CHAPTER XIV

CONCERNING THE DIFFERENTIALS OF FUNCTIONS IN CERTAIN CASES ONLY

337. If y were some function of x and this quantity of the variable x may be increased by the increment ω, so that x may change into $x+\omega$, then the function y may adopt this value

$$y+\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}+\frac{\omega^3 d^3y}{6dx^3}+\frac{\omega^4 d^4y}{24dx^4}+\text{etc.}$$

and thus may take this increment

$$\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}+\frac{\omega^3 d^3y}{6dx^3}+\frac{\omega^4 d^4y}{24dx^4}+\text{etc.},$$

as we have shown above [§ 48]. Whereby if there may be made $\omega = dx$, thus so that x may increase by its own differential *dx*, then the function y may take this increment

$$dy+\tfrac{1}{2}ddy+\tfrac{1}{6}d^3y+\tfrac{1}{24}d^4y+\text{etc.},$$

which will truly be the differential of y. Truly because any term of this series will have an infinite ratio to the following terms, all vanish before the first, thus so that *dy* truly may be taken in the customary manner as the differential of y. In a similar manner the true second, third, fourth, etc. differentials of y thus themselves are obtained

$$dd.y=ddy+\tfrac{3}{3}d^3y+\tfrac{7}{3\cdot4}d^4y+\tfrac{15}{3\cdot4\cdot5}d^5y+\tfrac{31}{3\cdot4\cdot5\cdot6}d^6y+\text{etc.}$$
$$d^3.y=d^3y+\tfrac{6}{4}d^4y+\tfrac{25}{4\cdot5}d^5y+\tfrac{90}{4\cdot5\cdot6}d^6y+\tfrac{301}{4\cdot5\cdot6\cdot7}d^7y+\text{etc.}$$
$$d^4.y=d^4y+\tfrac{10}{5}d^5y+\tfrac{65}{5\cdot6}d^6y+\tfrac{1050}{5\cdot6\cdot7}d^7y+\tfrac{1701}{5\cdot6\cdot7\cdot8}d^8y+\text{etc.}$$
$$d^5.y=d^5y+\tfrac{15}{6}d^6y+\tfrac{140}{6\cdot7}d^7y+\tfrac{1050}{6\cdot7\cdot8}d^8y+\tfrac{6951}{6\cdot7\cdot8\cdot9}d^9y+\text{etc.}$$
$$d^6.y=d^6y+\tfrac{21}{7}d^7y+\tfrac{266}{7\cdot8}d^8y+\tfrac{2646}{7\cdot8\cdot9}d^9y+\tfrac{22827}{7\cdot8\cdot9\cdot10}d^{10}y+\text{etc.}$$
$$\text{etc.},$$

[Thus,

dy takes an increment $= ddy + \frac{1}{2}d^3y + \frac{1}{6}d^4y + \frac{1}{24}d^5y +$ etc.;

$\frac{1}{2}ddy$ takes an increment $= \frac{1}{2}d^3y + \frac{1}{4}d^4y + \frac{1}{12}d^5y +$ etc.;

$\frac{1}{6}d^3y$ takes an increment $= \frac{1}{6}d^4y + \frac{1}{12}d^5y +$ etc.;

$\frac{1}{24}d^4y$ takes an increment $= \frac{1}{24}d^5y +$ etc.; etc. etc.

These fractions added together appropriately give the required values, which thus can be adjusted to produce Euler's scheme.]

which follow from §56, if there may be put dx in place of ω. Therefore these will be the complete differentials of y, clearly in which these same terms, which are ignored with respect to the first. But we may find these individual terms, if the function y may be differentiated continually on putting dx constant. Thus on putting $y = ax - xx$ on account of

$$dy = adx - 2xdx \text{ and } ddy = -2dx^2$$

the complete differentials of y itself will be

$$dy = adx - 2xdx - dx^2, \; ddy = -2dx^2;$$

but the following shall be zero.

338. Though moreover generally in these expressions of the differentials the terms following from the first before may be reckoned to be nothing, yet in special cases, in which the first term may vanish, this account may cease nor can the second term be ignored further. Thus in the example preceding, even if the differential of the formula $y = ax - xx$ in general is $= (a - 2x)dx$ with the term $-dx^2$ rejected, evidently which is infinitely less than the first $(a - 2x)dx$, yet here this condition clearly is understood, unless the first term by itself may vanish. On account of which if the differential of $y = ax - xx$ is sought in the case, in which $x = \frac{1}{2}a$, the decrement will be dx^2. But with this case alone excepted the differential of the function y will be always $= (a - 2x)dx$; for unless $x = \frac{1}{2}a$, the second term $-dx^2$ from the first before correctly can be ignored always. Nor indeed with the neglect of the term dx^2 even in the case $x = \frac{1}{2}a$ is it possible to induce an error; for the first differentials are accustomed to be compared between each other ; from which, because $dy = -dx^2$ in the case $x = \frac{1}{2}a$ will vanish before dx in the first differentials, likewise it is, that we may have in this case either $dy = 0$ or $dy = -dx^2$.

339. With *y* denoting some function of *x*, there shall be with the differentials taken continually

$$dy = pdx,\ \ dp = qdx,\ dq = rdx,\ dr = sdx \ \text{ etc.}$$

Hence the completed differentials therefore will be, in which nothing is ignored,

$$d.y = pdx + \tfrac{1}{2}qdx^2 + \tfrac{1}{6}rd^3x + \tfrac{1}{24}sdx^4 + \tfrac{1}{120}tdx^5 + \text{etc.}$$
$$d^2.y = qdx^2 + rdx^3 + \tfrac{7}{12}sdx^4 + \tfrac{1}{4}tdx^5 + \text{etc.}$$
$$d^3.y = rdx^3 + \tfrac{3}{2}sdx^4 + \tfrac{5}{4}tdx^5 + \text{etc.}$$
$$d^4.y = sdx^4 + 2tdx^5 + \text{etc.}$$
$$d^5.y = tdx^5 + \text{etc.}$$
$$\text{etc.}$$

Therefore unless the first terms of these expressions may vanish, these differentials alone of *y* will be shown; but if in a certain case the first term becomes $= 0$, then the following differential sought will be expressed. And if also the second term may vanish, then the third term will give the value of the differential sought; but if this may vanish, the fourth, and thus henceforth. From which it is thought the first differential of no function of *x* ever completely vanishes ; even if there becomes $p = 0$, in which case *dy* may be considered generally to vanish, then this differential will be expressed by a higher power of *dx*, as either by $\tfrac{1}{2}qdx^2$, or if also there shall be $q = 0$, by $\tfrac{1}{6}rdx^3$, and thus henceforth.

340. But nevertheless in these cases the differential of *y* in respect of some differential of the first order, with which it may be compared, correctly is ignored and may be reckoned as nothing, yet on many occasions it may help to know the true expression of this. Indeed from the complete form of the differential it can become evident at once, in which cases the function becomes a maximum or minimum. For if there were

$$d.y = pdx + \tfrac{1}{2}qdx^2 + \tfrac{1}{6}rdx^3 + \text{etc.},$$

in which *y* may meet with a maximum or minimum value, it is necessary that there shall be $p = 0$; therefore in this case there will be $dy = \tfrac{1}{2}qdx^2$ and the function *y*, if in place of *x* there may be put $x \pm dx$, will change into $y + \tfrac{1}{2}qdx^2$ and therefore will be a minimum, if q may have a positive value, but a maximum, if *q* may have a negative value. But if likewise there is made $q = 0$, there will be $dy = \tfrac{1}{6}rdx^3$ and the function *y* on putting $x \pm dx$ in place of x will change into $y \pm \frac{1}{6} rdx^3$ and neither a maximum nor a minimum will be produced in this case ; But if there also becomes $r = 0$, then on putting $x \pm dx$ in place of x the function y may become $= y + \frac{1}{24} sdx^4$, which show a maximum, if s were a negative quantity, truly a minimum, if s shall be a positive quantity. The other occasions, in which the complete expression of the differential may have a use, occur below.

341. We may put p to vanish in the case $x = a$, which comes about, if there were $p = (x-a)P$. But such a value will be produced, if there were $y = (x-a)^2 P + C$ with C denoting some positive quantity. For since there shall be

$$pdx = (x-a)^2 dP + 2(x-a)Pdx,$$

certainly there shall be $p = 0$ on putting $x = a$. Then therefore on account of

$$dpdx = qdx^2 = (x-a)^2 ddP + 4(x-a)dPdx + 2Pdx^2$$

on putting $x = a$ there becomes $qdx^2 = 2Pdx^2$ and thus the complete differential in this case $x = a$ will be

$$dy = Pdx^2,$$

unless perhaps also P may vanish on putting $x = a$, which cases I will consider later.

But it will be able to show the present cases more generally in this manner. Let there be

$$z = (x-a)^2 P + C$$

and y shall be some function of z, thus so that there becomes $dy = Zdz$ with Z denoting some function of $z = (x-a)^2 P + C$. Therefore there will be

$$dz = (x-a)^2 dP + 2(x-a)Pdx \text{ and } pdx = Z(x-a)^2 dP + 2Z(x-a)Pdx,$$

because the member shall be made $= 0$, if $x = a$; and in the same case with the terms ignored, which contain the factor $x = a$, there will be $qdx^2 = 2PZdx^2$ and thus in the case $x = a$ there becomes $dy = PZdx^2$, after which certainly in PZ everywhere in place of x there were put a. Whereby if y were some function of $z = (x-a)^2 P + C$, thus so that there shall be $dy = Zdz$, in the case $x = a$ there will be the differential

$$dy = PZdx^2 .$$

Therefore this function y becomes a maxima in the case $x = a$, if in the same case PZ is made a negative quantity, truly a minimum, if PZ is made a positive quantity.

342. If there were $p = (x-a)^2 P$, in the case $x = a$ also q will vanish; but such an expression for p arises, if there were

$$p = (x-a)^3 P + C .$$

Hence there will be

$$pdx = (x-a)^3 dP + 3(x-a)^2 Pdx,$$
$$qdx^2 = (x-a)^3 ddP + 6(x-a)^2 dPdx + 6(x-a)Pdx^2,$$

each member of which will vanish in the case $x = a$; but truly there will be the following

$$rdx^3 = (x-a)^3 d^3P + 9(x-a)^2 ddPdx + 18(x-a)dPdx^2 + 6Pdx^3 = 6Pdx^3$$

on putting $x = a$. Whereby since both p and q may vanish in the case $x = a$, there becomes

$$dy = \tfrac{1}{6} rdx^3 = Pdx^3 .$$

In a similar manner if there is put

$$z = (x-a)^3 P + C$$

and y were some function of z, thus so that there shall be $dy = Zdz$, on account of

$$dz = (x-a)^3 dP + 3(x-a)^2 Pdx$$

also there becomes $p = 0$ and $q = 0$ and there will be $rdx^3 = 6PZdx^3$; from which case $x = a$ there will be

$$dy = PZdx^3$$

Whereby that function y, even if in the case $x = a$ there is made $p = 0$, yet will it take either a maximum or minimum value.

