CHAPTER XV

CONCERNING THE VALUES OF FUNCTIONS WHICH MAY BE CONSIDERED TO BE INDETERMINATE IN CERTAIN CASES

355. If the fraction $\frac{P}{Q}$ were some function *y* of *x*, the numerator and the denominator of which likewise may vanish on putting a certain value in place of *x*, then in that case the fraction $\frac{P}{Q}$ may arise expressing the value of the function $y = \frac{0}{0}$; which expression thus may be considered indeterminate, since for each quantity either finite or infinite, or infinitely small it may become equal to, from that evidently in this case the value of *y* cannot be deduced. Yet meanwhile it is easily seen, because in addition in this case the function *y* takes a determined value always, whatever may be substituted for *x*, also in this case an indeterminate value of *y* cannot be possible. This is made clear from this example, if there were $y = \frac{aa-xx}{a-x}$, so that on making $x = a$ certainly there becomes $y = \frac{0}{0}$. But since with the numerator divided by the denominator it may become $y = a + x$, it is evident, if there is put $x = a$, to become $y = 2a$, thus so that in this case that fraction $\frac{0}{0}$ may be equivalent to the quantity *2a*.

356. Therefore since above we have shown that some ratio may intercede between the zeros, the ratio determined that the numerator may hold to the denominator must be investigated in examples of this kind. But since this may be unable to be seen in diverse situations from complete zeros , in place of these infinitely small quantities must be introduced ; which even if they may not differ in a significant ratio from zero, yet from the differences of these functions, which constitute the numerator and denominator, the value of the fraction may be elicited at once. Thus if this fraction $\frac{adx}{bdx}$ may be had, even if actually the numerator and denominator shall be $= 0$, yet it is evident the value of this fraction to be determined, surely $= \frac{a}{b}$. But if this fraction may be considered $\frac{adx^2}{bdx}$, the value of which is zero, just as the value $\frac{adx}{bdx^2}$ is infinitely great. Therefore if in place of the zeros, which frequently arise in the calculation, we may introduce infinitely small quantities, we may perceive thence this reward, so that soon we may know the ratio these may hold between themselves with zero, and without further doubt about the significance of expressions of this kind that may arise.

357. So that this may be rendered more plainly, we may put both the numerator as well as the denominator of the fraction $y = \frac{P}{Q}$ to vanish, if $x = a$ may be put in place. But to avoid these zeros, which cannot be compared with each other, we may put $x = a + dx$, which actually revert to the first position $x = a$ on account of $dx = 0$. Truly since, if in place of *x* there may be put $x+dx$, the functions *P* and *Q* may change into $P+dP$ and $Q+dQ$, $x=a+dx$ may satisfy the position, if in these values everywhere there may be put in place $x=a$, from which indeed we have assumed to vanish in the case *P* and *Q*. Hence if in place of *x* there may be put $a+dx$, the fraction $\frac{P}{Q}$ will be changed into this $\frac{dP}{dQ}$ which therefore will express the value of the function $y=\frac{P}{Q}$ in the case $x=a$, [as *P* and *Q* are both zero at this point]. And this expression cannot be more indeterminate, if indeed the differentials of the functions *P* and *Q* may be taken, as we have shown in the preceding chapter. For with this agreed upon the differentials *dP* and *dQ* under no circumstance will be able to be absolved to zero, but, unless they may be expressed by the differential *dx* itself, at lest they may be shown by the powers of this. But if therefore $dP=Rdx^m$ and $dQ=Sdx^n$ may be found and the value of the function $y=\frac{P}{Q}$ will be found in the case $x=a$ to be $=\frac{Rdx^m}{Sdx^n}$, which therefore will be finite and $=\frac{R}{S}$, if there were $m=n$; but if there shall be $m>n$, then the value of the fraction proposed actually will be $=0$; but if there shall be $m<n$, this value increases to infinity.

358. Therefore as often as a fraction of this kind $\frac{P}{Q}$ occurs, the numerator and denominator of which in a certain case, such as $x=a$, likewise may vanish, the value of this fraction in this case $x=a$ may be found by the following rule:

The differentials of the quantities P and Q may be sought in the case $x=a$ and these may be substituted in place of P and Q, with which done the fraction $\frac{dP}{dQ}$ will show the value of the fraction $\frac{P}{Q}$ sought.

If the differentials *dP* and *dQ* found by the customary method neither become infinite nor vanish in the case $x=a$, then these will be able to be retained ; but if both either become $=0$ or $=\infty$, then these differentials must be investigated in the case $x=a$ in the manner set out in the previous chapter. Generally also the calculation may be wonderfully shortened, if in the first place there may be put, $x-a=t$ or $x=a+t$, from which the fraction may be produced $\frac{P}{Q}$, the numerator and denominator of which vanish in the case $t=0$; for then the differentials *dP* and *dQ* will be considered, if *dt* may be substituted in place of *t* everywhere.

EXAMPLE 1

The value of this fraction $\frac{b-\sqrt{(bb-tt)}}{tt}$ *is sought in the case* $t=0$.

Because in this case $t=0$ both the numerator and denominator vanish, in place of t there may be written only dt, and the value sought will be expressed by this fraction $\frac{b-\sqrt{(bb-dt^2)}}{dt^2}$. Therefore since there shall be $\sqrt{(bb-dt^2)}=b-\frac{dt^2}{2b}$, that fraction will change into this $\frac{dt^2}{2bdt^2}=\frac{1}{2b}$. Hence the proposed fraction $\frac{b-\sqrt{(bb-dt^2)}}{dt^2}$ in the case $t=0$ takes the value $\frac{1}{2b}$.

EXAMPLE 2

The value of this fraction $\frac{\sqrt{(aa+ax+xx)}-\sqrt{(aa-ax+xx)}}{\sqrt{(a+x)}-\sqrt{(a-x)}}$ *is sought in the case* $x=0$.

Here again at once dx is able to be substituted in place of x ; with which done since there shall be

$$\sqrt{(aa+adx+dx^2)}=a+\tfrac{1}{2}dx+\tfrac{3dx^2}{8a},$$
$$\sqrt{(aa-adx+dx^2)}=a-\tfrac{1}{2}dx+\tfrac{3dx^2}{8a}$$

and

$$\sqrt{(a+dx)}=\sqrt{a}+\tfrac{dx}{2\sqrt{a}},$$
$$\sqrt{(a-dx)}=\sqrt{a}-\tfrac{dx}{2\sqrt{a}},$$

the numerator becomes $=dx$ and the denominator $=\frac{dx}{\sqrt{a}}$, from which the value of the proposed fraction sought will be $=\sqrt{a}$.

EXAMPLE 3

The value of this function is sought $\frac{x^3-4ax^2+7a^2x-2a^2\sqrt{(2ax-aa)}}{xx-2ax-aa+2a\sqrt{(2ax-aa)}}$ *in the case* $x=a$.

If the differentials may be taken in the customary manner and they may be substituted in place of the numerator and the denominator, there will be had

$$\frac{3xx-8ax+7a^2x-2a^3:\sqrt{(2ax-aa)}}{2x-2a+2a(a-x):\sqrt{(2ax-xx)}},$$

the numerator and denominator of which fraction again vanish, if there is put $x=a$. Whereby on account of the same reason in place of these again the differentials of these may be substituted, and there will be produced

$$\frac{6x-8a+2a^4:(2ax-aa)^{\frac{3}{2}}}{2-2a^3:(2ax-xx)^{\frac{3}{2}}},$$

the numerator and denominator of which again vanish in the case $x=a$. Therefore we may go on to substitute the differential of these in place of those

$$\frac{6-6a^5:(2ax-aa)^{\frac{5}{2}}}{6a^3(a-x):(2ax-xx)^{\frac{5}{2}}}=\frac{1-a^5:(2ax-aa)^{\frac{5}{2}}}{a^3(a-x):(2ax-xx)^{\frac{5}{2}}}$$

Truly here on putting $x=a$ again both the numerator as well as the denominator vanish. Therefore again with the differentials of those substituted in place there will arise

$$\frac{5a^6:(2ax-aa)^{\frac{7}{2}}}{-(5a^5-8a^4x+4a^3xx):(2ax-xx)^{\frac{7}{2}}}$$

Now finally there may be put a in place of x and there will be put in place this fraction to be determined $\frac{5:a}{-1:a^2}=-5a$, which is the value of the proposed fraction sought.

But if moreover, before this investigation were undertaken, there may be put $x=a+t$, the proposed fraction will be transformed into this

$$\frac{2a^3+2a^2t-att+t^3-2a^2\sqrt{(aa+2at)}}{-2aa+tt+2a\sqrt{(aa-tt)}};$$

where since it may receive the form $\frac{0}{0}$, if there may be put $t=0$, dt may be put in place of t and there will be

$$\frac{2a^3+2a^2dt-adt^2+dt^3-2a^2\sqrt{(aa+2adt)}}{-2aa+dt^2+2a\sqrt{(aa-dt^2)}}$$

Now the irrational formulas may be converted into series, which as far as these may be contained, so that the rational terms may no longer be cancelled by a member:

$$\sqrt{(aa+2adt)}=a+dt-\frac{dt^2}{2a}+\frac{dt^3}{2aa}-\frac{5dt^4}{8a^3},$$
$$\sqrt{(aa-dt^2)}=a-\frac{dt^2}{2a}-\frac{dt^4}{8a^3};$$

with which values substituted this fraction will be produced

$$\frac{5dt^4:4a}{-dt^4:4aa}=-5a,$$

which is the value of the fraction proposed now found before.

EXAMPLE 4

To find the value of this fraction $\frac{a+\sqrt{(2aa-2ax)}-\sqrt{(2ax-xx)}}{a-x+\sqrt{(aa-xx)}}$ *in the case* $x=a$.

This fraction will be produced with the differentials of these substituted in place of the numerator and denominator, which in the case of the proposed $x=a$ will be equal to :

$$\frac{-a:\sqrt{(2aa-2ax)}-(a-x):\sqrt{(2ax-xx)}}{-1-x:\sqrt{(aa-xx)}},$$

the numerator and denominator of which in the case $x=a$ are made infinite. Truly if each may be multiplied by $-\sqrt{(a-x)}$, there will be had

$$\frac{a:\sqrt{2}a+(a-x)^{\frac{3}{2}}:\sqrt{(2ax-xx)}}{\sqrt{(a-x)}+x:\sqrt{(a+x)}},$$

which on putting $x = a$ will give this value determined $\frac{a:\sqrt{2a}}{a:\sqrt{2a}} = 1$, which therefore is equal to the proposed fraction in the case $x = a$.

359. Therefore if the fraction $\frac{P}{Q}$ may be had, the numerator and denominator of which may vanish in the case $x = a$, the value of this can be assigned by the customary rules of differentiation nor is there a need for the differentials to recur which we have treated in the previous chapter. For from the differentials the proposed fraction $\frac{P}{Q}$ in the case $x = a$ will be equal to the fraction $\frac{dP}{dQ}$; if the numerator and denominator on putting $x = a$ may adopt finite values, the value of the proposed fraction will become known ; but if one becomes $= 0$ with the other remaining finite, then the fraction will be either $= 0$ or $= \infty$, according as either the numerator or denominator may vanish. But if either the one or the other becomes $= \infty$, which arises, if it may be divided by vanishing quantities in the case $x = a$, then on each requiring to be multiplied by these divisors that inconvenience may be removed, as came about in the last example. But if truly the numerator as well as the denominator again may vanish in the case $x = a$, then in turn, as was done initially, the differentials must be taken, thus so that this fraction $\frac{ddP}{ddQ}$ may be produced, which in the case proposed $x = a$ will be equal at this stage ; and if likewise anew in this fraction used it may arise, that there is made $\frac{0}{0}$, then in place of this $\frac{d^3P}{d^3Q}$ may be used in its place and so on thus, until a fraction may be come upon, which may show the determined value, either finite or infinite large or infinitely small. Thus in the third example it was required to progress to the fraction $\frac{d^4P}{d^4Q}$, before a value of the proposed fraction $\frac{P}{Q}$ was permitted to be assigned.

360. The use of this investigation may be evident in defining the sums of series, which we have elicited above (cap. II § 22), if there may be put $x = 1$. From these indeed, which have been treated there, there follows to be :

$$x + x^2 + x^3 + \cdots + x^n = \frac{x - x^{n+1}}{1-x}$$

$$x + x^3 + x^5 + \cdots + x^{2n-1} = \frac{x - x^{2n+1}}{1-xx}$$

$$x + 2x^2 + 3x^3 + \cdots + nx^n = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

$$x + 3x^3 + 5x^5 + \cdots + (2n-1)x^{2n-1} = \frac{x + x^3 - (2n+1)x^{2n+1} + (2n-1)x^{2n+3}}{(1-xx)^2}$$

$$x + 4x^3 + 9x^5 + \cdots + n^2x^n = \frac{x + x^2 - (2n+1)^2 x^{n+1} + (2nn+2n-1)x^{n+2} - nnx^{n+3}}{(1-x)^3}$$

etc.

