CHAPTER XII

CONCERNING THE USE OF DIFFERENTIATION IN INVESTIGATING EQUATIONS WITH REAL ROOTS

294. The nature of the maxima and minima reveals for us a way of knowing the character of the roots of equations, whether they shall be real or imaginary. Indeed let the proposed equation of some order be

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0,$$

the roots of which we may put to be *p, q, r, s, t* etc., thus so that *p* shall be the smallest, *q* that, which follows in an account of the magnitudes, and thus also the remaining roots shall be ordered properly following the order of the quantities ; clearly there shall be $q > p$, $r > q$, $s > r$, $t > s$ etc. Moreover we may assume all the roots of the equation to be real and the maximum exponent *n* likewise will be the number of the roots *p, q, r* etc. Also we may consider all these roots as unequal to each other; yet hence equal roots are not excluded, because therefore unequal roots, if the difference of these becomes infinitely small, become equal.

295. Because the proposed expression $x^n - Ax^{n-1} + \text{etc.}$ then only becomes equal to zero, when some value from *p, q, r* etc. is substituted in place of *x*, truly it will not vanish in all the remaining cases, and we may put generally

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \text{etc.} = z$$

thus so that it will be possible to regard *z* as a function of *x*. Now we may devise for *x* successively determined values to be substituted by beginning from the mimimum $x = -\infty$ and continually greater values to be arranged in place of *x* and it is evident *z* to be obtained to be the values come upon either greater or smaller than zero, nor the first value to be come upon, which may be put $x = p$; in which case there becomes $z = 0$. The values of *x* may be increased beyond *p* and the values of *z* become either positive or negative, then the value $x = q$ may be come upon; in which case again there will be $z = 0$. Therefore it is necessary that, since the values of *z* from 0 again will have approached to 0, meanwhile *z* will have had either a maximum or minimum value; clearly a maximum, if the values of *z*, were positive while *x* was moving between the limits *p* and *q*, and a minimum, if the values were negative. In a similar manner while *x* is increased from *q* as far as *r*, the function *z* reaches a maximum or minimum, without doubt a maximum, if before there were a minimum, and vice versa. For above [§ 263], we have seen the maxima and minima to follow each other alternately.

296. Whereby since between any two roots a case of x may arise, so that the function z is made a maximum or minimum, the number of maxima or minima which are implicated in the function z, will be less by one than the number of real roots; and thus indeed they will follow each other alternately, so that the maxima of z shall become positive numbers, the minima negative. But if in turn the function z may have a maximum or at least a positive value in the case $x = f$ and a minimum or at least a negative value in the case $x = g$, because, while the values of x cross over from f to g, the function z will change from positive into negative, it is necessary meanwhile that it will have passed through 0, and on account of this a root of x will be given retained between the limits f and g. But unless this condition may be present, so that the maxima and minima values of z are made alternately positive and negative, that conclusion does not follow. For if the minima of the function z may be given, which are positive also, it can happen, that the value of z may pass from a maximum to a minimum, but yet may not vanish in between. Moreover with these things said, even if not all the roots of the proposed equation were real, yet always between any two given maximum and minimum, even if the converse proposition generally may not be true, that between any two maxima or minima a real root may be contained; but it prevails with the added condition, that if the first value of z were positive, then the other is negative.

297. Because hence above we have seen the values of x, from which the function

$$z = x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.}$$

is made a maximum or minimum, to be the roots of this differential equation

$$\tfrac{dz}{dx} = nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + \text{etc.} = 0,$$

it is evident, if all the roots of the equation $z = 0$, the number of which is $= n$, were real, then also all the roots of the equation $\tfrac{dz}{dx} = 0$ are real. For since the function z may have just as many maxima or minima, as the number $n-1$ may contain units, it is necessary that the equation $\tfrac{dz}{dx} = 0$ may have just as many real roots ; and thus all the roots of this equation will be real. From which likewise it is evident the function z cannot have more maxima or minima than $n-1$. Therefore we have this rule extending the widest: If all the roots of the equation $z = 0$ were real, then also all the roots of the equation $\tfrac{dz}{dx} = 0$ will be had real. From which in turn it follows, if not all the roots of the equation $\tfrac{dz}{dx} = 0$ were real, then also not all the roots of the equation $z = 0$ may be real.

298. Because between any two roots of the equation $z = 0$ the real roots are given one case, in which the function z becomes a maximum or minimum, it follows, if the equation $z = 0$ may have two real roots, then necessarily the equation $\tfrac{dz}{dx} = 0$ will be having one real root.

Equally if the equation $z=0$ may have three real roots, then the equation $\frac{dz}{dx}=0$ certainly will have two real roots. And generally if the equation $z=0$ may have *m* real roots, it is necessary at a minimum, that there shall be $m-1$ real roots of the equation $\frac{dz}{dx}=0$. Whereby if the equation $\frac{dz}{dx}=0$ may have fewer real roots than $m-1$, then in turn the equation $z=0$ certainly will have fewer real roots than *m*. But it is required to beware, that the converse proposition may not be had to be true ; even if indeed the differerntial equation $\frac{dz}{dx}=0$ may have some or actually all its roots real, yet it does not follow that the equation $z=0$ is able to have any real root. For it can come about, that all the roots of the equation $\frac{dz}{dx}=0$ shall be real, while yet all the roots of the equation $z=0$ shall be imaginary.

299. Yet meanwhile, if the above condition mentioned may be added, the converse propositon will be able to be proposed thus, so that from the real roots of the equation $\frac{dz}{dx}=0$ it will be able certainly to know the number of the real roots of the equation $z=0$. For we may put in place $\alpha, \beta, \gamma, \delta$ etc. to be the real roots of the equation $\frac{dz}{dx}=0$, among which α shall be the greatest; truly the rest in turn may follow in order of magnitude. Therefore with these values substituted in place of *x* the function *z* will obtain either a maximum or minimum value alternately. But when the function *z* is made $=\infty$, if there may be put $x=\infty$, it is apparent the values of this must continually decrease, while the values of *x* are diminished from ∞ as far as to α ; from which, in the case $x=\alpha$, *z* is made a minimum. But if therefore in this case $x=\alpha$ the function *z* may adopt a negative value, so that a little before it were $=0$, and thus the root of the equation $z=0$ will be given a real root $x>\alpha$; but if on putting $x=\alpha$ the function *z* at this stage may retain a positive value, nowhere before would it be able to be smaller; for then previously too a minimum would be able to be given, before *x* may be diminished as far as α, which would be contrary to the hypothesis ; hence the equation $z=0$ will be able to have no real roots greater than α. Hence if we may put $x=\alpha$ there arises $z=\mathfrak{A}$, that it will be able to justify in this manner : If $\mathfrak{A}$ were a positive quantity, then the equation $z=0$ will have no real root greater than α ; but if $\mathfrak{A}$ were a negative quantity, then the equation $z=0$ always will have a single greater real root α, nor more.

300. Towards pursuing this criterion further :

if there may be put	there is made
$x=\alpha$	$z=\mathfrak{A}$
$x=\beta$	$z=\mathfrak{B}$
$x=\gamma$	$z=\mathfrak{C}$
$x=\delta$	$z=\mathfrak{D}$
$x=\varepsilon$ etc.	$z=\mathfrak{E}$ etc.

Therefore because $\mathfrak{A}$ should be a minimum, $\mathfrak{B}$ will be a maximum, and indeed if $\mathfrak{A}$ were positive , then $\mathfrak{B}$ also will be positive nor therefore will there be any real roots of the equation $z=0$ given between the limits α and β. Whereby if this equation shall have no real roots greater than α, nor will it have any, which may be greater than β. But if $\mathfrak{A}$ were a negative quantity, in which case a single root of the equation is given $x>\alpha$, it may be discerned, whether the value of $\mathfrak{B}$ shall be positive or negative. In the first case the root will be given $x>\beta$, truly in the latter case there will be no doubt that a root be retained within the limits α and β. In a similar manner while $\mathfrak{B}$ were a maximum, $\mathfrak{C}$ will be a minimum; whereby if $\mathfrak{B}$ had a negative value, $\mathfrak{C}$ will be much more negative and in this case no root will be held within the limits β and γ. But if $\mathfrak{C}$ were positive, the actual root will be given between the limits β and γ, if $\mathfrak{C}$ is made negative ; but if $\mathfrak{C}$ also were positive, then no root will be given held between the limits β and γ, and a further judgement will be required to be put in place in a similar manner.

301. So that these criteria may be understood more easily, I have included these in the following table.

The equation $z=0$ will have a real root, which may be held between the limits	if there were
$x=\infty$ and $x=\alpha$	$\mathfrak{A}=-$
$x=\alpha$ and $x=\beta$	$\mathfrak{A}=-$ and $\mathfrak{B}=+$
$x=\beta$ and $x=\gamma$	$\mathfrak{B}=+$ and $\mathfrak{C}=-$
$x=\gamma$ and $x=\delta$	$\mathfrak{C}=-$ and $\mathfrak{D}=+$
$x=\delta$ and $x=\varepsilon$	$\mathfrak{D}=+$ and $\mathfrak{E}=-$
etc.	etc.

Both the converses of these propositions and in negating the changes equally may be considered with all rigor. Clearly :

The equation $z=0$ will have no real root, which may be held within the limits	if there were not
$x=\infty$ and $x=\alpha$	$\mathfrak{A}=-$
$x=\alpha$ and $x=\beta$	$\mathfrak{A}=-$ and $\mathfrak{B}=+$
$x=\beta$ and $x=\gamma$	$\mathfrak{B}=+$ and $\mathfrak{C}=-$
$x=\gamma$ and $x=\delta$	$\mathfrak{C}=-$ and $\mathfrak{D}=+$
$x=\delta$ and $x=\varepsilon$	$\mathfrak{D}=+$ and $\mathfrak{E}=-$
etc.	etc.

Therefore with the aid of these rules from the roots of the equation $\frac{dz}{dx} = 0$, if these were known, not only the number of real roots of the equation $z = 0$ is deduced, but also the limits become known, between which these individual roots may be held.

EXAMPLE

This equation shall be proposed $x^4 - 14xx + 24x - 12 = 0$;
which whether it may have real roots and how many is sought.

The differential equation will be $4x^3 - 28x + 24 = 0$ or $x^3 - 7x + 6 = 0$, the roots of which are 1, 2 and – 3, which set out following the order of magnitudes will give

	from which there will be
$\alpha = 2$	$\mathfrak{A} = -4$
$\beta = 1$	$\mathfrak{B} = -1$
$\gamma = -3$	$\mathfrak{C} = -129.$

On account of the negative $\mathfrak{A}$ the proposed equation therefore will have a real root > 2, but on account of negative $\mathfrak{B}$ neither between the limits 2 and 1 nor between the limits 1 and – 3 will it have a real root. But since on putting $x = -3$ there becomes $z = \mathfrak{C} = -129$, and if there is put $x = -\infty$, there is made $z = +\infty$, it is necessary, that a real root is given contained between the limits -3 and $-\infty$. Therefore the proposed equation will have two real roots, the one $x > 2$, the other $x < -3$; from which two roots will be imaginary. In a similar manner therefore, it [*i.e.* the root] ought to be judged from the final maximum or minimum of the proposed equation, as from the first alone. Clearly if the proposed equation were of even order, the final will be either a maximum or a minimum (but here in this case a minimum), if it were negative, it will indicate a real root, if positive an imaginary root. But for equations of odd order, because on putting $x = -\infty$ there is made $z = -\infty$, if the final maximum were positive, a real root is indicated, but if negative, an imaginary root.