343. These differentials can be more easily found from the nature of the differentials themselves. For since the differential of y may arise, if y may be taken from the following next in place which is produced, if there may be put $x+dx$ in place of x, we may put in the first case, where there was

$$y=(x-a)^2 P+C,$$

and there will be $x+dx$ in place of x

$$y^{\mathrm{I}} =(x-a+dx)^2 \mathrm{P}^{\mathrm{I}}+C,$$

from which there becomes

$$dy=(x-a+dx)^2 P^{\mathrm{I}}+(x-a)^2 P$$

Therefore in the case, in which $x=a$, there will be $dy=P^1dx^2$, and since P^{I} to P may have the ratio of equality, there will be

$$dy=Pdx^2$$

In a similar manner if there were

$$z=(x-a)^2 P+C,$$

there will be $dz=Pdx^2$; whereby if y shall be some function of z, thus so that there shall be $dy=Zdz$, there will be

$$dy=PZdx^2$$

in the case, in which there is put $x=a$.
Then if there shall be

$$z=(x-a)^3 P+C,$$

there will be $z^{\mathrm{I}}=(x-a+dx)^3 P^{\mathrm{I}}+C$ and therefore in the case $x=a$ there arises

$$z^{\mathrm{I}}-z=dz=Pdx^3.$$

Hence if y were some function of z and $dy=Zdz$, there will be also in the case $x=a$ the differential $dy=PZdx^3$, if indeed in the functions P and Z in place of x, a may be substituted everywhere. Therefore because in this case there becomes $z=C$ and Z is a function of z, Z may become a constant quantity, such evidently a function of C, as such before it was of z.

344. Therefore if generally there were

$$y=(x-a)^n P+C,$$

because there is

$$y^{\mathrm{I}}=(x-a+dx)^n P^{\mathrm{I}} +C,$$

there becomes in the case $x = a$,

$$dy = Pdx^n;$$

from which if there were $n > 1$, this differential vanishes with respect of the other first differentials, which are homogeneous to *dx* itself [*i.e.* the lower powers of *dx*]. Therefore from the preceding it is evident the function *y* in the case $x = a$ becomes either a maximum or minimum, if *n* were an even number, if on putting $x = a$ and *P* becomes a positive quantity, then *y* will be made a minimum; but if *P* is made a negative quantity, then *y* is a maximum. Therefore in this manner an account of maxima and minima may be found more easily than by the method set out above, because there is no need to be progressing to higher differentials. Because truly if there shall be

$$z = (x-a)^n P + C$$

and *y* were some function of *z*, so that there shall be $dy = Zdz$, in the case $x = a$ the differential will be

$$dy = PZdx^n$$

But it is to be noted here *n* is to be taken for a positive number or greater than 0 ; if indeed *n* were a negative number, then on putting $x = 0$, $(x-a)^n$ may not vanish as we have assumed, but on that account becomes infinitely large.

345. Now we have seen with this agreed upon how the differential may be found much more readily than with the aid of series, by which before we have expressed the complete differential; if indeed *n* shall be a whole number, as many terms of that series must be gone through, as *n* may contain one. Truly if *n* shall be a fraction, then that series will never show a true differential at any time. For we may put to be

$$y = (x-a)^{\frac{3}{2}} + a\sqrt{a};$$

if we may consider the series

$$dy = pdx + \tfrac{1}{2}qdx^2 + \tfrac{1}{6}rd^3x + \tfrac{1}{24}sdx^4 + \text{etc.},$$

there is made

$$p = \tfrac{3}{2}\sqrt{(x-a)},\ \ q = \frac{3}{4\sqrt{(x-a)}},\ \ r = \frac{-3}{8(x-a)\sqrt{(x-a)}},\ \ s = \frac{9}{16(x-a)^2\sqrt{(x-a)}} \text{ etc.}$$

Whereby if there is put $x = a$, there becomes indeed $p = 0$, but all the following terms *q*, *r*, *s* etc. become infinite ; from which the value of the differential *dy* cannot generally be defined in this case. But truly the method deduced from the nature of the differentials leaves no doubt. For since there shall be $y = (x-a)^{\frac{3}{2}} + a\sqrt{a}$, on putting $x + dx$ in place of *x* it becomes $y^{\mathrm{I}} = (x-a+dx)^{\frac{3}{2}} + a\sqrt{a}$ and there will be, if there may be put $x = a$, $dy = dx\sqrt{dx}$. Therefore this differential will vanish before *dx;* but truly the second differentials homogeneous with dx^2 will vanish before that.

346. We may explain these cases somewhat more carefully, in which the exponent *n* is a fractional number, and there shall be

$$y = P\sqrt{(x-a)} + C;$$

on account of $y^{\mathrm{I}} = P\sqrt{(x-a+dx)} + C$ there becomes $dy = P\sqrt{dx}$ in the case $x = a$; from which this differential will hold an infinite ratio to *dx* and to the differentials homogeneous with *dx* . Hence it is apparent also, what account may be considered in this case concerning the maxima and minima. For since on putting $a + dx$ in place of *x*, *y* may change into

$$C + P\sqrt{dx},$$

on account of the ambiguous $\sqrt{dx}$ the function *y* may adopt twin values, the one greater than *C*, which it takes on putting $x = a$, the other smaller; from which in the case $x = a$ it becomes neither a maximum nor a minimum. Besides if *dx* may be taken negative, then the value of *y* accordingly is made imaginary. Likewise it is required to be considered, if there shall be $z = P\sqrt{(x-a)} + C$ and *y* some function of *z*, so that there shall be $dy = Zdz$; then indeed there will be $dy = PZ\sqrt{dx}$ in the case $x = a$.

347. If this function were proposed

$$y = (x-a)^{\frac{m}{n}} P + C,$$

the differential of which is sought in the case $x = a$, there will be as it has been deduced from the previous,

$$dy = Pdx^{\frac{m}{n}}.$$

On account of which if there were $m > n$, this differential will vanish before *dx*; but if there shall be $m < n$, ratio $\frac{dy}{dx}$ will be infinitely large. In addition truly if *n* shall be an even number, the differential *dy* will have a twin value, the one positive, the other negative ; and thus the function *y*, which in the case $x = a$ becomes $= C$, if there is put $x = a + dx$, will have two values, the one greater than *C*, the other truly smaller ; but if there may be put $x = a - dx$, then *y* accordingly may become imaginary ; from which in this case *y* is made neither a maximum nor a minimum. Now we may put the denominator *n* to be an odd number; the numerator m will be either even or odd. In the first place let the number m be even; because dy may retain the same value, if dx may be taken either positive or negative, it is evident the function y in the case $x = a$ becomes either a maximum or a minimum, provided in this case P were a negative or positive quantity. But if each number m and n were odd, the differential dy will change into its own negative on putting dx negative; and therefore in this case the function y will neither be a maximum nor a minimum, if there may be put $x = a$.

348. If the function y may depend on several terms of this kind, the individual terms of which shall be divisible by $x = a$, thus so that there shall be

$$y = (x-a)^m P + (x-a)^n Q + C,$$

then the differential of this in the case $x = a$ will be

$$dy = Pdx^m + Qdx^n;$$

in which expression, if there were $n > m$, the second term will vanish before the first, thus so that only $dy = Pdx^m$ may be produced. But if n shall be a fraction having an even denominator, then, even if Qdx^n may vanish before Pdx^m, yet generally it cannot be ignored. For from that it may be apparent, if dx may be taken negative, the value of dy becomes imaginary, which is not apparent from the single first term Pdx^m. Therefore since, if n shall be a fraction having an even denominator, dx cannot be taken negative, but if it may be taken positive, the term Qdx^n may give a twin value, the function $y = (x-a)^m P + (x-a)^n Q + C$, which in the case $x = a$ becomes $= C$, if there is put $x = a + dx$, will be

$$y = C + Pdx^m \pm Qdx^n;$$

each of which the value since it shall be either greater or smaller than C, according as P were either positive or negative quantity, the function y will be in the case $x = a$ either a minimum or a maximum of the second kind [§ 278].

349. Therefore in these cases the true differentials of functions cannot be found from the customary rules of differentiation ; evidently which prevail only, while the differential of the function is homogeneous with dx [*i.e.* of the same order]. But if in a certain case the differential of particular functions may be expressed by powers of this dx^n, then the rule will give 0 for this differential, if n were a number greater than unity; but truly the differential may be shown infinitely great, if n shall be an exponent less than unity. Thus if the differential may be sought of $y = \sqrt{(a-x)}$ in the case $x = a$, because there is

$dy = -\frac{dx}{\sqrt{(a-x)}}$, on making $x = a$ there will be produced $dy = -\frac{dx}{0}$. And if we may wish to call the following differentials in aid, all equally on account of the denominator $= 0$ increase to infinity, thus so that thence nothing will be able to be concluded. But truly in this case we have seen to be $dy = \sqrt{-dx}$ and thus imaginary. But if there may be put $x - dx$ in place of *x*, there will be $dy = \sqrt{dx}$ and accordingly it will be infinitely greater than *dx*, thus so that *dx* may vanish before *dy*. Whereby the customary rule also in this case will not lead to an error, since the value of *dy* may be shown to be infinite.

350. Therefore it is required to withdraw from the customary rule of differentiation, as often in the series

$$pdx + \tfrac{1}{2}qdx^2 + \tfrac{1}{6}rd^3x + \text{etc.},$$

by which the complete differential of the function *y* is expressed, the first term *p* either is made $= 0$ or it increases to infinity, and in that case the differential must be derived from first principles. Therefore as often as the differential of the function *y* is sought corresponding to a given a given value *x*, from which the letter *p* emerges either infinitely small or infinitely large, so many times is it required to return to the first principles of differentiation themselves. Truly with all the remaining cases, in which there becomes neither $p = 0$ nor $p = \infty$, the customary rule will give the true values of the differentials. Yet meanwhile the case mentioned before (§348) is not to be ignored, if the function *y* may contain a member of this kind $(x-a)^n Q$, with the fraction *n* present having an even denominator; for even if differentials smaller than Qdx^n may be present, before this differential may vanish with which, because Qdx^n may yet become imaginary if *dx* shall be negative, also Qdx^n may be changed into imaginary quantities with all remaining beyond this member ; the account of which circumstance will be considered chiefly for curves. Therefore I will explain particular cases of this kind in the adjoining examples, in which the true differential may not be indicated from the common rule.

EXAMPLE 1

The differential of the function $y = a + x - \sqrt{\left(xx + ax - x\sqrt{(2ax - xx)}\right)}$ *is sought in the case, in which there is put* $x = a$.

It is apparent the differential of this function in the case $x = a$ cannot be found by the customary rule from differentiation ; for there becomes

$$\frac{dy}{dx} = dx + \frac{-xdx - \frac{1}{2}adx + \frac{1}{2}dx\sqrt{(2ax-xx)} + (axdx - xxdx):\sqrt{(2ax-xx)}}{\sqrt{\left(xx+ax-x\sqrt{(2ax-xx)}\right)}} \;;$$

but on putting $x = a$ there will be $dy = dx - \frac{adx}{a} = 0$. Therefore we may begin from the principles of differentials and indeed in the first place on putting $x + dx$ in place of x it becomes

$$y^{\mathrm{I}} = a + x + dx$$
$$-\sqrt{\left(xx + 2xdx + dx^2 + ax + adx - (x+dx)\sqrt{\left(2ax - xx + 2adx - 2xdx - dx^2\right)}\right)}$$

But on putting $x = a$ there will be

$$y^{\mathrm{I}} = 2a + dx - \sqrt{\left(2aa + 3adx + dx^2 - (a+dx)\sqrt{\left(aa - dx^2\right)}\right)}.$$

Now since there shall be $\sqrt{\left(aa - dx^2\right)} = a - \frac{dx^2}{2a}$ (for the following terms can be ignored with care, because not all, which are infinitely great, maybe removed, as soon will be apparent), there will be

$$y^{\mathrm{I}} = 2a + dx - \sqrt{\left(aa + 2adx + \tfrac{3}{2}dx^2\right)}$$

and again on extracting the root there becomes

$$y^{\mathrm{I}} = 2a + dx - \left(a + dx + \frac{dx^2}{4a}\right) = a - \frac{dx^2}{4a}.$$

But in the case $x = a$ there will be $y = a$ *;* from which, since there shall be $y^{\mathrm{I}} = y + dy$, there will be obtained

$$dy = -\frac{dx^2}{4a} \;;$$

from which likewise it is evident the proposed function y is made a maximum, if there may be put $x = a$.

