CONCERNING THE CRITERIA OF IMAGINARY ROOTS

313. In the preceding chapter we have shown the manner of investigating each of the roots of an equation, thus so that with the benefit of that, if some equation may be proposed, it will be possible to find, how many real roots that may have, and how many imaginary roots. Indeed generally this investigation is put in place with most difficulty, since the differential equation thus has been provided, so that the roots of that cannot be shown. But nevertheless in these cases the same operation may be applied to the differential equation itself and the nature of the roots of this are to be investigated from the differential of this and hence the roots of this equation can be assigned approximately, yet the effort becomes most often excessively tiresome. On account of which in this matter, on many occasions, it suffices to know a criterion of such a kind, from the circumstances of which it will be able to conclude without risk the imaginary roots to be present in the equation proposed, even if from the absence of these criteria it cannot be inferred in turn that all the roots evidently are real. Which knowledge even if it is imperfect, yet frequently finds a use ; on account of which we have appointed the present chapter to explaining these criteria.

314. In the preceding chapter we have seen, if some equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0$$

may have all the roots real, then also the differential of this

$$\tfrac{dz}{dx} = nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + \text{etc.} = 0$$

is going to have all its roots real. Likewise truly we have shown, even if the differential equation may have all its roots real, yet thence it does not follow that all the roots of the proposed equation itself are to become real. Yet meanwhile, if the differential equation may have imaginary roots, then we always conclude correctly the proposed equation itself must have just as many imaginary roots. At least I can say; it can come about indeed, that the equation itself may have more imaginary roots. Therefore in this way more cannot be concluded from the differential equation than, if that may have imaginary roots, the proposed equation itself also must have imaginary roots, and indeed at least just as many.

315. If the proposed equation may be multiplied by some power x^m with *m* denoting a positive integer, then, because this new curve will have all the roots real, if indeed all the roots of the proposed equation were real, then the differential of this too, after it were divided by x^{m-1}, will have all real roots. Hence if this equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0$$

may have all real roots, then also this equation

$$(m+n)x^{n}-(m+n-1)Ax^{n-1}+(m+n-2)Bx^{n-2}-\text{etc.}=0$$

will have all the roots real. On account of the same reasoning, if this may be multiplied by x^{k} and differentiated again [and divided by x^{k-1}], the resulting equation

$$(m+n)(k+n)x^{n}-(m+n-1)(k+n-1)Ax^{n-1}+(m+n-2)(k+n-2)Bx^{n-2}-\text{etc.}=0$$

at this point will have real roots and thus, it is permitted to progress further as far as it may be wished. But if an equation of this kind may be taken to have imaginary roots, then likewise the proposed equation itself accordingly will be having just as many imaginary roots.

316. If the proposed equation, before it may be differentiated, may be multiplied by no power of x, then a criterion is deduced for an equation smaller by one degree. Thus if the proposed equation shall be

$$x^{n}-Ax^{n-1}+Bx^{n-2}-Cx^{n-3}+Dx^{n-4}-\text{etc.}=0$$

all the roots may be considered real, then also the differentials of all the orders of the radices will have real roots too. Whereby also the roots of all the following equations will be real

$$\begin{gathered}
nx^{n-1}-(n-1)Ax^{n-2}+(n-2)Bx^{n-3}-(n-3)Cx^{n-4}+\text{etc.}=0,\\
n(n-1)x^{n-2}-(n-1)(n-2)Ax^{n-3}+(n-2)(n-3)Bx^{n-4}-\text{etc.}=0,\\
n(n-1)(n-2)x^{n-3}-(n-1)(n-2)(n-3)Ax^{n-4}+\text{etc.}=0,\\
n(n-1)(n-2)(n-3)x^{n-4}-(n-1)(n-2)(n-3)(n-4)Ax^{n-5}+\text{etc.}=0,\\
\text{etc.,}
\end{gathered}$$

which equations may be recalled to the following forms

$$x^{n-1}-\frac{(n-1)}{n}Ax^{n-2}+\frac{(n-1)(n-2)}{n(n-1)}Bx^{n-3}-\frac{(n-1)(n-2)(n-3)}{n(n-1)(n-2)}Cx^{n-4}+\text{etc.}=0,$$

$$x^{n-2}-\frac{(n-2)}{n}Ax^{n-3}+\frac{(n-2)(n-3)}{n(n-1)}Bx^{n-4}-\frac{(n-2)(n-3)(n-4)}{n(n-1)(n-2)}Cx^{n-5}+\text{etc.}=0,$$

$$x^{n-3}-\frac{(n-3)}{n}Ax^{n-4}+\frac{(n-3)(n-4)}{n(n-1)}Bx^{n-5}-\frac{(n-3)(n-4)(n-5)}{n(n-1)(n-2)}Cx^{n-6}+\text{etc.}=0,$$

$$x^{n-4}-\frac{(n-4)}{n}Ax^{n-5}+\frac{(n-4)(n-5)}{n(n-1)}Ax^{n-6}-\frac{(n-4)(n-5)(n-6)}{n(n-1)(n-2)}Cx^{n-7}+\text{etc.}=0$$

etc.

317. Therefore in this manner a judgement can be returned on an equation of a given lesser order than that proposed itself. Thus if *m* were some number less than *n*, then, if all the roots of the proposed equation may be considered real, also all the roots of the order *m* of this equation will be real

$$x^{m}-\frac{m}{n}Ax^{m-1}+\frac{m(m-1)}{n(n-1)}Bx^{m-2}-\frac{m(m-1)(m-2)}{n(n-1)(n-2)}Cx^{m-3}+\text{etc.}=0.$$

Whereby if there is put $m=2$, this equation will be produced

$$x^{2}-\frac{2}{n}Ax+\frac{2\cdot 1}{n(n-1)}B=0,$$

the roots of which will be real, if indeed the proposed equation

$$x^{n}-Ax^{n-1}+Bx^{n-2}-Cx^{n-3}+\text{etc.}=0$$

may have all real roots. But since this quadratic equation cannot have real roots, unless there shall be $\frac{AA}{nn}>\frac{2\cdot 1}{n(n-1)}B,$ it follows that all the roots of the proposed equation cannot be real, unless there shall be $AA>\frac{2n}{n-1}B.$ On account of which if there were $AA<\frac{2n}{n-1}B$, this will be a certain sign that at least two roots of the proposed equation may be imaginary.

318. Hence therefore we have understood a necessary condition, by which the coefficients of the first three terms must be affected, if indeed all the roots of the proposed equation were real. And this is a criterion of this kind, as we have mentioned initially : clearly even if in the case $AA>\frac{2n}{n-1}B$ nothing for the realness of the roots may follow, but if there shall be $AA<\frac{2n}{n-1}B$, this still will be at any rate a sign of the two imaginary roots. Thus in order that all the roots may be real, it is required for the number *n* to be substituted successively for 2, 3, 4, 5 etc., as follows:

$$x^2 - Ax + B = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad A^2 > 4B$$
$$x^3 - Ax^2 + Bx - C = 0 \quad . \quad . \quad . \quad A^2 > \tfrac{6}{2}B$$
$$x^4 - Ax^3 + Bx^2 - Cx + D = 0 \quad . \quad . \quad A^2 > \tfrac{8}{3}B$$
$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0 \quad . \quad A^2 > \tfrac{10}{4}B.$$

Hence if the second term may be absent and the coefficient of the third term B shall be positive, so that the equation shall be of this kind

$$x^n + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0,$$

this cannot have all the roots real, but at least two will be imaginary.

319. Truly the criteria for the coefficients of the following terms can be elicited, if we may assess this equation carefully

$$1 - Ay + By^2 - Cy^3 + Dy^4 - \text{etc.} = 0$$

to have just as many real as imaginary roots , however many the proposed equation itself may contain. For this equation may arise from that, if there may be put $x = \frac{1}{y}$, thus so that from the roots of this equation likewise the roots of that may be had. Whereby if the equation proposed may have all the roots real, then also the differential of this reciprocal, evidently of this

$$-A + 2By - 3Cy^2 + 4Dy^3 - \text{etc.} = 0,$$

all the roots will be real. Again x may be substituted for $\frac{1}{y}$ in this and this equation will appear

$$Ax^{n-1} - 2Bx^{n-2} + 3Cx^{n-3} - 4Dx^{n-4} + \text{etc.} = 0,$$

the roots of which therefore all shall be real, if the roots of the proposed equation were such. Hence now it is apparent, if there were $n = 3$, to be necessary that there shall be $BB > 3AC$.

320. Moreover this equation may be differentiated further and there will be produced

$$Ax^{n-2}-\frac{2(n-2)}{n-1}Bx^{n-3}+\frac{3(n-2)(n-3)}{(n-1)(n-2)}Cx^{n-4}-\text{etc.}=0,$$
$$Ax^{n-3}-\frac{2(n-3)}{n-1}Bx^{n-4}+\frac{3(n-3)(n-4)}{(n-1)(n-2)}Cx^{n-5}-\text{etc.}=0,$$
$$Ax^{n-4}-\frac{2(n-4)}{n-1}Bx^{n-5}+\frac{3(n-4)(n-5)}{(n-1)(n-2)}Cx^{n-6}-\text{etc.}=0$$
etc.