But if now the sums of these series may be desired in the case, in which $x=1$, in theses expressions both the numerator and the denominator vanish. Therefore the values of these sums in the case $x=1$ will be able to be defined by the method set out here. Truly since the same sums may be agreed upon by other means, from the consensus the truth of this method will be further elucidated.

EXAMPLE 1

To define the value of this fraction $\frac{x-x^{n+1}}{1-x}$ in the case $x=1$, which will show the sum of the series $1+1+1+\cdots+1$ depending on n terms, which therefore will be $=n$.

Because in the case $x=1$ the numerator and denominator will vanish, the differentials may be substituted in place of these and there will be had

$$\frac{1-(n+1)x^n}{-1},$$

which on putting $x=1$ gives n for the sum of the series sought.

EXAMPLE 2

To define the value of the fraction $\frac{x-x^{2n+1}}{1-xx}$ in the case $x=1$, which will show the sum of the series $1+1+1+\cdots+1$ depending on n terms, which therefore will be $=n$.

With the differentials taken the proposed fraction may be changed into this :

$$\frac{1-(2n+1)x^{2n}}{-2x},$$

the value of which on putting $x=1$ will be $=n$.

EXAMPLE 3

To find the value of this fraction $\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$ in the case $x = 1$, which expresses the sum of the series $1+2+3+\cdots+n$, which agrees to be $=\frac{nn+n}{2}$.

With the differentials taken this fraction may be arrived at

$$\frac{1-(n+1)^2x^n+n(n+2)x^{n+1}}{-2(1-x)},$$

of which at this stage both the numerator as well as the denominator vanish in the case $x=1$. Hence the differentials may be taken anew, so that this fraction may be produced

$$\frac{-n(n+1)^2x^{n-1}+n(n+1)(n+2)x^n}{2},$$

which on putting $x=1$ will change into $\frac{n(n+1)}{2}=\frac{nn+n}{2}$, the sum of the series proposed.

EXAMPLE 4

To find the value of this fraction $\frac{x+x^3-(2n+1)x^{2n+1}+(2n-1)x^{2n+3}}{(1-xx)^2}$ *in the case* $x=1$, *which may express the sum of the series* $1+3+5+\cdots+(2n-1)$, *which agrees to be* $=nn$.

With the differentials in place of the numerator and denominator this fraction is come upon

$$\frac{1+3xx-(2n+1)^2x^{2n}+(2n-1)(2n+3)x^{2n+2}}{-4x(1-xx)};$$

which at this time likewise it may have the inconvenience, so that on putting $x=1$ it may become $\frac{0}{0}$, the differentials may be taken anew

$$\frac{6x-2n(2n+1)^2x^{2n-1}+(2n-1)(2n+2)(2n+3)x^{2n+1}}{-4+12xx},$$

which on putting $x=1$ will become

$$\frac{6-2n(2n+1)^2+(2n-1)(2n+2)(2n+3)}{8}=nn.$$

EXAMPLE 5
To find the value of this fraction

$$\frac{x+x^2-(2n+1)^2x^{n+1}+(2nn+2n-1)x^{n+2}-nnx^{n+3}}{(1-x)^3}$$

in the case $x=1$, *which will give the sum of the series* $1+4+9+\cdots+n^2$, *which agrees to be* $=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n$.

With the differentials of the numerator and denominator taken there becomes

$$\frac{1+2x-(n+1)^3x^n+(n+2)(2nn+2n-1)x^{n+1}-nn(n+3)x^{n+2}}{-3(1-x)^2}.$$

in which since the numerator and denominator on putting $x=1$ may vanish anew, the second differentials may be taken

$$\frac{2-n(n+1)^3x^{n-1}+(n+1)(n+2)(2nn+2n-1)x^n-n^2(n+2)(n+3)x^{n+1}}{6(1-x)}$$

Truly at this stage with the same inconvenient stopping it may precede to the third differentials, so that this fraction may be produced

$$\frac{-n(n-1)(n+1)^3x^{n-2}+n(n+1)(n+2)(2nn+2n-1)x^{n-1}-n^2(n+1)(n+2)(n+3)x^n}{-6},$$

which finally on putting $x=1$ will change into the determined form

$$\frac{-n(n-1)(n+1)^3+n(n+1)(n+2)(nn-n-1)}{-6}=\frac{n(n+1)(2n+1)}{6}=\tfrac{1}{3}n^3+\tfrac{1}{2}n^2+\tfrac{1}{6}n\,;$$

which is that value itself, by which we have found the series mentioned to be expressed.

EXAMPLE 6

Let this fraction be proposed $\frac{x^m-x^{m+n}}{1-x^{2p}}$*, the value of which it may be required to assign in the case* $x=1$.

Because this fraction had been produced from these two $\frac{x^m}{1+x^p}\cdot\frac{1-x^n}{1-x^p}$, but in the case $x=1$ the value of the first factor is $=\frac{1}{2}$, only there is a need, so that the value of the other factor $\frac{1-x^n}{1-x^p}$ is sought in the same case, which with the differentials taken will be $=\frac{nx^{n-1}}{px^{p-1}}=\frac{n}{p}$; from which the value of the proposed fraction in the case $x=1$ will be $=\frac{n}{2p}$. The same value will be produced, if the differentials are taken at once in the proposed fraction ; for there comes about

$$\frac{mx^{m-1}-(m+n)x^{m+n-1}}{-2px^{2p-1}},$$

the value of which on putting $x=1$ will be $=\frac{-n}{-2p}=\frac{n}{2p}$ as before.

361. The same method will be required to be used, if in the proposed fraction $\frac{P}{Q}$ whither the numerator or the denominator or each were a transcending quantity. In order that which operations may be set out more clearly, it has been considered to add the following examples.

EXAMPLE 1

Let this fraction be proposed $\frac{a^n-x^n}{la-lx}$, the value of which is sought in the case $x=a$.

With the differentials taken at once this fraction is come upon

$$\frac{-nx^{n-1}}{-1:x}=nx^n,$$

the value of which on putting $x=a$ will be na^n.

EXAMPLE 2

Let this fraction be proposed $\frac{lx}{\sqrt{(1-x)}}$, *the value of which is sought in the case* $x=1$.

With the differentials of the numerator and denominator taken there is produced

$$\frac{1:x}{-1:2\sqrt{(1-x)}}=\frac{-2\sqrt{(1-x)}}{x}\;;$$

the value of which on putting $x=1$ since it shall be $=0$, it follows that the fraction $\frac{lx}{\sqrt{(1-x)}}$ in the case $x=1$ vanishes.

EXAMPLE 3

Let this fraction be proposed $\frac{a-x-ala+alx}{a-\sqrt{(2ax-xx)}}$, *the value of which is sought on putting* $x=a$, *in which case the numerator and the denominator vanish.*

With the numerator and denominator differentials following the rule there will be

$$\frac{-1+a:x}{-(a-x):\sqrt{(2ax-xx)}}=\frac{(a-x)\sqrt{(2ax-xx)}}{-x(a-x)}\;;$$

where although the numerator and the denominator in the case $x=a$ vanish at this stage, yet, because each is divisible by $a-x$, this fraction will be had $-\sqrt{\frac{2a-x}{x}}$, the value of which in the case $x=a$ is determined and $=-1$; and therefore the proposed fraction will change into –1, if there may be put $x=a$.

EXAMPLE 4

Let this fraction be proposed $\frac{e^x - e^{-x}}{l(1+x)}$ *, the value of which is sought on putting* $x = 0$.

With the differentials taken this function will be had :

$$\frac{e^x + e^{-x}}{1:(1+x)} \ ,$$

which on putting $x = 0$ gives 2 for the value sought.

EXAMPLE 5

To find the value of this fraction $\frac{e^x - 1 - l(1+x)}{xx}$ *in the case, in which there is put* $x = 0$.

If in place of the numerator and the denominator the differentials of these may be substituted, this fraction may arise

$$\frac{e^x - 1:(1+x)}{2x} \ ,$$

which since at this stage it may change into $\frac{0}{0}$ if there may be put $x = 0$, the differentials are taken again, so that there may be had

$$\frac{e^x + 1:(1+x)^2}{2} \ ,$$

which on putting $x = 0$ gives $\frac{1+1}{2} = 1$. Because it is apparent likewise, if in place of x there may be substituted at once $0 + dx$, for since there shall be

$$e^{dx} = 1 + dx + \tfrac{1}{2}dx^2 + \text{etc. and } l(1+dx) = dx - \tfrac{1}{2}dx^2 + \text{ etc.,}$$

$$\frac{e^{dx} - 1 - l(1+dx)}{dx^2} = \frac{dx^2}{dx^2} = 1.$$

EXAMPLE 6

The value of the fraction $\frac{x^n}{lx}$ *is sought in the case, in which there is put* $x = \infty$.

So that this fraction may be reduced to the form which in this case may pass into $\frac{0}{0}$, it will be represented thus

$$\frac{1:lx}{1:x^n} \ ;$$

for thus in the case $x=\infty$ both the numerator as well as the denominator will vanish. Truly again there may be put $x=\frac{1}{y}$, thus so that in the case $x=\infty$ there becomes $y=0$, and thus the fraction y may be proposed

$$-\frac{1:ly}{y^n},$$

the value of which in the case $y=0$ must be investigated. But with the differentials taken there will be $\frac{1:y(ly)^2}{ny^{n-1}}=\frac{1:(ly)^2}{ny^n}$; which on putting $y=0$ since it may change into $\frac{0}{0}$, the differentials are taken anew and there will be $\frac{-2:(ly)^3}{n^2y^n}$; where because the same inconvenience is present, if the differentials are taken again, there will be produced $\frac{6:(ly)^4}{n^3y^n}$ and thus, as far as we may proceed, the same inconvenience always occurs. On account of which, so that we may elicit the value sought without this obstacle, let s be the value of the fraction $-\frac{1:ly}{y^n}$ in the case, in which there is put $y=0$, and since in the same case there shall be also

$$s=\frac{1:(ly)^2}{ny^n},$$

from that former equation there will be

$$ss=\frac{1:(ly)^2}{y^{2n}}$$

which divided by that will give

$$s=\frac{ny^n}{y^{2n}}=\frac{n}{y^n},$$

from which s is seen in the case $y=0$ to become infinite. Therefore the value of the fraction $-\frac{1:ly}{y^n}$ is made infinite in the case $y=0$ and thus on putting $y=dx$, $\frac{1}{ldx}$ to dx^n will be considered an infinite ratio, as above we have now hinted at in §351.

EXAMPLE 7

The value of the fraction $\frac{x^n}{e^{-1:dx}}$ is sought in the case $x=0$, so that both the numerator and the denominator vanish.

In this case let $\frac{x^n}{e^{-1:dx}}=s$; there will be also on taking the derivatives

$$s=\frac{nx^{n-1}}{e^{-1:dx}:xx}=\frac{nx^{n+1}}{e^{-1:dx}},$$

and because here it meets the same inconvenience and it recurs always, as far as the differentiations may be continued, we may make use of the previous remedy. The first equation gives

$$x^n = e^{-1:dx}s \quad \text{and} \quad x^{n(n+1)} = e^{-(n+1):x}s^{n+1};$$

the other equation gives

$$x^{n+1} = e^{-1:x}s:n,$$

from which there is made

$$x^{n(n+1)} = e^{-n:x}s:n^n,$$

which value to that the equation will give

$$e^{-1:x}sn^n = 1$$

and thus

$$s = \frac{1}{n^n e^{-1:x}} = \infty,$$

if $x = 0$. Whereby on putting x infinitely small the ratio dx^n to $e^{-1:dx}$ will be considered infinitely great, whatever finite number may be put in place for n ; from which it follows $e^{-1:dx}$ to be infinitely small , homogeneous with dx^m, if m were an infinitely great number.

EXAMPLE 8

The value of the fraction $\frac{1-\sin x+\cos x}{\sin x+\cos x-1}$ *is sought in the case, in which there is put* $x = \frac{\pi}{2}$ *or of the arc of* 90 *degrees.*

With the differentials taken this fraction will be obtained

$$\frac{-\cos x-\sin x}{\cos x-\sin x}$$

which on putting $x = \frac{\pi}{2}$ on account of $\sin x = 1$ and $\cos x = 0$ will change into 1, thus so that unity shall be the value of the proposed fraction. Because the same may be apparent without differentiation; for since there shall be $\cos x = \sqrt{(1+\sin x)(1-\sin x)}$, the proposed fraction will change into this

$$\frac{\sqrt{(1-\sin x)}+\sqrt{(1+\sin x)}}{\sqrt{(1+\sin x)}-\sqrt{(1-\sin x)}},$$

which evidently is made $=1$, if there shall become $\sin x = 1$.

EXAMPLE 9

To find the value of this expression $\frac{x^x - x}{1 - x + lx}$ *in the case in which there is put* $x = 1$.

