CHAPTER XVI

CONCERNING THE DIFFERENTIATION OF INEXPLICABLE FUNCTIONS

367. Here I call functions inexplicable, which are not able to be set out by determined expressions nor from the roots of equations, thus so that not only shall they not be algebraic, but also generally it shall be uncertain to what kind of transcending functions they may be related. An inexplicable function of this kind is

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots+\tfrac{1}{x},$$

which certainly depends on *x*, but unless *x* shall be a whole number, cannot be explained in any manner. In a similar manner this expression

$$1\cdot 2\cdot 3\cdot 4\cdots x$$

will be an inexplicable function of *x*, because, if *x* shall be any number, the value of which is unable be expressed either algebraically, or indeed by any sure kind of transcending quantity. Therefore generally the notion of such inexplicable functions can be derived from series. Indeed some proposed series shall be

$$\begin{array}{ccccccccccc} 1 & & 2 & & 3 & & 4 & & \cdots & & x \\ A & + & B & + & C & + & D & + & \cdots & + & X, \end{array}$$

the sum of which, if it is unable to be expressed by a finite formula, will behave as an inexplicable function of *x*, evidently

$$S = A+B+C+D+\cdots+X .$$

Similarly continued products from the terms of the series as

$$P = A\cdot B\cdot C\cdot D\cdots X$$

will show inexplicable functions of *x*, but which with the aid of logarithms are able to be returned to the prior form ; for there shall be

$$lP = lA+lB+lC+lD+\cdots+lX .$$

[We would now perhaps recognise Euler's inexplicable functions as a form of implicit functions, in which it is difficult or impossible to separate the variables, or in any case not worthwhile. It is apparent that Euler's inexplicable sum function for the case of a generalized harmonic series starts from the terms *A*, *B*, *C*, etc., the values of which are shown for the whole number index corresponding to the position in the series ; however a value *x* may be given, which is simply an integer treates as a real number, upon which the series terminates and the whole finite series is given by $S(x)=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{x}$, the last integral term being $\frac{1}{x}$, the following term in the series $S'(x)=S(x)+X'=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{x}+\frac{1}{x+1}$, etc., and thus *S* is the function of *x*; evidently if we set $x=n$, the customary harmonic series emerges as far as *n* . Note that if $x=0$, *S* is defined to be zero also. In general for any sum or product, the starting terms are *A*, *B*, *C*, etc.

Ed Sandifer has given a commentary on some aspects of this chapter in his *How Euler Did It* series of articles for the *AMA*, which it may be useful to read in conjunction with this translation, as part of Euler's work relates to the Beta and Gamma functions, and to the later Riemann zeta function, to be developed further in the next chapter; clearly a complete understanding of what the chapter contains, and it is certainly forward looking in nature, can only be found from a careful reading, as well as the previous chapters on which it depends. As Sandifer points out, little or no regard is paid to convergence of the series, which of course may be diverging, and hence the rearrangement of the terms in such series may be viewed with some suspicion; however, that is for the reader to decide.]

368. Therefore in this chapter I have put in place a method to explain the differentials of inexplicable functions of this kind requiring to be investigated. Which argument, although it may be considered to relate to the first part of this work, where the precepts of the differential calculus have been treated, yet, because it demands the more fertile knowledge of the principles of series, from which it is better to come upon in this other part of the work, by compelling to relinquish in this place the natural order that we have touched on. But since this investigation shall be wholly new not only shall it be free at this stage from any [previous] treatment, so that we may be able to resolve this part of the differential calculus, as we try rather to outline only the first elements of this. In addition truly I may propose several questions, the explanation of which may require the differentiation of functions of this kind, so that likewise the use of this treatment may be seen more clearly, which moreover without doubt will be much enlarged on later.

369. Before all it is necessary towards the differentiation of functions of this kind, that we shall investigate the values of these which they adopt if for *x* there may be put $x+\omega$. Therefore let there be

$$\begin{array}{ccccccccccc} & 1 & & 2 & & 3 & & 4 & \cdots & x & \\ S= & A & + & B & + & C & + & D & +\cdots+ & X, & \end{array}$$

and Σ may be put the value of S which it takes, if for *x* there may be put $x+\omega$, and Z shall be the value of the term corresponding to the index $x+\omega$. Therefore now the terms, which correspond to the indices $x+1, x+2, x+3$ etc., may be indicated by X', X'', X''' etc. and that, which it is agreed to indicate advanced to infinity $x+\infty$, by $X^{|\infty|}$. [Note : this notation does not mean that *X* has been raised to an infinite power, but indicates rather the form adopted by *X* at this stage.] And in a similar manner suitable terms with the indices $x+\omega+1, \quad x+\omega+2, \quad x+\omega+3$ etc. may be indicated by Z', Z'', Z'',etc. and let $Z^{|\infty|}$ correspond to the term $x+\omega+\infty$. With which in place there shall be

$$S' = S+X'$$
$$S'' = S+X'+X''$$
$$S''' = S+X'+X''+X'''$$
$$\text{etc.}$$
$$S^{|\infty|} = S+X'+X''+X'''+\cdots+X^{|\infty|}.$$

In a similar manner Σ also may be increased successively by the terms Z', Z'' etc., there will be

$$\Sigma' = \Sigma+Z'$$
$$\Sigma'' = \Sigma+Z'+Z''$$
$$\Sigma''' = \Sigma+Z'+Z''+Z'''$$
$$\text{etc.}$$
$$\Sigma^{|\infty|} = \Sigma+Z'+Z''+Z'''+\cdots+Z^{|\infty|}.$$

370. Now the nature of the series *S, S', S'', S'''* etc. is required to be considered carefully, of what kind it may become if continued to infinity; which if it may be formed into an arithmetic progression to infinity, will come about if the terms of the series X, X', X'', X''' etc. may converge to equality at infinity [*i.e.* they become equal to each other], thus so that the differences of the series S, S', S'' etc. finally are made equal, and in this case the quantities $S^{|\infty|}, \quad S^{|\infty+1|}, \quad S^{|\infty+2|}$ etc. will be in arithmetic progression, and since there shall be $\Sigma^{|\infty|} = S^{|\infty+\omega|}$, [as the term added to $S^{|\infty|}$ is $\omega\left(S^{|\infty+1|}-S^{|\infty|}\right)$, i.e. $\omega\times$common difference of the *A.P.*, where ω may be a finite or an infinitesimal quantity in general,] on account of

$$S^{|\infty+\omega|} = S^{|\infty|}+\omega\left(S^{|\infty+1|}-S^{|\infty|}\right) = \omega S^{|\infty+1|}+(1-\omega)S^{|\infty|}$$

there will be

$$\Sigma^{|\infty|} = \omega S^{|\infty+1|}+(1-\omega)S^{|\infty|}$$

But there is $S^{|\infty+1|} = S^{|\infty|} + X^{|\infty+1|}$, from which there is made

$$\Sigma^{|\infty|} = S^{|\infty|} + \omega X^{|\infty+1|}$$

from which this equation may be obtained

$$\Sigma + Z' + Z'' + Z''' + \cdots + Z^{|\infty|} = S + X' + X'' + X''' + \cdots + X^{|\infty|} + \omega X^{|\infty+1|},$$

from which the value Σ sought may be defined, which the function *S* adopts, while in that $x+\omega$ is substituted in place of *x*, and there will be

$$\Sigma = S + \omega X^{|\infty+1|} + X' + X'' + X''' + \text{etc.} \quad \text{to infinity}$$
$$- Z' - Z'' - Z''' - \text{etc.} \quad \text{to infinity.}$$

Whereby if the most infinite terms of the series A, B, C, D etc. may vanish [*i.e.* these terms furthest away at infinity], the term $\omega X^{|\infty+1|}$ will vanish and can be ignored.

371. Therefore the value of Σ is expressed by a new infinite series, which can be shown, if the general term of the series $A + B + C + \text{etc.}$ may be considered, from which the values of the terms Z', Z'', Z'', etc. can be defined. Therefore on putting ω infinitely small, since $\Sigma - S$ shall be the differential of the function *S*, this differential *dS* will be expressed by an infinite series. And if indeed higher powers of ω may not be ignored, the complete differential of this inexplicable function *S* will be had; we will illustrate this matter with the following examples, so that the nature of which may be made more clear on examination.

EXAMPLE 1

To find the [complete] differential of this inexplicable function

$$S = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{x}.$$

Because the general term *X* of this series is $= \frac{1}{x}$ and therefore

$$\begin{array}{l|l} X' = \frac{1}{x+1} & Z' = \frac{1}{x+1+\omega} \\ X'' = \frac{1}{x+2} & Z'' = \frac{1}{x+2+\omega} \\ X''' = \frac{1}{x+3} & Z''' = \frac{1}{x+3+\omega} \\ \text{etc.} & \text{etc.,} \end{array}$$

on account of

$$X^{|\infty+1|} = \frac{1}{x+\infty+1} = 0,$$

if there is put $x+\omega$ in place of *x*, the function *S* will change into Σ, so that there shall be

$$\begin{aligned}\Sigma = S &+ \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \text{etc.}\\ &- \frac{1}{x+1+\omega} - \frac{1}{x+2+\omega} - \frac{1}{x+3+\omega} - \text{etc.},\end{aligned}$$

or with these two terms collected into single terms there will be

$$\Sigma = S + \frac{\omega}{(x+1)(x+1+\omega)} + \frac{\omega}{(x+2)(x+2+\omega)} + \frac{\omega}{(x+3)(x+3+\omega)} + \text{etc.},$$

or since there shall be

$$\frac{1}{x+1+\omega} = \frac{1}{x+1} - \frac{\omega}{(x+1)^2} + \frac{\omega^2}{(x+1)^3} - \frac{\omega^3}{(x+1)^4} + \text{etc.}$$

$$\frac{1}{x+2+\omega} = \frac{1}{x+2} - \frac{\omega}{(x+2)^2} + \frac{\omega^2}{(x+2)^3} - \frac{\omega^3}{(x+2)^4} + \text{etc.}$$

etc.

there will be, with the following powers of ω ordered properly,

$$\begin{aligned}\Sigma = S &+ \omega\left(\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+4)^2} + \text{etc.}\right)\\ &- \omega^2\left(\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \frac{1}{(x+4)^3} + \text{etc.}\right)\\ &+ \omega^3\left(\frac{1}{(x+1)^4} + \frac{1}{(x+2)^4} + \frac{1}{(x+3)^4} + \frac{1}{(x+4)^4} + \text{etc.}\right)\\ &- \omega^4\left(\frac{1}{(x+1)^5} + \frac{1}{(x+2)^5} + \frac{1}{(x+3)^5} + \frac{1}{(x+4)^5} + \text{etc.}\right)\end{aligned}$$

etc.

Therefore on putting *dx* for ω we will obtain the complete differential of the proposed function *S*

$$dS = dx\left(\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\frac{1}{(x+4)^2}+\text{etc.}\right)$$
$$-dx^2\left(\frac{1}{(x+1)^3}+\frac{1}{(x+2)^3}+\frac{1}{(x+3)^3}+\frac{1}{(x+4)^3}+\text{etc.}\right)$$
$$+dx^3\left(\frac{1}{(x+1)^4}+\frac{1}{(x+2)^4}+\frac{1}{(x+3)^4}+\frac{1}{(x+4)^4}+\text{etc.}\right)$$
$$-dx^4\left(\frac{1}{(x+1)^5}+\frac{1}{(x+2)^5}+\frac{1}{(x+3)^5}+\frac{1}{(x+4)^5}+\text{etc.}\right)$$
$$\text{etc.}$$

EXAMPLE 2

To find the [complete] differential of this inexplicable function of x

$$S = 1+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}+\cdots+\tfrac{1}{2x-1}.$$

Because the general term of this series is $X = \frac{1}{2x-1}$, there will be

$$\begin{array}{r|l} X' = \frac{1}{2x+1} & Z' = \frac{1}{2x+1+2\omega} \\ X'' = \frac{1}{2x+3} & Z'' = \frac{1}{2x+3+2\omega} \\ X''' = \frac{1}{2x+5} & Z''' = \frac{1}{2x+5+2\omega} \\ \text{etc.} & \text{etc.,} \end{array}$$

On account of the infinite terms of this series vanishing and being equal to each other, if in place of *x* there may be put $x+\omega$,the value of S will produce,

$$\Sigma = S + \frac{1}{2x+1} + \frac{1}{2x+2} + \frac{1}{2x+3} + \text{etc.}$$
$$- \frac{1}{2x+1+2\omega} - \frac{1}{2x+3+2\omega} - \frac{1}{x+5+2\omega} - \text{etc.,}$$

or

$$\Sigma = S + \frac{2\omega}{(2x+1)(2x+1+2\omega)} + \frac{2\omega}{(2x+3)(2x+3+2\omega)} + \text{etc.,}$$

In truth if the individual terms may be resolved into a series following the powers of ω , there will be

$$\Sigma = S + 2\omega\left(\frac{1}{(2x+1)^2} + \frac{1}{(2x+3)^2} + \frac{1}{(2x+5)^2} + \text{etc.}\right)$$
$$-4\omega^2\left(\frac{1}{(2x+1)^3} + \frac{1}{(2x+3)^3} + \frac{1}{(2x+5)^3} + \text{etc.}\right)$$
$$+8\omega^3\left(\frac{1}{(2x+1)^4} + \frac{1}{(2x+3)^4} + \frac{1}{(2x+5)^4} + \text{etc.}\right)$$
$$-16\omega^4\left(\frac{1}{(2x+1)^5} + \frac{1}{(2x+3)^5} + \frac{1}{(2x+5)^5} + \text{etc.}\right)$$
$$\text{etc.}$$

Now there may be put dx for ω and the complete differential of the inexplicable function S proposed will be produced

$$dS = \quad 2dx\left(\frac{1}{(2x+1)^2} + \frac{1}{(2x+3)^2} + \frac{1}{(2x+5)^2} + \text{etc.}\right)$$
$$-4dx^2\left(\frac{1}{(2x+1)^3} + \frac{1}{(2x+3)^3} + \frac{1}{(2x+5)^3} + \text{etc.}\right)$$
$$+8dx^3\left(\frac{1}{(2x+1)^4} + \frac{1}{(2x+3)^4} + \frac{1}{(2x+5)^4} + \text{etc.}\right)$$
$$-16dx^4\left(\frac{1}{(2x+1)^5} + \frac{1}{(2x+3)^5} + \frac{1}{(2x+5)^5} + \text{etc.}\right)$$
$$\text{etc.}$$

EXAMPLE 3

To find the complete differential of this inexplicable function of x

$$S = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots + \frac{1}{x^n}$$

Since the general term of this series shall be $= \frac{1}{x^n}$, the infinite terms vanish and are equal to each other. And hence on account of

$$X' = \frac{1}{(x+1)^n} \qquad Z' = \frac{1}{(x+2+\omega)^n}$$
$$X'' = \frac{1}{(x+2)^n} \qquad Z'' = \frac{1}{(x+2+\omega)^n}$$
$$X''' = \frac{1}{(x+3)^n} \qquad Z''' = \frac{1}{(x+3+\omega)^n}$$
$$\text{etc.} \qquad \text{etc.,}$$

there will be

$$X' - Z' = \frac{n\omega}{(x+1)^{n+1}} - \frac{n(n+1)\omega^2}{2(x+1)^{n+2}} + \frac{n(n+1)(n+2)\omega^3}{6(x+1)^{n+3}} - \text{etc.}$$

$$X'' - Z'' = \frac{n\omega}{(x+2)^{n+1}} - \frac{n(n+1)\omega^2}{2(x+2)^{n+2}} + \frac{n(n+1)(n+2)\omega^3}{6(x+2)^{n+3}} - \text{etc.}$$

etc.

from which there may be found

$$\Sigma - S = n\omega\left(\frac{1}{(x+1)^{n+1}} + \frac{1}{(x+2)^{n+1}} + \frac{1}{(x+3)^{n+1}} + \text{etc.}\right)$$

$$-\frac{n(n+1)}{1\cdot 2}\omega^2\left(\frac{1}{(x+1)^{n+2}} + \frac{1}{(x+2)^{n+2}} + \frac{1}{(x+3)^{n+2}} + \text{etc.}\right)$$

$$+\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\omega^3\left(\frac{1}{(x+1)^{n+3}} + \frac{1}{(x+2)^{n+3}} + \frac{1}{(x+3)^{n+3}} + \text{etc.}\right)$$

etc.

Whereby on putting $\omega = dx$ the complete differential of the function sought S will be produced

$$dS = ndx\left(\frac{1}{(x+1)^{n+1}} + \frac{1}{(x+2)^{n+1}} + \frac{1}{(x+3)^{n+1}} + \text{etc.}\right)$$

$$-\frac{n(n+1)}{1\cdot 2}dx^2\left(\frac{1}{(x+1)^{n+2}} + \frac{1}{(x+2)^{n+2}} + \frac{1}{(x+3)^{n+2}} + \text{etc.}\right)$$

$$+\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}dx^3\left(\frac{1}{(x+1)^{n+3}} + \frac{1}{(x+2)^{n+3}} + \frac{1}{(x+3)^{n+3}} + \text{etc.}\right)$$

etc.