302. Therefore the rules for knowing the real and imaginary roots can be expressed conveniently in this manner. With some proposed equation $z = 0$ the differential of this may be considered $\frac{dz}{dx} = 0$, the real roots of which shall be set out following the order of the quantities $\alpha, \beta, \gamma, \delta$ etc.; then on putting

$$x = \alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta \text{ etc.}$$

there is made

$$z = \mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E},\ \mathfrak{F} \text{ etc.}$$

Now if the signs shall be $\quad - \quad + \quad - \quad + \quad - \quad +$ etc., the whole equation $z = 0$ will have real roots, as many as there may be had letters α, β, γ etc., and one extra. But if one from these slightly greater letters may not have a sign [change] written below, then two imaginary roots will be indicated. Thus if $\mathfrak{A}$ may have a + sign, then no root may be given held between the limits ∞ and β. If $\mathfrak{B}$ may have a $-$ sign, no root will be had between the limits α and γ, and if $\mathfrak{C}$ may have a + sign, there will be no root between the limits β and δ and thus henceforth. But generally besides the imaginary roots indicated in this manner, the equation $z = 0$ above will have just as many as the equation $\frac{dz}{dx} = 0$.

303. If it should arise, that some of the values $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ etc. may vanish [*i.e.* become equal to zero], then with that in place the equation $z = 0$ will have two equal roots. Evidently if there were $\mathfrak{A} = 0$, then it will have two roots equal to α; but if there shall be $\mathfrak{B} = 0$, the two roots will be $= \beta$. Indeed in this case the equation $z = 0$ will have one common root with the differential equation $\frac{dz}{dx} = 0$; but we have shown above [§ 245] this to be the indication of two equal roots. But if the equation $\frac{dz}{dx} = 0$ may have two or more equal roots, then, if the number of these were even, neither a maximum nor a minimum will be indicated; from which for the present arrangement equal roots for an even number can be ignored. But if the number of equal roots of the equation $\frac{dz}{dx} = 0$ were odd, then all besides one may be rejected in making a judgement, unless perhaps in that case also the function *z* itself may vanish. For if this may happen, the equation $z = 0$ also will have equal roots and indeed one more than the equation $\frac{dz}{dx} = 0$. Thus if there were $\frac{dz}{dx} = (x - \zeta)^n R$, thus so that this equation may have *n* roots equal to ζ, if there is put $x = \zeta$, z also may vanish, then the equation $z = 0$ will have $n + 1$ roots equal to ζ.

304. We shall apply these precepts to simpler equations and in the first place indeed we may begin with quadratics. Therefore let this equation be proposed

$$z = x^2 - Ax + B = 0;$$

the differential of this is

$$\frac{dz}{dx} = 2x - A,$$

with which made $= 0$ there will be

$$x = \tfrac{1}{2}A \text{ or } \alpha = \tfrac{1}{2}A.$$

This value may be substituted in place of *x* and there becomes

$$z = -\tfrac{1}{4}AA + B = \mathfrak{A}$$

from which we deduce, if this value of $\mathfrak{A}$ were negative, that is, if there shall be $AA > 4B$, the equation $xx - Ax + B = 0$ will have two real roots, the one greater than $\frac{1}{2}A$, the othere smaller. But if the value of $\mathfrak{A}$ were positive or $AA < 4B$, then both the roots of the proposed equation will be imaginary. But if there were $\mathfrak{A} = 0$ or $AA = 4B$, then the equation proposed will have two equal roots, clearly each $= \frac{1}{2}A$. Which since they may be well known from the nature of quadratic equations, the truth of these principles may be made very clear and likewise the usefulness of these in this matter is evident.

305. Therefore we may progress to cubic equations requiring to be found in a similar manner. Therefore let the proposed equation be

$$x^3 - Ax^2 + Bx - C = z = 0;$$

the differential of which, since it shall be

$$3xx - 2Ax + B = \frac{dz}{dx},$$

if this may be put $= 0$, there becomes

$$xx = \frac{2Ax - B}{3},$$

of which equation either both the roots shall be imaginary, or equal, or they shall be real and unequal. Therefore since there shall be

$$x = \frac{A \pm \sqrt{\left(A^2 - 3B\right)}}{3},$$

both the roots will be imaginary, if there were $AA < 3B$; in this case the proposed cubic equation will have a single real root, the other limits of which are not apparent except $+\infty$ and $-\infty$. Now both the roots may be equal to each other or $AA = 3B$; then there will be $x = \frac{A}{3}$. Therefore unless likewise there is made $z = 0$, these two roots cannot be considered for anything , and the equation will have a single real root as before ; but in the case $x = \frac{A}{3}$ likewise there is made $z = 0$, which arises, if there were $-\frac{2}{27}A^3 + \frac{1}{3}AB - C = 0$ or $C = \frac{1}{3}AB - \frac{2}{27}A^3$, this is, if there were $B = \frac{1}{3}A^2$ and $C = \frac{1}{27}A^3$, the equation will have three equal roots, evidently each one $= \frac{A}{3}$. Now we may set out the third case, in which both the roots of the differential equation are real and unequal to each other, which happens, if $AA > 3B$. Therefore let there be $AA = 3B + ff$ or $B = \frac{1}{3}AA - \frac{1}{3}ff$; both these roots will be

$$x=\frac{A\pm f}{3}$$

Therefore there becomes $\alpha=\frac{1}{3}A+\frac{1}{3}f$ and $\beta=\frac{1}{3}A-\frac{1}{3}f$. Therefore the values of *z*, $\mathfrak{A}$ and $\mathfrak{B}$, may be sought corresponding to these, and since both the roots may be present in this equation $xx=\frac{2}{3}Ax-\frac{1}{3}B$, there becomes

$$z=-\frac{1}{3}Axx+\frac{2}{3}Bx-C=-\frac{3}{9}AAx+\frac{1}{9}AB+\frac{2}{3}Bx-C.$$

And hence there becomes

$$\mathfrak{A}=-\frac{2}{27}A^3+\frac{1}{3}AB-\frac{2}{27}A^2f+\frac{2}{9}Bf-C=\frac{1}{27}A^3-\frac{1}{9}Aff-\frac{2}{27}f^3-C,$$
$$\mathfrak{B}=-\frac{2}{27}A^3+\frac{1}{3}AB+\frac{2}{27}A^2f-\frac{2}{9}Bf-C=\frac{1}{27}A^3-\frac{1}{9}Aff+\frac{2}{27}f^3-C$$

on account of $B=\frac{1}{3}AA-\frac{1}{3}ff$. Therefore if $\mathfrak{A}$ were a negative quantity, which happens, if there were $C>\frac{1}{27}A^3-\frac{1}{9}Aff-\frac{2}{27}f^3$, the equation $z=0$ will have one real root $>\alpha$, that is greater than $\frac{1}{3}A+\frac{1}{3}f$. Therefore we may put to be

$$C>\frac{1}{27}A^3-\frac{1}{9}Aff-\frac{2}{27}f^3 \quad \text{or there is} \quad C=\frac{1}{27}A^3-\frac{1}{9}Aff-\frac{2}{27}f^3+gg$$

and, as we have seen, the proposed cubic equation will have a real root $>\frac{1}{3}A+\frac{1}{3}f$. But what the kind of the remaining roots shall become, may be understood from the value $\mathfrak{B}$; moreover there is $\mathfrak{B}=\frac{4}{27}f^3-gg$; which if it were positive, the above equation will have two real roots, the first shall be held within the limits α and β, that is between $\frac{1}{3}A+\frac{1}{3}f$ and $\frac{1}{3}A-\frac{1}{3}f$, truly the other less than $\frac{1}{3}A-\frac{1}{3}f$. But if there were $gg>\frac{4}{27}f^3$ or negative $\mathfrak{B}$, then the equation will have two imaginary roots. But if there were $\mathfrak{B}=0$ or $\frac{4}{27}f^3=gg$, then the two roots come out equal, with each $=\beta=\frac{1}{3}A-\frac{1}{3}f$. And then if the value of $\mathfrak{A}$ were positive or $C<\frac{1}{27}A^3-\frac{1}{9}Aff-\frac{2}{27}f^3$, then the equation will have two imaginary roots and the third will be real and $<\frac{1}{3}A-\frac{1}{3}f$. And if the value of $\mathfrak{A}=0$, two roots will be $=\alpha$ with the remaining third one $<\frac{1}{3}A-\frac{1}{3}f$.

306. Therefore so that all the three roots of the cubic equation $x^3 - Ax^2 + Bx - C = 0$ shall be real, three conditions are required. In the first place, so that there shall be

$$B < \tfrac{1}{3}AA;$$

therefore there shall be $B = \tfrac{1}{3}AA - \tfrac{1}{3}ff$. In the second place, so that there shall be

$$C > \tfrac{1}{27}A^3 - \tfrac{1}{9}Aff - \tfrac{2}{27}f^3.$$

In the third place, so that there shall be

$$C < \tfrac{1}{27}A^3 - \tfrac{1}{9}Aff + \tfrac{2}{27}f^3.$$

Which two latter conditions may be reduced to this, so that C may be contained between these limits

$$\tfrac{1}{27}A^3 - \tfrac{1}{9}Aff - \tfrac{2}{27}f^3 \quad \text{and} \quad C\,\tfrac{1}{27}A^3 - \tfrac{1}{9}Aff + \tfrac{2}{27}f^3.$$

or between these limits

$$\tfrac{1}{27}(A+f)^2(A-2f) \quad \text{and} \quad \tfrac{1}{27}(A-f)^2(A+2f)$$

But if a single one of these conditions may be missing, the equation will have two imaginary roots. Thus if there were $A = 3, B = 2$, there will be $\tfrac{1}{3}ff = \tfrac{1}{3}AA - B = 1$ and $ff = 3$; from which this equaton $x^3 - 3xx + 2x - C = 0$ cannot have all the roots real, unless C may be held between the limits $-\frac{2\sqrt{3}}{9}$ and $+\frac{2\sqrt{3}}{9}$. Whereby if there were either

$$C < -\tfrac{2\sqrt{3}}{9} \text{ or } C < -0{,}3849, \quad \text{or} \quad C > +\tfrac{2\sqrt{3}}{9} \text{ or } C > 0{,}3849$$

or together $CC > \frac{4}{27}$, the equation will have a single real root.

307. Because in any equation the second term can be removed, we may put $A=0$, thus so that we may have the cubic equation

$$x^3+Bx-C=0$$

Therefore so that all three terms of this equation shall be real, it is necessary that in the first place there shall be $B<0$, or B must be a negative quantity. Therefore let $B=-kk$; there will be $ff=3kk$ and in addition it is required, that the quantity C may be contained within these limits $-\frac{2}{27}f^3$ and $+\frac{2}{27}f^3$, that is between these : $-\frac{2}{9}kk\sqrt{3kk}$ and $+\frac{2}{9}kk\sqrt{3kk}$. Therefore there will be $CC<\frac{4}{27}k^6$ or $CC<-\frac{4}{27}B^3$. Therefore from a single condition the nature of the cubic equations, which may have all three roots real, can be understood, provided that we may say that $4B^3+27CC$ is required to be a negative quantity. For thus now it may be required, that B shall be a negative quantity, because otherwise $4B^3+27CC$ cannot become negative. On account of which generally we may confirm that all three roots of the equation $x^3+Bx\pm C=0$ are to be considered real, if the quantity $4B^3+27CC$ were negative; but if this quantity were positive, then a singe root becomes real, the remaining two imaginary; but if there is made $4B^3+27CC=0$, then indeed all three roots become real, but two are equal to each other.

