CHAPTER XVII

CONCERNING THE INTERPOLATION OF SERIES

389. A series is said to be interpolated, provided that the terms of this may be assigned, which correspond to fractional or even to surd indices. Therefore if the general term of a series were known, the interpolation may be had without difficulty, since, whatever number may be substituted in place of the index x, that expression may give the corresponding term. Truly if a series were prepared thus, so that the general term cannot be shown in any way, then the interpolation of series of this kind generally is especially difficult nor mainly can terms corresponding to non integral indices be defined other than by infinite series. Therefore because in the preceding chapter we have determined the values corresponding to any indices of expressions of this kind, which are not able to be expressed in the customary manner, this treatment may bring a great usefulness to performing interpolations. Which use for this reason we may pursue more diligently here, which may follow on from the above chapter in this application.

390. Therefore let some proposed series be

$$\begin{matrix} 1 & & 2 & & 3 & & 4 & \cdots & x \\ A & + & B & + & C & + & D & +\cdots+ & X, \end{matrix}$$

of which the general term X shall be known, but the summatory term S may be hidden. Hence another series may be formed, the general term of which is equal to the summatory term of this series, and this new series will be

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & \\ A, & (A+B), & (A+B+C), & (A+B+C+D), & (A+B+C+D+E) & \text{etc.} \end{matrix}$$

and the general term, or corresponding to the indefinite index x, will be

$$= A+B+C+D+\cdots+X = S;$$

which since it may not be known explicitly, the interpolation of this new series will be troubled by the same difficulties, which we have mentioned before. Therefore towards interpolating this series it will be required to investigate the values of S, which it takes, if in place of x some non integral number may be substituted. For if x should be a whole number, then a fitting value of S may be found without difficulty, clearly by adding as many terms of the series $A+B+C+D+\text{etc.}$, as there are units contained in x.

391. Therefore so that those, which have been treated in the previous chapter, shall be called into use, we may put x to be a whole number, thus so that the value $S = A + B + C + \cdots + X$ corresponding to that shall be known, and we may seek the value Σ into which S may be transformed, if in place of x there may be written $x + \omega$ with some fraction ω present; and Σ will be the term of the proposed series interpolated, which corresponds to the index $x + \omega$; therefore from which the interpolation of this series shall be brought into view. Let Z be a term of the series A, B, C, D, E etc., which corresponds to the index $x + \omega$, and Z', Z'', Z'', etc. shall be the consecutive terms of this having the indices $x + \omega + 1$, $x + \omega + 2$, $x + \omega + 3$ etc. And in the first place indeed we may put the most infinite terms of the series A, B, C, D etc. to vanish. Therefore with these put in place the series

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \\ A, & (A+B), & (A+B+C), & (A+B+C+D), & (A+B+C+D+E) & \text{etc.,} \end{array}$$

the term of which corresponding to the index x is $S = A + B + C + \cdots + X$, it will be interpolated by seeking the term of this Σ, which may correspond to the fraction $x + \omega$; moreover it will be, as we have found,

$$\begin{aligned} \Sigma = S &+ X' + X'' + X''' + X'''' + \text{etc.} \\ &- Z' - Z'' - Z''' - Z'''' - \text{etc.} \end{aligned}$$

and thus the term sought Σ will be considered equal to this infinite series, which on account of

$$Z = X + \frac{\omega dX}{dx} + \frac{\omega^2 ddX}{1 \cdot 2 dx^2} + \frac{\omega^3 d^3 X}{1 \cdot 2 \cdot 3 dx^3} + \text{etc.}$$

may be transformed into this form, so that there shall be

$$\begin{aligned} \Sigma = S &- \frac{\omega}{dx} d.\left(X' + X'' + X''' + X'''' + \text{etc.} \right) \\ &- \frac{\omega^2}{2dx^2} dd.\left(X' + X'' + X''' + X'''' + \text{etc.} \right) \\ &- \frac{\omega^3}{6dx^3} d^3.\left(X' + X'' + X''' + X'''' + \text{etc.} \right) \\ &\text{etc.,} \end{aligned}$$

of which formulas that will be able to be used, which in any case may be seen to be more fitting.

392. We may take for A, B, C, D etc. any harmonic series

$$\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}+\frac{1}{a+3b}+\text{etc.},$$

the general term of which or corresponding to the index x is $=\frac{1}{a+(x-1)b}=X$. Hence this series shall be formed

$$\begin{matrix} 1 & 2 & 3 & 4 & \\ \frac{1}{a}, & \left(\frac{1}{a}+\frac{1}{a+b}\right), & \left(\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}\right), & \left(\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}+\frac{1}{a+3b}\right) & \text{etc.,} \end{matrix}$$

of which therefore the term corresponding to the index x will be

$$S=\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}+\frac{1}{a+3b}+\cdots+\frac{1}{a+(x-1)b}.$$

Now if Σ may denote the term of this series corresponding to the index $x+\omega$, on account of

$$Z=\frac{1}{a+(x+\omega-1)b}$$

$$\left.\begin{aligned} X'&=\frac{1}{a+bx}\\ X''&=\frac{1}{a+b+bx}\\ X'''&=\frac{1}{a+2b+bx}\\ &\text{etc.} \end{aligned}\right| \begin{aligned} Z'&=\frac{1}{a+bx+b\omega}\\ Z''&=\frac{1}{a+b+bx+b\omega}\\ Z'''&=\frac{1}{a+2b+bx+b\omega}\\ &\text{etc.,} \end{aligned}$$

and there may arise hence

$$\begin{aligned} \Sigma=S&+\frac{1}{a+bx}+\frac{1}{a+b+bx}+\frac{1}{a+2b+bx}+\text{etc.}\\ &-\frac{1}{a+bx+b\omega}-\frac{1}{a+b+bx+b\omega}-\frac{1}{a+2b+bx+b\omega}-\text{etc.;} \end{aligned}$$

but another expression will be of this kind

$$\Sigma = S + b\omega\left(\frac{1}{(a+bx)^2} + \frac{1}{(a+b+bx)^2} + \frac{1}{(a+2b+bx)^2} + \text{etc.}\right)$$
$$- b^2\omega^2\left(\frac{1}{(a+bx)^3} + \frac{1}{(a+b+bx)^3} + \frac{1}{(a+2b+bx)^3} + \text{etc.}\right)$$
$$+ b^3\omega^3\left(\frac{1}{(a+bx)^4} + \frac{1}{(a+b+bx)^4} + \frac{1}{(a+2b+bx)^4} + \text{etc.}\right)$$
etc.

EXAMPLE 1
Let this series be proposed

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1, & \left(1+\frac{1}{2}\right), & \left(1+\frac{1}{2}+\frac{1}{3}\right), & \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) \quad \text{etc.,} \end{array}$$

the terms of which may be required to be found, which correspond to fractional indices.

Therefore let there be $a = 1$ and $b = 1$; from which if the term corresponding to the integer x may be put

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x}$$

and the corresponding term to be indicated by the fraction $x + \omega$ may be called $= \Sigma$, there will be

$$\Sigma = S + \frac{1}{1+x} + \frac{1}{2+x} + \frac{1}{3+x} + \frac{1}{4+x} + \frac{1}{5+x} + \text{etc.}$$
$$- \frac{1}{1+x+\omega} - \frac{1}{2+x+\omega} - \frac{1}{3+x+\omega} - \frac{1}{4+x+\omega} - \frac{1}{5+x+\omega} - \text{etc.}$$

But it is to be noted, if the term corresponding to the fraction of the index ω, which we may put $= T$, from that the term of the index $x + \omega$ can be found easily; indeed there will be, if T', T'', T''' etc. may denote the terms with the corresponding indices $1+\omega, 2+\omega, 3+\omega$ etc.,

$$T' = T + \frac{1}{1+\omega}$$
$$T'' = T + \frac{1}{1+\omega} + \frac{1}{2+\omega}$$
$$T''' = T + \frac{1}{1+\omega} + \frac{1}{2+\omega} + \frac{1}{3+\omega}$$
etc.

from which it suffices to have investigated only those terms which correspond to indices ω less than one. Which in the end we may put $x = 0$; there will be also $S = 0$ and the term of the series T corresponding to the fraction of the index ω may be expressed thus

$$T = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.}$$
$$-\frac{1}{1+\omega} - \frac{1}{2+\omega} - \frac{1}{3+\omega} - \frac{1}{4+\omega} - \text{etc.}$$

or with these fractions changed into infinite series another expression will be produced

$$\begin{aligned} T = &+\omega\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\text{etc.}\right)\\ &-\omega^2\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\text{etc.}\right)\\ &+\omega^3\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\text{etc.}\right)\\ &-\omega^4\left(1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\text{etc.}\right)\\ &\text{etc.} \end{aligned}$$

which is extremely suitable for finding the approximate value of *T*.

Therefore there may be sought the term of the proposed series corresponding to the index $\frac{1}{2}$; which if it may be put $=T$, there will be

$$T = 1 - \frac{2}{3} + \frac{1}{2} - \frac{2}{5} + \frac{1}{3} - \frac{2}{7} + \frac{1}{4} - \frac{2}{9} + \text{etc.}$$

or

$$T = 2\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \text{etc.}\right),$$

the value of which series is $= 2 - 2l2$, and thus the term of the index $= \frac{1}{2}$ is able to be expressed finitely. Therefore the following terms, the indices of which are $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$ etc., thus will be able to be expressed

Index	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	
Term	$2-2l2,$	$2+\frac{2}{3}-2l2,$	$2+\frac{2}{3}+\frac{2}{5}-2l2$	$2+\frac{2}{3}+\frac{2}{5}+\frac{2}{7}-2l2$	etc.

EXAMPLE 2
This series shall be proposed

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \\ 1, & \left(1+\tfrac{1}{3}\right), & \left(1+\tfrac{1}{3}+\tfrac{1}{5}\right), & \left(1+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}\right) & \text{etc.,} \end{array}$$

of which it may be required to express the corresponding terms indicated by fractions.

Therefore there will be $a=1, b=2$; from which if there may be put the term corresponding to the whole integer x

$$S=1+\tfrac{1}{3}+\tfrac{1}{5}+\cdots+\tfrac{1}{2x-1}$$

and the term to be indicated by the fraction $x+\omega$ may be called $=\Sigma$, there will be

$$\begin{aligned} \Sigma = S &+ \tfrac{1}{1+2x} + \tfrac{1}{3+2x} + \tfrac{1}{5+2x} + \tfrac{1}{7+2x} + \text{etc.} \\ &- \tfrac{1}{1+2(x+\omega)} - \tfrac{1}{3+2(x+\omega)} - \tfrac{1}{5+2(x+\omega)} - \tfrac{1}{7+2(x+\omega)} - \text{etc.} \end{aligned}$$

Therefore since it may suffice to have assigned the terms with indices less than unity, there shall be $x=0$ and $S=0$; on account of which, if the term agreeing with the index ω may be put $=T$, there will be

$$\begin{aligned} T = 1 &+ \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{9} + \text{etc.} \\ &- \tfrac{1}{1+2\omega} - \tfrac{1}{3+2\omega} - \tfrac{1}{5+2\omega} - \tfrac{1}{7+\omega} - \tfrac{1}{9+\omega} - \text{etc.,} \end{aligned}$$

and if ω may be put to denote some number, because T is the term corresponding to the index ω, T will be the general term of the proposed series, which also may be expressed in this manner

$$T=\tfrac{2\omega}{1(1+2\omega)}+\tfrac{2\omega}{3(3+2\omega)}+\tfrac{2\omega}{5(5+2\omega)}+\tfrac{2\omega}{7(7+2\omega)}+\text{etc.}$$

or thus

$$T = 2\omega\left(1+\tfrac{1}{3^2}+\tfrac{1}{5^2}+\tfrac{1}{7^2}+\text{etc.}\right)$$
$$-4\omega^2\left(1+\tfrac{1}{3^3}+\tfrac{1}{5^3}+\tfrac{1}{7^3}+\text{etc.}\right)$$
$$+8\omega^3\left(1+\tfrac{1}{3^4}+\tfrac{1}{5^4}+\tfrac{1}{7^4}+\text{etc.}\right)$$
$$-16\omega^4\left(1+\tfrac{1}{3^5}+\tfrac{1}{5^5}+\tfrac{1}{7^5}+\text{etc.}\right)$$
$$\text{etc.}$$

We may put to be $\omega = \tfrac{1}{2}$ the term corresponding to this index will be

$$T = 1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\text{etc.} = l2$$

and there will be

Indices:	$\tfrac{1}{2}$	$\tfrac{3}{2}$	$\tfrac{5}{2}$	$\tfrac{7}{2}$	
Terms:	$l2,$	$\tfrac{1}{2}+l2,$	$\tfrac{1}{2}+\tfrac{1}{4}+l2,$	$\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{6}+l2$	etc.

If there shall be $\omega = \tfrac{1}{4}$, there will be

$$T = 1+\tfrac{1}{3}+\ \tfrac{1}{5}+\tfrac{1}{7}+\text{etc.}$$
$$-\tfrac{2}{3}-\tfrac{2}{7}-\tfrac{2}{11}-\tfrac{2}{15}-\text{etc.},$$

Or

$$T = 1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\text{etc.}-\tfrac{1}{2}l2 = \tfrac{\pi}{4}-\tfrac{1}{2}l2.$$

393. Therefore if the term of this general series

$$\tfrac{1}{a},\ \left(\tfrac{1}{a}+\tfrac{1}{a+b}\right),\ \left(\tfrac{1}{a}+\tfrac{1}{a+b}+\tfrac{1}{a+2b}\right)\ \text{etc.}$$

corresponding to the index $=\tfrac{1}{2}$ may be sought, there may be put $x = 0$ and $\omega = \tfrac{1}{2}$ in the expressions of the preceding paragraph and there is comes about $S = 0$ and the term sought corresponding to the index $\tfrac{1}{2}$ will be

$$\Sigma = \tfrac{1}{a}-\tfrac{2}{2a+b}+\tfrac{1}{a+b}-\tfrac{2}{2a+3b}+\tfrac{1}{a+2b}-\tfrac{2}{2a+5b}+\text{etc.}$$

or with terms producing greater uniformity there will be

$$\tfrac{1}{2}\Sigma = \tfrac{1}{2a}-\tfrac{1}{2a+b}+\tfrac{1}{2a+2b}-\tfrac{1}{2a+3b}+\tfrac{1}{a+4b}-\ \text{etc.};$$

in which the series may alternate with the sign + and – , the value of $\frac{1}{2}\Sigma$ may be expressed with the series converging more from the continued differences taken by the method set out above in § 8.

But the series of the differences will be

$$\frac{-b}{2a(2a+b)},\quad \frac{-b}{(2a+b)(2a+2b)},\quad \frac{-b}{(2a+2b)(2a+3b)} \text{ etc.}$$
$$\frac{2bb}{2a(2a+b)(2a+b)},\quad \frac{2bb}{(2a+b)(2a+2b)(2a+3b)} \text{ etc.}$$
$$\frac{-6b^3}{2a(2a+b)(2a+2b)(2a+3b)} \text{ etc.}$$
$$\text{etc.}$$

From which it is concluded to become

$$\tfrac{1}{2}\Sigma = \frac{1}{4a} + \frac{1b}{8a(2a+b)} + \frac{1\cdot 2bb}{16a(2a+b)(2a+2b)} + \frac{1\cdot 2\cdot 3b^3}{32a(2a+b)(2a+2b)(2a+3b)} + \text{etc.}$$

And hence therefore there will be had

$$\Sigma = \frac{1}{2a} + \frac{\frac{1}{2}b}{2a(2a+b)} + \frac{\frac{1}{2}\cdot\frac{2}{2}bb}{2a(2a+b)(2a+2b)} + \frac{\frac{1}{2}\cdot\frac{2}{2}\cdot\frac{3}{2}b^3}{2a(2a+b)(2a+2b)(2a+3b)} + \text{etc.},$$

which series will converge especially and the value of the term Σ is shown approximately by an easy effort.

394. But if in general the most infinite terms of the series A, B, C, D, E etc. may vanish and the term corresponding to the index ω were $= Z$ and of that following, which may correspond to the indices $\omega+1, \omega+2, \omega+3$ etc., shall be Z', Z'', Z''', Z'''' etc., if into the above (§ 391) there may be put $x = 0$, so that there shall be $S = 0$ and $X' = A,\ X'' = B,\ X''' = C$ etc., it will follow, if a series of this kind may be formed

$$\begin{matrix} 1 & 2 & 3 & 4 \\ A, & (A+B), & (A+B+C), & (A+B+C+D) \text{ etc.} \end{matrix}$$

and the term of that corresponding to the index ω may be put $= \Sigma$, to become

$$\Sigma = (A - Z') + (B - Z'') + (C - Z''') + (D - Z'''') + \text{etc.},$$

from which expression any intermediate terms will be able to be defined. But it may suffice to have investigated these terms according to the interpolation required to be performed, which may correspond to the indices ω smaller than unity. If indeed the term Σ were found corresponding to any index of this kind ω and those, which may agree with the indices $\omega+1, \omega+2, \omega+3$ etc., may be put $\Sigma', \Sigma'', \Sigma'''$, etc., there will be

$$\Sigma' = \Sigma + Z'$$
$$\Sigma'' = \Sigma + Z' + Z''$$
$$\Sigma''' = \Sigma + Z' + Z'' + Z'''$$
etc.