In place of the numerator and the denominator with the differentials of these substituted this fraction will be produced

$$\frac{x^x(1+lx)-1}{-1+1:x},$$

which since also now it may become $=\frac{0}{0}$ on putting $x=1$, the differential may be taken anew, so that there is produced

$$\frac{x^x(1+lx)^2+x^x:x}{-1:xx}$$

which on putting $x=1$, will change into -2, which is the value of the fraction sought in the case $x=1$.

362. Because here we have decided to examine all the expressions, which may be seen to receive indeterminate values in certain cases, not only these fractions which relate to these fractions $\frac{P}{Q}$, the numerator and the denominator of which vanish in a certain case, but here also fractions of such a kind, the numerator and the denominator of which become infinite in a certain case, are to be referred to at this point, because the values of these also therefore may be seen to be indeterminate. Evidently if P and Q were functions of this kind of *x*, so that in a certain case $x = a$ both become infinite and the fraction $\frac{P}{Q}$ adopt this form $\frac{\infty}{\infty}$, because infinite magnitudes and zeros can hold some relation between each other, hence the true smallest value is able to become known. Indeed this case can be recalled to the previous fraction $\frac{P}{Q}$ by being changed into this form $\frac{1:Q}{1:P}$, of which fraction now the numerator and the denominator vanish in the case $x = a$; and thus the value of this can be found in the manner treated before. But truly the value may be found without this transformation, if in place of x not *a*, but $a + dx$ may be substituted ; with which done a absolutely infinite values of this kind ∞ may not arise, but thus expressed by $\frac{1}{dx}$ or $\frac{A}{dx^n}$; which expressions even if they are equal to infinity and ∞, yet by comparison between *dx* or the powers of this in place the value sought may be deduced easily.

363. The products from two agreed factors relate to the same class also, one of which certainly vanishes in the case $x = a$, the other truly will go off to infinity; for since whatever quantity will be represented by a product of this kind $0 \cdot \infty$, the value of this may be seen to be indefinite. Let *PQ* be a product of this kind, in which if there may be put $x = a$, there becomes $P = 0$ and $Q = \infty$; the value of this may be found by the precepts treated before, if there may be put $Q = \frac{1}{R}$; for then the product *PQ* will be changed into the fraction $\frac{P}{R}$ of which both the numerator and the denominator vanish in the case $x = a$; and thus the value of this will be able to be investigated by the method found before.

Thus if the value of this product

$$(1-x)\operatorname{tang}\frac{\pi x}{2}$$

is sought in the case $x=1$, from which there becomes $1-x=0$ and $\operatorname{tang}\frac{\pi x}{2}=\infty$, that may be converted into this form

$$\frac{1-x}{\cot\frac{1}{2}\pi x},$$

the numerator and denominator of which vanish in the case $x=1$. Therefore since the differential of the numerator shall be $1-x=-dx$ and the differential of the denominator $\cot\frac{1}{2}\pi x=-\frac{\pi dx:2}{\left(\sin\frac{1}{2}\pi x\right)^{2}}$, in the case $x=1$ the value of the proposed fraction will be

$$=\frac{2}{\pi}\sin\frac{\pi x}{2}\cdot\sin\frac{\pi x}{2}=\frac{2}{\pi}$$

on account of $\sin\frac{\pi}{2}=1$.

364. But here in the first place expressions of such a kind are required to be refer to, which while a certain value may be attributed to *x*, may change into a form of this kind $\infty-\infty$; because indeed any two infinite quantities can differ from each other, it is evident in this case that the value of the expression cannot be determined, unless the difference between these two infinite quantities will be able to be assigned. Therefore this case occurs, if a function of the kind $P-Q$ may be proposed, in which on putting $x=a$ there may be made both $P=\infty$ as well as $Q=\infty$, in which case with the aid of the rule treated before the value sought will be found, but not so easily able to be assigned as before. For although, by putting $P-Q=f$ to be made in this case, there may be put in place $e^{P-Q}=e^{f}$, thus so that there shall be $e^{f}=\frac{e^{-Q}}{e^{-P}}$, where in the case $x=a$ both the numerator e^{-Q} as well as the denominator e^{-P} vanish, yet, if the rule treated before may be transferred at this stage, there becomes $e^{f}=\frac{e^{-Q}dQ}{e^{-P}dP}$, from which on account of $e^{f}=\frac{e^{-Q}}{e^{-P}}$ there may become $1=\frac{dQ}{dP}$ and thus the value of f sought itself hence does not become known. Indeed as often as *P* and *Q* are algebraic quantities, because these are unable to become infinite, unless they become fractions, the denominators of which vanish, then $P-Q$ can be gathered together into a single fraction, the denominator of which equally will vanish. With which done if the numerator also may vanish, the value may be defined in the manner explained above ; but if the numerator may not vanish, then the value of this actually will be infinite.

Thus if the value of this expression

$$\frac{1}{1-x}-\frac{2}{1-xx}$$

may be desired in the case $x=1$, because that will change into

$$\frac{-1+x}{1-xx}=\frac{-1}{1+x},$$

it is apparent the value sought to be $=-\frac{1}{2}$.

365. Truly if the functions P and Q were transcending, then generally this transformation may lead to a troublesome calculation. Therefore in this cases a direct method may be expedient to be used and in place of $x=a$, so that both the quantities P and Q depart to infinity, to put $x=a+\omega$ with some infinitely small amount ω arising, for which dx will be able to be taken. With which done if there may be made $P=\frac{A}{\omega}+B$ but $Q=\frac{A}{\omega}+C$, it is evident the function $P-Q$ is going to change into $B-C$, which will be a finite value. Therefore we will illustrate the account of functions of this kind, the values to be investigated by the following examples.

EXAMPLE 1

The value of this expression $\frac{x}{x-1}-\frac{1}{lx}$ *is sought, in the case in which there is put* $x=1$.

Because both $\frac{x}{x-1}$ as well as $\frac{1}{lx}$ will be made infinite on putting $x=1$, there may be considered $x=1+\omega$ and the proposed expression will be transformed into

$$\frac{1+\omega}{\omega}-\frac{1}{l(1+\omega)}$$

Therefore since there shall be

$$l(1+\omega)=\omega-\tfrac{1}{2}\omega^2+\tfrac{1}{3}\omega^3-\text{etc.}=\omega\left(1-\tfrac{1}{2}\omega+\tfrac{1}{3}\omega^2-\text{etc.}\right),,$$

there will be had

$$\frac{(1+\omega)\left(1-\frac{1}{2}\omega+\frac{1}{3}\omega^2-\text{etc.}\right)-1}{\omega\left(1-\frac{1}{2}\omega+\frac{1}{3}\omega^2-\text{etc.}\right)}=\frac{\frac{1}{2}\omega-\frac{1}{6}\omega^2+\text{etc.}}{\omega\left(1-\frac{1}{2}\omega+\frac{1}{3}\omega^2-\text{etc.}\right)}=\frac{\frac{1}{2}-\frac{1}{6}\omega+\text{etc.}}{1-\frac{1}{2}\omega+\frac{1}{3}\omega^2-\text{etc.}}$$

Now on putting ω infinitely small or $\omega=0$ it is evident the value sought to be $=\frac{1}{2}$.

EXAMPLE 2

With e denoting the number, the hyperbolic logarithm of which is $=1$, *and* π *the semi circumference of the circle, the radius of which is* = 1, *to investigate the value of this expression* $\frac{\pi x-1}{2xx}+\frac{\pi}{x\left(e^{2\pi x}-1\right)}$ *in the case x=0.*

That expression proposed shows the value of this series

$$\frac{1}{1+xx}+\frac{1}{4+xx}+\frac{1}{9+xx}+\frac{1}{16+xx}+\frac{1}{25+xx}+\text{etc.};$$

from which, if there may be put $x=0$, the sum of this series must be produced

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\text{etc.},$$

which it is agreed to be $=\frac{\pi\pi}{6}$. But by making $x=0$ the value of the proposed expression

$$\frac{\pi x-1}{2xx}+\frac{\pi}{x\left(e^{2\pi x}-1\right)}$$

may be seen to be especially indeterminate on account of all the infinite terms. Therefore there may be put $x=\omega$ with ω being an infinitely small quantity present and the first member $\frac{\pi x-1}{2xx}$ will change into

$$-\frac{1}{2\omega^2}+\frac{\pi}{2\omega}.$$

Then since there shall be

$$e^{2\pi\omega}-1=2\pi\omega+2\pi^2\omega^2+\tfrac{4}{3}\pi^3\omega^3+\text{etc.}$$

the other member $\frac{\pi}{x\left(e^{2\pi x}-1\right)}$ will change into

$$\frac{\pi}{\omega\left(2\pi\omega+2\pi^2\omega^2+\frac{4}{3}\pi^3\omega^3+\text{etc.}\right)}=\frac{1}{2\omega^2\left(1+\pi\omega+\frac{2}{3}\pi^2\omega^2+\text{etc.}\right)}.$$

But there is

$$\frac{1}{1+\pi\omega+\frac{2}{3}\pi^2\omega^2+\text{etc.}}=1-\pi\omega+\tfrac{1}{3}\pi^2\omega^2-\text{etc.},$$

from which the latter member shall become

$$=\frac{1}{2\omega^2}-\frac{\pi}{2\omega}+\frac{1}{6}\pi^2-\text{etc.};$$

to which if the first be added, there will be produced $\frac{1}{6}\pi^2$, which is the value of the proposed expression sought in the case $x=0$.

The same also can be prescribed by the method of fractions, the numerator and denominator of which vanish in a certain case; for the proposed expression may be changed into this fraction

$$\frac{\pi xe^{2\pi x}-e^{2\pi x}+\pi x+1}{2xxe^{2\pi x}-2xx}$$

the numerator and denominator of which vanish in the case $x=0$. Therefore with the differentials taken there arises

$$\frac{\pi e^{2\pi x}+2\pi\pi xe^{2\pi x}-2\pi e^{2\pi x}+\pi}{4xe^{2\pi x}+4\pi xxe^{2\pi x}-4x}$$

or if this

$$\frac{\pi-\pi e^{2\pi x}+2\pi\pi xe^{2\pi x}}{4xe^{2\pi x}+4\pi xxe^{2\pi x}-4x},$$

of which, if there is put $x=0$, at this stage the numerator and denominator vanish. Whereby with the differentials taken anew there will be had

$$\frac{-\pi\pi e^{2\pi x}+2\pi\pi e^{2\pi x}+4\pi^3 xe^{2\pi x}}{4e^{2\pi x}+8\pi xe^{2\pi x}+8\pi xe^{2\pi x}+8\pi^2 xxe^{2\pi x}-4}$$

or

$$\frac{\pi^3 xe^{2\pi x}}{e^{2\pi x}+4\pi xe^{2\pi x}+2\pi^2 x^2 e^{2\pi x}-1}$$

or

$$\frac{\pi^3 x}{1+4\pi x+2\pi^2 x^2-e^{-2\pi x}}$$

the numerator and denominator vanish at this stage in the case $x=0$. On account of which the differentials may be taken again

$$\frac{\pi^3}{4\pi+4\pi^2 x-2\pi e^{-2\pi x}},$$

which fraction on putting $x=0$ will change into $\frac{\pi^2}{6}$ as before.

EXAMPLE 3

With e and π retaining the same values the value of this expression $\frac{\pi}{4x}-\frac{\pi}{2x\left(e^{\pi x}+1\right)}$ is sought in the case $x=0$.

This expression may be changed into this

$$\frac{\pi e^{\pi x}-\pi}{4xe^{\pi x}+4x}$$

the numerator and denominator of which vanish in the case $x=0$. Therefore there may be put $x=\omega$, and since there shall be

$$e^{\pi\omega} = 1 + \pi\omega + \tfrac{1}{2}\pi^2\omega^2 + \tfrac{1}{6}\pi^3\omega^3 + \text{etc.},$$

the proposed formula may be changed into this

$$\frac{\pi^2\omega + \frac{1}{2}\pi^3\omega^2 + \frac{1}{6}\pi^4\omega^3 + \text{etc.}}{8\omega + 4\pi\omega^2 + 2\pi^2\omega^3 + \text{etc.}},$$

which on putting ω infinitely small gives the value $\frac{1}{8}\pi^2$, which is the value of the expression sought in the case $x = 0$. But indeed the expression proposed $\frac{\pi}{4x} - \frac{\pi}{2x(e^{\pi x}+1)}$ shows the sum of this series

$$\frac{1}{1+xx} + \frac{1}{4+xx} + \frac{1}{9+xx} + \frac{1}{16+xx} + \frac{1}{25+xx} + \text{etc.}$$

the sum of which on putting $x = 0$ everywhere will be $= \frac{1}{8}\pi^2$.

EXAMPLE 4

The value of this expression $\frac{1}{2xx} - \frac{\pi}{2x\,\text{tang}\,\pi x}$ *is sought in the case* $x = 0$.