372. From these also the sums of such series are to be interpolated or the values of the summatory terms can be shown, when the number of terms is not a whole number. Indeed if there may be put $x = 0$, also there will be $S = 0$ and Σ will express the sum of the terms, as many times as the number ω may contain units, even if this number ω shall not be an integer. Thus in the first example, if there may be put

$$\Sigma = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{\omega},$$

there will be

$$\Sigma = \frac{\omega}{1(1+\omega)} + \frac{\omega}{2(2+\omega)} + \frac{\omega}{3(3+\omega)} + \frac{\omega}{4(4+\omega)} + \text{etc.}$$

or

$$\begin{aligned}\Sigma = \omega&\left(1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}+\tfrac{1}{25}+\text{etc.}\right)\\ -\omega^2&\left(1+\tfrac{1}{2^3}+\tfrac{1}{3^3}+\tfrac{1}{4^3}+\tfrac{1}{5^3}+\text{etc.}\right)\\ +\omega^3&\left(1+\tfrac{1}{2^4}+\tfrac{1}{3^4}+\tfrac{1}{4^4}+\tfrac{1}{5^4}+\text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

Truly in the third example there will be

$$\Sigma = 1+\tfrac{1}{2^n}+\tfrac{1}{3^n}+\tfrac{1}{4^n}+\cdots+\tfrac{1}{\omega^n}.$$

And the value of Σ , if ω shall be a whole number or a fraction, will be expressed by a series in the following manner

$$\begin{aligned}\Sigma = n\omega&\left(\tfrac{1}{2^{n+1}}+\tfrac{1}{3^{n+1}}+\tfrac{1}{4^{n+1}}+\text{etc.}\right)\\ -\tfrac{n(n+1)}{1\cdot 2}\omega^2&\left(1+\tfrac{1}{2^{n+2}}+\tfrac{1}{3^{n+2}}+\tfrac{1}{4^{n+2}}+\text{etc.}\right)\\ +\tfrac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\omega^3&\left(1+\tfrac{1}{2^{n+3}}+\tfrac{1}{3^{n+3}}+\tfrac{1}{4^{n+3}}+\text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

373. These likewise can be adapted to the general series ; for since there shall be

$$\begin{matrix} & 1 & & 2 & & 3 & & 4 & \cdots & x \\ S = & A & + & B & + & C & + & D & +\cdots+ & X, \end{matrix}$$

and on putting $x+\omega$ in place of x , X may be changed into Z and S into Σ , there will be

$$Z = X + \frac{\omega dX}{dx} + \frac{\omega^2 ddX}{1\cdot 2dx^2} + \frac{\omega^3 d^3X}{1\cdot 2\cdot 3dx^3} + \text{etc.},$$

and because in a similar manner Z', Z'', Z''' etc. are expressed by X', X'', X''' etc.,
[Recall from previously that

$$\begin{aligned}\Sigma = \ & S+\omega X^{|\infty+1|}+X'+X''+X'''+\text{etc.} \quad \text{to infinity}\\ & -Z'-Z''-Z'''-\text{etc.} \quad \text{to infinity.}\end{aligned}$$]

there will be

$$\Sigma = S + \omega X^{|\infty+1|} - \tfrac{\omega}{dx} d.\left(X' + X'' + X''' + X'''' + \text{etc.}\right)$$
$$- \tfrac{\omega^2}{1\cdot 2dx^2} dd.\left(X' + X'' + X''' + X'''' + \text{etc.}\right) - \tfrac{\omega^3}{1\cdot 2\cdot 3dx^3} d^3.\left(X' + X'' + X''' + X'''' + \text{etc.}\right)$$

etc.,

and unless $X^{|\infty+1|}$ shall be $=0$, it may be expressed in such a manner that the consideration to infinity may be removed,

$$X^{|\infty+1|} = X' + \left(X'' - X'\right) + \left(X''' - X''\right) + \left(X'''' - X'''\right) + \text{etc.}$$

and therefore there will be

$$\Sigma = S + \omega X' + \omega\left(\left(X'' - X'\right) + \left(X''' - X''\right) + \left(X'''' - X'''\right) + \text{etc.}\right)$$
$$- \tfrac{\omega}{dx} d.\left(X' + X'' + X''' + X'''' + \text{etc.}\right)$$
$$- \tfrac{\omega^2}{2dx^2} dd.\left(X' + X'' + X''' + X'''' + \text{etc.}\right)$$
$$- \tfrac{\omega^3}{6dx^3} d^3.\left(X' + X'' + X''' + X'''' + \text{etc.}\right)$$

etc.

Therefore if there may be put $\omega = dx$, the complete differential of this sum may arise

$$S = A + B + C + \cdots + X$$

expressed thus

$$dS = X'dx + dx\left(\left(X'' - X'\right) + \left(X''' - X''\right) + \left(X'''' - X'''\right) + \text{etc.}\right)$$
$$- d.\left(X' + X'' + X''' + X'''' + \text{etc.}\right)$$
$$- \tfrac{1}{2} dd.\left(X' + X'' + X''' + X'''' + \text{etc.}\right)$$
$$- \tfrac{1}{6} d^3.\left(X' + X'' + X''' + X'''' + \text{etc.}\right)$$

etc.

374. We may put $x = 0$ to be in place; there will be made

$$X' = A, \quad X'' = B \quad \text{etc.}$$

and thus $X' + X'' + X''' + \text{etc.}$ will be an infinite series, the general term of which is = X. Then a series may be formed from these general terms

$$\tfrac{dX}{dx},\ \tfrac{ddX}{2dx^2},\ \tfrac{d^3X}{6d^3x},\ \tfrac{d^4X}{24d^4x} \quad \text{etc.},$$

of which the sums of the series continued to infinity shall be [Note the different summation sign $\mathcal{S}$, even fancier in the original, that has not been reproduced here]

$$\mathcal{S}X = \mathfrak{A}, \quad \mathcal{S}\tfrac{dX}{dx} = \mathfrak{B}, \quad \mathcal{S}\tfrac{ddX}{2dx^2} = \mathfrak{C}, \quad \mathcal{S}\tfrac{d^3X}{6dx^3} = \mathfrak{D} \text{ etc. };$$

and because on putting $x = 0$ there is made $S = 0$ also, and Σ will be the sum of the series

$$A + B + C + D + \cdots + Z$$

containing the terms ω; for Z is the term of the index ω, either ω shall be a whole number or a fraction. Whereby there will be had [from the last section]

$$\Sigma = \omega A + \omega\big((B-A)+(C-B)+(D-C)+\text{etc.}\big) - \omega\mathfrak{B} - \omega^2\mathfrak{C} - \omega^3\mathfrak{D} - \omega^4\mathfrak{E} - \text{etc.},$$

where the first series may be omitted only if the terms of the proposed series vanish.

375. Now we may write x in place of ω and Σ will be changed into S, thus so that there shall be

$$\begin{matrix} & 1 & & 2 & & 3 & & 4 & & \cdots & & x \\ S = & A & + & B & + & C & + & D & + & \cdots & + & X, \end{matrix}$$

and the same value of S now may be expressed by an infinite series in this way

$$S = Ax + x\big((B-A)+(C-B)+(D-C)+\text{etc.}\big) - \mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \mathfrak{F}x^5 - \text{etc.};$$

the value of which may be expressed equally distinctly, whether x shall be a whole number or a fraction, the differentials of S of each order hence can be shown easily :

$$\begin{aligned} \tfrac{dS}{dx} &= A + (B-A)+(C-B)+(D-C)+\text{etc.} \\ &\quad - \mathfrak{B} - 2\mathfrak{C}x - 3\mathfrak{D}x^2 - 4\mathfrak{E}x^3 - \text{etc.} \\ \tfrac{ddS}{2dx^2} &= -\mathfrak{C} - 3\mathfrak{D}x - 6\mathfrak{E}x^2 - 10\mathfrak{F}x^3 - \text{etc.} \\ \tfrac{d^3S}{6dx^3} &= -\mathfrak{D} - 4\mathfrak{E}x - 10\mathfrak{F}x^2 - 20\mathfrak{G}x^3 - \text{etc.} \\ \tfrac{d^4S}{24dx^4} &= -\mathfrak{E} - 5\mathfrak{F}x - 15\mathfrak{G}x^2 - 35\mathfrak{H}x^3 - \text{etc.} \end{aligned}$$

Whereby since the complete differential shall be

$$= dS + \tfrac{1}{2}ddS + \tfrac{1}{6}d^3S + \tfrac{1}{24}d^4S + \text{etc.},$$

the complete differential of the proposed function S will be

$$\begin{aligned} dS &= Adx + (B-A)dx + (C-B)dx + (D-C)dx + \text{etc.} \\ &-\mathfrak{B}dx - \mathfrak{C}\left(2xdx + dx^2\right) - \mathfrak{D}\left(3x^2dx + 3xdx^2 + dx^3\right) \\ &-\mathfrak{E}\left(4x^3dx + 6x^2dx2 + 4xdx^3 + dx^4\right) - \text{etc.} \end{aligned}$$

376. Therefore in this manner the complete differential of each inexplicable function S can be assigned, if the most infinite terms of the series

$$A + B + C + D + \text{etc}.$$

either may vanish or shall be equal to each other. For if indeed the most infinite terms were not $= 0$, then the sum of the series $\mathfrak{B}$, which may be formed from the general term $\frac{dX}{dx}$, will become infinite, but truly since with the series

$$A + (B-A) + (C-B) + (D-C) + \text{etc}.$$

taken together, it may constitute a finite sum. But it can happen, so that the terms of the series $A + (B-A) + (C-B) + (D-C) + \text{etc}$ thus may be increased to infinity, so that the sums may become infinitely great not only of the series $\mathfrak{B}$, but also of the series $\mathfrak{C}$, in which case it is not sufficient to be adding the series $A + (B-A) + (C-B) + (D-C) + \text{etc}.$; but because in this case the most infinite values of §370 considered, evidently $S^{|\infty|}$, $S^{|\infty+1|}$, $S^{|\infty+2|}$, are no longer in an arithmetic progression, as we may have assumed, an account of this progression will be required to be considered. Therefore just as we may assume the first differences of these terms to be equal, thus we may extend the method further, if we may be considering either the second or the third or further differences of these values finally as constants.
[Thus, the procedures are to be repeated for higher order terms at infinity, called most infinite, before constant differences are found.]

377. Therefore with the same reasoning retained, as we have used in §369, we may put the differences of the second values mentioned to be constants finally :

$$S^{|\infty|},\ S^{|\infty+1|},\ S^{|\infty+2|}$$

The first differences :

$$X^{|\infty+1|},\ X^{|\infty+2|}$$

The second differences :

$$X^{|\infty+2|}-X^{|\infty+3|}$$

Hence there will be

$$\Sigma^{|\infty|}=S^{|\infty+\omega|}+\omega X^{|\infty+1|}+\frac{\omega(\omega-1)}{1\cdot 2}\left(X^{|\infty+2|}-X^{|\infty+1|}\right)$$
$$=S^{|\infty|}-\frac{\omega(\omega-3)}{1\cdot 2}X^{|\infty+1|}+\frac{\omega(\omega-1)}{1\cdot 2}X^{|\infty+2|}.$$

On account of which we will have this equation,

$$\Sigma+Z'+Z''+Z'''+\cdots+Z^{|\infty|}$$
$$=S+X'+X''+X'''+\cdots+X^{|\infty|}-\frac{\omega(\omega-3)}{1\cdot 2}X^{|\infty+1|}+\frac{\omega(\omega-1)}{1\cdot 2}X^{|\infty+2|},$$

from which there is elicited

$$\begin{aligned}\Sigma=S&+X'+X''+X'''+X''''+\text{ etc. to infinity}\\ &-Z'-Z''-Z'''-Z''''-\text{ etc. to infinity}\\ &+\omega X^{|\infty+1|}+\frac{\omega(\omega-1)}{1\cdot 2}\left(X^{|\infty+2|}-X^{|\infty+1|}\right).\end{aligned}$$

But these most infinite terms thus can be represented, so that there shall be

$$\begin{aligned}\Sigma=S&+X'+X''+X'''+X''''+\text{etc.}\\ &-Z'-Z''-Z'''-Z''''-\text{ etc.}\\ &+\omega X'+\omega\begin{Bmatrix}+X''+X'''+X''''+X'''''+\text{etc.}\\ -X'-X''-X'''-X''''-\text{etc.}\end{Bmatrix}\\ &+\frac{\omega(\omega-1)}{1\cdot 2}X''-\frac{\omega(\omega-1)}{1\cdot 2}X'+\frac{\omega(\omega-1)}{1\cdot 2}\begin{Bmatrix}+X'''+X''''+X'''''+\text{etc.}\\ -2X''-2X'''-2X''''-\text{etc.}\\ +X+X'+X''+X'''+\text{etc.}\end{Bmatrix}\end{aligned}$$

from which the law may be apparent, by which this expression will be prepared, if the differences of the third or fourth orders finally were constants.

378. Therefore since there shall be, as we have shown above,

$$Z = X + \frac{\omega dX}{dx} + \frac{\omega^2 ddX}{1\cdot 2dx^2} + \frac{\omega^3 d^3X}{1\cdot 2\cdot 3dx^3} + \text{etc.},$$

if in place of Z', Z'', Z''' etc. hence we may substitute the values hence arising, the value of *S* follows, if in place of *x* there may be written $x+\omega$:

$$\begin{aligned}
\Sigma = S + \omega X' + \omega X' + \omega & \left\{ \begin{matrix} +X'' + X''' + X'''' + X''''' + \text{etc.} \\ -X' - X'' - X''' - X'''' - \text{etc.} \end{matrix} \right\} \\
+\frac{\omega(\omega-1)}{1\cdot 2}X'' - \frac{\omega(\omega-1)}{1\cdot 2}X' + \frac{\omega(\omega-1)}{1\cdot 2} & \left\{ \begin{matrix} +X''' + X'''' + X''''' + \text{etc.} \\ -2X'' - 2X''' - 2X'''' - \text{etc.} \\ +X + X' + X'' + X''' + \text{etc.} \end{matrix} \right\} \\
- \frac{\omega}{dx} d.\left(+X' + X'' + X''' + X'''' + \text{etc.} \right) & \\
- \frac{\omega^2}{2dx^2} d^2.\left(+X' + X'' + X''' + X'''' + \text{etc.} \right) & \\
+ \frac{\omega^3}{6dx^3} d^3.\left(+X' + X'' + X''' + X'''' + \text{etc.} \right) & \\
\text{etc.} &
\end{aligned}$$

If therefore in place of ω there may be put *dx*, the complete differential will be produced of the proposed inexplicable function proposed *S*, evidently

$$dZ = X'\,dx + dx\left\{\begin{matrix} +X''+X'''+X''''+X'''''+\text{etc.} \\ -X'-X''-X'''-X''''-\text{etc.}\end{matrix}\right\}$$

$$-X''\frac{dx(1-dx)}{1\cdot 2}+X'\frac{dx(1-dx)}{1\cdot 2}-\frac{dx(1-dx)}{1\cdot 2}\left\{\begin{matrix} +X'''+X''''+X'''''+\text{etc.} \\ -2X''-2X'''-2X''''-\text{etc.} \\ +X+X'+X''+X'''+\text{etc.}\end{matrix}\right\}$$

$$+X'''\frac{dx(1-dx)(2-dx)}{1\cdot 2\cdot 3}$$

$$-2X'''\frac{dx(1-dx)(2-dx)}{1\cdot 2\cdot 3}+\frac{dx(1-dx)(2-dx)}{1\cdot 2\cdot 3}\left\{\begin{matrix} +X''''+X'''''+\text{etc.} \\ -3X'''-3X''''-\text{etc.} \\ +3X''+3X'''+\text{etc.} \\ -X'-X''-\text{etc.}\end{matrix}\right\}$$

$$+X'\frac{dx(1-dx)(2-dx)}{1\cdot 2\cdot 3}$$

etc.

$$-d.\left(X'+X''+X'''+X''''+X'''''+\text{etc.}\right)$$
$$-\tfrac{1}{2}dd.\left(X'+X''+X'''+X''''+X'''''+\text{etc.}\right)$$
$$-\tfrac{1}{6}d^3.\left(X'+X''+X'''+X''''+X'''''+\text{etc.}\right)$$
$$-\tfrac{1}{24}d^4.\left(X'+X''+X'''+X''''+X'''''+\text{etc.}\right)$$

etc.,

which expression may appear the most general and, whatever differences should be constant at last, the differential sought will be shown. For this formula for the constant differences is the most convenient and likewise the law is apparent, if perhaps it shall be necessary to progress further.

379. But if the series $A+B+C+D+\text{etc.}$, from which the inexplicable function may be formed

$$\begin{matrix} & 1 & & 2 & & 3 & & 4 & \cdots & x \\ S = & A & + & B & + & C & + & D & +\cdots+ & X, \end{matrix}$$

were prepared thus, so that the most infinite terms of this may vanish there will be

$$\begin{aligned}dS = &-d.\left(X' + X'' + X''' + X'''' + \text{etc.}\right)\\ &-\tfrac{1}{2}dd.\left(X' + X'' + X''' + X'''' + \text{etc.}\right)\\ &-\tfrac{1}{6}d^3.\left(X' + X'' + X''' + X'''' + \text{etc.}\right)\\ &-\tfrac{1}{24}d^4.\left(X' + X'' + X''' + X'''' + \text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

But if the most infinite terms of that series may not become $= 0$, but yet may have vanishing differences, then to that above expression there must be added

$$dx\left\{\begin{matrix} & + X'' + X''' + X'''' + X''''' + \text{ etc.}\\ X' & \\ & - X' - X'' - X''' - X'''' - \text{etc.}\end{matrix}\right\}$$

Certainly if the second differences of the most infinite terms of this series $A + B + C + D + \text{etc.}$ may vanish at last, then it is required to add in addition

$$\frac{dx(dx-1)}{1\cdot 2}\left\{\begin{matrix} & + X''' + X'''' + X'''' \quad + \text{ etc.}\\ +X'' & \\ & - 2X'' - 2X'' - 2X'''' - \text{ etc.}\\ -X' & \\ & + X' + X'' + X''' + \text{ etc.}\end{matrix}\right\}$$

And if the differences finally of the third order mentioned were vanishing, then besides these now shown the above expressions must be added

$$\frac{dx(dx-1)(dx-2)}{1\cdot 2\cdot 3}\left\{\begin{matrix} & + X'''' + X''''' + X'''''' + \text{etc.}\\ +X'' & \\ & - 3X''' - 3X'''' - 3X''''' - \text{etc.}\\ -2X'' & \\ & + 3X'' + 3X''' + 3X'''' + \text{etc.}\\ +X' & \\ & - X' - X'' - X''' - \text{etc.}\end{matrix}\right\}$$

And thus again the above expressions will be prepared required to be added, if further differences of most infinite terms of the series $A + B + C + D + \text{etc.}$ may vanish at last. And hence thus, whatever series may be assumed, provided the most infinite terms may be produced finally to vanishing differences, from these it will be able to define the forms of the inexplicable functions.