EXAMPLE 1

To interpolate this series

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1, & \left(1+\frac{1}{4}\right), & \left(1+\frac{1}{4}+\frac{1}{9}\right), & \left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}\right) \end{array} \text{ etc.,}$$

Let Σ be the term of this series corresponding to the index ω, and since this series shall formed from the summation of this

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{etc.},$$

the term of which corresponding to the index ω is $=\frac{1}{\omega^2}$, there will be

$$\Sigma = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}$$
$$-\frac{1}{(1+\omega)^2} - \frac{1}{(2+\omega)^2} - \frac{1}{(3+\omega)^2} - \frac{1}{(4+\omega)^2} - \text{etc.}$$

But if therefore the term corresponding to the index $\frac{1}{2}$ of the proposed series may be sought, it will be required for $\omega = \frac{1}{2}$ to be put and there becomes

$$\Sigma = 1 - \frac{4}{9} + \frac{1}{4} - \frac{4}{25} + \frac{1}{9} - \frac{4}{49} + \text{etc.}$$

or

$$\Sigma = 4\left(\frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \frac{1}{36} - \frac{1}{49} + \text{etc}\right).$$

Therefore since there shall be

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{25} + \text{etc.} = \frac{\pi^2}{12}$$

there will be

$$\Sigma = 4\left(1 - \frac{\pi^2}{12}\right) = 4 - \frac{1}{3}\pi^2$$

which is the term corresponding to the index $\frac{1}{2}$. Hence therefore there will correspond

to the indices	$\frac{1}{2}$,	$\frac{3}{2}$,	$\frac{5}{2}$	etc.
the terms	$4-\frac{1}{3}\pi^2$,	$\frac{4}{1}+\frac{4}{9}-\frac{1}{3}\pi^2$,	$\frac{4}{1}+\frac{4}{9}+\frac{4}{25}-\frac{1}{3}\pi^2$	etc.

EXAMPLE 2

To interpolate this series

1	2	3	4	
1,	$\left(1+\frac{1}{9}\right)$,	$\left(1+\frac{1}{9}+\frac{1}{25}\right)$,	$\left(1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}\right)$	etc.

Let Σ be the term corresponding to any index ω, and since this series shall be formed from the summation of this

$$1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}+\text{etc.},$$

from which the term corresponding to the index ω becomes $Z=\frac{1}{(2\omega-1)^2}$, there will be

$$Z'=\frac{1}{(2\omega+1)^2},\quad Z''=\frac{1}{(2\omega+3)^2},\quad Z'''=\frac{1}{(2\omega+5)^2}\quad \text{etc.}$$

On account of which there will be had

$$\Sigma = 1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\text{etc.}$$
$$-\frac{1}{(1+2\omega)^2}-\frac{1}{(3+2\omega)^2}-\frac{1}{(5+2\omega)^2}-\frac{1}{(7+2\omega)^2}-\text{etc.}$$

We may put $\omega=\frac{1}{2}$ so that we may find the term of the proposed series corresponding to the index $=\frac{1}{2}$,
which will be

$$\Sigma = 1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\text{etc.}=\frac{\pi\pi}{12},$$

from which terms, which may be placed in the middle between any two given terms, will be expressed in the following manner. There will correspond to the indices $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$ etc.

the terms $\frac{\pi\pi}{12}$, $\frac{1}{4}+\frac{\pi\pi}{12}$, $\frac{1}{4}+\frac{1}{16}+\frac{\pi\pi}{12}$, $\frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\frac{\pi\pi}{12}$ etc.

EXAMPLE 3

To interpolate this series

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \\ 1, & \left(1+\frac{1}{2^n}\right), & \left(1+\frac{1}{2^n}+\frac{1}{3^n}\right), & \left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}\right) & \text{etc.} \end{array}$$

As before let Σ be the term corresponding to the index ω ; there will be $Z=\frac{1}{\omega^n}$ and

$$Z'=\frac{1}{(1+\omega)^2},\quad Z''=\frac{1}{(2+\omega)^2},\quad Z'''=\frac{1}{(3+\omega)^2}\quad \text{etc.}$$

hence there will be had

$$\begin{aligned} \Sigma = \; & 1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\text{etc.} \\ & -\frac{1}{(1+\omega)^2}-\frac{1}{(2+\omega)^2}-\frac{1}{(3+\omega)^2}-\frac{1}{(4+\omega)^2}-\text{etc.} \end{aligned}$$

Therefore if we may desire the term corresponding to the index $\frac{1}{2}$, that will be

$$=1-\frac{2^n}{3^n}+\frac{1}{2^n}-\frac{2^n}{5^n}+\frac{1}{3^n}-\frac{2^n}{7^n}+\text{etc.}$$

or

$$=2^n\left(\frac{1}{2^n}-\frac{1}{3^n}+\frac{1}{4^n}-\frac{1}{5^n}+\frac{1}{6^n}-\frac{1}{7^n}+\text{etc.}\right).$$

Whereby if there may be put

$$\mathfrak{N}=1-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\frac{1}{5^n}-\frac{1}{6^n}+\text{etc.}$$

the term of the proposed series will be $=2^n(1-\mathfrak{N})$, which corresponds to the index $\frac{1}{2}$*;* and hence there will correspond

to the indices $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$ etc.

the terms $2^n-2^n\mathfrak{N}$, $2^n+\frac{2^n}{3^n}-2^n\mathfrak{N}$, $2^n+\frac{2^n}{3^n}+\frac{2^n}{5^n}-2^n\mathfrak{N}$ etc.

EXAMPLE 4

To interpolate this series

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1, & \left(1+\frac{1}{3^n}\right), & \left(1+\frac{1}{3^n}+\frac{1}{5^n}\right), & \left(1+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}\right) \quad \text{etc.} \end{array}$$

Let Σ be the term, which may correspond to some index ω, and since there shall be $Z=\frac{1}{(2\omega-1)^n}$, there will be

$$Z'=\frac{1}{(2\omega+1)^n}, \quad Z''=\frac{1}{(2\omega+3)^n}, \quad Z'''=\frac{1}{(2\omega+5)^n} \quad \text{etc.}$$

and

$$\Sigma = 1+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\text{etc.}$$
$$-\frac{1}{(1+2\omega)^n}-\frac{1}{(3+2\omega)^n}-\frac{1}{(5+2\omega)^n}-\frac{1}{(7+2\omega)^n}-\text{etc.}$$

There may be put $\omega=\frac{1}{2}$ and the term will be produced corresponding to the index $\frac{1}{2}$

$$=1-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\frac{1}{5^n}-\frac{1}{6^n}+\text{etc.}=\mathfrak{N},$$

from which again there will be the remaining middle terms between the pairs of given terms

Indices:	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	etc.
Terms:	$\mathfrak{N}$,	$\frac{1}{2^n}+\mathfrak{N}$,	$\frac{1}{2^n}+\frac{1}{4^n}+\mathfrak{N}$	etc.

395. Now we may put the most infinite terms of the series A, B, C, D, E etc. not to vanish, from which the summation of the series may be formed by interpolation, but to be prepared thus, so that the differences of those may vanish, and let X be a term of that series corresponding to the index x and Z the term corresponding to the exponent $x+\omega$*;* then truly X', X'', X''', X'''' etc. shall be the following terms of X and Z', Z'', Z''', Z'''' etc. the following terms of Z. With which in place this series may be proposed requiring to be interpolated

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ A, & (A+B), & (A+B+C), & (A+B+C+D) \quad \text{etc.} \end{array}$$

of which the term corresponding to the index x shall be $=S$, but the term corresponding to the index $x+\omega$ shall be $=\Sigma$, and from these, which have been treated in the preceding chapter, there will be

$$\begin{aligned}\Sigma = S + X' + X'' + X''' + \text{etc.}\\ - Z' - Z'' - Z''' - \text{ etc.}\\ +\omega X' + \omega\begin{Bmatrix} +X'' + X''' + X'''' + \text{etc.}\\ -X' - X'' - X''' - \text{etc.}\end{Bmatrix}\end{aligned}$$

But because as before it will suffice to have investigated terms with the corresponding indices less than unity, we may put $x=0$, so that there shall be $S=0,\ X'=A,\ X''=B$ etc., and the term corresponding to the index w will be

$$\begin{aligned}\Sigma &= (A-Z') + (B-Z'') + (C-Z''') + (D-Z'''') \text{ etc.}\\ &+ \omega A + \omega\big((B-A) + (C-B) + (D-C) + (E-D) + \text{etc.}\big).\end{aligned}$$

Or if we may wish to express these in the manner received above, so that there is

$\Delta A = B - A,\ \Delta B = C - B$ etc., there will be had

$$\begin{aligned}\Sigma &= (A-Z') + (B-Z'') + (C-Z''') + (D-Z'''') \text{ etc.}\\ &+ \omega(A + \Delta A + \Delta B + \Delta C + \Delta D + \text{etc.}).\end{aligned}$$

396. But if the most infinite terms of the series A, B, C, D, E etc., from which summation the series may be formed requiring to be interpolated, neither may vanish themselves nor may they have vanishing first differences, then more series must be added to express the value of Σ, evidently until there may be come upon vanishing differences of the most infinite terms. Indeed as before let the term of the series A, B, C, D, E etc. be $=X$ corresponding to the index x and that following X', X'', X'', etc., but the term Z may correspond to the index $x+\omega$, and which may follow $Z', Z'' Z'''$ etc., and this series may be proposed

$$\begin{matrix} 1 & 2 & 3 & 4 & \\ A, & (A+B), & (A+B+C), & (A+B+C+D) & \text{etc.}\end{matrix}$$

of which the term corresponding to the index x shall be

$$S = A + B + C + D + \cdots + X,$$

truly the term Σ may correspond to the index $x+\omega$, thus so that

for the indices	the terms may correspond
$x+\omega+1$	$\Sigma' = \Sigma + Z'$
$x+\omega+2$	$\Sigma'' = \Sigma + Z' + Z''$
$x+\omega+3$	$\Sigma''' = \Sigma + Z' + Z'' + Z''$
etc.	etc.

If now the differences of the terms may be expressed thus, so that there shall be

$$\Delta X' = X'' - X', \quad \Delta X'' = X''' - X'', \quad \Delta X''' = X'''' - X''' \quad \text{etc.}$$
$$\Delta^2 X' = \Delta X'' - \Delta X', \quad \Delta^2 X'' = \Delta X''' - \Delta X'', \quad \Delta^2 X''' = \Delta X'''' - \Delta X''' \quad \text{etc.}$$
$$\Delta^3 X' = \Delta^2 X'' - \Delta^2 X', \quad \Delta^3 X'' = \Delta^2 X''' - \Delta^2 X'' \quad \text{etc.},$$

from § 377 the term Σ will be expressed in the following manner:

$$\begin{aligned}\Sigma = S &+ X' + X'' + X''' + X'''' + \text{etc.}\\ &- Z' - Z'' - Z''' - Z'''' - \text{etc.}\\ &+ \omega\left(X' + \Delta X' + \Delta X'' + \Delta X''' + \Delta X'''' + \text{etc.}\right)\\ &+ \tfrac{\omega(\omega-1)}{1\cdot 2}\left(\Delta X' + \Delta^2 X' + \Delta^2 X'' + \Delta^2 X''' + \Delta^2 X'''' + \text{etc}\right)\\ &+ \tfrac{\omega(\omega-1)(\omega-2)}{1\cdot 2\cdot 3}\left(\Delta^2 X' + \Delta^3 X' + \Delta^3 X'' + \Delta^3 X''' + \Delta^3 X'''' + \text{etc}\right)\\ &\qquad \text{etc.}\end{aligned}$$

397. As we have noted, it is sufficient that as many series of this kind may be added, until vanishing differences of the most infinite terms may be come upon ; if indeed we wish these series also to continue to infinity or at least to that point, until the differences of the finite terms vanish, then on account of

$$Z' = X' + \omega\Delta X' + \tfrac{\omega(\omega-1)}{1\cdot 2}\Delta^2 X' + \tfrac{\omega(\omega-1)(\omega-2)}{1\cdot 2\cdot 3}\Delta^3 X' + \text{etc.}$$

the whole expression found may be contracted into this

$$\Sigma = S + \omega X' + \tfrac{\omega(\omega-1)}{1\cdot 2}\Delta X' + \tfrac{\omega(\omega-1)(\omega-2)}{1\cdot 2\cdot 3}\Delta^2 X' + \text{etc.},$$

which has involved the term of summation of the series $A + B + C + D + \text{etc.}$; but which if it may be known, the interpolation may have no difficulty. Yet meanwhile also it will be permitted to use this formula, certainly which, as many times as it may be interrupted, shows some term expressed to be interpolated finitely and algebraically ; but if it may progress to infinity, generally it is better to use the first formula, in which an account of the most infinite terms is had. Indeed this, if there may be put $x=0$, so that Σ may denote some term corresponding to the index ω, on account of $S=0$, may adopt this form

$$\begin{aligned}\Sigma &= A + B + C + D + \text{etc.}\\ &\quad - Z' - Z'' - Z''' - Z'''' - \text{etc.}\\ &+\omega\left(A + \Delta A + \Delta B + \Delta C + \Delta D + \text{etc.}\right)\\ &+\frac{\omega(\omega-1)}{1\cdot 2}\left(\Delta A + \Delta^2 A + \Delta^2 B + \Delta^2 C + \Delta^2 D + \text{etc}\right)\\ &+\frac{\omega(\omega-1)(\omega-2)}{1\cdot 2\cdot 3}\left(\Delta^2 A + \Delta^3 A + \Delta^3 B + \Delta^3 C + \Delta^3 D + \text{etc}\right)\\ &\qquad\qquad \text{etc.}\end{aligned}$$

Or if there is put for the sake of brevity

$$\omega=\alpha,\quad \frac{\omega(\omega-1)}{1\cdot 2}=\beta,\quad \gamma=\frac{\omega(\omega-1)(\omega-2)}{1\cdot 2\cdot 3}\quad \text{etc.},$$

there will be

$$\begin{aligned}\Sigma &= \alpha A + \beta\Delta A + \gamma\Delta^2 A + \delta\Delta^3 A + \text{etc.}\\ &\quad + A + \alpha\Delta A + \beta\Delta^2 A + \gamma\Delta^3 A + \text{etc.} - Z'\\ &\quad + B + \alpha\Delta B + \beta\Delta^2 B + \gamma\Delta^3 B + \text{etc.} - Z''\\ &\quad + C + \alpha\Delta C + \beta\Delta^2 C + \gamma\Delta^3 C + \text{etc.} - Z'''\\ &\qquad\qquad \text{etc.},\end{aligned}$$

the number of which horizontal series indeed progresses to infinity, but agrees with any finite number of terms.