This formula proposed $\frac{1}{2xx} - \frac{\pi}{2x\,\text{tang}\,\pi x}$ expresses the sum of this infinite series

$$\frac{1}{1-xx} + \frac{1}{4-xx} + \frac{1}{9-xx} + \frac{1}{16-xx} + \text{etc.}$$

Therefore if there is put $x = 0$, the sum of the series must be produced

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.},$$

which is $= \frac{1}{6}\pi^2$. Because there is $\text{tang}\,\pi x = \frac{\sin\pi x}{\cos\pi x}$ the expression proposed may adopt this form

$$\frac{1}{2xx} - \frac{\pi\cos\pi x}{2x\sin\pi x} = \frac{\sin\pi x - \pi x\cos\pi x}{2xxx\sin\pi x},$$

the numerator and denominator of which vanish on putting $x = 0$. Therefore there may be put $z = \omega$, and since there shall be

$$\sin\pi\omega = \pi\omega - \tfrac{1}{6}\pi^3\omega^3 + \text{etc.},\quad \cos\pi\omega = 1 - \tfrac{1}{2}\pi^2\omega^2 + \text{etc.},$$

the proposed expression may become

$$\frac{\pi\omega-\frac{1}{6}\pi^3\omega^3+\text{etc.}-\pi\omega+\frac{1}{2}\pi^3\omega^3-\text{etc.}}{2\pi\omega^3-\frac{1}{3}\pi^3\omega^5}=\frac{\frac{1}{3}\pi^3\omega^3-\text{etc.}}{2\pi\omega^3-\text{etc.}},$$

which on account of ω being infinitely small gives $\frac{1}{6}\pi^2$.

EXAMPLE 5

Since the sum of this infinite series shall be $\frac{1}{1-xx}+\frac{1}{9-xx}+\frac{1}{25-xx}+\frac{1}{49-xx}+\text{etc.}=\frac{\pi\sin\frac{1}{2}\pi x}{4x\cos\frac{1}{2}\pi x}$, *to find the sum of that, if there were* $x=0$.

Because there is

$$\sin\tfrac{1}{2}\pi x=\tfrac{1}{2}\pi x-\tfrac{1}{48}\pi^3x^3+\text{etc.},\ \ \cos\tfrac{1}{2}\pi x=1-\tfrac{1}{8}\pi^2x^2+\text{etc.},$$

the proposed expression will be

$$=\frac{\frac{1}{2}\pi^2x-\frac{1}{48}\pi^4x^3+\text{etc.}}{4x-\frac{1}{2}\pi^2x^3+\text{etc.}}=\frac{\frac{1}{2}\pi^2-\frac{1}{48}\pi^4x^2+\text{etc.}}{4-\frac{1}{2}\pi^2x^2+\text{etc.}};$$

In which if there is made $x=0$, the value will be evidently $=\frac{1}{8}\pi^2$, which is the sum of the series

$$\frac{1}{1}+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\text{etc.}$$

has been shown in several ways above. But if a particular even number may be taken for *x*, the sum of the proposed series always is $=0$.

366. In these series, which we have treated in the two final examples, and with others containing a variable letter *x* values can be attributed to *x* of such a kind , that certain terms in the infinite series may increase, indeed in which cases the sum of the whole series may become infinite. Thus for the series

$$\frac{1}{1-xx}+\frac{1}{4-xx}+\frac{1}{9-xx}+\frac{1}{16-xx}+\text{etc.}$$

if a certain whole number is put in place for *x*, always one term on account of the denominator vanishing, shall become infinite and for that reason the sum of series itself may become infinite. But if moreover that infinite term may be removed from the series, then the remaining sum without doubt will be finite and may be expressed by the former sum with the infinite term extracted in this manner $\infty-\infty$; for which therefore the determined value will going to be had, here it will be able to be set out in this manner, as that may be seem more clearly from the adjoining examples.

EXAMPLE 1

To find the sum of the series $\frac{1}{1-xx}+\frac{1}{4-xx}+\frac{1}{9-xx}+\frac{1}{16-xx}+\text{etc.}$ *in the case* $x = 1$ *and with the first term removed, which in this case may be increased to infinity.*

Because in general the sum is

$$\frac{1}{2xx}-\frac{\pi}{2x\text{tang }\pi x}$$

the sum sought will be

$$\frac{1}{2xx}-\frac{\pi}{2x\text{tang }\pi x}-\frac{1}{1-xx}$$

on putting $x=1$. Let $x=1+\omega$ and there will be had for the sum sought

$$\frac{1}{2(1+2\omega+\omega\omega)}-\frac{\pi}{2(1+\omega)\text{tang }(\pi+\omega x)}+\frac{1}{2\omega+\omega\omega}$$

But there is

$$\text{tang}(\pi+\omega\pi)=\text{tang }\omega\pi \ =\pi\omega+\tfrac{1}{3}\pi^3\omega^3+\text{etc}.$$

From which , since the first term $\frac{1}{2xx}$ on putting $x=1$ may have the determined value $\frac{1}{2}$ only the two remaining terms are required to be considered, which will be

$$\frac{1}{\omega(2+\omega)}-\frac{\pi}{2\omega(1+\omega)\left(\pi+\frac{1}{3}\pi^3\omega^2\right)}=\frac{1}{\omega(2+\omega)}-\frac{1}{\omega(2+2\omega)\left(1+\frac{1}{3}\pi^2\omega^2\right)},$$

if indeed ω shall be infinitely small, in which case also the term $\frac{1}{3}\pi^2\omega^2$ can be ignored. But there may come about

$$\frac{\omega}{\omega(2+\omega)(2+2\omega)}=\frac{1}{4}$$

on putting $\omega=0$ and therefore this is $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$ the sum of the series

$$\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\text{etc.},$$

as it is agreed from elsewhere.

EXAMPLE 2

To find the sum of the series

$$\frac{1}{1-xx}+\frac{1}{4-xx}+\frac{1}{9-xx}+\frac{1}{16-xx}+\text{etc}.$$

in the case, in which for x there is put some integer n, and with that term $\frac{1}{nn-xx}$ *taken from the series, which shall be made infinite.*

Therefore this sum which is sought, will be expressed thus

$$\frac{1}{2xx}-\frac{\pi}{2x\operatorname{tang}\pi x}-\frac{1}{nn-xx},$$

if indeed $x=n$ is put in place, so that in a certain case the first term $\frac{1}{2xx}$ will change into $\frac{1}{2nn}$, truly the two remaining terms become infinite. Therefore there may be put $x=n+\omega$, and since there shall be $\operatorname{tang}(\pi n+\pi\omega)=\operatorname{tang}\pi\omega=\pi\omega$ on putting ω infinitely small, for the sum sought we will have

$$\frac{1}{2nn}-\frac{\pi}{2(n+\omega)\pi\omega}+\frac{1}{2n\omega+\omega\omega}$$

or

$$\frac{1}{2nn}-\frac{1}{\omega(2n+2\omega)}+\frac{1}{\omega(2n+\omega)}=\frac{1}{2nn}+\frac{1}{(2n+2\omega)(2n+\omega)},$$

from which, if there may become $\omega=0$, the sum sought will be produced

$$=\frac{1}{2nn}+\frac{1}{4nn}=\frac{3}{4nn}.$$

On account of which there will be

$$\frac{3}{4nn}=\frac{1}{1-nn}+\frac{1}{4-nn}+\frac{1}{9-nn}+\cdots+\frac{1}{(n-1)^2-nn}+\frac{1}{(n+1)^2-nn}+\frac{1}{(n+2)^2-nn}+\text{etc. to infinity},$$

or the sum of this infinite series will be

$$\frac{1}{(n+1)^2-nn}+\frac{1}{(n+2)^2-nn}+\frac{1}{(n+3)^2-nn}+\text{etc.}=\frac{3}{4nn}+\frac{1}{nn-1}+\frac{1}{nn-4}+\frac{1}{nn-9}+\cdots+\frac{1}{nn-(n-1)^2}.$$

EXAMPLE 3
To find the sum of this series

$$\frac{1}{1-xx}+\frac{1}{9-xx}+\frac{1}{25-xx}+\frac{1}{49-xx}+\text{etc.},$$

if there may be put $x=1$ *and the first term* $\frac{1}{1-xx}$ *may be taken away, which in this case will become infinite.*

Since the sum of this series shall be in general $=\frac{\pi\sin\frac{1}{2}\pi x}{4x\cos\frac{1}{2}\pi x}$, the sum sought will be

$$=\frac{\pi\sin\frac{1}{2}\pi x}{4x\cos\frac{1}{2}\pi x}-\frac{1}{1-xx},$$

if there may be put $x=1$. Truly because each term will become infinite, there may be put $x=1-\omega$, and since there shall be

$$\sin\left(\tfrac{1}{2}\pi-\tfrac{1}{2}\pi\omega\right)=\cos\tfrac{1}{2}\pi\omega\ =1-\tfrac{1}{8}\pi^2\omega^2$$

and

$$\cos\left(\tfrac{1}{2}\pi-\tfrac{1}{2}\pi\omega\right)=\sin\tfrac{1}{2}\pi\omega=\tfrac{1}{2}\pi\omega$$

on account of the infinitely small ω, this expression may be considered

$$\frac{\pi\left(1-\frac{1}{8}\pi^2\omega^2\right)}{4(1-\omega)\frac{1}{2}\pi\omega}-\frac{1}{2\omega-\omega\omega}=\frac{1}{\omega(2-2\omega)}-\frac{1}{\omega(2-\omega)}$$

which will become $=\frac{1}{4}$ on putting $\omega=0$, and therefore there is

$$\frac{1}{4}=\frac{1}{8}+\frac{1}{24}+\frac{1}{48}+\frac{1}{80}+\frac{1}{120}+\text{etc.}$$

EXAMPLE 4

To find the sum of this series

$$\frac{1}{1-xx}+\frac{1}{9-xx}+\frac{1}{25-xx}+\frac{1}{49-xx}+\text{etc.},$$

if for x there may be put some odd integer $2n-1$ and this term $\frac{1}{(2n-1)^2-xx}$, which in this case will be infinite, may be taken from the middle.

Therefore the sum which is sought will be,

$$=\frac{\pi\sin\frac{1}{2}\pi x}{4x\cos\frac{1}{2}\pi x}-\frac{1}{(2n-1)^2-xx}$$

on putting $x=2n-1$. Therefore we may put $x=2n-1-\omega$ with ω being some infinitely small quantity present and there becomes

$$\sin\tfrac{1}{2}\pi x=\sin\left(\tfrac{2n-1}{2}\pi-\tfrac{1}{2}\pi\omega\right)=\pm\cos\tfrac{1}{2}\pi\omega$$

where the upper sign will prevail, if *n* shall be an odd number, truly the lower, if it shall be even. In a similar manner there will be

$$\cos\tfrac{1}{2}\pi x=\cos\left(\tfrac{2n-1}{2}\pi-\tfrac{1}{2}\pi\omega\right)=\pm\sin\tfrac{1}{2}\pi\omega\ ;$$

and thus, if n shall be either even or odd, there will be

$$\frac{\sin\frac{1}{2}\pi x}{\cos\frac{1}{2}\pi x}=\frac{1}{\operatorname{tang}\frac{1}{2}\pi\omega}=\frac{1}{\frac{1}{2}\pi\omega}.$$

Hence the sum sought may be expressed thus

$$\frac{1}{2\omega(2n-1-\omega)}-\frac{1}{\omega(2(2n-1)-\omega)}$$

and therefore it will be $=\frac{1}{4(2n-1)^2}$. Thus, if there shall be $n=2$, there will be

$$\frac{1}{36}=-\frac{1}{8}+\frac{1}{16}+\frac{1}{40}+\frac{1}{72}+\frac{1}{112}+\text{etc.},$$

the truth of which summation is agreed from elsewhere.

CAPUT XV

DE VALORIBUS FUNCTIONUM QUI CERTIS CASIBUS VIDENTUR INDETERMINATI

355. Si functio ipsius x quaecunque y fuerit fractio $\frac{P}{Q}$, cuius numerator ac denominator posito loco x certo quodam valore simul evanescant, tum isto casu fractio $\frac{P}{Q}$ valorem functionis y exprimens evadet $=\frac{0}{0}$; quae expressio cum cuique quantitati sive finitae sive infinitae sive infinite parvae possit esse aequalis, ex ea prorsus valor ipsius y hoc casu colligi nequit atque ideo videtur indeterminatus. Interim tamen facile perspicitur, quia praeter hunc casum functio y perpetuo valorem determinatum recipit, quicquid pro x substituatur, etiam hoc casu valorem ipsius y indeterminatum esse non posse. Manifestum hoc fiet vel ex hoc exemplo, si fuerit $y=\frac{aa-xx}{a-x}$, quo facto $x=a$ fit utique $y=\frac{0}{0}$. Cum autem numeratore per denominatorem diviso fiat $y=a+x$, evidens est, si ponatur $x=a$, fore $y=2a$, ita ut hoc casu fractio illa $\frac{0}{0}$ aequivaleat quantitati $2a$.