380. If there may be put $x=0$, there will become $X'=A,\quad X''=B,\quad X'''=C$ etc. Whereby as $A+B+C+D+$ etc. is the series, the general term of which is X, if from the general terms

$$\frac{dX}{dx},\quad \frac{ddX}{2dx^2},\quad \frac{d^3X}{6dx^3},\quad \frac{d^4X}{24dx^4}\quad \text{etc.}$$

in a similar manner an infinite series may be formed by the letters $\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D}$ etc. respectively, the sum ω of the terms of the series

$$A+B+C+D+\text{etc.}$$

may be expressed thus, so that there shall be likewise, ω shall be either a whole number or otherwise. Therefore may write x for ω, so that there shall be

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & \cdots\ x \\ S= & A & +\ B & +\ C & +\ D & +\cdots+\ X, \end{array}$$

and if the most infinite terms of this series may vanish, there will be

$$S=-\mathfrak{B}x-\mathfrak{C}x^2-\mathfrak{D}x^3-\mathfrak{E}x^4-\text{etc.}$$

But if the most infinite terms perhaps may have constant first differences, then to that value above this must be added

$$x\left\{\begin{array}{l} \quad +B+C+D+E+\text{etc.} \\ A \\ \quad -A-\ B-\ C-D-\text{etc.} \end{array}\right\}$$

But if the second order of the most infinite terms may vanish at last, then besides there must be added

$$\frac{x(x-1)}{1\cdot 2}\left\{\begin{array}{l} \quad +\ C+\ D+\ \ E+\ \ F+\text{etc.} \\ +B \\ \quad -2B-2C-2D-2E-\text{etc.} \\ -A \\ \quad +A+\ B+\ \ C+\ \ D+\text{etc.} \end{array}\right\}$$

Finally if the third differences were vanishing, then in addition there must be added this infinite series

$$\frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}\left\{\begin{array}{l} \quad + D + \ E \ + \ F + \ G + \text{etc.} \\ + C \\ \quad - 3C - 3D - 3E - 3F - \text{etc.} \\ -2B \\ \quad + 3B + 3C + 3D + 3E + \text{etc.} \\ + A \\ \quad - A \ - \ B \ - \ C \ - D - \text{etc.} \\ \qquad\qquad \text{etc.} \end{array}\right\}$$

381. We may apply this reasoning also to another kind of inexplicable functions, which are agreed to produce some number of terms of the proposed series $A+B+C+D+\text{etc.}$, and there shall be

$$\begin{array}{c} \ \ 1 \ \ 2 \ \ 3 \ \ 4\cdots \ x \\ S = A\cdot B\cdot C\cdot D\cdots X\,, \end{array}$$

and in the first place there is sought the value Σ, into which S may be changed, if in place of x there may be written $x+\omega$; moreover we may put Z as before to be a term of the series $A+B+C+D+\text{etc.}$, the index of which shall be $=x+\omega$, as X may correspond to the index x. Therefore so that this case may be reduced to the preceding, we may take the logarithms and there will be

$$lS = lA + lB + lC + lD + \cdots + lX\ .$$

But if now the most infinite terms of this series may vanish, it will be the same method which we have used before, on using

$$\begin{aligned} l\Sigma = \ & lS + lX' + lX'' + lX''' + \text{etc.} \\ & \quad - lZ' - \ lZ'' - lZ''' - \text{etc.} \end{aligned}$$

and hence by returning to numbers there will be

$$\Sigma = S\cdot\tfrac{X'}{Z'}\cdot\tfrac{X''}{Z''}\cdot\tfrac{X'''}{Z'''}\cdot\tfrac{X''''}{Z''''}\cdot\text{etc.};$$

therefore which expression may prevail, if the most infinite terms of the series A, B, C, D etc. may be equal to unity. But if the logarithms of the most infinite terms of this series do not vanish, but yet there may be had vanishing differences, then to that series which we have found for lS, in addition there must be added this series

$$\omega lX' + \omega\left(l\tfrac{X''}{X'} + l\tfrac{X'''}{X''} + l\tfrac{X''''}{X'''} + \text{etc.}\right)$$

and thus with numbers taken there will be had

$$\Sigma = S \cdot X'^{\omega} \cdot \frac{X''^{\omega} X'^{1-\omega}}{Z'} \cdot \frac{X'''^{\omega} X''^{1-\omega}}{Z''} \cdot \frac{X''''^{\omega} X'''^{1-\omega}}{Z'''} \cdot \text{etc.}$$

382. But if therefore we may put $x = 0$, in which case there is made $S = 1$ and $X' = A,\ X'' = B,\ X''' = C$ etc., Σ will denote the product ω of the terms of this series A, B, C, D etc. Therefore if for ω we may write *x*, so that Σ may obtain the value, which before we had attributed to *S*, thus so that there shall be

$$\begin{array}{c} 1\ \ 2\ \ 3\ \ 4 \cdots\ x \\ S = A \cdot B \cdot C \cdot D \cdots X, \end{array}$$

because now Z', Z'', Z''' etc. are changed into X', X'', X''' etc., if the logarithms of the most infinite terms of this series A, B, C, D, E etc. may vanish, *S* may be expressed in this manner

$$S = \frac{A}{X'} \cdot \frac{B}{X''} \cdot \frac{C}{X'''} \cdot \frac{D}{X''''} \cdot \frac{E}{X'''''} \cdot \text{etc.}$$

But if the difference of the logarithms of the most infinite terms of the series A, B, C, D etc. may vanish at last, then this function *S* may be expressed in the following manner, so that there shall be

$$S = A^{x} \cdot \frac{B^{x} A^{1-x}}{X'} \cdot \frac{C^{x} B^{1-x}}{X''} \cdot \frac{D^{x} C^{1-x}}{X'''} \cdot \frac{E^{x} D^{1-x}}{X''''} \cdot \text{etc.};$$

if the second differences of these logarithms shall vanish at last, from the preceding it may be deduced easily in addition factors of this kind must be added; which case we have ignored here, since usually it may hardly ever occur . Moreover I will show the use of these expressions in the business of interpolation in the following chapter.

383. Therefore here since the differentiation of inexplicable functions of this kind shall be proposed chiefly, we may investigate the differential of this function

$$S = A \cdot B \cdot C \cdot D \cdots X .$$

Accordingly we may resume the equation found before

$$\begin{aligned} l\Sigma = {} & lS + lX' + lX'' + lX''' + \text{etc.} \\ & - lZ' - lZ'' - lZ''' - \text{etc.} \end{aligned}$$

and since *lZ* may arise from *lX*, if there is put $x+\omega$ in place of *x*, there will be

$$lZ = lX + \tfrac{\omega}{dx} d.lX + \tfrac{\omega^2}{2dx^2} dd.lX + \tfrac{\omega^3}{6dx^3} d^3.lX + \text{etc.}$$

with which values substituted for lZ', lZ'', lZ''' etc. there will be had

$$\begin{aligned} l\Sigma = lS &- \tfrac{\omega}{dx} d.\left(lX' + lX'' + lX''' + lX'''' + \text{etc.}\right) \\ &- \tfrac{\omega^2}{2dx^2} dd.\left(lX' + lX'' + lX''' + lX'''' + \text{etc.}\right) \\ &- \tfrac{\omega^3}{6dx^3} d^3.\left(lX' + lX'' + lX''' + lX'''' + \text{etc.}\right) \\ &\text{etc.} \end{aligned}$$

Now there may be put $\omega = dx$ and there will become $l\Sigma = lS + d.lS$ and thus there will be

$$\begin{aligned} \tfrac{dS}{S} = &-d.\left(lX' + lX'' + lX''' + lX'''' + \text{etc.}\right) \\ &- \tfrac{1}{2} dd.\left(lX' + lX'' + lX''' + lX'''' + \text{etc.}\right) \\ &- \tfrac{1}{6} d^3.\left(lX' + lX'' + lX''' + lX'''' + \text{etc.}\right) \\ &\text{etc.,} \end{aligned}$$

which formula prevails, if the logarithms of the most infinite terms of the series A, B, C, D etc. may vanish ; but if these may not vanish, vanishing differences may yet be had, then in addition this series must be added to the preceding complete differential

$$dxlX' + dx\left(l\tfrac{X''}{X'} + l\tfrac{X'''}{X''} + l\tfrac{X''''}{X''} + \text{etc.}\right),$$

so that the complete differential may be obtained.

384. Also at this point another way can be seen to be better. There may be put $x = 0$, in which case *lS* will change into 0. Then the series may be formed, the general terms of which shall be

$$lX, \quad \tfrac{d.lX}{dx}, \quad \tfrac{dd.lX}{2dx^2}, \quad \tfrac{d^3.lX}{6dx^3} \quad \text{etc.},$$

and the sums of the infinite series of these shall be respectively $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. There may be written *x* for ω, so that there shall be $\Sigma = S$, and there will be

$$lS = -\mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \text{etc.},$$

if indeed the logarithms of the most infinite terms of the series A, B, C, D etc. may vanish, the general term of which is X ; but if the differences of these logarithm may vanish at last, there will be

$$lS = xlA + x\left(l\tfrac{B}{A} + l\tfrac{C}{B} + l\tfrac{D}{C} + l\tfrac{E}{D} + \text{etc.}\right)$$
$$-\mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \text{etc.}$$

And so hence the differential of lS will be

$$\tfrac{dS}{S} = dxlA + dx\left(l\tfrac{B}{A} + l\tfrac{C}{B} + l\tfrac{D}{C} + l\tfrac{E}{D} + \text{etc.}\right)$$
$$-\mathfrak{B}dx - 2\mathfrak{C}xdx - 3\mathfrak{D}x^2dx - 4\mathfrak{E}x^3dx - \text{etc.}$$

But if the complete differential may be required, there will be this

$$\tfrac{dS}{S} = dxlA + dx\left(l\tfrac{B}{A} + l\tfrac{C}{B} + l\tfrac{D}{C} + l\tfrac{E}{D} + \text{etc.}\right)$$
$$-\mathfrak{B}dx - \mathfrak{C}\left(2xdx + dx^2\right) - \mathfrak{D}\left(3xxdx + 3xdx^2 + dx^3\right) - \text{etc.}$$

Towards showing the use of which formulas we may add the following examples, which we will resolve in each way.

EXAMPLE 1
To find the [complete] differential of this inexplicable function

$$S = \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdot\tfrac{7}{8}\cdot\tfrac{2x-1}{2x}.$$

Here before everything it is to be noted the most infinite terms of these factors change into unity and thus the logarithms of these vanish. Therefore since there shall be $X = \tfrac{2x-1}{2x}$, there will be

$$X' = \tfrac{2x+1}{2x+2}, \quad X'' = \tfrac{2x+3}{2x+4}, \quad X''' = \tfrac{2x+5}{2x+6} \quad \text{etc.}$$

and generally

$$X^{|n|} = \tfrac{2x+2n-1}{2x+2n};$$

from which there will be

$$lX^{|n|} = l(2x+2n-1) - l(2x+2n)$$

$$d.lX^{|n|} = \frac{2dx}{2x+2n-1} - \frac{2dx}{2x+2n}$$

$$dd.lX^{|n|} = -\frac{4dx^2}{(2x+2n-1)^2} + \frac{4dx^2}{(2x+2n)^2}$$

$$d^3.lX^{|n|} = +\frac{2\cdot 2\cdot 4dx^3}{(2x+2n-1)^3} - \frac{2\cdot 2\cdot 4dx^3}{(2x+2n)^3}$$

$$d^4.lX^{|n|} = +\frac{2\cdot 2\cdot 4\cdot 6dx^4}{(2x+2n-1)^4} - \frac{2\cdot 2\cdot 4\cdot 6dx^4}{(2x+2n)^4}$$

etc.;

from which the complete differential will be

$$\frac{dS}{S} = -2dx\left\{\begin{array}{l} \frac{1}{2x+1} + \frac{1}{2x+3} + \frac{1}{2x+5} + \text{etc.} \\ -\frac{1}{2x+2} - \frac{1}{2x+4} - \frac{1}{2x+6} - \text{etc.} \end{array}\right\}$$

$$+\frac{8}{3}dx^3\left\{\begin{array}{l} \frac{1}{(2x+1)^2} + \frac{1}{(2x+3)^2} + \frac{1}{(2x+5)^2} + \text{etc.} \\ -\frac{1}{(2x+2)^2} - \frac{1}{(2x+4)^2} - \frac{1}{(2x+6)^2} - \text{etc.} \end{array}\right\}$$

$$-\frac{4}{2}dx^2\left\{\begin{array}{l} \frac{1}{(2x+1)^3} + \frac{1}{(2x+3)^3} + \frac{1}{(2x+5)^3} + \text{etc.} \\ -\frac{1}{(2x+2)^3} - \frac{1}{(2x+4)^3} - \frac{1}{(2x+6)^3} - \text{etc.} \end{array}\right\}$$

But if only the first differential may be sought, that will be

$$\frac{dS}{S} = -2dx\cdot\left(\frac{1}{(2x+1)(2x+2)} + \frac{1}{(2x+3)(2x+4)} + \frac{1}{(2x+5)(2x+6)} + \text{etc.}\right),$$

thus the same may be investigated treated by the other method of § 394. Since there shall be

$$lX = l\frac{2x-1}{2x},$$

there will be

$$\frac{d.lX}{dx} = \frac{2}{2x-1} - \frac{1}{x}, \quad \frac{dd.lX}{2dx^2} = -\frac{2}{(2x-1)^2} + \frac{1}{2xx},$$

$$\frac{d^3.lX}{6dx^3} = \frac{8}{3(2x-1)^3} - \frac{1}{3x^3} \quad \text{etc.}$$

and thus there will come about

$$\mathfrak{A} = l\tfrac{1}{2} + l\tfrac{3}{4} + l\tfrac{5}{6} + l\tfrac{7}{8} + \text{ etc.}$$

$$\mathfrak{B} = \left\{\begin{matrix} \tfrac{2}{1} + \tfrac{2}{3} + \tfrac{2}{5} + \tfrac{2}{7} + \tfrac{2}{9} + \text{etc.} \\ -\tfrac{2}{2} - \tfrac{2}{4} - \tfrac{2}{6} - \tfrac{2}{8} - \tfrac{2}{10} - \text{etc.} \end{matrix}\right\} = 2l2$$

$$\mathfrak{C} = -\tfrac{4}{2}\left\{\begin{matrix} \tfrac{1}{1} + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \text{etc.} \\ -\tfrac{1}{2^2} - \tfrac{1}{4^2} - \tfrac{1}{6^2} - \tfrac{1}{8^2} - \text{etc.} \end{matrix}\right\}$$

$$\mathfrak{D} = \tfrac{8}{3}\left\{\begin{matrix} \tfrac{1}{1} + \tfrac{1}{3^3} + \tfrac{1}{5^3} + \tfrac{1}{7^3} + \text{etc.} \\ -\tfrac{1}{2^3} - \tfrac{1}{4^3} - \tfrac{1}{6^3} - \tfrac{1}{8^3} - \text{etc.} \end{matrix}\right\}$$

$$\mathfrak{E} = -\tfrac{16}{4}\left\{\begin{matrix} \tfrac{1}{1} + \tfrac{1}{3^4} + \tfrac{1}{5^4} + \tfrac{1}{7^4} + \text{etc.} \\ -\tfrac{1}{2^4} - \tfrac{1}{4^4} - \tfrac{1}{6^4} - \tfrac{1}{8^4} - \text{etc.} \end{matrix}\right\}$$

etc.

or there will be

$$\mathfrak{B} = +\tfrac{2}{1}\left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \text{etc.}\right)$$

$$\mathfrak{C} = -\tfrac{4}{2}\left(1 - \tfrac{1}{2^2} + \tfrac{1}{3^2} - \tfrac{1}{4^2} + \tfrac{1}{5^2} - \text{etc.}\right)$$

$$\mathfrak{D} = +\tfrac{8}{3}\left(1 - \tfrac{1}{2^3} + \tfrac{1}{3^3} - \tfrac{1}{4^3} + \tfrac{1}{5^3} - \text{etc.}\right)$$

$$\mathfrak{E} = -\tfrac{16}{4}\left(1 - \tfrac{1}{2^4} + \tfrac{1}{3^4} - \tfrac{1}{4^4} + \tfrac{1}{5^4} - \text{etc.}\right)$$

etc.

With which values found substituted there will be

$$\begin{aligned} dS = &-2dx\left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \text{etc.}\right) \\ &+ 4xdx\left(1 - \tfrac{1}{2^2} + \tfrac{1}{3^2} - \tfrac{1}{4^2} + \tfrac{1}{5^2} - \text{etc.}\right) \\ &- 8x^2\,dx\left(1 - \tfrac{1}{2^3} + \tfrac{1}{3^3} - \tfrac{1}{4^3} + \tfrac{1}{5^3} - \text{etc.}\right) \\ &+ 16x^3dx\left(1 - \tfrac{1}{2^4} + \tfrac{1}{3^4} - \tfrac{1}{4^4} + \tfrac{1}{5^4} - \text{etc.}\right) \end{aligned}$$

etc.

If therefore there shall be $x = 0$, in which case there becomes $lS = 0$ and $S = 1$, there will be $dS = -2dxl2$.

EXAMPLE 2
To find the [complete] differential of this inexplicable function

$$S = 1\cdot 2\cdot 3\cdot 4\cdots x.$$

The terms of this series 1, 2, 3, 4 etc. thus increase to infinity, so that the difference of the logarithms may vanish ; indeed there is

$$l(\infty+1) - l\infty = l\left(1+\tfrac{1}{\infty}\right) = \tfrac{1}{\infty} = 0.$$

Therefore since there shall be $X = x$, there will be

$$X' = x+1,\ \ X'' = x+2,\ \ X''' = x+3 \ \text{ etc.};$$

but again on account of $lX = lx$ there becomes

$$d.lX = \tfrac{dx}{x},\ \ dd.lX = -\tfrac{dx^2}{x^2},\ \ d^3.lX = \tfrac{2dx^3}{x^3},\ \ d^4.lX = -\tfrac{2\cdot 3dx^4}{x^4} \ \text{ etc.};$$

from which if the final logarithms vanish, there becomes

$$\begin{aligned}\tfrac{dS}{S} = -dx&\left(\tfrac{1}{x+1} + \tfrac{1}{x+2} + \tfrac{1}{x+3} + \tfrac{1}{x+4} + \text{etc.}\right)\\ +\tfrac{dx^2}{2}&\left(\tfrac{1}{(x+1)^2} + \tfrac{1}{(x+2)^2} + \tfrac{1}{(x+3)^2} + \tfrac{1}{(x+4)^2} + \text{etc.}\right)\\ -\tfrac{dx^3}{3}&\left(\tfrac{1}{(x+1)^3} + \tfrac{1}{(x+2)^3} + \tfrac{1}{(x+3)^3} + \tfrac{1}{(x+4)^3} + \text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

But since the differences of the logarithms finally may vanish, in addition this expression must be added

$$dxl(x+1) + dx\left(l\tfrac{x+2}{x+1} + l\tfrac{x+3}{x+2} + l\tfrac{x+4}{x+3} + l\tfrac{x+5}{x+4} + \text{etc.}\right).$$

Because truly there is

$$\begin{aligned} l\tfrac{x+2}{x+1} &= \tfrac{1}{x+1} - \tfrac{1}{2(x+1)^2} + \tfrac{1}{3(x+1)^3} - \tfrac{1}{4(x+1)^4} + \text{etc.},\\ l\tfrac{x+3}{x+2} &= \tfrac{1}{x+2} - \tfrac{1}{2(x+2)^2} + \tfrac{1}{3(x+2)^3} - \tfrac{1}{4(x+2)^4} + \text{etc.}\\ &\text{etc.}\end{aligned}$$

the true complete differential will be

$$\frac{dS}{S} = dxl(x+1) - \tfrac{1}{2}\left(dx - dx^2\right)\left(\tfrac{1}{(x+1)^2} + \tfrac{1}{(x+2)^2} + \tfrac{1}{(x+3)^2} + \tfrac{1}{(x+4)^2} + \text{etc.}\right)$$
$$- \tfrac{1}{3}\left(dx - dx^3\right)\left(\tfrac{1}{(x+1)^3} + \tfrac{1}{(x+2)^3} + \tfrac{1}{(x+3)^3} + \tfrac{1}{(x+4)^3} + \text{etc.}\right)$$
$$+ \tfrac{1}{4}\left(dx - dx^4\right)\left(\tfrac{1}{(x+1)^4} + \tfrac{1}{(x+2)^4} + \tfrac{1}{(x+3)^4} + \tfrac{1}{(x+4)^4} + \text{etc.}\right)$$
$$- \tfrac{1}{5}\left(dx - dx^5\right)\left(\tfrac{1}{(x+1)^5} + \tfrac{1}{(x+2)^5} + \tfrac{1}{(x+3)^5} + \tfrac{1}{(x+4)^5} + \text{etc.}\right)$$
etc.