EXAMPLE

To interpolate this series

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \frac{1}{2}, & \left(\frac{1}{2}+\frac{2}{3}\right), & \left(\frac{1}{2}+\frac{2}{3}+\frac{3}{4}\right), & \left(\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}\right) \quad \text{etc.}\end{array}$$

Let the term of this series corresponding to the index ω be $=\Sigma$, and since that may arise from the summation of this series $\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5}$ etc., there will be $Z=\frac{\omega}{\omega+1}$, and because the most infinite terms have their first differences vanishing, only the first differences are to be taken, which will be on account of

$$A = \tfrac{1}{2}, \quad B = \tfrac{2}{3}, \quad C = \tfrac{3}{4}, \quad D = \tfrac{4}{5} \text{ etc.}$$
$$\Delta A = \tfrac{1}{2\cdot 3}, \quad \Delta B = \tfrac{1}{3\cdot 4} \quad \Delta C = \tfrac{1}{4\cdot 5} \text{ etc.}$$

Hence therefore there will be had

$$\begin{aligned}\Sigma = \tfrac{\omega}{2} &+ \tfrac{1}{2} + \tfrac{2}{3} + \tfrac{3}{4} + \tfrac{4}{5} + \text{etc.}\\ &+ \tfrac{\omega}{2\cdot 3} + \tfrac{\omega}{3\cdot 4} + \tfrac{\omega}{4\cdot 5} + \tfrac{\omega}{5\cdot 6} + \text{etc.}\\ &- \tfrac{\omega+1}{\omega+2} - \tfrac{\omega+2}{\omega+3} - \tfrac{\omega+3}{\omega+4} - \tfrac{\omega+4}{\omega+5} - \text{etc.}\end{aligned}$$

or on account of

$$\tfrac{\omega}{2} + \tfrac{\omega}{2\cdot 3} + \tfrac{\omega}{3\cdot 4} + \tfrac{\omega}{4\cdot 5} + \text{etc.} = \omega$$

there will be

$$\begin{aligned}\Sigma = \omega &+ \tfrac{1}{2} + \tfrac{2}{3} + \tfrac{3}{4} + \tfrac{4}{5} + \text{etc.}\\ &- \tfrac{\omega+1}{\omega+2} - \tfrac{\omega+2}{\omega+3} - \tfrac{\omega+3}{\omega+4} - \tfrac{\omega+4}{\omega+5} - \text{etc.}\end{aligned}$$

If therefore the term is sought corresponding to the index $\tfrac{1}{2}$, this will be

$$\Sigma = \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{3}{5} + \tfrac{2}{3} - \tfrac{5}{7} + \tfrac{3}{4} - \tfrac{7}{9} + \tfrac{4}{5} - \tfrac{9}{11} + \text{etc.}$$

or

$$\Sigma = \tfrac{1}{2} - \tfrac{1}{2\cdot 5} - \tfrac{1}{3\cdot 7} - \tfrac{1}{4\cdot 9} - \tfrac{1}{5\cdot 11} - \tfrac{1}{6\cdot 13} - \text{etc.}$$

and thus

$$\tfrac{1}{2}\Sigma = \tfrac{1}{4} - \tfrac{1}{4\cdot 5} - \tfrac{1}{6\cdot 7} - \tfrac{1}{8\cdot 9} - \tfrac{1}{10\cdot 11} - \tfrac{1}{12\cdot 13} - \text{etc.}$$

or

$$\begin{aligned}\tfrac{1}{2}\Sigma = \tfrac{1}{4} &- \tfrac{1}{4} - \tfrac{1}{6} - \tfrac{1}{8} - \tfrac{1}{10} - \tfrac{1}{12} - \text{etc.}\\ &+ \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} + \tfrac{1}{13} + \text{etc.}\end{aligned}$$

Whereby, since there shall be

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \text{etc.} = l2,$$

there will be

$$\tfrac{1}{2}\Sigma = l2 - 1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} = l2 - \tfrac{7}{12}$$

and thus

$$\Sigma = 2l2 - \tfrac{7}{6}.$$

398. We may now go on to series requiring to be interpolated, the terms of which have been constructed from factors, and this shall be the most general series proposed

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ A, & AB, & ABC, & ABCD, & ABCDE \end{array} \quad \text{etc.},$$

the term of which corresponding to the index ω shall be $=\Sigma$. Therefore $l2$ will be the term corresponding to the index ω in this series

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ lA, & (lA+lB), & (lA+lB+lC), & (lA+lB+lC+lD) \end{array} \quad \text{etc.},$$

But if therefore we may put the most infinite terms to vanish and the term of the series *A, B, C, D, E* etc. corresponding to the index ω to be Z and with the indices following $\omega+1, \omega+2, \omega+3, \omega+4$ etc. to be corresponding to Z', Z'', Z''', Z'''' etc., there will be from the above demonstrations

$$\begin{aligned} l\Sigma = lA \;+lB \;+lC \;+lD \;&+\text{etc.} \\ -lZ'-lZ''-lZ'''-lZ''''&-\text{etc.} \end{aligned}$$

Hence therefore there will be had on progressing to numbers

$$\Sigma = \tfrac{A}{Z'} \cdot \tfrac{B}{Z''} \cdot \tfrac{C}{Z'''} \cdot \tfrac{D}{Z''''} \cdot \text{etc.}$$

399. But if the logarithms of the most infinite terms of the series *A, B, C, D* etc. may not vanish, but they may have vanishing differences, there will be as we have seen,

$$\begin{aligned} l\Sigma = lA \;+lB \;+lC \;+lD \;&+\text{etc.} \\ -lZ'-lZ''-lZ'''-lZ''''&-\text{etc.} \\ +\omega lA+\omega\left(l\tfrac{B}{A}+l\tfrac{C}{B}+l\tfrac{D}{C}+\text{etc.}\right)& \end{aligned}$$

and hence on proceeding to numbers from logarithms there comes about

$$\Sigma = A^{\omega} \cdot \tfrac{A^{1-\omega}B^{\omega}}{Z'} \cdot \tfrac{B^{1-\omega}C^{\omega}}{Z''} \cdot \tfrac{C^{1-\omega}D^{\omega}}{Z'''} \cdot \tfrac{D^{1-\omega}E^{\omega}}{Z''''} \cdot \text{etc.}$$

But if the second differences of these most infinite logarithms may vanish at last, there will be

$$\begin{aligned} l\Sigma = {} & lA + lB + lC + lD + \text{etc.} \\ & - lZ' - lZ'' - lZ''' - lZ'''' - \text{etc.} \\ & + \omega\left(lA + l\tfrac{B}{A} + l\tfrac{C}{B} + l\tfrac{D}{C} + l\tfrac{E}{D} + \text{etc.}\right) \\ & + \tfrac{\omega(\omega-1)}{1\cdot 2}\left(l\tfrac{B}{A} + l\tfrac{AC}{B^2} + l\tfrac{BD}{C^2} + l\tfrac{CE}{D^2}\, l\tfrac{DF}{E^2} + \text{etc.}\right). \end{aligned}$$

And from these there will be obtained

$$\Sigma = A^{\frac{\omega(3-\omega)}{2}} \cdot B^{\frac{\omega(\omega-1)}{1\cdot 2}} \cdot \frac{A^{\frac{(\omega-1)(\omega-2)}{1\cdot 2}} B^{\omega(2-\omega)} C^{\frac{\omega(\omega-1)}{1\cdot 2}}}{Z'} \cdot \frac{B^{\frac{(\omega-1)(\omega-2)}{1\cdot 2}} C^{\omega(2-\omega)} D^{\frac{\omega(\omega-1)}{1\cdot 2}}}{Z''} \cdot \text{ etc.},$$

which, if $\omega < 1$, thus may be expressed more conveniently

$$\Sigma = \frac{A^{\frac{\omega(3-\omega)}{2}}}{B^{\frac{\omega(1-\omega)}{1\cdot 2}}} \cdot \frac{A^{\frac{\omega(1-\omega)(2-\omega)}{1\cdot 2}} B^{\omega(2-\omega)}}{C^{\frac{\omega(1-\omega)}{1\cdot 2}} Z'} \cdot \frac{B^{\frac{\omega(1-\omega)(2-\omega)}{1\cdot 2}} C^{\omega(2-\omega)}}{D^{\frac{\omega(1-\omega)}{1\cdot 2}} Z''} \cdot \text{etc.}$$

400. We may adapt this interpolation to that series

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \\ \frac{a}{b}, & \frac{a(a+c)}{b(b+c)}, & \frac{a(a+c)(a+2c)}{b(b+c)(b+2c)}, & \frac{a(a+c)(a+2c)(a+3c)}{b(b+c)(b+2c)(b+3c)} & \text{etc.,} \end{array}$$

the factors of which have been selected from this series

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \\ \frac{a}{b}, & \frac{a+c}{b+c}, & \frac{(a+2c)}{(b+2c)}, & \frac{(a+3c)}{(b+3c)} & \text{etc.,} \end{array}$$

of which the logarithms of the most infinite terms are $= 0$. Therefore there will be

$$Z = \frac{a-c+c\omega}{b-c+c\omega}, \quad Z' = \frac{a+c\omega}{b+c\omega} \quad \text{etc.}$$

Hence, if the term of that series corresponding to the index ω may be put $= \Sigma$, there will be from § 398

$$\Sigma = \frac{a(b+c\omega)}{b(a+c\omega)} \cdot \frac{(a+c)(b+c+c\omega)}{(b+c)(a+c+c\omega)} \cdot \frac{(a+2c)(b+2c+c\omega)}{(b+2c)(a+2c+c\omega)} \cdot \text{etc.}$$

Whereby if the term corresponding to the index $\frac{1}{2}$ may be desired, on making $\omega = \frac{1}{2}$ there will be

$$\Sigma = \frac{a(2b+c)}{b(2a+c)} \cdot \frac{(a+c)(2b+3c)}{(b+c)(2a+3c)} \cdot \frac{(a+2c)(2b+5c)}{(b+2c)(2a+5c)} \cdot \text{etc.}$$

EXAMPLE

To interpolate this series

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & \\ \frac{1}{2}, & \frac{1\cdot3}{2\cdot4}, & \frac{1\cdot3\cdot5}{2\cdot4\cdot6}, & \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}, & \frac{1\cdot3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot10} & \text{etc.} \end{matrix}$$

Since here there shall be $a = 1$, $b = 2$, et $c = 2$, if the term corresponding to some $\omega = \Sigma$, there will be

$$\Sigma = \frac{1(2+2\omega)}{2(1+2\omega)} \cdot \frac{3(4+2\omega)}{4(3+2\omega)} \cdot \frac{5(6+2\omega)}{6(5+2\omega)} \cdot \frac{7(8+2\omega)}{8(7+2\omega)} \cdot \text{etc.}$$

Hence if the terms, which correspond to the indices $\omega+1$, $\omega+2$, $\omega+3$ etc., may be put Σ', Σ'', Σ''' etc., there will be

$$\Sigma' = \frac{1+2\omega}{2+2\omega} \cdot \Sigma$$

$$\Sigma'' = \frac{1+2\omega}{2+2\omega} \cdot \frac{3+2\omega}{4+2\omega} \cdot \Sigma$$

$$\Sigma''' = \frac{1+2\omega}{2+2\omega} \cdot \frac{3+2\omega}{4+2\omega} \cdot \frac{5+2\omega}{6+2\omega} \cdot \Sigma$$

etc.

And thus if the term may be desired corresponding to the index $\frac{1}{2}$, on making $\omega = \frac{1}{2}$ there will be

$$\Sigma = \frac{1\cdot3}{2\cdot2} \cdot \frac{3\cdot5}{4\cdot4} \cdot \frac{5\cdot7}{6\cdot6} \cdot \frac{7\cdot9}{8\cdot8} \cdot \frac{9\cdot11}{10\cdot10} \cdot \text{etc.}$$

Truly on putting $\pi =$ semi–circumference of the circle, the radius of which is $= 1$, we have shown the above to be

$$\pi = 2 \cdot \frac{2\cdot2}{1\cdot3} \cdot \frac{4\cdot4}{3\cdot5} \cdot \frac{6\cdot6}{5\cdot7} \cdot \frac{8\cdot8}{7\cdot9} \cdot \text{etc.}$$

Because of this, the intermediate terms for the indices $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$ etc. will be able to be expressed by the periphery of a circle in this manner

$$\text{Indices: } \quad \tfrac{1}{2} \qquad \tfrac{3}{2} \qquad \tfrac{5}{2} \qquad \tfrac{7}{2}$$

$$\text{Terms: } \quad \tfrac{2}{\pi}, \quad \tfrac{2}{3}\cdot\tfrac{2}{\pi}, \quad \tfrac{2\cdot 4}{3\cdot 5}\cdot\tfrac{2}{\pi}, \quad \tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\tfrac{2}{\pi} \quad \text{etc.}$$

Which same interpolation is come upon in the *Arithmetica infinitorum* of Wallis [Book I, p. 355].

401. We may now consider such a series

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ a, & a(a+b), & a(a+b)(a+2b), & a(a+b)(a+2b)(a+3b) \end{array} \text{ etc.,}$$

the factors of which constitute this arithmetic progression

$$a,\ (a+b),\ (a+2b),\ (a+3b),\ (a+4b) \text{ etc.}$$

the most infinite terms of which thus have been prepared, so that the differences of the logarithms of these may vanish. Since there shall be

$$Z = a - b + b\omega$$

and

$$Z' = a + b\omega, \quad Z'' = a + b + b\omega, \quad Z''' = a + 2b + b\omega \quad \text{etc.},$$

if Σ may denote the term of the proposed series, of which the index is $=\omega$, there will be

$$\Sigma = a^{\omega} \cdot \frac{a^{1-\omega}(a+b)^{\omega}}{a+b\omega} \cdot \frac{(a+b)^{1-\omega}(a+2b)^{\omega}}{a+b+b\omega} \cdot \frac{(a+2b)^{1-\omega}(a+3b)^{\omega}}{a+2b+b\omega} \cdot \text{etc.}$$

And with this value found, if ω may denote some fractional number less than one, the terms corresponding to the following indices $1+\omega,\ 2+\omega,\ 3+\omega$ etc. thus may be determined, so that there shall be

$$\Sigma' = (a+b\omega)\Sigma$$
$$\Sigma'' = (a+b\omega)(a+b+b\omega)\Sigma$$
$$\Sigma''' = (a+b\omega)(a+b+b\omega)(a+2b+b\omega)\Sigma$$
$$\text{etc.}$$

Whereby if the term corresponding to the index $\frac{1}{2}$, on making $\omega = \frac{1}{2}$ there will be

$$\Sigma = a^{\frac{1}{2}} \cdot \frac{a^{\frac{1}{2}}(a+b)^{\frac{1}{2}}}{a+\frac{1}{2}b} \cdot \frac{(a+b)^{\frac{1}{2}}(a+2b)^{\frac{1}{2}}}{a+\frac{3}{2}b} \cdot \frac{(a+2b)^{\frac{1}{2}}(a+3b)^{\frac{1}{2}}}{a+\frac{5}{2}b} \cdot \text{etc.}$$

and thus with the squares taken

$$\Sigma^2 = a \cdot \frac{a(a+b)}{(a+\frac{1}{2}b)(a+\frac{1}{2}b)} \cdot \frac{(a+b)(a+2b)}{(a+\frac{3}{2}b)(a+\frac{3}{2}b)} \cdot \frac{(a+2b)(a+3b)}{(a+\frac{5}{2}b)(a+\frac{5}{2}b)} \cdot \text{etc.}$$

402. There may be put in the series, which we have treated above in §400,

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \frac{f}{g}, & \frac{f(f+h)}{g(g+h)}, & \frac{f(f+h)(f+2h)}{g(g+h)(g+2h)}, & \frac{f(f+h)(f+2h)(f+3h)}{g(g+h)(g+2h)(g+3h)} \text{ etc.} \end{array}$$

the term corresponding to the index $\frac{1}{2}$, $= \Theta$; there will be

$$\Theta = \frac{f(f+\frac{1}{2}h)}{g(g+\frac{1}{2}h)} \cdot \frac{(f+h)(f+\frac{3}{2}h)}{(g+h)(g+\frac{3}{2}h)} \cdot \frac{(f+2h)(f+\frac{5}{2}h)}{(g+2h)(g+\frac{5}{2}h)} \cdot \text{ etc.;}$$

now there may be considered

$$f = a, \quad g = a + \tfrac{1}{2}b \text{ and } h = b;$$

there will be

$$\Theta = \frac{a(a+b)}{(a+\frac{1}{2}b)(a+\frac{1}{2}b)} \cdot \frac{(a+b)(a+2b)}{(a+\frac{3}{2}b)(a+\frac{3}{2}b)} \cdot \text{etc.}$$

and thus there comes about $\Sigma^2 = a\Theta$ and $\Sigma = \sqrt{a\Theta}$. On account of which if the term corresponding to the index $\frac{1}{2}$ of this series

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ a, & a(a+b), & a(a+b)(a+2b), & a(a+b)(a+2b)(a+3b) \text{ etc.,} \end{array}$$

may be placed $= \Sigma$ and the term corresponding to the index $\frac{1}{2}$ of this series

$$\begin{array}{ccc} 1 & 2 & 3 \\ \frac{a}{(a+\frac{1}{2}b)}, & \frac{a(a+b)}{(a+\frac{1}{2}b)(a+\frac{3}{2}b)}, & \frac{a(a+b)(a+2b)}{(a+\frac{1}{2}b)(a+\frac{3}{2}b)(a+\frac{5}{2}b)} \text{ etc.} \end{array}$$

may be put $= \Theta$, there will be $\Sigma = \sqrt{a\Theta}$.

Therefore since here the term of the single numerators corresponding to the index $\frac{1}{2}$ shall be $= \Sigma$, if in the series of the denominators the term corresponding to the index $\frac{1}{2}$ may be put equal to Λ, there will be $\Theta = \frac{\Sigma}{\Lambda}$; but there is $\Theta = \frac{\Sigma^2}{a}$, from which there comes about $\Sigma = \frac{a}{\Lambda}$ or $\Sigma\Lambda = a$, from which theorems the interpolation of series of this kind may be well illustrated.

