \cdot \cos \omega.$$

From which there is found :

$$\Omega^2 + q^2 = 4n^2 f^{4n} \sin^2 \omega,$$

$$\mathfrak{P}q - p\Omega = \frac{2nf^{m+2n}}{g^m} \left(\pm \cos m\varphi \cdot \sin \omega \cdot \cos \omega + \sin m\varphi \cdot \sin^2 \omega \right),$$

or if

$$\mathfrak{P}q - p\Omega = \pm \frac{2nf^{m+2n}}{g^m} \sin \omega \cdot \cos(m\varphi \mp \omega)$$

or

$$\mathfrak{P}q - p\Omega = \pm \frac{2nf^{m+2n}}{g^m} \sin \omega \cdot \cos \left(\frac{2km\pi \pm (m-n)\omega}{n} \right),$$

$$\mathfrak{P}\Omega + pq = \frac{2nf^{m+2n}}{g^m} \left(-\cos m\varphi \cdot \sin^2 \omega \pm \sin m\varphi \cdot \cos \omega \right),$$

$$\mathfrak{P}\Omega + pq = \pm \frac{2nf^{m+2n}}{g^m} \sin \omega \cdot \sin(m\varphi \mp \omega)$$

or

$$\mathfrak{P}\Omega + pq = \pm \frac{2nf^{m+2n}}{g^m} \sin \omega \cdot \sin \left(\frac{2km\pi \pm (m-n)\omega}{n} \right).$$

Hence from the factor of the denominator

$$ff - 2fgx \cos \frac{2k\pi \pm \omega}{n} + ggxx$$

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2
Chapter 18

Translated and annotated by Ian Bruce.

1249

this simple fraction arises

$$\frac{\pm f \sin \frac{2k\pi \pm \omega}{n} \cdot \cos \frac{2km\pi \pm (m-n)\omega}{n} \pm \sin \frac{2km\pi \pm (m-n)\omega}{n} (gx - f \cos \frac{2k\pi \pm \omega}{n})}{nf^{2n-m} g^{m-1} \sin \omega (ff - 2fgx \cos \frac{2k\pi \pm \omega}{n} + ggxx)}$$

or

$$\frac{\pm gx \sin \frac{2km\pi \pm (m-n)\omega}{n} \pm f \sin \frac{2k(m-1)\pi \pm (m-n-1)\omega}{n}}{nf^{2n-m} g^{m-1} \sin \omega (ff - 2fgx \cos \frac{2k\pi \pm \omega}{n} + ggxx)}$$

EXAMPLE

To resolve this fraction $\frac{x^{m-1}}{f^{2n} - 2f^n g^n x^n \cos \omega + g^{2n} x^{2n}}$ into its simple fractions.

These simple fractions sought thus are :

$$\begin{aligned} & \frac{f \sin \frac{\omega}{n} \cdot \cos \frac{(m-n)\omega}{n} + \sin \frac{(m-n)\omega}{n} (gx - f \cos \frac{\omega}{n})}{nf^{2n-m} g^{m-1} \sin \omega (ff - 2fgx \cos \frac{\omega}{n} + ggxx)} \\ & - \frac{f \sin \frac{2\pi - \omega}{n} \cdot \cos \frac{2m\pi - (m-n)\omega}{n} + \sin \frac{2m\pi - (m-n)\omega}{n} (gx - f \cos \frac{2\pi - \omega}{n})}{nf^{2n-m} g^{m-1} \sin \omega (ff - 2fgx \cos \frac{2\pi - \omega}{n} + ggxx)} \\ & + \frac{f \sin \frac{2\pi + \omega}{n} \cdot \cos \frac{2m\pi + (m-n)\omega}{n} + \sin \frac{2m\pi + (m-n)\omega}{n} (gx - f \cos \frac{2k\pi + \omega}{n})}{nf^{2n-m} g^{m-1} \sin \omega (ff - 2fgx \cos \frac{2k\pi + \omega}{n} + ggxx)} \\ & - \frac{f \sin \frac{4\pi - \omega}{n} \cdot \cos \frac{4m\pi - (m-n)\omega}{n} + \sin \frac{4m\pi - (m-n)\omega}{n} (gx - f \cos \frac{4\pi - \omega}{n})}{nf^{2n-m} g^{m-1} \sin \omega (ff - 2fgx \cos \frac{4\pi - \omega}{n} + ggxx)} \\ & + \frac{f \sin \frac{4\pi + \omega}{n} \cdot \cos \frac{4m\pi + (m-n)\omega}{n} + \sin \frac{4m\pi + (m-n)\omega}{n} (gx - f \cos \frac{4k\pi + \omega}{n})}{nf^{2n-m} g^{m-1} \sin \omega (ff - 2fgx \cos \frac{4k\pi + \omega}{n} + ggxx)} \\ & \text{etc.} \end{aligned}$$

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1250

and thus with these progressing to that point until the number of these fraction becomes equal to n . If m should be a number either greater than $2n - 1$ or a negative number, in the first case the integral parts, and in the second case the above fractions are to be added , which are easily found in the manner shown before.

CAPUT XVIII

**DE USU CALCULI DIFFERENTIALIS
IN RESOLUTIONE FRACTIONUM**

403. Methodus fractionem quamvis propositam in fractiones simplices resolvendi, quam in *Introductione* exposuimus, etsi per se satis est facilis, tamen ope calculi differentialis ita perfici potest, ut saepenumero multo minori negotio in usum vocari possit. Praecipue vero si denominator fractionis resolvendae fuerit indefiniti gradus, methodus ante exposita plerumque non mediocriter impeditur, dum loco quantitatis incognitae substitutio valoris, quem ex quopiam factore induit, fieri debet. Imprimis autem his casibus divisio denominatoris per factorem iam inventum nimis fit molesta. Quae operatio, si calculus differentialis in subsidium vocetur, evitari poterit, ita ut non opus sit alterum denominatoris factorem, qui oritur, si denominator per factorem iam cognitum dividatur, nosse. Hunc autem usum praestat methodus determinandi valorem fractionis, cuius numerator ac denominator certo casu ambo evanescunt; cuius beneficia quemadmodum resolutio fractionum iam supra tradita commodior et tractabilior reddi

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1251

queat, hoc capite doceamus simulque finem huic libro, in quo usum calculi differentialis in Analysi exposuimus, imponamus.

404. Si igitur proposita fuerit fractio quaecunque $\frac{P}{Q}$ cuius numerator ac denominator sint functiones variabilis quantitatis x rationales et integrae, primum videndum est, utrum x in numeratore P tot pluresve dimensiones habeat quam in denominatore Q . Quod si eveniat, complectetur fractio $\frac{P}{Q}$ in se partem integram huius formae $A + Bx + Cx^2 + \text{etc.}$, quae divisionis ope inde erui poterit; pars reliqua erit fractio eundem denominatorem Q habens, sed cuius numerator erit functio, puta R , pauciores ipsius x dimensiones continens quam denominator Q , ita ut ulterior resolutio instituenda sit in fractione $\frac{R}{Q}$. Interim tamen non opus est nosse hunc novum numeratorem R , sed eaedem fractiones simplices, quas fractio $\frac{R}{Q}$ suppeditatura esset, elici possunt immediate ex fractione proposita $\frac{P}{Q}$, prouti iam supra notavimus.

405. Praeter partem integram igitur, si quam continet fractio $\frac{P}{Q}$, erui debent fractiones simplices, quarum denominatores sint vel binomiales huius formae $f + gx$ vel trinomiales huiusmodi $ff + 2x \cos \varphi \sqrt{fg} + gxx$ vel eiusmodi formularum quadrata cubive seu altiores potestates. Hique denominatores omnes erunt factores denominatoris Q , ita ut quilibet denominatoris ipsius Q factor praebeat fractionem simplicem. Scilicet si denominator Q factorem habeat $f + gx$, ex eo nascetur fractio simplex huiusmodi

$$\frac{\mathfrak{A}}{f + gx},$$

sin autem factor fuerit $(f + gx)^2$, binae fractiones

$$\frac{\mathfrak{A}}{(f + gx)^2} + \frac{\mathfrak{B}}{f + gx}.$$

Atque ex denominatoris Q factore cubico $(f + gx)^3$ orientur tres fractiones simplices huius formae

$$\frac{\mathfrak{A}}{(f + gx)^3} + \frac{\mathfrak{B}}{(f + gx)^2} + \frac{\mathfrak{C}}{f + gx}$$

et ita porro. Quodsi autem denominator Q factorem habuerit trinomialem huiusmodi $ff - 2xfg \cos \varphi + gxx$, ex eo oriatur fractio simplex talis formae

$$\frac{\mathfrak{A} + \alpha x}{ff - 2xfg \cos \varphi + gxx},$$

et si duo huiusmodi factores fuerint aequales uti $(ff - 2xfg \cos \varphi + gxx)^2$,

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1252

hinc prodibunt duae fractiones

$$\frac{\mathfrak{A}+ax}{(ff-2xfg \cos \varphi+gxx)^2} + \frac{\mathfrak{B}+bx}{ff-2xfg \cos \varphi+gxx}.$$

Huiusmodi autem factor cubicus $(ff - 2fgx \cos \varphi + gxx)^3$ dabit tres fractiones simplices, biquadratus quatuor et ita porro.

406. Resolutio ergo fractionis cuiuscunque $\frac{P}{Q}$ ita instituat. Quaerantur primo omnes factores tam simplices seu binomiales quam trinomiales denominatoris Q , et si qui fuerint inter se aequales, ii probe notentur et instar unius habeantur. Tum ex singulis his denominatoris factoribus elicantur fractiones simplices vel modo iam supra ostenso vel eo, quem hic sumus tradituri et qui pro lubitu in locum prioris substitui poterit. Quo facto aggregatum omnium istarum fractionum simplicium una cum parte integra, si quam continet fractio proposita $\frac{P}{Q}$, huius valorem exhaurient. Inventionem quidem factorum denominatoris Q hic tanquam cognitam assumimus, cum pendeat a resolutione aequationis $Q = 0$, methodumque hic trademus per calculum differentialem pro dato quovis denominatoris factore fractionem simplicem inde ortam definiendi. Quod, cum istarum fractionum simplicium denominatores iam habeantur, praestabitur, si numeratorem cuiusque fractionis investigare doceamus.