308. We may progress to biquadratic equations, in which also we may put the second term to be missing. Therefore let there be

$$x^4+Bx^2-Cx+D=0.$$

We may put in place $x=\frac{1}{u}$ and there will be

$$1+Bu^2-Cu^3+Du^4=0,$$

the differential equation of which is

$$2Bu-3Cu^2+4Du^3=0,$$

which has one root $u=0$; then indeed there will be

$$uu=\frac{6Cu-4B}{8D}$$

and

$$u=\frac{3C\pm\sqrt{(9CC-32BD)}}{8D}.$$

Therefore in order that all four roots shall be real, in the first place, it is required that there shall be $9CC > 32BD$. Therefore we may put to be $9CC = 32BD + 9ff$; there will be $u = \frac{3C \pm 3f}{8D}$. Here we are able to assume always for C to be a positive quantity ; for unless it were such, on putting $u = -v$ such will emerge. But soon we will demonstrate that all the roots cannot be real, unless B shall be a negative quantity. Therefore let $B = -gg$ and there will be

$$9CC = 9ff - 32ggD \text{ and } u = \frac{3C \pm 3f}{8D}.$$

And two cases are required to be considered, as D shall be a positive or negative quantity.

I. Let D be a positive quantity and there will be $f > C$ and the three roots of u will be arranged according to the order of the quantity

$$1.\ u = \frac{3C+3f}{8D}, \quad 2.\ u = 0, \quad 3.\ u = \frac{3C-3f}{8D}.$$

But the equation

$$u^4 - \frac{Cu^3}{D} + \frac{Bu^2}{D} + \frac{1}{D} = 0$$

with these values substituted in place of u will give the following three values

$$\mathfrak{A} = \frac{27(C+f)^3(C-3f)}{4096D^4} + \frac{1}{D}, \quad \mathfrak{B} = \frac{1}{D}, \quad \mathfrak{C} = \frac{27(C-f)^3(C+3f)}{4096D^4} + \frac{1}{D},$$

of which the first and the third ought to be negative; indeed each on account of positive C and $C < f$ becomes less than $\frac{1}{D}$. And thus there is required to be

$$\frac{1}{D} < \frac{27(C+f)^3(3f-C)}{4096D^4} \quad \text{and} \quad \frac{1}{D} < \frac{27(f-C)^3(C+3f)}{4096D^4}$$

or

$$4096D^3 < 27(C+f)^3(3f-C) \quad \text{and} \quad 4096D^3 < 27(f-C)^3(C+3f).$$

But the first quantity is alway far greater than the second; from which it suffices, if there were $D^3 < \frac{27}{4096}(f-C)^3(C+3f)$ with $B = \frac{9CC-9f}{32ff}$ and $f > C$ and also $D > 0.$ If therefore the quantity D were positive, C positive, B negative, so that there shall be $f > C$, and $D^3 < \frac{27}{4096}(f-C)^3(C+3f)$, that is $D < \frac{3}{16}(f-C)\sqrt[3]{(3f+C)}$, then the equation will have all the roots real. But if there were $D > \frac{3}{16}(f-C)\sqrt[3]{(3f+C)}$, but yet

$D < \frac{3}{16}(f+C)\sqrt[3]{(3f-C)}$, then two roots will be real and two imaginary. But if in particular there were $D > \frac{3}{16}(f+C)\sqrt[3]{(3f-C)}$, then all four roots will be imaginary.

II. Let D be a negative quantity, consider $=-F$, with C remaining positive and B negative; on account of $B = \frac{9CC-9ff}{32D} = \frac{9ff-9CC}{32F}$ there will be $C > f$. Therefore since there shall be $u = \frac{3C \pm 3f}{8D} = -\frac{3C \pm 3f}{8F}$, the three values of u following the order of magnitudes put in place will be

$$1.\ u = 0, \quad 2.\ u = -\frac{3C-3f}{8F}, \quad 3.\ u = -\frac{3C+3f}{8F},$$

which will give the following values

$$\mathfrak{A} = -\frac{1}{F}, \quad \mathfrak{B} = \frac{27(C-f)^3(C+3f)}{4096F^4} - \frac{1}{F}, \quad \mathfrak{C} = \frac{27(C+f)^3(C-3f)}{4096F^4} - \frac{1}{F}.$$

Therefore since $\mathfrak{A}$ shall be a negative quantity, the equation now certainly will have one and therefore also two real roots. But in order that all the roots shall be real, it is required that $\mathfrak{B}$ shall be a positive quantity and thus $27(C-f)^3(C+3f) > 4096F^3$; then truly it is necessary, that $\mathfrak{C}$ shall be a negative quantity or $27(C+f)^3(C-3f) < 4096F^3$. On account of which so that all the roots become real, it is required that F^3 may be contained between these limits

$$\tfrac{27}{4096}(C+f)^3(C-3f) \quad \text{and} \quad \tfrac{27}{4096}(C-f)^3(C+3f)$$

or so that F may be contained between these limits

$$\tfrac{3}{16}(C+f)\sqrt[3]{(C-3f)} \quad \text{and} \quad \tfrac{3}{16}(C-f)\sqrt[3]{(C+3f)};$$

and unless F may be contained within these limits, two roots will be imaginary.

III. Now we may put B to be a positive quantity and equally D to be positive; on account of $B = \frac{9CC-9ff}{32D}$ there will be $C > f$ and since there shall be $u = \frac{3C \pm 3f}{8D}$, the roots arranged in the order of of magnitude will be

$$1.\ u = \frac{3(C+f)}{8D}, \quad 2.\ u = \frac{3(C-f)}{8D}, \quad 3.\ u = 0,$$

from which the following values arise

$$\mathfrak{A}=\frac{27(C+f)^3(C-3f)}{4096D^4}+\frac{1}{D},\quad \mathfrak{B}=\frac{27(C-f)^3(C+3f)}{4096D^4}+\frac{1}{D},\quad \mathfrak{C}=\frac{1}{D}\ ;$$

where since $\mathfrak{C}$ shall be a positive quantity, certainly two roots will be imaginary. But if $\mathfrak{A}$ were negative , which arised if $4096D^3<27(C+f)^3(3f-C)$, two roots will be real ; if there were $4096D^3>27(C+f)^3(3f-C)$, then all four roots will be imaginary.

IV. B may remain positive, but D shall be negative $=-F$; on account of $B=\frac{9ff-9CC}{32F}$ there will be $f>C$ and on account of $u=-\frac{3C\pm3f}{8F}$, the three roots u set out following the order of magnitude will be

$$1.\ u=\frac{3(f-C)}{8F},\quad 2.\ u=0,\quad 3.\ u=-\frac{3(C+f)}{8F},$$

from which these values arise

$$\mathfrak{A}=-\frac{27(f-C)^3(C+3f)}{4096F^4}-\frac{1}{F},\quad \mathfrak{B}=-\frac{1}{F},\quad \mathfrak{C}=-\frac{27(C+f)^3(3f-C)}{4096F^4}-\frac{1}{F}$$

were on account of $\mathfrak{A}$ and $\mathfrak{C}$ negative clearly the equation has two real roots, but on account of negative $\mathfrak{B}$ two roots are imaginary.

309. If therefore we may put the letters B, C, D to denote positive quantities, the following different cases arise requiring to be decided, which on account of $f=\sqrt{\left(CC-\frac{32}{9}BD\right)}$ may be reduced here.

I. If the equation shall be $x^4-Bx^2\pm Cx+D=0$, all the roots will be real, if there were

$$D>\frac{3}{16}\left(\sqrt{\left(CC+\frac{32}{9}BD\right)}-C\right)\sqrt[3]{\left(3\sqrt{\left(CC+\frac{32}{9}BD\right)}+C\right)}\ ;$$

two roots will be real and two imaginary, if there were

$$D>\frac{3}{16}\left(\sqrt{\left(CC+\frac{32}{9}BD\right)}-C\right)\sqrt[3]{\left(3\sqrt{\left(CC+\frac{32}{9}BD\right)}+C\right)},$$

but

$$D < \tfrac{3}{16}\left(\sqrt{\left(CC + \tfrac{32}{9}BD\right)} + C\right)\sqrt[3]{\left(3\sqrt{\left(CC + \tfrac{32}{9}BD\right)} - C\right)}\ ;$$

but all the roots will be imaginary, if there were

$$D > \tfrac{3}{16}\left(\sqrt{\left(CC + \tfrac{32}{9}BD\right)} + C\right)\sqrt[3]{\left(3\sqrt{\left(CC + \tfrac{32}{9}BD\right)} - C\right)}\ .$$

II. If the equation shall be $x^4 - Bx^2 \pm Cx - D = 0$, two roots always shall be real; the remaining two roots also will be real, if the quantity D may be held within these limits

$$D > \tfrac{3}{16}\left(\sqrt{\left(CC - \tfrac{32}{9}BD\right)} + C\right)\sqrt[3]{\left(C - 3\sqrt{\left(CC - \tfrac{32}{9}BD\right)}\right)}\ ,$$
$$D < \tfrac{3}{16}\left(C - \sqrt{\left(CC - \tfrac{32}{9}BD\right)}\right)\sqrt[3]{\left(C + 3\sqrt{\left(CC - \tfrac{32}{9}BD\right)}\right)}\ ;$$

but unless D may be contained within these limits, the remaining two roots will be imaginary.

III. If the equation shall be $x^4 + Bx^2 \pm \mathrm{C}x + D = 0$, two roots always shall be imaginary; truly the remaining two will be real, if there were

$$D < \tfrac{3}{16}\left(\sqrt{\left(CC - \tfrac{32}{9}BD\right)} + C\right)\sqrt[3]{\left(3\sqrt{\left(CC - \tfrac{32}{9}BD\right)} - C\right)}$$

truly the remaining two also will be imaginary, if there were

$$D > \tfrac{3}{16}\left(\sqrt{\left(CC - \tfrac{32}{9}BD\right)} + C\right)\sqrt[3]{\left(3\sqrt{\left(CC - \tfrac{32}{9}BD\right)} - C\right)}$$

IV. If the equation shall be $x^4 + Bx^2 \pm\ Cx - D = 0$, two roots of which equation always shall be real, truly the remaining two always imaginary.

EXAMPLE 1

If this equation may be proposed $x^4 - 2xx + 3x + 4 = 0$, the nature of the roots is sought, whether they shall be real or imaginary.

Because this example pertains to the first case, there is $B = 2,\quad C = 3\quad$ and $\quad D = 4$; from which

$$CC+\tfrac{32}{9}BD=9+\tfrac{32\cdot 8}{9}=\tfrac{337}{9} \text{ and } \sqrt{\left(CC+\tfrac{32}{9}BD\right)}=\tfrac{\sqrt{337}}{3},$$

from which the conditions, so that all the roots shall be real, are

$$4<\tfrac{3}{16}\left(3+\tfrac{\sqrt{337}}{3}\right)\sqrt[3]{\left(\sqrt{337}-3\right)}=\tfrac{1}{16}\left(9+\sqrt{337}\right)\sqrt[3]{\left(\sqrt{337}-3\right)},$$
$$4<\tfrac{3}{16}\left(\tfrac{\sqrt{337}}{3}-3\right)\sqrt[3]{\left(\sqrt{337}+3\right)}=\tfrac{1}{16}\left(\sqrt{337}-9\right)\sqrt[3]{\left(\sqrt{337}+3\right)}.$$

Therefore it ought to be examined by the use of approximations, whether there shall be $4<\tfrac{69}{16}$ and $4<\tfrac{24}{16}$; whereby since only the first condition may be considered correct, the equation will have two real and two imaginary roots.