356. Quoniam ergo supra ostendimus inter cyphras rationem quamcunque intercedere posse, in huiusmodi exemplis ratio determinata, quam numerator ad denominatorem teneat, investigari debet. Cum, autem in cyphris absolutis ista diversitas perspici nequeat, earum loco quantitates infinite parvae introduci debent; quae etsi ratione significationis a cyphra non differunt, tamen ex diversis earum functionibus, quae numeratorem et denominatorem constituunt, valor fractionis sponte elucet. Sic si habeatur ista fractio $\frac{adx}{bdx}$, etiamsi revera numerator et denominator sit $=0$, tamen patet valorem huius fractionis esse determinatum, nempe $=\frac{a}{b}$. Sin autem habeatur haec fractio $\frac{adx^2}{bdx}$, huius valor erit nullus, quemadmodum huius valor $\frac{adx}{bdx^2}$ est infinite magnus. Si igitur loco nihilorum, quae saepenumero in calculum ingrediuntur, infinite parva introducamus, hunc inde fructum percipiemus, ut rationem, quam illa nihila inter se tenent, mox cognoscamus nullumque amplius dubium circa significationern huiusmodi expressionum supersit.

357. Quo haec planiora reddantur, ponamus fractionis $y=\frac{P}{Q}$ tam numeratorem quam denominatorem evanescere, si statuatur $x=a$. Ad haec autem nihila, quae inter se comparari non possunt, evitanda ponamus $x=a+dx$, quae positio revera in priorem $x=a$ recidit ob $dx=0$. Cum vero, si loco x ponatur $x+dx$, functiones P et Q abeant in $P+dP$ et $Q+dQ$, positioni $x=a+dx$ satisfiet, si in his valoribus ubique statuatur $x=a$, quo quidem casu P et Q evanescere assumuntur. Hinc si loco x ponatur $a+dx$, fractio $\frac{P}{Q}$

transmutabitur in hanc $\frac{dP}{dQ}$ quae propterea valorem functionis $y = \frac{P}{Q}$ exprimit casu $x = a$. Haecque expressio indeterminata amplius esse non poterit, siquidem functionum *P* et *Q* differentialia vera sumantur, uti in capite praecedente docuimus. Hoc enim pacto differentialia *dP* et *dQ* nunquam in nihilum absolutum abeunt, sed, nisi per differentiale *dx* ipsum exprimantur, saltem per eius potestates exhibebuntur. Quodsi igitur reperiatur $dP = Rdx^m$ et $dQ = Sdx^n$, erit functionis $y = \frac{P}{Q}$ casu $x = a$ valor $= \frac{Rdx^m}{Sdx^n}$, qui propterea erit finitus et $= \frac{R}{S}$, si fuerit $m = n$; sin autem sit $m > n$, tum valor fractionis propositae revera erit $= 0$; at si sit $m < n$, iste valor in infinitum excrescit.

358. Quoties ergo huiusmodi fractio occurrit $\frac{P}{Q}$ cuius numerator et denominator certo casu, puta $x = a$, simul evanescant, valor istius fractionis hoc casu $x = a$ per sequentem regulam invenietur:

Quaerantur quantitatum P et Q differentialia casu $x = a$ eaque loco ipsarum P et Q substituantur, quo facto fractio $\frac{dP}{dQ}$ exhibebit valorem fractionis $\frac{P}{Q}$ quaesitum.

Si differentialia *dP* et *dQ* methodo consueta inventa neque infinita fiant neque evanescant casu $x = a$, tum ea retineri poterunt; sin autem ambo vel $= 0$ fiant vel $= \infty$, tum modo in praecedente capite exposito haec differentialia completa casu $x = a$ investigari debent. Plerumque etiam calculus mirifice contrahitur, si antea ponatur $x - a = t$ seu $x = a + t$, quo prodeat fractio $\frac{P}{Q}$ cuius numerator ac denominator evanescunt casu $t = 0$; tum enim differentialia *dP* et *dQ* habebuntur, si ubique *dt* loco *t* substituatur.

EXEMPLUM 1

Quaeratur valor fractionis huius $\frac{b - \sqrt{(bb - tt)}}{tt}$ *casu* $t = 0$.

Quoniam hoc casu $t = 0$ et numerator et denominator evanescit, loco *t* tantum scribatur *dt* atque valor quaesitus exprimetur hac fractione $\frac{b - \sqrt{(bb - dt^2)}}{dt^2}$. Cum vero sit $\sqrt{\left(bb - dt^2\right)} = b - \frac{dt^2}{2b}$, ista fractio abit in hanc $\frac{dt^2}{2bdt^2} = \frac{1}{2b}$. Hinc fracta proposita $\frac{b - \sqrt{(bb - dt^2)}}{dt^2}$ casu $t = 0$ recipit hunc valorem $\frac{1}{2b}$.

EXEMPLUM 2

Quaeratur valor huius fractionis $\frac{\sqrt{(aa + ax + xx)} - \sqrt{(aa - ax + xx)}}{\sqrt{(a + x)} - \sqrt{(a - x)}}$ *casu* $x = 0$.

Hic iterum statim *dx* loco *x* substitui potest; quo facto cum sit

$$\sqrt{\left(aa+adx+dx^2\right)}=a+\tfrac{1}{2}dx+\tfrac{3dx^2}{8a},$$
$$\sqrt{\left(aa-adx+dx^2\right)}=a-\tfrac{1}{2}dx+\tfrac{3dx^2}{8a}$$

atque

$$\sqrt{(a+dx)}=\sqrt{a}+\tfrac{dx}{2\sqrt{a}},$$
$$\sqrt{(a-dx)}=\sqrt{a}-\tfrac{dx}{2\sqrt{a}},$$

fiet numerator $=dx$ et denominator $=\frac{dx}{\sqrt{a}}$, ex quo fractionis propositae valor quaesitus erit $=\sqrt{a}$.

EXEMPLUM 3

Quaeratur valor huius fractionis $\frac{x^3-4ax^2+7a^2x-2a^2\sqrt{(2ax-aa)}}{xx-2ax-aa+2a\sqrt{(2ax-aa)}}$ *casu* $x=a$.

Si more consueto differentialia sumantur et in loca numeratoris ac denominatoris substituantur, habebitur

$$\frac{3xx-8ax+7a^2x-2a^3:\sqrt{(2ax-aa)}}{2x-2a+2a(a-x):\sqrt{(2ax-xx)}},$$

cuius fractionis numerator ac denominator denuo evanescunt, si ponatur $x=a$.
Quare ob eandem rationem eorum loco denuo ipsorum differentialia substituantur prodibitque

$$\frac{6x-8a+2a^4:(2ax-aa)^{\frac{3}{2}}}{2-2a^3:(2ax-xx)^{\frac{3}{2}}},$$

cuius numerator ac denominator iterum casu $x=a$ evanescunt. Pergamus ergo eorum loco ipsorum differentialia substituere

$$\frac{6-6a^5:(2ax-aa)^{\frac{5}{2}}}{6a^3(a-x):(2ax-xx)^{\frac{5}{2}}}=\frac{1-a^5:(2ax-aa)^{\frac{5}{2}}}{a^3(a-x):(2ax-xx)^{\frac{5}{2}}}$$

Verum et hic posito $x=a$ denuo tam numerator quam denominator evanescunt.
Porro igitur differentialibus ipsorum loco substitutis orietur

$$\frac{5a^6:(2ax-aa)^{\frac{7}{2}}}{-\left(5a^5-8a^4x+4a^3xx\right):(2ax-xx)^{\frac{7}{2}}}$$

Nunc denique loco *x* ponatur *a* prodibitque haec fractio determinata $\frac{5:a}{-1:a^2}=-5a$, qui est valor quaesitus fractionis propositae.

Quodsi autem, antequam haec investigatio suscipiatur, ponatur $x=a+t$, fractio proposita transmutabitur in hanc

$$\frac{2a^3+2a^2t-att+t^3-2a^2\sqrt{(aa+2at)}}{-2aa+tt+2a\sqrt{(aa-tt)}};$$

quae cum recipiat formam $\frac{0}{0}$, si ponatur $t=0$, ponatur *dt* loco *t* et erit

$$\frac{2a^3+2a^2dt-adt^2+dt^3-2a^2\sqrt{(aa+2adt)}}{-2aa+dt^2+2a\sqrt{\left(aa-dt^2\right)}}$$

Convertantur iam formulae irrationales in series, quae eousque continuentur, quoad termini a membro rationali non amplius destruantur:

$$\sqrt{(aa+2adt)}=a+dt-\tfrac{dt^2}{2a}+\tfrac{dt^3}{2aa}-\tfrac{5dt^4}{8a^3},$$
$$\sqrt{\left(aa-dt^2\right)}=a-\tfrac{dt^2}{2a}-\tfrac{dt^4}{8a^3};$$

quibus valoribus substitutis prodibit fractio haec

$$\frac{5dt^4:4a}{-dt^4:4aa}=-5a,$$

qui est valor fractionis propositae iam ante inventus.

EXEMPLUM 4

Invenire valorem huius fractionis $\frac{a+\sqrt{(2aa-2ax)}-\sqrt{(2ax-xx)}}{a-x+\sqrt{(aa-xx)}}$ *casu* $x=a$.

Substitutis in loca numeratoris et denominatoris eorum differentialibus prodibit haec fractio, quae casu $x=a$ ipsi propositae erit aequalis:

$$\frac{-a:\sqrt{(2aa-2ax)}-(a-x):\sqrt{(2ax-xx)}}{-1-x:\sqrt{(aa-xx)}},$$

cuius numerator ac denominator casu $x=a$ fiunt infiniti. Verum si uterque per $-\sqrt{(a-x)}$ multiplicetur, habebitur

$$\frac{a:\sqrt{2}a+(a-x)^{\frac{3}{2}}:\sqrt{(2ax-xx)}}{\sqrt{(a-x)}+x:\sqrt{(a+x)}},$$

quae posito $x=a$ dabit hunc valorem determinatum $\frac{a:\sqrt{2}a}{a:\sqrt{2}a}=1$ qui propterea aequalis est fractioni propositae casu $x=a$.

359. Si igitur habeatur fractio $\frac{P}{Q}$, cuius numerator et denominator casu $x = a$ evanescat, eius valor per consuetas differentiandi regulas assignari poterit neque opus erit ad differentialia, quae capite praecedente tractavimus, recurrere. Sumtis enim differentialibus fractio proposita $\frac{P}{Q}$ casu $x=a$ aequalis erit fractioni $\frac{dP}{dQ}$; cuius si numerator et denominator posito $x=a$ induant valores finitos, cognoscetur valor fractionis propositae; sin autem alter fiat $=0$ manente altero finito, tum fractio erit vel $=0$ vel $=\infty$, prout vel numerator evanescat vel denominator. At si alteruter vel uterque fiat $=\infty$, quod evenit, si dividantur per quantitates casu $x=a$ evanescentes, tum multiplicando utrumque per hos divisores istud incommodum tolletur, uti in exemplo postremo evenit. Quodsi vero tam numerator quam denominator casu $x=a$ denuo evanescat, tum iterum, uti initio factum est, differentialia erunt capienda, ita ut haec fractio $\frac{ddP}{ddQ}$ prodeat, quae casu $x=a$ propositae adhuc erit aequalis; et si idem rursus in hac fractione usu veniat, ut fiat $\frac{0}{0}$, tum in eius locum surrogetur haec $\frac{d^3P}{d^3Q}$ atque ita porro, donec ad fractionem perveniatur, quae valorem determinatum exhibeat, sive finitum sive infinite magnum sive infinite parvum. Sic in exemplo tertio oportebat ad fractionem $\frac{d^4P}{d^4Q}$ progredi, antequam valorem fractionis propositae $\frac{P}{Q}$ assignari licuerit.

360. Usus huius investigationis elucet in definiendis summis serierum, quas supra (cap. II § 22) eruimus, si ponatur $x=1$. Ex iis enim, quae ibi tradita sunt, sequitur fore:

$$x + x^2 + x^3 + \cdots + x^n = \frac{x - x^{n+1}}{1-x}$$

$$x + x^3 + x^5 + \cdots + x^{2n-1} = \frac{x - x^{2n+1}}{1-xx}$$

$$x + 2x^2 + 3x^3 + \cdots + nx^n = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

$$x + 3x^3 + 5x^5 + \cdots + (2n-1)x^{2n-1} = \frac{x + x^3 - (2n+1)x^{2n+1} + (2n-1)x^{2n+3}}{(1-xx)^2}$$

$$x + 4x^3 + 9x^5 + \cdots + n^2 x^n = \frac{x + x^2 - (2n+1)^2 x^{n+1} + (2nn+2n-1)x^{n+2} - nnx^{n+3}}{(1-x)^3}$$

etc.

Quodsi nunc harum serierum summae desiderentur casu, quo $x = 1$, in expressionibus istis tam numerator quam denominator evanescunt. Valores ergo harum summarum casu $x = 1$ methodo hic exposita definiri poterunt. Quoniam vero eaedem summae aliunde constant, ex consensu veritas huius methodi magis elucebit.