Therefore generally, if *m* shall be a number less than *n*, there will be

$$Ax^{m}-\frac{2m}{n-1}Bx^{m-1}+\frac{3m(m-1)}{(n-1)(n-2)}Cx^{m-2}-\text{etc.}=0.$$

If now there may be put $m=2$, this equation will be had

$$Ax^{2}-\frac{4}{n-1}Bx+\frac{6}{(n-1)(n-2)}C=0\,;$$

so that the roots of which shall be real, there is required to be $\frac{4BB}{(n-1)^2}>\frac{6AC}{(n-1)(n-2)}$. Whereby if the proposed equation may have all the roots real, there will be

$$BB>\frac{3(n-1)}{2(n-2)}AC\,.$$

And if there were $BB<\frac{3(n-1)}{2(n-2)}AC$ this is a certain sign that the proposed equation has at least two imaginary roots. Therefore if there shall be $n=3$, the criterion shall be $BB>3AC$; if $n=4$, it will be $BB>\frac{3\cdot 3}{2\cdot 2}AC$; if $n=5$, it will be $BB>\frac{3\cdot 4}{2\cdot 3}AC$, and thus henceforth.

321. So that we may transfer this criterion to the following coefficients, we may resume the differential equation found in *y*

$$-A+2By-3Cy^{2}+4Dy^{3}-5Ey^{4}+\text{etc.}=0$$

and we may differentiate this anew, so that we may have

$$2B-6Cy+12Dy^{2}-20Ey^{3}+\text{etc.}=0\,,$$

which on restoring $\frac{1}{x}$ in place of y will give

$$Bx^{n-2} - 3Cx^{n-3} + 6Dx^{n-4} - 10Ex^{n-5} + \text{etc.} = 0,$$

from the further differentiation of which these equations follow

$$Bx^{n-3} - \frac{3(n-3)}{n-2}Cx^{n-4} + \frac{6(n-3)(n-4)}{(n-2)(n-3)}Dx^{n-5} - \text{etc.} = 0$$

and generally

$$Bx^{m} - \frac{3m}{n-2}Cx^{m-1} + \frac{6m(m-1)}{(n-2)(n-3)}Dx^{m-2} - \text{etc.} = 0.$$

But if therefore we may put $m = 2$, the quadratic equation will be produced

$$Bx^{2} - \frac{2 \cdot 3}{n-2}Cx + \frac{6 \cdot 2}{(n-2)(n-3)}D = 0,$$

the roots of which will be real, if there were $\frac{9CC}{(n-2)^2} > \frac{6 \cdot 2BD}{(n-2)(n-3)}$ or

$$CC > \frac{4(n-2)}{3(n-3)}BD.$$

Whereby if the proposed equation shall have all the roots real, there will be $CC > \frac{4(n-2)}{3(n-3)}BD$, and if this condition be deficient, the equation certainly at least will have two imaginary roots.

322. If we may differentiate the above equation $2B - 6Cy + 12Dy^2 - \text{etc.} = 0$ anew, there will be produced

$$-6C + 24Dy - 60Ey^2 + \text{etc.} = 0$$

or

$$C - 4Dy + 10Ey^2 - 20Fy^3 + \text{etc.} = 0,$$

which on restoring x in place of $\frac{1}{y}$ will change into

$$Cx^{n-3} - 4Dx^{n-4} + 10Ex^{n-5} - 20Fx^{n-6} + \text{etc.} = 0$$

from the further differentiation of which

$$Cx^{n-4} - \tfrac{4(n-4)}{n-3}\ Dx^{n-5} + \tfrac{10(n-4)(n-5)}{(n-3)(n-4)} Ex^{n-6} - \text{etc.} = 0,$$

$$Cx^{n-5} - \tfrac{4(n-5)}{n-3}\ Dx^{n-6} + \tfrac{10(n-5)(n-6)}{(n-3)(n-4)} Ex^{n-7} - \text{etc.} = 0$$

follow, and generally

$$Cx^{m} - \tfrac{4m}{n-3}\ Dx^{m-1} + \tfrac{10m(m-1)}{(n-3)(n-4)} Ex^{m-2} - \text{etc.} = 0\,.$$

We may put $m = 2$ and there will be

$$Cx^{2} - \tfrac{2\cdot 4}{n-3}\ Dx + \tfrac{2\cdot 10}{(n-3)(n-4)} E = 0\,,$$

from which, if the roots of this shall be real, it follows to be

$$\tfrac{4\cdot 4}{(n-3)^2} DD > \tfrac{2\cdot 10}{(n-3)(n-4)} CE \quad \text{or} \quad DD > \tfrac{5(n-3)}{4(n-4)} CE\,.$$

323. Now from these the relation of all the coefficients is considered sufficiently. Generally therefore, if this equation

$$x^{n} - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + \text{etc.} = 0$$

may have all the roots real, there will be

$$AA > \tfrac{2n}{1(n-1)} B$$

$$BB > \tfrac{3(n-1)}{2(n-2)} AC$$

$$CC > \tfrac{4(n-2)}{3(n-3)} BD$$

$$DD > \tfrac{5(n-3)}{4(n-4)} CE$$

$$EE > \tfrac{6(n-4)}{5(n-5)} DF$$

etc.

Of which conditions if one be missing, the equation will have at least two imaginary roots. And if in turn they may not depend on that criterion, it is seen easily, however many of these may not agree, an equal equivalent number of imaginary roots be given. But although all these conditions may have a place in a certain equation, yet it does not follow thence that no imaginary roots be given ; but rather it can happen, that by this obstacle not all the roots shall be imaginary. Therefore it is to be warned, lest more can be attributed to these criteria, than can be given from the strength of the principles, from which they have been deduced.

324. But it appears easily that not only the individual criteria which fail, can indicate two imaginary roots; for in an equation of n dimensions, because $n+1$ terms may be had and besides the first and the last terms, criteria may be taken from the individual terms, and all together $n-1$ criteria will be had; nor yet, if the individuals may fail, the equation will be able to have $2n-2$ imaginary roots, because generally therefore it may have only n roots. Moreover one criterion always brings to light two imaginary roots, and because it can come about, that two criteria of this kind can show no more roots of the kind, it may be evident, whether these two criteria may be contiguous or not ; in the first case the number of imaginary roots will not be increased, indeed in the latter, because the criteria do involve different letters, each one will show two imaginary roots. Thus, even if there were

$$AA > \tfrac{2n}{1(n-1)}B \quad \text{and} \quad BB > \tfrac{3(n-1)}{2(n-2)}AC\,,$$

yet hence by necessity four imaginary roots may not indicated, but whether perhaps it may indicate the same two. But if indeed there were

$$AA < \tfrac{2n}{1(n-1)}B \quad \text{and} \quad CC > \tfrac{4(n-3)}{3(n-3)}BD$$

with $BB > \tfrac{3(n-1)}{2(n-2)}AC$ present, four imaginary roots will be indicated.

325. Therefore from these criteria of the imaginary roots immediately following each other, no more may follows than from one ; but if they may proceed with that order disturbed, so that between any two of the criteria one or more may be put in place in the opposite sense, then from each one two imaginary roots can be concluded. Which consideration gives rise to the following rule. The coefficients of the criteria found before, of the individual terms of the proposed equation besides the first and last, may be recorded in this manner

$$\begin{array}{cccccc} & \tfrac{2n}{1(n-1)} & \tfrac{3(n-1)}{2(n-2)} & \tfrac{4(n-2)}{3(n-3)} & \tfrac{5(n-3)}{4(n-4)} & \text{etc.} \\ x^n & -Ax^{n-1} & +Bx^{n-2} & -Cx^{n-3} & +Dx^{n-4} & -\text{etc.}=0 \\ + & . & . & . & . & \text{etc.} \end{array}$$

Then the square of each coefficient may be examined, whether it shall be greater or less than the fraction written down by the product of the adjacent coefficients multiplied together ; in the first case a + sign may be subscribed for the term, in the latter case a – sign ; truly with the first and last terms a + sign may be subscribed always. With which done, as often as variations occur of the these subscripted signs, the equation may be agreed to have at least just as many imaginary roots.

326. This is the rule found by Newton regarding the imaginary roots of each equation to be examined; but concerning which it is to be understood properly, which we have recorded now, how often it can come about, that an equation may have more imaginary roots, then are detected by this method. Hence others have provided a service, so that other similar rules may be found, which give the number of imaginary rules more precisely, thus so that truly the number of roots of this kind often may exceed less, than the rule may show. Within this kind the rule of Campbell stands out especially, adjoined to Newton's *Universal Arithmetic*, which therefore may be agreed upon to be explained here, even if it shall not be perfect. Moreover it depends on this lemma: If there were the quantities $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. and the number of which shall be *m*, the sum of these quantities may be put

$$\alpha+\beta+\gamma+\delta+\text{etc.}=S,$$

the sum of the squares

$$\alpha^2+\beta^2+\gamma^2+\delta^2+\text{etc.}=V;$$

and certainly there shall be $V>0$. But that shall be produced from the two

$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\text{etc.}=\tfrac{SS-V}{2},$$

there will be $(m-1)V>SS-V$ or $mV>SS$. For if the squares may be taken of the differences between two quantities, the sum of these will be

$$\begin{aligned}&(\alpha-\beta)^2+(\alpha-\gamma)^2+(\alpha-\delta)^2+(\beta-\gamma)^2+(\beta-\delta)^2\text{ etc.}\\&=(m-1)\left(\alpha^2+\beta^2+\gamma^2+\delta^2+\text{etc.}\right)-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\text{etc.})\\&=(m-1)V-2\tfrac{SS-V}{2}=mV-SS.\end{aligned}$$