EXAMPLE 2

To find the differential of this function $y = 2ax - xx + a\sqrt{(aa - xx)}$
in the case in which there is put $x = a$.

With the differentiation made in the usual manner

$$dy = 2adx - 2xdx - \frac{axdx}{\sqrt{(aa-xx)}},$$

because on putting $x = a$ it will change into infinity neither therefore may it be indicated in this manner. Truly all the differentials of the following orders become infinite, thus so that from these nor indeed from the series $pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \text{etc.}$ may the true value of the differential be able to be found. Therefore we may put $x + dx$ in place of x and we will have

$$y^{\mathrm{I}} = 2ax - xx + 2adx - 2xdx - dx^2 + a\sqrt{\left(aa - xx - 2xdx - dx^2\right)}$$

and on putting $x = a$ there will be

$$y^{\mathrm{I}} = aa - dx^2 + a\sqrt{\left(-2adx - dx^2\right)}.$$

But in the same case there becomes $y = aa$; from which there will be $dy = -dx^2 + a\sqrt{-2adx}$, and since dx^2 may vanish before $\sqrt{-2adx}$, there will be

$$dy = a\sqrt{-2adx}.$$

Whereby if the differential dx may be taken positive, dy will be imaginary ; but if for x there may be written $x - dx$, there will be

$$dy = a\sqrt{2adx};$$

since there shall be a twofold value of this, the one positive, the other negative, the function y in the case $x = a$ neither is made a maximum nor a minimum.

EXAMPLE 3

To find the differential of the function $y = 3aax - 3axx + x^3 + (a-x)^2\sqrt[3]{\left(a^3 - x^3\right)}$,
in the case in which there is put $x = a$.

Because this function may be transformed into this case

$$y = a^3 - (a-x)^3 + (a-x)^{\frac{7}{3}}\sqrt[3]{(aa + ax + xx)},$$

on putting $x = a + dx$ it becomes

$$y^{\mathrm{I}} = a^3 + dx^3 - dx^{\frac{7}{3}}\sqrt[3]{3aa}$$

and in the same case there is $y = a^3$. Hence there will be $dy = dx^3 - dx^{\frac{7}{3}}\sqrt[3]{3aa}$, and since dx^3 may vanish before $dx^{\frac{7}{3}}$, there will be

$$dy = -dx^{\frac{7}{3}}\sqrt[3]{3aa} \;;$$

therefore $x = a$ shall make y neither a maximum nor a minimum.

EXAMPLE 4

To find the differential of the function $y = \sqrt{x} + \sqrt[4]{x^3} = \left(1 + \sqrt[4]{x}\right)\sqrt{x}$, *in the case* $x = 0$.

Because the case $x = 0$ is proposed and with that there shall be $y = 0$, in place of x only dx may be written and there will be considered

$$dy = dx^{\frac{1}{2}} + dx^{\frac{3}{4}} \text{ or } dy = \left(1 + \sqrt[4]{dx}\right)\sqrt{dx} \;;$$

from which it appears the first dx cannot accept a negative value. Then truly, even if the other $\sqrt{dx}$ twin value may have taken itself away, the one positive, the other negative, yet in this case, because the root $\sqrt[4]{dx}$ of this occurs, only the positive can be taken. But truly each $\sqrt[4]{dx}$ requires to be indicated and there will be

$$dy = \sqrt{dx} \pm \sqrt[4]{dx^3} \text{ and } y^{\mathrm{I}} = \sqrt{dx} \pm \sqrt[4]{dx^3}$$

on account of $y = 0$. Therefore since each value of y^{I} shall be greater than of y, in the case $x = 0$, y becomes a minimum. But because the function $y = \sqrt{x} + \sqrt[4]{x^3}$ may not include this, $y = -\sqrt{x} + \sqrt[4]{x^3}$, it will be apparent each should lead to rationality. For the first changed into this form $y - \sqrt{x} = \sqrt[4]{x^3}$ and squared gives $y^2 - 2y\sqrt{x} + x = x\sqrt{x}$ or $y^2 + x = (x + 2y)\sqrt{x}$, which squared again gives

$$y^4 - 2yyx - 4xxy + xx - x^3 = 0 .$$

..

Truly the other $y + \sqrt{x} = \sqrt[4]{x^3}$ will give $y^2 + x = (x - 2y)\sqrt{x}$ and again

$$y^4 - 2yyx + 4xxy + xx - x^3 = 0 ,$$

which is different from the former. But truly the other member $\sqrt[4]{x^3}$ retains the ambiguity of sign. On account of which this circumstance is required to be noted properly, because, even if the roots of equal powers generally each include the sign + and – , yet this ambiguity may cease, if in the same expression of the same roots further roots of even powers may occur ; evidently which will become imaginary, if the first roots may be taken negative. And from this source maxima and minima follow of the second kind, when such may not be considered to be present.

EXAMPLE 5

To find the differential of the function

$$y = a + \sqrt{(x-f)} + (x-f)\sqrt[4]{(x-f)} + (x-f)^2\sqrt[8]{(x-f)}$$

in the case in which there is put $x = f$.

We may put $x - f = t$, and since there shall be $y = a + \sqrt{t} + t\sqrt[4]{t} + tt\sqrt[8]{t}$, the differential of this is sought in the case $t = 0$, so that there is made $y = a$. Therefore on putting $t + dt$ or $0 + dt$ in place of *t* there comes about

$$y^{\mathrm{I}} = y + dy = a + \sqrt{dt} + dt\sqrt[4]{dt} + dt^2\sqrt[8]{dt}$$

and thus there will be had

$$dy = \sqrt{dt} + dt\sqrt[4]{dt} + dt^2\sqrt[8]{dt};$$

Where in the first place the differential *dt* cannot be taken negative, so that *dy* may not become imaginary. Then truly not only $\sqrt{dt}$, but also $\sqrt[4]{dt}$ cannot be taken negative; for $\sqrt[8]{dt}$ may become imaginary; from which the differential *dy* may only have a twin value

$$dy = \sqrt{dt} + dt\sqrt[4]{dt} \pm dt^2\sqrt[8]{dt}$$

of which since each shall be greater than zero, it follows the function *y* becomes a minimum of the second kind on putting $t = 0$ or $x = f$. Therefore nevertheless in these cases the terms $\sqrt[4]{dt}$ and $\sqrt[8]{dt}$ may vanish before $\sqrt{dt}$, yet an account of these is required to be had, if a multiplicity of the values may be considered, so that imaginary numbers may be avoided.

EXAMPLE 6

To find the differential of the function
$$y = ax + bxx + (x - f)^n + (x - f)^{m+\frac{1}{2}n} \textit{ in the case } x = f.$$

If there may be put $x = f$, there is made $y = af + bff$, and if in place of x there may be put $x + dx$ or $f + dx$, the nearby value will be produced

$$y^{\mathrm{I}} = af + bff + adx + 2bfdx + bdx^2 + dx^n + dx^{m+\frac{1}{2}n},$$

thus so that there shall be
$$dy = adx + 2bfdx + bdx^2 + dx^n + dx^m \sqrt{dx^n}$$

Therefore unless the number n shall be even, the differential dx is unable to be taken negative. But the final term $dx^m \sqrt{dx^n}$ has an ambiguous sign ; from which the value of y^{I} will be twofold, each greater than of y itself, if indeed $a + 2bf$ were a positive quantity and the exponents n and $m + \frac{1}{2}n$ were greater than unity. Therefore the value of the function y in the case $x = f$ becomes a minimum and this comes about, if n shall be a whole number or a fraction, provided the denominator in the first case and the number itself in the second case were not even.

351. But this differential method required to be deduced from first principles has a special use with transcending functions, since in certain cases the differential found by the customary manner either vanishes or may be seen to increase to infinity. But here examples of such a kind occur of the infinitely large and infinitely small, which never can be found in the algebraic cases. Since indeed, if i may denote an infinite number, then li also shall be indeed infinite, but yet holding an infinitely small ratio to this number i itself and accordingly to any power i^n, however small the exponent n may be decided, the fraction $\frac{li}{i^n}$ will be infinitely small; nor will it be able to become finite, before the fraction n is made infinitely small. Therefore li will be homogeneous with i^n, if the exponent n were infinitely small. Now we may put $i = \frac{1}{\omega}$ with an infinitely small quantity ω arising ; $-l\omega$ will be homogeneous with $\frac{1}{\omega^n}$, if the exponent n shall be infinitely small, and thus $-\frac{1}{l\omega}$ will be homogeneous with ω^n; and hence $-\frac{1}{ldx}$ will be infinitely small on being compared with dx^n with n becoming an infinitely small fraction. Thus if there were $y = -\frac{1}{lx}$, the differential of y in the case $x = 0$ will be $= -\frac{1}{ldx} = dx^n$ and thus dy will hold an infinite ratio to *dx* and to whatever power of *dx*; and before $-\frac{1}{ldx}$, in general all the powers of *dx* vanish, however small the exponents of these should be.

352. Then also we have seen, if *a* were a number greater than one and *i* were infinite, as then a^i becomes infinite to such a high order, so that before that not only *i*, but also any power of *i* may vanish; neither can i^n emerge homogeneous with a^i, as the exponent *n* were increased to infinity. Now let $i = \frac{1}{\omega}$, thus so that ω may denote an infinitely small amount; $a^{\frac{1}{\omega}}$ will be homogeneous with $\frac{1}{\omega^n}$ with the number *n* arising infinitely great and thus $a^{\frac{-1}{\omega}}$ or $\frac{1}{a^{1:\omega}}$ will be infinitely small in comparison with ω^n. Hence $\frac{1}{a^{1:dx}}$ will be infinitely small, because moreover it vanishes before all the powers of *dx*, since it shall be homogeneous with the power dx^n arising with the number *n* infinitely great. Whereby if the differential of $y = \frac{1}{a^{1:x}}$ is sought in the case $x = 0$, because there becomes $y = 0$, there will be $dy = \frac{1}{a^{1:dx}}$ and thus is an infinite number of times smaller than any power of *dx* raised.

353. But if *a* shall be a number smaller than one, then because $\frac{1}{a}$ becomes greater than one, the question is reduced to the preceding case. Clearly if the expression $a^{\frac{1}{\omega}}$ may be considered, that on putting $a = 1:b$ will be changed into $b^{-\frac{1}{\omega}}$ or $\frac{1}{b^{1:\omega}}$ which will be homogeneous with ω^n on account of $b > 1$ with the number *n* arising infinitely great. Therefore from these premises we will be able to resolve the following examples.

EXAMPLE 1

To find the differential of the function $y = xx - \frac{1}{lx}$ *in the case* $x = 0$.

Because on putting $x = 0$ there becomes $y = 0$, if we may put $x + dx$ or $0 + dx$ in place of *x*, there is made

$$y^{\mathrm{I}} = dy = dx^2 - \frac{1}{ldx}.$$

But since $-\frac{1}{ldx}$ shall be homogeneous with dx^n with *n* denoting an infinitely small number, before that dx^2 will vanish and there will be

$$dy = -\frac{1}{ldx} = dx^n.$$

But truly because the logarithms of negative numbers are imaginary, *dx* shall be unable to be taken negative and thus in the case $x = 0$ it will be the minimum function *y*, but neither relating to the first nor the second kind. Evidently it does not relate to the first kind, because y has no nearby preceding values, but yet is smaller than the following values, if x may be considered greater than zero. Thus it does not relate to the second kind, because the following values, with which it may be compared, are not twins ; thus accordingly a third kind of maximum or minimum emerges, which has a place only with logarithmic and transcending functions, but under no circumstances does it appear in algebraic functions ; concerning which it is developed further in the following part about curved lines.