EXEMPLUM 1

Definire valorem huius fractionis $\frac{x - x^{n+1}}{1-x}$ *casu* $x = 1$, *qui exhibebit summam seriei* $1 + 1 + 1 + \cdots + 1$ *ex n terminis constantis, quae propterea erit* $= n$.

Quoniam casu $x = 1$ numerator ac denominator evanescit, substituantur differentialia in eorum locum habebiturque

$$\frac{1 - (n+1)x^n}{-1},$$

quae posito $x = 1$ dat n pro summa seriei quaesita.

EXEMPLUM 2

Definire valorem fractionis $\frac{x - x^{2n+1}}{1-xx}$ *casu* $x = 1$, *qui exhibebit summam seriei* $1 + 1 + 1 + \cdots + 1$ *ex n terminis constantis, quae propterea erit* $= n$.

Sumtis differentialibus fractio proposita transmutatur in hanc

$$\frac{1-(2n+1)x^{2n}}{-2x},$$

cuius valor posito $x=1$ erit $=n$.

EXEMPLUM 3

Invenire valorem huius fractionis $\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$ *casu* $x=1$, *qui exprimet summam seriei* $1+2+3+\cdots+n$ *, quam constat esse* $=\frac{nn+n}{2}$.

Sumtis differentialibus pervenietur ad hanc fractionem

$$\frac{1-(n+1)^2x^n+n(n+2)x^{n+1}}{-2(1-x)},$$

cuius adhuc tam numerator quam denominator casu $x=1$ evanescit. Hinc denuo differentialia sumantur, ut prodeat haec fractio

$$\frac{-\,n(n+1)^2x^{n-1}+n(n+1)(n+2)x^n}{2},$$

quae posito $x=1$ abit in $\frac{n(n+1)}{2}=\frac{nn+n}{2}$, summam seriei propositae.

EXEMPLUM 4

Invenire valorem huius fractionis $\frac{x+x^3-(2n+1)x^{2n+1}+(2n-1)x^{2n+3}}{(1-xx)^2}$ *casu* $x=1$, *qui exprimet summam seriei* $1+3+5+\cdots+(2n-1)$, *quam constat esse* $=nn$.

Substitutis differentialibus in loca numeratoris et denominatoris provenit haec fractio

$$\frac{1+3xx-(2n+1)^2x^{2n}+(2n-1)(2n+3)x^{2n+2}}{-4x(1-xx)};$$

quae cum adhuc idem incommodum habeat, ut posito $x=1$ abeat in $\frac{0}{0}$, denuo differentialia sumantur

$$\frac{6x-2n(2n+1)^2x^{2n-1}+(2n-1)(2n+2)(2n+3)x^{2n+1}}{-4+12xx},$$

quae posito $x=1$ abit in

$$\frac{6-2n(2n+1)^2+(2n-1)(2n+2)(2n+3)}{8}=nn.$$

EXEMPLUM 5

Invenire valorem huius fractionis

$$\frac{x+x^2-(2n+1)^2x^{n+1}+(2nn+2n-1)x^{n+2}-nnx^{n+3}}{(1-x)^3}$$

casu $x=1$, *qui dabit summam seriei* $1+\ 4+9+\cdots+n^2$, *quam constat esse* $=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n$.

Sumtis numeratoris ac denominatoris differentialibus fiet

$$\frac{1+2x-(n+1)^3x^n+(n+2)(2nn+2n-1)x^{n+1}-nn(n+3)x^{n+2}}{-3(1-x)^2}.$$

in qua cum numerator ac denominator posito $x=1$ denuo evanescat, differentialia secunda sumantur

$$\frac{2-n(n+1)^3x^{n-1}+(n+1)(n+2)(2nn+2n-1)x^n-n^2(n+2)(n+3)x^{n+1}}{6(1-x)}$$

Eodem vero adhuc subsistente incommodo ad differentialia tertia procedatur, ut prodeat haec fractio

$$\frac{-\ n(n-1)(n+1)^3x^{n-2}+n(n+1)(n+2)(2nn+2n-1)x^{n-1}-n^2(n+1)(n+2)(n+3)x^n}{-6},$$

quae tandem posito $x=1$ abit in hanc formam determinatam

$$\frac{-n(n-1)(n+1)^3+n(n+1)(n+2)(nn-n-1)}{-6}=\frac{n(n+1)(2n+1)}{6}=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n;$$

qui est ille ipse valor, quo seriem memoratam exprimi invenimus.

EXEMPLUM 6

Sit proposita ista fractio $\frac{x^m-x^{m+n}}{1-x^{2p}}$, *cuius valorem casu* $x=1$ *assignari oporteat.*

Quoniam haec fractio est productum ex his duabus $\frac{x^m}{1+x^p}\cdot\frac{1-x^n}{1-x^p}$, prioris autem factoris casu $x=1$ valor est $=\frac{1}{2}$, tantum opus est, ut alterius factoris $\frac{1-x^n}{1-x^p}$ valor eodem casu quaeratur, qui sumtis differentialibus erit $=\frac{nx^{n-1}}{px^{p-1}}=\frac{n}{p}$; unde fractionis propositae valor casu $x=1$ erit $=\frac{n}{2p}$. Idem valor prodit, si immediate differentialia in fractione proposita capiantur; fiet enim

$$\frac{mx^{m-1}-(m+n)x^{m+n-1}}{-2px^{2p-1}},$$

cuius valor posito $x=1$ erit $=\frac{-n}{-2p}=\frac{n}{2p}$ ut ante.

361. Eadem methodo erit utendum, si in fractione proposita $\frac{P}{Q}$ vel numerator vel denominator vel uterque fuerit quantitas transcendens. Quae operationes quo clarius explicentur, sequentia exempla adiicere visum est.

EXEMPLUM 1

Sit proposita ista fractio $\frac{a^n-x^n}{la-lx}$, cuius *valor quaeratur casu* $x=a$.

Sumtis differentialibus statim pervenitur ad hanc fractionem

$$\frac{-nx^{n-1}}{-1:x}=nx^n,$$

cuius valor posito $x=a$ erit na^n.

EXEMPLUM 2

Sit proposita ista fractio $\frac{lx}{\sqrt{(1-x)}}$, *cuius valor quaeratur casu* $x=1$.

Sumtis differentialibus numeratoris et denominatoris prodit

$$\frac{1:x}{-1:2\sqrt{(1-x)}}=\frac{-2\sqrt{(1-x)}}{x};$$

cuius valor posito $x=1$ cum sit $=0$, sequitur fractionem $\frac{lx}{\sqrt{(1-x)}}$ casu $x=1$ evanescere.

EXEMPLUM 3

Sit proposita ista fractio $\frac{a-x-ala+alx}{a-\sqrt{(2ax-xx)}}$, *cuius valor quaeratur posito* $x=a$, *quo casu numerator et denominator evanescunt.*

Differentiatis secundum regulam numeratore ac denominatore erit

$$\frac{-1+a:x}{-(a-x):\sqrt{(2ax-xx)}}=\frac{(a-x)\sqrt{(2ax-xx)}}{-x(a-x)}\;;$$

ubi etsi numerator ac denominator casu $x=a$ adhuc evanescit, tamen, quia uterque divisibilis est per $a-x$, habebitur ista fractio $-\sqrt{\frac{2a-x}{x}}$, cuius valor casu $x=a$ est determinatus atque $=-1$; abitque igitur fractio proposita in –1, si ponatur $x=a$.

EXEMPLUM 4

Sit proposita ista fractio $\frac{e^x-e^{-x}}{l(1+x)}$, *cuius valor quaeratur posito* $x=0$.

Sumtis differentialibus habebitur ista functio

$$\frac{e^x+e^{-x}}{1:(1+x)}\,,$$

quae posito $x=0$ dat 2 pro valore quaesito.

EXEMPLUM 5

Invenire valorem huius fractionis $\frac{e^x-1-l(1+x)}{xx}$ *casu, quo ponitur* $x=0$.

Si loco numeratoris ac denominatoris eorum differentialia substituantur, orietur haec fractio

$$\frac{e^x-1:(1+x)}{2x}\,,$$

quae cum adhuc abeat in $\frac{0}{0}$ si ponatur $x=0$, denuo differentialia sumantur, ut habeatur

$$\frac{e^x+1:(1+x)^2}{2}\,,$$

quae posito $x=0$ praebet $\frac{1+1}{2}=1$. Quod idem patet, si loco x statim $0+dx$ substituatur, cum enim sit

$$e^{dx}=1+dx+\tfrac{1}{2}dx^2+\text{etc. et }\;l(1+dx)=dx-\tfrac{1}{2}dx^2+\text{ etc.},$$

$$\frac{e^{dx}-1-l(1+dx)}{dx^2}=\frac{dx^2}{dx^2}=1.$$

EXEMPLUM 6

Quaeratur valor fractionis $\frac{x^n}{lx}$ *casu, quo ponitur* $x=\infty$.

Quo ista fractio ad formam, quae hoc casu transeat in $\frac{0}{0}$, reducatur, ita repraesentetur

$$\frac{1:lx}{1:x^n}\ ;$$

sic enim casu $x=\infty$ tam numerator quam denominator evanescet. Ponatur vero porro $x=\frac{1}{y}$, ita ut casu $x=\infty$ fiat $y=0$, atque proponetur ista y fractio

$$-\frac{1:ly}{y^n},$$

cuius valor casu $y=0$ investigari debet. Sumtis autem differentialibus erit $\frac{1:y(ly)^2}{ny^{n-1}}=\frac{1:(ly)^2}{ny^n}$; quae posito $y=0$ cum abeat in $\frac{0}{0}$, sumantur denuo differentialia eritque $\frac{-2:(ly)^3}{n^2y^n}$; ubi quia idem incommodum adest, si porro differentialia sumantur, prodibit $\frac{6:(ly)^4}{n^3y^n}$ sicque, quousque procedamus, perpetuo idem incommodum occurret. Quamobrem, ut hoc non obstante valorem quaesitum eruamus, sit s valor fractionis $-\frac{1:ly}{y^n}$ casu, quo ponitur $y=0$, et cum eadem casu sit quoque

$$s=\frac{1:(ly)^2}{ny^n},$$

erit ex illa aequatione

$$ss=\frac{1:(ly)^2}{y^{2n}}$$

quae per istam divisa dabit

$$s=\frac{ny^n}{y^{2n}}=\frac{n}{y^n},$$

ex qua perspicitur casu $y=0$ fieri s infinitum. Fit ergo fractionis $-\frac{1:ly}{y^n}$ valor casu $y=0$ infinitus ideoque posito $y=dx$ habebit $\frac{1}{ldx}$ ad dx^n rationem infinitam, uti iam supra innuimus.

EXEMPLUM 7

Quaeratur valor fractionis $\frac{x^n}{e^{-1:dx}}$ *casu* $x=0$, *quo tam numerator quam denominator evanescit.*

Sit hoc casu $\frac{x^n}{e^{-1:dx}} = s$; erit sumtis differentialibus quoque

$$s = \frac{nx^{n-1}}{e^{-1:dx} : xx} = \frac{nx^{n+1}}{e^{-1:dx}},$$

et quia hic idem incommodum occurrit perpetuoque recurrit, quousque differentiationes continuentur, remedio ante adhibito utamur. Prior aequatio dat

$$x^n = e^{-1:dx} s \quad \text{et} \quad x^{n(n+1)} = e^{-(n+1):x} s^{n+1};$$

altera aequatio dat

$$x^{n+1} = e^{-1:x} s : n,$$

unde fit

$$x^{n(n+1)} = e^{-n:x} s : n^n,$$

qui valor illi aequatus dabit

$$e^{-1:x} s n^n = 1$$

ideoque

$$s = \frac{1}{n^n e^{-1:x}} = \infty,$$

si $x = 0$. Quare posito x infinite parvo habebit dx^n ad $e^{-1:dx}$ rationem infinite magnam, quicunque numerus finitus pro n statuatur; unde sequitur $e^{-1:dx}$ esse infinite parvum homogeneum cum dx^m, si m fuerit numerus infinite magnus.

EXEMPLUM 8

Quaeratur valor $\frac{1-\sin x+\cos x}{\sin x+\cos x-1}$ *casu, quo ponitur* $x = \frac{\pi}{2}$ *seu arcui* 90 *graduum.*

Sumtis differentialibus obtinebitur haec fractio

$$\frac{-\cos x-\sin x}{\cos x-\sin x}$$

quae posito $x = \frac{\pi}{2}$ ob $\sin x = 1$ et $\cos x = 0$ abit in 1, ita ut unitas sit valor quaesitus fractionis propositae. Quod idem patet sine differentiatione; cum enim sit $\cos x = \sqrt{(1+\sin x)(1-\sin x)}$, fractio proposita abit in hanc

$$\frac{\sqrt{(1-\sin x)}+\sqrt{(1+\sin x)}}{\sqrt{(1+\sin x)}-\sqrt{(1-\sin x)}},$$

quae fit evidenter $=1$, si fiat $\sin x = 1$.

EXEMPLUM 9

Invenire valorem huius expressionis $\frac{x^x - x}{1 - x + lx}$ *casu, quo ponitur* $x = 1$.