Therefore since the sum of the real squares shall be positive always, there will be

$$mV-SS>0 \text{ and thus } mV>SS$$

327. With this lemma presented if this equation may be considered

$$x^n-Ax^{n-1}+Bx^{n-2}-Cx^{n-3}+Dx^{n-4}-Ex^{n-5}+Fx^{n-6}-\text{etc.}=0$$

and all the real roots of that, which shall be *a*, *b*, *c*, *d*, *e* etc., were *n* in number, as agreed upon from the nature of the equation,

	the number of terms
$A = a + b + c + d + \text{etc.}$	n
$B = ab + ac + ad + bc + bd + \text{etc.}$	$\frac{n(n-1)}{1\cdot 2}$
$C = abc + abd + abe + acd + bcd + \text{etc.}$	$\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$
$D = abcd + abce + abde + \text{etc.}$	$\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}$
etc	

Now the squares of the individual terms of this series may be taken and there may be put

$$\begin{aligned}
&P = a^2 + b^2 + c^2 + d^2 + \text{etc.},\\
&Q = a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + \text{etc.},\\
&R = a^2b^2c^2 + a^2b^2d^2 + a^2b^2e^2 + a^2c^2d^2 + \text{etc.},\\
&S = a^2b^2c^2d^2 + a^2b^2c^2e^2 + a^2b^2d^2e^2 + \text{etc.}\\
&\qquad\text{etc.};
\end{aligned}$$

there will be, from the nature of the combinations,

$$\begin{aligned}
&P = A^2 - 2B,\\
&Q = B^2 - 2AC + 2D,\\
&R = C^2 - 2BD + 2AE - 2F,\\
&S = D^2 - 2CE + 2BF - 2AG + 2H\\
&\qquad\text{etc.}
\end{aligned}$$

328. Therefore from the strength of the lemmas presented we will have

$$\begin{aligned}
&nP > AA,\\
&\tfrac{n(n-1)}{1\cdot 2} Q > BB,\\
&\tfrac{(n-1)(n-2)}{1\cdot 2\cdot 3} R > CC,\\
&\tfrac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} S > DD\\
&\qquad\text{etc.}
\end{aligned}$$

But if therefore in place of P, Q, R etc. the values found before may be substituted, we will obtain the following properties of the real roots

$$nAA - 2nB > AA \text{ or } AA > \tfrac{2n}{n-1}B,$$

$$\tfrac{n(n-1)}{1\cdot 2}BB - \tfrac{2n(n-1)}{1\cdot 2}AC + \tfrac{2n(n-1)}{1\cdot 2}D > BB \text{ or } BB > \frac{\frac{2n(n-1)}{1\cdot 2}}{\frac{n(n-1)}{1\cdot 2}-1}(AC - D)$$

and in a similar manner the following equations may be given

$$CC > \frac{\frac{2n(n-1)(n-2)}{1\cdot 2\cdot 3}}{\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}-1}(BD - AE + F),$$

$$DD > \frac{\frac{2n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}}{\frac{2n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}-1}(CE - BF + AG - H).$$

Hence therefore the square of each coefficient may be compared not only with the nearby product, but also with the rectangles of two of equal distance on both sides, thus so that the signs of these rectangles still may be changed in turn.

329. Therefore before the first and the last of the individual terms of the equation fractions must be inscribed, the numerators of which are parts of the binomial raised to the same power doubled, truly the denominators the same parts diminished by one. Thus by considering quadratics, cubics, biquadratics, etc., if all the roots of these were real, there will be,

$$\tfrac{4}{1}$$

$$x^2 - Ax + B = 0; \quad A^2 > 4B.$$

For the cubic equation

$$\tfrac{6}{2} \qquad \tfrac{6}{2}$$

$$x^3 - Ax^2 + Bx - C = 0$$

there will be

$$A^2 > 3B \text{ and } B^2 > 3AC.$$

For the biquadratic equation

$$\tfrac{8}{3} \qquad \tfrac{12}{5} \qquad \tfrac{8}{3}$$

$$x^4 - Ax^3 + Bx^2 - Cx + D = 0$$

there will be

$$A^2 > \tfrac{8}{3}B, \quad B^2 > \tfrac{12}{5}(AC - D), \quad C^2 > \tfrac{8}{3}BD.$$

For the equation of the fifth power

$$\begin{matrix} & \tfrac{10}{4} & \tfrac{20}{9} & \tfrac{20}{9} & \tfrac{10}{4} & \\ x^5 & -Ax^4 & +Bx^3 & -Cx^2 & +Dx & -E=0 \end{matrix}$$

there will be

$$AA > \tfrac{10}{4}B,\quad B^2 > \tfrac{20}{9}\left(AC-D\right),\ C^2 > \tfrac{20}{9}\left(BD-AE\right)\quad \text{and}\quad D^2 > \tfrac{10}{4}CE.$$

For the equation of the sixth power

$$\begin{matrix} & \tfrac{12}{5} & \tfrac{30}{14} & \tfrac{40}{19} & \tfrac{30}{14} & \tfrac{12}{5} & \\ x^6 & -Ax^5 & +Bx^4 & -Cx^3 & +Dx^2 & -Ex & +F=0 \end{matrix}$$

will be

$$A^2 > \tfrac{12}{5}B,\ B^2 > \tfrac{30}{14}\left(AC-D\right),\ C^2 > \tfrac{40}{19}\left(BD-AE\right),$$
$$D^2 > \tfrac{30}{14}\left(CE-BF\right),\ E^2 > \tfrac{12}{5}DF.$$

etc.

330. Therefore if a certain criterion fails, that will be an indication at least two imaginary roots are present in the equation proposed. But when, if they all fail, the equation cannot have imaginary roots more than twice, in a similar manner the criterion for these cases will be required to be resolved by the Newtonian rules, which we have indicated before. Evidently if the quadratic terms of each were greater than the inscribed fraction by the product of the adjacent terms and with each of the equidistant sides multiplied, then to this term there is subscribed the + sign , the opposite truly the – sign ; and truly always for the first and final terms there will be subscribed the + sign. With which put in place the order of these subscribed signs may be inspected, and as often as a variation occurs, so often will an imaginary root be indicated. Therefore as often as this rule indicates more rules than the Newtonian method, so often also will it approach the truth. Yet meanwhile it can happen, that the equation may have more imaginary roots than may be indicated by each rule.

331. Therefore we may be mistaken, if from these criteria with perfect signs of the real and imaginary roots such as we may wish to use, because it can happen therefore, that the equation may have more imaginary roots than these criteria indicate; but the error in that may be greater, where the proposed equation were of a higher order. For in a quadratic equation these criteria thus are in agreement with the truth, so that, if no imaginary roots may be indicated, also the equation shall not be about to have any. But a cubic equation can have two imaginary roots, even if neither rule (both moreover in this case are in agreement at this stage) may show these. Therefore on starting to investigate these cases, this general cubic equation shall be proposed

3 3

$$x^3 - Ax^2 + Bx - C = 0;$$

In which if there were $AA > 3B$ and $BB > 3AC$, neither rule may indicate imaginary roots. But above (§ 306) we have seen from that, that no imaginary roots may be present, to be required in the first place that there shall be $B < \frac{1}{3}AA$, which condition also both rules require. Therefore let there be $B = \frac{1}{3}AA - \frac{1}{3}ff$ and it is necessary that C may be contained within these limits

$$\tfrac{1}{27}A^3 - \tfrac{1}{9}Aff - \tfrac{2}{27}f^3 \quad \text{and} \quad \tfrac{1}{27}A^3 - \tfrac{1}{9}Aff + \tfrac{2}{27}f^3.$$

But each rule postulates only, that there shall be $C < \frac{BB}{3A}$, that is

$$C < \tfrac{1}{27}A^3 - \tfrac{2}{27}Aff + \frac{f^4}{27A}.$$

Which condition can have a place, even if C may not be contained within the said limits.

332. Now let there be

$$C = \tfrac{1}{27}A^3 - \tfrac{2}{27}Aff + \frac{f^4}{27A} - gg$$

and the rules will indicate no imaginary roots. Yet meanwhile there will be two imaginary roots present, if there were either

$$\tfrac{1}{27}A^3 - \tfrac{2}{27}Aff + \frac{f^4}{27A} - gg < \tfrac{1}{27}A^3 - \tfrac{1}{9}Aff - \tfrac{2}{27}f^3$$

or

$$\tfrac{1}{27}A^3 - \tfrac{2}{27}Aff + \frac{f^4}{27A} - gg > \tfrac{1}{27}A^3 - \tfrac{1}{9}Aff + \tfrac{2}{27}f^3.$$

Therefore if there were either

$$gg > \frac{(ff + Af)^2}{27A} \quad \text{or} \quad gg < \frac{(Af - ff)^2}{27A},$$

the equation will have two imaginary cube roots, even if neither rule may indicate these. But here we have assumed A to be a positive quantity ; for if it should be negative on putting $x = -y$ the equation may be transformed into a form of this kind, in which A shall be positive. Hence infinitely many cubic equations can be formed, which may have two imaginary roots, even if they may not be indicated by the rule.

For there may be $gg = \frac{(ff+Af)^2}{27A} + hh$; there will be

$$C = \frac{(ff-AA)^2}{27A} - gg = \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3 - hh \quad \text{and} \quad B = \frac{1}{3}AA - \frac{1}{3}ff$$

Or there shall be $gg = \frac{(Af-ff)^2}{27A} - hh$ with $hh < \frac{(Af-ff)^2}{27A}$ present; there will be

$$C = \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3 + hh \quad \text{and} \quad B = \frac{1}{3}AA - \frac{1}{3}ff\ .$$

In each case there will be produced an equation having two imaginary roots neither being indicated by the rule. For argument's sake we may put $A = 4,\ f = 1$; there will be $B = 5$ and on account of $gg = \frac{25}{108} + hh$ there will be

$$C = \frac{225}{108} - \frac{25}{108} - hh = \frac{50}{27} - hh\ .$$

Whereby if there shall be $C < \frac{50}{27}$, the equation $x^3 - 4x^2 + 5x - C = 0$ always will have two imaginary roots. But on taking $gg = \frac{1}{12} - hh$ there will become $hh < \frac{1}{12}$ and there is made

$$C = \frac{25}{12} - \frac{1}{12} + hh = 2 + hh\ .$$

Let there be $hh = \frac{1}{16}$ and the equation $x^3 - 4xx + 5x - \frac{33}{16} = 0$ will have two imaginary roots, even if it may be produced by no rules.