EXAMPLE 2

To find the differential of the function $y=(a-x)^n - x^n(la-lx)^n$ *in the case in which* $x=a$.

This differential, if n shall be a whole number, can be found from the general formula

$$dy = pdx + \tfrac{1}{2}qdx^2 + \tfrac{1}{6}rdx^3 + \text{etc.};$$

for there will be

$$pdx = -n(a-x)^{n-1}dx - nx^{n-1}(la-lx)^n + nx^{n-1}(la-lx)^{n-1}dx$$

which value vanishes everywhere on putting $x = a$; for even if there shall be $n = 1$, there will be

$$pdx = -dx + dx = 0.$$

Therefore if we may progress further, there will be

$$\tfrac{1}{2}qdx^2 = \tfrac{n(n-1)}{1\cdot 2}(a-x)^{n-2}dx^2 - \tfrac{n(n-1)}{1\cdot 2}\ x^{n-2}(la-lx)^n + \tfrac{n^2}{2}x^{n-2}dx^2(la-lx)^{n-1}$$
$$+\tfrac{n(n-1)}{1\cdot 2}x^{n-2}dx^2(la-lx)^{n-1} - \tfrac{n(n-1)}{1\cdot 2}\ x^{n-2}(la-lx)^{n-2}.$$

Hence therefore, if there were $n=1$, there will be $\tfrac{1}{2}qdx^2 = \tfrac{dx^2}{2a}$ on putting $x=a$. In a similar manner, if there shall be $n=2$, it might be required to go on to the third term $\tfrac{1}{6}rdx^3$, and thus henceforth. Therefore from the principles of differentiation themselves we will use more easily, and since on putting $x=a$ there becomes $y=0$, if we may put $x+dx$ or $a+dx$ in place of x, there will be

$$y^{\mathrm{I}} = (-dx)^n - (a+dx)^n\left(la - l(a+dx)\right)^n = y+dy = dy$$

on account of $y=0$. Truly there is

$$l(a+dx)=la+\tfrac{dx}{a}-\tfrac{dx^2}{2a^2}+\tfrac{dx^3}{3a^3}-\text{etc.},$$

from which there becomes

$$dy=(-dx)^n-\left(a^n+na^{n-1}dx+\tfrac{n(n-1)}{1\cdot 2}a^{n-2}dx^2+\text{etc.}\right)\left(-\tfrac{dx}{a}+\tfrac{dx^2}{2a^2}-\tfrac{dx^3}{3a^3}+\text{etc.}\right)^n=\tfrac{n}{2a}(-dx)^{n+1}.$$

Therefore in the case $x=a$ the differential of the proposed formula sought *dy* will be, as follows:

if $n=1$	$dy=\frac{dx^2}{2a}$, as we found before,
" $n=2$	$dy=-\frac{2dx^3}{2a}$
" $n=3$	$dy=\frac{3dx^4}{2a}$
" $n=4$	$dy=-\frac{4dx^5}{2a}$
etc.	etc.

Therefore if *n* were an odd number, the function *y* in the case $x=a$ becomes a minimum, but if *n* shall be an even number, it becomes neither a maximum nor a minimum; which likewise prevails, if *n* were a fraction having an odd denominator. But if *n* were a fraction having an even denominator, then *dx* must be taken negative, lest we may come upon imaginary ; and also on account of the ambiguity signified the function comes out neither a maximum nor a minimum.

EXAMPLE 3

To find the differential of the function $y=x^x$ *in the case* $x=\frac{1}{e}$ *with e denoting the number, of which the hyperbolic logarithm is* $=1$.

Because there becomes in general $dy=x^x dx(lx+1)$, this differential vanishes in the case $x=\frac{1}{e}$ or $lx=-1$. Therefore this differential may be compared with the general form $pdx+\frac{1}{2}qdx^2+\text{etc.}$; there will be $p=x^x(lx+1)$ and $q=x^x(lx+1)^2+x^{x-1}$ and on putting $lx=-1$ or $x=\frac{1}{e}$ there will be

$$q=\left(\tfrac{1}{e}\right)^{\frac{1-e}{e}}=e^{\frac{e-1}{e}}.$$

Whereby the sought differential will be

$$dy=\tfrac{1}{2}e^{(e-1):e}dx^2$$

and therefore the function $y = x^x$ will appear a minimum in the case $x = \frac{1}{e}$.

EXAMPLE 4

To find the differential of the function $y = x^n + e^{-1:x}$ *on the case in which* $x = 0$.

Because on making $x = 0$ there comes about $y = 0$, if there may be put $x = 0 + dx$, there will be

$$y^{\mathrm{I}} = dy = dx^n + \frac{1}{e^{1:dx}}.$$

But we have seen $\frac{1}{e^{1:dx}}$ to be homogeneous with the infinite power of dx or with dx^{∞}, and thus may vanish before dx^n, thus so that there shall be

$$dy = dx^n.$$

354. Because in certain cases using the first differentials it comes about that they may not be produced by the accustomed rule of differentiation, likewise too with differentials of the second and of the third and of higher orders for these cases it arises, from which in the form for the complete differential

$$dy = pdx + \tfrac{1}{2}qdx^2 + \tfrac{1}{6}rd^3x + \tfrac{1}{24}sdx^4 + \text{etc.}$$

some of the quantities *q*, *r*, s etc. either vanish or become infinite. Evidently since there shall be

$$dd.y = qdx^2 + rdx^3 + \tfrac{7}{12}sdx^4 + \text{etc.},$$

if in which case there is made $q = 0$, then there will be $ddy = rdx^3$; but if in the same case r may vanish also, then there will be $ddy = \frac{7}{12}sdx^4$, and thus so forth. But if either q, r or s etc. becomes infinite, then from this series clearly the second differential cannot be found, but it will be required to take refuge according to the principles of differentials ; clearly on putting $x + dx$ in place of x the value y^{I} may be sought and on putting $x + 2dx$ in place of x the value of y^{II}, from which done there will be the true value of the second differential

$$ddy = dy^{\mathrm{I}} - dy = dy^{\mathrm{II}} - 2dy^{\mathrm{I}} + y.$$

In a similar manner if a question may be proposed about the third differential, then in addition in *y* in place of *x* there may be written $x+3dx$ and with the value y^{III} found there will be

$$d^3y = y^{\mathrm{III}} - 3y^{\mathrm{II}} + 3y^{\mathrm{I}} - y$$

and thus henceforth. Which cases we will illustrate by the following examples.

EXAMPLE 1

To find the differential of the following function $y = \frac{aa-xx}{aa+xx}$ *in the case, in which there is put* $x = \frac{a}{\sqrt{3}}$.

By seeking the complete differential of y from the form

$$dy = pdx + \tfrac{1}{2}qdx^2 + \tfrac{1}{6}rd^3x + \tfrac{1}{24}sdx^4 + \text{etc.}$$

the following values will be produced for *p, q, r,* s etc.

$$p = -\frac{4aax}{(aa+xx)^2},\ q = \frac{-4a^4+12aaxx}{(aa+xx)^3} \quad \text{and} \quad r = \frac{48a^4x-48aax^3}{(aa+xx)^4}.$$

Now since there shall be

$$ddy = rdx^3 + \tfrac{7}{12}sdx^4 + \text{etc.}$$

on account of $q=0$ in the case $x=-a$ and in the same case there shall be $r = \frac{27\sqrt{3}}{8a^3}$, the second differential sought becomes

$$ddy = \frac{27dx^2\sqrt{3}}{8a^3}.$$

EXAMPLE 2

To find the third differential of the function $y = \frac{aa-xx}{aa+xx}$ *in the case* $x=a$.

On seeking as before the complete differential

$$dy = pdx + \tfrac{1}{2}qdx^2 + \tfrac{1}{6}rd^3x + \tfrac{1}{24}sdx^4 + \text{etc.},$$

because $d^3y = rdx^3 + \tfrac{3}{2}sdx^4 + \text{etc.}$ is the third differential [see §337], on account of

$$r = \frac{48a^4x - 48aax^3}{(aa+xx)^4}$$

there becomes $r = 0$ in the case $x = a$; whereby it is required to progress to the value s, which will be

$$s = \frac{48a^4x - 144aaxx}{(aa+xx)^4} - \frac{8x\left(48a^4x - 48aax^3\right)}{(aa+xx)^5} \;;$$

therefore on making $x = a$ there will be $s = -\frac{96a^4}{2^4a^8} = -\frac{6}{a^4}$; from which in this case there will be

$$d^3y = -\frac{9dx^4}{a^4}.$$

EXAMPLE 3

To find the differentials of any order of the function $y = ax^m + bx^n$ in the case $x = 0$.

On putting successively $x + dx,\ x + 2dx,\ x + 3dx$ etc. in place of x the following values of the function y will be

$$y^{\mathrm{I}} = a(x + dx)^m + b(x + dx)^n,$$
$$y^{\mathrm{II}} = a(x + 2dx)^m + b(x + 2dx)^n,$$
$$y^{\mathrm{III}} = a(x + 3dx)^m + b(x + 3dx)^n$$

etc.

Therefore on putting $x = 0$ there will be $y = 0$ and the differentials of this will be

$$dy = adx^m + bdx^n,$$
$$ddy = \left(2^m - 2\right)adx^m + \left(2^n - 2\right)bdx^n,$$
$$d^3y = \left(3^m - 3\cdot 2^m + 3\right)adx^m + \left(3^n - 3\cdot 2^n + 3\right)bdx^n,$$
$$d^4y = \left(4^m - 4\cdot 3^m + 6\cdot 2^m - 4\right)adx^m + \left(4^n - 4\cdot 3^n + 6\cdot 2^n - 4\right)bdx^n$$

etc.

Therefore if the exponent n were greater than m, the second terms in these expressions vanish before the first. Yet meanwhile an account of these will be had, if n were a fractional number, so that the cases are able to be decided, in which these differentials either become imaginary or ambiguous. Truly the setting out of higher cases will be agreed to be reserved for the teaching of curves. [This did not appear in print until around 1860!]