407. Ponamus ergo fractionis $\frac{P}{Q}$ denominatorem Q factorem habere $f + gx$, ita ut sit $Q = (f + gx) S$ neque vero hic alter factor S insuper eundem factorem $f + gx$ contineat. Sit fractio simplex ex isto factore orta $= \frac{\mathfrak{A}}{f+gx}$ et complementum huiusmodi formam habeat $\frac{V}{S}$, ita ut sit

$$\frac{\mathfrak{A}}{f+gx} + \frac{V}{S} = \frac{P}{Q}.$$

Erit ergo

$$\frac{V}{S} = \frac{P}{Q} - \frac{\mathfrak{A}}{f+gx} = \frac{P - \mathfrak{A}S}{(f+gx)S}$$

ideoque

$$V = \frac{P - \mathfrak{A}S}{f+gx}$$

Cum igitur V sit functio integra ipsius x , necesse est, ut $P - \mathfrak{A}S$ sit divisibile per $f + gx$; ac propterea, si ponatur $f + gx = 0$ seu $x = \frac{-f}{g}$, expressio $P - \mathfrak{A}S$ evanescet. Fiat ergo $x = \frac{-f}{g}$, et cum sit $P - \mathfrak{A}S = 0$, erit $\mathfrak{A} = \frac{P}{S}$, uti iam supra invenimus. Cum autem sit

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1253

$$S = \frac{Q}{f+gx}, \text{ fiet}$$

$$\mathfrak{A} = \frac{(f+gx)P}{Q}$$

si ubique ponatur $f + gx = 0$ seu $x = \frac{-f}{g}$. Quoniam vera hoc casu tam numerator $(f + gx)P$ quam denominator Q evanescit, per ea, quae de valore huiusmodi fractionum investigando exposuimus, erit

$$\mathfrak{A} = \frac{(f+gx)dP + Pgdx}{dQ},$$

si quidem ponatur $x = \frac{-f}{g}$. Hoc autem casu ob $(f + gx)dP = 0$ erit

$$\mathfrak{A} = \frac{Pgdx}{dQ}$$

sicque per differentiationem valor numeratoris & expedite reperitur.

407. Ponamus ergo fractionis $\frac{P}{Q}$ denominatorem Q factorem habere $f + gx$, ita ut sit $Q = (f + gx)S$ neque vero hic alter factor S insuper eundem factorem $f + gx$ contineat. Sit fractio simplex ex isto factore orta = $\frac{\mathfrak{A}}{f+gx}$ et complementum huiusmodi formam habeat $\frac{V}{S}$, ita ut sit

$$\frac{\mathfrak{A}}{f+gx} + \frac{V}{S} = \frac{P}{Q}$$

Erit ergo

$$\frac{V}{S} = \frac{P}{Q} - \frac{\mathfrak{A}}{f+gx} = \frac{P}{(f+gx)S} - \frac{\mathfrak{A}}{f+gx} = \frac{P - \mathfrak{A}S}{(f+gx)S}$$

ideoque

$$V = \frac{P - \mathfrak{A}S}{(f+gx)},$$

Cum igitur V sit functio integra ipsius x , necesse est, ut $P - \mathfrak{A}S$ sit divisibile per $f + gx$; ac propterea, si ponatur $f + gx = 0$ seu $x = \frac{-f}{g}$, expressio $P - \mathfrak{A}S$ evanescet. Fiat ergo $x = \frac{-f}{g}$, et cum sit $P - \mathfrak{A}S = 0$, erit $\mathfrak{A} = \frac{P}{S}$, uti iatu supra invenimus.

Cum autem sit $S = \frac{Q}{f+gx}$ fiet

$$\mathfrak{A} = \frac{(f + gx)P}{Q}$$

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1254

si ubique ponatur $f + gx = 0$ seu $x = \frac{-f}{g}$. Quoniam vera hoc casu tam numerator $(f + gx)P$ quam denominator Q evanescit, per ea, quae de valore huiusmodi fractionum investigando exposuimus, erit

$$\mathfrak{A} = \frac{(f + gx)dP + Pgdx}{dQ}$$

si quidem ponatur $x = \frac{-f}{g}$. Hoc autem casu ob $(f + gx)dP = 0$ erit

$$\mathfrak{A} = \frac{Pgdx}{dQ}$$

sicque per differentiationem valor numeratoris \mathfrak{A} expediter reperitur.

408. Si igitur fractionis propositae $\frac{P}{Q}$ denominator Q factorem habeat simplicem $f + gx$, ex eo orietur fractio simplex

$$\frac{\mathfrak{A}}{f + gx}$$

existente $\mathfrak{A} = \frac{Pgdx}{dQ}$, postquam hic ubique loco x valor $\frac{-f}{g}$ ex aequatione $f + gx = 0$

oriundus fuerit substitutus. Hoc ergo modo non necesse est, ut ante quaeratur alter denominatoris Q factor S , qui oritur, si Q per $f + gx$ dividatur. Hinc, si Q non in factoribus exprimatur, hanc divisionem saepe non parum molestam, praecipue si x in denominatore Q habeat exponentes indefinitos, omittere poterimus, cum valor \mathfrak{A} ipsius ex formula $\frac{Pgdx}{dQ}$

obteneatur. Sin autem denominator Q iam in factoribus fuerit expressus, ita ut inde valor ipsius S sponte pateat, tum praefenda erit altera expressio, qua invenimus $\mathfrak{A} = \frac{P}{S}$

ponendo pariter ubique $x = \frac{-f}{g}$. Sicque pro inveniendone valore ipsius \mathfrak{A} quovis casu ea formula adhiberi poterit, quae commodior et expeditior videatur. Usus autem novae formulae aliquot exemplis illustrabimus.

EXEMPLUM 1

Sit proposita ista fractio $\frac{x^9}{1+x^{17}}$ cuius fractionem simplicem ex denominatoris factore $1 + x$ oriundam definiri oporteat.

Quoniam hic est $Q = 1 + x^{17}$, cuius etsi factor $1 + x$ constat, tamen, si, uti prima methodus postulat, per eum dividere velimus, prodiret

$$S = 1 - x + xx - x^3 + \dots + x^{16}$$

Commodius igitur utemur nova formula $\mathfrak{A} = \frac{gPdx}{dQ}$; quia itaque est $f = 1$, $g = 1$ et $P = x^9$,

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1255

ob $dQ = 17x^{16}dx$ fiet $\mathfrak{A} = \frac{x^9}{17x^{16}} = \frac{1}{17x^7}$ posito $x = -1$, unde fit $\mathfrak{A} = -\frac{1}{17}$ et fractio simplex ex denominatoris factore $1+x$ oriunda erit $\frac{-1}{17(1+x)}$.

EXEMPLUM 2

Proposita fractione $\frac{x^m}{1-x^{2n}}$ fractionem simplicem ex denominatoris factore $1-x$ oriundam investigare.

Ob factorem propositum $1-x$ erit $f = 1$ et $g = -1$. Tum vero denominator $Q = 1 - x^{2n}$ dat $dQ = -2nx^{2n-1}dx$, unde propter $P = x^m$ obtinebitur $\mathfrak{A} = \frac{-x^m}{-2nx^{2n-1}}$. Positoque, ex aequatione $1-x = 0$, $x = 1$ fiet $\mathfrak{A} = \frac{1}{2n}$, ita ut fractio simplex futura sit haec

$$\frac{1}{2n(1-x)}$$

EXEMPLUM 3

Proposita fractione $\frac{x^m}{1-4x^k+3x^n}$ eius fractionem simplicem ex denominatoris factore $1-x$ oriundam determinare.

Hic ergo fit $f = 1$, $g = -1$, $P = x^m$, et $Q = 1 - 4x^k + 3x^n$ et $\frac{dQ}{dx} = -4kx^{k-1} + 3nx^{n-1}$; unde fit $\mathfrak{A} = \frac{-x^m}{-4kx^{k-1} + 3nx^{n-1}}$ et posito $x = 1$ erit $\mathfrak{A} = \frac{1}{4k-3n}$. Fractio ergo simplex ex isto denominatoris factore simplici $1-x$ oriunda erit

$$\frac{1}{(4k-3n)(1-x)}$$

409. Ponamus nunc fractionis $\frac{P}{Q}$ denominatorem Q factorem habere quadratum $(f+gx)^2$ et fractiones simplices hinc oriundas esse

$$\frac{\mathfrak{A}}{(f+gx)^2} + \frac{\mathfrak{B}}{f+gx}$$

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1256

Sit $Q = (f + gx)^2 S$ et complementum $= \frac{V}{S}$; ita ut sit

$$\frac{V}{S} = \frac{P}{Q} - \frac{\mathfrak{A}}{(f+gx)^2} - \frac{\mathfrak{B}}{f+gx} \text{ et } V = \frac{P - \mathfrak{A}S - \mathfrak{B}(f+gx)S}{(f+gx)^2}.$$

Quia nunc Vest functio integra, necesse est, ut sit $P - \mathfrak{A}S - \mathfrak{B}(f + gx)S$

divisibile per $(f + gx)^2$; et cum S factorem $f + gx$ amplius non contineat,

quoque haec expressio $\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f + gx)$ divisibilis erit per $(f + gx)^2$

ideoque facto $f + gx = 0$ seu $x = -\frac{f}{g}$ non solum ipsa, sed etiam eius differentiale

$d.\frac{P}{S} - \mathfrak{B}gdx$ evanescet. Fiat ergo $x = -\frac{f}{g}$ eritque ex priori aequatione $\mathfrak{A} = \frac{P}{S}$, ex

posteriori vero erit $\mathfrak{B} = \frac{1}{gdx} d.\frac{P}{S}$; quibus valoribus inventis habebuntur fractiones

quaesitae

$$\frac{\mathfrak{A}}{(f+gx)^2} + \frac{\mathfrak{B}}{f+gx}.$$

EXEMPLUM

*Proposita fractione $\frac{x^m}{1-4x^3+3x^4}$ cuius denominator factorem habet $(1-x)^2$,
invenire fractiones simplices hinc oriundas.*