But if we may wish to express this differential in the other way, because there is

$$lX = lx, \quad \frac{d.lX}{dx} = \frac{1}{x}, \quad \frac{dd.lX}{2dx^2} = -\frac{1}{2x^2}, \quad \frac{d^3.lX}{6dx^3} = \frac{1}{3x^3}, \quad \frac{d^4.lX}{24dx^4} = -\frac{1}{x^4} \quad \text{etc.},$$

we will have the following series

$$\mathfrak{A} = \quad l1 + l2 + l3 + l4 + l5 + \text{etc.}$$
$$\mathfrak{B} = \ 1\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \text{etc.}\right)$$
$$\mathfrak{C} = -\tfrac{1}{2}\left(1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \tfrac{1}{5^2} + \text{etc.}\right)$$
$$\mathfrak{D} = +\tfrac{1}{3}\left(1 + \tfrac{1}{2^3} + \tfrac{1}{3^3} + \tfrac{1}{4^3} + \tfrac{1}{5^3} + \text{etc.}\right)$$
$$\mathfrak{E} = -\tfrac{1}{4}\left(1 + \tfrac{1}{2^4} + \tfrac{1}{3^4} + \tfrac{1}{4^4} + \tfrac{1}{5^4} + \text{etc.}\right)$$
etc.

Hence on account of $lA = l1 = 0$, there becomes from §384

$$lS = \ x\left(l\tfrac{2}{1} + l\tfrac{3}{2} + l\tfrac{4}{3} + l\tfrac{5}{4} + \text{etc.}\right)$$
$$- \ x\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \text{etc.}\right)$$
$$+ \tfrac{1}{2}x^2\left(1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \text{etc.}\right)$$
$$- \tfrac{1}{3}x^3\left(1 + \tfrac{1}{2^3} + \tfrac{1}{3^3} + \tfrac{1}{4^3} + \text{etc.}\right)$$
$$+ \tfrac{1}{4}x^4\left(1 + \tfrac{1}{2^4} + \tfrac{1}{3^4} + \tfrac{1}{4^4} + \text{etc.}\right)$$
etc.

But the two first series, by which *x* has been multiplied, even if each may have an infinite sum, yet both together have a finite sum. For even if *n* terms of each may be taken, there will be produced

$$l(n+1)-1-\tfrac{1}{2}-\tfrac{1}{3}-\tfrac{1}{4}-\cdots-\tfrac{1}{n}.$$

But we have found above (§142a)

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots+\tfrac{1}{n}=\text{Const.}+ln+\tfrac{1}{2n}-\tfrac{\mathfrak{A}}{2n^2}+\tfrac{\mathfrak{B}}{4n^4}-\text{etc.}$$

and this gives the constant $=0{,}5772156649015325$. But if therefore there may be put $n=\infty$, there will be

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots+\tfrac{1}{\infty}=\text{Const.}+l\infty,$$

from which the value of both of those series continued to infinity will be

$$=l(\infty+1)-\text{Const.}-l\infty=-\text{ Const.}$$

From which there will be

$$\begin{aligned}
lS &= -x\cdot 0{,}5772156649015325\\
&+\tfrac{1}{2}x^2\left(1+\tfrac{1}{2^2}+\tfrac{1}{3^2}+\tfrac{1}{4^2}+\text{etc.}\right)\\
&-\tfrac{1}{3}x^3\left(1+\tfrac{1}{2^3}+\tfrac{1}{3^3}+\tfrac{1}{4^3}+\text{etc.}\right)\\
&+\tfrac{1}{4}x^4\left(1+\tfrac{1}{2^4}+\tfrac{1}{3^4}+\tfrac{1}{4^4}+\text{etc.}\right)\\
&\qquad\text{etc.}
\end{aligned}$$

from which the differentials of each order will be found easily. Indeed there shall be

$$\begin{aligned}
\tfrac{dS}{S} &= -dx\cdot 0{,}5772156649015325\\
&+xdx\left(1+\tfrac{1}{2^2}+\tfrac{1}{3^2}+\tfrac{1}{4^2}+\tfrac{1}{5^2}+\text{etc.}\right)\\
&-x^2dx\left(1+\tfrac{1}{2^3}+\tfrac{1}{3^3}+\tfrac{1}{4^3}+\tfrac{1}{5^3}+\text{etc.}\right)\\
&+x^3dx\left(1+\tfrac{1}{2^4}+\tfrac{1}{3^4}+\tfrac{1}{4^4}+\tfrac{1}{5^4}+\text{etc.}\right)\\
&\qquad\text{etc.}
\end{aligned}$$

But if these series may be gathered into one, there will be

$$\frac{dS}{S} = -dx \cdot 0{,}5772156649015325 + \frac{xdx}{1(1+x)} + \frac{xdx}{2(2+x)} + \frac{xdx}{3(3+x)} + \frac{xdx}{4(4+x)} + \text{etc.}$$

Whereby if there shall be $x = 0$, there becomes

$$\frac{dS}{S} = -dx \cdot 0{,}5772156649015325.$$

From the first expression in this case truly there will be

$$\begin{aligned}\frac{dS}{S} = &-\tfrac{1}{2}dx\left(1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \text{etc.}\right)\\ &+\tfrac{1}{3}dx\left(1 + \tfrac{1}{2^3} + \tfrac{1}{3^3} + \tfrac{1}{4^3} + \text{etc.}\right)\\ &-\tfrac{1}{4}dx\left(1 + \tfrac{1}{2^4} + \tfrac{1}{3^4} + \tfrac{1}{4^4} + \text{etc.}\right)\\ &+\tfrac{1}{5}dx\left(1 + \tfrac{1}{2^5} + \tfrac{1}{3^5} + \tfrac{1}{4^5} + \text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

385. Hence also therefore the differentials of inexplicable functions of this kind in any specific case are able to be shown, because therefore here we have elicited the complete differentials. On account of which if such functions may arise in expressions, which may be seen to be indeterminate, of such a kind as we have treated in the preceding chapter, we will be able to determine the values by the same method, as may be understood from the adjoining examples.

EXAMPLE 1

To determine the value of this expression from that case, when there is put $x = 1$

$$\frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{x}}{x(x-1)} - \frac{1}{(x-1)(2x-1)}.$$

We may put

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots + \tfrac{1}{x} = S;$$

from § 371 there will be

$$\begin{aligned}S = &\,x\left(1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \text{etc.}\right)\\ &-x^2\left(1 + \tfrac{1}{2^3} + \tfrac{1}{3^3} + \tfrac{1}{4^3} + \text{etc.}\right)\\ &+x^3\left(1 + \tfrac{1}{2^4} + \tfrac{1}{3^4} + \tfrac{1}{4^4} + \text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

or since there shall be also

$$S = +1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\text{etc.}$$
$$-\tfrac{1}{1+x}-\tfrac{1}{2+x}-\tfrac{1}{3+x}-\tfrac{1}{4+x}-\tfrac{1}{5+x}-\text{etc.},$$

if any of the upper terms may be combined with the preceding lower, there will be produced

$$S=1+\tfrac{x-1}{2(1+x)}+\tfrac{x-1}{3(2+x)}+\tfrac{x-1}{4(3+x)}+\text{etc.},$$

which expression, because there must be put in place $x=1$, is more convenient. Therefore let $x=1+\omega$ and there becomes

$$S=1+\tfrac{\omega}{2(2+\omega)}+\tfrac{\omega}{3(3+\omega)}+\tfrac{\omega}{4(4+\omega)}+\text{etc.}$$

or

$$\begin{aligned}S=1&+\omega\left(\tfrac{1}{2^2}+\tfrac{1}{3^2}+\tfrac{1}{4^2}+\tfrac{1}{5^2}+\text{etc.}\right)=1+\mathfrak{B}\omega\\ &-\omega^2\left(\tfrac{1}{2^3}+\tfrac{1}{3^3}+\tfrac{1}{4^3}+\tfrac{1}{5^3}+\text{etc.}\right)\qquad-\mathfrak{C}\omega^2\\ &+\omega^3\left(\tfrac{1}{2^4}+\tfrac{1}{3^4}+\tfrac{1}{4^4}+\tfrac{1}{5^4}+\text{etc.}\right)\qquad+\mathfrak{D}\omega^3\\ &\qquad\qquad\text{etc.}\qquad\qquad\qquad\qquad\text{etc.}\end{aligned}$$

Therefore the whole expression on putting $x=1+\omega$ will change into this

$$\tfrac{1+\mathfrak{B}\omega-\mathfrak{C}\omega^2+\mathfrak{D}\omega^3-\text{etc.}}{\omega(1+\omega)}-\tfrac{1}{\omega(1+2\omega)}$$

or

$$\tfrac{\omega+\mathfrak{B}\omega+2\mathfrak{B}\omega^2-\mathfrak{C}\omega^2-\text{etc.}}{\omega(1+\omega)(1+2\omega)}=\tfrac{1+\mathfrak{B}+2\mathfrak{B}\omega-\mathfrak{C}\omega-\text{etc.}}{(1+\omega)(1+2\omega)}.$$

Now there may be put $\omega=0$ and the value of the proposed expression in the case $x=1$ will be

$$=1+\mathfrak{B}=1+\tfrac{1}{2^2}+\tfrac{1}{3^2}+\tfrac{1}{4^2}+\text{etc.};$$

which series since it shall be $=\tfrac{1}{2}\pi^2$, it follows that the value of the series sought shall be $=\tfrac{1}{2}\pi^2$.

EXAMPLE 2
To find the value of this expression

$$\frac{2x-xx}{(x-1)^2}+\frac{\pi\pi x}{6(x-1)}-\frac{(2x-1)\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{x}\right)}{x(x-1)^2}$$

in the case, in which there is put $x=1$.

There may be put $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{x}=S$ and there may be considered $x=1+\omega$*;* there becomes, as we have found in the previous example,

$$S=1+\mathfrak{B}\omega-\mathfrak{C}\omega^2+\mathfrak{D}\omega^3-\text{etc.}$$

with arising

$$\mathfrak{B}=\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\text{etc.}=\frac{1}{6}\pi\pi-1$$
$$\mathfrak{C}=\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\text{etc.}$$
$$\mathfrak{D}=\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\text{etc.}$$
etc.

Therefore on putting $x=1+\omega$ the proposed expression may adopt this form

$$\frac{1-\omega\omega}{\omega\omega}+\frac{(1+\mathfrak{B})(1+\omega)}{\omega}-\frac{(1+2\omega)\left(1+\mathfrak{B}\omega-\mathfrak{C}\omega^2+\mathfrak{D}\omega^3-\text{ etc.}\right)}{(1+\omega)\omega^2},$$

which reduced to the same denominator $2(1+\omega)$ will become

$$\frac{1+\omega-\omega^2-\omega^3+\omega+2\omega^2+\omega^3+\mathfrak{B}\omega(1+2\omega+\omega\omega)-1-\mathfrak{B}\omega+\mathfrak{C}\omega^2-\mathfrak{D}\omega^3-2\omega-2\mathfrak{B}\omega^2+2\mathfrak{C}\omega^3-\text{etc.}}{\omega^2(1+\omega)},$$

which is reduced to this form

$$\frac{\omega^2+\mathfrak{C}\omega^2+\mathfrak{B}\omega^3+2\mathfrak{C}\omega^3-\mathfrak{D}\omega^3+\text{ etc.}}{\omega^2(1+\omega)}.$$

Now there is made $\omega=0$ and there will be produced $1+\mathfrak{C}$. On account of which the value of the proposed expression in the case $x=1$ will be $=1+\mathfrak{C}$ and thus may be expressed by the same series

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\text{etc.};$$

the sum of which value sought, since it will be shown neither by logarithms nor by the periphery of the circle, cannot be assigned even now by any finite way. Therefore the use from these two examples can be seen clearly enough, which the differentiation of inexplicable functions can have in the doctrine of series.

386. In the method treated here requiring inexplicable functions to be differentiated, we have assumed the most infinite terms of the series of the series A, B, C, D, E etc. to be either $=0$ or to have vanishing differences only; of which if neither may be appropriate, this method will not be allowed to be used. On this account I will explain another method not restricted to this condition, as the general summation of series from the general term demands and a further explanation is provided above [Chap.V]. Therefore the letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}$ etc. may denote the Bernoulli numbers shown in §122, and let this inexplicable function be proposed

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & x \\ S = A & + B & + C & + D & + \cdots + & X, \end{array}$$

and because above (in Chapter V of Part II, §130) we have shown to be

$$S = \int Xdx + \tfrac{1}{2}X + \frac{\mathfrak{A}dX}{1\cdot 2dx} - \frac{\mathfrak{B}d^3X}{1\cdot 2\cdot 3\cdot 4dx^3} + \frac{\mathfrak{C}d^5X}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6dx^5} - \text{etc.},$$

hence the differential of this function S is shown easily; for there will be

$$dS = Xdx + \tfrac{1}{2}dX + \frac{\mathfrak{A}ddX}{1\cdot 2dx} - \frac{\mathfrak{B}d^4X}{1\cdot 2\cdot 3\cdot 4dx^3} + \frac{\mathfrak{C}d^6X}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6dx^5} - \text{etc.}$$

387. But if the proposed progression be joined together with a geometrical progression, in which case the most infinite terms of this nowhere may be reduced to constant differences and therefore there is no place for the method first found, then the method offered in §174 may bring about a remedy. For if this function be proposed

$$S = Ap + Bp^2 + Cp^3 + Dp^4 + \cdots + Xp^x,$$

the values of the letters $\alpha, \beta, \gamma, \delta$ etc. may be sought, so that there shall be

$$\frac{p-1}{p-e^u} = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \text{etc.},$$

with which found, as we have shown these in §173, there will be

$$S = \frac{p}{p-1}\cdot p^x \left(X - \frac{\alpha dX}{dx} + \frac{\beta ddX}{dx^2} - \frac{\gamma d^3X}{dx^3} + \frac{\delta d^3X}{dx^4} - \text{etc.} \right) \pm \text{Constant},$$

which sum may be returned $=0$, if there may be put $x=0$, or which may be satisfactory for some other case. Therefore with the differential taken this constant will depart from the calculation and there will be

$$dS = \frac{p}{p-1}\cdot p^x dxlp\left(X - \frac{\alpha dX}{dx} + \frac{\beta ddX}{dx^2} - \frac{\gamma d^3X}{dx^3} + \frac{\delta d^3X}{dx^4} - \text{etc.}\right)$$
$$+\frac{p}{p-1}\cdot p^x\left(dX - \frac{\alpha ddX}{dx} + \frac{\beta d^3X}{dx^2} - \frac{\gamma d^4X}{dx^3} + \text{etc.}\right)$$

or

$$dS = \frac{p^{x+1}}{p-1}\left(Xdxlp - (\alpha lp - 1)dX + (\beta lp - \alpha)\frac{ddX}{dx} - (\gamma lp - \beta)\frac{d^3X}{dx^2} + \text{etc.}\right),$$

which is the differential sought of the proposed function *S*.

388. But if the proposed inexplicable function may depend on factors and the most infinite logarithms of these may have constant or less differences, then also by this method it will be able to show always the differential of the function.
For let there be

$$\begin{matrix} 1 & 2 & 3 & 4\cdots & x \\ S = A\cdot & B\cdot & C\cdot & D\cdots & X. \end{matrix}$$

Because hence there shall be

$$lS = lA + lB + lC + lD + \cdots + lX\ ,$$

from the above method on calling on the Bernoulli numbers to help there will be

$$lS = \int dxlX + \tfrac{1}{2}lX + \frac{\mathfrak{A}d.lX}{1\cdot 2dx} - \frac{\mathfrak{B}d^3.lX}{1\cdot 2\cdot 3\cdot 4dx^3} + \text{etc.}$$

from which expression the differential shall be

$$\frac{dS}{S} = dxlX + \tfrac{1}{2}d.lX + \frac{\mathfrak{A}dd.lX}{1\cdot 2dx} - \frac{\mathfrak{B}d^4.lX}{1\cdot 2\cdot 3\cdot 4dx} + \frac{\mathfrak{C}d^6.lX}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6dx} - \frac{\mathfrak{D}d^8.lX}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8dx} + \text{etc.}$$

Hence if there were $X = x$, so that there shall be $S = 1\cdot 2\cdot 3\cdot 4\cdots x$, there becomes with the application made

$$\frac{dS}{S} = dxlx + \frac{dx}{2x} - \frac{\mathfrak{A}dx}{2xx} + \frac{\mathfrak{B}dx}{4x^4} - \frac{\mathfrak{C}dx}{6x^6} + \frac{\mathfrak{D}dx}{8x^8} - \text{etc.},$$

which form, if x shall be a very large number, may be used more conveniently than those, which we have found before.