EXAMPLE 2

This equation shall be proposed $x^4-9xx+12x-4=0$.

Which since it may pertain to the second case, it will have two real roots. Towards finding the nature of the rest, on account of $B=9$, $C=12$ and $D=4$ there will be

$$\sqrt{\left(CC-\tfrac{32}{9}BD\right)}=\sqrt{\left(144-32\cdot 4\right)}=4.$$

And thus it may be seen, whether there shall be

$$4>\tfrac{3}{16}\cdot 16\sqrt[3]{0}, \quad \text{that is} \quad 4>0,$$

and

$$4<\tfrac{3}{16}\cdot 8\sqrt[3]{24}, \quad \text{that is} \quad 4<3\sqrt[3]{3},$$

since each of which comes about, the proposed equation will have real roots.

EXAMPLE 3

Let this equation be proposed $x^4+xx-2x+6=0$.

Which since it may pertain to the third case, two roots evidently will be imaginary. Then truly there is $B=1$, $C=2$ and $D=6$ and thus

$$\sqrt{\left(CC-\tfrac{32}{9}BD\right)}=\sqrt{\left(4-\tfrac{64}{3}\right)};$$

which since it shall be an imaginary quantity, the two remaining roots certainly will be imaginary also.

EXAMPLE 4

Let this equation be proposed $x^4-4x^3+8x^2-16x+20=0.$

In the first place the second term may be eliminated; on substituting $x=y+1$ there becomes

$$\begin{aligned} x^4 &= y^4+4y^3+6yy+4y+1\\ -4x^3 &= -4y^3-12y^2-12y-4\\ +8x^2 &= +8y^2+16y+8\\ -16x &= -16y-16\\ +20 &= +20 \end{aligned}$$

Hence $$y^4+2yy-8y+9=0;$$

which since it may relate to the third case, it will have two imaginary roots. Then indeed on account of $B=2,\quad C=8,\quad D=9$ there will be

$$\sqrt{\left(CC-\tfrac{32}{9}BD\right)}=\sqrt{(64-64)}=0.$$

Hence $D=9$ may be compared with $\frac{3}{16}\cdot 8\sqrt[3]{-8}=-3$. Therefore since there shall be $D=9>-3$, the two remaining roots also will be imaginary.

EXAMPLE 5

Let this equation be proposed $x^4-4x^3-7x^2+34x-24=0$,
the roots of which are agreed to be 1, 2, 4 *and* – 3.

But if moreover we may apply the rules, with the second term removed on putting $x=y+1$ there becomes

$$y^3-13yy+12y+0=0,$$

which since compared with the second case gives $B=13,\quad C=12,\quad D=0$. Therefore there must be $D>\frac{3}{16}\cdot 24\sqrt[3]{-24}$ or $0>-9\sqrt[3]{3}$ and $D<0$; therefore since D shall not be greater than 0, the equation is indicated to have four real roots. For if there shall be $D=0$, the other equation becomes $D<\frac{3}{16}\left(\frac{16BD}{9C}\right)\sqrt[3]{4C}$ and thus $1<\frac{B}{3C}\sqrt[3]{4C}$ or $27CC<4B^3$;

indeed $27\cdot 144 < 4.13^3$ or $36\cdot 27 < 13^3$.

310. The work becomes especially difficult, if we wish to carry over the criteria to equations of higher order, because the roots of the differential equations cannot therefore generally be shown; but as often as it may be allowed to assign these roots, from the principles treated it is easily deduced, how many real and imaginary roots the proposed equation may have. Hence with all equations, which may be established from three terms, it will be possible to define whether the roots shall be real or imaginary. For let this general equation be proposed

$$x^{m+n} + Ax^n + B = 0 = z.$$

The differential of this is taken

$$\tfrac{dz}{dx} = (m+n)x^{m+n-1} + nAx^{n-1};$$

with which in the first place on putting the equation equal to nothing, $x^{n-1} = 0$; from which if *n* were an odd number, the root zero arises showing a maximum or minimum ; but if n shall be an even number, one root being led to in the computatin will be $x = 0$. Then indeed there will be $(m+n)x^m + nA = 0$; which equation has no real roots, if *m* shall be and even number and *A* a positive quantity. Hence the following cases will be required to be considered.

I. Let *m* be an even number and *n* and odd numbe and the root $x = 0$ will not prevail. Therefore if *A* were a positive quantity, so that it it will not have a root showing a maximum or minimum; from which on account of the odd number $m+n$ the equation proposed will have a single real root. But if *A* were a negative quantity, consider $A = -E$, $x = \pm\sqrt[m]{\tfrac{nE}{m+n}}$ from which

$$\alpha = +\sqrt[m]{\tfrac{nE}{m+n}} \quad \text{et} \quad \beta = -\sqrt[m]{\tfrac{nE}{m+n}}.$$

From which values there becomes

$$\mathfrak{A} = \left(x^m - E\right)x^n + B = -\tfrac{mE}{m+n}\left(\tfrac{nE}{m+n}\right)^{n:m} + B$$

and

$$\mathfrak{B} = +\tfrac{mE}{m+n}\left(\tfrac{nE}{m+n}\right)^{n:m} + B$$

Therefore if $\mathfrak{A}$ were a negative quantity or

$$\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} > B$$

the equation will have one real root $>\alpha$. If in addition there were

$$B > -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m}$$

that is, both conditions on being formed into one if there were

$$(m+n)^{m+n}B^m < m^m n^n E^{m+n},$$

then the equation will have three real roots, and unless this condition may be considered true, a single root of the equation will be real. These conditions prevail for the equation $x^{m+n} - Ex^n + B = 0$, if m were an even number and n odd; where if E were a negative number, the equation always will have a real root.

II. Let both the numbers m and n be odd, so that $m+n$ shall be an even number and there shall be no root $x = 0$ may enter into the computation. Because there is $(m+n)x^m + nA = 0$, there will be $x = -\sqrt[m]{\frac{nA}{m+n}}$; which single root shall be $=\alpha$, there is made

$$\mathfrak{A} = \frac{mA}{m+n}x^n + B = -\frac{mA}{m+n}\left(\frac{nA}{m+n}\right)^{n:m} + B.$$

Which value if it were negative, the proposed equation will have two real roots, on the contrary none. Therefore the proposed equation $x^{m+n} + \mathrm{A}x^n + B = 0$ will have two real roots, if there were

$$m^m n^n A^{m+n} > (m+n)^{m+n}B^m;$$

but if there were

$$m^m n^n A^{m+n} < (m+n)^{m+n}B^m,$$

in short no root will be real.

III. Let both the numbers m and n be even ; equally $m+n$ will be an even number and a single maximum or minimum root $x = 0$ will be given; which shall be single, if A were a positive quantity, which with $\alpha = 0$ made there will be $\mathfrak{A} = B$. Whereby if B also were a positive quantity, the equation will have no real root ; but if B shall be a negative quantity, two real roots will be had and no more, if indeed A were a positive quantity. But we may put A to be a negative quantity or $A = -E$; there will be $x = \pm\sqrt[m]{\frac{nE}{m+n}}$ and we will have three maxima or minima, surely

$$\alpha = +\sqrt[m]{\tfrac{nE}{m+n}}, \quad \beta = 0, \quad \gamma = -\sqrt[m]{\tfrac{nE}{m+n}}.$$

From which the values of $z = x^{m+n} - Ex^n + B$ correspond

$$\mathfrak{A} = -\tfrac{mE}{m+n}\left(\tfrac{nE}{m+n}\right)^{n:m} + B, \quad \mathfrak{B} = B, \quad \mathfrak{C} = -\tfrac{mE}{m+n}\left(\tfrac{nE}{m+n}\right)^{n:m} + B$$

Therefore if B shall be a negative quantity, on account of negative $\mathfrak{A}$ and $\mathfrak{B}$ the equation will have only two real roots, therefore because also $\mathfrak{B} = B$ is made negative. But if B were a positive quantity, the equation will have four real roots, if there shall be

$$(m+n)^{m+n} B^m < m^m n^n E^{m+n}.$$

But it will have no real root, if there were

$$(m+n)^{m+n} B^m > m^m n^n E^{m+n}.$$

IV. *m* shall be an odd number and *n* an even number and the root $x = 0$ will give a maximum or minimum. Therefore truly there will be $x = -\sqrt[m]{\tfrac{nA}{m+n}}$. If therefore A shall be a positive number, there becomes $\alpha = 0$ and $\beta = -\sqrt[m]{\tfrac{nA}{m+n}}$ and hence

$$\mathfrak{A} = B \quad \text{and} \quad \mathfrak{B} = \tfrac{mA}{m+n}\left(\tfrac{nA}{m+n}\right)^{n:m} + B.$$

Whereby if B were a negative quantity, consider $B = -F$, and the above may become

$$m^m n^n A^{m+n} > (m+n)^{m+n} F^m,$$

the equation will have three real roots ; otherwise only one shall be real. But if A shall be a negative quantity, consider $A = -E$, there becomes $x = +\sqrt[m]{\tfrac{nE}{m+n}}$ and

$$\alpha = \sqrt[m]{\tfrac{nE}{m+n}} \quad \text{and} \quad \beta = 0,$$

to which there corresponds

$$\mathfrak{A} = -\tfrac{mE}{m+n}\left(\tfrac{nE}{m+n}\right)^{n:m} + B \quad \text{and} \quad \mathfrak{B} = B.$$

Whereby the equation will have four real roots, if B were a positive quantity and

$$m^m n^n E^{m+n} > (m+n)^{m+n} B^m;$$

unless a place may be found for which property, the equation will have a single real root.

311. All the coefficients shall be $=1$ and with μ and ν denoting whole numbers the following thus will be decided on :

$$x^{2\mu+2\nu-1} + x^{2\nu-1} \pm 1 = 0$$

will have a single real root.

$$x^{2\mu+2\nu-1} - x^{2\nu-1} \pm 1 = 0$$

will have three real roots, if there were

$$(2\mu+2\nu-1)^{2\mu+2\nu-1} < (2\mu)^{2\mu}(2\nu-1)^{2\nu-1}$$

which since it can happen nowhere, the equation always will have a single real root.

$$x^{2\mu+2\nu} \pm x^{2\nu-1} - 1 = 0$$

has two real roots.

$$x^{2\mu+2\nu} \pm x^{2\nu-1} + 1 = 0$$

has no real roots.

$$x^{2\mu+2\nu} \pm x^{2\nu} + 1 = 0$$

has no real roots.

$$x^{2\mu+2\nu} \pm x^{2\nu} - 1 = 0$$

has two real roots.

$$x^{2\mu+2\nu+1} + x^{2\nu} \pm 1 = 0$$

has a single real root.

$$x^{2\mu+2\nu+1} - x^{2\nu} \pm 1 = 0$$

has a single real root.

Because moreover in the third case both the exponents are even, it can be reduced to a simpler form on putting $xx = y$ and thus in this case it may be omitted. With which done it can be affirmed that no agreed equation with three terms is able to have more than three real roots.

EXAMPLE

The cases may be sought, in which this equation $x^5 \pm Ax^2 \pm B = 0$, may have three real roots.