Cum hic sit $f = 1, g = -1, P = x^m$ et $Q = 1 - 4x^3 + 3x^4$, erit $S = 1 + 2x + 3xx$,

$$\frac{P}{S} = \frac{x^m}{1+2x+3xx} \text{ et } d.\frac{P}{S} = \frac{mx^{m-1}dx + 2(m-1)x^m dx + 3(m-2)x^{m+1} dx}{(1+2x+3xx)^2}$$

Hinc posito $x = 1$ erit

$$\mathfrak{A} = \frac{1}{6} \text{ et } \mathfrak{B} = -1 \cdot \frac{6m-8}{36} = \frac{4-3m}{18};$$

unde fractiones quaesitae erunt

$$\frac{1}{6(1-x)^2} + \frac{4-3m}{18(1-x)}.$$

410. Habeat fractionis $\frac{P}{Q}$ denominator Q tres factores simplices aequales

seu sit $Q = (f + gx)^3 S$ sintque fractiones simplices ex hoc factore cubico

$(f + gx)^3$ oriundae hae

$$\frac{\mathfrak{A}}{(f+gx)^3} + \frac{\mathfrak{B}}{(f+gx)^2} + \frac{\mathfrak{C}}{(f+gx)}$$

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

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1257

complementum vero harum fractionum ad fractionem propositam $\frac{P}{Q}$ constituendam

sit $\frac{V}{S}$ eritque

$$V = \frac{P - \mathfrak{A}S - \mathfrak{B}S(f+gx) - \mathfrak{C}S(f+gx)^2}{(f+gx)^3}$$

Quare haec expressio $\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f+gx) - \mathfrak{C}(f+gx)^2 = 0$ divisibilis erit per $(f+gx)^3$

; unde posito $f+gx=0$ seu $x = \frac{-f}{g}$ non solum ipsa haec expressio, sed etiam eius

differentiale primum et secundum evadet = 0. Erit scilicet ponendo $x = \frac{-f}{g}$

$$\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f+gx) - \mathfrak{C}(f+gx)^2 = 0$$

$$d.\frac{P}{S} - \mathfrak{B}gdx - 2\mathfrak{C}gdx(f+gx) = 0$$

$$dd.\frac{P}{S} - 2\mathfrak{C}g^2dx^2 = 0.$$

Ex prima aequatione ergo erit

$$\mathfrak{A} = \frac{P}{S}.$$

Ex secunda vero erit

$$\mathfrak{B} = \frac{1}{gdx} d.\frac{P}{S}.$$

Ex tertia denique definitur

$$\mathfrak{C} = \frac{1}{2g^2dx^2} dd.\frac{P}{S}.$$

411. Generaliter ergo, si fractionis $\frac{P}{Q}$ denominator Q factorem habeat $(f+gx)^n$, ita ut

sit $Q = (f+gx)^n S$, positis fractionibus simplicibus ex hoc factore $(f+gx)^n$ oriundis his

$$\frac{\mathfrak{A}}{(f+gx)^n} + \frac{\mathfrak{B}}{(f+gx)^{n-1}} + \frac{\mathfrak{C}}{(f+gx)^{n-2}} + \frac{\mathfrak{D}}{(f+gx)^{n-3}} + \frac{\mathfrak{E}}{(f+gx)^{n-4}} + \text{etc.},$$

quo ad ultimam, cuius denominator est $f+gx$, perveniatur, si ratiocinium ut ante instituat, reperietur haec expressio

$$\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f+gx) - \mathfrak{C}(f+gx)^2 - \mathfrak{D}(f+gx)^3 - \mathfrak{E}(f+gx)^4 - \text{etc.}$$

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1258

divisibilis esse debere per $(f + gx)^n$; hinc tam ipsa quam singula eius differentialia usque ad gradum $n - 1$ casu $x = \frac{-f}{g}$ evanescere debebunt. Ex quibus aequationibus concludetur fore ponendo ubique $x = \frac{-f}{g}$

$$\mathfrak{A} = \frac{P}{S}$$

$$\mathfrak{B} = \frac{1}{1gdx} d \cdot \frac{P}{S}$$

$$\mathfrak{C} = \frac{1}{1 \cdot 2g^2 dx^2} dd \cdot \frac{P}{S}$$

$$\mathfrak{D} = \frac{1}{1 \cdot 2 \cdot 3g^3 dx^3} d^3 \cdot \frac{P}{S}$$

$$\mathfrak{E} = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4g^4 dx^4} d^4 \cdot \frac{P}{S}$$

etc.

Ubi quidem notandum est differentialia ista ipsius $\frac{P}{S}$ ante capi oportere,

quam loco x ponatur $\frac{-f}{g}$; alias enim variabilitas ipsius x tolleretur.

412. Facilius ergo hoc modo isti numeratores $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. exprimuntur quam eo modo, qui in *Introductione* est traditus, et saepenumero quoque hac nova ratione. eorum valores expeditius reperiuntur. Quae comparatio quo facilius institui queat, valores litterarum $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. priori modo definiamus.

Posito $x = \frac{-f}{g}$ Statuatur relicto x variabili

	$\mathfrak{A} = \frac{P}{S}$	$\frac{P - \mathfrak{A}S}{f + gx} = \mathfrak{B},$
erit	$\mathfrak{B} = \frac{\mathfrak{B}}{S}$	$\frac{\mathfrak{B} - \mathfrak{B}S}{f + gx} = \mathfrak{C},$
erit	$\mathfrak{C} = \frac{\mathfrak{C}}{S}$	$\frac{\mathfrak{C} - \mathfrak{C}S}{f + gx} = \mathfrak{D},$
erit	$\mathfrak{D} = \frac{\mathfrak{D}}{S}$	$\frac{\mathfrak{D} - \mathfrak{D}S}{f + gx} = \mathfrak{E},$
erit	$\mathfrak{E} = \frac{\mathfrak{E}}{S}$	et ita porro.

413. Quodsi autem fractionis $\frac{P}{Q}$ denominator Q non omnes faectores simplices habeat reales, tum bini imaginariorum iunctim sumantur, quorum productum erit reale. Sit ergo denominatoris Q factor $ff - 2fgx \cos \varphi + ggxx$, qui positus = 0 dat hunc duplicem valorem imaginarium

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1259

$$x = \frac{f}{g} \cos \varphi \pm \frac{f}{g\sqrt{-1}} \sin \varphi$$

ex quo erit

$$x^n = \frac{f^n}{g^n} \cos n\varphi \pm \frac{f^n}{g^n\sqrt{-1}} \sin n\varphi$$

Ponamus esse $Q = (ff - 2fgx \cos \varphi + ggxx)S$ atque S praeterea per

$ff - 2fgx \cos \varphi + ggxx$ non esse divisibile. Sit fractio ex isto factore denominatoris oriunda

$$\frac{\mathfrak{A} + \alpha x}{ff - 2fgx \cos \varphi + ggxx}$$

et complementum ad propositam $\frac{P}{Q}$ sit $= \frac{V}{S}$; erit

$$V = \frac{P - (\mathfrak{A} + \alpha x)S}{ff - 2fgx \cos \varphi + ggxx}$$

unde $P - (\mathfrak{A} + \alpha x)S$ ac propterea quoque $\frac{P}{S} - \mathfrak{A} - \alpha x$ divisibile erit per

$ff - 2fgx \cos \varphi + ggxx$. Evanescet ergo $\frac{P}{S} - \mathfrak{A} - \alpha x$, si ponatur

$ff - 2fgx \cos \varphi + ggxx = 0$, hoc est, si ponatur vel

$$x = \frac{f}{g} \cos \varphi + \frac{f}{g\sqrt{-1}} \sin \varphi$$

vel

$$x = \frac{f}{g} \cos \varphi - \frac{f}{g\sqrt{-1}} \sin \varphi$$

414. Quoniam P et S sunt functiones integrae ipsius x , fiat in utroque seorsim utraque substitutio; et quia pro quavis potestate ipsius x , puta x^n , binomium hoc

$$x^n = \frac{f^n}{g^n} \cos n\varphi \pm \frac{f^n}{g^n\sqrt{-1}} \sin n\varphi$$

substitui debet, ponamus primo ubique $\frac{f^n}{g^n} \cos n\varphi$ pro x^n hocque facto abeat P

in \mathfrak{P} et S in \mathfrak{S} . Deinde ponatur ubique $\frac{f^n}{g^n} \sin n\varphi$ pro x^n hocque facto abeat

P in \mathfrak{p} et S in \mathfrak{s} ; ubi notandum est ante has substitutiones utramque functionem P et S penitus debere evolvi, ita ut, si forte factoribus sint implicatae, ii per actualem multiplicationem tollantur. His valoribus \mathfrak{P} , \mathfrak{p} , \mathfrak{S} , \mathfrak{s} inventis manifestum erit, si ponatur