333. But general equations can be formed of this kind, in which neither rules may show imaginary roots, even if yet most often two or more shall be present. This will come about, if two similar signs may follow each other always, as

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + Fx^{n-6} - \text{etc.} = 0$$

or

$$x^n + Ax^{n-1} - Bx^{n-2} - Cx^{n-3} + Dx^{n-4} + Ex^{n-5} - Fx^{n-6} - \text{etc.} = 0\ ;$$

here by each rule at no time will an imaginary root be produced. But because most often roots of this kind are able to be contained, either shown from the cubic equation $x^3 - Ax^2 - Bx + C = 0$, which on putting $ff = AA + 3B$ always has two imaginary roots, if there were either

$$-C < \tfrac{1}{27}A^3 - \tfrac{1}{9}Aff - \tfrac{2}{27}f^3 \quad \text{or} \quad -C > \tfrac{1}{27}A^3 - \tfrac{1}{9}Aff + \tfrac{2}{27}f^3.$$

Yet meanwhile these cases also are allowed to be elicited from rules, if the equation may be transformed into another equation with the aid of a substitution. There may be put $x = y + k$ and there is made

$$\left.\begin{array}{r} y^3 + 3ky^2 + 3kky + k^3 \\ -\ Ayy - 2Aky - Akk \\ -By \quad -Bk \\ +C \end{array}\right\} = 0,$$

which examined following the rules will give at once indeed

$$(3k - A)^2 > 3(3kk - 2Ak - B);$$

but from which there shall be

$$(3kk - 2Ak - B)^2 > 3(3k - A)\left(k^3 - Akk - Bk + C\right),$$

because there is another criterion, it is necessary, that there shall be

$$BB + 3AC + (AB - 9C)k + (AA + 3B)kk > 0,$$

whatever value may be attributed to k itself. Therefore k may be taken thus, so that this expression may arrive at a minimum value, which happens on putting $k = \frac{9C - AB}{2(AA + 3B)}$ and if this expression at this point were > 0, it will be probable the proposed equation has no imaginary roots. But there is made

$$BB + 3AC - \frac{(AB - 9C)^2}{2(AA + 3B)} + \frac{(AB - 9C)^2}{4(AA + 3B)} > 0$$

or

$$BB + 3AC > \frac{(AB - 9C)^2}{4(AA + 3B)}.$$

Therefore since there shall be $B = \frac{1}{3}ff - \frac{1}{3}AA$, there will be

$$4ff\left(\tfrac{1}{9}f^4 - \tfrac{2}{9}AAff + \tfrac{1}{9}A^4 + 3AC\right) > \left(\tfrac{1}{3}Aff - \tfrac{1}{3}A^3 - 9C\right)^2$$

or

$$4f^6 - 8A^2f^4 + 4A^4ff + 108ACff > A^2f^4 - 2A^4f^2 - 54ACff + A^6 + 54A^3C + 729CC$$

or

$$4f^6 > 9A^2f^4 - 6A^4ff - 162ACff + A^6 + 54A^3C + 729CC,$$

from which with the factors taken it will become

$$\left(2f^3 + A^3 - 3Af^2 + 27C\right)\left(2f^3 - A^3 + 3Af^2 - 27C\right) > 0.$$

And hence the imaginary roots rules will be shown, if there were either

$$C > -\tfrac{1}{27}A^3 + \tfrac{1}{9}Af^2 - \tfrac{2}{27}f^3 \quad \text{and} \quad C > -\tfrac{1}{27}A^3 + \tfrac{1}{9}Af^2 + \tfrac{2}{27}f^3$$

or

$$C < -\tfrac{1}{27}A^3 + \tfrac{1}{9}Af^2 - \tfrac{2}{27}f^3 \quad \text{and} \quad C < -\tfrac{1}{27}A^3 + \tfrac{1}{9}Af^2 + \tfrac{2}{27}f^3$$

Which are the same conditions, which we found above [§ 306]. Therefore it is apparent suitable rules for the transformation of the proposed equation treated in this chapter thus may be perfected, so that they may not differ from the truth, even if they may be changed.

334. Also from these principles Harriot's rules can be shown, from which any equation may be predicted to have as many positive roots as there may be given changes of signs, truly so many negatives, just as many successions of the same sign may be given; which rule indeed prevails only for real roots. [At this stage, mathematicians were averse to negative numbers, and usually they considered only equations with positive roots; Euler has completed the rules for negative values; see *e.g.* Briggs's and Harriot's translations on this website]
Therefore we may put this equation

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0$$

to have all real positive roots and the differential of this

$$nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - \text{etc.} = 0$$

not only will have all its roots also real and positive, but also the roots of this constitute the limits of the roots of that equation. Therefore truly on putting $x = \frac{1}{y}$ this equation

$$1 - Ay + By^2 - Cy^3 + Dy^4 - \text{etc.} = 0$$

will have positive real roots also, but the reciprocals of that, so that thus, which roots in that equation may be maximas, these in this equation become minimas. With these in place, if that proposed equation may be continually differentiated, at last it may came to an equation of the first order, which will be $x-\frac{1}{n}A=0$ (§ 317), the root of which at this stage will be positive, and thus the coefficient of the second term will have a $-$ sign, as we have assumed. But if this coefficient may have a + sign, then certainly it may follow that the proposed equation does not have all the roots positive, but one at least to be negative, and indeed that, which may correspond to the limits hitherto induced.

335. If the proposed equation may be converted into its reciprocal and differentiated, then again indeed x may be restored and the differentiations continued, at last a simple equation will be come upon, which from § 320 will be of this kind $Ax-\frac{2}{n-1}B=0$, therefore the root of this also must be positive, if indeed the proposed equation may have all its roots real and positive, and hence the second and third terms will have different signs. But if therefore these two terms may have similar signs, at least one negative root will be indicated corresponding to the limit indicated by this signed equation, which difference will be from the previous equation indicated, therefore because here the roots have been converted into their reciprocals once ; from which it is concluded, if the three terms of the equation will have had equal initial signs, then two negative roots are indicated.

336. In a similar manner if conversions and differentiations following § 321 may be put in place and they may be continued until the simple equation $Bx-\frac{3}{n-2}C=0$ is arrived at, and the root of this equation must be positive, if indeed all the roots of the proposed equation were such; from which if the third and fourth terms may have equal signs, one negative root will be indicated. And thus continually, if any two contiguous terms were affected with equal signs, one single negative root is produced ; and thus, however many successive signs of this kind there were, the equation will have at least just as many negative roots, because these individual criteria are referred to different limits. But if the proposed equation may be put to have all the roots negative, then, because the roots of all the differential equations deduced from that must equally be negative, all the terms must be affected with the same signs. Whereby if two contiguous terms may have different signs, from these one positive root at least may be concluded. And in a similar manner, however many variations of the signs of the two terms occur in the equation, at least just as many positive roots have been said to be present. Therefore since the equation may have just as many roots, as there may be given combinations of two contiguous signs, and no more, it follows any equation, of which all the roots are real, to have just as many positive roots, as there will have been variations of the contiguous signs, truly just as many negatives, as there will have been successions of the same sign.

[This chapter fits more naturally into the Theory of Equations; and it is interesting as it shows the contemporary thinking of Euler about locating the number of real and imaginary roots of polynomials; the ideas were set out originally by Descartes, following a more implicit treatment by Harriot, Newton's Rule of Signs followed, how much of the present chapter is Euler's own work is hard for me to say; Sturm's Theorem followed later. See, for example, Ch. 7 of Turnbull's little book *Theory of Equations.* Pub. by Oliver & Boyd, for an elementary discussion. Turnbull however does not mention the present chapter in this book.]

CAPUT XIII

DE CRITERTIS RADICUM IMAGINARIARUM

313. In capite praecedenti modum exhibuimus naturam radicum cuiusque aequationis explorandi, ita ut eius beneficio, si proponatur aequatio quaecunque, inveniri possit, quot ea radices habeat reales et quot imaginarias. Plerumque quidem haec investigatio difficillime instituitur, cum aequatio differentialis ita est comparata, ut eius radices exhiberi nequeant. Quanquam autem his casibus eadem operatio ad aequationem differentialem ipsam accommodari eiusque radicum natura ex ipsius differentiali indagari hincque illius radices proxime assignari possent, tamen labor nimium saepissime fieret molestus. Quamobrem in hoc negotio saepenumero sufficit eiusmodi criteria nosse, ex quorum praesentia tuto concludi possit inesse in aequatione proposita radices imaginarias, etiamsi ex eorum absentia vicissim inferri nequeat omnes prorsus radices esse reales. Quae cognitio etei est imperfecta, tamen frequenter usu non destituitur; quocirca his criteriis explicandis praesens caput destinavimus.