EXAMPLE 1

Let this proposed series be interpolated

$$1,\ 1\cdot 2,\ 1\cdot 2\cdot 3,\ 1\cdot 2\cdot 3\cdot 4\ \text{ etc}.$$

Because here there is $a = 1$ and $b = 1$, if the term corresponding to the index ω may be put $= \Sigma$, there will be

$$\Sigma = \frac{1^{1-\omega}\cdot 2^{\omega}}{1+\omega}\cdot\frac{2^{1-\omega}\cdot 3^{\omega}}{2+\omega}\cdot\frac{3^{1-\omega}\cdot 4^{\omega}}{3+\omega}\cdot\frac{4^{1-\omega}\cdot 5^{\omega}}{4+\omega}\cdot\text{etc.}$$

Here always for ω a fraction less than one can be taken; indeed nevertheless the interpolation may be extended through the whole series. For if the terms corresponding to the indices $1+\omega,\ 2+\omega,\ 3+\omega$ etc. may be put $\Sigma',\ \Sigma'',\ \Sigma'''$ etc., there will be

$$\begin{aligned}\Sigma' &= (1+\omega)\Sigma\\ \Sigma'' &= (1+\omega)(1+2\omega)\Sigma\\ \Sigma''' &= (1+\omega)(1+2\omega)(1+3\omega)\Sigma\\ &\text{etc.}\end{aligned}$$

Therefore the term corresponding to the proposed index $\frac{1}{2}$ will be

$$\Sigma = \frac{1^{\frac{1}{2}}\cdot 2^{\frac{1}{2}}}{1\frac{1}{2}}\cdot\frac{2^{\frac{1}{2}}\cdot 3^{\frac{1}{2}}}{2\frac{1}{2}}\cdot\frac{3^{\frac{1}{2}}\cdot 4^{\frac{1}{2}}}{3\frac{1}{2}}\cdot\text{etc.}$$

or

$$\Sigma^2 = \frac{2\cdot 4}{3\cdot 3}\cdot\frac{4\cdot 6}{5\cdot 5}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot\frac{8\cdot 10}{9\cdot 9}\cdot\text{etc.}$$

From which, since there shall be

$$\pi = 2\cdot\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\text{etc.},$$

there will be

$$\Sigma^2 = \frac{\pi}{4} \quad\text{and}\quad \Sigma = \frac{\sqrt{\pi}}{2}$$

and hence there will correspond

to the indices $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$ etc.

the terms $\frac{\sqrt{\pi}}{2}$, $\frac{3}{2}\cdot\frac{\sqrt{\pi}}{2}$, $\frac{3\cdot 5}{2\cdot 2}\cdot\frac{\sqrt{\pi}}{2}$, $\frac{3\cdot 5\cdot 7}{2\cdot 2\cdot 2}\cdot\frac{\sqrt{\pi}}{2}$ etc.

EXAMPLE 2

Let this proposed series be interpolated

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1, & 1\cdot 3, & 1\cdot 3\cdot 5, & 1\cdot 3\cdot 5\cdot 7 \quad \text{etc.} \end{array}$$

Because here there is $a=1,\ b=2$, if the term corresponding to the index ω may be put $=\Sigma$, there will be

$$\Sigma=\frac{1^{1-\omega}\cdot 3^{\omega}}{1+2\omega}\cdot\frac{3^{1-\omega}\cdot 5^{\omega}}{3+2\omega}\cdot\frac{5^{1-\omega}\cdot 7^{\omega}}{5+2\omega}\cdot\text{etc.}$$

and the terms following in order thus will be prepared :

$$\begin{aligned} \Sigma' &=(1+2\omega)\Sigma \\ \Sigma'' &=(1+2\omega)(3+2\omega)\Sigma \\ \Sigma''' &=(1+2\omega)(3+2\omega)(5+3\omega)\Sigma \\ &\text{etc.} \end{aligned}$$

If therefore the term may be desired of the proposed series corresponding to the index $\frac{1}{2}$ and it may be called $=\Sigma$, there will be

$$\Sigma=\frac{\sqrt{1\cdot 3}}{2}\cdot\frac{\sqrt{3\cdot 5}}{4}\cdot\frac{\sqrt{5\cdot 7}}{6}\cdot\frac{\sqrt{7\cdot 9}}{8}\cdot\text{etc.},$$

therefore

$$\Sigma^2=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{7\cdot 9}{8\cdot 8}\cdot\text{etc.}=\frac{2}{\pi}$$

and thus there will be had $\Sigma=\sqrt{\frac{2}{\pi}}$. But there will correspond

to the indices $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, etc.

the terms $\sqrt{\frac{2}{\pi}}$, $2\cdot\sqrt{\frac{2}{\pi}}$, $2\cdot 4\sqrt{\frac{2}{\pi}}$, $2\cdot 4\cdot 6\sqrt{\frac{2}{\pi}}$ etc.

But if therefore the first series and this may be multiplied in turn, so that this series may be had

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1^2, & 1^2\cdot 2\cdot 3, & 1^2\cdot 2\cdot 3^2\cdot 5, & 1^2\cdot 2\cdot 3^2\cdot 4\cdot 5\cdot 7 & 1^2\cdot 2\cdot 3^2\cdot 4\cdot 5^2\cdot 7\cdot 9 \end{matrix} \text{ etc.,}$$

the term of this corresponding to the index $\frac{1}{2}$ will be $=\frac{\sqrt{\pi}}{2}\cdot\sqrt{\frac{2}{\pi}}=\frac{1}{\sqrt{2}}$; which may be seen easily, if this form may be given to that series

$$\begin{matrix} 1 & 2 & 3 & 4 \\ \frac{1\cdot 2}{2}, & \frac{1\cdot 2\cdot 3\cdot 4}{2^2}, & \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{2^3}, & \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}{2^4} \end{matrix} \quad \text{etc.,}$$

of which the term corresponding to the index $\frac{1}{2}$ evidently is $=\frac{1}{\sqrt{2}}$.

EXAMPLE 3

Let this proposed series be interpolated

$$\begin{matrix} 1 & 2 & 3 & 4 \\ \frac{n}{1}, & \frac{n(n-1)}{1\cdot 2}, & \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}, & \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} \end{matrix} \quad \text{etc.}$$

The numerators and denominators of this series may be considered separately, and since the numerators shall be

$$\begin{matrix} 1 & 2 & 3 & 4 \\ n, & n(n-1), & n(n-1)(n-2), & n(n-1)(n-2)(n-3) \end{matrix} \quad \text{etc.,}$$

arising from the application made $a=n$ and $b=-1$, from which the term of this series corresponding to the index ω will be

$$=n^{\omega}\cdot\frac{n^{1-\omega}(n-1)^{\omega}}{n-\omega}\cdot\frac{(n-1)^{1-\omega}(n-2)^{\omega}}{n-1-\omega}\cdot\frac{(n-2)^{1-\omega}(n-3)^{\omega}}{n-2-\omega}\cdot\text{etc.,}$$

but which expression on account of the factors becoming negative nothing certain can be shown. Therefore the proposed series may be transformed on putting for the sake of brevity $1\cdot 2\cdot 3\cdots n=N$ into this

$$\begin{matrix} 1 & 2 & 3 \\ \frac{N}{1\cdot 1\cdot 2\cdot 3\cdots(n-1)}, & \frac{N}{1\cdot 2\cdot 1\cdot 2\cdot 3\cdots(n-2)}, & \frac{N}{1\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3\cdots(n-3)} \end{matrix} \quad \text{etc.;}$$

the denominators of which since they depend on two factors, may put in place this second series

$$\begin{matrix} 1 & 2 & 3 & \\ 1\cdot 2\cdot 3\cdots(n-1), & 1\cdot 2\cdot 3\cdots(n-2), & 1\cdot 2\cdot 3\cdots(n-3) & \text{etc.,} \end{matrix}$$

the term of which corresponding to the index ω agrees with the corresponding term of this series

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & \\ 1, & 1\cdot 2, & 1\cdot 2\cdot 3, & 1\cdot 2\cdot 3\cdot 4, & 1\cdot 2\cdot 3\cdot 4\cdot 5 & \text{etc.,} \end{matrix}$$

for the index $n-\omega$, which is

$$\frac{1^{1-n+\omega}\cdot 2^{n-\omega}}{1+n-\omega}\cdot\frac{2^{1-n+\omega}\cdot 3^{n-\omega}}{2+n-\omega}\cdot\frac{3^{1-n+\omega}\cdot 4^{n-\omega}}{3+n-\omega}\cdot\text{etc.}$$

But if the term of this series corresponding to the index $1-\omega$ shall be $=\Theta$; there will be

$$\Theta=\frac{1^{\omega}\cdot 2^{1-\omega}}{2-\omega}\cdot\frac{2^{\omega}\cdot 3^{1-\omega}}{3-\omega}\cdot\frac{3^{\omega}\cdot 4^{1-\omega}}{4-\omega}\cdot\text{etc.}$$

and since there may correspond

$$\begin{matrix} \text{to the indices} & 1-\omega, & 2-\omega, & 3-\omega & \text{etc.} \\ \text{the terms} & \Theta, & (2-\omega)\Theta, & (2-\omega)(3-\omega)\Theta & \text{etc.,} \end{matrix}$$

to the index $n-\omega$ there will correspond this term

$$(2-\omega)(3-\omega)(4-\omega)\cdots(\mathrm{n}-\omega)\Theta.$$

Then the other factors of those denominators constitute this series

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & \\ 1, & 1\cdot 2, & 1\cdot 2\cdot 3, & 1\cdot 2\cdot 3\cdot 4, & 1\cdot 2\cdot 3\cdot 4\cdot 5 & \text{etc.;} \end{matrix}$$

if the term corresponding to the index ω may be put $=\Lambda$, there will be

$$\Lambda=\frac{1^{1-\omega}\cdot 2^{\omega}}{1+\omega}\cdot\frac{2^{1-\omega}\cdot 3^{\omega}}{2+\omega}\cdot\frac{3^{1-\omega}\cdot 4^{\omega}}{3+\omega}\cdot\text{etc.}$$

With which found, if the term Σ of this proposed series

$$\begin{matrix} 1 & 2 & 3 & 4 \\ \frac{n}{1}, & \frac{n(n-1)}{1\cdot 2}, & \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}, & \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} \end{matrix} \text{ etc.}$$

corresponding to the index ω were put in place, there will be

$$\Sigma = \frac{N}{\Lambda\cdot(2-\omega)(3-\omega)(4-\omega)\cdots(\mathrm{n}-\omega)\Theta}.$$

But now there is

$$\frac{N}{(2-\omega)(3-\omega)(4-\omega)\cdots(\mathrm{n}-\omega)} = \frac{2}{2-\omega}\cdot\frac{3}{3-\omega}\cdot\frac{4}{4-\omega}\cdots\frac{n}{\mathrm{n}-\omega}$$

and

$$\Lambda\Theta = \frac{1\cdot 2}{(1+\omega)(2-\omega)}\cdot\frac{2\cdot 3}{(2+\omega)(3-\omega)}\cdot\frac{3\cdot 4}{(3+\omega)(4-\omega)}\cdot\text{etc.}$$

From which the term corresponding to the index ω sought will be

$$\Sigma = \frac{2}{2-\omega}\cdot\frac{3}{3-\omega}\cdot\frac{4}{4-\omega}\cdot\frac{5}{5-\omega}\cdots\frac{n}{\mathrm{n}-\omega}\cdot\frac{(1+\omega)(2-\omega)}{1\cdot 2}\cdot\frac{(2+\omega)(3-\omega)}{2\cdot 3}\cdot\frac{(3+\omega)(4-\omega)}{3\cdot 4}\cdot\text{etc. to infinity.}$$

Therefore for the index $\frac{1}{2}$ there will correspond that term

$$\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}\cdot\frac{10}{9}\cdot\frac{12}{11}\cdots\frac{2n}{2n-1}\cdot\frac{3\cdot 3}{2\cdot 4}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot\frac{77}{6\cdot 8}\cdot\frac{9\cdot 9}{8\cdot 10}\cdot\text{etc.},$$

which may be reduced to

$$\frac{4\cdot 6\cdot 8\cdot 10\cdots 2n}{3\cdot 5\cdot 7\cdot 9\cdots(2n-1)}\cdot\frac{4}{\pi},$$

or there will be

$$= \frac{2}{\pi}\cdot\frac{2\cdot 4\cdot 6\cdot 8\cdot 10\cdots 2n}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdots(2n-1)}.$$

If there were $n = 2$, that interpolated series will be produced

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1, & 2, & 1, & 0, & 0, & 0, & 0 \text{ etc.,} \end{matrix}$$

therefore the term of which corresponding to the index $\frac{1}{2}$ is $= \frac{16}{3\pi}$.

EXAMPLE 4

There may be sought the term corresponding to the index $=\frac{1}{2}$ *in this series*

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 1, & +\frac{1}{2}, & -\frac{1\cdot 1}{2\cdot 4}, & +\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}, & -\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8} \quad \text{etc.} \end{array}$$

This series arises from the preceding, if there may be put $n=\frac{1}{2}$, and therefore the term sought will be, which shall be $=\Sigma$,

$$\Sigma=\frac{2}{\pi}\cdot\frac{2\cdot 4\cdot 6\cdot 8\cdots 2n}{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}$$

on putting $n=\frac{1}{2}$. There may be put

$$\frac{2\cdot 4\cdot 6\cdot 8\cdots 2n}{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}=\Theta,$$

if there shall be $n=\frac{1}{2}$, and Θ will be the term corresponding to the index $\frac{1}{2}$ in this series

$$\frac{2}{1},\ \frac{2\cdot 4}{1\cdot 3},\ \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5},\ \frac{2\cdot 4\cdot 6\cdot 8}{1\cdot 3\cdot 5\cdot 7}\ \text{etc.},$$

which from above will produce $=\frac{\pi}{2}$. On account of which the term of the proposed series corresponding to the index $\frac{1}{2}$ which is sought, will be $=1$. But since in that series, if the term corresponding to some ω may be put equal to $=\Sigma$, following that there will be $\Sigma'=\frac{1-2\omega}{2+2\omega}\Sigma$, the proposed series thus will be interpolated by the middle terms placed between:

$$\begin{array}{lcccccccc} \text{Indices:} & 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & 3 & \frac{7}{2} \\ \text{Termini:} & 1, & 1, & \frac{1}{2}, & 0, & \frac{-1\cdot 1}{2\cdot 4}, & 0, & \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}, & 0\ \text{etc.} \end{array}$$

EXAMPLE 5

If n were some fractional number, to find the term corresponding to the index ω in the series

$$\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \\ 1 & \frac{n}{1}, & \frac{n(n-1)}{1\cdot 2}, & \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}, & \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} & \text{etc.,} \end{array}$$

If we may compare the expression

$$\frac{2}{2-\omega}\cdot\frac{3}{3-\omega}\frac{4}{4-\omega}\cdots\frac{n}{n-\omega}$$

with § 400,

there becomes $a=1,\ c=1,\ b=1-\omega$ and on putting n in place of ω everywhere, there will be

$$\frac{1}{1-\omega}\cdot\frac{2}{2-\omega}\cdot\frac{3}{3-\omega}\cdots\frac{n}{n-\omega}=\frac{1(1-\omega+n)}{(1-\omega)(1+n)}\cdot\frac{2(2-\omega+n)}{(2-\omega)(2+n)}\cdot\text{etc.,}$$

from which the term sought corresponding to the index ω, if it may be put $=\Sigma$, will be

$$\Sigma=\frac{(1-\omega+n)\cdot 2}{(1+n)(2-\omega)}\cdot\frac{(2-\omega+n)3}{(2+n)(3-\omega)}\cdot\text{etc.}\cdot\frac{(1+\omega)(2-\omega)}{1\cdot 2}\cdot\frac{(2+\omega)(3-\omega)}{2\cdot 3}\text{etc.}$$

and thus

$$\Sigma=\frac{(1+\omega)(1+n-\omega)}{1(1+n)(2-\omega)}\cdot\frac{(2+\omega)(2+n-\omega)}{2(2+n)}\cdot\frac{(3+\omega)(3+n-\omega)}{3(3+n)}\cdot\text{etc.:}$$

therefore as often as $n-\omega$ should be a whole number, the value of Σ can be expressed rationally.
Thus if there shall be $n=\omega$, there will be $\Sigma=1$;
if $n=1+\omega$, there will be $\Sigma=n$;
if $n=2+\omega$, there will be $\Sigma=\frac{n(n-1)}{1\cdot 2}$;
if $n=3+\omega$, there will be $\Sigma=\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$.
etc.

But if $\omega-n$ were always a positive integer, there will be always $\Sigma=0$.

CAPUT XVII

DE INTERPOLATIONE SERIERUM

389. Series interpolari dicitur, dum eius termini assignantur, qui respondent indicibus fractis vel etiam surdis. Si igitur seriei terminus generalis fuerit cognitus, interpolatio nullam habet difficultatem, cum, quicunque numerus loco indicis x substituatur, ista expressio praebeat terminum respondentem. Verum si series ita fuerit comparata, ut eius terminus generalis nullo modo exhiberi queat, tum interpolatio huiusmodi serierum plerumque est maxime difficilis neque maximam partem termini indicibus non integris respondentes aliter nisi per series infinitas definiri possunt. Quoniam ergo in capite praecedente huius modi expressionum, quae more consueto finite exprimi non possunt, valores quibuscunque indicibus respondentes determinavimus, ea tractatio maximam afferet utilitatem ad interpolationes perficiendas. Quam ob causam usum, qui ex superiori capite in hoc negotium redundat, hic diligentius prosequemur.

390. Sit ergo proposita series quaecunque

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & x \\ A & +\,B & +\,C & +\,D & +\cdots+ & X, \end{array}$$

cuius terminus generalis X sit cognitus, summatorius autem S lateat. Hinc formetur alia series, cuius terminus generalis aequetur illius seriei termino summatorio, eritque ista nova series

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ A, & (A+B), & (A+B+C), & (A+B+C+D), & (A+B+C+D+E) \quad \text{etc.} \end{array}$$

eiusque terminus generalis seu indici indefinito x respondens erit

$$= A+B+C+D+\cdots+X = S;$$

qui cum explicite non sit cognitus, interpolatio huius novae seriei iisdem difficultatibus erit obnoxia, quas ante meminimus. Ad hanc ergo seriem interpolandam investigari oportet valores ipsius S, quos recipit, si loco x numeri quicunque non integri substituantur. Si enim x esset numerus integer, tum conveniens ipsius S valor sine difficultate reperiretur, additione scilicet tot terminorum seriei $A+B+C+D+\text{etc.}$, quot x contineat unitates.