$x = \frac{f}{g} \cos \varphi \pm \frac{f}{g\sqrt{-1}} \sin \varphi$, functionem P abituram esse in $\mathfrak{P} \pm \frac{\mathfrak{p}}{\sqrt{-1}}$ et functionem S

abituram esse in $\mathfrak{S} \pm \frac{\mathfrak{s}}{\sqrt{-1}}$. Hinc cum $\frac{P}{S} - \mathfrak{A} - \alpha x$ seu $P - (\mathfrak{A} + \alpha x)S$ utroque casu

evanescere debeat, erit

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1260

$$\mathfrak{P} \pm \frac{\mathfrak{p}}{\sqrt{-1}} = \left(\mathfrak{A} + \frac{\mathfrak{a}f}{g} \cos \varphi \pm \frac{\mathfrak{a}f}{g\sqrt{-1}} \sin \varphi \right) \left(\mathfrak{G} \pm \frac{\mathfrak{s}}{\sqrt{-1}} \right)$$

unde ob signa ambigua haa duae aequationes orientur

$$\mathfrak{P} = \mathfrak{A}\mathfrak{G} + \frac{\mathfrak{a}f\mathfrak{G}}{g} \cos \varphi - \frac{\mathfrak{a}f\mathfrak{s}}{g} \sin \varphi$$

$$\mathfrak{p} = \mathfrak{A}\mathfrak{s} + \frac{\mathfrak{a}f\mathfrak{s}}{g} \cos \varphi + \frac{\mathfrak{a}f\mathfrak{G}}{g} \sin \varphi,$$

ex quibus eliminando \mathfrak{A} eruitur

$$\mathfrak{G}\mathfrak{p} - \mathfrak{s}\mathfrak{P} = \frac{\mathfrak{a}f(\mathfrak{G}^2 + \mathfrak{s}^2)}{g} \sin \varphi$$

ideoque erit

$$\mathfrak{a} = \frac{g(\mathfrak{G}\mathfrak{p} - \mathfrak{s}\mathfrak{P})}{f(\mathfrak{G}^2 + \mathfrak{s}^2) \sin \varphi}$$

Deinde eliminando $\sin \varphi$ erit

$$\mathfrak{G}\mathfrak{P} + \mathfrak{s}\mathfrak{p} = (\mathfrak{G}^2 + \mathfrak{s}^2) \left(\mathfrak{A} + \frac{\mathfrak{a}f}{g} \cos \varphi \right)$$

Ergo

$$\mathfrak{A} = \frac{\mathfrak{G}\mathfrak{P} + \mathfrak{s}\mathfrak{p}}{\mathfrak{G}^2 + \mathfrak{s}^2} - \frac{(\mathfrak{G}\mathfrak{p} - \mathfrak{s}\mathfrak{P}) \cos \varphi}{(\mathfrak{G}^2 + \mathfrak{s}^2) \sin \varphi}$$

415. Cum iam sit

$$S = \frac{Q}{ff - 2fgx \cos \varphi + ggxx}$$

quia posito

$$ff - 2fgx \cos \varphi + ggxx = 0$$

tam numerator quam denominator evanescent, erit hoc casu

$$S = \frac{dQ:dx}{2ggx - 2fg \cos \varphi}$$

Ponamus nunc, si ubique substituat $\frac{f^n}{g^n} \cos n\varphi$, functionem $\frac{dQ}{dx}$ abire

in \mathfrak{Q} , sin autem statuatur $\frac{f^n}{g^n} \sin n\varphi$, eam abire in \mathfrak{q} ; atque manifestum

est, si ponatur $x = \frac{f}{g} \cos \varphi \pm \frac{f}{g\sqrt{-1}} \sin \varphi$, functionem $\frac{dQ}{dx}$ abire in $\mathfrak{Q} \pm \frac{\mathfrak{q}}{\sqrt{-1}}$.

Ex quo functio S abibit in

$$\frac{\mathfrak{Q} \pm \mathfrak{q} \sqrt{-1}}{\pm 2fg \sin \varphi \sqrt{-1}}$$

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1261

Cum ergo sit $S = \mathfrak{S} \pm \frac{\mathfrak{s}}{\sqrt{-1}}$ eodem valore pro x posito, habebitur

$$\mathfrak{Q} \pm \frac{\mathfrak{q}}{\sqrt{-1}} = \pm \frac{2fg\mathfrak{S}}{\sqrt{-1}} \sin \varphi - 2fg\mathfrak{s} \sin \varphi$$

Erit ergo

$$\mathfrak{s} = \frac{-\mathfrak{Q}}{2fg \sin \varphi} \text{ et } \mathfrak{S} = \frac{\mathfrak{q}}{2fg \sin \varphi}$$

Hisque valoribus substitutis fiet

$$\mathfrak{a} = \frac{2gg(\mathfrak{p}\mathfrak{q} + \mathfrak{P}\mathfrak{Q})}{\mathfrak{Q}^2 + \mathfrak{q}^2}$$

et

$$\mathfrak{A} = \frac{2fg(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q}) \sin \varphi}{\mathfrak{Q}^2 + \mathfrak{q}^2} - \frac{2fg(\mathfrak{p}\mathfrak{q} + \mathfrak{P}\mathfrak{Q}) \cos \varphi}{\mathfrak{Q}^2 + \mathfrak{q}^2}$$

416. Hinc ergo idonea obtinetur ratio ex quovis factore secundae potestatis fractionem simplicem formandi hicque, cum ipse fractionis propositae denominator in computo retineatur, divisionem, qua valor litterae S definiri deberet et quae saepe non parum est molesta, evitamus. Si igitur fractionis $\frac{P}{Q}$ denominator Q factorem habeat talem

$ff - 2fgx \cos \varphi + ggxx = 0$, sequenti modo fractio simplex ex hoc factore oriunda, quam fingamus

$$= \frac{\mathfrak{A} + \mathfrak{a}x}{ff - 2fgx \cos \varphi + ggxx}$$

definietur. Ponatur $x = \frac{f}{g} \cos \varphi$ et pro quavis ipsius x potestate x^n scribatur $\frac{f^n}{g^n} \cos n\varphi$;

quo facto abeat P in \mathfrak{P} et functio $\frac{dQ}{dx}$ in \mathfrak{Q} . Deinde ibidem ponatur $x = \frac{f}{g} \sin \varphi$

et potestas eius quaevis $x^n = \frac{f^n}{g^n} \sin n\varphi$ abeatque P in \mathfrak{p} et $\frac{dQ}{dx}$ in \mathfrak{q} . Inventisque hoc

modo valoribus litterarum $\mathfrak{P}, \mathfrak{Q}, \mathfrak{p}$ et \mathfrak{q} quantitates \mathfrak{A} et \mathfrak{a} ita definientur, ut sit

$$\mathfrak{A} = \frac{2fg(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q}) \sin \varphi}{\mathfrak{Q}^2 + \mathfrak{q}^2} - \frac{2fg(\mathfrak{p}\mathfrak{q} + \mathfrak{P}\mathfrak{Q}) \cos \varphi}{\mathfrak{Q}^2 + \mathfrak{q}^2}$$

$$\mathfrak{a} = \frac{2gg(\mathfrak{p}\mathfrak{q} + \mathfrak{P}\mathfrak{Q})}{\mathfrak{Q}^2 + \mathfrak{q}^2}$$

Fractio ergo ex denominatoris Q factore $ff - 2fgx \cos \varphi + ggxx$ oriunda erit

$$\frac{2fg(\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q}) \sin \varphi + 2g(\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q})(gx - f \cos \varphi)}{(\mathfrak{Q}^2 + \mathfrak{q}^2)(ff - 2fgx \cos \varphi + ggxx)}.$$

EXEMPLUM 1

**EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2**

Chapter 18

Translated and annotated by Ian Bruce.

1262

Si proposita fuerit haec fractio $\frac{x^m}{a+bx^n}$, cuius denominator $a+bx^n$ factorem habeat hunc $ff - 2fgx \cos \varphi + ggxx$, invenire fractionem simplicem huic factori convenientem.

Quoniam hic est $P = x^m$ et $Q = a + bx^n$, erit

$$\frac{dQ}{dx} = nbx^n$$

unde fiet

$$\begin{aligned} \mathfrak{P} &= \frac{f^m}{g^m} \cos m\varphi, & \mathfrak{p} &= \frac{f^n}{g^n} \sin m\varphi, \\ \mathfrak{Q} &= \frac{nbf^{n-1}}{g^{n-1}} \cos(n-1)\varphi, & \mathfrak{q} &= \frac{nbf^{n-1}}{g^{n-1}} \sin(n-1)\varphi. \end{aligned}$$

Ex his erit

$$\begin{aligned} \mathfrak{Q}^2 + \mathfrak{q}^2 &= \frac{n^2 b^2 f^{2(n-1)}}{g^{2(n-1)}}, \\ \mathfrak{P}\mathfrak{Q} - \mathfrak{p}\mathfrak{q} &= \frac{nbf^{m+n-1}}{g^{m+n-1}} \sin(n-m-1)\varphi, \end{aligned}$$

atque

$$\mathfrak{P}\mathfrak{Q} + \mathfrak{p}\mathfrak{q} = \frac{nbf^{m+n-1}}{g^{m+n-1}} \cos(n-m-1)\varphi.$$

Quamobrem erit fractio simplex quaesita

$$\frac{2g^{n-m}(f \sin \varphi \cdot \sin(n-m-1)\varphi + gx \cos(n-m-1)\varphi - f \cos \varphi \cdot \cos(n-m-1)\varphi)}{nbf^{n-m-1}(ff - 2fgx \cos \varphi + ggxx)}$$

seu

$$\frac{2g^{n-m}(gx \cos(n-m-1)\varphi - f \cos(n-m)\varphi)}{nbf^{n-m-1}(ff - 2fgx \cos \varphi + ggxx)}.$$

EXEMPLUM 2

Sit proposita haec fractio $\frac{1}{x^m(a+bx^n)}$ cuius denominator factorem

habeat $ff - 2fgx \cos \varphi + ggxx$; invenire fractionem simplicem inde oriundam.

Cum sit $P = 1$ et $Q = ax^m + bx^{m+n}$, erit

$$\frac{dQ}{dx} = max^{m-1} + (m+n)bx^{m+n-1}$$

ideoque posito $x^n = \frac{f^n}{g^n} \cos n\varphi$, ob $P = x^0$, $\mathfrak{P} = 1$ et

**EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2**

Chapter 18

Translated and annotated by Ian Bruce.