Because this equation relates to the fourth case, it is apparent the quantities A and B must be given opposite signs. Whereby unless a form of this kind may be had, it will have a single real root ; but if the proposed equation were of this kind $x^5 \pm Ax^2 \mp B = 0$, so that this may have three real roots, it is necessary that there shall be $3^3 2^2 A^5 > 5^5 B^3$ or $A^5 > \frac{3125}{108} B^3$. But if therefore there were $B = 1$, there may be required to be $A^5 > \frac{3125}{108}$ or $A > 1,960132$. Therefore if there shall be $A = 2$, this equation $x^5 - 2x^2 + 1 = 0$ has three real roots; since one of which shall be $x = 1$, it follows this biquadratic equation $x^4 + x^3 + x^2 - x - 1 = 0$ has four real roots. Because indeed then from these given precepts it can be understood, as well as from these, which have been shown in the above book [*i.e.* Euler's *Introductio*], it is evident, where we have shown, whatever the equation of even order, of which the final term shall be a negative number, always to have two real roots.

312. From these principles also the equations, which depend on four terms, will be able to be judged, provided the roots of the differential equation are able to be shown conveniently, which happens, if the exponents of *x* either in the three first or in the three last terms shall be in arithmetical progression. But since this judgement undertaken in general may lead to more cases, we will resolve that by some examples.

EXAMPLE 1

Let this equation be proposed $x^7 - 2x^5 + x^3 - a = 0.$

On making $z = x^7 - 2x^5 + x^3 - a$ there will be

$$\frac{dz}{dx} = 7x^6 - 10x^4 + 3xx$$

from which equation in the first place on setting equal to zero $xx = 0$, which double root for zero we may consider. Then truly there will be $7x^4 = 10x^2 - 3$, from which there is made $x^2 = \frac{5 \pm 2}{7}$, and four values for *x* emerge, which following the magnitude of the order the following values for *z* will be given :

$\alpha = 1$	$\mathfrak{A} = -a$
$\beta = +\sqrt{\frac{3}{7}}$	$\mathfrak{B} = \frac{48}{343}\sqrt{\frac{3}{7}} - a$
$\gamma = -\sqrt{\frac{3}{7}}$	$\mathfrak{C} = \frac{-48}{343}\sqrt{\frac{3}{7}} - a$
$\delta = -1$	$\mathfrak{D} = -a.$

Therefore if a were a positive number, there will be either $a > \frac{48}{343}\sqrt{\frac{3}{7}}$ or $a < \frac{48}{343}\sqrt{\frac{3}{7}}$; in the first case on account of all the negative quantities $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ the equation proposed will have a single real root $x > 1$. In the latter case, if $a < \frac{48}{343}\sqrt{\frac{3}{7}}$ the equation will have three real roots, the first $x > 1$, the second contained between the limits 1 and $\sqrt{\frac{3}{7}}$ and the third between the limits $+\sqrt{\frac{3}{7}}$ and $-\sqrt{\frac{3}{7}}$.

But if a shall be a negative quantity, on putting $x = -y$ the equation is reduced to the first form. Therefore so that the proposed equation may have three real roots, it is necessary that there shall be $a < 0{,}0916134$ or $a < \frac{1}{11}$.

EXAMPLE 2

Let this equation be proposed $ax^8 - 3x^6 + 10x^3 - 12 = 0.$

Because here the exponents of the last three terms are in arithmetical progression, there is put $x = \frac{1}{y}$ and the equation will be transformed into this

$$a - 3y^2 + 10y^5 - 12y^8 = 0;$$

therefore there is put

$$z = 12y^8 - 10y^5 + 3y^2 - a = 0$$

and there will be on differentiation

$$\frac{dz}{dx} = 96y^7 - 50y^4 + 6y = 0,$$

from which equation first there is made $y = 0$; then truly there will be

$$y^6 = \frac{50y^3 - 6}{96} \quad \text{and} \quad y^3 = \frac{25 \pm 7}{96}$$

and thus either $y = \sqrt[3]{\frac{1}{3}}$ vel $y = \sqrt[3]{\frac{3}{16}}$. Therefore with these three roots arranged following the magnitude the corresponding values of z thus will be had themselves :

$$\begin{array}{l|c} \alpha=\sqrt[3]{\frac{1}{3}} & \mathfrak{A}=\sqrt[3]{\frac{1}{9}}-a \\ \beta=\sqrt[3]{\frac{3}{16}} & \mathfrak{B}=\frac{99}{64}\sqrt[3]{\frac{9}{256}}-a=\frac{99}{256}\sqrt[3]{\frac{9}{4}}-a \\ \gamma=0 & \mathfrak{C}=-a. \end{array}$$

But if therefore there were $a>\sqrt[3]{\frac{1}{9}}$, the proposed equation will have two real roots, the one $>\sqrt[3]{\frac{1}{3}}$, the other <0; or in addition these above will have two real roots, if likewise $\mathfrak{B}$ were a positive quantity, that is , if there were $a<\frac{99}{256}\sqrt[3]{\frac{9}{4}}$. On account of which the proposed equation will have real roots, if the quantity a may be held within the limits $\sqrt[3]{\frac{1}{9}}$ and $\frac{99}{256}\sqrt[3]{\frac{9}{4}}$; which limits are approximately 0,48075 and 0,50674. Therefore on putting $a=\frac{1}{2}$ this equation $x^8-6x^6+20x^3-24=0$ has four roots within the limits $\infty,\ \sqrt[3]{\frac{16}{3}},\ \sqrt[3]{3},\ 0,\ -\infty$; therefore three shall be positive and one negative .

CAPUT XII

DE USU DIFFERENTIALIUM IN INVESTIGANDIS RADICIBUS REALIBUS AEQUATIONUM

294. Natura maximorum ac minimorum viam nobis patefacit ad indolem radicum aequationum, utrum sint reales an imaginariae, cognoscendam. Sit enim proposita aequatio cuiuscunque ordinis

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.} = 0,$$

cuius radices ponamus esse *p, q, r, s, t* etc., ita ut *p* sit minima, *q* ea, quae ratione magnitudinis sequitur, sicque et reliquae radices secundum ordinem quantitatis sint dispositae; scilicet sit $q > p$, $r > q$, $s > r$, $t > s$ etc. Assumamus autem omnes radices aequationis esse reales eritque exponens maximus *n* simul numerus radicum *p, q, r* etc. Consideremus quoque has radices omnes tanquam inter se inaequales; hinc tamen aequales radices non excluduntur, propterea quod radices inaequales, si earum differentia abeat in infinite parvam, fiant aequales.

295. Quoniam proposita expressio $x^n - Ax^{n-1} + \text{etc.}$ tum solum fit nihilo aequalis, cum loco *x* aliquis valor ex *p*, *q*, *r* etc. substituitur, reliquis vero casibus omnibus non evanescit, ponamus generatim

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \text{etc.} = z$$

ita ut *z* spectari possit tanquam functio ipsius *x*. Fingamus nunc pro *x* successive substitui valores determinatos incipiendo a minimo $x = -\infty$ atque continuo maiores in locum ipsius *x* collocari perspicuumque est *z* nacturum hinc esse valores vel nihilo maiores vel nihilo minores neque prius esse evaniturum, quam ponatur $x = p$; quo casu fiet $z = 0$. Augeantur valores ipsius *x* ultra *p* atque valores ipsius *z* vel affirmativi vel negativi fient, donec perveniatur ad valorem $x = q$; quo casu iterum erit $z = 0$. Necesse ergo est, ut, cum valores ipsius *z* ab 0 iterum ad 0 accesserint, interea *z* habuerit valorem vel maximum vel minimum, maximum scilicet, si valores ipsius *z*, dum *x* intra limites *p* et *q* versabatur, fuerint affirmativi, minimum, si fuerint negativi. Simili modo dum *x* ultra *q* ad *r* usque augetur, functio *z* maximum vel minimum attinget, maximum nimirum, si ante fuerit minimum, et contra. Supra enim [§ 263] vidimus maxima et minima se mutuo alternatim excipere.

296. Quare cum inter binas quasvis radices ipsius *x* existat casus, quo functio *z* fit maximum vel minimum, erit numerus maximorum et minimorum, quae in functione *z* implicantur, unitate minor quam numerus radicum realium; atque ita quidem alternatim se excipient, ut maximi ipsius z valores sint affirmativi, minimi negativi. Quodsi vicissim functio z habeat maximum vel saltem valorem affirmativum casu $x = f$ atque minimum seu saltem negativum casu $x = g$, quoniam, dum valores ipsius x ab f ad g transeunt, functio z ab affirmativo abit in negativum, necesse est, ut interea per 0 transierit, et hanc ob rem dabitur radix ipsius x intra limites f et g contenta. Nisi autem haec conditio adsit, ut valores maximi minimique ipsius z fiant alternatim affirmativi et negativi, illa conclusio non sequitur. Si enim dentur functionis z minima, quae quoque sint affirmativa, fieri potest, ut valor ipsius z a maximo ad sequens minimum transeat, cum tamen interea non evanescat. Ceterum ex dictis intelligitur, etiamsi aequationis propositae non omnes radices fuerint reales, tamen semper inter binas quasque dari maximum vel minimum, etiamsi propositio conversa generatim non valeat, ut inter bina quaevis maxima seu minima radix realis contineatur; valet autem adiecta conditione, si alter valor ipsius z fuerit affirmativus, alter negativus.

297. Quoniam ergo supra vidimus valores ipsius x, quibus functio

$$z = x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.}$$

fit maximum vel minimum, esse radices aequationis differentialis huius

$$\tfrac{dz}{dx} = nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + \text{etc.} = 0,$$

manifestum est, si aequationis $z = 0$ omnes radices, quarum numerus est $= n$, fuerint reales, tum quoque omnes radices aequationis $\tfrac{dz}{dx} = 0$ fore reales. Cum enim functio z tot habeat maxima vel minima, quot numerus $n-1$ continet unitates, necesse est, ut aequatio $\tfrac{dz}{dx} = 0$ totidem habeat radices reales; ideoque omnes eius radices erunt reales. Ex quo simul perspicitur functionem z plura maxima minimave habere non posse quam $n-1$. Habemus ergo hanc regulam latissime patentem: Si aequationis $z = 0$ omnes radices fuerint reales, tum quoque aequatio $\tfrac{dz}{dx} = 0$ omnes radices habebit reales. Unde vicissim sequitur, si aequationis $\tfrac{dz}{dx} = 0$ non omnes radices fuerint reales, tum quoque non omnes aequationis $z = 0$ radices reales fore.

298. Quia inter binas quasvis aequationis $z = 0$ radices reales datur unus casus, quo functio z fit maximum vel minimum, sequitur, si aequatio $z = 0$ duas habeat radices reales, tum aequationem $\tfrac{dz}{dx} = 0$ necessario unam radicem habituram esse realem. Pariter si aequatio $z = 0$ tres habeat radices reales, tum aequatio $\tfrac{dz}{dx} = 0$ certo duas habebit radices reales. Atque generatim si aequatio $z = 0$ habeat m radices reales, necesse est, ut aequationis

$\frac{dz}{dx}=0$ ad minimum sint *m* -1 radices reales. Quare si aequatio $\frac{dz}{dx}=0$ pauciores habeat radices reales quam $m-1$, tum vicissim aequatio $z=0$ certo pauciores quam *m* habebit radices reales. Cavendum autem est, ne propositio conversa pro vera habeatur; etiamsi enim aequatio differentialis $\frac{dz}{dx}=0$ aliquot vel adeo omnes radices suas habeat reales, tamen non sequitur aequationem $z=0$ ullam habituram esse radicem realem. Fieri enim potest, ut aequationis $\frac{dz}{dx}=0$ omnes radices sint reales, cum tamen aequationis $z=0$ omnes radices sint imaginariae.