391. Quo igitur ea, quae in capite praecedente sunt tradita, in usum vocari possint, ponamus x esse numerum integrum, ita ut valor ei respondens $S = A+B+C+\cdots+X$ sit cognitus, et quaeramus valorem Σ in quem S transmutetur, si loco x scribatur $x+\omega$ existente ω fractione quacunque; eritque Σ terminus seriei propositae interpolandae, qui respondet indici $x+\omega$; quo ergo invento interpolatio huius seriei erit in promtu. Sit Z terminus seriei A, B, C, D, E etc., qui respondet indici $x+\omega$, sintque Z', Z'', Z'', etc. termini eius consecutivi indices habentes $x+\omega+1$, $x+\omega+2$, $x+\omega+3$ etc. Ac primo quidem ponamus seriei A, B, C, D etc. terminos infinitesimos evanescere. His ergo positis series

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \\ A, & (A+B), & (A+B+C), & (A+B+C+D), & (A+B+C+D+E) & \text{etc.,} \end{array}$$

cuius terminus indici x respondens est $S=A+B+C+\cdots+X$, interpolabitur quaerendo eius terminum Σ, qui indici fracto $x+\omega$ respondeat;
erit autem, uti invenimus,

$$\begin{aligned} \Sigma = S &+ X'+X''+X'''+X''''+\text{etc.} \\ &- Z'-Z''-Z'''-Z''''-\text{etc.} \end{aligned}$$

sicque habebitur series infinita isti termino quaesito Σ aequalis, quae ob

$$Z = X + \frac{\omega dX}{dx} + \frac{\omega^2 ddX}{1\cdot 2dx^2} + \frac{\omega^3 d^3 X}{1\cdot 2\cdot 3dx^3} + \text{etc.}$$

in hanc formam transmutatur, ut sit

$$\begin{aligned} \Sigma = S &- \frac{\omega}{dx} d.\left(X'+X''+X'''+X''''+\text{etc.}\right) \\ &- \frac{\omega^2}{2dx^2} dd.\left(X'+X''+X'''+X''''+\text{etc.}\right) \\ &- \frac{\omega^3}{6dx^3} d^3.\left(X'+X''+X'''+X''''+\text{etc.}\right) \\ &\text{etc.,} \end{aligned}$$

quarum formularum ea, quae quovis casu commodior videatur, adhiberi poterit.

392. Sumamus pro A, B, C, D etc. seriem harmonicam quamcunque

$$\frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} + \text{etc.,}$$

cuius terminus generalis seu indici x respondens est $= \frac{1}{a+(x-1)b} = X$. Hinc formata sit ista series

$$\begin{matrix} 1 & 2 & 3 & 4 \\ \frac{1}{a}, & \left(\frac{1}{a}+\frac{1}{a+b}\right), & \left(\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}\right), & \left(\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}+\frac{1}{a+3b}\right) \end{matrix} \text{ etc.,}$$

cuius propterea terminus indici x respondens erit

$$S=\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}+\frac{1}{a+3b}+\cdots+\frac{1}{a+(x-1)b}.$$

Si iam Σ denotet terminum istius seriei indici $x+\omega$ respondentem, ob

$$Z=\frac{1}{a+(x+\omega-1)b}$$

$$\left.\begin{aligned} X'&=\frac{1}{a+bx}\\ X''&=\frac{1}{a+b+bx}\\ X'''&=\frac{1}{a+2b+bx}\\ &\text{etc.} \end{aligned}\right| \begin{aligned} Z'&=\frac{1}{a+bx+b\omega}\\ Z''&=\frac{1}{a+b+bx+b\omega}\\ Z'''&=\frac{1}{a+2b+bx+b\omega}\\ &\text{etc.,} \end{aligned}$$

hincque orietur

$$\begin{aligned} \Sigma=S &+\frac{1}{a+bx}+\frac{1}{a+b+bx}+\frac{1}{a+2b+bx}+\text{etc.}\\ &-\frac{1}{a+bx+b\omega}-\frac{1}{a+b+bx+b\omega}-\frac{1}{a+2b+bx+b\omega}-\text{etc.}; \end{aligned}$$

altera expressio autem erit huiusmodi

$$\begin{aligned} \Sigma=S &+b\omega\left(\frac{1}{(a+bx)^2}+\frac{1}{(a+b+bx)^2}+\frac{1}{(a+2b+bx)^2}+\text{etc.}\right)\\ &-b^2\omega^2\left(\frac{1}{(a+bx)^3}+\frac{1}{(a+b+bx)^3}+\frac{1}{(a+2b+bx)^3}+\text{etc.}\right)\\ &+b^3\omega^3\left(\frac{1}{(a+bx)^4}+\frac{1}{(a+b+bx)^4}+\frac{1}{(a+2b+bx)^4}+\text{etc.}\right)\\ &\text{etc.} \end{aligned}$$

EXEMPLUM 1

Proposita sit ista series

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1, & \left(1+\frac{1}{2}\right), & \left(1+\frac{1}{2}+\frac{1}{3}\right), & \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) \quad \text{etc.,} \end{array}$$

cuius terminos, qui indicibus fractis respondent, inveniri oporteat.

Erit ergo $a=1$ et $b=1$; unde si terminus indici integro x respondens ponatur

$$S=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{x}$$

terminusque indici fracto $x+\omega$ respondens vocetur $=\Sigma$, erit

$$\begin{aligned}\Sigma = S &+ \frac{1}{1+x} + \frac{1}{2+x} + \frac{1}{3+x} + \frac{1}{4+x} + \frac{1}{5+x} + \text{etc.} \\ &- \frac{1}{1+x+\omega} - \frac{1}{2+x+\omega} - \frac{1}{3+x+\omega} - \frac{1}{4+x+\omega} - \frac{1}{5+x+\omega} - \text{etc.}\end{aligned}$$

Notandum autem est, si inventus fuerit terminus respondens indici fracto ω, quem ponamus $=T$, ex eo terminum indicis $x+\omega$ facile inveniri posse; erit enim, si T', T'', T''' etc. denotent terminos indicibus $1+\omega, 2+\omega, 3+\omega$ etc. respondentes,

$$\begin{aligned} T' &= T+\frac{1}{1+\omega} \\ T'' &= T+\frac{1}{1+\omega}+\frac{1}{2+\omega} \\ T''' &= T+\frac{1}{1+\omega}+\frac{1}{2+\omega}+\frac{1}{3+\omega} \\ &\text{etc.} \end{aligned}$$

unde sufficit eos tantum terminos, qui respondent indicibus ω unitate minoribus, investigasse. Quem in finem ponamus $x=0$; erit quoque $S=0$ atque terminus seriei T indici fracto ω respondens ita exprimetur

$$\begin{aligned} T = &\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.} \\ &- \frac{1}{1+\omega} - \frac{1}{2+\omega} - \frac{1}{3+\omega} - \frac{1}{4+\omega} - \text{etc.} \end{aligned}$$

vel his fractionibus in series infinitas conversis prodibit altera expressio

$$\begin{aligned}T=&+\omega\left(1+\tfrac{1}{2^2}+\tfrac{1}{3^2}+\tfrac{1}{4^2}+\tfrac{1}{5^2}+\text{etc.}\right)\\&-\omega^2\left(1+\tfrac{1}{2^3}+\tfrac{1}{3^3}+\tfrac{1}{4^3}+\tfrac{1}{5^3}+\text{etc.}\right)\\&+\omega^3\left(1+\tfrac{1}{2^4}+\tfrac{1}{3^4}+\tfrac{1}{4^4}+\tfrac{1}{5^4}+\text{etc.}\right)\\&-\omega^4\left(1+\tfrac{1}{2^5}+\tfrac{1}{3^5}+\tfrac{1}{4^5}+\tfrac{1}{5^5}+\text{etc.}\right)\\&\text{etc.}\end{aligned}$$

quae ad valorem ipsius T proxime inveniendum perquam est apta.

Quaeratur, ergo propositae seriei terminus respondens indici $\frac{1}{2}$; qui si ponatur $=T$, erit

$$T=1-\tfrac{2}{3}+\tfrac{1}{2}-\tfrac{2}{5}+\tfrac{1}{3}-\tfrac{2}{7}+\tfrac{1}{4}-\tfrac{2}{9}+\text{etc.}$$

seu

$$T=2\left(\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{4}-\tfrac{1}{5}+\tfrac{1}{6}-\text{etc.}\right),$$

cuius seriei valor est $=2-2l2$, sicque terminus indicis $=\frac{1}{2}$ finite exprimi potest. Erunt ergo termini sequentes, quorum indices sunt $\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2}$ etc., ita expressi

Ind.	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	
Term.	$2-2l2$,	$2+\frac{2}{3}-2l2$,	$2+\frac{2}{3}+\frac{2}{5}-2l2$	$2+\frac{2}{3}+\frac{2}{5}+\frac{2}{7}-2l2$	etc.

EXEMPLUM 2

Proposita sit ista series

$$\begin{array}{cccc}1 & 2 & 3 & 4\\ 1, & \left(1+\frac{1}{3}\right), & \left(1+\frac{1}{3}+\frac{1}{5}\right), & \left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\right) \quad \text{etc.,}\end{array}$$

cuius terminos indicibus fractis respondentes exprimere oporteat.

Erit ergo $a=1, b=2$; unde si terminus indici integro x respondens ponatur

$$S=1+\tfrac{1}{3}+\tfrac{1}{5}+\cdots+\tfrac{1}{2x-1}$$

terminusque indici fracto $x+\omega$ vocetur $=\Sigma$, erit

$$\Sigma = S + \frac{1}{1+2x} + \frac{1}{3+2x} + \frac{1}{5+2x} + \frac{1}{7+2x} + \text{etc.}$$
$$- \frac{1}{1+2(x+\omega)} - \frac{1}{3+2(x+\omega)} - \frac{1}{5+2(x+\omega)} - \frac{1}{7+2(x+\omega)} - \text{etc.}$$

Cum igitur sufficiat terminos indicibus unitate minoribus assignasse, sit $x = 0$ et $S = 0$; quocirca, si terminus indici ω conveniens ponatur $= T$, erit

$$T = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \text{etc.}$$
$$- \frac{1}{1+2\omega} - \frac{1}{3+2\omega} - \frac{1}{5+2\omega} - \frac{1}{7+\omega} - \frac{1}{9+\omega} - \text{etc.},$$

et si ω numerum quemcunque denotare ponatur, quoniam T est terminus indici ω respondens, erit T terminus generalis seriei propositae, qui etiam hoc modo exprimetur

$$T = \frac{2\omega}{1(1+2\omega)} + \frac{2\omega}{3(3+2\omega)} + \frac{2\omega}{5(5+2\omega)} + \frac{2\omega}{7(7+2\omega)} + \text{etc.}$$

vel ita

$$T = 2\omega\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.}\right)$$
$$- 4\omega^2\left(1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \text{etc.}\right)$$
$$+ 8\omega^3\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \text{etc.}\right)$$
$$- 16\omega^4\left(1 + \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \text{etc.}\right)$$
$$\text{etc.}$$

Ponamus esse $\omega = \frac{1}{2}$ erit terminus huic indici respondens

$$T = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \text{etc.} = l2$$

eruntque

Indices:	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	
Termini:	$l2,$	$\frac{1}{2} + l2,$	$\frac{1}{2} + \frac{1}{4} + l2,$	$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + l2$	etc.

Si sit $\omega = \frac{1}{4}$, erit

$$T = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \text{etc.}$$
$$- \frac{2}{3} - \frac{2}{7} - \frac{2}{11} - \frac{2}{15} - \text{etc.},$$

Sive

$$T = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{etc.} - \frac{1}{2}l2 = \frac{\pi}{4} - \frac{1}{2}l2.$$

393. Quodsi ergo huius seriei generalis

$$\frac{1}{a},\ \left(\frac{1}{a}+\frac{1}{a+b}\right),\ \left(\frac{1}{a}+\frac{1}{a+b}+\frac{1}{a+2b}\right)\ \text{etc.}$$

quaeratur terminus respondens indici $=\frac{1}{2}$, ponatur in expressionibus paragraphi praecedentis $x=0$ et $\omega=\frac{1}{2}$ fietque $S=0$ et terminus indici $\frac{1}{2}$ respondens quaesitus erit

$$\Sigma=\frac{1}{a}-\frac{2}{2a+b}+\frac{1}{a+b}-\frac{2}{2a+3b}+\frac{1}{a+2b}-\frac{2}{2a+5b}+\text{etc.}$$

sive terminis ad maiorem uniformitatem perductis erit

$$\frac{1}{2}\Sigma=\frac{1}{2a}-\frac{1}{2a+b}+\frac{1}{2a+2b}-\frac{1}{2a+3b}+\frac{1}{a+4b}-\ \text{etc.};$$

in qua serie cum signa + et – alternentur, sumendis continuis differentiis per methodum supra [§ 8] expositam valor ipsius $\frac{1}{2}\Sigma$ per seriem magis convergentem exprimetur.

Erunt autem differentiarum series

$$\frac{-b}{2a(2a+b)},\quad \frac{-b}{(2a+b)(2a+2b)},\quad \frac{-b}{(2a+2b)(2a+3b)}\ \text{etc.}$$

$$\frac{2bb}{2a(2a+b)(2a+b)},\ \frac{2bb}{(2a+b)(2a+2b)(2a+3b)}\ \text{etc.}$$

$$\frac{-6b^3}{2a(2a+b)(2a+2b)(2a+3b)}\ \text{etc.}$$

etc.

Ex quibus concluditur fore

$$\frac{1}{2}\Sigma=\frac{1}{4a}+\frac{1b}{8a(2a+b)}+\frac{1\cdot 2bb}{16a(2a+b)(2a+2b)}+\frac{1\cdot 2\cdot 3b^3}{32a(2a+b)(2a+2b)(2a+3b)}+\text{etc.}$$

Hincque ergo habebitur

$$\Sigma=\frac{1}{2a}+\frac{\frac{1}{2}b}{2a(2a+b)}+\frac{\frac{1}{2}\cdot\frac{2}{2}bb}{2a(2a+b)(2a+2b)}+\frac{\frac{1}{2}\cdot\frac{2}{2}\cdot\frac{3}{2}b^3}{2a(2a+b)(2a+2b)(2a+3b)}+\text{etc.},$$

quae series maxime convergit atque valorem termini Σ facili labore proxime exhibet.

394. Quodsi autem in genere seriei A, B, C, D, E etc. termini infinitesimi evanescant terminusque indici ω respondens fuerit $= Z$ eiusque sequentes, qui indicibus $\omega+1, \omega+2, \omega+3$ etc. respondeant, sint Z', Z'', Z''', Z'''' etc., si in superioribus (§ 391) ponatur $x = 0$, ut sit $S = 0$ et $X' = A, X'' = B, X''' = C$ etc., sequetur, si formetur huiusmodi series

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ A, & (A+B), & (A+B+C), & (A+B+C+D) \text{ etc.} \end{array}$$

eiusque terminus indici ω respondens ponatur $= \Sigma$, fore

$$\Sigma = (A - Z') + (B - Z'') + (C - Z''') + (D - Z'''') + \text{etc.},$$

ex qua expressione termini quicunque intermedii definiri poterunt. Sufficiet autem ad interpolationem perficiendam eos terminos investigasse, qui respondeant indicibus ω unitate minoribus. Si enim terminus Σ indici huiusmodi cuicunque ω respondens fuerit repertus iique, qui conveniant indicibus $\omega+1, \omega+2, \omega+3$ etc., ponantur $\Sigma', \Sigma'', \Sigma'''$, etc., erit

$$\begin{aligned} \Sigma' &= \Sigma + Z' \\ \Sigma'' &= \Sigma + Z' + Z'' \\ \Sigma''' &= \Sigma + Z' + Z'' + Z''' \\ &\text{etc.} \end{aligned}$$

EXEMPLUM 1

Interpolare hanc seriem

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1, & \left(1+\frac{1}{4}\right), & \left(1+\frac{1}{4}+\frac{1}{9}\right), & \left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}\right) \text{ etc.,} \end{array}$$

Sit Σ huius seriei terminus respondens indici ω, et cum haec series formata sit ex summatione huius

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.},$$

cuius terminus indici ω respondens est $= \frac{1}{\omega^2}$, erit

$$\Sigma = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.}$$
$$-\frac{1}{(1+\omega)^2} - \frac{1}{(2+\omega)^2} - \frac{1}{(3+\omega)^2} - \frac{1}{(4+\omega)^2} - \text{etc.}$$

Quodsi ergo seriei propositae quaeratur terminus indici $\frac{1}{2}$ respondens, poni debebit $\omega = \frac{1}{2}$ fietque

$$\Sigma = 1 - \frac{4}{9} + \frac{1}{4} - \frac{4}{25} + \frac{1}{9} - \frac{4}{49} + \text{etc.}$$

sive

$$\Sigma = 4\left(\frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \frac{1}{36} - \frac{1}{49} + \text{etc}\right).$$

Cum igitur sit

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{25} + \text{etc.} = \frac{\pi^2}{12}$$

erit

$$\Sigma = 4\left(1 - \frac{\pi^2}{12}\right) = 4 - \frac{1}{3}\pi^2$$

qui est terminus indici $\frac{1}{2}$ respondens. Hinc ergo respondebunt indicibus

indicibus	$\frac{1}{2}$,	$\frac{3}{2}$,	$\frac{5}{2}$	etc.
termini	$4 - \frac{1}{3}\pi^2$,	$\frac{4}{1} + \frac{4}{9} - \frac{1}{3}\pi^2$,	$\frac{4}{1} + \frac{4}{9} + \frac{4}{25} - \frac{1}{3}\pi^2$	etc.