CAPUT XIV

DE DIFFERENTlALIBUS FUNCTIONUM IN CERTIS TANTUM CASIBUS

337. Si y fuerit functio quaecunque ipsius x atque haec quantitas variabilis x augeatur incremento ω, ut x abeat in $x+\omega$, tum functio y induet hunc valorem

$$y+\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}+\frac{\omega^3 d^3y}{6dx^3}+\frac{\omega^4 d^4y}{24dx^4}+\text{etc.}$$

ideoque capiet hoc incrementum

$$\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}+\frac{\omega^3 d^3y}{6dx^3}+\frac{\omega^4 d^4y}{24dx^4}+\text{etc.}$$

uti supra demonstravimus. Quare si fiat $\omega=dx$, ita ut x suo differentiali dx crescat, tum functio y incrementum accipiet

$$dy+\tfrac{1}{2}ddy+\tfrac{1}{6}d^3y+\tfrac{1}{24}d^4y+\text{etc.},$$

quod erit verum differentiale ipsius y. Quoniam vero huius seriei quilibet terminus ad sequentes habet rationem infinitam, prae primo omnes evanescunt, ita ut dy more consueto sumtum praebeat verum differentiale ipsius y.Simili modo vera differentialia secunda, tertia, quarta etc. ipsius y ita se habebunt

$$dd.y=ddy+\tfrac{3}{3}d^3y+\tfrac{7}{3\cdot4}d^4y+\tfrac{15}{3\cdot4\cdot5}d^5y+\tfrac{31}{3\cdot4\cdot5\cdot6}d^6y+\text{etc.}$$
$$d^3.y=d^3y+\tfrac{6}{4}d^4y+\tfrac{25}{4\cdot5}d^5y+\tfrac{90}{4\cdot5\cdot6}d^6y+\tfrac{301}{4\cdot5\cdot6\cdot7}d^7y+\text{etc.}$$
$$d^4.y=d^4y+\tfrac{10}{5}d^5y+\tfrac{65}{5\cdot6}d^6y+\tfrac{1050}{5\cdot6\cdot7}d^7y+\tfrac{1701}{5\cdot6\cdot7\cdot8}d^8y+\text{etc.}$$
$$d^5.y=d^5y+\tfrac{15}{6}d^6y+\tfrac{140}{6\cdot7}d^7y+\tfrac{1050}{6\cdot7\cdot8}d^8y+\tfrac{6951}{6\cdot7\cdot8\cdot9}d^9y+\text{etc.}$$
$$d^6.y=d^6y+\tfrac{21}{7}d^7y+\tfrac{266}{7\cdot8}d^8y+\tfrac{2646}{7\cdot8\cdot9}d^9y+\tfrac{22827}{7\cdot8\cdot9\cdot10}d^{10}y+\text{etc.}$$
$$\text{etc.},$$

quae sequuntur ex § 56, si loco ω ponatur dx. Erunt ergo haec differentialia ipsius y completa, quippe in quibus ne ii quidem termini, qui respectu primi evanescunt, negliguntur. Inveniuntur autem singuli isti termini, si functio y continuo differentietur ponendo dx constans. Sic posito $y=ax-xx$ ob

$$dy = adx - 2xdx \text{ et } ddy = -2dx^2$$

erunt ipsius y differentialia completa

$$dy = adx - 2xdx - dx^2, \ ddy = -2dx^2;$$

sequentia autem sunt nulla.

338. Quanquam autem generatim in his expressionibus differentialium sequentes termini prae primis pro nihilo reputantur, tamen in casibus specialibus, quibus ipse terminus primus evanescit, haec ratio cessat neque terminus secundus amplius negligi poterit. Sic in exemplo praecedente etiamsi formulae $y = ax - xx$ differentiale in genere est $=(a-2x)dx$ reiecto termino $-dx^2$, quippe qui est infinities minor quam primus $(a-2x)dx$, hic tamen ista conditio manifesto subintelligitur, nisi primus terminus per se evanescat. Quocirca si ipsius $y = ax - xx$ quaeratur differentiale casu, quo $x = \frac{1}{2}a$, decrementum erit dx^2. Hoc autem solo casu excepto perpetuo functionis y differentiale erit $=(a-2x)dx$; nisi enim sit $x = \frac{1}{2}a$, terminus secundus $-dx^2$ prae primo semper recte negligitur. Neque vero neglectio termini dx^2 etiam in casu $x = \frac{1}{2}a$ in errorem inducere potest; comparari enim differentialia prima inter se solent; unde, quia $dy = -dx^2$ casu $x = \frac{1}{2}a$ prae differentialibus primis dx evanescit, perinde est, sive hoc casu habeamus $dy = 0$ sive $dy = -dx^2$.

339. Denotante y functionem quamcunque ipsius x, sit differentialibus continuis sumtis

$$dy = pdx, \ dp = qdx, \ dq = rdx, \ dr = sdx \text{ etc.}$$

Hinc ergo differentialia completa, in quibus nihil negligatur, ipsius y erunt

$$\begin{aligned}
d.y &= pdx + \tfrac{1}{2}qdx^2 + \tfrac{1}{6}rd^3x + \tfrac{1}{24}sdx^4 + \tfrac{1}{120}tdx^5 + \text{etc.}\\
d^2.y &= qdx^2 + rdx^3 + \tfrac{7}{12}sdx^4 + \tfrac{1}{4}tdx^5 + \text{etc.}\\
d^3.y &= rdx^3 + \tfrac{3}{2}sdx^4 + \tfrac{5}{4}tdx^5 + \text{etc.}\\
d^4.y &= sdx^4 + 2tdx^5 + \text{etc.}\\
d^5.y &= tdx^5 + \text{etc.}\\
&\text{etc.}
\end{aligned}$$

Nisi ergo primi termini harum expressionum evanescant, ii soli differentialia ipsius y exhibebunt; sin autem quopiam casu primus terminus fiat $=0$, tum sequens differentiale quaesitum exprimet. Atque si etiam secundus terminus evanescat, tum tertius terminus valorem differentialis quaesiti praebebit; sin autem et hic evanescat, quartus, et ita deinceps. Unde intelligitur nullius functionis ipsius x differentiale primum unquam penitus evanescere; etiamsi enim fiat $p=0$, quo casu vulgo dy evanescere censetur, tum hoc differentiale per altiorem ipsius dx potestatem exprimetur, uti vel per $\frac{1}{2}qdx^2$ vel, si etiam sit $q=0$, per $\frac{1}{6}rdx^3$, et ita porro.

340. Quanquam autem his casibus differentiale ipsius y respectu aliorum differentialium primorum, quibuscum comparatur, recte negligitur atque pro nihilo reputatur, tamen saepenumero eius veram expressionem nosse iuvat. Ex completa enim differentialis forma statim perspici potest, quibus casibus data functio fiat maximum vel minimum. Si enim fuerit

$$d.y = pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \text{etc.},$$

quo y nanciscatur maximum minimumve valorem, necesse est, ut sit $p=0$; erit ergo hoc casu $dy = \frac{1}{2}qdx^2$ et functio y, si loco x ponatur $x \pm dx$, abit in $y + \frac{1}{2}qdx^2$ eritque propterea minima, si q habeat valorem affirmativum, at maxima, si q habeat valorem negativum. At si simul fiat $q=0$, erit $dy = \frac{1}{6}rdx^3$ et functio y ponendo $x \pm dx$ loco x abibit in $y \pm \frac{1}{6}rdx^3$ neque hoc casu maximum neque minimum prodit; sin autem fiat et $r=0$, tum posito $x \pm dx$ loco x functio y evadet $= y + \frac{1}{24}sdx^4$, quae maximum exhibet, si s fuerit quantitas negativa, minimum vero, si s sit quantitas affirmativa. Aliae occasiones, quibus differentialium completa expressio usum habet, infra occurrent.

341. Ponamus p evanescere casu $x=a$, quod evenit, si fuerit $p=(x-a)P$. Talis autem valor prodit, si fuerit $y=(x-a)^2 P + C$ denotante C quantitatem constantem quamcunque. Cum enim sit

$$pdx = (x-a)^2 dP + 2(x-a)Pdx,$$

erit utique $p=0$ posito $x=a$. Tum ergo ob

$$dpdx = qdx^2 = (x-a)^2 ddP + 4(x-a)dPdx + 2Pdx^2$$

posito $x=a$ fiet $qdx^2 = 2Pdx^2$ atque differentiale completum hoc casu $x=a$ erit

$$dy = Pdx^2,$$

nisi forte et P evanescat posito $x=a$, quos casus postea contemplabor.

Praesens autem casus generalius hoc modo exhiberi potest. Sit

$$z=(x-a)^2 P+C$$

atque *y* sit functio quaecunque ipsius *z*, ita ut fiat $dy=Zdz$ denotante *Z* functionem quamcunque ipsius $z=(x-a)^2 P+C$. Erit ergo

$$dz=(x-a)^2 dP+2(x-a)Pdx \text{ et } pdx=Z(x-a)^2 dP+2Z(x-a)Pdx,$$

quod membrum fit $=0$, si $x=a$; eodemque casu neglectis terminis, qui continent factorem $x=a$, erit $qdx^2=2PZdx^2$ ideoque casu $x=a$ fiet $dy=PZdx^2$, postquam in *PZ* ubique loco *x* positum fuerit *a*. Quare si fuerit *y* functio quaecunque ipsius $z=(x-a)^2 P+C$, ita ut sit $dy=Zdz$, erit casu $x=a$ differentiale

$$dy=PZdx^2.$$

Fiet ergo haec functio *y* maxima casu *x* = *a*, si eodem casu fiat *PZ* quantitas negativa, minima vero, si *PZ* fiat quantitas affirmativa.

342. Si fuerit $p=(x-a)^2 P$, casu $x=a$ quoque *q* evanescit; talis autem expressio pro *p* oritur, si fuerit

$$p=(x-a)^3 P+C.$$

Erit ergo

$$pdx=(x-a)^3 dP+3(x-a)^2 Pdx,$$
$$qdx^2=(x-a)^3 ddP+6(x-a)^2 dPdx+6(x-a)Pdx^2,$$

quorum utrumque membrum casu $x=a$ evanescit; at vero sequens erit

$$rdx^3=(x-a)^3 d^3P+9(x-a)^2 ddPdx+18(x-a)dPdx^2+6Pdx^3=6Pdx^3$$

posito $x=a$. Quare cum et *p* et *q* casu $x=a$ evanescat, fiet

$$dy=\tfrac{1}{6}rdx^3=Pdx^3.$$

Simili modo si ponatur

$$z=(x-a)^3 P+C$$

fueritque *y* functio quaecunque ipsius *z*, ita ut sit $dy = Zdz$, ob

$$dz = (x-a)^3 dP + 3(x-a)^2 Pdx$$

fiet quoque $p = 0$ et $q = 0$ eritque $rdx^3 = 6PZdx^3$; unde casu $x = a$ erit

$$dy = PZdx^3$$

Quare ista functio *y*, etiamsi casu $x = a$ fiat $p = 0$, tamen neque maximum neque minimum valorem recipit.

343. Haec differentialia facilius inveniri possunt ex ipsa differentialium natura. Cum enim differentiale ipsius *y* oriatur, si *y* a statu sequenti proximo subtrahatur, qui prodit, si loco *x* ponatur $x + dx$, ponamus casu primo, quo erat

$$y = (x-a)^2 P + C,$$

$x + dx$ loco *x* eritque

$$y^{\mathrm{I}} = (x-a+dx)^2 \mathrm{P}^{\mathrm{I}} + C,$$

unde fiet

$$dy = (x-a+dx)^2 P^{\mathrm{I}} + (x-a)^2 P$$

Casu igitur, quo $x = a$, erit $dy = P^1 dx^2$, et cum P^{I} ad *P* rationem aequalitatis habeat, erit

$$dy = Pdx^2$$

Simili modo si fuerit

$$z = (x-a)^2 P + C,$$

erit $dz = Pdx^2$; quare si sit *y* functio quaecunque ipsius *z*, ita ut sit $dy = Zdz$, erit

$$dy = PZdx^2$$

casu, quo ponitur $x = a$.
Deinde si sit

$$z = (x-a)^3 P + C,$$

erit $z^{\mathrm{I}} = (x-a+dx)^3 P^{\mathrm{I}} + C$ et propterea casu $x = a$ fiet

$$z^{\mathrm{I}} - z = dz = Pdx^3.$$

Hinc si fuerit *y* functio quaecunque ipsius *z* atque $dy = Zdz$, erit quoque casu $x = a$ differentiale $dy = PZdx^3$, siquidem in functionibus *P* et *Z* loco *x* ubique substituatur *a*. Quoniam vero hoc casu fit $z = C$ atque *Z* est functio ipsius *z*, evadet *Z* quantitas constans, talis scilicet functio ipsius *C*, qualis ante erat ipsius *z*.