CAPUT XVI

DE DIFFERENTIATIONE FUNCTIONUM INEXPLICABILIUM

367. Functiones inexplicabiles hic voco, quae neque expressionibus determinatis neque per aequationum radices explicari possunt, ita ut non solum non sint algebraicae, sed etiam plerumque incertum sit, ad quod genus transcendentium pertineant. Huiusmodi functio inexplicabilis est

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots+\tfrac{1}{x},$$

quae utique ab x pendet, at, nisi x sit numerus integer, nullo modo explicari potest. Simili modo haec expressio

$$1\cdot 2\cdot 3\cdot 4\cdots x$$

erit functio inexplicabilis ipsius x, quoniam, si x sit numerus quicunque, eius valor non solum non algebraice, sed ne quidem per ullum certum quantitatum transcendentium genus exprimi potest. Generatim ergo talium functionum inexplicabilium notio ex seriebus derivari potest. Sit enim proposita series quaecunque

$$\begin{array}{ccccccccccc} 1 & & 2 & & 3 & & 4 & & \cdots & & x \\ A & + & B & + & C & + & D & + & \cdots & + & X, \end{array}$$

cuius summa, si formula finita exprimi nequeat, praebebit functionem inexplicabilem ipsius x, nempe

$$S=A+B+C+D+\cdots+X\ .$$

Similiter continua producta ex terminis serierum uti

$$P=A\cdot B\cdot C\cdot D\cdots X$$

exhibebunt functiones inexplicabiles ipsius x, quae autem ope logarithmorum ad formam priorem revocari possunt; erit enim

$$lP=lA+lB+lC+lD+\cdots+lX\ .$$

368. Hoc igitur capite methodum explicare constitui huiusmodi functionum inexplicabilium differentialia investigandi. Quod argumentum, quamvis ad primam huius operis partem, ubi praecepta calculi differentialis sunt tradita, pertinere videatur, tamen, quoniam uberiorem doctrinae serierum cognitionem postulat, ad quam in hac altera parte pervenire licuit, ordinem naturalem relinquere coacti hoc loco attingamus. Cum autem haec investigatio prorsus sit nova neque a quoquam adhuc tractata, tantum abest, ut hanc calculi differentialis partem absolvere queamus, ut potius prima tantum eius elementa adumbrare conemur. Praeterea vero nonnullas quaestiones proponam, quarum enodatio differentiationem huiusmodi functionum inexplicabilium requirat, quo simul usus huius tractationis, qui autem in posterum sine dubio multo amplior erit, clarius perspiciatur.

369. Ad huiusmodi functiones inexplicabiles differentiandas ante omnia necesse est, ut earum valores investigemus, quos induunt, si pro x ponatur $x+\omega$. Sit igitur

$$\begin{array}{cccccccccc} & 1 & & 2 & & 3 & & 4 & \cdots & x \\ S= & A & + & B & + & C & + & D & +\cdots+ & X, \end{array}$$

atque ponatur Σ valor ipsius S, quem recipit, si pro x ponatur $x+\omega$, sitque Z terminus seriei respondens indici $x+\omega$. Iam igitur termini, qui respondent indicibus $x+1, x+2, x+3$ etc., indicentur per X', X'', X''' etc. atque is, qui convenit indici infinito $x+\infty$, per $X^{|\infty|}$. Similique modo termini competentes indicibus $x+\omega+1,\quad x+\omega+2,\quad x+\omega+3$ etc. indicentur per Z', Z'', Z'', etc. et sit $Z^{|\infty|}$ terminus respondens indici $x+\omega+\infty$. Quibus positis erit

$$\begin{aligned} S' &= S+X' \\ S'' &= S+X'+X'' \\ S''' &= S+X'+X''+X''' \\ &\text{etc.} \\ S^{|\infty|} &= S+X'+X''+X'''+\cdots+X^{|\infty|}. \end{aligned}$$

Simili modo cum etiam Σ successive terminis Z', Z'' etc. augeatur, erit

$$\begin{aligned} \Sigma' &= \Sigma+Z' \\ \Sigma'' &= \Sigma+Z'+Z'' \\ \Sigma''' &= \Sigma+Z'+Z''+Z''' \\ &\text{etc.} \\ \Sigma^{|\infty|} &= \Sigma+Z'+Z''+Z'''+\cdots+Z^{|\infty|}. \end{aligned}$$

370. Nunc natura seriei *S, S', S'', S'''* etc. est perpendenda, qualis futura sit, si in infinitum continuetur; quae si in infinito cum progressione arithmetica confundatur, quod fit, si termini seriei X, X', X'', X''' etc. in infinito ad aequalitatem convergant, ita ut differentiae seriei S, S', S'' etc. tandem fiant aequales, hoc casu quantitates $S^{|\infty|}$, $S^{|\infty+1|}$, $S^{|\infty+2|}$ etc. erunt in arithmetica progressione, et cum sit $\Sigma^{|\infty|} = S^{|\infty+\omega|}$, ob

$$S^{|\infty+\omega|} = S^{|\infty|} + \omega\left(S^{|\infty+1|} - S^{|\infty|}\right) = \omega S^{|\infty+1|} + (1-\omega)S^{|\infty|}$$

erit

$$\Sigma^{|\infty|} = \omega S^{|\infty+1|} + (1-\omega)S^{|\infty|}$$

At est $S^{|\infty+1|} = S^{|\infty|} + X^{|\infty+1|}$, unde fit

$$\Sigma^{|\infty|} = S^{|\infty|} + \omega X^{|\infty+1|}$$

ex quo obtinebitur haec aequatio

$$\Sigma + Z' + Z'' + Z''' + \cdots + Z^{|\infty|} = S + X' + X'' + X''' + \cdots + X^{|\infty|} + \omega X^{|\infty+1|},$$

ex qua definitur valor quaesitus Σ, quem induit functio *S*, dum in ea $x+\omega$ loco *x* substituitur, eritque

$$\begin{aligned}\Sigma = S + \omega X^{|\infty+1|} &+ X' + X'' + X''' + \text{etc.} \quad \text{in infinitum}\\ &- Z' - Z'' - Z''' - \text{etc.} \quad \text{in infinitum.}\end{aligned}$$

Quare si seriei *A, B, C, D* etc. termini infinitesimi evanescant, terminus $\omega X^{|\infty+1|}$ evanescit et omitti potest.

371. Exprimitur ergo valor ipsius Σ per novam seriem infinitam, quae exhiberi potest, si seriei $A + B + C + \text{etc.}$ habeatur terminus generalis, ex quo valores terminorum Z', Z'', Z'', etc. definiri queant. Posito ergo ω infinite parvo, cum sit $\Sigma - S$ differentiale functionis *S*, hoc differentiale *dS* per seriem infinitam exprimetur. Atque si nequidem altiores potestates ipsius ω negligantur, habebitur differentiale completum functionis huius inexplicabilis *S;* cuius natura quo clarius ob oculos ponatur, sequentibus exemplis hoc negotium illustrabimus.

EXEMPLUM 1

Invenire differentiale huius functionis inexplicabilis

$$S = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{x}.$$

Quoniam huius seriei terminus generalis X est $= \frac{1}{x}$ ac propterea

$$\begin{array}{c|c} X' = \frac{1}{x+1} & Z' = \frac{1}{x+1+\omega} \\ X'' = \frac{1}{x+2} & Z'' = \frac{1}{x+2+\omega} \\ X''' = \frac{1}{x+3} & Z''' = \frac{1}{x+3+\omega} \\ \text{etc.} & \text{etc.,} \end{array}$$

ob

$$X^{|\infty+1|} = \frac{1}{x+\infty+1} = 0,$$

si loco x ponatur $x+\omega$, functio S abibit in Σ, ut sit

$$\begin{aligned} \Sigma = S + \frac{1}{x+1} &+ \frac{1}{x+2} + \frac{1}{x+3} + \text{etc.} \\ - \frac{1}{x+1+\omega} &- \frac{1}{x+2+\omega} - \frac{1}{x+3+\omega} - \text{etc.,} \end{aligned}$$

sive binis his terminis in singulos colligendis erit

$$\Sigma = S + \frac{\omega}{(x+1)(x+1+\omega)} + \frac{\omega}{(x+2)(x+2+\omega)} + \frac{\omega}{(x+3)(x+3+\omega)} + \text{etc.,}$$

seu cum sit

$$\frac{1}{x+1+\omega} = \frac{1}{x+1} - \frac{\omega}{(x+1)^2} + \frac{\omega^2}{(x+1)^3} - \frac{\omega^3}{(x+1)^4} + \text{etc.}$$

$$\frac{1}{x+2+\omega} = \frac{1}{x+2} - \frac{\omega}{(x+2)^2} + \frac{\omega^2}{(x+2)^3} - \frac{\omega^3}{(x+2)^4} + \text{etc.}$$

etc.

erit seriebus secundum potestates ipsius ω dispositis

$$\begin{aligned}\Sigma = S + \omega&\left(\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\frac{1}{(x+4)^2}+\text{etc.}\right)\\ -\omega^2&\left(\frac{1}{(x+1)^3}+\frac{1}{(x+2)^3}+\frac{1}{(x+3)^3}+\frac{1}{(x+4)^3}+\text{etc.}\right)\\ +\omega^3&\left(\frac{1}{(x+1)^4}+\frac{1}{(x+2)^4}+\frac{1}{(x+3)^4}+\frac{1}{(x+4)^4}+\text{etc.}\right)\\ -\omega^4&\left(\frac{1}{(x+1)^5}+\frac{1}{(x+2)^5}+\frac{1}{(x+3)^5}+\frac{1}{(x+4)^5}+\text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

Posito ergo dx pro ω obtinebimus functionis propositae S differentiale completum

$$\begin{aligned}dS = dx&\left(\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\frac{1}{(x+4)^2}+\text{etc.}\right)\\ -dx^2&\left(\frac{1}{(x+1)^3}+\frac{1}{(x+2)^3}+\frac{1}{(x+3)^3}+\frac{1}{(x+4)^3}+\text{etc.}\right)\\ +dx^3&\left(\frac{1}{(x+1)^4}+\frac{1}{(x+2)^4}+\frac{1}{(x+3)^4}+\frac{1}{(x+4)^4}+\text{etc.}\right)\\ -dx^4&\left(\frac{1}{(x+1)^5}+\frac{1}{(x+2)^5}+\frac{1}{(x+3)^5}+\frac{1}{(x+4)^5}+\text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

EXEMPLUM 2

Invenire differentiale huius functionis inexplicabilis ipsius x

$$S = 1+\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}+\cdots+\tfrac{1}{2x-1}.$$

Quia huius seriei terminus generalis est $X = \frac{1}{2x-1}$, erit

$$\begin{array}{c|c} X' = \frac{1}{2x+1} & Z' = \frac{1}{2x+1+2\omega} \\ X'' = \frac{1}{2x+3} & Z'' = \frac{1}{2x+3+2\omega} \\ X''' = \frac{1}{2x+5} & Z''' = \frac{1}{2x+5+2\omega} \\ \text{etc.} & \text{etc.,} \end{array}$$

Ob terminos huius seriei infinitesimos evanescentes et aequales prodibit valor ipsius S, si loco x ponatur $x+\omega$,

$$\Sigma = S + \frac{1}{2x+1} + \frac{1}{2x+2} + \frac{1}{2x+3} + \text{etc.}$$
$$- \frac{1}{2x+1+2\omega} - \frac{1}{2x+3+2\omega} - \frac{1}{x+5+2\omega} - \text{etc.},$$

seu

$$\Sigma = S + \frac{2\omega}{(2x+1)(2x+1+2\omega)} + \frac{2\omega}{(2x+3)(2x+3+2\omega)} + \text{etc.},$$

Verum si singuli termini in series secundum dimensiones ipsius ω resolvantur, erit

$$\begin{aligned}\Sigma = S &+ 2\omega\left(\frac{1}{(2x+1)^2} + \frac{1}{(2x+3)^2} + \frac{1}{(2x+5)^2} + \text{etc.}\right)\\ &- 4\omega^2\left(\frac{1}{(2x+1)^3} + \frac{1}{(2x+3)^3} + \frac{1}{(2x+5)^3} + \text{etc.}\right)\\ &+ 8\omega^3\left(\frac{1}{(2x+1)^4} + \frac{1}{(2x+3)^4} + \frac{1}{(2x+5)^4} + \text{etc.}\right)\\ &- 16\omega^4\left(\frac{1}{(2x+1)^5} + \frac{1}{(2x+3)^5} + \frac{1}{(2x+5)^5} + \text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

Ponatur nunc dx pro ω atque prodibit differentiale completum functionis inexplicabilis S propositae

$$\begin{aligned}dS = &\ 2dx\left(\frac{1}{(2x+1)^2} + \frac{1}{(2x+3)^2} + \frac{1}{(2x+5)^2} + \text{etc.}\right)\\ &- 4dx^2\left(\frac{1}{(2x+1)^3} + \frac{1}{(2x+3)^3} + \frac{1}{(2x+5)^3} + \text{etc.}\right)\\ &+ 8dx^3\left(\frac{1}{(2x+1)^4} + \frac{1}{(2x+3)^4} + \frac{1}{(2x+5)^4} + \text{etc.}\right)\\ &- 16dx^4\left(\frac{1}{(2x+1)^5} + \frac{1}{(2x+3)^5} + \frac{1}{(2x+5)^5} + \text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

EXEMPLUM 3

Invenire differentiale completum functionis huius inexplicabilis ipsius x

$$S = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots + \frac{1}{x^n}$$

Cum huius seriei terminus generalis sit $= \frac{1}{x^n}$, erunt termini infinitesimi evanescentes et inter se aequales. Hincque ob

$$\begin{array}{r|l} X' = \frac{1}{(x+1)^n} & Z' = \frac{1}{(x+2+\omega)^n} \\ X'' = \frac{1}{(x+2)^n} & Z'' = \frac{1}{(x+2+\omega)^n} \\ X''' = \frac{1}{(x+3)^n} & Z''' = \frac{1}{(x+3+\omega)^n} \\ \text{etc.} & \text{etc.,} \end{array}$$

erit

$$X' - Z' = \frac{n\omega}{(x+1)^{n+1}} - \frac{n(n+1)\omega^2}{2(x+1)^{n+2}} + \frac{n(n+1)(n+2)\omega^3}{6(x+1)^{n+3}} - \text{etc.}$$

$$X'' - Z'' = \frac{n\omega}{(x+2)^{n+1}} - \frac{n(n+1)\omega^2}{2(x+2)^{n+2}} + \frac{n(n+1)(n+2)\omega^3}{6(x+2)^{n+3}} - \text{etc.}$$

etc.

ex quibus invenitur

$$\begin{aligned} \Sigma - S = n\omega & \left(\frac{1}{(x+1)^{n+1}} + \frac{1}{(x+2)^{n+1}} + \frac{1}{(x+3)^{n+1}} + \text{etc.} \right) \\ & - \frac{n(n+1)}{1\cdot 2}\omega^2 \left(\frac{1}{(x+1)^{n+2}} + \frac{1}{(x+2)^{n+2}} + \frac{1}{(x+3)^{n+2}} + \text{etc.} \right) \\ & + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\omega^3 \left(\frac{1}{(x+1)^{n+3}} + \frac{1}{(x+2)^{n+3}} + \frac{1}{(x+3)^{n+3}} + \text{etc.} \right) \\ & \text{etc.} \end{aligned}$$

Quare posito $\omega = dx$ prodibit differentiale completum functionis S quaesitum

$$dS = ndx\left(\frac{1}{(x+1)^{n+1}} + \frac{1}{(x+2)^{n+1}} + \frac{1}{(x+3)^{n+1}} + \text{etc.}\right)$$
$$-\frac{n(n+1)}{1\cdot 2}dx^2\left(\frac{1}{(x+1)^{n+2}} + \frac{1}{(x+2)^{n+2}} + \frac{1}{(x+3)^{n+2}} + \text{etc.}\right)$$
$$+\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}dx^3\left(\frac{1}{(x+1)^{n+3}} + \frac{1}{(x+2)^{n+3}} + \frac{1}{(x+3)^{n+3}} + \text{etc.}\right)$$
$$\text{etc.}$$

372. Ex his quoque summae istarum serierum interpolari seu valores terminorum summatoriorum exhiberi possunt, quando numerus terminorum non est numerus integer. Si enim ponatur $x = 0$, erit quoque $S = 0$ atque Σ exprimet summam terminorum, quot numerus ω continet unitates, etiamsi iste numerus *w* non sit integer. Ita in exemplo primo si ponatur

$$\Sigma = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{\omega},$$

erit

$$\Sigma = \frac{\omega}{1(1+\omega)} + \frac{\omega}{2(2+\omega)} + \frac{\omega}{3(3+\omega)} + \frac{\omega}{4(4+\omega)} + \text{etc.}$$

sive

$$\Sigma = \omega\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.}\right)$$
$$-\omega^2\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.}\right)$$
$$+\omega^3\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.}\right)$$
$$\text{etc.}$$

In exemplo vero tertio erit

$$\Sigma = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots + \frac{1}{\omega^n}.$$

Valorque ipsius Σ, sive ω sit numerus integer sive fractus, per series sequenti modo exprimetur

$$\Sigma = n\omega\left(\frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{4^{n+1}} + \text{etc.}\right)$$
$$-\frac{n(n+1)}{1\cdot 2}\omega^2\left(1 + \frac{1}{2^{n+2}} + \frac{1}{3^{n+2}} + \frac{1}{4^{n+2}} + \text{etc.}\right)$$
$$+\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\omega^3\left(1 + \frac{1}{2^{n+3}} + \frac{1}{3^{n+3}} + \frac{1}{4^{n+3}} + \text{etc.}\right)$$
$$\text{etc.}$$