299. Interim tamen, si conditio supra memorata adiiciatur, propositio conversa ita proponi poterit, ut ex radicibus realibus aequationis $\frac{dz}{dx}=0$ numerus radicum realium aequationis $z=0$ certo cognosci possit. Ponamus enim $\alpha,\ \beta,\ \gamma,\ \delta$ etc. esse radices reales aequationis $\frac{dz}{dx}=0$, inter quas α sit maxima; reliquae vero ordine magnitudinis se invicem sequantur. His igitur valoribus loco *x* substitutis functio *z* obtinebit vel maximos vel minimos valores alternatim. Cum autem functio *z* fiat $=\infty$, si ponatur $x=\infty$, patet eius valores continuo decrescere debere, dum valores ipsius *x* ab ∞ usque ad α diminuuntur; ex quo, casu $x=\alpha$, fiet *z* minimum. Quodsi ergo hoc casu $x=\alpha$ functio *z* valorem induat negativum, necesse est, ut ante alicubi fuerit $=0$, sicque aequationis $z=0$ radix dabitur realis $x>\alpha$; sin autem posito $x=\alpha$ functio *z* adhuc retineat valorem affirmativum, ante nusquam potuit esse minor; alias enim quoque daretur minimum, antequam *x* ad α usque diminueretur, quod esset contra hypothesin; hinc aequatio $z=0$ nullam habere poterit radicem realem maiorem quam α. Si ergo ponamus posito $x=\alpha$ fieri $z=\mathfrak{A}$, hoc modo iudicari poterit: Si fuerit $\mathfrak{A}$ quantitas affirmativa, tum aequatio $z=0$ nullam habebit radicem realem α maiorem; sin autem $\mathfrak{A}$ fuerit quantitas negativa, tum ,aequatio $z=0$ unam perpetuo habebit radicem realem α maiorem neque plures.

300. Ad hoc iudicium ulterius persequendum

si ponatur	fiat
$x=\alpha$	$z=\mathfrak{A}$
$x=\beta$	$z=\mathfrak{B}$
$x=\gamma$	$z=\mathfrak{C}$
$x=\delta$	$z=\mathfrak{D}$
$x=\varepsilon$	$z=\mathfrak{E}$
etc.	etc.

Quia ergo $\mathfrak{A}$ fuit minimum, erit $\mathfrak{B}$ maximum, et quidem si $\mathfrak{A}$ fuerit affirmativum, erit quoque $\mathfrak{B}$ affirmativum neque ergo inter limites α et β dabitur radix realis aequationis $z=0$. Quare si haec aequatio nullam habeat radicem realem α maiorem, neque ullam habebit, quae esset maior quam β. Sin autem $\mathfrak{A}$ fuerit quantitas negativa, quo casu una datur aequationis radix $x>\alpha$, dispiciatur, utrum valor ipsius $\mathfrak{B}$ sit affirmativus an negativus. Priori casu dabitur radix $x > \beta$, posteriori vero nulla dabitur radix intra limites α et β contenta. Simili modo cum $\mathfrak{B}$ fuerit maximum, erit $\mathfrak{C}$ minimum; quare si $\mathfrak{B}$ habuerit valorem negativum, multo magis $\mathfrak{C}$ erit negativum nullaque hoc casu dabitur radix intra limites β et γ contenta. At si $\mathfrak{C}$ fuerit affirmativum, radix dabitur realis inter limites β et γ, si $\mathfrak{C}$ fiat negativum; sin autem $\mathfrak{C}$ quoque sit affirmativum, tum nulla dabitur radix inter limites β et γ contenta similique modo iudicium ulterius erit instituendum.

301. Quo haec iudicia facilius intelligantur, ea in sequenti tabella complexus sum.

Aequatio $z = 0$ unam habebit radicem realem, quae continetur intra limites	si fuerit
$x = \infty$ et $x = \alpha$	$\mathfrak{A} = -$
$x = \alpha$ et $x = \beta$	$\mathfrak{A} = -$ et $\mathfrak{B} = +$
$x = \beta$ et $x = \gamma$	$\mathfrak{B} = +$ et $\mathfrak{C} = -$
$x = \gamma$ et $x = \delta$	$\mathfrak{C} = -$ et $\mathfrak{D} = +$
$x = \delta$ et $x = \varepsilon$	$\mathfrak{D} = +$ et $\mathfrak{E} = -$
etc.	etc.

Harumque propositionum conversae et in negantes transmutatae pariter in omni rigore locum obtinent. Scilicet

Aequatio $z = 0$ nullam habebit radicem realem, quae continetur intra limites	si non fuerit
$x = \infty$ et $x = \alpha$	$\mathfrak{A} = -$
$x = \alpha$ et $x = \beta$	$\mathfrak{A} = -$ et $\mathfrak{B} = +$
$x = \beta$ et $x = \gamma$	$\mathfrak{B} = +$ et $\mathfrak{C} = -$
$x = \gamma$ et $x = \delta$	$\mathfrak{C} = -$ et $\mathfrak{D} = +$
$x = \delta$ et $x = \varepsilon$	$\mathfrak{D} = +$ et $\mathfrak{E} = -$
etc.	etc.

Ope harum ergo regularum ex radicibus aequationis $\frac{dz}{dx} = 0$, si eae fuerint cognitae, non solum numerus radicum realium aequationis $z = 0$ colligitur, sed etiam limites innotescunt, intra quos singulae istae radices contineantur.

EXEMPLUM

Sit proposita ista aequatio $x^4 - 14xx + 24x - 12 = 0$;
quae an habeat radices reales et quot, quaeritur.

Aequatio differentialis erit $4x^3 - 28x + 24 = 0$ seu $x^3 - 7x + 6 = 0$, cuius radices sunt 1, 2 et – 3, quae secundum ordinem magnitudinis dispositae dabunt

	unde erit
$\alpha = 2$	$\mathfrak{A} = -4$
$\beta = 1$	$\mathfrak{B} = -1$
$\gamma = -3$	$\mathfrak{C} = -129.$

Ob $\mathfrak{A}$ negativum ergo aequatio proposita habebit radicem realem > 2, at ob $\mathfrak{B}$ negativum neque inter limites 2 et 1 neque inter limites 1 et – 3 radicem habebit realem. Cum autem posito $x = -3$ fiat $z = \mathfrak{C} = -129$, ac si statuatur $x = -\infty$, fiat $z = +\infty$, necesse est, ut radix detur realis inter limites -3 et $-\infty$ contenta. Habebit ergo aequatio proposita duas radices reales, alteram $x > 2$, alteram $x < -3$; ex quo duae radices erunt imaginariae. Simili modo ergo ex ultimo aequationis propositae maximo vel minimo iudicari debet, quo ex primo solo. Scilicet si aequatio proposita fuerit ordinis paris, ultimum sive maximum sive minimum (erit autem hoc casu minimum), si fuerit negativum, radicem realem, sin affirmativum, radicem imaginariam indicat. At pro aequationibus imparium graduum, quia posito $x = -\infty$ fit $z = -\infty$, si ultimum maximum fuerit affirmativum, radix realis, sin negativum, imaginaria indicatur.

302. Regula ergo pro cognoscendis radicibus realibus et imaginariis hoc modo commode exprimi poterit. Proposita aequatione quacunque $z = 0$ consideretur eius differentialis $\frac{dz}{dx} = 0$, cuius radices reales secundum ordinem quantitatis dispositae sint $\alpha, \beta, \gamma, \delta$ etc.; tum posito

$$x = \alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon,\ \zeta \text{ etc.}.$$

fiat

$$z = \mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{D},\ \mathfrak{E},\ \mathfrak{F} \text{ etc.}$$

Jam si signa sint $\quad - \quad + \quad - \quad + \quad - \quad + \quad \text{etc.},$

tot aequatio $z = 0$ habebit radices reales, quot habentur litterae α, β, γ etc., et insuper unam. Sin autem una ex his litteris maiusculis non habeat signum infra scriptum, tum binae radices imaginariae indicabuntur. Ita si $\mathfrak{A}$ haberet signum +, tum nulla daretur radix intra limites ∞ et β contenta. Si $\mathfrak{B}$ habeat signum –, nulla dabitur radix inter limites α et γ, et si $\mathfrak{C}$ habeat signum +, nulla erit radix inter limites β et δ et ita porro. Generatim autem praeter radices imaginarias hoc modo indicatas aequatio $z=0$ insuper tot habebit imaginarias quot aequatio $\frac{dz}{dx}=0$.

303. Si eveniat, ut valorum $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ etc. aliquis evanescat, tum eo loco aequatio $z=0$ duas habebit radices aequales. Scilicet si fuerit $\mathfrak{A}=0$, tum habebit duas radices ipsi α aequales; sin sit $\mathfrak{B}=0$, duae erunt radices $=\beta$. Hoc enim casu aequatio $z=0$ unam habebit radicem communem cum aequatione differentiali $\frac{dz}{dx}=0$; supra autem [§ 245] demonstravimus hoc esse indicium duarum radicum aequalium. Sin autem aequatio $\frac{dz}{dx}=0$ duas pluresve radices habeat aequales, tum, si earum numerus fuerit par, neque maximum neque minimum indicabitur; unde pro praesenti instituto radices aequales numero pares negligi poterunt. Sin autem numerus radicum aequalium aequationis $\frac{dz}{dx}=0$ fuerit impar, tum omnes praeter unam in formatione iudicii reiiciendae sunt, nisi forte hoc casu ipsa quoque functio *z* evanescat. Si enim hoc eveniat, aequatio $z=0$ quoque habebit radices aequales et quidem una plures quam aequatio $\frac{dz}{dx}=0$. Sic si fuerit $\frac{dz}{dx}=(x-\zeta)^n R$, ita ut haec aequatio habeat *n* radices aequales ipsi ζ, si posito $x=\zeta$ quoque evanescat *z*, tum aequatio $z=0$ habebit $n+1$ radices aequales ipsi ζ.

304. Applicemus haec praecepta ad aequationes simpliciores ac primo quidem a quadratica incipiamus. Sit igitur proposita haec aequatio

$$z=x^2-Ax+B=0;$$

erit eius differentialis

$$\frac{dz}{dx}=2x-A,$$

qua facta $=0$ erit

$$x=\tfrac{1}{2}A \text{ seu } \alpha=\tfrac{1}{2}A.$$

Substituatur hic valor loco *x* fietque

$$z=-\tfrac{1}{4}AA+B=\mathfrak{A}$$

unde colligimus, si iste valor ipsius $\mathfrak{A}$ fuerit negativus, hoc est, si sit $AA>4B$, aequationem $xx-Ax+B=0$ habituram esse duas radices reales, alteram maiorem quam $\frac{1}{2}A$, alteram minorem. Sin autem valor ipsius $\mathfrak{A}$ fuerit affirmativus seu $AA<4B$, tum ambae aequationis propositae radices erunt imaginariae. At si fuerit $\mathfrak{A}=0$ seu $AA=4B$, tum aequatio proposita habebit duas radices aequales, utramque scilicet $=\frac{1}{2}A$. Quae cum ex natura aequationum quadraticarum sint notissima, veritas horum principiorum non mediocriter illustratur simulque eorum utilitas in hoc negotio perspicitur.