344. Si igitur generaliter fuerit

$$y = (x-a)^n P + C,$$

quia est

$$y^{\mathrm{I}} = (x-a+dx)^n P^{\mathrm{I}} + C,$$

casu $x = a$ fiet

$$dy = Pdx^n;$$

unde si fuerit $n > 1$, hoc differentiale respectu aliorum differentialium primorum, quae ipsi *dx* sunt homogenea, evanescet. Ex praecedentibus ergo manifestum est functionem *y* fieri casu $x = a$ vel maximam vel minimam, si fuerit *n* numerus par; tum enim, si posito $x = a$ fiat *P* quantitas affirmativa, fiet *y* minimum; sin autem *P* fit quantitas negativa, fiet *y* maximum. Hocque ergo modo ratio maximorum et minimorum multo facilius invenitur quam methodo supra exposita, quia non opus est ad differentialia altiora progredi. Quodsi vero sit

$$z = (x-a)^n P + C$$

atque *y* fuerit functio quaecunque ipsius *z*, ut sit $dy = Zdz$, erit casu $x = a$ differentiale

$$dy = PZdx^n$$

Notandum autem est hic *n* sumi pro numero affirmativo seu 0 maiore; si enim *n* esset numerus negativus, tum posito $x = 0$ non evanesceret $(x-a)^n$, uti assumsimus, sed adeo fieret infinite magnum.

345. Iam vidimus hoc pacta differentiale multo expeditius inveniri quam ope seriei, qua ante differentiale completum expressimus; si enim sit *n* numerus integer, tot seriei illius termini perlustrari deberent, quot *n* contineat unitates. Verum si *n* sit numerus fractus, tum series ista nequidem verum differentiale unquam exhibebit. Ponamus enim esse

$$y = (x-a)^{\frac{3}{2}} + a\sqrt{a};$$

si seriem

$$dy = pdx + \tfrac{1}{2}qdx^2 + \tfrac{1}{6}rd^3x + \tfrac{1}{24}sdx^4 + \text{etc.}$$

spectemus, fiet

$$p=\tfrac{3}{2}\sqrt{(x-a)},\ \ q=\frac{3}{4\sqrt{(x-a)}},\ \ r=\frac{-3}{8(x-a)\sqrt{(x-a)}},\ \ s=\frac{9}{16(x-a)^2\sqrt{(x-a)}}\ \text{etc.}$$

Quare si ponatur $x=a$, fiet quidem $p=0$, at sequentes termini omnes *q, r,s* etc. evadent infiniti; unde valor differentialis *dy* hoc casu omnino definiri non potest. At vero methodus ex ipsa differentialium natura deducta nullum dubium relinquit. Cum enim sit $y=(x-a)^{\frac{3}{2}}+a\sqrt{a}$, posito $x+dx$ loco *x* fiet $y^{\mathrm{I}}=(x-a+dx)^{\frac{3}{2}}+a\sqrt{a}$ eritque, si $x=a$ ponatur, $dy=dx\sqrt{dx}$. Evanescit ergo hoc differentiale prae *dx;* at vero differentialia secunda cum dx^2 homogenea prae eo evanescent.

346. Evolvamus hos casus, quibus exponens *n* est numerus fractus, aliquanto accuratius sitque

$$y=P\sqrt{(x-a)}+C\,;$$

ob $y^{\mathrm{I}}=P\sqrt{(x-a+dx)}+C$ fiet $dy=P\sqrt{dx}$ casu $x=a$; unde hoc differentiale ad *dx* et ad differentialia cum *dx* homogenea rationem tenebit infinitam. Hinc etiam patet, quid hoc casu de ratione maximi ac minimi sit tenendum. Cum enim posito $a+dx$ loco *x* abeat *y* in

$$C+P\sqrt{dx}\,,$$

ob $\sqrt{dx}$ ambiguum functio *y* geminum induet valorem, alterum maiorem quam *C*, quem recipit posito $x=a$, alterum minorem; unde casu $x=a$ neque maximum neque minimum fiet. Praeterea si *dx* capiatur negative, tum valor ipsius *y* adeo fiet imaginarius. Idem tenendum est, si sit $z=P\sqrt{(x-a)}+C$ et *y* functio quaecunque ipsius *z*, ut sit $dy=Zdz$; tum enim erit $dy=PZ\sqrt{dx}$ casu $x=a$.

347. Si proposita fuerit ista functio

$$y=(x-a)^{\frac{m}{n}}P+C\,,$$

cuius differentiale quaeritur casu $x=a$, erit, uti ex antecedentibus colligitur,

$$dy=Pdx^{\frac{m}{n}}.$$

Quocirca si fuerit $m>n$, hoc differentiale prae *dx* evanescet; sin autem sit $m<n$, ratio $\frac{dy}{dx}$ erit infinite magna. Praeterea vero si *n* sit numerus par, differentiale *dy* geminum habebit valorem, alterum affirmativum, alterum negativum; sicque functio *y*, quae casu $x=a$ fit $=C$, si ponatur $x=a+dx$, binos habebit valores, alterum maiorem quam *C*, alterum

vero minorem; sin autem poneretur $x = a - dx$, tum *y* adeo fieret imaginarium; unde hoc casu *y* neque maximum fit neque minimum. Ponamus nunc denominatorem *n* esse numerum imparem; erit numerator *m* vel par vel impar. Sit primo *m* numerus par; quia *dy* eundem valorem retinet, sive *dx* sumatur affirmative sive negative, perspicuum est functionem *y* casu $x = a$ fieri sive maximam sive minimam, prout hoc casu fuerit *P* vel quantitas negativa vel affirmativa. Sin autem uterque numerus *m* et *n* fuerit impar, differentiale *dy* in sui negativum abibit posito *dx* negativo; hocque ergo casu functio *y* neque maximum erit neque minimum, si ponatur $x = a$.

348. Si functio *y* ex pluribus huiusmodi terminis, quorum singuli sint divisibiles per $x = a$, constet, ita ut sit

$$y = (x-a)^m P + (x-a)^n Q + C,$$

tum eius differentiale casu $x = a$ erit

$$dy = Pdx^m + Qdx^n;$$

in qua expressione, si fuerit $n > m$, terminus secundus prae primo evanescit, ita ut tantum prodeat $dy = Pdx^m$. Sin autem *n* sit fractio denominatorem habens parem, tum, etiamsi Qdx^n prae Pdx^m evanescat, tamen omnino negligi non potest. Ex eo enim apparet, si capiatur *dx* negative, valorem ipsius *dy* fieri imaginarium, quod ex solo termino primo Pdx^m non patet. Cum ergo, si *n* sit fractio denominatorem habens parem, *dx* negative accipi nequeat, sin autem affirmative capiatur, terminus Qdx^n geminum praebeat valorem, functio $y = (x-a)^m P + (x-a)^n Q + C$, quae casu $x = a$ fit $= C$, si ponatur $x = a + dx$, erit

$$y = C + Pdx^m \pm Qdx^n;$$

quorum valorum uterque cum vel maior sit vel minor quam *C*, prout *P* fuerit quantitas vel affirmativa vel negativa, erit functio *y* casu $x = a$ vel minimum vel maximum secundae speciei [§ 278].

349. His igitur casibus differentialia functionum vera non per regulas differentiationis consuetas inveniri possunt; quippe quae tantum valent, quamdiu differentiale functionis est homogeneum cum *dx*. Sin autem casu quopiam singulari differentiale functionis exprimatur per eius potestatem dx^n, tum regula praebet pro hoc differentiali 0, si *n* fuerit numerus unitate maior; at vero differentiale exhibet infinite magnum, si *n* sit exponens unitate minor. Sic si ipsius $y = \sqrt{(a-x)}$ differentiale quaeratur casu $x = a$, quia est $dy = -\frac{dx}{\sqrt{(a-x)}}$, facto $x = a$ prodit $dy = -\frac{dx}{0}$. Atque si differentialia sequentia in subsidium vocare velimus, omnia pariter ob denominatores $=0$ in infinitum excrescunt, ita ut inde nihil concludi possit. At vero hoc casu vidimus esse $dy=\sqrt{-dx}$ atque adeo imaginarium. Sin autem loco x ponatur $x-dx$, erit $dy=\sqrt{dx}$ atque adeo erit infinities maius quam dx, ita ut dx prae dy evanescat. Quare regula consueta etiam hoc casu in errorem non inducit, cum valorem ipsius dy infinitum exhibeat.

350. A regula ergo consueta differentiationis recedendum est, quoties in serie

$$pdx+\tfrac{1}{2}qdx^2+\tfrac{1}{6}rd^3x+\text{etc.},$$

qua differentiale completum functionis y exprimitur, primus terminus p vel fit $=0$ vel in infinitum excrescit, eoque casu differentiale ex primis principiis derivari debet. Quoties ergo functionis y differentiale quaeritur dato ipsius x valori respondens, quo littera p vel infinite parva evadit vel infinite magna, toties recurrendum est ad ipsa prima differentiationis principia. Omnibus vero reliquis casibus, quibus fit neque $p=0$ neque $p=\infty$, consueta regula veros differentialis valores praebebit. Interim tamen casus ante (§ 348) memoratus non est negligendus, si functio y contineat huiusmodi membrum ex $(x-a)^n Q$ existente n fractione denominatorem parem habente; etiamsi enim adsint differentialia inferiora quam Qdx^n, prae quibus hoc evanescat, tamen, quoniam Qdx^n, si sit dx negativum, fit imaginarium, hoc membrum Qdx^n reliqua omnia, prae quibus evanescit, quoque transmutat in imaginaria; cuius circumstantiae ratio potissimum in lineis erit habenda. Huiusmodi ergo casus particulares, quibus verum differentiale communi regula non indicatur, in adiunctis exemplis explicabo.

EXEMPLUM 1

Quaeratur differentiale functionis $y=a+x-\sqrt{\left(xx+ax-x\sqrt{(2ax-xx)}\right)}$ *casu, quo ponitur* $x=a$.

Differentiale istius functionis casu $x=a$ per regulam receptam non reperiri ex differentiatione patet; fit enim

$$\frac{dy}{dx}=dx+\frac{-xdx-\frac{1}{2}adx+\frac{1}{2}dx\sqrt{(2ax-xx)}+(axdx-xxdx):\sqrt{(2ax-xx)}}{\sqrt{\left(xx+ax-x\sqrt{(2ax-xx)}\right)}};$$

posito autem $x=a$ erit $dy=dx-\frac{adx}{a}=0$. Ordiamur ergo a principiis differentialis ac primo quidem posito $x+dx$ loco x fiet

$$y^{\mathrm{I}} = a + x + dx$$
$$-\sqrt{\left(xx + 2xdx + dx^2 + ax + adx - (x+dx)\sqrt{\left(2ax - xx + 2adx - 2xdx - dx^2\right)}\right)}$$

Posito autem $x = a$ erit

$$y^{\mathrm{I}} = 2a + dx - \sqrt{\left(2aa + 3adx + dx^2 - (a+dx)\sqrt{\left(aa - dx^2\right)}\right)}.$$

Iam cum sit $\sqrt{\left(aa - dx^2\right)} = a - \frac{dx^2}{2a}$ (sequentes enim termini tuto negligli poterunt, quia non omnes, qui sunt infinities maiores, destruentur, ut mox patebit), erit

$$y^{\mathrm{I}} = 2a + dx - \sqrt{\left(aa + 2adx + \tfrac{3}{2}dx^2\right)}$$

porroque radicem extrahendo fiet

$$y^{\mathrm{I}} = 2a + dx - \left(a + dx + \frac{dx^2}{4a}\right) = a - \frac{dx^2}{4a}.$$

At casu $x = a$ erit $y = a$ *;* unde, cum sit $y^{\mathrm{I}} = y + dy$, obtinebitur

$$dy = -\frac{dx^2}{4a} \;;$$

ex quo simul perspicitur functionem propositam y fieri maximum, si ponatur $x = a$.