1263

$$\mathfrak{Q} = \frac{maf^{m-1}}{g^{m-1}} \cos(m-1)\varphi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}} \cos(m+n-1)\varphi$$

et [posito $x^n = \frac{f^n}{g^n} \sin n\varphi$] $\mathfrak{p} = 0$ atque

$$\mathfrak{q} = \frac{maf^{m-1}}{g^{m-1}} \sin(m-1)\varphi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}} \sin(m+n-1)\varphi.$$

Ergo

$$\mathfrak{A}^2 + \mathfrak{q}^2 = \frac{m^2 a^2 f^{2(m-1)}}{g^{m-1}} + \frac{2m(m+n)abf^{2m+n-2}}{g^{2m+n-2}} \cos n\varphi + \frac{(m+n)^2 b^2 f^{2(m+n-1)}}{g^{2(m+n-1)}}.$$

Quodsi vero est $ff - 2fgx \cos \varphi + ggxx$ divisor ipsius $a + bx^n$, erit

$$a + \frac{bf^n}{g^n} \cos n\varphi = 0 \text{ et } \frac{bf^n}{g^n} \sin n\varphi = 0, \text{ unde } aa = \frac{bbf^{2n}}{g^{2n}}$$

Erit ergo

$$\mathfrak{A}^2 + \mathfrak{q}^2 = \frac{(m+n)^2 bbf^{2(m+n-1)}}{g^{2(m+n-1)}} - \frac{m(2n+m)aa f^{2(m-1)}}{g^{2(m-1)}} = \frac{nnaaf^{2(m-1)}}{g^{2(m-1)}} = \frac{nnbbf^{2(m+n-1)}}{g^{2(m+n-1)}}$$

Deinde vero erit

$$\begin{aligned} \mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} &= \frac{maf^{m-1}}{g^{m-1}} \sin(m-1)\varphi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}} \sin(m+n-1)\varphi \\ &= \frac{bf^{m+n-1}}{g^{m+n-1}} \left((m+n) \sin(m+n-1)\varphi - m \cos n\varphi \cdot \sin(m-1)\varphi \right) \\ &= \frac{bf^{m+n-1}}{g^{m+n-1}} \left(n \cos n\varphi \cdot \sin(m-1)\varphi + (m+n) \sin n\varphi \cdot \cos(m-1)\varphi \right) \end{aligned}$$

et

$$\mathfrak{P}\mathfrak{q} + \mathfrak{p}\mathfrak{Q} = \frac{bf^{m+n-1}}{g^{m+n-1}} \left((m+n) \cos(m+n-1)\varphi - m \cos n\varphi \cdot \cos(m-1)\varphi \right)$$

Vel cum $ff - 2fg \cos \varphi + ggxx$ sit quoque divisor ipsius $ax^{m-1} + bx^{m+n-1}$, erit

$$\frac{af^{m-1}}{g^{m-1}} \cos(m-1)\varphi + \frac{bf^{m+n-1}}{g^{m+n-1}} \cos(m+n-1)\varphi = 0$$

et

$$\frac{af^{m-1}}{g^{m-1}} \sin(m-1)\varphi + \frac{bf^{m+n-1}}{g^{m+n-1}} \sin(m+n-1)\varphi = 0,$$

unde erit

$$\mathfrak{Q} = \frac{nbf^{m+n-1}}{g^{m+n-1}} \cos(m+n-1)\varphi \text{ et } \mathfrak{q} = \frac{nbf^{m+n-1}}{g^{m+n-1}} \sin(m+n-1)\varphi$$

seu

**EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2**

Chapter 18

Translated and annotated by Ian Bruce.

1264

$$\Omega = \frac{-naf^{m-1}}{g^{m-1}} \cos(m-1)\varphi \quad \text{et} \quad \mathfrak{q} = \frac{-naf^{m-1}}{g^{m-1}} \sin(m-1)\varphi$$

Ex quibus resultabit fractio quaesita

$$\frac{2g^m(f \cos m\varphi - gx \cos(m-1)\varphi)}{naf^{m-1}(ff - 2fgx \cos \varphi + ggxx)}$$

Quae formula ex priori exemplo sequitur, si ponatur m negativum, unde non opus fuisset hunc easum peculiarem constituisse.

EXEMPLUM 3

Si huius fractionis $\frac{x^m}{a+bx^n+cx^{2n}}$ denominator habuerit factorem $ff - 2fgx \cos \varphi + ggxx$, fractionem simplicem investigare ex hoc factore oriundam.

Si $ff - 2fgx \cos \varphi + ggxx$ est factor denominatorisa $a + bx^n + cx^{2n}$, erit, ut supra ostendimus,

$$a + \frac{bf^n}{g^n} \cos n\varphi + \frac{cf^{2n}}{g^{2n}} \cos 2n\varphi = 0 \quad \text{et} \quad \frac{bf^n}{g^n} \sin n\varphi + \frac{cf^{2n}}{g^{2n}} \sin 2n\varphi = 0$$

Cum igitur sit $P = x^m$ et $Q = a + bx^n + cx^{2n}$, erit

$$\frac{dQ}{dx} = nbx^{n-1} + 2ncx^{2n-1}$$

unde efficitur

$$\begin{aligned} \mathfrak{P} &= \frac{f^m}{g^m} \cos m\varphi \quad \text{and} \quad \mathfrak{p} = \frac{f^m}{g^m} \sin m\varphi \\ \Omega &= \frac{nbf^{n-1}}{g^{n-1}} \cos(n-1)\varphi + \frac{2ncf^{2n-1}}{g^{2n-1}} \cos(2n-1)\varphi, \\ \mathfrak{q} &= \frac{nbf^{n-1}}{g^{n-1}} \sin(n-1)\varphi + \frac{2ncf^{2n-1}}{g^{2n-1}} \sin(2n-1)\varphi. \end{aligned}$$

Quamobrem habebimus

$$\Omega^2 + \mathfrak{q}^2 = \frac{n^2 f^{2(n-1)}}{g^{2(n-1)}} \left(bb + \frac{4bcf^n}{g^n} \cos n\varphi + \frac{4ccf^{2n}}{g^{2n}} \right)$$

At ex duabus prioribus aequationibus est

$$\frac{f^{2n}}{g^{2n}} \left(bb + \frac{2bcf^n}{g^n} \cos n\varphi + \frac{ccf^{2n}}{g^{2n}} \right) = aa$$

ideoque

$$\frac{4bcf^n}{g^n} \cos n\varphi = \frac{2g^{2n}aa}{f^{2n}} - 2bb - \frac{2ccf^{2n}}{g^{2n}}$$

quo valore ibi substituto erit

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1265

$$\mathfrak{A}^2 + \mathfrak{Q}^2 = \frac{n^2 f^{2n-2}}{g^{2n-2}} \left(\frac{2aag^{2n}}{f^{2n}} - bb + \frac{2ccf^{2n}}{g^{2n}} \right)$$

seu

$$\mathfrak{A}^2 + \mathfrak{Q}^2 = \frac{n^2 (2aag^{4n} - bbf^{2n} g^{2n} + 2ccf^{4n})}{ffg^{4n-2}}.$$

Deinde erit

$$\mathfrak{P}\mathfrak{Q} - \mathfrak{P}\mathfrak{Q} = \frac{nbf^{m+n-1}}{g^{m+n-1}} \sin(n-m-1)\varphi + \frac{2ncf^{m+2n-1}}{g^{m+2n-1}} \sin(2n-m-1)\varphi,$$

$$\mathfrak{P}\mathfrak{Q} + \mathfrak{P}\mathfrak{Q} = \frac{nbf^{m+n-1}}{g^{m+n-1}} \cos(n-m-1)\varphi + \frac{2ncf^{m+2n-1}}{g^{m+2n-1}} \cos(2n-m-1)\varphi.$$

Quibus valoribus inventis erit fractio simplex quaesita

$$\frac{2fg(\mathfrak{P}\mathfrak{Q} - \mathfrak{P}\mathfrak{Q})\sin\varphi + 2g(\mathfrak{P}\mathfrak{Q} + \mathfrak{P}\mathfrak{Q})(gx - f\cos\varphi)}{(\mathfrak{Q}^2 + \mathfrak{A}^2)(ff - 2fgx\cos\varphi + ggxx)}$$

417. Hae autem fractiones facilius exprimentur, si ipsos denominatorum factores determinemus. Sit igitur denominator fractionis propositae

$$a + bx^n;$$

cuius factor trinomialis si ponatur

$$ff - 2fgx\cos\varphi + ggxx,$$

erit, uti in *Introductione* ostendimus,

$$a + \frac{bf^n}{g^n} \cos n\varphi = 0 \quad \text{et} \quad \frac{bf^n}{g^n} \sin n\varphi = 0$$

cum igitur sit $\sin n\varphi = 0$, erit vel $n\varphi = (2k-1)\pi$ vel $n\varphi = 2k\pi$; priori casu erit $\cos n\varphi = -1$, posteriori $\cos n\varphi = +1$. Si ergo a et b sint quantitates affirmativae, prior

casus solus locum habebit, quo fit $a = \frac{bf^n}{g^n} a$ ac propterea

$$f = a^{\frac{1}{n}} \quad \text{and} \quad g = b^{\frac{1}{n}}$$

Retineamus autem loco harum quantitatum irrationalium litteras f et g seu ponamus potius $a = f^n$ et $b = g^n$, ita ut factores investigari debeant huius functionis

$$f^n + g^n x^n.$$

Cum igitur sit $\varphi = \frac{(2k-1)\pi}{n}$, ibi k numerum quemcunque affirmativum integrum

designare potest, at vero maiores numeri pro k non sunt sumendi, quam qui reddant $\frac{2k-1}{n}$

unitate minorem; hinc fractionis propositae $f^n + g^n x^n$ factores erunt sequentes

EULER'S INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1266

$$ff - 2fgx \cos \frac{\pi}{n} + ggxx$$

$$ff - 2fgx \cos \frac{3\pi}{n} + ggxx$$

$$ff - 2fgx \cos \frac{5\pi}{n} + ggxx$$

etc,

ubi notandum est, si n sit numerus impar, unum factorem haberi binomium hunc $f + gx$; sin autem n sit numerus par, nullus factor aderit binomius.

EXEMPLUM 1

Besolvere hanc fractionem $\frac{x^m}{f^n + g^n x^n}$, in suas fractiones simplices.