373. Haec eadem quoque ad seriem generalem accommodari possunt; cum enim sit

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & \cdots & x \\ S = A & + B & + C & + D & + \cdots + & X, \end{array}$$

atque posito $x+\omega$ loco x abeat X in Z et S in Σ, erit

$$Z = X + \frac{\omega dX}{dx} + \frac{\omega^2 ddX}{1\cdot 2dx^2} + \frac{\omega^3 d^3 X}{1\cdot 2\cdot 3dx^3} + \text{etc.},$$

et quia simili modo Z', Z'', Z''' etc. per X', X'', X''' etc. exprimuntur, erit

$$\Sigma = S + \omega X^{|\infty+1|} - \frac{\omega}{dx} d.\left(X' + X'' + X''' + X'''' + \text{etc.}\right)$$
$$- \frac{\omega^2}{1\cdot 2dx^2} dd.\left(X' + X'' + X''' + X'''' + \text{etc.}\right) - \frac{\omega^3}{1\cdot 2\cdot 3dx^3} d^3.\left(X' + X'' + X''' + X'''' + \text{etc.}\right)$$
$$\text{etc.},$$

et nisi $X^{|\infty+1|}$ sit $=0$, hoc modo exprimi poterit, ut consideratio infiniti tollatur,

$$X^{|\infty+1|} = X' + \left(X'' - X'\right) + \left(X''' - X''\right) + \left(X'''' - X'''\right) + \text{etc.}$$

eritque ergo

$$\Sigma = S + \omega X' + \omega\left(\left(X'' - X'\right) + \left(X''' - X''\right) + \left(X'''' - X'''\right) + \text{etc.}\right)$$
$$- \frac{\omega}{dx} d.\left(X' + X'' + X''' + X'''' + \text{etc.}\right)$$
$$- \frac{\omega^2}{2dx^2} dd.\left(X' + X'' + X''' + X'''' + \text{etc.}\right)$$
$$- \frac{\omega^3}{6dx^3} d^3.\left(X' + X'' + X''' + X'''' + \text{etc.}\right)$$
$$\text{etc.}$$

Si ergo ponatur $\omega = dx$, orietur differentiale completum ipsius

$$S = A + B + C + \cdots + X$$

ita expressum

$$dS = X'dx + dx\big((X'' - X') + (X''' - X'') + (X'''' - X''') + \text{etc.}\big)$$
$$- d.(X' + X'' + X''' + X'''' + \text{etc.})$$
$$- \tfrac{1}{2}dd.(X' + X'' + X''' + X'''' + \text{etc.})$$
$$- \tfrac{1}{6}d^3.(X' + X'' + X''' + X'''' + \text{etc.})$$
$$\text{etc.}$$

374. Ponamus esse $x = 0$; fiet

$$X' = A, \quad X'' = B \quad \text{etc.}$$

ideoque $X' + X'' + X''' + \text{etc.}$ erit series infinita, cuius terminus generalis est = X. Formentur deinde series ex his terminis generalibus

$$\frac{dX}{dx},\ \frac{ddX}{2dx^2},\ \frac{d^3X}{6d^3x},\ \frac{d^4X}{24d^4x}\ \text{ etc.},$$

quarum serierum in infinitum continuatarum summae sint

$$\mathcal{S}X = \mathfrak{A}, \quad \mathcal{S}\frac{dX}{dx} = \mathfrak{B}, \quad \mathcal{S}\frac{ddX}{2dx^2} = \mathfrak{C}, \quad \mathcal{S}\frac{d^3X}{6dx^3} = \mathfrak{D} \ \text{ etc. };$$

et quia posito $x = 0$ fit quoque $S = 0$, et Σ erit summa seriei

$$A + B + C + D + \cdots + Z$$

continentis ω terminos; est enim Z terminus indicis ω, sive ω sit numerus integer sive fractus. Quare habebitur

$$\Sigma = \omega A + \omega\big((B - A) + (C - B) + (D - C) + \text{etc.}\big) - \omega\mathfrak{B} - \omega^2\mathfrak{C} - \omega^3\mathfrak{D} - \omega^4\mathfrak{E} - \text{etc.},$$

ubi prima series praetermitti potest, si seriei propositae termini tandem evanescant.

375. Scribamus nunc x loco ω abibitque Σ in S, ita ut sit

$$\begin{array}{ccccccccccc} & 1 & & 2 & & 3 & & 4 & & \cdots & x \\ S = & A & + & B & + & C & + & D & + & \cdots + & X, \end{array}$$

atque idem ipsius S valor iam per seriem infinitam exprimetur hoc modo

$$S = Ax + x\big((B - A) + (C - B) + (D - C) + \text{etc.}\big) - \mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \mathfrak{F}x^5 - \text{etc.};$$

cuius valor cum aeque distincte exprimatur, sive x sit numerus integer sive fractus, differentialia ipsius S cuiusque ordinis hinc facile exhiberi possunt:

$$\frac{dS}{dx} = A + (B-A) + (C-B) + (D-C) + \text{etc.}$$
$$- \mathfrak{B} - 2\mathfrak{C}x - 3\mathfrak{D}x^2 - 4\mathfrak{E}x^3 - \text{etc.}$$
$$\frac{ddS}{2dx^2} = -\mathfrak{C} - 3\mathfrak{D}x - 6\mathfrak{E}x^2 - 10\mathfrak{F}x^3 - \text{etc.}$$
$$\frac{d^3S}{6dx^3} = -\mathfrak{D} - 4\mathfrak{E}x - 10\mathfrak{F}x^2 - 20\mathfrak{G}x^3 - \text{etc.}$$
$$\frac{d^4S}{24dx^4} = -\mathfrak{E} - 5\mathfrak{F}x - 15\mathfrak{G}x^2 - 35\mathfrak{H}x^3 - \text{etc.}$$

Quare cum differentiale completum sit

$$= dS + \tfrac{1}{2}ddS + \tfrac{1}{6}d^3S + \tfrac{1}{24}d^4S + \text{etc.},$$

erit functionis propositae S differentiale completum

$$dS = Adx + (B-A)dx + (C-B)dx + (D-C)dx + \text{etc.}$$
$$-\mathfrak{B}dx - \mathfrak{C}\left(2xdx + dx^2\right) - \mathfrak{D}\left(3x^2dx + 3xdx^2 + dx^3\right)$$
$$-\mathfrak{E}\left(4x^3dx + 6x^2dx2 + 4xdx^3 + dx^4\right) - \text{etc.}$$

376. Hoc ergo modo functionis cuiusque inexplicabilis S differentiale assignari potest, si seriei

$$A + B + C + D + \text{etc}.$$

termini infinitesimi vel evanescant vel inter se sint aequales. Quodsi enim huius seriei termini infinitesimi non fuerint $= 0$, tum summa seriei $\mathfrak{B}$, quae ex termino generali $\frac{dX}{dx}$ formatur, fiet infinita, at vero cum serie

$$A + (B-A) + (C-B) + (D-C) + \text{etc}.$$

coniuncta summam finitam constituet. At fieri potest, ut termini seriei $A + (B-A) + (C-B) + (D-C) + \text{etc}$ ita in infinitum augeantur, ut non solum seriei $\mathfrak{B}$, sed etiam seriei $\mathfrak{C}$ summa fiat infinite magna, quo casu non sufficit seriem $A + (B-A) + (C-B) + (D-C) + \text{etc}$. adiecisse; sed quoniam hoc casu valores infinitesimi

§ 370 considerati, nempe $S^{|\infty|}, S^{|\infty+1|}, S^{|\infty+2|}$, non amplius in arithmetica sunt progressione, uti assumseramus, huius progressionis ratio erit habenda. Quemadmodum ergo assumsimus horum terminorum differentias primas esse aequales, ita methodum amplius extendemus, si horum valorum differentias demum secundas vel tertias vel ulteriores constantes statuamus.

377. Retento ergo eodem ratiocinio, quo § 369 sumus usi, ponamus memoratorum valorum differentias demum secundas esse constantes

$$S^{|\infty|}, S^{|\infty+1|}, S^{|\infty+2|}$$

Differentiae primae

$$X^{|\infty+1|}, X^{|\infty+2|}$$

Differentiae secundae

$$X^{|\infty+2|} - X^{|\infty+3|}$$

Hinc erit

$$\begin{aligned}\Sigma^{|\infty|} &= S^{|\infty+\omega|} + \omega X^{|\infty+1|} + \frac{\omega(\omega-1)}{1\cdot 2}\left(X^{|\infty+2|} - X^{|\infty+1|}\right)\\ &= S^{|\infty|} - \frac{\omega(\omega-3)}{1\cdot 2} X^{|\infty+1|} + \frac{\omega(\omega-1)}{1\cdot 2} X^{|\infty+2|}.\end{aligned}$$

Quamobrem habebimus hanc aequationem ,

$$\begin{gathered}\Sigma + Z' + Z'' + Z''' + \cdots + Z^{|\infty|}\\ = S + X' + X'' + X''' + \cdots + X^{|\infty|} - \frac{\omega(\omega-3)}{1\cdot 2} X^{|\infty+1|} + \frac{\omega(\omega-1)}{1\cdot 2} X^{|\infty+2|},\end{gathered}$$

ex qua elicitur

$$\begin{aligned}\Sigma = S &+ X' + X'' + X''' + X'''' + \text{ etc. in infinitum}\\ &- Z' - Z'' - Z''' - Z'''' - \text{ etc. in infinitum}\\ &+ \omega X^{|\infty+1|} + \frac{\omega(\omega-1)}{1\cdot 2}\left(X^{|\infty+2|} - X^{|\infty+1|}\right).\end{aligned}$$

Termini autem isti infinitesimi ita repraesentari poterunt, ut sit

$$\begin{aligned}\Sigma = S &+ X' + X'' + X''' + X'''' + \text{etc.}\\ &- Z' - Z'' - Z''' - Z'''' - \text{etc.}\end{aligned}$$

$$+\omega X' + \omega\left\{\begin{matrix}+X'' + X''' + X'''' + X''''' + \text{etc.}\\ -X' - X'' - X''' - X'''' - \text{etc.}\end{matrix}\right\}$$

$$+\frac{\omega(\omega-1)}{1\cdot 2}X'' - \frac{\omega(\omega-1)}{1\cdot 2}X' + \frac{\omega(\omega-1)}{1\cdot 2}\left\{\begin{matrix}+X''' + X'''' + X''''' + \text{etc.}\\ -2X'' - 2X''' - 2X'''' - \text{etc.}\\ +X + X' + X'' + X''' + \text{etc.}\end{matrix}\right\}$$

unde simul lex patet, qua haec expressio erit comparata, si differentiae demum tertiae vel quartae vel ulteriores fuerint constantes.

378. Cum igitur sit, ut supra demonstravimus,

$$Z = X + \frac{\omega dX}{dx} + \frac{\omega^2 ddX}{1\cdot 2dx^2} + \frac{\omega^3 d^3X}{1\cdot 2\cdot 3dx^3} + \text{etc.},$$

si loco Z', Z'', Z''' etc. valores hinc oriundos substituamus, erit valor ipsius *S*, si loco *x* scribatur $x+\omega$, sequens:

$$\Sigma = S + \omega X' + \omega X' + \omega\left\{\begin{matrix}+X'' + X''' + X'''' + X''''' + \text{etc.}\\ -X' - X'' - X''' - X'''' - \text{etc.}\end{matrix}\right\}$$

$$+\frac{\omega(\omega-1)}{1\cdot 2}X'' - \frac{\omega(\omega-1)}{1\cdot 2}X' + \frac{\omega(\omega-1)}{1\cdot 2}\left\{\begin{matrix}+X''' + X'''' + X''''' + \text{etc.}\\ -2X'' - 2X''' - 2X'''' - \text{etc.}\\ +X + X' + X'' + X''' + \text{etc.}\end{matrix}\right\}$$

$$\begin{aligned}&-\frac{\omega}{dx}d.\left(+X' + X'' + X''' + X'''' + \text{etc.}\right)\\ &-\frac{\omega^2}{2dx^2}d^2.\left(+X' + X'' + X''' + X'''' + \text{etc.}\right)\\ &+\frac{\omega^3}{6dx^3}d^3.\left(+X' + X'' + X''' + X'''' + \text{etc.}\right)\\ &\qquad\qquad\text{etc.}\end{aligned}$$

Si ergo loco ω ponatur *dx*, prodibit differentiale completum functionis inexplicabilis propositae *S*, scilicet

$$dZ = X'\,dx + dx\left\{\begin{matrix} +X'' + X''' + X'''' + X''''' + \text{etc.} \\ -X' - X'' - X''' - X'''' - \text{etc.}\end{matrix}\right\}$$

$$-X''\frac{dx(1-dx)}{1\cdot 2} + X'\frac{dx(1-dx)}{1\cdot 2} - \frac{dx(1-dx)}{1\cdot 2}\left\{\begin{matrix} +X''' + X'''' + X''''' + \text{etc.} \\ -2X'' - 2X''' - 2X'''' - \text{etc.} \\ +X + X' + X'' + X''' + \text{etc.}\end{matrix}\right\}$$

$$+X'''\frac{dx(1-dx)(2-dx)}{1\cdot 2\cdot 3}$$

$$-2X'''\frac{dx(1-dx)(2-dx)}{1\cdot 2\cdot 3} + \frac{dx(1-dx)(2-dx)}{1\cdot 2\cdot 3}\left\{\begin{matrix} +X'''' + X''''' + \text{etc.} \\ -3X''' - 3X'''' - \text{etc.} \\ +3X'' + 3X''' + \text{etc.} \\ -X' - X'' - \text{etc.}\end{matrix}\right\}$$

$$+X'\frac{dx(1-dx)(2-dx)}{1\cdot 2\cdot 3}$$

etc.

$$-d.\left(X' + X'' + X''' + X'''' + X''''' + \text{etc.}\right)$$
$$-\tfrac{1}{2}dd.\left(X' + X'' + X''' + X'''' + X''''' + \text{etc.}\right)$$
$$-\tfrac{1}{6}d^3.\left(X' + X'' + X''' + X'''' + X''''' + \text{etc.}\right)$$
$$-\tfrac{1}{24}d^4.\left(X' + X'' + X''' + X'''' + X''''' + \text{etc.}\right)$$

etc.,

quae expressio latissime patet et, quotaecunque demum differentiae fuerint constantes, differentiale quaesitum exhibet. Accommodata enim est haec formula ad differentias constantes et simul lex patet, si forte ulterius progredi necesse sit.

379. Quodsi series $A + B + C + D + \text{etc.}$, ex qua formatur functio inexplicabilis

$$\begin{matrix} & 1 & & 2 & & 3 & & 4 & & \cdots & & x \\ S = & A & + & B & + & C & + & D & + & \cdots & + & X, \end{matrix}$$

ita fuerit comparata, ut eius termini infinitesimi evanescant, tum, uti iam notavimus, erit

$$
\begin{aligned}
dS = & -d.(X' + X'' + X''' + X'''' + \text{etc.}) \\
& -\tfrac{1}{2}dd.(X' + X'' + X''' + X'''' + \text{etc.}) \\
& -\tfrac{1}{6}d^3.(X' + X'' + X''' + X'''' + \text{etc.}) \\
& -\tfrac{1}{24}d^4.(X' + X'' + X''' + X'''' + \text{etc.}) \\
& \text{etc.}
\end{aligned}
$$

Sin autem illius seriei termini infinitesimi non sint $=0$, sed tamen differentias habeant evanescentes, tum ad istam expressionem insuper addi debet

$$
dx\left\{\begin{array}{l} \quad + X'' + X''' + X'''' + X''''' + \text{ etc.} \\ X' \\ \quad - X' - X'' - X''' - X'''' - \text{etc.} \end{array}\right\}
$$

Verum si terminorum infinitesimorum huius seriei $A+B+C+D+\text{etc.}$ differentiae demum secundae evanescant, tum praeterea adiici oportet

$$
\frac{dx(dx-1)}{1\cdot 2}\left\{\begin{array}{l} \qquad + X''' + X'''' + X''''' + \text{ etc.} \\ +X'' \\ \qquad -2X'' - 2X'' - 2X'''' - \text{ etc.} \\ -X' \\ \qquad + X' + X'' + X''' + \text{ etc.} \end{array}\right\}
$$

Atque si memoratorum terminorum infinitesimorum differentiae demum tertiae fuerint evanescentes, tum praeter has iam exhibitas expressiones insuper addi debet

$$
\frac{dx(dx-1)(dx-2)}{1\cdot 2\cdot 3}\left\{\begin{array}{l} \qquad + X'''' + X''''' + X'''''' + \text{etc.} \\ +X'' \\ \qquad -3X''' - 3X'''' - 3X''''' - \text{etc.} \\ -2X'' \\ \qquad +3X'' + 3X''' + 3X'''' + \text{etc.} \\ +X' \\ \qquad - X' - X'' - X''' - \text{etc.} \end{array}\right\}
$$

Sicque porro expressiones insuper addendae erunt comparatae, si ulteriores demum differentiae terminorum infinitesimorum seriei $A+B+C+D+\text{etc.}$ evanescant. Hincque adeo, quaecunque series assumatur, dummodo eius termini infinitesimi tandem ad differentias evanescentes perducantur, functionis inexplicabilis ex ea formatae differentiale definiri poterit.