314. In capite igitur praecedenti vidimus, si aequatio quaecunque

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0$$

omnes radices habeat reales, tum etiam eius differentialem

$$\tfrac{dz}{dx} = nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + \text{etc.} = 0$$

omnes suas radices habituram esse reales. Simul vero ostendimus, etiamsi aequatio differentialis omnes habeat radices reales, tamen inde non sequi ipsius aequationis propositae omnes radices futuras esse reales. Interim tamen, si aequatio differentialis habeat radices imaginarias, tum semper recte concludimus aequationem ipsam propositam ad minimum totidem habere debere radices imaginarias. Ad minimum dico; fieri enim potest, ut ipsa aequatio plures habeat radices imaginarias. Hoc ergo modo ex aequatione differentiali plus concludi non potest quam, si ea habeat radices imaginarias, ipsam propositam aequationem eiusmodi radices quoque habere debere, et quidem ad minimum totidem.

315. Si aequatio proposita multiplicetur per potestatem quamcunque x^m denotante m numerum integrum affirmativum, tum, quia haec nova aequatio omnes radices habebit reales, si quidem propositae radices omnes fuerint reaIes, tum quoque eius differentialis, postquam per x^{m-1} fuerit divisa, radices erunt reales omnes. Hinc si haec aequatio

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0$$

omnes radices habeat reales, tum quoque ista aequatio

$$(m+n)x^n - (m+n-1)Ax^{n-1} + (m+n-2)Bx^{n-2} - \text{etc.} = 0$$

omnes radices habebit reales. Ob eandem rationem, si haec multiplicetur per x^k et denuo differentietur, aequatio resultans

$$(m+n)(k+n)x^n - (m+n-1)(k+n-1)Ax^{n-1} + (m+n-2)(k+n-2)Bx^{n-2} - \text{etc.} = 0$$

omnes adhuc radices habebit reales sicque, quousque libuerit, ulterius progredi licet. Sin autem huiusmodi aequatio radices imaginarias habere deprehendatur, tum simul certum erit ipsam aequationem propositam saltem totidem radices imaginarias esse habituram.

316. Si aequatio proposita, antequam differentiatur, per nullam potestatem ipsius *x* multiplicetur, tum iudicium ad aequationem uno gradu inferiorem deducitur. Ita si aequatio proposita

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0$$

omnes radices habeat reales, tum quoque eius differentiales omnium ordinum omnes radices habebunt reales. Quare et sequentium aequationum omnium radices erunt reales

$$nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + \text{etc.} = 0,$$
$$n(n-1)x^{n-2} - (n-1)(n-2)Ax^{n-3} + (n-2)(n-3)Bx^{n-4} - \text{etc.} = 0,$$
$$n(n-1)(n-2)x^{n-3} - (n-1)(n-2)(n-3)Ax^{n-4} + \text{etc.} = 0,$$
$$n(n-1)(n-2)(n-3)x^{n-4} - (n-1)(n-2)(n-3)(n-4)Ax^{n-5} + \text{etc.} = 0,$$
$$\text{etc.},$$

quae aequationes ad sequentes formas revocantur

$$x^{n-1}-\frac{(n-1)}{n}Ax^{n-2}+\frac{(n-1)(n-2)}{n(n-1)}Bx^{n-3}-\frac{(n-1)(n-2)(n-3)}{n(n-1)(n-2)}Cx^{n-4}+\text{etc.}=0,$$
$$x^{n-2}-\frac{(n-2)}{n}Ax^{n-3}+\frac{(n-2)(n-3)}{n(n-1)}Bx^{n-4}-\frac{(n-2)(n-3)(n-4)}{n(n-1)(n-2)}Cx^{n-5}+\text{etc.}=0,$$
$$x^{n-3}-\frac{(n-3)}{n}Ax^{n-4}+\frac{(n-3)(n-4)}{n(n-1)}Bx^{n-5}-\frac{(n-3)(n-4)(n-5)}{n(n-1)(n-2)}Cx^{n-6}+\text{etc.}=0,$$
$$x^{n-4}-\frac{(n-4)}{n}Ax^{n-5}+\frac{(n-4)(n-5)}{n(n-1)}Ax^{n-6}-\frac{(n-4)(n-5)(n-6)}{n(n-1)(n-2)}Cx^{n-7}+\text{etc.}=0$$
etc.

317. Hoc igitur modo iudicium ad aequationem dati gradus inferioris, quam est ipsa proposita, reduci potest. Sic si m fuerit numerus quicunque minor quam n, tum, si aequatio proposita omnes radices habeat reales, tum quoque huius aequationis gradus m omnes radices erunt reales

$$x^{m}-\frac{m}{n}Ax^{m-1}+\frac{m(m-1)}{n(n-1)}Bx^{m-2}-\frac{m(m-1)(m-2)}{n(n-1)(n-2)}Cx^{m-3}+\text{etc.}=0.$$

Quare si ponatur $m=2$, prodibit ista aequatio

$$x^{2}-\frac{2}{n}Ax+\frac{2\cdot 1}{n(n-1)}B=0,$$

cuins radices debebunt esse reales, si quidem aequatio proposita

$$x^{n}-Ax^{n-1}+Bx^{n-2}-Cx^{n-3}+\text{etc.}=0$$

omnes habeat radices reales. Cum autem ista aequatio quadratica radices reales habere nequeat, nisi sit $\frac{AA}{nn}>\frac{2\cdot 1}{n(n-1)}B$, sequitur aequationis propositae radices omnes reales esse non posse, nisi sit $AA>\frac{2n}{n-1}B.$ Quamobrem si, fuerit $AA<\frac{2n}{n-1}B$, hoc certum erit signum aequationis propositae ad minimum duas radices fore imaginarias.

318. Hinc ergo assecuti sumus affectionem necessariam, qua coefficientes trium primorum terminorum affecti esse debent, si quidem aequationis propositae omnes radices fuerint reales. Hocque est eiusmodi criterium, uti initio meminimus: scilicet etiamsi casu $AA>\frac{2n}{n-1}B$ nihil pro realitate radicum sequatur, at si sit $AA<\frac{2n}{n-1}B$, hoc tamen certum sit signum duarum saltem radicum imaginariarum. Sic ut omnes radices sint reales, successive pro n numero 2, 3, 4, 5 etc. substituendo requiritur, ut sequitur:

$$
\begin{array}{ll}
x^2 - Ax + B = 0 \quad . \quad . \quad . \quad . \quad . \quad . & A^2 > 4B \\
x^3 - Ax^2 + Bx - C = 0 \quad . \quad . \quad . & A^2 > \frac{6}{2}B \\
x^4 - Ax^3 + Bx^2 - Cx + D = 0 \quad . \quad . & A^2 > \frac{8}{3}B \\
x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0 \quad . & A^2 > \frac{10}{4}B.
\end{array}
$$

Hinc si terminus secundus desit tertiique coefficiens B sit affirmativus, ut aequatio sit huiusmodi

$$x^n + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0,$$

haec omnes radices reales habere nequit, sed ad minimum duae erunt imaginariae.

319. Huiusmodi vero criteria pro coefficientibus sequentium terminorum erui possunt, si perpendamus aequationem hanc

$$1 - Ay + By^2 - Cy^3 + Dy^4 - \text{etc.} = 0$$

totidem habere radices tam reales quam imaginarias, quot ipsa aequatio proposita contineat. Haec enim aequatio ex illa oritur, si ponatur $x = \frac{1}{y}$ ita ut ex radicibus huius aequationis simul radices illius habeantur. Quare si aequatio proposita omnes radices habeat reales, tum quoque reciprocae istius differentialis, scilicet huius

$$-A + 2By - 3Cy^2 + 4Dy^3 - \text{etc.} = 0,$$

radices omnes erunt reales. Substituatur in hac iterum x pro $\frac{1}{y}$ atque emerget ista aequatio

$$Ax^{n-1} - 2Bx^{n-2} + 3Cx^{n-3} - 4Dx^{n-4} + \text{etc.} = 0,$$

cuius radices propterea omnes erunt reales, si radices aequationis propositae fuerint tales. Hinc iam patet, si fuerit $n = 3$, necesse esse, ut sit $BB > 3AC$.

320. Differentietur autem ista aequatio ulterius atque prodibunt

$$Ax^{n-2} - \frac{2(n-2)}{n-1}Bx^{n-3} + \frac{3(n-2)(n-3)}{(n-1)(n-2)}Cx^{n-4} - \text{etc.} = 0,$$
$$Ax^{n-3} - \frac{2(n-3)}{n-1}Bx^{n-4} + \frac{3(n-3)(n-4)}{(n-1)(n-2)}Cx^{n-5} - \text{etc.} = 0,$$
$$Ax^{n-4} - \frac{2(n-4)}{n-1}Bx^{n-5} + \frac{3(n-4)(n-5)}{(n-1)(n-2)}Cx^{n-6} - \text{etc.} = 0$$
etc.

Generaliter ergo, si *m* sit numerus minor quam *n*, erit

$$Ax^{m} - \frac{2m}{n-1}Bx^{m-1} + \frac{3m(m-1)}{(n-1)(n-2)}Cx^{m-2} - \text{etc.} = 0.$$

Si iam ponatur $m = 2$, habebitur ista aequatio

$$Ax^{2} - \frac{4}{n-1}Bx + \frac{6}{(n-1)(n-2)}C = 0\,;$$

cuius radices sint reales, oportet esse $\frac{4BB}{(n-1)^2} > \frac{6AC}{(n-1)(n-2)}$. Quare si aequatio proposita omnes habeat radices reales, erit

$$BB > \frac{3(n-1)}{2(n-2)}AC\,.$$

Atque si fuerit $BB < \frac{3(n-1)}{2(n-2)}AC$ hoc certum est signum aequationem propositam ad minimum duas habere radices imaginarias. Si igitur sit $n = 3$, criterium erit $BB > 3AC$; si sit $n = 4$, erit $BB > \frac{3\cdot 3}{2\cdot 2}AC$; si $n = 5$, erit $BB > \frac{3\cdot 4}{2\cdot 3}AC$, et ita porro.