Cum denominatoris unusquisque factor trinomialis contineatur in hac forma

$$ff - 2fgx \cos \frac{(2k-1)\pi}{n} + ggxx,$$

erit in paragraphi praecedentis exemplo $a = f^n$, $b = g^n$, and $\varphi = \frac{(2k-1)\pi}{n}$

unde erit

$$\sin(n-m-1)\varphi = \sin(m+1)\varphi = \sin \frac{(m+1)(2k-1)\pi}{n},$$

et

$$\cos(n-m-1)\varphi = -\cos(m+1)\varphi = -\cos \frac{(m+1)(2k-1)\pi}{n}.$$

Hinc ex isto factore oritur fractio simplex haec

$$\frac{2f \sin \frac{(2k-1)\pi}{n} \sin \frac{(m+1)(2k-1)\pi}{n} - 2 \cos \frac{(m+1)(2k-1)\pi}{n} \left(gx - f \cos \frac{(2k-1)\pi}{n} \right)}{nf^{n-m-1} g^m \left(ff - 2fgx \cos \frac{(2k-1)\pi}{n} + ggxx \right)}$$

Quamobrem fractio proposita resolvetur! in has simplices

$$\begin{aligned} & \frac{2f \sin \frac{\pi}{n} \sin \frac{(m+1)\pi}{n} - 2 \cos \frac{(m+1)\pi}{n} \left(gx - f \cos \frac{\pi}{n} \right)}{nf^{n-m-1} g^m \left(ff - 2fgx \cos \frac{\pi}{n} + ggxx \right)} \\ & + \frac{2f \sin \frac{3\pi}{n} \sin \frac{3(m+1)\pi}{n} - 2 \cos \frac{3(m+1)\pi}{n} \left(gx - f \cos \frac{3\pi}{n} \right)}{nf^{n-m-1} g^m \left(ff - 2fgx \cos \frac{3\pi}{n} + ggxx \right)} \\ & + \frac{2f \sin \frac{5\pi}{n} \sin \frac{5(m+1)\pi}{n} - 2 \cos \frac{5(m+1)\pi}{n} \left(gx - f \cos \frac{5\pi}{n} \right)}{nf^{n-m-1} g^m \left(ff - 2fgx \cos \frac{5\pi}{n} + ggxx \right)} \\ & \text{etc.} \end{aligned}$$

Si ergo n fuerit numerus par in hoc modo omnes oriuntur fractiones simplices;

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1267

sin autem n sit numerus impar, ob factorem binomium $f+gx$ ad fractiones hoc modo resultantes insuper addi debet haec

$$\frac{\pm 1}{nf^{n-m-1}g^m(f+gx)}$$

ubi signum $+$ valet, si m fuerit numerus par, contra signum si m fuerit numerus malor quam n , tum ad has fractiones accedent insuper partes integrae huiusmodi

$$Ax^{m-n} + Bx^{m-2n} + Cx^{m-3n} + Dx^{m-4n} + \text{etc.},$$

quamdiu exponentes manent affirmativi, eritque

$$Ag^n = 1 \quad \text{ergo} \quad A = \frac{1}{g^n}$$

$$Af^n + Bg^n = 0 \quad B = -\frac{f^n}{g^{2n}}$$

$$Bf^n + Cg^n = 0 \quad C = +\frac{f^{2n}}{g^{3n}}$$

$$Cf^n + Dg^n = 0 \quad D = -\frac{f^{3n}}{g^{4n}}$$

etc. etc.

EXEMPLUM 2

Resolvere hanc fractionem $\frac{1}{x^m(f^n+g^n x^n)}$ in suas fractiones simplices.

Quod ad factores ipsius $f^n + g^n x^n$ attinet, ex iis oriuntur eadem fractiones, quas exemplo praecedente eruimus, dummodo ibi sumatur m negative; superest igitur tantum, ut fractiones simplices ex denominatoris altero factore x^n definiamus, quod hoc modo commodissime fit. Statuatur fractio proposita

$$= \frac{\mathfrak{A}}{x^m} + \frac{\mathfrak{N}x^{n-m}}{(f^n + g^n x^n)}$$

eritque

$$\mathfrak{A}f^n = 1 \quad \text{ergo} \quad \mathfrak{A} = \frac{1}{f^n}$$

$$\mathfrak{A}g^n + \mathfrak{N} = 0 \quad \mathfrak{N} = \frac{g^n}{f^n}.$$

Si $n - m$ adhuc fuerit numerus negativus, simili modo erit operandum, ita ut, si m fuerit numerus quantumvis magnus, resultent huiusmodi fractiones simplices

$$\frac{\mathfrak{A}}{x^m} + \frac{\mathfrak{B}}{x^{m-n}} + \frac{\mathfrak{C}}{x^{m-2n}} + \frac{\mathfrak{D}}{x^{m-3n}} + \text{etc.},$$

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2
Chapter 18

Translated and annotated by Ian Bruce.

1268

cuius seriei tot termini sunt sumendi, quot habentur ipsius x exponentes affirmativi in denominatore. Eritque

$$\begin{aligned} \mathfrak{A}f^n = 1 \quad \text{ergo} \quad \mathfrak{A} &= \frac{1}{f^n} \\ \mathfrak{A}g^n + \mathfrak{B}f^n = 0 \quad \mathfrak{B} &= -\frac{g^n}{f^{2n}} \\ \mathfrak{B}g^n + \mathfrak{C}f^n = 0 \quad \mathfrak{C} &= +\frac{g^{2n}}{f^{3n}} \\ \mathfrak{C}g^n + \mathfrak{D}f^n = 0 \quad \mathfrak{D} &= -\frac{g^{3n}}{f^{4n}} \\ \text{etc.} \quad \quad \quad \quad \quad \quad \quad \quad & \text{etc.} \end{aligned}$$

Fractio ergo proposita omnino in has fractiones simplices resolvetur

$$\begin{aligned} & \frac{1}{f^n x^m} - \frac{g^n}{f^{2n} x^{m-n}} + \frac{g^{2n}}{f^{3n} x^{m-2n}} - \frac{g^{3n}}{f^{4n} x^{m-3n}} + \text{etc.} \\ & - \frac{2fg^m \sin \frac{\pi}{n} \sin \frac{(m-1)\pi}{n} + 2g^m \cos \frac{(m-1)\pi}{n} (gx - f \cos \frac{\pi}{n})}{nf^{n+m-1} (ff - 2fgx \cos \frac{\pi}{n} + ggxx)} \\ & - \frac{2fg^m \sin \frac{3\pi}{n} \sin \frac{3(m-1)\pi}{n} + 2g^m \cos \frac{3(m-1)\pi}{n} (gx - f \cos \frac{3\pi}{n})}{nf^{n+m-1} (ff - 2fgx \cos \frac{3\pi}{n} + ggxx)} \\ & - \frac{2fg^m \sin \frac{5\pi}{n} \sin \frac{5(m-1)\pi}{n} + 2g^m \cos \frac{5(m-1)\pi}{n} (gx - f \cos \frac{5\pi}{n})}{nf^{n+m-1} (ff - 2fgx \cos \frac{5\pi}{n} + ggxx)} \\ & \text{etc.} \end{aligned}$$

Quibus formulis, si n fuerit numerus impar, ob $f + gx$ factorem denominatoris insuper adiici debet

$$\frac{\pm g^m}{nf^{m+m-1} (f + gx)},$$

ubi signorum ambiguum + superius valet, si m fuerit numerus par, inferius vero, si m impar.

418. Consideremus nunc quoque formulam $a + bx^n$, si b sit numerus negativus, sitque proposita haec functio

$$f^n - g^n x^n,$$

cuius primo semper erit factor $f - gx$; atque si n sit numerus par, quoque $f + gx$ eius erit factor. Reliqui vero erunt trinomiales; quorum forma generalis si ponatur

$$ff - 2fgx \cos \varphi + ggxx$$

erit

**EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2**

Chapter 18

Translated and annotated by Ian Bruce.

1269

$$f^n - f^n \cos n\varphi = 0 \quad \text{et} \quad f^n \sin \varphi = 0$$

sive

$$\sin n\varphi = 0 \quad \text{et} \quad \cos n\varphi = 1.$$

Quibus ut satisfiat, oportet esse $n\varphi = 2k\pi$ existente k numero quocunque integro atque propterea erit $\varphi = \frac{2k\pi}{n}$. Factor ergo generalis erit

$$ff - 2fgx \cos \frac{2k\pi}{n} + ggxx;$$

sumendo ergo pro $2k$ omnes numeros pares exponente n minores prodibunt factores trinomiales omnes

$$ff - 2fgx \cos \frac{2\pi}{n} + ggxx$$

$$ff - 2fgx \cos \frac{4\pi}{n} + ggxx$$

$$ff - 2fgx \cos \frac{6\pi}{n} + ggxx$$

etc.

EXEMPLUM 1

Resolve hanc fractionem $\frac{x^m}{f^n - g^n x^n}$ in suas fractiones simplices.