Loco numeratoris et denominatoris eorum differentialibus substitutis prodibit ista fractio

$$\frac{x^x(1+lx)-1}{-1+1:x}$$

quae cum etiam nunc fiat $= \frac{0}{0}$ posito $x = 1$, sumantur denuo differentialia, ut prodeat

$$\frac{x^x(1+lx)^2 + x^x : x}{-1 : xx}$$

quae posito $x = 1$, abit in -2, qui est valor fractionis propositae casu $x = 1$.

362. Quoniam hic omnes expressiones, quae quibusdam casibus indeterminatos valores recipere videntur, pertractare constituimus, huc non solum pertinent eae fractiones $\frac{P}{Q}$, quarum numerator ac denominator certo casu evanescunt, sed etiam eiusmodi fractiones, quarum numerator ac denominator certo casu fiunt infiniti, huc sunt referendae, propterea quod earum valores aeque indeterminati videntur. Si scilicet *P* et *Q* eiusmodi fuerint functiones ipsius *x*, ut casu quopiam $x = a$ ambae fiant infinitae fractioque $\frac{P}{Q}$ induat hanc formam $\frac{\infty}{\infty}$, quoniam infinita aeque ac cyphrae inter se rationem quamcunque tenere possunt, hinc valor verus minime cognosci potest. Hic quidem casus ad praecedentem revocari potest fractionem $\frac{P}{Q}$ in hanc formam $\frac{1:Q}{1:P}$ transmutando, cuius fractionis nunc numerator ac denominator casu $x = a$ evanescunt; ideoque eius valor modo ante tradito inveniri potest. At vero quoque sine hac transformatione valor invenietur, si loco *x* non *a*, sed $a + dx$ substituatur; quo facto non eiusmodi infinita absoluta ∞ provenient, sed ita erunt expressa $\frac{1}{dx}$ vel $\frac{A}{dx^n}$; quae expressiones etsi sunt aeque infinitae ac ∞, tamen comparatione inter *dx* eiusve potestates instituta valor quaesitus facile colligetur.

363. Ad eandem classem quoque pertinent producta ex duobus factoribus constantia, quorum alter certo casu $x = a$ evanescit, alter vero in infinitum abit; cum enim quaevis quantitas per huiusmodi productum $0 \cdot \infty$ repraesentari possit, eius valor indefinitus videtur. Sit *PQ* huiusmodi productum, in quo, si ponatur $x = a$, fiat $P = 0$ et $Q = \infty$; eius valor per praecepta ante tradita invenietur, si ponatur $Q = \frac{1}{R}$; tum enim productum *PQ* transmutabitur in fractionem $\frac{P}{R}$ cuius numerator ac denominator ambo casu $x = a$ evanescunt; ideoque eius valor methodo ante exposita investigari poterit.

Sic si quaeratur valor huius producti

$$(1-x)\tan\frac{\pi x}{2}$$

casu $x=1$, quo fit $1-x=0$ et $\operatorname{tang}\frac{\pi x}{2}=\infty$, convertatur id in hanc fractionem

$$\frac{1-x}{\cot\frac{1}{2}\pi x},$$

cuius numerator ac denominator casu $x=1$ evanescunt. Cum igitur sit differentiale numeratoris $1-x=-dx$ et differentiale denominatoris $\cot\frac{1}{2}\pi x=-\frac{\pi dx:2}{\left(\sin\frac{1}{2}\pi x\right)^2}$, casu $x=1$ valor fractionis propositae erit

$$=\frac{2}{\pi}\sin\frac{\pi x}{2}\cdot\sin\frac{\pi x}{2}=\frac{2}{\pi}$$

ob $\sin\frac{\pi}{2}=1$.

364. Imprimis autem huc sunt referendae eiusmodi expressiones, quae, dum ipsi *x* certus quidam valor tribuitur, abeunt in huiusmodi formam $\infty-\infty$; quoniam enim duo infinita quavis quantitate finita inter se discrepare possunt, manifestum est hoc casu valorem expressionis non determinari, nisi differentia inter illa duo infinita assignari possit. Iste ergo casus occurrit, si proponatur huiusmodi functio $P-Q$, in qua posito $x=a$ fiat tam $P=\infty$ quam $Q=\infty$, quo casu ope regulae ante traditae valor quaesitus non tam facile assignari potest. Etsi enim, posito hoc casu fieri $P-Q=f$, statuatur $e^{P-Q}=e^f$, ita ut sit $e^f=\frac{e^{-Q}}{e^{-P}}$, ubi casu $x=a$ tam numerator e^{-Q} quam denominator e^{-P} evanescit, tamen, si regula ante tradita huc transferatur, fiet $e^f=\frac{e^{-Q}dQ}{e^{-P}dP}$, unde ob $e^f=\frac{e^{-Q}}{e^{-P}}$ fieret $1=\frac{dQ}{dP}$ ideoque valor quaesitus ipsius *f* hinc non innotescit. Quoties quidem *P* et *Q* sunt quantitates algebraicae, quoniam hae infinitae fieri nequeunt, nisi sint fractiones, quarum denominatores evanescunt, tum $P-Q$ in unicam fractionem colligi poterit, cuius denominator pariter evanescet. Quo facto si etiam numerator evanescat, valor modo supra explicato definietur; sin autem numerator non evanescat, tum eius valor revera erit infinitus.

Sic si huius expressionis

$$\frac{1}{1-x}-\frac{2}{1-xx}$$

valor desideretur casu $x=1$, quia ea abit in

$$\frac{-1+x}{1-xx}=\frac{-1}{1+x},$$

patet valorem quaesitum esse $=-\frac{1}{2}$.

365. Verum si functiones *P* et *Q* fuerint transcendentes, tum plerumque haec transformatio ad calculum molestissimum perduceret. Expediet ergo his casibus methodo directa uti atque loco $x = a$, quo ambae quantitates P et Q in infinitum abeunt, poni $x = a + \omega$ existente ω quantitate infinite parva, pro qua dx accipi poterit. Quo facto si fiat $P = \frac{A}{\omega} + B$ at $Q = \frac{A}{\omega} + C$, manifestum est functionem $P - Q$ abituram esse in $B - C$, qui erit valor finitus. Rationem igitur huiusmodi functionum valores investigandi sequentibus exemplis illustrabimus.

EXEMPLUM 1

Quaeratur valor huius expressionis $\frac{x}{x-1} - \frac{1}{lx}$ *casu, quo ponitur* $x = 1$.

Quoniam tam $\frac{x}{x-1}$ quam $\frac{1}{lx}$ fit infinitum posito $x = 1$, statuatur $x = 1 + \omega$ atque expressio proposita transformabitur in

$$\frac{1+\omega}{\omega} - \frac{1}{l(1+\omega)}$$

Cum igitur sit

$$l(1+\omega) = \omega - \frac{1}{2}\omega^2 + \frac{1}{3}\omega^3 - \text{etc.} = \omega\left(1 - \frac{1}{2}\omega + \frac{1}{3}\omega^2 - \text{etc.}\right),,$$

habebitur

$$\frac{(1+\omega)\left(1-\frac{1}{2}\omega+\frac{1}{3}\omega^2-\text{etc.}\right)-1}{\omega\left(1-\frac{1}{2}\omega+\frac{1}{3}\omega^2-\text{etc.}\right)} = \frac{\frac{1}{2}\omega-\frac{1}{6}\omega^2+\text{etc.}}{\omega\left(1-\frac{1}{2}\omega+\frac{1}{3}\omega^2-\text{etc.}\right)} = \frac{\frac{1}{2}-\frac{1}{6}\omega+\text{etc.}}{1-\frac{1}{2}\omega+\frac{1}{3}\omega^2-\text{etc.}}$$

Posito nunc ω infinite parvo seu $\omega = 0$ manifestum est valorem quaesitum esse $= \frac{1}{2}$.

EXEMPLUM 2

Denotantibus e numerum, cuius logarithmus hyperbolicus est $= 1$, *et* π *semicircumferentiam circuli, cuius radius est* = 1, *investigare valorem huius expressionis* $\frac{\pi x-1}{2xx} + \frac{\pi}{x\left(e^{2\pi x}-1\right)}$ *casu x=0.*

Expressio ista proposita exhibet summam huius seriei

$$\frac{1}{1+xx} + \frac{1}{4+xx} + \frac{1}{9+xx} + \frac{1}{16+xx} + \frac{1}{25+xx} + \text{etc.};$$

unde, si ponatur $x = 0$, prodire debet summa seriei huius

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \text{etc.},$$

quam constat esse $= \frac{\pi\pi}{6}$. Facto autem $x = 0$ expressionis propositae

$$\frac{\pi x-1}{2xx}+\frac{\pi}{x\left(e^{2\pi x}-1\right)}$$

valor maxime videtur indeterminatus ob omnes terminos infinitos. Ponatur ergo $x=\omega$ existente ω quantitate infinite parva atque membrum prius $\frac{\pi x-1}{2xx}$ abit in

$$-\frac{1}{2\omega^2}+\frac{\pi}{2\omega}.$$

Cum deinde sit

$$e^{2\pi\omega}-1=2\pi\omega+2\pi^2\omega^2+\tfrac{4}{3}\pi^3\omega^3+\text{etc.}$$

alterum membrum $\frac{\pi}{x\left(e^{2\pi x}-1\right)}$ abit in

$$\frac{\pi}{\omega\left(2\pi\omega+2\pi^2\omega^2+\frac{4}{3}\pi^3\omega^3+\text{etc.}\right)}=\frac{1}{2\omega^2\left(1+\pi\omega+\frac{2}{3}\pi^2\omega^2+\text{etc.}\right)}.$$

At est

$$\frac{1}{1+\pi\omega+\frac{2}{3}\pi^2\omega^2+\text{etc.}}=1-\pi\omega+\tfrac{1}{3}\pi^2\omega^2-\text{etc.},$$

unde posterius membrum fit

$$=\frac{1}{2\omega^2}-\frac{\pi}{2\omega}+\tfrac{1}{6}\pi^2-\text{etc.};$$

ad quod si prius addatur, prodit $\frac{1}{6}\pi^2$, qui est valor quaesitus expressionis propositae casu $x=0$.

Idem quoque per methodum fractionum, quarum numerator ac denominator certo casu evanescunt, praestari potest; expressio enim proposita in hanc fractionem transmutatur

$$\frac{\pi xe^{2\pi x}-e^{2\pi x}+\pi x+1}{2xxe^{2\pi x}-2xx}$$

cuius numerator ac denominator casu $x=0$ evanescunt. Sumtis ergo differentialibus oritur

$$\frac{\pi e^{2\pi x}+2\pi\pi xe^{2\pi x}-2\pi e^{2\pi x}+\pi}{4xe^{2\pi x}+4\pi xxe^{2\pi x}-4x}$$

sive haec

$$\frac{\pi-\pi e^{2\pi x}+2\pi\pi xe^{2\pi x}}{4xe^{2\pi x}+4\pi xxe^{2\pi x}-4x}$$

cuius, si ponatur $x=0$, adhuc numerator ac denominator evanescunt. Quare sumtis denuo differentialibus habebitur

$$\frac{-\pi\pi e^{2\pi x}+2\pi\pi e^{2\pi x}+4\pi^3 xe^{2\pi x}}{4e^{2\pi x}+8\pi xe^{2\pi x}+8\pi xe^{2\pi x}+8\pi^2 xxe^{2\pi x}-4}$$

seu

$$\frac{\pi^3 xe^{2\pi x}}{e^{2\pi x}+4\pi xe^{2\pi x}+2\pi^2 x^2 e^{2\pi x}-1}$$

seu

$$\frac{\pi^3 x}{1+4\pi x+2\pi^2 x^2-e^{-2\pi x}}$$

cuius numerator ac denominator adhuc evanescunt casu $x=0$. Quocirca iterum differentialia sumantur

$$\frac{\pi^3}{4\pi+4\pi^2 x-2\pi e^{-2\pi x}},$$

quae fractio posito $x=0$ abit in $\frac{\pi^2}{6}$ ut ante.

EXEMPLUM 3

Retinentibus e et π eosdem valores quaeratur valor expressionis huius casu $x=0$

$$\frac{\pi}{4x}-\frac{\pi}{2x\left(e^{\pi x}+1\right)}.$$

Expressio haec transmutatur in hanc

$$\frac{\pi e^{\pi x}-\pi}{4xe^{\pi x}+4x}$$

cuius numerator ac denominator casu $x=0$ evanescunt. Ponatur ergo $x=\omega$, et cum sit

$$e^{\pi\omega}=1+\pi\omega+\tfrac{1}{2}\pi^2\omega^2+\tfrac{1}{6}\pi^3\omega^3+\text{etc.},$$

formula proposita transmutatur in hanc

$$\frac{\pi^2\omega+\frac{1}{2}\pi^3\omega^2+\frac{1}{6}\pi^4\omega^3+\text{etc.}}{8\omega+4\pi\omega^2+2\pi^2\omega^3+\text{etc.}},$$

quae posito ω infinite parvo statim dat valor $\frac{1}{8}\pi^2$, qui est valor quaesitus expressionis propositae casu $x=0$. At vero expressio proposita $\frac{\pi}{4x}-\frac{\pi}{2x\left(e^{\pi x}+1\right)}$ exhibet summam huius seriei

$$\frac{1}{1+xx}+\frac{1}{4+xx}+\frac{1}{9+xx}+\frac{1}{16+xx}+\frac{1}{25+xx}+\text{etc.}$$

cuius summa posito $x=0$ utique fit $=\frac{1}{8}\pi^2$.