321. Ut haec criteria ad sequentes coefficientes transferamus, resumamus aequationem differentialem in *y* inventam

$$-A + 2By - 3Cy^{2} + 4Dy^{3} - \text{etc.} = 0$$

hancque denuo diffierentiemus, ut habeamus

$$2B - 6Cy + 12Dy^{2} - 20Ey^{3} + \text{etc.} = 0\,,$$

quae restituto $\frac{1}{x}$ loco *y* dabit

$$Bx^{n-2} - 3Cx^{n-3} + 6Dx^{n-4} - 10Ex^{n-5} + \text{etc.} = 0\,,$$

ex cuius ulterioris differentiatione sequuntur hae aequationes

$$Bx^{n-3} - \frac{3(n-3)}{n-2}Cx^{n-4} + \frac{6(n-3)(n-4)}{(n-2)(n-3)}Dx^{n-5} - \text{etc.} = 0$$

et generaliter

$$Bx^{m} - \frac{3m}{n-2}Cx^{m-1} + \frac{6m(m-1)}{(n-2)(n-3)}Dx^{m-2} - \text{etc.} = 0.$$

Quodsi igitur ponamus $m = 2$, prodibit aequatio quadrata

$$Bx^{2} - \frac{2\cdot 3}{n-2}Cx + \frac{6\cdot 2}{(n-2)(n-3)}D = 0\,,$$

cuius radices erunt reales, si fuerit $\frac{9CC}{(n-2)^2} > \frac{6\cdot 2BD}{(n-2)(n-3)}$ seu

$$CC > \frac{4(n-2)}{3(n-3)}BD\,.$$

Quare si aequatio proposita omnes radices habeat reales, erit $CC > \frac{4(n-2)}{3(n-3)}BD$, atque si haec conditio deficiat, aequatio certo duas ad minimum habebit radices imaginarias.

322. Si aequationem superiorem $2B - 6Cy + 12Dy^2 - \text{etc.} = 0$ denuo differentiemus, prodibit

$$-\ 6C + 24Dy - 60Ey^2 + \text{etc.} = 0$$

sive

$$C - 4Dy + 10Ey^2 - 20Fy^3 + \text{etc.} = 0\,,$$

quae restituto x loco $\frac{1}{y}$ abibit in hanc

$$Cx^{n-3} - 4Dx^{n-4} + 10Ex^{n-5} - 20Fx^{n-6} + \text{etc.} = 0$$

ex cuius ulteriori differentiatione sequuntur

$$Cx^{n-4}-\tfrac{4(n-4)}{n-3}\,Dx^{n-5}+\tfrac{10(n-4)(n-5)}{(n-3)(n-4)}Ex^{n-6}-\text{etc.}=0,$$

$$Cx^{n-5}-\tfrac{4(n-5)}{n-3}\,Dx^{n-6}+\tfrac{10(n-5)(n-6)}{(n-3)(n-4)}Ex^{n-7}-\text{etc.}=0$$

et generaliter

$$Cx^{m}-\tfrac{4m}{n-3}\,Dx^{m-1}+\tfrac{10m(m-1)}{(n-3)(n-4)}Ex^{m-2}-\text{etc.}=0\,.$$

Ponamus $m=2$ eritque

$$Cx^{2}-\tfrac{2\cdot 4}{n-3}\,Dx+\tfrac{2\cdot 10}{(n-3)(n-4)}E=0\,,$$

ex qua, si eius radices sint reales, sequitur fore

$$\tfrac{4\cdot 4}{(n-3)^2}DD>\tfrac{2\cdot 10}{(n-3)(n-4)}CE\quad\text{seu}\quad DD>\tfrac{5(n-3)}{4(n-4)}CE\,.$$

323. Ex his iam satis perspicitur relatio omnium coefficientium. Generatim ergo, si aequatio haec

$$x^n-Ax^{n-1}+Bx^{n-2}-Cx^{n-3}+Dx^{n-4}-Ex^{n-5}+\text{etc.}=0$$

omnes radices habeat reales, erit

$$AA>\tfrac{2n}{1(n-1)}B$$

$$BB>\tfrac{3(n-1)}{2(n-2)}AC$$

$$CC>\tfrac{4(n-2)}{3(n-3)}BD$$

$$DD>\tfrac{5(n-3)}{4(n-4)}CE$$

$$EE>\tfrac{6(n-4)}{5(n-5)}DF$$

etc.

Quarum conditionum si una desit, aequatio ad minimum duas habebit radices imaginarias. Atque si ista criteria a se invicem non pendeant, facile perspicitur, quotquot eorum non conveniant, totidem dari paria radicum imaginariarum. Quamvis autem hae conditiones omnes in quapiam aequatione locum habeant, tamen inde non sequitur nullas dari radices imaginarias; quin potius evenire potest, ut hoc non obstante omnes radices sint imaginariae.

Cavendum ergo est, ne his criteriis plus tribuatur, quam ipsis vi principiorum, unde sunt deducta, tribui potest.

324. Facile autem apparet non singula criteria, quae deficiunt, binas radices imaginarias indicare posse; in aequatione enim *n* dimensionum, quia habentur $n+1$ termini atque ex singulis praeter primum et ultimum criterium desumi potest, omnino criteria habebuntur $n-1$; neque tamen, si singula deficiant, aequatio $2n-2$ radices imaginarias habere poterit, propterea quod omnino tantum *n* habeat radices. Unum autem criterium semper duas radices imaginarias patefacit, et quia fieri potest, ut duo criteria huiusmodi radicum non plures ostendant, videndum est, utrum haec duo criteria sint contigua necne; priori casu numerus radicum imaginariarum non augebitur, posteriori vero, quia criteria litteras prorsus diversas involvunt, unumquodque binas radices imaginarias monstrabit. Ita, etiamsi fuerit

$$AA > \frac{2n}{1(n-1)}B \quad \text{et} \quad BB > \frac{3(n-1)}{2(n-2)}AC,$$

tamen hinc non necessario quatuor radices imaginariae indicantur, sed utrumque fortasse easdem binas indicat. Quodsi vero fuerit

$$AA < \frac{2n}{1(n-1)}B \quad \text{et} \quad CC > \frac{4(n-3)}{3(n-3)}BD$$

existente $BB > \frac{3(n-1)}{2(n-2)}AC$, quatuor radices imaginariae indicabuntur.

325. Ex criteriis ergo radicum imaginariarum se immediate insequentibus plus non sequitur quam ex uno; sin autem ea ordine interrupto procedant, ut inter bina quaeque criterium unum vel plura contraria interiaceant, tum ex unoquoque binae radices imaginariae concludi poterunt. Quae consideratio sequentem regulam suppeditat. Aequationis propositae singulis terminis praeter primum et ultimum inscribantur coefficientes criteriorum ante inventi, hoc modo

$$\begin{array}{cccccc} & \frac{2n}{1(n-1)} & \frac{3(n-1)}{2(n-2)} & \frac{4(n-2)}{3(n-3)} & \frac{5(n-3)}{4(n-4)} & \text{etc.} \\ x^n & -Ax^{n-1} & +Bx^{n-2} & -Cx^{n-3} & +Dx^{n-4} & -\text{etc.}=0 \\ + & . & . & . & . & \text{etc.} \end{array}$$

Tum examinetur quadratum cuiusque coefficientis, utrum sit maius an minus quam fractio inscripta per productum adiacentium coefficientium multiplicata; priori casu termino subscribatur signum +, posteriori signum – ; primo vero termino et ultimo perpetuo signum + subscribatur. Quo facto, quot signorum horum subscriptorum variationes occurrunt, totidem radices imaginarias aequatio ad minimum habere censenda erit.

326. Haec est regula a NEUTONO inventa ad radices imaginarias cuiusque aequationis explorandas; de qua autem probe tenendum est, quod iam annotavimus, saepenumero fieri posse, ut aequatio plures habeat radices imaginarias, quam hac methodo deteguntur. Hinc alii operam dederunt, ut similes regulas alias invenirent, quae numerum radicum imaginariarum exactius praeberent, ita ut verus istiusmodi radicum numerus minus saepe eum, quem regula ostendat, excederet. In hoc genere imprimis prostat regula CAMPBELLI *Arithmeticae* NEUTONI *universali* subiuncta, quam propterea hic explicari conveniet, etiamsi non sit perfecta. Nititur autem hoc lemmate: Si fuerint $\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon$ etc. quantitates earumque numerus sit *m*, ponatur summa harum quantitatum

$$\alpha+\beta+\gamma+\delta+\text{etc.}=S,$$

summa quadratorum

$$\alpha^2+\beta^2+\gamma^2+\delta^2+\text{etc.}=V;$$

erit utique $V>0$. Sed eum sit productum ex binis

$$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\text{etc.}=\tfrac{SS-V}{2},$$

erit $(m-1)V>SS-V$ seu $mV>SS$. Nam si differentiarum inter binas. quantitates quadrata sumantur, erit eorum summa

$$\begin{aligned}&(\alpha-\beta)^2+(\alpha-\gamma)^2+(\alpha-\delta)^2+(\beta-\gamma)^2+(\beta-\delta)^2\text{ etc.}\\&=(m-1)\left(\alpha^2+\beta^2+\gamma^2+\delta^2+\text{etc.}\right)-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\text{etc.})\\&=(m-1)V-2\tfrac{SS-V}{2}=mV-SS.\end{aligned}$$