Quoniam denominatoris factor est $f - gx$, inde orietur fractio huiusmodi $\frac{\mathfrak{A}}{f - gx}$; ad cuius

numeratorem inveniendum ponatur $x^m = P$ et $f^n - g^n x^n = Q$;
erit

$$dQ = -ng^n x^{n-1} dx$$

fietque

$$\mathfrak{A} = \frac{-gx^m}{-ng^n x^{n-1}} = \frac{x^m}{ng^{n-1} x^{n-1}}$$

posito $x = \frac{f}{g}$. Ergo erit $\mathfrak{A} = \frac{1}{nf^{n-m-1} g^m}$ hincque fractio simplex ex factore $f - gx$ orta erit

$$\frac{1}{nf^{n-m-1} g^m (f - gx)}$$

Si n sit numerus par, quia tum denominatoris factor quoque est $f + gx$, ponatur fractio simplex inde oriunda = $\frac{\mathfrak{A}}{f + gx}$; erit

$$\mathfrak{A} = \frac{-gx^m}{ng^n x^{n-1}} = \frac{-x^m}{ng^{n-1} x^{n-1}}$$

posito $x = \frac{-f}{g}$. Fiet ergo ob $n-1$ numerum imparem $ng^{n-1} x^{n-1} = -f^{n-1}$;

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1270

at erit $x^m = \frac{\pm f^m}{g^m}$, ubi signum superius valet, si m fuerit numerus par, inferius, si m sit

numerus impar. Quare cum sit $\mathfrak{A} = \frac{\mp 1}{nf^{n-m-1}g^m}$ erit fractio simplex ex factore $f + gx$ oriunda haec

$$\mathfrak{A} = \frac{\mp 1}{nf^{n-m-1}g^m(f+gx)}.$$

Deinde cum factorum trinomialium forma generalis sit

$$ff - 2fgx \cos \frac{2k\pi}{n} + ggxx,$$

si comparatio cum exemplo 1 § 416 instituat, erit $a = f^n$, $b = -g^n$ et $\varphi = \frac{2k\pi}{n}$;

unde

$$\sin n\varphi = 0 \text{ et } \cos n\varphi = 1$$

atque

$$\sin(n-m-1)\varphi = -\sin(m+1)\varphi = -\sin \frac{2k(m+1)\pi}{n}$$

et

$$\cos(n-m-1)\varphi = \cos(m+1)\varphi = \cos \frac{2k(m+1)\pi}{n}$$

Ex quibus erit fractio simplex hinc oriunda

$$\frac{2f \sin \frac{2k\pi}{n} \cdot \sin \frac{2k(m+1)\pi}{n} - 2 \cos \frac{2k(m+1)\pi}{n} (gx - f \cos \frac{2k\pi}{n})}{nf^{n-m-1}g^m(ff - 2fgx \cos \frac{2k\pi}{n} + ggxx)}$$

Hanc ob rem fractiones simplices quaesitae erunt

$$\begin{aligned} & \frac{1}{nf^{n-m-1}g^m(f-gx)} \\ & + \frac{2f \sin \frac{2\pi}{n} \cdot \sin \frac{2(m+1)\pi}{n} - 2 \cos \frac{2(m+1)\pi}{n} (gx - f \cos \frac{2\pi}{n})}{nf^{n-m-1}g^m(ff - 2fgx \cos \frac{2\pi}{n} + ggxx)} \\ & + \frac{2f \sin \frac{4\pi}{n} \cdot \sin \frac{4(m+1)\pi}{n} - 2 \cos \frac{4(m+1)\pi}{n} (gx - f \cos \frac{4\pi}{n})}{nf^{n-m-1}g^m(ff - 2fgx \cos \frac{4\pi}{n} + ggxx)} \\ & + \frac{2f \sin \frac{6\pi}{n} \cdot \sin \frac{6(m+1)\pi}{n} - 2 \cos \frac{6(m+1)\pi}{n} (gx - f \cos \frac{6\pi}{n})}{nf^{n-m-1}g^m(ff - 2fgx \cos \frac{6\pi}{n} + ggxx)} \end{aligned}$$

etc.,

quibus, si n fuerit numerus par, insuper addi debet haec fractio

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1271

$$\mathfrak{A} = \frac{\mp 1}{nf^{n-m-1}g^m(f+gx)},$$

cuius signum superius – est sumendum, si m fuerit numerus par, inferius, si impar.
Praeterea vero, si m sit numerus non minor quam n , adiciendae sunt partes integrae

$$Ax^{m-n} + Bx^{m-2n} + Cx^{m-3n} + Dx^{m-4n} + \text{etc.},$$

quamdiu exponentes non fuerint negativi, eritque

$$-Ag^n = 1 \quad \text{seu} \quad A = \frac{-1}{g^n}$$

$$Af^n - Bg^n = 0 \quad B = -\frac{f^n}{g^{2n}}$$

$$Bf^n - Cg^n = 0 \quad C = -\frac{f^{2n}}{g^{3n}}$$

$$Cf^n - Dg^n = 0 \quad D = -\frac{f^{3n}}{g^{4n}}$$

etc. etc.

EXEMPLUM 2

Resolve hanc fractionem $\frac{1}{x^m(f^n - g^n x^n)}$ in suas fractiones simplices.

Fractiones, quae ex denominatoris factore $f^n - g^n x^n$ oriuntur, eadem erunt quae ante, dummodo in illis formulis m negative accipiatur. Quare ad alterum factorem x^m est respiciendum; ex quo si ponamus has fractiones resultare

$$\frac{\mathfrak{A}}{x^m} + \frac{\mathfrak{B}}{x^{m-n}} + \frac{\mathfrak{C}}{x^{m-2n}} + \frac{\mathfrak{D}}{x^{m-3n}} + \text{etc.},$$

quae series eousque est continuanda, donec exponentes ipsius x fiant negativi, erit

$$\mathfrak{A}f^n = 1 \quad \text{ergo} \quad \mathfrak{A} = \frac{1}{f^n}$$

$$\mathfrak{B}f^n - \mathfrak{A}g^n = 0 \quad \mathfrak{B} = \frac{g^n}{f^{2n}}$$

$$\mathfrak{C}f^n - \mathfrak{B}g^n = 0 \quad \mathfrak{C} = \frac{g^{2n}}{f^{3n}}$$

$$\mathfrak{D}f^n - \mathfrak{C}g^n = 0 \quad \mathfrak{D} = \frac{g^{3n}}{f^{4n}}$$

etc. etc.

Fractio ergo proposita resolvetur in has fractiones simplices

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1272

$$\begin{aligned} & \frac{1}{f^n x^m} + \frac{g^n}{f^{2n} x^{m-n}} + \frac{g^{2n}}{f^{3n} x^{m-2n}} + \frac{g^{3n}}{f^{4n} x^{m-3n}} + \text{etc.} \\ & + \frac{g^m}{nf^{n+m-1}(f-gx)} \\ & - \frac{2fg^m \sin \frac{2\pi}{n} \sin \frac{2(m-1)\pi}{n} + 2g^m \cos \frac{2(m-1)\pi}{n} (gx - f \cos \frac{2\pi}{n})}{nf^{n+m-1} (ff - 2fgx \cos \frac{2\pi}{n} + ggxx)} \\ & - \frac{2fg^m \sin \frac{4\pi}{n} \sin \frac{4(m-1)\pi}{n} + 2g^m \cos \frac{4(m-1)\pi}{n} (gx - f \cos \frac{4\pi}{n})}{nf^{n+m-1} (ff - 2fgx \cos \frac{4\pi}{n} + ggxx)} \\ & - \frac{2fg^m \sin \frac{6\pi}{n} \sin \frac{6(m-1)\pi}{n} + 2g^m \cos \frac{6(m-1)\pi}{n} (gx - f \cos \frac{6\pi}{n})}{nf^{n+m-1} (ff - 2fgx \cos \frac{6\pi}{n} + ggxx)} \\ & \text{etc.} \end{aligned}$$

quibus, si n fuerit numerus par, insuper addi debet haec fractio

$$\frac{\mp g^m}{nf^{n+m-1}(f-gx)}$$

quae autem praetermittitur, si n fuerit numerus impar. Signorum ambiguum vero superius – valet, si m sit numerus par, inferius vero +, si m sit numerus impar.

419. Hoc ergo modo omnes fractiones, quarum denominator ex duobus constat membris huiusmodi $a + bx^n$, in fractiones simplices resolvuntur. At si denominator constet tribus huiusmodi membris $a + bx^n + cx^{2n}$, tum primum videndum est, utrum is in duos factores reales prioris formae resolvi possit. Hoc enim si eveniat, resolutio in fractiones simplices modo ante exposito institui poterit. Si enim proponatur huiusmodi fractio

$$\frac{x^m}{(f^n + g^n x^n)(f^n + h^n x^n)},$$

ea primum in duas fractiones transformabitur huiusmodi

$$\frac{\alpha x^m}{f^n + g^n x^n} + \frac{\beta x^m}{f^n + h^n x^n}$$

eritque

$$\alpha f^n + \beta f^n = 1 \quad \text{et} \quad \alpha h^n + \beta g^n = 0$$

unde fit

$$\alpha = \frac{1}{f^n} - \beta = -\frac{\beta g^n}{h^n}$$

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1273

ideoque habebitur

$$\beta = \frac{h^n}{f^n(h^n - g^n)} \quad \text{et} \quad \alpha = \frac{g^n}{f^n(g^n - h^n)}$$

Si exponent m fuerit maior quam n , transmutatio in sequentes fractiones erit commodior

$$\frac{\alpha x^{m-n}}{f^n + g^n x^n} + \frac{\beta x^{m-n}}{f^n + h^n x^n}$$

qua fit

$$\alpha + \beta = 0 \quad \text{et} \quad \alpha h^n + \beta g^n = 1$$

ideoque

$$\alpha = \frac{1}{h^n - g^n} \quad \text{et} \quad \beta = \frac{1}{g^n - h^n}.$$

Utra autem transformatio adhibeatur, utraque fractio hoc modo oriunda methodo ante exposita resolvetur in suas fractiones simplices, quae iunctim sumtae fractioni propositae erunt aequales.

420. Simili modo methodus hactenus tradita sufficiet, si denominator ex pluribus membris constet huiusmodi

$$a + bx^n + cx^{2n} + dx^{3n} + ex^{4n} + \text{etc.},$$

dummodo is in factores formae $f^n \pm g^n x^n$ resolvi queat. Ponamus enim occurrere hanc fractionem in suas fractiones simplices resolvendam

$$\frac{x^m}{(a-x^n)(b-x^n)(c-x^n)(d-x^n) \text{ etc.}}$$

Haec primum resolvetur in has

$$\frac{Ax^m}{a-x^n} + \frac{Bx^m}{b-x^n} + \frac{Cx^m}{c-x^n} + \frac{Dx^m}{d-x^n} + \text{etc.},$$

quarum numeratores sequenti modo determinabuntur, ut sit

$$A = \frac{1}{(b-a)(c-a)(d-a) \text{ etc.}}$$

$$B = \frac{1}{(a-b)(c-b)(d-b) \text{ etc.}}$$

$$C = \frac{1}{(a-c)(b-c)(d-c) \text{ etc.}}$$

etc.

Hac ergo praeparatione facta singulae istae fractiones methodo ante exposita in suas fractiones simplices resoiventur; quae cunctae in unam summam erunt colligendae.