380. Si ponatur $x=0$, fiet $X'=A,\quad X''=B,\quad X'''=C$ etc. Quare uti $A+B+C+D+$ etc. est series, cuius terminus generalis est X, si ex terminis generalibus

$$\frac{dX}{dx},\quad \frac{ddX}{2dx^2},\quad \frac{d^3X}{6dx^3},\quad \frac{d^4X}{24dx^4}\quad \text{etc.}$$

simili modo formentur series infinitae earumque summae denotentur per litteras $\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D}$ etc. respective, summa ω terminorum seriei

$$A+B+C+D+\text{etc.}$$

ita exprimetur, ut perinde sit, sive ω sit numerus integer sive secus. Scribamus ergo x pro w, ut sit

$$\begin{matrix} & 1 & & 2 & & 3 & & 4 & & \cdots & & x \\ S= & A & + & B & + & C & + & D & + & \cdots & + & X, \end{matrix}$$

atque si huius seriei termini infinitesimi evanescant, erit

$$S=-\mathfrak{B}x-\mathfrak{C}x^2-\mathfrak{D}x^3-\mathfrak{E}x^4-\text{etc.}$$

At si termini infinitesimi differentias saltem primas habeant constantes, tum ad hunc valorem insuper addi debet hic

$$x\left\{\begin{matrix} & +B+C+D+E+\text{etc.} \\ A & \\ & -A-\ B-\ C-D-\text{etc.} \end{matrix}\right\}$$

Sin autem illorum terminorum infinitesimorum differentiae demum secundae evanescant, tum praeterea addi debet

$$\frac{x(x-1)}{1\cdot 2}\left\{\begin{matrix} & +\ C+\ D+\ \ E+\ \ F+\text{etc.} \\ +B & \\ & -2B-2C-2D-2E-\text{etc.} \\ -A & \\ & +A\ +\ B\ +\ \ C+\ \ D+\text{etc.} \end{matrix}\right\}$$

Si differentiae demum tertiae fuerint evanescentes, tum insuper adiici debet haec series infinita

$$\frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}\left\{\begin{array}{l} \quad\; + D + \;\; E \;+\;\; F + \; G + \text{etc.} \\ + C \\ \quad\; - 3C - 3D - 3E - 3F - \text{etc.} \\ -2B \\ \quad\; + 3B + 3C + 3D + 3E + \text{etc.} \\ + A \\ \quad\; - A \;-\; B \;-\;\; C \;- D - \text{etc.} \\ \qquad\qquad\quad \text{etc.} \end{array}\right\}$$

381. Accommodemus haec quoque ad alterum functionum inexplicabilium genus, quae constant continuo producto terminorum aliquot seriei propositae
$A + B + C + D + \text{etc.}$, sitque

$$\begin{array}{c} 1 \;\; 2 \;\; 3 \;\; 4 \cdots \; x \\ S = A \cdot B \cdot C \cdot D \cdots X , \end{array}$$

et quaeratur primo valor Σ, in quem S transmutatur, si loco x scribatur $x + \omega$; ponamus autem ut ante esse Z terminum seriei $A + B + C + D + \text{etc.}$, cuius index sit $= x + \omega$, uti X respondet indici x. Quo ergo hunc casum ad praecedentem reducamus, sumamus logarithmos eritque

$$lS = lA + lB + lC + lD + \cdots + lX .$$

Quodsi iam huius seriei termini infinitesimi evanescant, erit eandem methodum, qua ante usi sumus, adhibendo

$$\begin{aligned} l\Sigma = \;& lS + lX' + lX'' + lX''' + \text{etc.} \\ & \quad - lZ' - \; lZ'' - lZ''' - \text{etc.} \end{aligned}$$

hincque ad numeros regrediendo erit

$$\Sigma = S \cdot \tfrac{X'}{Z'} \cdot \tfrac{X''}{Z''} \cdot \tfrac{X'''}{Z'''} \cdot \tfrac{X''''}{Z''''} \cdot \text{etc.};$$

quae ergo expressio valet, si seriei A, B, C, D etc. termini infinitesimi unitati aequentur. Sin autem logarithmi terminorum infinitesimorum huius seriei non evanescant, at tamen differentias habeant evanescentes, tum ad illam seriem, quam pro lS invenimus, insuper addi debet haec series

$$\omega l X' + \omega\left(l\tfrac{X''}{X'} + l\tfrac{X'''}{X''} + l\tfrac{X''''}{X'''} + \text{etc.}\right)$$

sicque numeris sumendis habebitur

$$\Sigma = S \cdot X'^{\omega} \cdot \frac{X''^{\omega} X'^{1-\omega}}{Z'} \cdot \frac{X'''^{\omega} X''^{1-\omega}}{Z''} \cdot \frac{X''''^{\omega} X'''^{1-\omega}}{Z'''} \cdot \text{etc.}$$

382. Quodsi ergo ponamus $x=0$, quo casu fit $S=1$ et $X'=A,\ X''=B,\ X'''=C$ etc., Σ denotabit productum ω terminorum huius seriei A,B,C,D etc. Si igitur pro ω scribamus x, ut Σ obtineat valorem, quem ante ipsi S tribueramus, ita ut sit

$$\begin{array}{ccccccccccc} & & 1 & & 2 & & 3 & & 4 & \cdots & x \\ S & = & A & + & B & + & C & + & D & +\cdots+ & X, \end{array}$$

quia nunc Z', Z'', Z''' etc. abeunt in X', X'', X''' etc., si logarithmi terminorum infinitesimorum istius seriei A, B, C, D, E etc. evanescant, exprimetur
S hoc modo

$$S = \tfrac{A}{X'} \cdot \tfrac{B}{X''} \cdot \tfrac{C}{X'''} \cdot \tfrac{D}{X''''} \cdot \tfrac{E}{X'''''} \cdot \text{etc.}$$

Sin autem differentiae demum logarithmorum terminorum infinitesimorum seriei A, B, C, D etc. evanescant, tum ista functio S sequenti modo exprimetur, ut sit

$$S = A^{x} \cdot \tfrac{B^{x}A^{1-x}}{X'} \cdot \tfrac{C^{x}B^{1-x}}{X''} \cdot \tfrac{D^{x}C^{1-x}}{X'''} \cdot \tfrac{E^{x}D^{1-x}}{X''''} \cdot \text{etc.};$$

si illorum logarithmorum differentiae secundae demum sint evanescentes, ex praecedentibus facile colligitur, cuiusmodi factores insuper addi debeant; quem casum, cum vix occurrere soleat, hic praetermittamus. Ceterum usum harum expressionum in interpolationis negotio capite sequente ostendam.

383. Hic igitur cum differentiatio huiusmodi functionum inexplicabilium potissimum sit proposita, investigemus differentiale huius functionis

$$S = A \cdot B \cdot C \cdot D \cdots X.$$

Ad hoc resumamus aequationem ante inventam

$$\begin{aligned} l\Sigma = {} & lS + lX' + lX'' + lX''' + \text{etc.} \\ & - lZ' - lZ'' - lZ''' - \text{etc.} \end{aligned}$$

et cum lZ oriatur ex lX, si loco x ponatur $x+\omega$, erit

$$lZ = lX + \frac{\omega}{dx} d.lX + \frac{\omega^2}{2dx^2} dd.lX + \frac{\omega^3}{6dx^3} d^3.lX + \text{etc.}$$

quibus valoribus pro lZ', lZ'', lZ''' etc. substitutis habebitur

$$\begin{aligned} l\Sigma = lS &- \frac{\omega}{dx} d.\left(lX' + lX'' + lX''' + lX'''' + \text{etc.}\right) \\ &- \frac{\omega^2}{2dx^2} dd.\left(lX' + lX'' + lX''' + lX'''' + \text{etc.}\right) \\ &- \frac{\omega^3}{6dx^3} d^3.\left(lX' + lX'' + lX''' + lX'''' + \text{etc.}\right) \\ &\text{etc.} \end{aligned}$$

Ponatur nunc $\omega = dx$ fietque $l\Sigma = lS + d.lS$ ideoque erit

$$\begin{aligned} \frac{dS}{S} = &-d.\left(lX' + lX'' + lX''' + lX'''' + \text{etc.}\right) \\ &- \frac{1}{2} dd.\left(lX' + lX'' + lX''' + lX'''' + \text{etc.}\right) \\ &- \frac{1}{6} d^3.\left(lX' + lX'' + lX''' + lX'''' + \text{etc.}\right) \\ &\text{etc.,} \end{aligned}$$

quae formula valet, si logarithmi terminorum infinitesimorum seriei A, B, C, D etc. evanescant; sin autem ipsi non evanescant, attamen differentias habeant evanescentes, tum ad praecedentem differentialis completi expressionem insuper addi debet haec series

$$dxlX' + dx\left(l\frac{X''}{X'} + l\frac{X'''}{X''} + l\frac{X''''}{X'''} + \text{etc.}\right),$$

ut obtineatur differentiale completum.

384. Idem adhuc alio modo praestari potest. Ponatur $x = 0$, quo casu abit lS in 0. Tum formentur series, quarum termini generales sint

$$lX, \quad \frac{d.lX}{dx}, \quad \frac{dd.lX}{2dx^2}, \quad \frac{d^3.lX}{6dx^3} \quad \text{etc.},$$

harumque serierum infinitarum summae sint respective $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc. Scribatur x pro ω, ut sit $\Sigma = S$, eritque

$$lS = -\mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \text{etc.},$$

siquidem logarithmi terminorum infinitesimorum seriei A, B, C, D etc., cuius terminus generalis est X, evanescant; at si horum logarithmorum differentiae demum evanescant, erit

$$lS = xlA + x\left(l\tfrac{B}{A} + l\tfrac{C}{B} + l\tfrac{D}{C} + l\tfrac{E}{D} + \text{etc.}\right)$$
$$-\mathfrak{B}x - \mathfrak{C}x^2 - \mathfrak{D}x^3 - \mathfrak{E}x^4 - \text{etc.}$$

Hincque adeo differentiale ipsius lS erit

$$\tfrac{dS}{S} = dxlA + dx\left(l\tfrac{B}{A} + l\tfrac{C}{B} + l\tfrac{D}{C} + l\tfrac{E}{D} + \text{etc.}\right)$$
$$-\mathfrak{B}dx - 2\mathfrak{C}xdx - 3\mathfrak{D}x^2dx - 4\mathfrak{E}x^3dx - \text{etc.}$$

At si differentiale completum desideretur, erit id

$$\tfrac{dS}{S} = dxlA + dx\left(l\tfrac{B}{A} + l\tfrac{C}{B} + l\tfrac{D}{C} + l\tfrac{E}{D} + \text{etc.}\right)$$
$$-\mathfrak{B}dx - \mathfrak{C}\left(2xdx + dx^2\right) - \mathfrak{D}\left(3xxdx + 3xdx^2 + dx^3\right) - \text{etc.}$$

Ad quarum formularum. usum ostendendum sequentia exempla adiicimus, quae utroque modo resolvemus.

EXEMPLUM 1

Invenire differentiale huius functionis inexplicabilis

$$S = \tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdot\tfrac{7}{8}\cdot\tfrac{2x-1}{2x}.$$

Hic ante omnia notandum est terminos infinitesimos horum factorum abire in unitates ideoque eorum logarithmos evanescere. Cum igitur sit $X = \tfrac{2x-1}{2x}$, erit

$$X' = \tfrac{2x+1}{2x+2}, \quad X'' = \tfrac{2x+3}{2x+4}, \quad X''' = \tfrac{2x+5}{2x+6} \quad \text{etc.}$$

et generaliter

$$X^{|n|} = \tfrac{2x+2n-1}{2x+2n};$$

unde erit

$$lX^{|n|} = l(2x+2n-1) - l(2x+2n)$$
$$d.lX^{|n|} = \frac{2dx}{2x+2n-1} - \frac{2dx}{2x+2n}$$
$$dd.lX^{|n|} = -\frac{4dx^2}{(2x+2n-1)^2} + \frac{4dx^2}{(2x+2n)^2}$$
$$d^3.lX^{|n|} = +\frac{2\cdot2\cdot4dx^3}{(2x+2n-1)^3} - \frac{2\cdot2\cdot4dx^3}{(2x+2n)^3}$$
$$d^4.lX^{|n|} = +\frac{2\cdot2\cdot4\cdot6dx^4}{(2x+2n-1)^4} - \frac{2\cdot2\cdot4\cdot6dx^4}{(2x+2n)^4}$$
etc.;

unde erit differentiale completum

$$\frac{dS}{S} = -2dx\left\{\begin{matrix} \frac{1}{2x+1} + \frac{1}{2x+3} + \frac{1}{2x+5} + \text{etc.} \\ -\frac{1}{2x+2} - \frac{1}{2x+4} - \frac{1}{2x+6} - \text{etc.} \end{matrix}\right\}$$
$$+\frac{8}{3}dx^3\left\{\begin{matrix} \frac{1}{(2x+1)^2} + \frac{1}{(2x+3)^2} + \frac{1}{(2x+5)^2} + \text{etc.} \\ -\frac{1}{(2x+2)^2} - \frac{1}{(2x+4)^2} - \frac{1}{(2x+6)^2} - \text{etc.} \end{matrix}\right\}$$
$$-\frac{4}{2}dx^2\left\{\begin{matrix} \frac{1}{(2x+1)^3} + \frac{1}{(2x+3)^3} + \frac{1}{(2x+5)^3} + \text{etc.} \\ -\frac{1}{(2x+2)^3} - \frac{1}{(2x+4)^3} - \frac{1}{(2x+6)^3} - \text{etc.} \end{matrix}\right\}$$

Quodsi autem tantum differentiale primum quaeratur, erit id

$$\frac{dS}{S} = -2dx\cdot\left(\frac{1}{(2x+1)(2x+2)} + \frac{1}{(2x+3)(2x+4)} + \frac{1}{(2x+5)(2x+6)} + \text{etc.}\right),$$

quod idem altera methodo § 394 tradita ita investigatur. Cum sit

$$lX = l\frac{2x-1}{2x},$$

erit

$$\frac{d.lX}{dx} = \frac{2}{2x-1} - \frac{1}{x}, \quad \frac{dd.lX}{2dx^2} = -\frac{2}{(2x-1)^2} + \frac{1}{2xx},$$
$$\frac{d^3.lX}{6dx^3} = \frac{8}{3(2x-1)^3} - \frac{1}{3x^3} \quad \text{etc.}$$

ideoque fiet

$$\mathfrak{A} = l\tfrac{1}{2} + l\tfrac{3}{4} + l\tfrac{5}{6} + l\tfrac{7}{8} + \text{ etc.}$$

$$\mathfrak{B} = \left\{ \begin{array}{l} \tfrac{2}{1} + \tfrac{2}{3} + \tfrac{2}{5} + \tfrac{2}{7} + \tfrac{2}{9} + \text{etc.} \\ -\tfrac{2}{2} - \tfrac{2}{4} - \tfrac{2}{6} - \tfrac{2}{8} - \tfrac{2}{10} - \text{etc.} \end{array} \right\} = 2l2$$

$$\mathfrak{C} = -\tfrac{4}{2} \left\{ \begin{array}{l} \tfrac{1}{1} + \tfrac{1}{3^2} + \tfrac{1}{5^2} + \tfrac{1}{7^2} + \text{etc.} \\ -\tfrac{1}{2^2} - \tfrac{1}{4^2} - \tfrac{1}{6^2} - \tfrac{1}{8^2} - \text{etc.} \end{array} \right\}$$

$$\mathfrak{D} = \tfrac{8}{3} \left\{ \begin{array}{l} \tfrac{1}{1} + \tfrac{1}{3^3} + \tfrac{1}{5^3} + \tfrac{1}{7^3} + \text{etc.} \\ -\tfrac{1}{2^3} - \tfrac{1}{4^3} - \tfrac{1}{6^3} - \tfrac{1}{8^3} - \text{etc.} \end{array} \right\}$$

$$\mathfrak{E} = -\tfrac{16}{4} \left\{ \begin{array}{l} \tfrac{1}{1} + \tfrac{1}{3^4} + \tfrac{1}{5^4} + \tfrac{1}{7^4} + \text{etc.} \\ -\tfrac{1}{2^4} - \tfrac{1}{4^4} - \tfrac{1}{6^4} - \tfrac{1}{8^4} - \text{etc.} \end{array} \right\}$$

etc.

sive erit

$$\mathfrak{B} = +\tfrac{2}{1}\left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \text{etc.}\right)$$
$$\mathfrak{C} = -\tfrac{4}{2}\left(1 - \tfrac{1}{2^2} + \tfrac{1}{3^2} - \tfrac{1}{4^2} + \tfrac{1}{5^2} - \text{etc.}\right)$$
$$\mathfrak{D} = +\tfrac{8}{3}\left(1 - \tfrac{1}{2^3} + \tfrac{1}{3^3} - \tfrac{1}{4^3} + \tfrac{1}{5^3} - \text{etc.}\right)$$
$$\mathfrak{E} = -\tfrac{16}{4}\left(1 - \tfrac{1}{2^4} + \tfrac{1}{3^4} - \tfrac{1}{4^4} + \tfrac{1}{5^4} - \text{etc.}\right)$$

etc.

Quibus valoribus inventis substitutis erit

$$\begin{aligned} dS = &-2dx\left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \text{etc.}\right) \\ &+4xdx\left(1 - \tfrac{1}{2^2} + \tfrac{1}{3^2} - \tfrac{1}{4^2} + \tfrac{1}{5^2} - \text{etc.}\right) \\ &-8x^2\,dx\left(1 - \tfrac{1}{2^3} + \tfrac{1}{3^3} - \tfrac{1}{4^3} + \tfrac{1}{5^3} - \text{etc.}\right) \\ &+16x^3dx\left(1 - \tfrac{1}{2^4} + \tfrac{1}{3^4} - \tfrac{1}{4^4} + \tfrac{1}{5^4} - \text{etc.}\right) \end{aligned}$$

etc.

Si igitur sit $x = 0$, quo casu fit $lS = 0$ et $S = 1$, erit $dS = -2dxl2$.