EXEMPLUM 2

Interpolare hanc seriem

1	2	3	4	
1,	$\left(1 + \frac{1}{9}\right)$,	$\left(1 + \frac{1}{9} + \frac{1}{25}\right)$,	$\left(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49}\right)$	etc.

Sit Σ terminus respondens indici cuicunque ω, et cum haec series formata sit ex summatione huius

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \text{etc.},$$

ex qua fit terminus indici ω respondens $Z = \frac{1}{(2\omega-1)^2}$, erit

$$Z' = \frac{1}{(2\omega+1)^2}, \quad Z'' = \frac{1}{(2\omega+3)^2}, \quad Z''' = \frac{1}{(2\omega+5)^2} \quad \text{etc.}$$

Quamobrem habebitur

$$\begin{aligned}\Sigma = \; & 1 + \tfrac{1}{9} + \tfrac{1}{25} + \tfrac{1}{49} + \text{etc.}\\ & - \frac{1}{(1+2\omega)^2} - \frac{1}{(3+2\omega)^2} - \frac{1}{(5+2\omega)^2} - \frac{1}{(7+2\omega)^2} - \text{etc.}\end{aligned}$$

Ponamus $\omega = \frac{1}{2}$ ut inveniamus terminum seriei propositae respondentem indici $= \frac{1}{2}$, qui erit

$$\Sigma = 1 - \tfrac{1}{4} + \tfrac{1}{9} - \tfrac{1}{16} + \tfrac{1}{25} - \tfrac{1}{36} + \text{etc.} = \tfrac{\pi\pi}{12},$$

ex quo termini, qui medium interiacent inter binos quosvis datos, sequenti modo exprimentur. Respondebunt

indicibus	$\frac{1}{2}$,	$\frac{3}{2}$,	$\frac{5}{2}$,	$\frac{7}{2}$	etc.
termini	$\frac{\pi\pi}{12}$,	$\frac{1}{4} + \frac{\pi\pi}{12}$,	$\frac{1}{4} + \frac{1}{16} + \frac{\pi\pi}{12}$,	$\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \frac{\pi\pi}{12}$	etc.

EXEMPLUM 3

Interpolare hanc seriem

1	2	3	4	
1,	$\left(1 + \frac{1}{2^n}\right)$,	$\left(1 + \frac{1}{2^n} + \frac{1}{3^n}\right)$,	$\left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n}\right)$	etc.

Sit ut ante Σ terminus indici ω respondens; erit $Z = \frac{1}{\omega^n}$ et

$$Z' = \frac{1}{(1+\omega)^2}, \quad Z'' = \frac{1}{(2+\omega)^2}, \quad Z''' = \frac{1}{(3+\omega)^2} \quad \text{etc.}$$

hincque habebitur

$$\begin{aligned}\Sigma = \; & 1 + \tfrac{1}{2^n} + \tfrac{1}{3^n} + \tfrac{1}{4^n} + \text{etc.}\\ & - \frac{1}{(1+\omega)^2} - \frac{1}{(2+\omega)^2} - \frac{1}{(3+\omega)^2} - \frac{1}{(4+\omega)^2} - \text{etc.}\end{aligned}$$

Si igitur desideretur terminus indici $\frac{1}{2}$ respondens, erit is

$$= 1 - \tfrac{2^n}{3^n} + \tfrac{1}{2^n} - \tfrac{2^n}{5^n} + \tfrac{1}{3^n} - \tfrac{2^n}{7^n} + \text{etc.}$$

seu

$$= 2^n \left(\tfrac{1}{2^n} - \tfrac{1}{3^n} + \tfrac{1}{4^n} - \tfrac{1}{5^n} + \tfrac{1}{6^n} - \tfrac{1}{7^n} + \text{etc.} \right).$$

Quare si ponatur

$$\mathfrak{N} = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \text{etc.}$$

erit seriei propositae terminus, qui indici $\frac{1}{2}$ respondet, $= 2^n\left(1 - \mathfrak{N}\right)$; hincque respondebunt

$$\begin{array}{lllll} \text{indicibus} & \frac{1}{2}, & \frac{3}{2}, & \frac{5}{2} & \text{etc.} \\ \text{termini} & 2^n - 2^n\mathfrak{N}, & 2^n + \frac{2^n}{3^n} - 2^n\mathfrak{N}, & 2^n + \frac{2^n}{3^n} + \frac{2^n}{5^n} - 2^n\mathfrak{N} & \text{etc.} \end{array}$$

EXEMPLUM 4

Interpolare hanc seriem

$$\begin{array}{lllll} 1 & 2 & 3 & 4 & \\ 1, & \left(1 + \frac{1}{3^n}\right), & \left(1 + \frac{1}{3^n} + \frac{1}{5^n}\right), & \left(1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n}\right) & \text{etc.} \end{array}$$

Sit Σ terminus, qui indici cuicunque ω respondeat, et cum sit $Z = \frac{1}{(2\omega - 1)^n}$, erit

$$Z' = \frac{1}{(2\omega + 1)^n}, \quad Z'' = \frac{1}{(2\omega + 3)^n}, \quad Z''' = \frac{1}{(2\omega + 5)^n} \quad \text{etc.}$$

atque

$$\begin{aligned} \Sigma = \; & 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \text{etc.} \\ & - \frac{1}{(1+2\omega)^n} - \frac{1}{(3+2\omega)^n} - \frac{1}{(5+2\omega)^n} - \frac{1}{(7+2\omega)^n} - \text{etc.} \end{aligned}$$

Ponatur $\omega = \frac{1}{2}$ et prodibit terminus indici $\frac{1}{2}$ respondens

$$= 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \text{etc.} = \mathfrak{N},$$

ex quo porro erunt reliqui termini inter binos datos medii

$$\begin{array}{lllll} \text{Indices:} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \text{etc.} \\ \text{Termini:} & \mathfrak{N}, & \frac{1}{2^n} + \mathfrak{N}, & \frac{1}{2^n} + \frac{1}{4^n} + \mathfrak{N} & \text{etc.} \end{array}$$

395. Ponamus nunc seriei *A*, *B*, *C*, *D*, *E* etc., ex cuius summatione series interpolanda formatur, terminos infinitesimos non evanescere, sed ita esse comparatos, ut eorum differentiae evanescant, sitque *X* huius seriei terminus respondens indici *x* et *Z* terminus respondens exponenti $x+\omega$; tum vero sint X', X'', X''', X'''' etc. termini ipsum X sequentes et Z', Z'', Z''', Z'''' etc. termini ipsum Z sequentes. Quibus positis proponatur haec series interpolanda

$$\begin{matrix} 1 & 2 & 3 & 4 \\ A, & (A+B), & (A+B+C), & (A+B+C+D) \text{ etc.} \end{matrix}$$

cuius terminus indici x respondens sit $=S$, at terminus indici $x+\omega$ respondens sit $=\Sigma$, eritque ex iis, quae capite praecedente sunt tradita,

$$\begin{aligned} \Sigma = S + X' + X'' + X''' + \text{etc.} \\ - Z' - Z'' - Z''' - \text{ etc.} \\ +\omega X' + \omega \begin{Bmatrix} +X'' + X''' + X'''' + \text{etc.} \\ -X' - X'' - X''' - \text{etc.} \end{Bmatrix} \end{aligned}$$

Quia autem ut ante sufficit terminos indicibus unitate minoribus respondentes investigasse, ponamus $x=0$, ut sit $S=0, X'=A, X''=B$ etc., eritque terminus indici *w* respondens

$$\begin{aligned} \Sigma = (A-Z') + (B-Z'') + (C-Z''') + (D-Z'''') \text{ etc.} \\ + \omega A + \omega\big((B-A) + (C-B) + (D-C) + (E-D) + \text{etc.}\big). \end{aligned}$$

Vel si differentias has more supra recepto exprimere velimus, quo est

$\Delta A = B - A,\ \Delta B = C - B$ etc., habebitur

$$\begin{aligned} \Sigma = (A-Z') + (B-Z'') + (C-Z''') + (D-Z'''') \text{ etc.} \\ + \omega(A + \Delta A + \Delta B + \Delta C + \Delta D + \text{etc.}). \end{aligned}$$

396. Sin autem seriei *A, B, C, D, E* etc., ex cuius summatione series interpolanda formatur, termini infinitesimi neque ipsi evanescant neque differentias primas habeant evanescentes, tum plures series ad valorem ipsius Σ exprimendum adiici debebunt, quoad scilicet ad differentias terminorum infinitesimorum evanescentes perveniatur. Sit enim ut ante seriei *A, B, C, D, E* etc. terminus indici x respondens $=X$ eumque sequentes X', X'', X'', etc., indici autem $x+\omega$ respondeat terminus Z, quem sequantur $Z', Z'' Z'''$ etc., atque proponatur haec series

$$\begin{matrix} 1 & 2 & 3 & 4 \\ A, & (A+B), & (A+B+C), & (A+B+C+D) \text{ etc.} \end{matrix}$$

cuius terminus indici x respondens sit

$$S = A + B + C + D + \cdots + X ,$$

indici vero $x+\omega$ respondeat terminus Σ, ita ut

indicibus	respondeant termini
$x+\omega+1$	$\Sigma' = \Sigma + Z'$
$x+\omega+2$	$\Sigma'' = \Sigma + Z' + Z''$
$x+\omega+3$	$\Sigma''' = \Sigma + Z' + Z'' + Z''$
etc.	etc.

Si iam differentiae terminorum ita exprimantur, ut sit

$$\Delta X' = X'' - X', \quad \Delta X'' = X''' - X'', \quad \Delta X''' = X'''' - X''' \quad \text{etc.}$$
$$\Delta^2 X' = \Delta X'' - \Delta X', \quad \Delta^2 X'' = \Delta X''' - \Delta X'', \quad \Delta^2 X''' = \Delta X'''' - \Delta X''' \quad \text{etc.}$$
$$\Delta^3 X' = \Delta^2 X'' - \Delta^2 X', \quad \Delta^3 X'' = \Delta^2 X''' - \Delta^2 X'' \quad \text{etc.},$$

ex § 377 terminus Σ sequenti modo exprimetur:

$$\begin{aligned}\Sigma = S &+ X' + X'' + X''' + X'''' + \text{etc.}\\ &- Z' - Z'' - Z''' - Z'''' - \text{etc.}\\ &+\omega\left(X' + \Delta X' + \Delta X'' + \Delta X''' + \Delta X'''' + \text{etc.}\right)\\ &+\tfrac{\omega(\omega-1)}{1\cdot 2}\left(\Delta X' + \Delta^2 X' + \Delta^2 X'' + \Delta^2 X''' + \Delta^2 X'''' + \text{etc}\right)\\ &+\tfrac{\omega(\omega-1)(\omega-2)}{1\cdot 2\cdot 3}\left(\Delta^2 X' + \Delta^3 X' + \Delta^3 X'' + \Delta^3 X''' + \Delta^3 X'''' + \text{etc}\right)\\ &\qquad\text{etc.}\end{aligned}$$

397. Sufficit, uti iam notavimus, tot huiusmodi series adiecisse, donec ad terminorum infinitesimorum differentias evanescentes perveniatur; si enim has ipsas series quoque in infinitum continuare velimus vel eousque saltem, donec terminorum finitorum differentiae evanescant, tum ob

$$Z' = X' + \omega\Delta X' + \tfrac{\omega(\omega-1)}{1\cdot 2}\Delta^2 X' + \tfrac{\omega(\omega-1)(\omega-2)}{1\cdot 2\cdot 3}\Delta^3 X' + \text{etc.}$$

tota expressio inventa contrahetur in hanc

$$\Sigma = S + \omega X' + \frac{\omega(\omega-1)}{1\cdot 2}\Delta X' + \frac{\omega(\omega-1)(\omega-2)}{1\cdot 2\cdot 3}\Delta^2 X' + \text{etc.},$$

quae terminum summatorium seriei $A + B + C + D + \text{etc.}$ involvit; qui autem si esset cognitus, interpolatio nullam haberet difficultatem. Interim tamen et hac formula uti licebit, quippe quae, quoties abrumpitur, quemvis terminum interpolandum finite et algebraice expressum exhibet; sin autem in infinitum progrediatur, plerumque praestat priorem formulam adhibere, in qua ratio terminorum infinitesimorum habetur. Haec vero, si ponatur $x = 0$, ut Σ denotet terminum indici ω respondentem, ob $S = 0$ hanc formam induet

$$\begin{aligned}
\Sigma = &\ A + B + C + D + \text{etc.} \\
&- Z' - Z'' - Z''' - Z'''' - \text{etc.} \\
&+ \omega\left(A + \Delta A + \Delta B + \Delta C + \Delta D + \text{etc.}\right) \\
&+ \frac{\omega(\omega-1)}{1\cdot 2}\left(\Delta A + \Delta^2 A + \Delta^2 B + \Delta^2 C + \Delta^2 D + \text{etc}\right) \\
&+ \frac{\omega(\omega-1)(\omega-2)}{1\cdot 2\cdot 3}\left(\Delta^2 A + \Delta^3 A + \Delta^3 B + \Delta^3 C + \Delta^3 D + \text{etc}\right) \\
&\qquad\text{etc.}
\end{aligned}$$

Vel si ponatur brevitatis gratia

$$\omega = \alpha, \quad \frac{\omega(\omega-1)}{1\cdot 2} = \beta, \quad \gamma = \frac{\omega(\omega-1)(\omega-2)}{1\cdot 2\cdot 3} \quad \text{etc.},$$

erit

$$\begin{aligned}
\Sigma = &\ \alpha A + \beta\Delta A + \gamma\Delta^2 A + \delta\Delta^3 A + \text{etc.} \\
&+ A + \alpha\Delta A + \beta\Delta^2 A + \gamma\Delta^3 A + \text{etc.} - Z' \\
&+ B + \alpha\Delta B + \beta\Delta^2 B + \gamma\Delta^3 B + \text{etc.} - Z'' \\
&+ C + \alpha\Delta C + \beta\Delta^2 C + \gamma\Delta^3 C + \text{etc.} - Z''' \\
&\qquad\text{etc.},
\end{aligned}$$

quarum serierum horizontalium numerus in infinitum quidem progreditur, at quaelibet finito terminorum numero constat.