Cum igitur summa quadratorum realium sit semper affirmativa, erit

$$mV-SS>0 \text{ ideoque } mV>SS$$

327. Hoc lemmate praemisso si habeatur haec aequatio

$$x^n-Ax^{n-1}+Bx^{n-2}-Cx^{n-3}+Dx^{n-4}-Ex^{n-5}+Fx^{n-6}-\text{etc.}=0$$

eiusque omnes radices fuerint reales numero *n*, quae sint *a, b, c, d, e* etc., erit, uti constat ex natura aequationum,

	numerus terminorum
$A=a+b+c+d+\text{etc.}$	n

$B = ab + ac + ad + bc + bd + \text{etc.}$	$\frac{n(n-1)}{1\cdot 2}$
$C = abc + abd + abe + acd + bcd + \text{etc.}$	$\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$
$D = abcd + abce + abde + \text{etc.}$	$\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}$
etc	

Sumantur iam singulorum harum serierum terminorum quadrata ac ponatur

$$P = a^2 + b^2 + c^2 + d^2 + \text{etc.},$$
$$Q = a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + \text{etc.},$$
$$R = a^2b^2c^2 + a^2b^2d^2 + a^2b^2e^2 + a^2c^2d^2 + \text{etc.},$$
$$S = a^2b^2c^2d^2 + a^2b^2c^2e^2 + a^2b^2d^2e^2 + \text{etc.}$$
$$\text{etc.};$$

erit ex natura combinationum

$$P = A^2 - 2B,$$
$$Q = B^2 - 2AC + 2D,$$
$$R = C^2 - 2BD + 2AE - 2F,$$
$$S = D^2 - 2CE + 2BF - 2AG + 2H$$
$$\text{etc.}$$

328. Vi igitur lemmatis praemissi habebimus

$$nP > AA,$$
$$\frac{n(n-1)}{1\cdot 2}Q > BB,$$
$$\frac{(n-1)(n-2)}{1\cdot 2\cdot 3}R > CC,$$
$$\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}S > DD$$
$$\text{etc.}$$

Quodsi ergo loco *P*, *Q*, *R* etc. valores ante inventi substituantur, obtinebimus sequentes radicum realium proprietates

$$nAA - 2nB > AA \text{ seu } AA > \frac{2n}{n-1}B,$$

$$\frac{n(n-1)}{1\cdot 2}BB - \frac{2n(n-1)}{1\cdot 2}AC + \frac{2n(n-1)}{1\cdot 2}D > BB \text{ sive } BB > \frac{\frac{2n(n-1)}{1\cdot 2}}{\frac{n(n-1)}{1\cdot 2}-1}(AC - D)$$

similique modo aequationes sequentes praebent

$$CC > \frac{\frac{2n(n-1)(n-2)}{1\cdot 2\cdot 3}}{\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}-1}(BD - AE + F),$$

$$DD > \frac{\frac{2n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}}{\frac{2n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}-1}(CE - BF + AG - H).$$

Hinc ergo cuiusque coefficientis quadratum non solum cum producto proxime adiacentium comparatur, sed etiam cum rectangulis binorum quorumque utrinque aeque distantium, ita tamen, ut horum rectangulorum signa alternatim mutentur.

329. Singulis igitur aequationis terminis praeter primum et ultimum inscribi debent fractiones, quarum numeratores sint unciae binomii ad similem dignitatem elevati duplicatae, denominatores vero eaedem unciae unitate minutae. Ita considerando aequationes quadratas, cubicas , biquadratas etc., si earum radices omnes fuerint reales, erit

$$\overset{\frac{4}{1}}{}$$
$$x^2 - Ax + B = 0; \quad A^2 > 4B.$$

Pro aequatione cubica

$$\qquad\frac{6}{2}\qquad\frac{6}{2}$$
$$x^3 - Ax^2 + Bx - C = 0$$

erit

$$A^2 > 3B \text{ et } B^2 > 3AC.$$

Pro aequatione biquadrata

$$\frac{8}{3}\qquad\frac{12}{5}\qquad\frac{8}{3}$$
$$x^4 - Ax^3 + Bx^2 - Cx + D = 0$$

erit

$$A^2 > \tfrac{8}{3}B, \quad B^2 > \tfrac{12}{5}(AC - D), \quad C^2 > \tfrac{8}{3}BD.$$

Pro aequatione potestatis quintae

$$\begin{array}{ccccc} \tfrac{10}{4} & \tfrac{20}{9} & \tfrac{20}{9} & \tfrac{10}{4} \end{array}$$

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$$

erit

$$AA > \tfrac{10}{4}B,\quad B^2 > \tfrac{20}{9}\left(AC - D\right),\quad C^2 > \tfrac{20}{9}\left(BD - AE\right) \quad \text{et} \quad D^2 > \tfrac{10}{4}CE.$$

Pro aequatione potestatis sextae

$$\begin{array}{ccccc} \tfrac{12}{5} & \tfrac{30}{14} & \tfrac{40}{19} & \tfrac{30}{14} & \tfrac{12}{5} \end{array}$$

$$x^6 - Ax^5 + Bx^4 - Cx^3 + Dx^2 - Ex + F = 0$$

erit

$$A^2 > \tfrac{12}{5}B,\ B^2 > \tfrac{30}{14}\left(AC - D\right),\ C^2 > \tfrac{40}{19}\left(BD - AE\right),$$
$$D^2 > \tfrac{30}{14}\left(CE - BF\right),\ E^2 > \tfrac{12}{5}DF.$$

etc.

330. Si igitur quodpiam criterium fallat, id erit indicium duas ad minimum inesse radices imaginarias in aequatione proposita. Cum autem, si singula fallant, aequatio ideo non duplo plures habere queat radices imaginarias, simili modo iudicium his casibus erit absolvendum, quem ante pro NEUTONIANA regula indicavimus. Scilicet si cuiusque termini quadratum maius fuerit quam fractio inscripta per producta terminorum adiacentium et utrinque aequidistantium multiplicata, tum isti termino subscribatur signum +, contra vero signum – ; primo vero et ultimo termino constanter subscribatur signum +. Quo facto inspiciatur ordo signorum horum subscriptorum, et quoties occurrit variatio, toties radix imaginaria indicabitur. Quoties ergo haec regula plures radices imaginarias indicat quam NEUTONIANA, toties quoque ad veritatenl magis accedit. Interim tamen fieri potest, ut aequatio plures habeat radices imaginarias, quam per utramque regulam indicantur.

331. Falleremur ergo, si his criteriis tanquam perfectis signis radicum realium et imaginariarum uti vellemus, propterea quod fieri potest, ut aequatio plures habeat radices imaginarias, quam haec criteria indicant; error autem eo maior esse posset, quo altioris gradus fuerit aequatio proposita. Nam in aequatione quadrata haec criteria ita veritati sunt consentanea, ut, si nullas radices imaginarias indicent, etiam aequatio nullas sit habitura. Aequatio autem cubica duas radices imaginarias habere potest, etiamsi neutra regula (ambae autem hoc casu adhuc conveniunt) eas exhibeat. Hos igitur casus investigaturis sit proposita haec aequatio cubica generalis

$$\begin{array}{cc} 3 & 3 \end{array}$$

$$x^3 - Ax^2 + Bx - C = 0;$$

In qua si fuerit $AA > 3B$ et $BB > 3AC$, neutra regula radices imaginarias indicat. Supra autem (§ 306) vidimus ad id, ut nullae radices imaginariae adsint, requiri primo, ut sit $B < \frac{1}{3}AA$, quam conditionem quoque ambae regulae requirunt. Sit igitur $B = \frac{1}{3}AA - \frac{1}{3}ff$ atque necesse est, ut C contineatur intra hos limites

$$\tfrac{1}{27}A^3 - \tfrac{1}{9}Aff - \tfrac{2}{27}f^3 \quad \text{et} \quad \tfrac{1}{27}A^3 - \tfrac{1}{9}Aff + \tfrac{2}{27}f^3 .$$

Utraque autem regula tantum postulat, ut sit $C < \frac{BB}{3A}$, hoc est

$$C < \tfrac{1}{27}A^3 - \tfrac{2}{27}Aff + \tfrac{f^4}{27A} .$$

Quae conditio locum habere potest, etiamsi C non intra dictos limites contineatur.

332. Sit enim

$$C = \tfrac{1}{27}A^3 - \tfrac{2}{27}Aff + \tfrac{f^4}{27A} - gg$$

atque regulae nullas radices imaginarias indicabunt. Interim tamen inerunt duae radices imaginariae, si fuerit vel

$$\tfrac{1}{27}A^3 - \tfrac{2}{27}Aff + \tfrac{f^4}{27A} - gg < \tfrac{1}{27}A^3 - \tfrac{1}{9}Aff - \tfrac{2}{27}f^3$$

vel

$$\tfrac{1}{27}A^3 - \tfrac{2}{27}Aff + \tfrac{f^4}{27A} - gg > \tfrac{1}{27}A^3 - \tfrac{1}{9}Aff + \tfrac{2}{27}f^3 .$$

Si igitur fuerit vel

$$gg > \frac{(ff + Af)^2}{27A} \quad \text{vel} \quad gg < \frac{(Af - ff)^2}{27A},$$

aequatio cubica duas habebit radices imaginarias, etiamsi neutra regula eas indicet. Sumimus autem hic esse A quantitatem affirmativam; si enim esset negativa, ponendo $x = -y$ aequatio in eiusmodi formam transmutaretur, in qua A esset affirmativa. Hinc infinitae aequationes cubicae formari possunt, quae habeant duas radices imaginarias, etiamsi per regulam non indicentur.