EXEMPLUM 2

Invenire differentiale huius functionis inexplicabilis

$$S = 1 \cdot 2 \cdot 3 \cdot 4 \cdots x.$$

Huius seriei 1, 2, 3, 4 etc. termini in infinitum ita crescunt, ut logarithmorum differentiae evanescant; est enim

$$l(\infty + 1) - l\infty = l\left(1 + \tfrac{1}{\infty}\right) = \tfrac{1}{\infty} = 0..$$

Cum igitur sit $X = x$, erit

$$X' = x + 1,\ \ X'' = x + 2,\ \ X''' = x + 3 \ \text{ etc.};$$

porro autem ob $lX = lx$ fiet

$$d.lX = \tfrac{dx}{x},\ \ dd.lX = -\tfrac{dx^2}{x^2},\ \ d^3.lX = \tfrac{2dx^3}{x^3},\ \ d^4.lX = -\tfrac{2\cdot 3dx^4}{x^4} \ \text{ etc.};$$

unde si logarithmi ultimi evanescerent, foret

$$\begin{aligned} \tfrac{dS}{S} = -dx&\left(\tfrac{1}{x+1} + \tfrac{1}{x+2} + \tfrac{1}{x+3} + \tfrac{1}{x+4} + \text{etc.}\right) \\ +\tfrac{dx^2}{2}&\left(\tfrac{1}{(x+1)^2} + \tfrac{1}{(x+2)^2} + \tfrac{1}{(x+3)^2} + \tfrac{1}{(x+4)^2} + \text{etc.}\right) \\ -\tfrac{dx^3}{3}&\left(\tfrac{1}{(x+1)^3} + \tfrac{1}{(x+2)^3} + \tfrac{1}{(x+3)^3} + \tfrac{1}{(x+4)^3} + \text{etc.}\right) \\ &\text{etc.} \end{aligned}$$

At cum differentiae demum logarithmorum evanescant, insuper addi debet haec expressio

$$dxl(x+1) + dx\left(l\tfrac{x+2}{x+1} + l\tfrac{x+3}{x+2} + l\tfrac{x+4}{x+3} + l\tfrac{x+5}{x+4} + \text{etc.}\right).$$

Quia vero est

$$\begin{aligned} l\tfrac{x+2}{x+1} &= \tfrac{1}{x+1} - \tfrac{1}{2(x+1)^2} + \tfrac{1}{3(x+1)^3} - \tfrac{1}{4(x+1)^4} + \text{etc.}, \\ l\tfrac{x+3}{x+2} &= \tfrac{1}{x+2} - \tfrac{1}{2(x+2)^2} + \tfrac{1}{3(x+2)^3} - \tfrac{1}{4(x+2)^4} + \text{etc.} \\ &\text{etc.} \end{aligned}$$

erit verum differentiale completum

$$\begin{aligned}\frac{dS}{S} = dxl(x+1) - \tfrac{1}{2}\left(dx - dx^2\right)&\left(\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+4)^2} + \text{etc.}\right)\\ - \tfrac{1}{3}\left(dx - dx^3\right)&\left(\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \frac{1}{(x+4)^3} + \text{etc.}\right)\\ + \tfrac{1}{4}\left(dx - dx^4\right)&\left(\frac{1}{(x+1)^4} + \frac{1}{(x+2)^4} + \frac{1}{(x+3)^4} + \frac{1}{(x+4)^4} + \text{etc.}\right)\\ - \tfrac{1}{5}\left(dx - dx^5\right)&\left(\frac{1}{(x+1)^5} + \frac{1}{(x+2)^5} + \frac{1}{(x+3)^5} + \frac{1}{(x+4)^5} + \text{etc.}\right)\\ &\text{etc.}\end{aligned}$$

Sin autem altero modo differentiale hoc exprimere velimus, quia est

$$lX = lx,\quad \frac{d.lX}{dx} = \frac{1}{x},\quad \frac{dd.lX}{2dx^2} = -\frac{1}{2x^2},\quad \frac{d^3.lX}{6dx^3} = \frac{1}{3x^3},\quad \frac{d^4.lX}{24dx^4} = -\frac{1}{x^4}\quad \text{etc.},$$

habebuntur sequentes series

$$\begin{aligned}\mathfrak{A} &= \quad l1 + l2 + l3 + l4 + l5 + \text{etc.}\\ \mathfrak{B} &= \ 1\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \text{etc.}\right)\\ \mathfrak{C} &= -\tfrac{1}{2}\left(1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \tfrac{1}{5^2} + \text{etc.}\right)\\ \mathfrak{D} &= +\tfrac{1}{3}\left(1 + \tfrac{1}{2^3} + \tfrac{1}{3^3} + \tfrac{1}{4^3} + \tfrac{1}{5^3} + \text{etc.}\right)\\ \mathfrak{E} &= -\tfrac{1}{4}\left(1 + \tfrac{1}{2^4} + \tfrac{1}{3^4} + \tfrac{1}{4^4} + \tfrac{1}{5^4} + \text{etc.}\right)\\ &\qquad\qquad \text{etc.}\end{aligned}$$

Hinc ob $lA = l1 = 0$ fiet ex § 384

$$\begin{aligned} lS = & \ x\left(l\tfrac{2}{1} + l\tfrac{3}{2} + l\tfrac{4}{3} + l\tfrac{5}{4} + \text{etc.}\right) \\ & - x\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \text{etc.}\right) \\ & + \tfrac{1}{2}x^2\left(1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \text{etc.}\right) \\ & - \tfrac{1}{3}x^3\left(1 + \tfrac{1}{2^3} + \tfrac{1}{3^3} + \tfrac{1}{4^3} + \text{etc.}\right) \\ & + \tfrac{1}{4}x^4\left(1 + \tfrac{1}{2^4} + \tfrac{1}{3^4} + \tfrac{1}{4^4} + \text{etc.}\right) \\ & \text{etc.} \end{aligned}$$

Binae autem primae series, per quas x est multiplicatum, etiamsi utraque habeat summam infinitam, tamen ambae simul summam habent finitam. Si enim utriusque n termini capiantur, prodibit

$$l(n+1) - 1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{4} - \cdots - \tfrac{1}{n}.$$

At supra (§142a) invenimus esse

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots + \tfrac{1}{n} = \text{Const.} + ln + \tfrac{1}{2n} - \tfrac{\mathfrak{A}}{2n^2} + \tfrac{\mathfrak{B}}{4n^4} - \text{etc.}$$

haecque constans prodit $= 0,5772156649015325$. Quodsi ergo ponatur $n = \infty$, erit

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots + \tfrac{1}{\infty} = \text{Const.} + l\infty,$$

unde binarum illarum serierum in infinitum continuatarum valor erit

$$= l(\infty + 1) - \text{Const.} - l\infty = -\ \text{Const.}$$

Ex quo erit

$$\begin{aligned} lS = & -x \cdot 0,5772156649015325 \\ & + \tfrac{1}{2}x^2\left(1 + \tfrac{1}{2^2} + \tfrac{1}{3^2} + \tfrac{1}{4^2} + \text{etc.}\right) \\ & - \tfrac{1}{3}x^3\left(1 + \tfrac{1}{2^3} + \tfrac{1}{3^3} + \tfrac{1}{4^3} + \text{etc.}\right) \\ & + \tfrac{1}{4}x^4\left(1 + \tfrac{1}{2^4} + \tfrac{1}{3^4} + \tfrac{1}{4^4} + \text{etc.}\right) \\ & \text{etc.} \end{aligned}$$

unde differentialia cuiusque ordinis facile reperiuntur. Erit enim

$$\frac{dS}{S} = -dx \cdot 0{,}5772156649015325$$
$$+xdx\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\text{etc.}\right)$$
$$-x^2dx\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\text{etc.}\right)$$
$$+x^3dx\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\text{etc.}\right)$$
$$\text{etc.}$$

At si hae series in unam colligantur, erit

$$\frac{dS}{S} = -dx \cdot 0{,}5772156649015325 + \frac{xdx}{1(1+x)} + \frac{xdx}{2(2+x)} + \frac{xdx}{3(3+x)} + \frac{xdx}{4(4+x)} + \text{etc.}$$

Quare si sit $x = 0$, fiet

$$\frac{dS}{S} = -dx \cdot 0{,}5772156649015325.$$

Ex priori vero expressione hoc casu erit

$$\frac{dS}{S} = -\frac{1}{2}dx\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{etc.}\right)$$
$$+\frac{1}{3}dx\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\text{etc.}\right)$$
$$-\frac{1}{4}dx\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.}\right)$$
$$+\frac{1}{5}dx\left(1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\text{etc.}\right)$$
$$\text{etc.}$$

385. Hinc ergo etiam huiusmodi functionum inexplicabilium differentialia quovis casu speciali exhiberi possunt, propterea quod hic differentialia completa eruimus. Quamobrem si tales functiones ingrediantur in expressiones, quae indeterminatae videntur, cuiusmodi capite praecedente tractavimus, valores eadem methodo definiri poterunt, uti ex adiunctis exemplis intelligetur.

EXEMPLUM 1

Determinare valorem huius expressionis

$$\frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{x}}{x(x-1)} - \frac{1}{(x-1)(2x-1)}$$

eo casu, quando ponitur $x = 1$.

Ponamus

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{x} = S;$$

erit ex § 372

$$\begin{aligned} S = x&\left(1+\tfrac{1}{2^2}+\tfrac{1}{3^2}+\tfrac{1}{4^2}+\text{etc.}\right) \\ -x^2&\left(1+\tfrac{1}{2^3}+\tfrac{1}{3^3}+\tfrac{1}{4^3}+\text{etc.}\right) \\ +x^3&\left(1+\tfrac{1}{2^4}+\tfrac{1}{3^4}+\tfrac{1}{4^4}+\text{etc.}\right) \\ &\text{etc.} \end{aligned}$$

seu cum sit quoque

$$S = 1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\text{etc.}-\tfrac{1}{1+x}-\tfrac{1}{2+x}-\tfrac{1}{3+x}-\tfrac{1}{4+x}-\tfrac{1}{5+x}-\text{etc.},$$

si quivis terminus superioris seriei cum praecedente inferioris combinetur, prodibit

$$S = 1+\tfrac{x-1}{2(1+x)}+\tfrac{x-1}{3(2+x)}+\tfrac{x-1}{4(3+x)}+\text{etc.},$$

quae expressio, quoniam poni debet $x=1$, est commodior. Sit ergo $x=1+\omega$ fietque

$$S = 1+\tfrac{\omega}{2(2+\omega)}+\tfrac{\omega}{3(3+\omega)}+\tfrac{\omega}{4(4+\omega)}+\text{etc.}$$

sive

$$\begin{aligned} S = 1+\omega&\left(\tfrac{1}{2^2}+\tfrac{1}{3^2}+\tfrac{1}{4^2}+\tfrac{1}{5^2}+\text{etc.}\right) = 1+\mathfrak{B}\omega \\ -\omega^2&\left(\tfrac{1}{2^3}+\tfrac{1}{3^3}+\tfrac{1}{4^3}+\tfrac{1}{5^3}+\text{etc.}\right) \qquad -\mathfrak{C}\omega^2 \\ +\omega^3&\left(\tfrac{1}{2^4}+\tfrac{1}{3^4}+\tfrac{1}{4^4}+\tfrac{1}{5^4}+\text{etc.}\right) \qquad +\mathfrak{D}\omega^3 \\ &\text{etc.} \qquad\qquad\qquad\qquad \text{etc.} \end{aligned}$$

Tota ergo expressio posito $x=1+\omega$ abibit in hanc

$$\frac{1+\mathfrak{B}\omega-\mathfrak{C}\omega^2+\mathfrak{D}\omega^3-\text{etc.}}{\omega(1+\omega)}-\frac{1}{\omega(1+2\omega)}$$

seu

$$\frac{\omega+\mathfrak{B}\omega+2\mathfrak{B}\omega^2-\mathfrak{C}\omega^2-\text{etc.}}{\omega(1+\omega)(1+2\omega)} = \frac{1+\mathfrak{B}+2\mathfrak{B}\omega-\mathfrak{C}\omega-\text{etc.}}{(1+\omega)(1+2\omega)}.$$

Ponatur nunc $\omega=0$ atque expressionis propositae valor casu $x=1$ erit

$$=1+\mathfrak{B}=1+\tfrac{1}{2^2}+\tfrac{1}{3^2}+\tfrac{1}{4^2}+\text{etc.};$$

quae series cum sit $=\tfrac{1}{2}\pi^2$, sequitur valorem quaesitum esse $=\tfrac{1}{2}\pi^2$.

EXEMPLUM 2

lnvenire valorem huius expressionis

$$\frac{2x-xx}{(x-1)^2}+\frac{\pi\pi x}{6(x-1)}-\frac{(2x-1)\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{x}\right)}{x(x-1)^2}$$

casu, quo ponitur $x=1$.

Ponatur $1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots+\tfrac{1}{x}=S$ statuaturque $x=1+\omega$*;* fiet, ut in exemplo praecedente invenimus,

$$S=1+\mathfrak{B}\omega-\mathfrak{C}\omega^2+\mathfrak{D}\omega^3-\text{etc.}$$

existente

$$\mathfrak{B}=\tfrac{1}{2^2}+\tfrac{1}{3^2}+\tfrac{1}{4^2}+\tfrac{1}{5^2}+\text{etc.}=\tfrac{1}{6}\pi\pi-1$$
$$\mathfrak{C}=\tfrac{1}{2^3}+\tfrac{1}{3^3}+\tfrac{1}{4^3}+\tfrac{1}{5^3}+\text{etc.}$$
$$\mathfrak{D}=\tfrac{1}{2^4}+\tfrac{1}{3^4}+\tfrac{1}{4^4}+\tfrac{1}{5^4}+\text{etc.}$$
etc.

Posito ergo $x=1+\omega$ expressio proposita induet hanc formam

$$\frac{1-\omega\omega}{\omega\omega}+\frac{(1+\mathfrak{B})(1+\omega)}{\omega}-\frac{(1+2\omega)\left(1+\mathfrak{B}\omega-\mathfrak{C}\omega^2+\mathfrak{D}\omega^3-\text{ etc.}\right)}{(1+\omega)\omega^2},$$

quae ad eandem denominationem $2(1+\omega)$ perducta fit

$$\frac{1+\omega-\omega^2-\omega^3+\omega+2\omega^2+\omega^3+\mathfrak{B}\omega(1+2\omega+\omega\omega)-1-\mathfrak{B}\omega+\mathfrak{C}\omega^2-\mathfrak{D}\omega^3-2\omega-2\mathfrak{B}\omega^2+2\mathfrak{C}\omega^3-\text{etc.}}{\omega^2(1+\omega)},$$

quae reducitur ad hanc formam

$$\frac{\omega^2+\mathfrak{C}\omega^2+\mathfrak{B}\omega^3+2\mathfrak{C}\omega^3-\mathfrak{D}\omega^3+\text{ etc.}}{\omega^2(1+\omega)}.$$

Fiat nunc $\omega=0$ atque prodibit $1+\mathfrak{C}$. Quocirca expressionis propositae valor casu $x=1$ erit $=1+\mathfrak{C}$ ideoque per hanc seriem exprimetur

$$1+\tfrac{1}{2^3}+\tfrac{1}{3^3}+\tfrac{1}{4^3}+\tfrac{1}{5^3}+\text{etc.};$$

cuius summa cum neque per logarithmos neque per peripheriam circuli exhiberi possit, valor quaesitus etiamnum alio modo finite assignari non potest. Ex his ergo duobus exemplis usus, quem differentiatio functionum inexplicabilium in doctrina serierum habere potest, satis luculenter perspicitur.

386. In methodo hic tradita functiones inexplicabiles differentiandi assumsimus seriei A, B, C, D, E etc. terminos infinitesimos vel esse $=0$ vel differentias tandem evanescentes habere; quorum si neutrum contingat, ista methodo uti non licebit. Hanc ob rem aliam exponam methodum huic conditioni non adstrictam, quam summatio generalis serierum ex termino generali petita et supra [cap. V] fusius explicata suppeditat. Denotent igitur litterae $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}$ etc. numeros BERNOULLIANOS § 122 exhibitos sitque functio inexplicabilis proposita haec

$$\begin{array}{ccccccccc} & 1 & & 2 & & 3 & & 4 & \cdots & x \\ S = & A & + & B & + & C & + & D & +\cdots+ & X, \end{array}$$

et quia supra (§130) ostendimus fore

$$S = \int Xdx + \tfrac{1}{2}X + \frac{\mathfrak{A}dX}{1\cdot 2dx} - \frac{\mathfrak{B}d^3X}{1\cdot 2\cdot 3\cdot 4dx^3} + \frac{\mathfrak{C}d^5X}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6dx^5} - \text{etc.},$$

hinc facile erit istius functionis S differentiale exhibere; erit enim

$$dS = Xdx + \tfrac{1}{2}dX + \frac{\mathfrak{A}ddX}{1\cdot 2dx} - \frac{\mathfrak{B}d^4X}{1\cdot 2\cdot 3\cdot 4dx^3} + \frac{\mathfrak{C}d^6X}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6dx^5} - \text{etc.}$$

387. Sin autem progressio proposita coniuncta sit cum geometrica, quo casu termini eius infinitesimi nunquam ad differentias constantes reducuntur ac propterea methodus prior locum invenit nullum, tum methodus §174 tradita medelam afferet. Si enim proposita sit haec functio

$$S = Ap + Bp^2 + Cp^3 + Dp^4 + \cdots + Xp^x,$$

quaerantur valores litterarum α, β, γ, δ etc., ut sit

$$\frac{p-1}{p-e^u} = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \text{etc.},$$

quibus inventis, uti eos § 173 exhibuimus, erit

$$S = \frac{p}{p-1}\cdot p^x\left(X - \frac{\alpha dX}{dx} + \frac{\beta ddX}{dx^2} - \frac{\gamma d^3X}{dx^3} + \frac{\delta d^3X}{dx^4} - \text{etc.}\right) \pm \text{Constante},$$

quae summam reddat $=0$, si ponatur $x=0$, seu quae cuiquam alii casui satisfaciat. Sumto ergo differentiali haec constans ex computo abibit eritque

$$dS=\frac{p}{p-1}\cdot p^x dxlp\left(X-\frac{\alpha dX}{dx}+\frac{\beta ddX}{dx^2}-\frac{\gamma d^3X}{dx^3}+\frac{\delta d^3X}{dx^4}-\text{etc.}\right)$$
$$+\frac{p}{p-1}\cdot p^x\left(dX-\frac{\alpha ddX}{dx}+\frac{\beta d^3X}{dx^2}-\frac{\gamma d^4X}{dx^3}+\text{etc.}\right)$$

sive

$$dS=\frac{p^{x+1}}{p-1}\left(Xdxlp-(\alpha lp-1)dX+(\beta lp-\alpha)\frac{ddX}{dx}-(\gamma lp-\beta)\frac{d^3X}{dx^2}+\text{etc.}\right),$$

quod est differentiale quaesitum functionis propositae *S*.

388. Sin autem functio inexplicabilis proposita ex factoribus constet eorumque logarithmi infinitesimi differentias habeant constantes sive minus, tum hac quoque methodo differentiale functionis perpetuo exhiberi poterit.
Sit enim

$$\begin{matrix}1 & 2 & 3 & 4\cdots & x\end{matrix}$$
$$S=A\cdot B\cdot C\cdot D\cdots X.$$

Quia hinc fit

$$lS=lA+lB+lC+lD+\cdots+lX,$$

methodo superiori numeros BERNOULLIANOS in subsidium vocando erit

$$lS=\int dxlX+\frac{1}{2}lX+\frac{\mathfrak{A}d.lX}{1\cdot 2dx}-\frac{\mathfrak{B}d^3.lX}{1\cdot 2\cdot 3\cdot 4dx^3}+\text{etc.}$$

qua expressione differentiata fit

$$\frac{dS}{S}=dxlX+\frac{1}{2}d.lX+\frac{\mathfrak{A}dd.lX}{1\cdot 2dx}-\frac{\mathfrak{B}d^4.lX}{1\cdot 2\cdot 3\cdot 4dx}+\frac{\mathfrak{C}d^6.lX}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6dx}-\frac{\mathfrak{D}d^8.lX}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8dx}+\text{etc.}$$

Hinc si fuerit $X=x$, ut sit

$$S=1\cdot 2\cdot 3\cdot 4\cdots x,$$

fiet applicatione facta

$$\frac{dS}{S}=dxlx+\frac{dx}{2x}-\frac{\mathfrak{A}dx}{2xx}+\frac{\mathfrak{B}dx}{4x^4}-\frac{\mathfrak{C}dx}{6x^6}+\frac{\mathfrak{D}dx}{8x^8}-\text{etc.},$$

quae forma, si *x* sit numerus valde magnus, commodius usurpatur quam eae, quas ante invenimus.