EXEMPLUM 4

Quaeratur valor huius expressionis $\frac{1}{2xx}-\frac{\pi}{2x\text{tang}\,\pi x}$ *casu* $x=0$.

Formula haec proposita $\frac{1}{2xx}-\frac{\pi}{2x\text{tang}\,\pi x}$ exprimit summam huius seriei infinitae

$$\frac{1}{1-xx}+\frac{1}{4-xx}+\frac{1}{9-xx}+\frac{1}{16-xx}+\text{etc.}$$

Si igitur ponatur $x=0$, prodire debet summa seriei

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\text{etc.},$$

quae est $=\frac{1}{6}\pi^2$. Quoniam est $\text{tang}\,\pi x=\frac{\sin\pi x}{\cos\pi x}$ expressio proposita induet hanc formam

$$\frac{1}{2xx}-\frac{\pi\cos\pi x}{2x\sin\pi x}=\frac{\sin\pi x-\pi x\cos\pi x}{2xxx\sin\pi x},$$

cuius numerator ac denominator evanescit posito $x=0$. Ponatur ergo
$z=\omega$, et cum sit

$$\sin\pi\omega=\pi\omega-\frac{1}{6}\pi^3\omega^3+\text{etc.},\ \cos\pi\omega=1-\frac{1}{2}\pi^2\omega^2+\text{etc.},$$

expressio proposita fiet

$$\frac{\pi\omega-\frac{1}{6}\pi^3\omega^3+\text{etc.}-\pi\omega+\frac{1}{2}\pi^3\omega^3-\text{etc.}}{2\pi\omega^3-\frac{1}{3}\pi^3\omega^5}=\frac{\frac{1}{3}\pi^3\omega^3-\text{etc.}}{2\pi\omega^3-\text{etc.}},$$

quae ob *w* infinite parvum dat $\frac{1}{6}\pi^2$.

EXEMPLUM 5

Cum sit summa huius seriei infinitae $\frac{1}{1-xx}+\frac{1}{9-xx}+\frac{1}{25-xx}+\frac{1}{49-xx}+\text{etc.}=\frac{\pi\sin\frac{1}{2}\pi x}{4x\cos\frac{1}{2}\pi x}$, *invenire eius summam, si fuerit* $x=0$.

Quia est

$$\sin\tfrac{1}{2}\pi x=\tfrac{1}{2}\pi x-\tfrac{1}{48}\pi^3x^3+\text{etc.},\ \cos\tfrac{1}{2}\pi x=1-\tfrac{1}{8}\pi^2x^2+\text{etc.},$$

erit expressio proposita

$$=\frac{\frac{1}{2}\pi^2 x-\frac{1}{48}\pi^4 x^3+\text{etc.}}{4x-\frac{1}{2}\pi^2 x^3+\text{etc.}}=\frac{\frac{1}{2}\pi^2-\frac{1}{48}\pi^4 x^2+\text{etc.}}{4-\frac{1}{2}\pi^2 x^2+\text{etc.}};$$

In qua si fiat $x=0$, valor erit manifesto $=\frac{1}{8}\pi^2$, quam esse summam seriei

$$\frac{1}{1}+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\text{etc.}$$

supra pluribus modis est demonstratum. Sin autem pro x sumatur numerus par quicunque, summa seriei propositae semper est $=0$.

366. In his seriebus, quas binis ultimis exemplis tractavimus, aliisque litteram variabilem x continentibus ipsi x eiusmodi valores tribui possunt, ut quidam termini in infinitum excrescant, quibus quidem casibus summa totius seriei fiet infinita. Sic seriei

$$\frac{1}{1-xx}+\frac{1}{4-xx}+\frac{1}{9-xx}+\frac{1}{16-xx}+\text{etc.}$$

si pro x ponatur numerus quicunque integer, unus perpetuo terminus ob denominatorem evanescentem fit infinitus hancque ob causam ipsa seriei summa infinita evadet. Quodsi autem iste terminus infinitus ex serie tollatur, tum summa reliqua sine dubio erit finita exprimeturque summa priori infinita termino isto infinito mulctata hoc modo $\infty-\infty$; quemnam ergo habitura sit valorem determinatum, modo hic exposito inveniri poterit, id quod clarius ex subiunctis exemplis perspicietur.

EXEMPLUM 1

Invenire summam seriei $\frac{1}{1-xx}+\frac{1}{4-xx}+\frac{1}{9-xx}+\frac{1}{16-xx}+\text{etc.}$ *casu* $x=1$ *et demto termino primo, qui hoc casu in infinitum augetur.*

Quia in genere summa est

$$\frac{1}{2xx}-\frac{\pi}{2x\operatorname{tang}\pi x}$$

erit summa quaesita

$$\frac{1}{2xx}-\frac{\pi}{2x\operatorname{tang}\pi x}-\frac{1}{1-xx}$$

posito $x=1$. Sit $x=1+\omega$ et habebitur pro summa quaesita

$$\frac{1}{2(1+2\omega+\omega\omega)}-\frac{\pi}{2(1+\omega)\operatorname{tang}(\pi+\omega x)}+\frac{1}{2\omega+\omega\omega}$$

At est

$$\operatorname{tang}(\pi+\omega\pi)=\operatorname{tang}\omega\pi=\pi\omega+\tfrac{1}{3}\pi^3\omega^3+\text{etc}.$$

Unde, cum primus terminus $\frac{1}{2xx}$ posito $x=1$ determinatum habeat valorem $\frac{1}{2}$ duo reliqui tantum termini sunt spectandi, qui erunt

$$\frac{1}{\omega(2+\omega)}-\frac{\pi}{2\omega(1+\omega)\left(\pi+\frac{1}{3}\pi^3\omega^2\right)}=\frac{1}{\omega(2+\omega)}-\frac{1}{\omega(2+2\omega)\left(1+\frac{1}{3}\pi^2\omega^2\right)},$$

siquidem ω sit infinite parvum, quo casu etiam terminus $\frac{1}{3}\pi^2\omega^2$ negligi poterit. Proveniet autem

$$\frac{\omega}{\omega(2+\omega)(2+2\omega)}=\frac{1}{4}$$

posito $\omega=0$ estque ergo $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$ summa serei

$$\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\text{etc.},$$

uti aliunde constat.

EXEMPLUM 2

Invenire summam seriei

$$\frac{1}{1-xx}+\frac{1}{4-xx}+\frac{1}{9-xx}+\frac{1}{16-xx}+\text{etc}.$$

casu, quo pro x ponitur numerus quicunque integer n, et demto ex serie,termino illo $\frac{1}{nn-xx}$*, qui fit infinitus.*

Summa ergo haec, quae quaeritur, ita erit expressa

$$\frac{1}{2xx}-\frac{\pi}{2x\operatorname{tang}\pi x}-\frac{1}{nn-xx},$$

siquidem statuatur $x=n$, quo quidem casu primus terminus $\frac{1}{2xx}$ abit in $\frac{1}{2nn}$, bini vero reliqui ambo fiunt infiniti. Ponatur ergo $x=n+\omega$, et cum sit $\operatorname{tang}(\pi n+\pi\omega)=\operatorname{tang}\pi\omega=\pi\omega$ posito ω infinite parvo, habebimus pro summa quaesita

$$\frac{1}{2nn}-\frac{\pi}{2(n+\omega)\pi\omega}+\frac{1}{2n\omega+\omega\omega}$$

seu

$$\frac{1}{2nn}-\frac{1}{\omega(2n+2\omega)}+\frac{1}{\omega(2n+\omega)}=\frac{1}{2nn}+\frac{1}{(2n+2\omega)(2n+\omega)}$$

unde, si fiat $\omega=0$, prodibit summa quaesita

$$=\frac{1}{2nn}+\frac{1}{4nn}=\frac{3}{4nn}.$$

Quocirca erit

$$\frac{3}{4nn}=\frac{1}{1-nn}+\frac{1}{4-nn}+\frac{1}{9-nn}+\cdots+\frac{1}{(n-1)^2-nn}+\frac{1}{(n+1)^2-nn}+\frac{1}{(n+2)^2-nn}+\text{etc. in infinitum}$$

sive erit istius seriei infinitae summa

$$\frac{1}{(n+1)^2-nn}+\frac{1}{(n+2)^2-nn}+\frac{1}{(n+3)^2-nn}+\text{etc.}=\frac{3}{4nn}+\frac{1}{nn-1}+\frac{1}{nn-4}+\frac{1}{nn-9}+\cdots+\frac{1}{nn-(n-1)^2}.$$

EXEMPLUM 3

Invenire summam huius seriei

$$\frac{1}{1-xx}+\frac{1}{9-xx}+\frac{1}{25-xx}+\frac{1}{49-xx}+\text{etc.},$$

si ponatur $x=1$ *atque terminus primus* $\frac{1}{1-xx}$ *qui hoc casu fit infinitus, auferatur.*

Cum huius seriei summa sit in genere $=\frac{\pi\sin\frac{1}{2}\pi x}{4x\cos\frac{1}{2}\pi x}$, erit summa quaesita

$$=\frac{\pi\sin\frac{1}{2}\pi x}{4x\cos\frac{1}{2}\pi x}-\frac{1}{1-xx},$$

si ponatur $x=1$. Quia vero uterque terminus fit infinitus, ponatur $x=1-\omega$, et cum sit

$$\sin\left(\tfrac{1}{2}\pi-\tfrac{1}{2}\pi\omega\right)=\cos\tfrac{1}{2}\pi\omega=1-\tfrac{1}{8}\pi^2\omega^2$$

et

$$\cos\left(\tfrac{1}{2}\pi-\tfrac{1}{2}\pi\omega\right)=\sin\tfrac{1}{2}\pi\omega=\tfrac{1}{2}\pi\omega$$

ob ω infinite parvum, habebitur ista expressio

$$\frac{\pi\left(1-\frac{1}{8}\pi^2\omega^2\right)}{4(1-\omega)\frac{1}{2}\pi\omega}-\frac{1}{2\omega-\omega\omega}=\frac{1}{\omega(2-2\omega)}-\frac{1}{\omega(2-\omega)}$$

quae fit $=\frac{1}{4}$ posito $\omega=0$, estque propterea

$$\frac{1}{4}=\frac{1}{8}+\frac{1}{24}+\frac{1}{48}+\frac{1}{80}+\frac{1}{120}+\text{etc.}$$

EXEMPLUM 4

Invenire summam seriei huius

$$\frac{1}{1-xx}+\frac{1}{9-xx}+\frac{1}{25-xx}+\frac{1}{49-xx}+\text{etc.},$$

si pro x ponatur numerus quicunque integer impar $2n-1$ *isque terminus* $\frac{1}{(2n-1)^2-xx}$, *qui hoc casu fit infinitus, e medio tollatur*.

Erit ergo summa, quae quaeritur,

$$=\frac{\pi\sin\frac{1}{2}\pi x}{4x\cos\frac{1}{2}\pi x}-\frac{1}{(2n-1)^2-xx}$$

posito $x=2n-1$. Statuamus ergo $x=2n-1-\omega$ existente ω infinite parvo fietque

$$\sin\tfrac{1}{2}\pi x=\sin\left(\tfrac{2n-1}{2}\pi-\tfrac{1}{2}\pi\omega\right)=\pm\cos\tfrac{1}{2}\pi\omega$$

ubi signum superius valet, si sit n numerus impar, inferius vero, si sit par. Simili modo erit

$$\cos\tfrac{1}{2}\pi x=\cos\left(\tfrac{2n-1}{2}\pi-\tfrac{1}{2}\pi\omega\right)=\pm\sin\tfrac{1}{2}\pi\omega\ ;$$

ideoque, sive n sit par sive impar, erit

$$\frac{\sin\frac{1}{2}\pi x}{\cos\frac{1}{2}\pi x}=\frac{1}{\operatorname{tang}\frac{1}{2}\pi\omega}=\frac{1}{\frac{1}{2}\pi\omega}.$$

Hinc summa quaesita ita exprimetur

$$\frac{1}{2\omega(2n-1-\omega)}-\frac{1}{\omega(2(2n-1)-\omega)}$$

eritque propterea $\frac{1}{4(2n-1)^2}$. Sic, si sit $n=2$, erit

$$\frac{1}{36}=-\frac{1}{8}+\frac{1}{16}+\frac{1}{40}+\frac{1}{72}+\frac{1}{112}+\text{etc.},$$

cuius summationis veritas aliunde constat.