Sit enim $gg = \frac{(ff + Af)^2}{27A} + hh$; erit

$$C = \frac{(ff - AA)^2}{27A} - gg = \tfrac{1}{27}A^3 - \tfrac{1}{9}Aff - \tfrac{2}{27}f^3 - hh \quad \text{et} \quad B = \tfrac{1}{3}AA - \tfrac{1}{3}ff$$

Vel sit $gg = \frac{(Af - ff)^2}{27A} - hh$ existente $hh < \frac{(Af - ff)^2}{27A}$; erit

$$C = \tfrac{1}{27}A^3 - \tfrac{1}{9}Aff + \tfrac{2}{27}f^3 + hh \quad \text{et} \quad B = \tfrac{1}{3}AA - \tfrac{1}{3}ff .$$

Utroque casu prodibit aequatio duas habens radices imaginarias neutra regula indicandas. Ponamus verbi gratia $A = 4,\ f = 1$; erit $B = 5$ et ob $gg = \tfrac{25}{108} + hh$
erit

$$C = \tfrac{225}{108} - \tfrac{25}{108} - hh = \tfrac{50}{27} - hh .$$

Quare si sit $C < \tfrac{50}{27}$, aequatio $x^3 - 4x^2 + 5x - C = 0$ semper habebit duas radices imaginarias. At sumto $gg = \tfrac{1}{12} - hh$ debebit esse $hh < \tfrac{1}{12}$ fietque

$$C = \tfrac{25}{12} - \tfrac{1}{12} + hh = 2 + hh .$$

Sit $hh = \tfrac{1}{16}$ atque aequatio $x^3 - 4xx + 5x - \tfrac{33}{16} = 0$ duas habebit radices imaginarias, etiamsi nulla regulis prodatur.

333. Quin etiam eiusmodi aequationes generales formari possunt, in quibus neutra regula radices imaginarias exhibeat, etiamsi tamen saepissime duae pluresve insint. Evenit hoc, si perpetuo duo signa similia se mutuo excipiant,
uti

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + Fx^{n-6} - \text{etc.} = 0$$

vel

$$x^n + Ax^{n-1} - Bx^{n-2} - Cx^{n-3} + Dx^{n-4} + Ex^{n-5} - Fx^{n-6} - \text{etc.} = 0 ;$$

hic utraque regula nullam unquam radicem imaginariam prodit. Quod autem saepissime huiusmodi radices continere queant, vel ex aequatione cubica elucet $x^3 - Ax^2 - Bx + C = 0$, quae posito $ff = AA + 3B$ semper habet duas radices imaginarias, si fuerit vel

$$-C < \tfrac{1}{27}A^3 - \tfrac{1}{9}Aff - \tfrac{2}{27}f^3 \quad \text{vel} \quad -C > \tfrac{1}{27}A^3 - \tfrac{1}{9}Aff + \tfrac{2}{27}f^3 .$$

Interim tamen et hos casus ex regulis elicere licet, si aequatio ope substitutionis in aliam formam transformetur. Ponatur $x = y + k$ fietque

$$\left.\begin{aligned} &y^3 + 3ky^2 + 3kky + k^3 \\ &\quad - Ayy - 2Aky - Akk \\ &\qquad\qquad - By \quad - Bk \\ &\qquad\qquad\qquad + C \end{aligned}\right\} = 0,$$

quae secundum regulas examinata dabit primo quidem sponte

$$(3k-A)^2 > 3(3kk-2Ak-B);$$

at quo sit

$$(3kk-2Ak-B)^2 > 3(3k-A)\left(k^3-Akk-Bk+C\right),$$

quod est alterum criterium, necesse est, ut sit

$$BB+3AC+(AB-9C)k+(AA+3B)kk>0,$$

quicunque valor ipsi k tribuatur. Sumatur ergo k ita, ut haec expressio minimum valorem adipiscatur, quod fiet ponendo $k=\frac{9C-AB}{2(AA+3B)}$ et si ista expressio adhuc fuerit >0, probabile erit aequationem propositam nullas habere radices imaginarias. Fiet autem

$$BB+3AC-\frac{(AB-9C)^2}{2(AA+3B)}+\frac{(AB-9C)^2}{4(AA+3B)}>0$$

seu

$$BB+3AC>\frac{(AB-9C)^2}{4(AA+3B)}.$$

Cum ergo sit $B=\frac{1}{3}ff-\frac{1}{3}AA$, erit

$$4ff\left(\tfrac{1}{9}f^4-\tfrac{2}{9}AAff+\tfrac{1}{9}A^4+3AC\right)>\left(\tfrac{1}{3}Aff-\tfrac{1}{3}A^3-9C\right)^2$$

seu

$$4f^6-8A^2f^4+4A^4ff+108ACff>A^2f^4-2A^4f^2-54ACff+A^6+54A^3C+729CC$$

vel

$$4f^6>9A^2f^4-6A^4ff-162ACff+A^6+54A^3C+729CC,$$

unde factoribus sumtis esse debebit

$$\left(2f^3+A^3-3Af^2+27C\right)\left(2f^3-A^3+3Af^2-27C\right)>0.$$

Hincque regulae radices imaginarias ostendent, si fuerit vel

$$C > -\tfrac{1}{27}A^3 + \tfrac{1}{9}Af^2 - \tfrac{2}{27}f^3 \quad \text{et} \quad C > -\tfrac{1}{27}A^3 + \tfrac{1}{9}Af^2 + \tfrac{2}{27}f^3$$

vel

$$C < -\tfrac{1}{27}A^3 + \tfrac{1}{9}Af^2 - \tfrac{2}{27}f^3 \quad \text{et} \quad C < -\tfrac{1}{27}A^3 + \tfrac{1}{9}Af^2 + \tfrac{2}{27}f^3$$

Quae sunt eaedem conditiones, quas supra [§ 306] invenimus. Patet ergo idonea aequationis propositae transmutatione regulas hoc capite traditas ita perfici posse, ut a veritate non dissideant, etiamsi convertantur.

334. Ex his principiis quoque regula HARRIOTTI, qua quaelibet aequatio tot radices affirmativas habere praedicatur, quot dentur signorum variationes, tot vero negativas, quot dentur eiusdem signi successiones, demonstrari potest; quae quidem regula pro radicibus tantum realibus valet. Ponamus ergo aequationem

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0$$

omnes radices habere reales atque affirmativas atque eius differentialis

$$nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - \text{etc.} = 0$$

non solum omnes suas radices quoque habebit reales et affirmativas, sed etiam huius radices constituent limites radicum illius aequationis. Praeterea vero posito $x = \frac{1}{y}$ haec aequatio

$$1 - Ay + By^2 - Cy^3 + Dy^4 - \text{etc.} = 0$$

omnes quoque radices habebit reales affirmativas, sed reciprocas illius, ita ut, quae radices in illa aequatione sint maximae, hae in ista fiant minimae. His positis, si illa aequatio proposita continuo differentietur, donec ad aequationem primi ordinis perveniatur, quae erit $x - \frac{1}{n}A = 0$ (§ 317), huius radix adhuc erit affirmativa ideoque coefficiens secundi termini habebit signum –, uti assumsimus. Sin autem iste coefficiens haberet signum +, tum certo sequeretur aequationem propositam non omnes radices habere affirmativas, sed unam ad minimum fore negativam, et quidem eam, quae limitibus hucusque perductis respondeat.

335. Si aequatio proposita in sui reciprocam eonvertatur et differentietur, tum vero iterum x restituatur atque differentiationes continuentur, donec perveniatur ad aequationem simplicem, quae ex § 320 erit huiusmodi $Ax - \frac{2}{n-1}B = 0$, huius propterea radix quoque debet esse affirmativa, si quidem proposita orones suas radices habeat reales affirmativas, hincque secundus et tertius terminus diversa signa habebunt. Quodsi ergo hi duo termini similia habeant signa, ad minimum una radix negativa indicabitur respondens limiti hac aequatione signato, qui diversus erit a limite praecedente aequatione indicato, propterea quod hic radices semel sunt in suas reciprocas conversae; unde concluditur, si tres termini aequationis initiales paria habuerint signa, tum duas radices negativas indicari.

336. Simili modo si conversiones et differentiationes secundum § 321 instituantur atque eousque continuentur, donec ad aequationem simplicem $Bx - \frac{3}{n-2}C = 0$ perveniatur, et huius aequationis radix esse debet affirmativa, siquidem propositae aequationis omnes radices fuerint tales; unde si termini tertius et quartus paria habeant signa, indicabitur una radix negativa. Sicque perpetuo, si duo quicunque termini contigui aequalibus signis fuerint affecti, una radix negativa proditur; ideoque, quotcunque fuerint eiusdem signi successiones, totidem ad minimum aequatio proposita habebit radices negativas, quoniam haec singula criteria ad diversos limites referuntur. Quodsi autem aequatio proposita omnes radices negativas habere ponatur, tum, quia radices omnium aequationum differentialium ex ea deductarum debent esse pariter negativae, omnes termini aequalibus signis affecti esse debebunt. Quare si duo termini contigui diversa haheant signa, ex iis una minimum radix affirmativa concludetur. Atque simili modo, quotcunque in aequatione occurrant binorum terminorum variationes signorum, totidem ad minimum radices affirmativae inesse dicendae sunt. Cum igitur aequatio omnis tot habeat radices, quot dantur duorum signorum contiguorum combinationes, neque plures, sequitur quamvis aequationem, cuius omnes radices sint reales, tot habere radices affirmativas, quot fuerint signorum contiguorum variationes, tot vero negativas, quot fuerint eiusdem signi successiones.