EXEMPLUM 2

Invenire differentiale huius functionis $y = 2ax - xx + a\sqrt{(aa - xx)}$ *casu, quo ponitur* $x = a$.

Facta differentiatione more consueto fit

$$dy = 2adx - 2xdx - \frac{axdx}{\sqrt{(aa - xx)}},$$

quod posito $x = a$ in infinitum abit neque ergo hoc modo indicatur. Differentialia vero sequentium ordinum pariter omnia fient infinita, ita ut ex iis nequidem ex serie $pdx+\frac{1}{2}qdx^2+\frac{1}{6}rdx^3+\text{etc.}$ verus valor differentialis inveniri queat. Ponamus ergo $x+dx$ loco x atque habebimus

$$y^{\mathrm{I}}=2ax-xx+2adx-2xdx-dx^2+a\sqrt{\left(aa-xx-2xdx-dx^2\right)}$$

et posito $x=a$ erit

$$y^{\mathrm{I}}=aa-dx^2+a\sqrt{\left(-2adx-dx^2\right)}\,.$$

At eodem casu fit $y=aa$; unde erit $dy=-dx^2+a\sqrt{-2adx}$, et cum dx^2 prae $\sqrt{-2adx}$ evanescat, erit

$$dy=a\sqrt{-2adx}\,.$$

Quare si differentiale *dx* affirmative capiatur, erit *dy* imaginarium; sin autem pro *x* scribatur $x-dx$, erit

$$dy=a\sqrt{2adx}\,;$$

cuius cum duplex sit valor, alter affirmativus, alter negativus, functio *y* casu $x=a$ neque maxima fiet neque minima.

EXEMPLUM 3

Invenire differentiale functionis $y=3aax-3axx+x^3+(a-x)^2\sqrt[3]{\left(a^3-x^3\right)}$ *casu, quo ponitur* $x=a$.

Quoniam haec functio in istam formam transformatur

$$y=a^3-(a-x)^3+(a-x)^{\frac{7}{3}}\sqrt[3]{\left(aa+ax+xx\right)}\,,$$

posito $x=a+dx$ fit

$$y^{\mathrm{I}}=a^3+dx^3-dx^{\frac{7}{3}}\sqrt[3]{3aa}$$

eodemque casu est $y=a^3$. Erit ergo $dy=dx^3-dx^{\frac{7}{3}}\sqrt[3]{3aa}$, et cum dx^3 evanescat prae $dx^{\frac{7}{3}}$, erit

$$dy=-dx^{\frac{7}{3}}\sqrt[3]{3aa}\;;$$

ergo $x=a$ functio *y* neque maximum fit neque minimum.

EXEMPLUM 4

Invenire differentiale functionis $y=\sqrt{x}+\sqrt[4]{x^3}=\left(1+\sqrt[4]{x}\right)\sqrt{x}$ *casu* $x=0$.

Quoniam casus $x=0$ proponitur eoque sit $y=0$, loco x tantum dx scribatur et habebitur

$$dy=dx^{\frac{1}{2}}+dx^{\frac{3}{4}} \text{ seu } dy=\left(1+\sqrt[4]{dx}\right)\sqrt{dx};$$

unde primum patet dx negative accipi non posse. Tum vero, etiamsi alias $\sqrt{dx}$ geminum valorem prae se ferat, alterum affirmativum, alterum negativum, tamen hoc casu, quia eius radix $\sqrt[4]{dx}$ occurrit, non nisi affirmative accipi potest. At vero $\sqrt[4]{dx}$ utrumque significatum recipit eritque

$$dy=\sqrt{dx}\pm\sqrt[4]{dx^3} \text{ et } y^{\mathrm{I}}=\sqrt{dx}\pm\sqrt[4]{dx^3}$$

ob $y=0$. Cum igitur uterque ipsius y^{I} valor maior sit quam ipius y, sequitur casu $x=0$ fieri y minimum. Quod autem functio $y=\sqrt{x}+\sqrt[4]{x^3}$ non complectatur hanc $y=-\sqrt{x}+\sqrt[4]{x^3}$, utramque ad rationalitatem perducendo patebit. Prior enim fusa in hanc formam $y-\sqrt{x}=\sqrt[4]{x^3}$ et quadrata dat $y^2-2y\sqrt{x}+x=x\sqrt{x}$ seu $y^2+x=(x+2y)\sqrt{x}$, quae denuo quadrata praebet

$$y^4-2yyx-4xxy+xx-x^3=0.$$

..

Altera vero $y+\sqrt{x}=\sqrt[4]{x^3}$ dabit $y^2+x=(x-2y)\sqrt{x}$ et porro

$$y^4-2yyx+4xxy+xx-x^3=0,$$

quae ab illa est diversa. At vero alterum membrum $\sqrt[4]{x^3}$ ambiguitatem signi retinet. Quamobrem ista circumstantia probe est notanda, quod, etiamsi communiter radices potestatum parium utrumque signum + et – includant, tamen haec ambiguitas cesset, si in eadem expressione earumdem radicum ulteriores radices potestatum parium occurrant; quippe quae fierent imaginariae, si radices priores negative acciperentur. Atque ex hoc fonte maxima et minima secundae speciei sequuntur, quando talia non locum habere videantur.

EXEMPLUM 5

Invenire differentiale functionis

$$y = a + \sqrt{(x-f)} + (x-f)\sqrt[4]{(x-f)} + (x-f)^2\sqrt[8]{(x-f)}$$

casu, quo ponitur $x = f$.

Ponamus $x - f = t$, et cum sit $y = a + \sqrt{t} + t\sqrt[4]{t} + tt\sqrt[8]{t}$, huius differentiale quaeritur casu $t = 0$, quo fit $y = a$. Posito ergo $t + dt$ seu $0 + dt$ loco t fiet

$$y^{\mathrm{I}} = y + dy = a + \sqrt{dt} + dt\sqrt[4]{dt} + dt^2\sqrt[8]{dt}$$

ideoque habebitur

$$dy = \sqrt{dt} + dt\sqrt[4]{dt} + dt^2\sqrt[8]{dt}\ ;$$

Ubi primo patet differentiale *dt* negative accipi non posse, quin *dy* fiat imaginarium. Tum verum non solum $\sqrt{dt}$, sed nequidem $\sqrt[4]{dt}$ negative accipi potest; fieret enim $\sqrt[8]{dt}$ imaginarium; unde differentiale *dy* geminum tantum habet valorem

$$dy = \sqrt{dt} + dt\sqrt[4]{dt} \pm dt^2\sqrt[8]{dt}$$

quorum cum uterque maior sit nihilo, sequitur functionem *y* fieri minimum secundae speciei posito $t = 0$ seu $x = f$. Quanquam ergo his casibus termini $\sqrt[4]{dt}$ et $\sqrt[8]{dt}$ prae primo $\sqrt{dt}$ evanescant, tamen eorum ratio est habenda, si multiplicitas valorum spectetur, ut imaginaria evitentur.

EXEMPLUM 6

Invenire differen#ale functionis $y = ax + bxx + (x-f)^n + (x-f)^{m+\frac{1}{2}n}$ *casu* $x = f$.

Si ponatur $x = f$, fiet $y = af + bff$, et si loco *x* ponatur $x + dx$ seu $f + dx$, prodibit valor proximus

$$y^{\mathrm{I}} = af + bff + adx + 2bfdx + bdx^2 + dx^n + dx^{m+\frac{1}{2}n},$$

ita ut sit

$$dy = adx + 2bfdx + bdx^2 + dx^n + dx^m\sqrt{dx^n}$$

Nisi ergo sit *n* numerus par, differentiale *dx* negative sumi nequit. Ultimus autem terminus $dx^m\sqrt{dx^n}$ signum habet ambiguum; unde valor ipsius y^{I} erit duplex, uterque maior quam ipsius *y*, si quidem $a + 2bf$ fuerit quantitas affirmativa atque exponentes *n* et $m + \frac{1}{2}n$

unitate fuerint maiores. Fiet ergo valor functionis *y* casu $x = f$ minimus hocque evenit, sive *n* sit numerus integer sive fractus, dummodo numerator hoc casu et ipse numerus ino casu non fuerit par.

351. Imprimis autem haec methodus differentialia ex ipsis principiis deducendi usum habet in functionibus transcendentibus, cum quibusdam casibus differentiale more consueto inventum vel evanescit vel in infinitum excrescere videtur. Occurrunt autem hic eiusmodi infinitorum et infinite parvorum species, quae in algebraicis nunquam inveniuntur. Cum enim, si *i* denotet numerum infinitum, *li* sit quoque infinitus quidem, sed tamen ad ipsum numerum *i* eiusque adeo potestatem quamcunque i^n, quamtumvis exiguus statuatur exponens *n*, rationem tenens infinite parvam, erit fractio $\frac{li}{i^n}$ infinite parva neque ante finita esse poterit, quam exponens *n* fiat infinite parvus. Erit ergo *li* homogeneum cum i^n, si exponens *n* fuerit infinite parvus. Ponamus nunc $i = \frac{1}{\omega}$ existente ω quantitate infinite parva; erit $-l\omega$ homogeneum cum $\frac{1}{\omega^n}$, si exponens *n* sit infinite parvus, ideoque $-\frac{1}{l\omega}$ homogeneum erit cum ω^n; hincque $-\frac{1}{ldx}$ erit infinite parvum comparandum cum dx^n existente *n* fractione infinite parva. Ita si fuerit $y = -\frac{1}{ldx}$, differentiale ipsius *y* casu $x = 0$ erit $= -\frac{1}{ldx} = dx^n$ ideoque *dy* ad *dx* atque ad quamcunque ipsius *dx* potestatem tenebit rationem infinitam; atque prae $-\frac{1}{ldx}$ evanescunt omnes omnino potestates ipsius *dx*, quantumvis exigui fuerint earum exponentes.

352. Deinde quoque vidimus, si *a* fuerit numerus unitate maior et *i* infinitus, tum a^i fore infinitum tam excelsi gradus, ut prae eo non solum *i*, sed etiam quaevis ipsius *i* potestas evanescat; neque i^n ante homogeneum cum a^i evadet, quam exponens *n* in infinitum fuerit auctus. Sit nunc $i = \frac{1}{\omega}$, ita ut ω infinite parvum denotet; erit $a^{\frac{1}{\omega}}$ homogeneum cum $\frac{1}{\omega^n}$ existente *n* numero infinite magno ideoque $a^{\frac{-1}{\omega}}$ seu $\frac{1}{a^{1:\omega}}$ erit infinite parvum comparandum cum ω^n. Hinc $\frac{1}{a^{1:dx}}$ erit infinite parvum, quod autem prae omnibus ipsius *dx* potestatibus evanescit, cum homogeneum sit cum potestate dx^n existente *n* numero infinite magno. Quare si quaeratur differentiale ipsius $y = \frac{1}{a^{1:x}}$ casu $x = 0$, quoniam fit $y = 0$, erit $dy = \frac{1}{a^{1:dx}}$ ideoque infinities minus est quam potestas quantumvis alta ipsius *dx*.

353. Sin autem *a* sit numerus unitate minor, tum, quia $\frac{1}{a}$ fit unitate maior, quaestio ad casum praecedentem reducitur. Scilicet si habeatur expressio $a^{\frac{1}{\omega}}$, ea ponendo $a = 1:b$

transmutabitur in $b^{-\frac{1}{\omega}}$ seu $\frac{1}{b^{1:\omega}}$ quae homogenea erit ob $b > 1$ cum ω^n existente n numero infinite magno. His igitur praemissis sequentia exempla resolvere poterimus.