EXEMPLUM

Interpolare hanc seriem

$$\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\frac{1}{2}, & \left(\frac{1}{2}+\frac{2}{3}\right), & \left(\frac{1}{2}+\frac{2}{3}+\frac{3}{4}\right), & \left(\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}\right) \quad \text{etc.}
\end{array}$$

Sit huius seriei terminus indici ω respondens $=\Sigma$, et cum ea oriatur ex summatione huius seriei $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$ etc., erit $Z = \frac{\omega}{\omega+1}$, et quia termini infinitesimi differentias suas primas iam habent evanescentes, differentiae tantum primae sunt accipiendae, quae erunt ob

$$A = \tfrac{1}{2}, \quad B = \tfrac{2}{3}, \quad C = \tfrac{3}{4}, \quad D = \tfrac{4}{5} \text{ etc.}$$
$$\Delta A = \tfrac{1}{2\cdot 3}, \quad \Delta B = \tfrac{1}{3\cdot 4} \quad \Delta C = \tfrac{1}{4\cdot 5} \text{ etc.}$$

Hinc ergo habebitur

$$\begin{aligned}\Sigma = \tfrac{\omega}{2} &+ \tfrac{1}{2} + \tfrac{2}{3} + \tfrac{3}{4} + \tfrac{4}{5} + \text{etc.}\\ &+ \tfrac{\omega}{2\cdot 3} + \tfrac{\omega}{3\cdot 4} + \tfrac{\omega}{4\cdot 5} + \tfrac{\omega}{5\cdot 6} + \text{etc.}\\ &- \tfrac{\omega+1}{\omega+2} - \tfrac{\omega+2}{\omega+3} - \tfrac{\omega+3}{\omega+4} - \tfrac{\omega+4}{\omega+5} - \text{etc.}\end{aligned}$$

seu ob

$$\tfrac{\omega}{2} + \tfrac{\omega}{2\cdot 3} + \tfrac{\omega}{3\cdot 4} + \tfrac{\omega}{4\cdot 5} + \text{etc.} = \omega$$

erit

$$\begin{aligned}\Sigma = \omega &+ \tfrac{1}{2} + \tfrac{2}{3} + \tfrac{3}{4} + \tfrac{4}{5} + \text{etc.}\\ &- \tfrac{\omega+1}{\omega+2} - \tfrac{\omega+2}{\omega+3} - \tfrac{\omega+3}{\omega+4} - \tfrac{\omega+4}{\omega+5} - \text{etc.}\end{aligned}$$

Si ergo quaeratur terminus indici $\frac{1}{2}$ respondens, erit is

$$\Sigma = \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{3}{5} + \tfrac{2}{3} - \tfrac{5}{7} + \tfrac{3}{4} - \tfrac{7}{9} + \tfrac{4}{5} - \tfrac{9}{11} + \text{etc.}$$

seu

$$\Sigma = \tfrac{1}{2} - \tfrac{1}{2\cdot 5} - \tfrac{1}{3\cdot 7} - \tfrac{1}{4\cdot 9} - \tfrac{1}{5\cdot 11} - \tfrac{1}{6\cdot 13} - \text{etc.}$$

ideoque

$$\tfrac{1}{2}\Sigma = \tfrac{1}{4} - \tfrac{1}{4\cdot 5} - \tfrac{1}{6\cdot 7} - \tfrac{1}{8\cdot 9} - \tfrac{1}{10\cdot 11} - \tfrac{1}{12\cdot 13} - \text{etc.}$$

seu

$$\begin{aligned}\tfrac{1}{2}\Sigma = \tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{6} - \tfrac{1}{8} - \tfrac{1}{10} - \tfrac{1}{12} - \text{etc.}\\ + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{9} + \tfrac{1}{11} + \tfrac{1}{13} + \text{etc.}\end{aligned}$$

Quare, cum sit

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \text{etc.} = l2,$$

erit

$$\tfrac{1}{2}\Sigma = l2 - 1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} = l2 - \tfrac{7}{12}$$

ideoque

$$\Sigma = 2l2 - \tfrac{7}{6}.$$

398. Pergamus nunc ad series interpolandas, quarum termini ex factoribus sunt conflati, sitque proposita haec series generalissima

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ A, & AB, & ABC, & ABCD, & ABCDE \quad \text{etc.}, \end{array}$$

cuius terminus indici ω respondens sit $=\Sigma$. Erit ergo $l2$ terminus respondens indici ω in hac serie

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ lA, & (lA+lB), & (lA+lB+lC), & (lA+lB+lC+lD) \quad \text{etc.}, \end{array}$$

Quodsi ergo ponamus huius seriei terminos infinitesimos evanescere atque seriei *A*, *B*, *C*, *D*, *E* etc. terminum indici ω respondentem esse *Z* eiusque sequentes indicibus $\omega+1, \omega+2, \omega+3, \omega+4$ etc. respondentes esse Z', Z'', Z''', Z'''' etc., erit ex supra demonstratis

$$\begin{aligned} l\Sigma = {} & lA + lB + lC + lD + \text{etc.} \\ & - lZ' - lZ'' - lZ''' - lZ'''' - \text{etc.} \end{aligned}$$

Hinc igitur ad numeros progrediendo habebitur

$$\Sigma = \frac{A}{Z'} \cdot \frac{B}{Z''} \cdot \frac{C}{Z'''} \cdot \frac{D}{Z''''} \cdot \text{etc.}$$

399. Quodsi autem terminorum infinitesimorum seriei *A*, *B*, *C*, *D* etc. logarithmi non evanescant, sed habeant differentias evanescentes, erit, uti vidimus,

$$\begin{aligned} l\Sigma = {} & lA + lB + lC + lD + \text{etc.} \\ & - lZ' - lZ'' - lZ''' - lZ'''' - \text{etc.} \\ & + \omega lA + \omega\left(l\tfrac{B}{A} + l\tfrac{C}{B} + l\tfrac{D}{C} + \text{etc.}\right) \end{aligned}$$

hincque ad numeros a logarithmis procedendo fiet

$$\Sigma = A^{\omega} \cdot \frac{A^{1-\omega}B^{\omega}}{Z'} \cdot \frac{B^{1-\omega}C^{\omega}}{Z''} \cdot \frac{C^{1-\omega}D^{\omega}}{Z'''} \cdot \frac{D^{1-\omega}E^{\omega}}{Z''''} \cdot \text{etc.}$$

At si illorum logarithmorum infinitesimorum differentiae demum secundae evanescant, erit

$$\begin{aligned} l\Sigma = &\, lA + lB + lC + lD + \text{etc.} \\ &- lZ' - lZ'' - lZ''' - lZ'''' - \text{etc.} \\ &+ \omega\left(lA + l\tfrac{B}{A} + l\tfrac{C}{B} + l\tfrac{D}{C} + l\tfrac{E}{D} + \text{etc.}\right) \\ &+ \frac{\omega(\omega-1)}{1\cdot 2}\left(l\tfrac{B}{A} + l\tfrac{AC}{B^2} + l\tfrac{BD}{C^2} + l\tfrac{CE}{D^2}\, l\tfrac{DF}{E^2} + \text{etc.}\right). \end{aligned}$$

Ex his itaque obtinebitur

$$\Sigma = A^{\frac{\omega(3-\omega)}{2}} \cdot B^{\frac{\omega(\omega-1)}{1\cdot 2}} \cdot \frac{A^{\frac{(\omega-1)(\omega-2)}{1\cdot 2}} B^{\omega(2-\omega)} C^{\frac{\omega(\omega-1)}{1\cdot 2}}}{Z'} \cdot \frac{B^{\frac{(\omega-1)(\omega-2)}{1\cdot 2}} C^{\omega(2-\omega)} D^{\frac{\omega(\omega-1)}{1\cdot 2}}}{Z''} \cdot \text{ etc.},$$

quae, si $\omega < 1$, commodius ita exprimetur

$$\Sigma = \frac{A^{\frac{\omega(3-\omega)}{2}}}{B^{\frac{\omega(1-\omega)}{1\cdot 2}}} \cdot \frac{A^{\frac{\omega(1-\omega)(2-\omega)}{1\cdot 2}} B^{\omega(2-\omega)}}{C^{\frac{\omega(1-\omega)}{1\cdot 2}} Z'} \cdot \frac{B^{\frac{\omega(1-\omega)(2-\omega)}{1\cdot 2}} C^{\omega(2-\omega)}}{D^{\frac{\omega(1-\omega)}{1\cdot 2}} Z''} \cdot \text{etc.}$$

400. Accommodemus hanc interpolationem ad istam seriem

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \\ \frac{a}{b}, & \frac{a(a+c)}{b(b+c)}, & \frac{a(a+c)(a+2c)}{b(b+c)(b+2c)}, & \frac{a(a+c)(a+2c)(a+3c)}{b(b+c)(b+2c)(b+3c)} & \text{etc.}, \end{array}$$

cuius factores desumti sunt ex hac serie

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & \\ \frac{a}{b}, & \frac{a+c}{b+c}, & \frac{(a+2c)}{(b+2c)}, & \frac{(a+3c)}{(b+3c)} & \text{etc.}, \end{array}$$

cuius terminorum infinitesimorum logarithmi sunt $=0$. Erit ergo

$$Z = \frac{a-c+c\omega}{b-c+c\omega}, \quad Z' = \frac{a+c\omega}{b+c\omega} \quad \text{etc.}$$

Hinc, si illius seriei terminus indici ω respondens ponatur $=\Sigma$, erit ex § 398

$$\Sigma = \frac{a(b+c\omega)}{b(a+c\omega)} \cdot \frac{(a+c)(b+c+c\omega)}{(b+c)(a+c+c\omega)} \cdot \frac{(a+2c)(b+2c+c\omega)}{(b+2c)(a+2c+c\omega)} \cdot \text{etc.}$$

Quare si desideretur terminus indici $\frac{1}{2}$ respondens, facto $\omega = \frac{1}{2}$ erit

$$\Sigma = \frac{a(2b+c)}{b(2a+c)} \cdot \frac{(a+c)(2b+3c)}{(b+c)(2a+3c)} \cdot \frac{(a+2c)(2b+5c)}{(b+2c)(2a+5c)} \cdot \text{etc.}$$

EXEMPLUM

Interpolare hanc seriem

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \frac{1}{2}, & \frac{1\cdot3}{2\cdot4}, & \frac{1\cdot3\cdot5}{2\cdot4\cdot6}, & \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}, & \frac{1\cdot3\cdot5\cdot7\cdot9}{2\cdot4\cdot6\cdot8\cdot10} \quad \text{etc.} \end{array}$$

Cum hic sit $a = 1,\ b = 2,$ et $c = 2$, si terminus indici cuicunque ω respondens $= \Sigma$, erit

$$\Sigma = \frac{1(2+2\omega)}{2(1+2\omega)} \cdot \frac{3(4+2\omega)}{4(3+2\omega)} \cdot \frac{5(6+2\omega)}{6(5+2\omega)} \cdot \frac{7(8+2\omega)}{8(7+2\omega)} \cdot \text{etc.}$$

Hinc si termini, qui indicibus $\omega+1,\ \omega+2,\ \omega+3$ etc. respondent, ponantur $\Sigma',\ \Sigma'',\ \Sigma'''$ etc., erit

$$\Sigma' = \frac{1+2\omega}{2+2\omega} \cdot \Sigma$$

$$\Sigma'' = \frac{1+2\omega}{2+2\omega} \cdot \frac{3+2\omega}{4+2\omega} \cdot \Sigma$$

$$\Sigma''' = \frac{1+2\omega}{2+2\omega} \cdot \frac{3+2\omega}{4+2\omega} \cdot \frac{5+2\omega}{6+2\omega} \cdot \Sigma$$

etc.

Si itaque desideretur terminus indici $\frac{1}{2}$ respondens, facto $\omega = \frac{1}{2}$ erit

$$\Sigma = \frac{1\cdot3}{2\cdot2} \cdot \frac{3\cdot5}{4\cdot4} \cdot \frac{5\cdot7}{6\cdot6} \cdot \frac{7\cdot9}{8\cdot8} \cdot \frac{9\cdot11}{10\cdot10} \cdot \text{etc.}$$

Verum posito $\pi =$ semicircumferentiae circuli, cuius radius est $= 1$, supra ostendimus esse

$$\pi = 2 \cdot \frac{2\cdot2}{1\cdot3} \cdot \frac{4\cdot4}{3\cdot5} \cdot \frac{6\cdot6}{5\cdot7} \cdot \frac{8\cdot8}{7\cdot9} \cdot \text{etc.}$$

Hanc ob rem termini intermedii indicibus $\frac{1}{2},\ \frac{3}{2},\ \frac{5}{2}$ etc. per peripheriam circuli exprimi poterunt hoc modo

$$\begin{array}{lcccc} \text{Indices:} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} \\ \text{Termini:} & \frac{2}{\pi}, & \frac{2}{3} \cdot \frac{2}{\pi}, & \frac{2\cdot4}{3\cdot5} \cdot \frac{2}{\pi}, & \frac{2\cdot4\cdot6}{3\cdot5\cdot7} \cdot \frac{2}{\pi} \quad \text{etc.} \end{array}$$

Quam eandem interpolationem WALLISIUS in *Arithmetica infinitorum* invenit.

401. Consideremus nunc istam seriem

$$\begin{matrix} 1 & 2 & 3 & 4 \\ a, & a(a+b), & a(a+b)(a+2b), & a(a+b)(a+2b)(a+3b) \end{matrix} \text{ etc.,}$$

cuius factores hanc progressionem arithmeticam constituunt

$$a,\ (a+b),\ (a+2b),\ (a+3b),\ (a+4b) \text{ etc.}$$

cuiusque termini infinitesimi ita sunt comparati, ut eorum logarithmorum differentiae evanescant. Cum igitur sit

$$Z = a - b + b\omega$$

et

$$Z' = a + b\omega,\quad Z'' = a + b + b\omega,\quad Z''' = a + 2b + b\omega \quad \text{etc.},$$

si Σ denotet terminum seriei propositae, cuius index est $=\omega$, erit

$$\Sigma = a^{\omega}\cdot\frac{a^{1-\omega}(a+b)^{\omega}}{a+b\omega}\cdot\frac{(a+b)^{1-\omega}(a+2b)^{\omega}}{a+b+b\omega}\cdot\frac{(a+2b)^{1-\omega}(a+3b)^{\omega}}{a+2b+b\omega}\cdot\text{etc.}$$

Hocque valore invento, si ω denotet numerum quemvis fractum unitate minorem, termini sequentes indicibus $1+\omega,\ 2+\omega,\ 3+\omega$ etc. respondentes ita determinabuntur, ut sit

$$\begin{aligned} \Sigma' &= (a+b\omega)\Sigma \\ \Sigma'' &= (a+b\omega)(a+b+b\omega)\Sigma \\ \Sigma''' &= (a+b\omega)(a+b+b\omega)(a+2b+b\omega)\Sigma \\ &\text{etc.} \end{aligned}$$

.

Quare si desideretur terminus indici $\frac{1}{2}$ respondens, facto $\omega = \frac{1}{2}$ erit

$$\Sigma = a^{\frac{1}{2}}\cdot\frac{a^{\frac{1}{2}}(a+b)^{\frac{1}{2}}}{a+\frac{1}{2}b}\cdot\frac{(a+b)^{\frac{1}{2}}(a+2b)^{\frac{1}{2}}}{a+\frac{3}{2}b}\cdot\frac{(a+2b)^{\frac{1}{2}}(a+3b)^{\frac{1}{2}}}{a+\frac{5}{2}b}\cdot\text{etc.}$$

ideoque sumtis quadratis

$$\Sigma^2 = a\cdot\frac{a(a+b)}{(a+\frac{1}{2}b)(a+\frac{1}{2}b)}\cdot\frac{(a+b)(a+2b)}{(a+\frac{3}{2}b)(a+\frac{3}{2}b)}\cdot\frac{(a+2b)(a+3b)}{(a+\frac{5}{2}b)(a+\frac{5}{2}b)}\cdot\text{etc.}$$

402. Ponatur in serie, quam supra § 400 tractavimus,

$$\begin{matrix} 1 & 2 & 3 & 4 \\ \frac{f}{g}, & \frac{f(f+h)}{g(g+h)}, & \frac{f(f+h)(f+2h)}{g(g+h)(g+2h)}, & \frac{f(f+h)(f+2h)(f+3h)}{g(g+h)(g+2h)(g+3h)} \end{matrix} \text{ etc.}$$

terminus indici $\frac{1}{2}$ respondens $=\Theta$; erit

$$\Theta = \frac{f(f+\frac{1}{2}h)}{g(g+\frac{1}{2}h)} \cdot \frac{(f+h)(f+\frac{3}{2}h)}{(g+h)(g+\frac{3}{2}h)} \cdot \frac{(f+2h)(f+\frac{5}{2}h)}{(g+2h)(g+\frac{5}{2}h)} \cdot \text{ etc.;}$$

statuatur nunc

$$f = a, \quad g = a+\tfrac{1}{2}b \text{ et } h = b;$$

erit

$$\Theta = \frac{a(a+b)}{(a+\frac{1}{2}b)(a+\frac{1}{2}b)} \cdot \frac{(a+b)(a+2b)}{(a+\frac{3}{2}b)(a+\frac{3}{2}b)} \cdot \text{etc.}$$

ideoque fiet $\Sigma^2 = a\Theta$ et $\Sigma = \sqrt{a\Theta}$. Quocirca si huius seriei

$$\begin{matrix} 1 & 2 & 3 & 4 \\ a, & a(a+b), & a(a+b)(a+2b), & a(a+b)(a+2b)(a+3b) \end{matrix} \text{ etc.,}$$

terminus indici $\frac{1}{2}$ respondens statuatur $=\Sigma$ atque huius seriei

$$\begin{matrix} 1 & 2 & 3 \\ \frac{a}{(a+\frac{1}{2}b)}, & \frac{a(a+b)}{(a+\frac{1}{2}b)(a+\frac{3}{2}b)}, & \frac{a(a+b)(a+2b)}{(a+\frac{1}{2}b)(a+\frac{3}{2}b)(a+\frac{5}{2}b)} \end{matrix} \text{ etc.}$$

terminus indici $\frac{1}{2}$ respondens ponatur $=\Theta$, erit $\Sigma = \sqrt{a\Theta}$.

Cum igitur hic seriei solorum numeratorum terminus indici $\frac{1}{2}$ respondens sit $=\Sigma$, si in serie denominatorum terminus indici $\frac{1}{2}$ respondens ponatur Λ, erit $\Theta = \frac{\Sigma}{\Lambda}$; at est $\Theta = \frac{\Sigma^2}{a}$, unde fiet $\Sigma = \frac{a}{\Lambda}$ seu $\Sigma\Lambda = a$, quibus theorematibus interpolatio huiusmodi serierum non mediocriter illustratur.