305. Progrediamur ergo ad aequationes cubicas simili modo inquirendas. Sit ergo proposita aequatio

$$x^3 - Ax^2 + Bx - C = z = 0;$$

cuius differentialis cum sit

$$3xx - 2Ax + B = \frac{dz}{dx},$$

si haec ponatur $=0$, fiet

$$xx = \frac{2Ax - B}{3},$$

cuius aequationis vel ambae radices sunt imaginariae vel aequales vel reales inaequales. Cum igitur hinc sit

$$x = \frac{A \pm \sqrt{\left(A^2 - 3B\right)}}{3},$$

ambae radices erunt imaginariae, si fuerit $AA < 3B$ *;* hoc ergo casu aequatio cubica proposita unicam habebit radicem realem, cuius alii limites non patent praeter $+\infty$ et $-\infty$. Sint iam ambae radices inter se aequales seu $AA = 3B$; erit $x = \frac{A}{3}$. Nisi ergo simul fiat $z = 0$, hae duae radices pro nulla reputari debebunt habebitque aequatio ut ante unicam radicem realem; sin autem casu $x = \frac{A}{3}$ simul fiat $z = 0$, quod evenit, si fuerit $-\frac{2}{27}A^3 + \frac{1}{3}AB - C = 0$ seu $C = \frac{1}{3}AB - \frac{2}{27}A^3$, hoc est, si fuerit $B = \frac{1}{3}A^2$ et $C = \frac{1}{27}A^3$, aequatio habebit tres radices aequales, singulas scilicet $= \frac{A}{3}$. Evolvamus nunc tertium casum, quo ambae radices aequationis differentialis sunt reales et inter se inaequales, quod evenit, si $AA > 3B$. Sit ergo $AA = 3B + ff$ seu $B = \frac{1}{3}AA - \frac{1}{3}ff$; erunt ambae illae radices

$$x = \frac{A \pm f}{3}$$

Fiet ergo $\alpha = \frac{1}{3}A + \frac{1}{3}f$ et $\beta = \frac{1}{3}A - \frac{1}{3}f$. Quaerantur ergo valores ipsius z his respondentes $\mathfrak{A}$ et $\mathfrak{B}$, et cum ambae radices contineantur in hac aequatione $xx = \frac{2}{3}Ax - \frac{1}{3}B$, fiet

$$z = -\frac{1}{3}Axx + \frac{2}{3}Bx - C = -\frac{3}{9}AAx + \frac{1}{9}AB + \frac{2}{3}Bx - C.$$

Hinc itaque oritur

$$\mathfrak{A} = -\frac{2}{27}A^3 + \frac{1}{3}AB - \frac{2}{27}A^2 f + \frac{2}{9}Bf - C = \frac{1}{27}A^3 - \frac{1}{9}Aff - \frac{2}{27}f^3 - C,$$
$$\mathfrak{B} = -\frac{2}{27}A^3 + \frac{1}{3}AB + \frac{2}{27}A^2 f - \frac{2}{9}Bf - C = \frac{1}{27}A^3 - \frac{1}{9}Aff + \frac{2}{27}f^3 - C$$

ob $B=\frac{1}{3}AA-\frac{1}{3}ff$. Si igitur fuerit $\mathfrak{A}$ quantitas negativa, quod evenit, si fuerit $C>\frac{1}{27}A^3-\frac{1}{9}Aff-\frac{2}{27}f^3$, aequatio $z=0$ unam habebit radicem realem $>\alpha$, hoc est maiorem quam $\frac{1}{3}A+\frac{1}{3}f$. Ponamus ergo esse

$$C>\frac{1}{27}A^3-\frac{1}{9}Aff-\frac{2}{27}f^3 \quad \text{seu esse} \quad C=\frac{1}{27}A^3-\frac{1}{9}Aff-\frac{2}{27}f^3+gg$$

atque, ut vidimus, aequatio proposita cubica habebit radicem realem $>\frac{1}{3}A+\frac{1}{3}f$. Quales autem futurae sint reliquae radices, ex valore $\mathfrak{B}$ intelligetur; erit autem $\mathfrak{B}=\frac{4}{27}f^3-gg$; qui si fuerit affirmativus, aequatio insuper duas habebit radices reales, priorem intra limites α et β, hoc est intra $\frac{1}{3}A+\frac{1}{3}f$ et $\frac{1}{3}A-\frac{1}{3}f$, contentam, alteram vero minorem quam $\frac{1}{3}A-\frac{1}{3}f$. Sin autem fuerit $gg>\frac{4}{27}f^3$ seu $\mathfrak{B}$ negativum, tum aequatio habebit duas radices imaginarias. At si fuerit $\mathfrak{B}=0$ seu $\frac{4}{27}f^3=gg$, tum duae radices evadent aequales, utraque $=\beta=\frac{1}{3}A-\frac{1}{3}f$. Denique si sit valor ipsius $\mathfrak{A}$ affirmativus seu $C<\frac{1}{27}A^3-\frac{1}{9}Aff-\frac{2}{27}f^3$, tum aequatio duas habebit radices imaginarias tertiaque erit realis et $<\frac{1}{3}A-\frac{1}{3}f$. Atque si sit valor ipsius $\mathfrak{A}=0$, duae erunt radices aequales $=\alpha$ ex manente tertia $<\frac{1}{3}A-\frac{1}{3}f$.

306. Quo igitur aequationis cubicae $x^3-Ax^2+Bx-C=0$ omnes tres radices sint reales, requiruntur tres conditiones. Primo, ut sit

$$B<\frac{1}{3}AA;$$

sit ergo $B=\frac{1}{3}AA-\frac{1}{3}ff$. Secundo, ut sit

$$C>\frac{1}{27}A^3-\frac{1}{9}Aff-\frac{2}{27}f^3.$$

Tertio, ut sit

$$C<\frac{1}{27}A^3-\frac{1}{9}Aff+\frac{2}{27}f^3.$$

Quae duae posteriores conditiones eo redeunt, ut C contineatur intra hos limites

$$\frac{1}{27}A^3-\frac{1}{9}Aff-\frac{2}{27}f^3 \quad \text{et} \quad C\,\frac{1}{27}A^3-\frac{1}{9}Aff+\frac{2}{27}f^3.$$

seu intra hos limites

$$\tfrac{1}{27}\left(A+f\right)^2\left(A-2f\right) \quad \text{et} \quad \tfrac{1}{27}\left(A-f\right)^2\left(A+2f\right)$$

Quodsi ergo harum conditionum unica desit, aequatio duas habebit radices imaginarias. Sic si fuerit $A=3, B=2$, erit $\frac{1}{3}ff=\frac{1}{3}AA-B=1$ et $ff=3$; unde ista aequatio $x^3-3xx+2x-C=0$ omnes radices reales habere nequit, nisi C contineatur intra limites $-\frac{2\sqrt{3}}{9}$ et $+\frac{2\sqrt{3}}{9}$. Quare si fuerit vel

$$C<-\tfrac{2\sqrt{3}}{9} \text{ seu } C<-0{,}3849 \quad \text{vel} \quad C>+\tfrac{2\sqrt{3}}{9} \text{ seu } C>0{,}3849$$

aut coniunctim $CC>\frac{4}{27}$ aequatio unicam habebit radicem realem.

307. Quoniam in omni aequatione secundus terminus tolli potest, ponamus esse $A=0$, ita ut habeamus hanc aequationem cubicam

$$x^3+Bx-C=0$$

Ut igitur huius aequationis omnes tres radices sint reales, necesse est, ut primo sit $B<0$, seu B debet esse quantitas negativa. Sit ergo $B=-kk$; erit $ff=3kk$ atque insuper requiritur, ut quantitas C contineatur intra hos limites $-\frac{2}{27}f^3$ et $+\frac{2}{27}f^3$, hoc est inter hos $-\frac{2}{9}kk\sqrt{3kk}$ et $+\frac{2}{9}kk\sqrt{3kk}$. Erit ergo $CC<\frac{4}{27}k^6$ seu $CC<-\frac{4}{27}B^3$. Unica ergo conditione natura aequationum cubicarum, quae omnes tres radices habeant reales, comprehendi poterit, dum dicemus esse oportere

$$4B^3+27CC$$

quantitatem negativam. Sic enim iam postulatur, ut sit B quantitas negativa, quia alioquin $4B^3+27CC$ negativum fieri non posset. Quocirca generatim affirmamus aequationem $x^3+Bx\pm C=0$ omnes tres radices habituram esse reales, si fuerit $4B^3+27CC$ quantitas negativa; sin autem haec quantitas fuerit affirmativa, tum unicam fore realem, reliquas binas imaginarias; at si fiat $4B^3+27CC=0$, tum omnes quidem radices futuras esse reales, at binas inter se aequales.

308. Progrediamur ad aequationes biquadratas, in quibus etiam secundum terminum deesse ponamus. Sit ergo

$$x^4+Bx^2-Cx+D=0.$$

Statuamus $x=\frac{1}{u}$ eritque

$$1+Bu^2-Cu^3+Du^4=0,$$

cuius aequatio differentialis est

$$2Bu-3Cu^2+4Du^3=0,$$

quae unam habet radicem $u=0$; tum vero erit

$$uu=\frac{6Cu-4B}{8D}$$

et

$$u=\frac{3C\pm\sqrt{(9CC-32BD)}}{8D}.$$

Ut igitur omnes quatuor radices sint reales, primo requiritur, ut sit $9CC>32BD$. Ponamus ergo esse $9CC=32BD+9ff$; erit $u=\frac{3C\pm3f}{8D}$. Hic C semper pro quantitate affirmativa sumere poterimus; nisi enim talis fuerit, ponendo $u=-v$ talis evadet. Mox autem demonstrabimus omnes radices reales esse non posse, nisi sit B quantitas negativa. Sit ergo $B=-gg$ eritque

$$9CC=9ff-32ggD \text{ et } u=\frac{3C\pm3f}{8D}.$$

Atque duo casus erunt perpendendi, prout D sit quantitas affirmativa vel negativa.

I. Sit D quantitas affirmativa eritque $f>C$ ac tres ipsius u radices secundum quantitatis ordinem dispositae erunt

$$1.\ u=\frac{3C+3f}{8D},\quad 2.\ u=0,\quad 3.\ u=\frac{3C-3f}{8D}.$$

Aequatio autem

$$u^4-\frac{Cu^3}{D}+\frac{Bu^2}{D}+\frac{1}{D}=0$$

his valoribus loco u substitutis dabit sequentes tres valores

$$\mathfrak{A}=\frac{27(C+f)^3(C-3f)}{4096D^4}+\frac{1}{D},\quad \mathfrak{B}=\frac{1}{D},\quad \mathfrak{C}=\frac{27(C-f)^3(C+3f)}{4096D^4}+\frac{1}{D},$$

quorum primus ac tertius debet esse negativus; uterque quidem ob C affirmativum et $C<f$ fit minor quam $\frac{1}{D}$. Oportet itaque esse

$$\frac{1}{D}<\frac{27(C+f)^3(3f-C)}{4096D^4} \quad \text{et} \quad \frac{1}{D}<\frac{27(f-C)^3(C+3f)}{4096D^4}$$

seu

$$4096D^3 < 27(C+f)^3(3f-C) \quad \text{et} \quad 4096D^3 < 27(f-C)^3(C+3f).$$

At prior quantitas semper longe maior est posteriori; unde sufficit, si fuerit $D^3 < \frac{27}{4096}(f-C)^3(C+3f)$ existente $B = \frac{9CC-9f}{32ff}$ et $f > C$ atque $D > 0$. Si igitur fuerit D quantitas affirmativa, C affirmativa, B negativa, ut sit $f > C$, atque $D^3 < \frac{27}{4096}(f-C)^3(C+3f)$, hoc est $D < \frac{3}{16}(f-C)\sqrt[3]{(3f+C)}$, tum aequatio omnes radices habebit reales. Sin autem fuerit $D > \frac{3}{16}(f-C)\sqrt[3]{(3f+C)}$, attamen $D < \frac{3}{16}(f+C)\sqrt[3]{(3f-C)}$, tum duae radices erunt reales et duae imaginariae. At si adeo fuerit $D > \frac{3}{16}(f+C)\sqrt[3]{(3f-C)}$, tum omnes quatuor radices erunt imaginariae.