EXEMPLUM 1

Invenire differentiale functionis $y = xx - \frac{1}{lx}$ *casu* $x = 0$.

Quoniam posito $x = 0$ fit $y = 0$, si ponamus $x + dx$ seu $0 + dx$ loco x, fiet

$$y^{\mathrm{I}} = dy = dx^2 - \frac{1}{ldx}.$$

Cum autem $-\frac{1}{ldx}$ homogeneum sit cum dx^n denotante n numerum infinite parvum, prae eo dx^2 evanescet eritque

$$dy = -\frac{1}{ldx} = dx^n.$$

At vero quia logarithmi numerorum negativorum sunt imaginarii, dx negative accipi non poterit eritque adeo casu $x = 0$ functio y minimum, sed neque ad primam neque ad secundam speciem pertinens. Ad primam scilicet speciem non pertinet, quia y nullos habet valores antecedentes proximos, sed tantum minus est valoribus sequentibus, si x nihilo maius statuatur. Ad secundam autem speciem ideo non pertinet, quia valores sequentes, quibuscum comparatur, non sunt gemini; sic itaque prodit tertia species maximorum minimorumve, quae in functionibus logarithmicis et transcendentibus tantum locum habet, in algebraicis autem nunquam occurrit; de qua in sequente parte de lineis curvis fusius agetur.

EXEMPLUM 2

Invenire differentiale functionis $y = (a - x)^n - x^n (la - lx)^n$ *casu, quo* $x = a$.

Differentiale hoc, si n sit numerus integer, ex formula generali

$$dy = pdx + \frac{1}{2}qdx^2 + \frac{1}{6}rdx^3 + \text{etc}.$$

inveniri potest; erit enim

$$pdx = -n(a - x)^{n-1}dx - nx^{n-1}(la - lx)^n + nx^{n-1}(la - lx)^{n-1}dx$$

qui valor posito $x = a$ utique evanescit; nam etiamsi sit $n = 1$, erit

$$pdx = -dx + dx = 0.$$

Si igitur ulterius progrediamur, erit

$$\tfrac{1}{2}qdx^2 = \tfrac{n(n-1)}{1\cdot 2}(a-x)^{n-2}dx^2 - \tfrac{n(n-1)}{1\cdot 2}\; x^{n-2}(la-lx)^n + \tfrac{n^2}{2}x^{n-2}dx^2(la-lx)^{n-1}$$
$$+\tfrac{n(n-1)}{1\cdot 2}x^{n-2}dx^2(la-lx)^{n-1} - \tfrac{n(n-1)}{1\cdot 2}\; x^{n-2}(la-lx)^{n-2}.$$

Hinc ergo, si fuert $n=1$,erit $\tfrac{1}{2}qdx^2 = \tfrac{dx^2}{2a}$ posito $x=a$. Simili modo, si sit $n=2$, ad terminum tertium $\tfrac{1}{6}rdx^3$ esset pergendum et ita porro. Facilius ergo utemur ipsis differentiationis principiis, et cum posito $x=a$ fiat $y=0$, si ponamus $x+dx$ seu $a+dx$ loco *x*, erit

$$y^{\mathrm{I}} = (-dx)^n - (a+dx)^n\left(la\ -l(a+dx)\right)^n = y+dy = dy$$

ob $y=0$. Est vero

$$l(a+dx) = la + \tfrac{dx}{a} - \tfrac{dx^2}{2a^2} + \tfrac{dx^3}{3a^3} - \text{etc.},$$

unde fit

$$dy = (-dx)^n - \left(a^n + na^{n-1}dx + \tfrac{n(n-1)}{1\cdot 2}a^{n-2}dx^2 + \text{etc.}\right)\left(-\tfrac{dx}{a} + \tfrac{dx^2}{2a^2} - \tfrac{dx^3}{3a^3} + \text{etc.}\right)^n = \tfrac{n}{2a}(-dx)^{n+1}.$$

Casu igitur $x=a$ erit formulae propositae differentiale quaesitum *dy*, ut sequitur:

si $n=1$	$dy = \tfrac{dx^2}{2a}$, ut ante invenimus,
si $n=2$	$dy = -\tfrac{2dx^3}{2a}$
si $n=3$	$dy = \tfrac{3dx^4}{2a}$
si $n=4$	$dy = -\tfrac{4dx^5}{2a}$
etc.	etc.

Si ergo *n* fuerit numerus impar, functio *y* casu $x=a$ fit minimum, sin autem *n* sit numerus par, neque maximum neque minimum; quod idem valet, si *n* fuerit fractio denominatorem habens imparem. Sin autem *n* fuerit fractio denominatorem habens parem, tum *dx* negative accipi debet, ne in imaginaria incidamus; et ob ambiguitatem significationis functio quoque neque maxima neque minima evadet.

EXEMPLUM 3

Invenire differentiale functionis $y = x^x$ *casu* $x = \tfrac{1}{e}$ *denotante e numerum, cuius logarithmus hyperbolicus est* = 1.

Quia fit in genere $dy = x^x dx(lx+1)$, hoc differentiale casu $x = \frac{1}{e}$ seu $lx = -1$ evanescit. Comparetur ergo hoc differentiale cum forma generali $pdx + \frac{1}{2}qdx^2 + \text{etc.}$; erit

$$p = x^x(lx+1) \text{ et } q = x^x(lx+1)^2 + x^{x-1}$$

et posito $lx = -1$ seu $x = \frac{1}{e}$ erit

$$q = \left(\tfrac{1}{e}\right)^{\frac{1-e}{e}} = e^{\frac{e-1}{e}}.$$

Quare differentiale quaesitum erit

$$dy = \tfrac{1}{2}e^{(e-1):e}dx^2$$

evaditque ergo functio $y = x^x$ minimum casu $x = \frac{1}{e}$.

EXEMPLUM 4

Invenire differentiale functionis huius $y = x^n + e^{-1:x}$ casu, quo $x = 0$.

Quia facto $x = 0$ fit $y = 0$, si ponatur $x = 0 + dx$, erit

$$y^{\mathrm{I}} = dy = dx^n + \frac{1}{e^{1:dx}}.$$

Vidimus autem $\frac{1}{e^{1:dx}}$ homogeneum esse cum potestate ipsius *dx* infinita seu cum dx^∞ ideoque prae dx^n evanescet, ita ut sit

$$dy = dx^n.$$

354. Quod in differentialibus primis certis casibus usu venit, ut consueta differentiationis regula non prodeant, idem quoque in differentialibus secundi ac tertii superiorumque ordinum evenit iis casibus, quibus in forma differentiali completa

$$dy = pdx + \tfrac{1}{2}qdx^2 + \tfrac{1}{6}rd^3x + \tfrac{1}{24}sdx^4 + \text{etc.}$$

quantitatum *q, r,* s etc. nonnullae vel evanescunt vel in infinitum abeunt. Scilicet cum sit

$$dd.y = qdx^2 + rdx^3 + \tfrac{7}{12}sdx^4 + \text{etc.},$$

si quo casu fiat $q = 0$, tum erit $ddy = rdx^3$; sin autem eodem casu et *r* evanescat, tum erit $ddy = \frac{7}{12}sdx^4$, et ita porro. Sin autem vel *q* vel *r* vel *s* etc. fiat infinitum, tum ex ista serie differentiale secundum prorsus inveniri nequit, sed confugiendum erit ad principia differentialium; scilicet ponendo $x+dx$ loco x quaeratur valor y^{I} et ponendo $x+2dx$ loco x valor ipsius y^{II}, quo facto erit verus valor differentialis secundi

$$ddy = dy^{\mathrm{I}} - dy = dy^{\mathrm{II}} - 2dy^{\mathrm{I}} + y\,.$$

Simili modo si de differentiali tertio quaestio proponatur, tum praeterea in y loco x scribatur $x+3dx$ inventoque valore y^{III} erit

$$d^3y = y^{\mathrm{III}} - 3y^{\mathrm{II}} + 3y^{\mathrm{I}} - y$$

sicque deinceps. Quos casus sequentibus exemplis illustrabimus.

EXEMPLUM 1

Invenire differentiale secundum funcitonis $y=\frac{aa-xx}{aa+xx}$ *casu, quo ponitur* $x=\frac{a}{\sqrt{3}}$.

Quaerendo differentiale completum ipsius y ex forma

$$dy = pdx + \tfrac{1}{2}qdx^2 + \tfrac{1}{6}rd^3x + \tfrac{1}{24}sdx^4 + \text{etc.}$$

prodibunt pro p, q, r, s etc. sequentes valores

$$p = -\frac{4aax}{(aa+xx)^2},\ q = \frac{-4a^4+12aaxx}{(aa+xx)^3} \quad \text{atque} \quad r = \frac{48a^4x-48aax^3}{(aa+xx)^4}\,.$$

Cum nunc sit

$$ddy = rdx^3 + \tfrac{7}{12}sdx^4 + \text{etc.}$$

ob $q=0$ casu $x=-a$ eodemque casu sit $r=\frac{27\sqrt{3}}{8a^3}$, fiet differentiale secundum quaesitum

$$ddy = \frac{27dx^2\sqrt{3}}{8a^3}\,.$$

EXEMPLUM 2

Invenire differentiale tertium functionis $y=\frac{aa-xx}{aa+xx}$ *casu* $x=a$.

Quaerendo ut ante differentiale completum

$$dy = pdx + \tfrac{1}{2}qdx^2 + \tfrac{1}{6}rd^3x + \tfrac{1}{24}sdx^4 + \text{etc.},$$

quia est differentiale tertium $d^3y = rdx^3 + \frac{3}{2}sdx^4 + \text{etc.}$, ob

$$r = \frac{48a^4x - 48aax^3}{(aa+xx)^4}$$

fiet $r = 0$ casu $x = a$; quare ad valorem s est progrediendum, qui erit

$$s = \frac{48a^4x - 144aaxx}{(aa+xx)^4} - \frac{8x(48a^4x - 48aax^3)}{(aa+xx)^5} \;;$$

facto ergo $x = a$ erit $s = -\frac{96a^4}{2^4a^8} = -\frac{6}{a^4}$; unde hoc casu erit

$$d^3y = -\frac{9dx^4}{a^4}.$$

EXEMPLUM 3

Invenire differentialia cuiusque gradus functionis $y = ax^m + bx^n$ *casu* $x = 0$.

Ponendo successive $x + dx,\ x + 2dx,\ x + 3dx$ etc. loco x valores sequentes functionis y erunt

$$y^{\text{I}} = a(x + dx)^m + b(x + dx)^n,$$
$$y^{\text{II}} = a(x + 2dx)^m + b(x + 2dx)^n,$$
$$y^{\text{III}} = a(x + 3dx)^m + b(x + 3dx)^n$$

etc.

Posito ergo $x = 0$ erit $y = 0$ eiusque differentialia erunt

$$dy = adx^m + bdx^n,$$
$$ddy = (2^m - 2)adx^m + (2^n - 2)bdx^n,$$
$$d^3y = (3^m - 3\cdot 2^m + 3)adx^m + (3^n - 3\cdot 2^n + 3)bdx^n,$$
$$d^4y = (4^m - 4\cdot 3^m + 6\cdot 2^m - 4)adx^m + (4^n - 4\cdot 3^n + 6\cdot 2^n - 4)bdx^n$$

etc.

Si igitur exponens n fuerit maior quam m, termini secundi in his expressionibus evanescunt prae primis. Interim tamen eorum ratio erit habenda, si n fuerit numerus fractus, ut casus, quibus haec differentialia vel fiunt imaginaria vel ambigua, diiudicari queant. Ulteriorem vero horum casuum evolutionem in doctrinam de lineis curvis reservari convenit.