EXEMPLUM 1

Sit proposita haec series interpolanda

$$1,\ 1\cdot 2,\ 1\cdot 2\cdot 3,\ 1\cdot 2\cdot 3\cdot 4\quad \text{etc}.$$

Quia hic est $a=1$ et $b=1$, si terminus indici ω respondens ponatur $=\Sigma$, erit

$$\Sigma=\frac{1^{1-\omega}\cdot 2^{\omega}}{1+\omega}\cdot\frac{2^{1-\omega}\cdot 3^{\omega}}{2+\omega}\cdot\frac{3^{1-\omega}\cdot 4^{\omega}}{3+\omega}\cdot\frac{4^{1-\omega}\cdot 5^{\omega}}{4+\omega}\cdot\text{etc.}$$

Hic pro ω semper fractio unitate minor accipi potest; nihilominus enim interpolatio per totam seriem extendetur. Nam si termini indicibus $1+\omega,\ 2+\omega,\ 3+\omega$ etc. respondentes ponantur $\Sigma',\ \Sigma'',\ \Sigma'''$ etc., erit

$$\Sigma' =(1+\omega)\Sigma$$
$$\Sigma'' =(1+\omega)(1+2\omega)\Sigma$$
$$\Sigma''' =(1+\omega)(1+2\omega)(1+3\omega)\Sigma$$
etc.

Seriei ergo propositae terminus indici $\frac{1}{2}$ respondens erit

$$\Sigma=\frac{1^{\frac{1}{2}}\cdot 2^{\frac{1}{2}}}{1\frac{1}{2}}\cdot\frac{2^{\frac{1}{2}}\cdot 3^{\frac{1}{2}}}{2\frac{1}{2}}\cdot\frac{3^{\frac{1}{2}}\cdot 4^{\frac{1}{2}}}{3\frac{1}{2}}\cdot\text{etc.}$$

sive

$$\Sigma^2=\frac{2\cdot 4}{3\cdot 3}\cdot\frac{4\cdot 6}{5\cdot 5}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot\frac{8\cdot 10}{9\cdot 9}\cdot\text{etc.}$$

Unde, cum sit

$$\pi=2\cdot\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\text{etc.},$$

erit

$$\Sigma^2=\frac{\pi}{4}\quad\text{et}\quad\Sigma=\frac{\sqrt{\pi}}{2}$$

hincque respondebunt

indicibus	$\frac{1}{2}$,	$\frac{3}{2}$,	$\frac{5}{2}$,	$\frac{7}{2}$	etc.
termini	$\frac{\sqrt{\pi}}{2}$,	$\frac{3}{2}\cdot\frac{\sqrt{\pi}}{2}$,	$\frac{3\cdot 5}{2\cdot 2}\cdot\frac{\sqrt{\pi}}{2}$,	$\frac{3\cdot 5\cdot 7}{2\cdot 2\cdot 2}\cdot\frac{\sqrt{\pi}}{2}$	etc.

EXEMPLUM 2

Sit proposita haec series interpolanda

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1, & 1\cdot 3, & 1\cdot 3\cdot 5, & 1\cdot 3\cdot 5\cdot 7 \quad \text{etc.} \end{array}$$

Quia hic est $a=1,\ b=2$, si terminus indici ω respondens ponatur $=\Sigma$, erit

$$\Sigma = \tfrac{1^{1-\omega}\cdot 3^{\omega}}{1+2\omega}\cdot\tfrac{3^{1-\omega}\cdot 5^{\omega}}{3+2\omega}\cdot\tfrac{5^{1-\omega}\cdot 7^{\omega}}{5+2\omega}\cdot\text{etc.}$$

terminique ordine sequentes ita erunt comparati:

$$\begin{aligned} \Sigma' &= (1+2\omega)\Sigma \\ \Sigma'' &= (1+2\omega)(3+2\omega)\Sigma \\ \Sigma''' &= (1+2\omega)(3+2\omega)(5+3\omega)\Sigma \\ &\text{etc.} \end{aligned}$$

Si igitur seriei propositae desideretur terminus indici $\frac{1}{2}$ respondens isque vocetur $=\Sigma$, erit

$$\Sigma = \tfrac{\sqrt{1\cdot 3}}{2}\cdot\tfrac{\sqrt{3\cdot 5}}{4}\cdot\tfrac{\sqrt{5\cdot 7}}{6}\cdot\tfrac{\sqrt{7\cdot 9}}{8}\cdot\text{etc.},$$

ergo

$$\Sigma^2 = \tfrac{1\cdot 3}{2\cdot 2}\cdot\tfrac{3\cdot 5}{4\cdot 4}\cdot\tfrac{5\cdot 7}{6\cdot 6}\cdot\tfrac{7\cdot 9}{8\cdot 8}\cdot\text{etc.} = \tfrac{2}{\pi}$$

ideoque habebitur $\Sigma=\sqrt{\frac{2}{\pi}}$. At respondebunt

$$\begin{array}{lcccc} \text{indicibus} & \frac{1}{2}, & \frac{3}{2}, & \frac{5}{2}, & \frac{7}{2}, \quad \text{etc.} \\ \text{termini} & \sqrt{\frac{2}{\pi}}, & 2\cdot\sqrt{\frac{2}{\pi}}, & 2\cdot 4\sqrt{\frac{2}{\pi}}, & 2\cdot 4\cdot 6\sqrt{\frac{2}{\pi}} \quad \text{etc.} \end{array}$$

Quodsi ergo prior series et haec invicem multiplicentur, ut habeatur haec series

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1^2, & 1^2\cdot 2\cdot 3, & 1^2\cdot 2\cdot 3^2\cdot 5, & 1^2\cdot 2\cdot 3^2\cdot 4\cdot 5\cdot 7 & 1^2\cdot 2\cdot 3^2\cdot 4\cdot 5^2\cdot 7\cdot 9 \ \text{etc.,} \end{array}$$

eius terminus indici $\frac{1}{2}$ respondens erit $=\frac{\sqrt{\pi}}{2}\cdot\sqrt{\frac{2}{\pi}}=\frac{1}{\sqrt{2}}$; quod facile perspicitur, si isti seriei haec forma tribuatur

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \frac{1\cdot 2}{2}, & \frac{1\cdot 2\cdot 3\cdot 4}{2^2}, & \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{2^3}, & \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}{2^4} \end{array} \text{ etc.,}$$

cuius terminus indici $\frac{1}{2}$ respondens manifesto est $=\frac{1}{\sqrt{2}}$.

EXEMPLUM 3

Sit ista series proposita interpolanda

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \frac{n}{1}, & \frac{n(n-1)}{1\cdot 2}, & \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}, & \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} \end{array} \text{ etc.}$$

Considerentur huius seriei numeratores ac denominatores seorsim, et cum numeratores sint

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ n, & n(n-1), & n(n-1)(n-2), & n(n-1)(n-2)(n-3) \end{array} \text{ etc.,}$$

fiet applicatione facta $a=n$ et $b=-1$, unde huius seriei terminus indici ω respondens erit

$$=n^{\omega}\cdot\frac{n^{1-\omega}(n-1)^{\omega}}{n-\omega}\cdot\frac{(n-1)^{1-\omega}(n-2)^{\omega}}{n-1-\omega}\cdot\frac{(n-2)^{1-\omega}(n-3)^{\omega}}{n-2-\omega}\cdot\text{etc.,}$$

quae autem expressio ob factores in negativos abeuntes nihil certi monstrat. Transformetur ergo series proposita ponendo brevitatis gratia $1\cdot 2\cdot 3\cdots n=N$ in hanc

$$\begin{array}{ccc} 1 & 2 & 3 \\ \frac{N}{1\cdot 1\cdot 2\cdot 3\cdots(n-1)}, & \frac{N}{1\cdot 2\cdot 1\cdot 2\cdot 3\cdots(n-2)}, & \frac{N}{1\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3\cdots(n-3)} \end{array} \text{ etc.,}$$

cuius denominatores cum constent duobus factoribus, alteri constituent hanc seriem

$$\begin{array}{ccc} 1 & 2 & 3 \\ 1\cdot 2\cdot 3\cdots(n-1), & 1\cdot 2\cdot 3\cdots(n-2), & 1\cdot 2\cdot 3\cdots(n-3) \end{array} \text{ etc.,}$$

cuius terminus indici ω respondens convenit cum termino huius seriei

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 1, & 1\cdot 2, & 1\cdot 2\cdot 3, & 1\cdot 2\cdot 3\cdot 4, & 1\cdot 2\cdot 3\cdot 4\cdot 5 \end{array} \text{ etc.,}$$

indici $n-\omega$ respondente, qui est

$$\frac{1^{1-n+\omega}\cdot 2^{n-\omega}}{1+n-\omega}\cdot\frac{2^{1-n+\omega}\cdot 3^{n-\omega}}{2+n-\omega}\cdot\frac{3^{1-n+\omega}\cdot 4^{n-\omega}}{3+n-\omega}\cdot\text{etc.}$$

Sit autem huius seriei terminus indici $1-\omega$ respondens $=\Theta$; erit

$$\Theta=\frac{1^{\omega}\cdot 2^{1-\omega}}{2-\omega}\cdot\frac{2^{\omega}\cdot 3^{1-\omega}}{3-\omega}\cdot\frac{3^{\omega}\cdot 4^{1-\omega}}{4-\omega}\cdot\text{etc.}$$

atque cum respondeant

indicibus	$1-\omega$,	$2-\omega$,	$3-\omega$	etc.
termini	Θ,	$(2-\omega)\Theta$,	$(2-\omega)(3-\omega)\Theta$	etc.,

indici $n-\omega$ respondebit hic terminus

$$(2-\omega)(3-\omega)(4-\omega)\cdots(\mathrm{n}-\omega)\Theta.$$

Deinde illorum denominatorum alteri factores constituent hanc seriem

1	2	3	4	5	
1,	$1\cdot 2$,	$1\cdot 2\cdot 3$,	$1\cdot 2\cdot 3\cdot 4$,	$1\cdot 2\cdot 3\cdot 4\cdot 5$	etc. ;

si terminus indici ω respondens ponatur $=\Lambda$, erit

$$\Lambda=\frac{1^{1-\omega}\cdot 2^{\omega}}{1+\omega}\cdot\frac{2^{1-\omega}\cdot 3^{\omega}}{2+\omega}\cdot\frac{3^{1-\omega}\cdot 4^{\omega}}{3+\omega}\cdot\text{etc.}$$

Quibus inventis, si ipsius seriei propositae

1	2	3	4	
$\frac{n}{1}$,	$\frac{n(n-1)}{1\cdot 2}$,	$\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$,	$\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}$	etc.

terminus indici ω respondens ponatur Σ, erit

$$\Sigma=\frac{N}{\Lambda\cdot(2-\omega)(3-\omega)(4-\omega)\cdots(\mathrm{n}-\omega)\Theta}.$$

At vero est

$$\frac{N}{(2-\omega)(3-\omega)(4-\omega)\cdots(\mathrm{n}-\omega)}=\frac{2}{2-\omega}\cdot\frac{3}{3-\omega}\cdot\frac{4}{4-\omega}\cdots\frac{n}{\mathrm{n}-\omega}$$

atque

$$\Lambda\Theta=\frac{1\cdot 2}{(1+\omega)(2-\omega)}\cdot\frac{2\cdot 3}{(2+\omega)(3-\omega)}\cdot\frac{3\cdot 4}{(3+\omega)(4-\omega)}\cdot\text{etc.}$$

Ex quibus terminus indici ω respondens quaesitus erit

$$\Sigma = \frac{2}{2-\omega}\cdot\frac{3}{3-\omega}\cdot\frac{4}{4-\omega}\cdot\frac{5}{5-\omega}\cdots\frac{n}{n-\omega}\cdot\frac{(1+\omega)(2-\omega)}{1\cdot 2}\cdot\frac{(2+\omega)(3-\omega)}{2\cdot 3}\cdot\frac{(3+\omega)(4-\omega)}{3\cdot 4}\cdot \text{etc. in infinitum.}$$

Indici ergo $\frac{1}{2}$ respondebit iste terminus

$$\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7}\cdot\frac{10}{9}\cdot\frac{12}{11}\cdots\frac{2n}{2n-1}\cdot\frac{3\cdot 3}{2\cdot 4}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot\frac{77}{6\cdot 8}\cdot\frac{9\cdot 9}{8\cdot 10}\cdot \text{etc.},$$

qui reducitur ad

$$\frac{4\cdot 6\cdot 8\cdot 10\cdots 2n}{3\cdot 5\cdot 7\cdot 9\cdots(2n-1)}\cdot\frac{4}{\pi},$$

seu erit

$$= \frac{2}{\pi}\cdot\frac{2\cdot 4\cdot 6\cdot 8\cdot 10\cdots 2n}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdots(2n-1)}.$$

Si fuerit $n = 2$, prodibit ista series interpolanda

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1, & 2, & 1, & 0, & 0, & 0, & 0 \text{ etc.,} \end{array}$$

cuius propterea terminus indici $\frac{1}{2}$ respondens est $= \frac{16}{3\pi}$.

EXEMPLUM 4

Quaeratur terminus respondens indici $= \frac{1}{2}$ *in hac serie*

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 1, & +\frac{1}{2}, & -\frac{1\cdot 1}{2\cdot 4}, & +\frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}, & -\frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8} \quad \text{etc.} \end{array}$$

Oritur haec series ex praecedente, si ponatur $n = \frac{1}{2}$, eritque propterea terminus quaesitus, qui sit $= \Sigma$,

$$\Sigma = \frac{2}{\pi}\cdot\frac{2\cdot 4\cdot 6\cdot 8\cdots 2n}{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}$$

posito $n = \frac{1}{2}$. Ponatur

$$\frac{2\cdot 4\cdot 6\cdot 8\cdots 2n}{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)} = \Theta,$$

si sit $n = \frac{1}{2}$, eritque Θ terminus respondens indici $\frac{1}{2}$ in hac serie

$$\frac{2}{1},\ \frac{2\cdot 4}{1\cdot 3},\ \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5},\ \frac{2\cdot 4\cdot 6\cdot 8}{1\cdot 3\cdot 5\cdot 7} \text{ etc.,}$$

qui ex superioribus prodit $= \frac{\pi}{2}$. Quocirca seriei propositae terminus indici $\frac{1}{2}$ respondens, qui quaeritur, erit $=1$. Quoniam autem in ista serie, si terminus indici cuicunque ω respondens ponatur $= \Sigma$, sequens eum erit $\Sigma' = \frac{1-2\omega}{2+2\omega}\Sigma$, series proposita ita mediis terminis interiiciendis interpolabitur:

$$\begin{array}{lcccccccc} \text{Indices:} & 0 & \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & 3 & \frac{7}{2} \\ \text{Termini:} & 1, & 1, & \frac{1}{2}, & 0, & \frac{-1\cdot 1}{2\cdot 4}, & 0, & \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}, & 0 \text{ etc.} \end{array}$$

EXEMPLUM 5

Si n fuerit numerus quicunque fractus, invenire terminum indici ω respondentem in serie

$$\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ 1 & \frac{n}{1}, & \frac{n(n-1)}{1\cdot 2}, & \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}, & \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} \text{ etc.,} \end{array}$$

Si expressionem

$$\frac{2}{2-\omega}\cdot\frac{3}{3-\omega}\frac{4}{4-\omega}\cdots\frac{n}{n-\omega}$$

cum § 400 comparemus, fit $a=1,\ c=1,\ b=1-\omega$ ibique loco ω posito n erit

$$\frac{1}{1-\omega}\cdot\frac{2}{2-\omega}\cdot\frac{3}{3-\omega}\cdots\frac{n}{n-\omega} = \frac{1(1-\omega+n)}{(1-\omega)(1+n)}\cdot\frac{2(2-\omega+n)}{(2-\omega)(2+n)}\cdot\text{etc.,}$$

unde terminus quaesitus indici ω respondens, si ponatur $= \Sigma$, erit

$$\Sigma = \frac{(1-\omega+n)\cdot 2}{(1+n)(2-\omega)}\cdot\frac{(2-\omega+n)3}{(2+n)(3-\omega)}\cdot\text{etc.}\cdot\frac{(1+\omega)(2-\omega)}{1\cdot 2}\cdot\frac{(2+\omega)(3-\omega)}{2\cdot 3}\text{etc.}$$

ideoque

$$\Sigma = \frac{(1+\omega)(1+n-\omega)}{1(1+n)(2-\omega)}\cdot\frac{(2+\omega)(2+n-\omega)}{2(2+n)}\cdot\frac{(3+\omega)(3+n-\omega)}{3(3+n)}\cdot\text{etc.:}$$

quoties ergo $n-\omega$ fuerit numerus integer, valor ipsius Σ rationaliter exprimi potest.
Sic si sit $n=\omega$, erit $\Sigma = 1$;
si $n = 1+\omega$, erit $\Sigma = n$;
si $n = 2+\omega$, erit $\Sigma = \frac{n(n-1)}{1\cdot 2}$;
si $n = 3+\omega$, erit $\Sigma = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$ etc.

At si fuerit $\omega - n$ numerus integer affirmativus, erit semper $\Sigma = 0$.