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1274

421. Quodsi vero huiusmodi denominator

$$a + bx^n + cx^{2n} + dx^{3n} + ex^{4n} + \text{etc.},$$

non omnes factores formae $f^n + g^n x^n$ habeant reales, bini imaginarii erunt coniungendi. Ponamus ergo huiusmodi binorum factorum productum esse

$$f^{2n} - 2f^n g^n x^n \cos \omega + g^{2n} x^{2n},$$

et cum haec expressio nullos habeat factores simplices reales, ponamus factores trinomiales in hac forma generali contineri

$$ff - 2fgx \cos \varphi + ggxx,$$

quorum numerus erit $= n$. Posito ergo $x^n = \frac{f^n}{g^n} \cos n\varphi$; oriatur haec aequatio

$$1 - 2 \cos \omega \cdot \cos n\varphi + \cos 2n\varphi = 0.$$

Deinde posito $x^n = \frac{f^n}{g^n} \sin n\varphi$ erit quoque

$$-2 \cos \omega \cdot \sin n\varphi + \sin 2n\varphi = 0,$$

quae divisa per $\sin n\varphi$ dat $\cos n\varphi = \cos \omega$ sicque simul priori aequationi satisfit. Erit ergo $n\varphi = 2k\pi \pm \omega$ denotante k numerum quemvis integrum ideoque erit $\varphi = \frac{2k\pi \pm \omega}{n}$ et factores omnes continebuntur in hac forma

$$ff - 2fgx \cos \frac{2k\pi \pm \omega}{n} + ggxx,$$

unde sequentes habebuntur factores

$$ff - 2fgx \cos \frac{\omega}{n} + ggxx$$

$$ff - 2fgx \cos \frac{2\pi - \omega}{n} + ggxx$$

$$ff - 2fgx \cos \frac{2\pi + \omega}{n} + ggxx$$

$$ff - 2fgx \cos \frac{4\pi - \omega}{n} + ggxx$$

$$ff - 2fgx \cos \frac{4\pi + \omega}{n} + ggxx$$

etc.,

quorum tot sunt sumendi, donec eorum numerus fiat $= n$.

422. Si igitur proponatur ista fractio in suas fractiones simplices resolvenda

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1275

$$\frac{x^{m-1}}{f^{2n} - 2f^n g^n x^n \cos \omega + g^{2n} x^{2n}},$$

quoniam denominatoris factor trinomialis quicunque continetur in hac forma

$$ff - 2fgx \cos \varphi + ggxx$$

existente $\varphi = \frac{2k\pi \pm \omega}{n}$, consideretur ista fractio

$$\frac{x^m}{f^{2n}x - 2f^n g^n x^{n+1} \cos \omega + g^{2n} x^{2n+1}}$$

illi aequalis ac ponatur numerator $x^m = P$ ac denominator

$$f^{2n}x - 2f^n g^n x^{n+1} \cos \omega + g^{2n} x^{2n+1} = Q;$$

erit

$$\frac{dQ}{dx} = f^{2n} - 2f^n(n+1)g^n x^n \cos \omega + (2n+1)g^{2n} x^{2n}.$$

Hinc ponendo

$$x^n = \frac{f^n}{g^n} \cos n\varphi$$

erit

$$\mathfrak{P} = \frac{f^m}{g^m} \cos m\varphi \quad \text{seu} \quad \mathfrak{P} = \frac{f^m}{g^m} \cos \frac{m(2k\pi \pm \omega)}{n}$$

et

$$\mathfrak{Q} = f^{2n} (1 - 2(n+1) \cos \omega \cdot \cos n\varphi + (2n+1) \cos 2n\varphi).$$

Cum autem sit $\cos n\varphi = \cos \omega$, erit

$$\cos 2n\varphi = 2 \cos^2 \omega - 1$$

ideoque

$$\mathfrak{Q} = f^{2n} (-2n + 2n \cos^2 \omega) = -2nf^{2n} \sin^2 \omega.$$

Deinde posito

$$x^n = \frac{f^n}{g^n} \sin n\varphi$$

fiet

$$\mathfrak{p} = \frac{f^m}{g^m} \sin m\varphi = \frac{f^m}{g^m} \sin \frac{m(2k\pi \pm \omega)}{n}$$

et

$$\mathfrak{q} = -f^{2n} (2(n+1) \cos \omega \cdot \sin n\varphi - (2n+1) \sin 2n\varphi);$$

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

Translated and annotated by Ian Bruce.

1276

ob

$$\sin 2n\varphi = 2 \sin n\varphi \cdot \cos n\varphi = 2 \cos \omega \cdot \sin n\varphi$$

erit

$$q = 2nf^{2n} \cos \omega \cdot \sin n\varphi.$$

Cum autem sit $n\varphi = 2k\pi \pm \omega$, erit $\sin n\varphi = \pm \sin \omega$ et

$$q = \pm 2nf^{2n} \sin \omega \cdot \cos \omega.$$

His inventis erit

$$\begin{aligned} \Omega^2 + q^2 &= 4n^2 f^{4n} \sin^2 \omega \\ \mathfrak{P}q - p\Omega &= \frac{2nf^{m+2n}}{g^m} \left(\pm \cos m\varphi \cdot \sin \omega \cdot \cos \omega + \sin m\varphi \cdot \sin^2 \omega \right) \end{aligned}$$

sive

$$\mathfrak{P}q - p\Omega = \pm \frac{2nf^{m+2n}}{g^m} \sin \omega \cdot \cos(m\varphi \mp \omega)$$

seu

$$\begin{aligned} \mathfrak{P}q - p\Omega &= \pm \frac{2nf^{m+2n}}{g^m} \sin \omega \cdot \cos \left(\frac{2km\pi \pm (m-n)\omega}{n} \right), \\ \mathfrak{P}\Omega + pq &= \frac{2nf^{m+2n}}{g^m} \left(-\cos m\varphi \cdot \sin^2 \omega \pm \sin m\varphi \cdot \cos \omega \right), \\ \mathfrak{P}\Omega + pq &= \pm \frac{2nf^{m+2n}}{g^m} \sin \omega \cdot \sin(m\varphi \mp \omega) \end{aligned}$$

seu

$$\mathfrak{P}\Omega + pq = \pm \frac{2nf^{m+2n}}{g^m} \sin \omega \cdot \sin \left(\frac{2km\pi \pm (m-n)\omega}{n} \right).$$

Hinc ex denominatoris factore

$$ff - 2fgx \cos \frac{2k\pi \pm \omega}{n} + ggxx$$

nascitur ista fractio simplex

$$\frac{\pm f \sin \frac{2k\pi \pm \omega}{n} \cdot \cos \frac{2km\pi \pm (m-n)\omega}{n} \pm \sin \frac{2km\pi \pm (m-n)\omega}{n} (gx - f \cos \frac{2k\pi \pm \omega}{n})}{nf^{2n-m} g^{m-1} \sin \omega (ff - 2fgx \cos \frac{2k\pi \pm \omega}{n} + ggxx)}$$

seu

$$\frac{\pm gx \sin \frac{2km\pi \pm (m-n)\omega}{n} \pm f \sin \frac{2k(m-1)\pi \pm (m-n-1)\omega}{n}}{nf^{2n-m} g^{m-1} \sin \omega (ff - 2fgx \cos \frac{2k\pi \pm \omega}{n} + ggxx)}$$

EULER'S
INSTITUTIONUM CALCULI DIFFERENTIALIS PART 2

Chapter 18

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1277

EXEMPLUM

Resolvete hanc fractionem $\frac{x^{m-1}}{f^{2n}-2f^ng^n x^n \cos \omega + g^{2n} x^{2n}}$ in suas fractiones simplices.

Istae fractiones simplices quaesitae ergo erunt

$$\begin{aligned} & \frac{f \sin \frac{\omega}{n} \cdot \cos \frac{(m-n)\omega}{n} + \sin \frac{(m-n)\omega}{n} (gx - f \cos \frac{\omega}{n})}{nf^{2n-m} g^{m-1} \sin \omega (ff - 2fgx \cos \frac{\omega}{n} + ggxx)} \\ & - \frac{f \sin \frac{2\pi-\omega}{n} \cdot \cos \frac{2m\pi-(m-n)\omega}{n} + \sin \frac{2m\pi-(m-n)\omega}{n} (gx - f \cos \frac{2\pi-\omega}{n})}{nf^{2n-m} g^{m-1} \sin \omega (ff - 2fgx \cos \frac{2\pi-\omega}{n} + ggxx)} \\ & + \frac{f \sin \frac{2\pi+\omega}{n} \cdot \cos \frac{2m\pi+(m-n)\omega}{n} + \sin \frac{2m\pi+(m-n)\omega}{n} (gx - f \cos \frac{2k\pi+\omega}{n})}{nf^{2n-m} g^{m-1} \sin \omega (ff - 2fgx \cos \frac{2k\pi+\omega}{n} + ggxx)} \\ & - \frac{f \sin \frac{4\pi-\omega}{n} \cdot \cos \frac{4m\pi-(m-n)\omega}{n} + \sin \frac{4m\pi-(m-n)\omega}{n} (gx - f \cos \frac{4\pi-\omega}{n})}{nf^{2n-m} g^{m-1} \sin \omega (ff - 2fgx \cos \frac{4\pi-\omega}{n} + ggxx)} \\ & + \frac{f \sin \frac{4\pi+\omega}{n} \cdot \cos \frac{4m\pi+(m-n)\omega}{n} + \sin \frac{4m\pi+(m-n)\omega}{n} (gx - f \cos \frac{4k\pi+\omega}{n})}{nf^{2n-m} g^{m-1} \sin \omega (ff - 2fgx \cos \frac{4k\pi+\omega}{n} + ggxx)} \\ & \text{etc.} \end{aligned}$$

sicque eousque erit progrediendum, quoad harum fractionum numerus fuerit n . Si m fuerit numerus vel maior quam $2n - 1$ vel numerus negativus, priori casu partes integrae, posteriori vero fractiones insuper sunt adiiciendae, quae modo ante exposito facile inveniuntur.