II. Sit D quantitas negativa, puta $=-F$, manente C affirmativa ac B negativa; ob $B = \frac{9CC-9ff}{32D} = \frac{9ff-9CC}{32F}$ erit $C > f$. Cum igitur sit $u = \frac{3C\pm 3f}{8D} = -\frac{3C\pm 3f}{8F}$, tres valores ipsius u secundem ordinem magnitudinis dispositi erunt

$$1.\ u = 0, \quad 2.\ u = -\frac{3C-3f}{8F}, \quad 3.\ u = -\frac{3C+3f}{8F},$$

qui dabunt sequentes valores

$$\mathfrak{A} = -\frac{1}{F}, \quad \mathfrak{B} = \frac{27(C-f)^3(C+3f)}{4096F^4} - \frac{1}{F}, \quad \mathfrak{C} = \frac{27(C+f)^3(C-3f)}{4096F^4} - \frac{1}{F}.$$

Cum igitur $\mathfrak{A}$ sit quantitas negativa, aequatio iam certo unam ac propterea quoque duas habebit radices reales. Ut autem omnes radices sint reales, oportet, ut $\mathfrak{B}$ sit quantitas affirmativa ideoque $27(C-f)^3(C+3f) > 4096F^3$; tum vero necesse est, ut sit $\mathfrak{C}$ quantitas negativa seu $27(C+f)^3(C-3f) < 4096F^3$. Quocirca ut omnes radices fiant reales, requiritur, ut F^3 contineatur intra hos limites

$$\frac{27}{4096}(C+f)^3(C-3f) \quad \text{et} \quad \frac{27}{4096}(C-f)^3(C+3f)$$

seu ut F contineatur intra limites

$$\frac{3}{16}(C+f)\sqrt[3]{(C-3f)} \quad \text{et} \quad \frac{3}{16}(C-f)\sqrt[3]{(C+3f)};$$

et nisi F contineatur intra hos limites, duae radices erunt imaginariae.

III. Ponamus iam B esse quantitatem affirmativam et D pariter affirmativam; ob $B = \frac{9CC - 9ff}{32D}$ erit $C > f$ et cum sit $u = \frac{3C \pm 3f}{8D}$, radices ordine magnitudinis dispositae erunt

$$1.\ u = \frac{3(C+f)}{8D}, \quad 2.\ u = \frac{3(C-f)}{8D}, \quad 3.\ u = 0,$$

unde sequentes oriuntur valores

$$\mathfrak{A} = \frac{27(C+f)^3(C-3f)}{4096D^4} + \frac{1}{D}, \quad \mathfrak{B} = \frac{27(C-f)^3(C+3f)}{4096D^4} + \frac{1}{D}, \quad \mathfrak{C} = \frac{1}{D}\ ;$$

ubi cum $\mathfrak{C}$ sit quantitas affirmativa, certo duae radices erunt imaginariae. Sin autem fuerit $\mathfrak{A}$ negativum, quod evenit, si $4096D^3 < 27(C+f)^3(3f - C)$, duae radices erunt reales; sin fuerit $4096D^3 > 27(C+f)^3(3f - C)$, tum omnes quatuor radices erunt imaginariae.

IV. Maneat B affirmativum, sit autem D negativum $= -F$; ob $B = \frac{9ff - 9CC}{32F}$ erit $f > C$ et ob $u = -\frac{3C \pm 3f}{8F}$ tres ipsius u radices secundum ordinem magnitudinis dispositae erunt

$$1.\ u = \frac{3(f-C)}{8F}, \quad 2.\ u = 0, \quad 3.\ u = -\frac{3(C+f)}{8F},$$

unde isti valores nascuntur

$$\mathfrak{A} = -\frac{27(f-C)^3(C+3f)}{4096F^4} - \frac{1}{F}, \quad \mathfrak{B} = -\frac{1}{F}, \quad \mathfrak{C} = -\frac{27(C+f)^3(3f-C)}{4096F^4} - \frac{1}{F}$$

ubi ob $\mathfrak{A}$ et $\mathfrak{C}$ negativa aequatio certo duas habet radices reales, at ob $\mathfrak{B}$ negativum duae radices erunt imaginariae.

309. Si igitur ponamus litteras B, C, D quantitates affirmativas denotare, sequentes oriuntur casus diversi diiudicandi, qui ob $f = \sqrt{\left(CC - \frac{32}{9}BD\right)}$ huc redeunt.

I. Si aequatio sit $x^4 - Bx^2 \pm Cx + D = 0$, omnes radices erunt reales, si fuerit

$$D > \tfrac{3}{16}\left(\sqrt{\left(CC+\tfrac{32}{9}BD\right)}-C\right)\sqrt[3]{\left(3\sqrt{\left(CC+\tfrac{32}{9}BD\right)}+C\right)};$$

duae radices erunt reales duaeque imaginariae, si fuerit

$$D > \tfrac{3}{16}\left(\sqrt{\left(CC+\tfrac{32}{9}BD\right)}-C\right)\sqrt[3]{\left(3\sqrt{\left(CC+\tfrac{32}{9}BD\right)}+C\right)},$$

at

$$D < \tfrac{3}{16}\left(\sqrt{\left(CC+\tfrac{32}{9}BD\right)}+C\right)\sqrt[3]{\left(3\sqrt{\left(CC+\tfrac{32}{9}BD\right)}-C\right)};$$

omnes autem radices erunt imaginariae, si fuerit

$$D > \tfrac{3}{16}\left(\sqrt{\left(CC+\tfrac{32}{9}BD\right)}+C\right)\sqrt[3]{\left(3\sqrt{\left(CC+\tfrac{32}{9}BD\right)}-C\right)}.$$

II. Si aequatio sit $x^4 - Bx^2 \pm Cx - D = 0$, duae radices semper sunt reales; reliquae binae quoque erunt reales, si quantitas *D* contineatur intra hos limites

$$D > \tfrac{3}{16}\left(\sqrt{\left(CC-\tfrac{32}{9}BD\right)}+C\right)\sqrt[3]{\left(C-3\sqrt{\left(CC-\tfrac{32}{9}BD\right)}\right)},$$

$$D < \tfrac{3}{16}\left(C-\sqrt{\left(CC-\tfrac{32}{9}BD\right)}\right)\sqrt[3]{\left(C+3\sqrt{\left(CC-\tfrac{32}{9}BD\right)}\right)};$$

nisi autem *D* contineatur intra hos limites, duae reliquae radices erunt imaginariae.

III. Si aequatio sit $x^4 + Bx^2 \pm Cx + D = 0$, duae radices semper erunt imaginariae; reliquae vero duae erunt reales, si fuerit

$$D < \tfrac{3}{16}\left(\sqrt{\left(CC-\tfrac{32}{9}BD\right)}+C\right)\sqrt[3]{\left(3\sqrt{\left(CC-\tfrac{32}{9}BD\right)}-C\right)}$$

reliquae vero duae quoque erunt imaginariae, si fuerit

$$D > \tfrac{3}{16}\left(\sqrt{\left(CC-\tfrac{32}{9}BD\right)}+C\right)\sqrt[3]{\left(3\sqrt{\left(CC-\tfrac{32}{9}BD\right)}-C\right)}$$

IV. Si aequatio sit $x^4 + Bx^2 \pm Cx - D = 0$, huius aequationis duae radices semper erunt reales, duae reliquae vero semper imaginariae.

EXEMPLUM 1

Si proponatur haec aequatio $x^4 - 2xx + 3x + 4 = 0$, *quaeratur natura radicum, utrum sint reales an imaginariae.*

Quia hoc exemplum ad casum primum pertinet, est $B = 2,\quad C = 3\quad$ et $\quad D = 4$; unde

$$CC + \tfrac{32}{9}BD = 9 + \tfrac{32\cdot 8}{9} = \tfrac{337}{9} \text{ et } \sqrt{\left(CC + \tfrac{32}{9}BD\right)} = \tfrac{\sqrt{337}}{3},$$

unde conditiones, ut omnes radices sint reales, sunt

$$4 < \tfrac{3}{16}\left(3 + \tfrac{\sqrt{337}}{3}\right)\sqrt[3]{\left(\sqrt{337} - 3\right)} = \tfrac{1}{16}\left(9 + \sqrt{337}\right)\sqrt[3]{\left(\sqrt{337} - 3\right)},$$
$$4 < \tfrac{3}{16}\left(\tfrac{\sqrt{337}}{3} - 3\right)\sqrt[3]{\left(\sqrt{337} + 3\right)} = \tfrac{1}{16}\left(\sqrt{337} - 9\right)\sqrt[3]{\left(\sqrt{337} + 3\right)}.$$

Adhibitis approximationibus examinari debet ergo, utrum sit $4 < \frac{69}{16}$ et $4 < \frac{24}{16}$; quare cum prior tantum conditio locum habeat, aequatio habebit duas radices reales et duas imaginarias.

EXEMPLUM 2

Proposita sit haec aequatio $x^4 - 9xx + 12x - 4 = 0$.

Quae cum pertineat ad casum secundum, duas habebit radices reales. Ad reliquarum naturam investigandam ob $B = 9,\quad C = 12\quad$ et $\quad D = 4$ erit

$$\sqrt{\left(CC - \tfrac{32}{9}BD\right)} = \sqrt{\left(144 - 32\cdot 4\right)} = 4.$$

Ideoque videndum est, utrum sit

$$4 > \tfrac{3}{16}\cdot 16\sqrt[3]{0}, \quad \text{hoc est} \quad 4 > 0,$$

et

$$4 < \tfrac{3}{16}\cdot 8\sqrt[3]{24}, \quad \text{hoc est} \quad 4 < 3\sqrt[3]{3},$$

quorum utrumque cum eveniat, aequatio proposita quatuor habebit radices reales.

EXEMPLUM 3

Proposita sit haec aequatio $x^4 + xx - 2x + 6 = 0$.

Quae cum pertineat ad casum tertium, duae radices certo erunt imaginariae. Tum vero est $B = 1$, $C = 2$ et $D = 6$ ideoque

$$\sqrt{\left(CC - \tfrac{32}{9}BD\right)} = \sqrt{\left(4 - \tfrac{64}{3}\right)};$$

quae quantitas cum sit imaginaria, et duae reliquae radices certo erunt imaginariae.

EXEMPLUM 4

Sit proposita aequatio haec $x^4 - 4x^3 + 8x^2 - 16x + 20 = 0$.

Eliminetur primo secundus terminus; substituendo $x = y + 1$ fiet

$$\begin{aligned} x^4 &= y^4 + 4y^3 + 6yy + 4y + 1 \\ -4x^3 &= \quad -4y^3 - 12y^2 - 12y - 4 \\ +8x^2 &= \qquad\quad + 8y^2 + 16y + 8 \\ -16x &= \qquad\qquad\quad - 16y - 16 \\ +20 &= \underline{\qquad\qquad\qquad\qquad + 20} \end{aligned}$$

Ergo
$$y^4 + 2yy - 8y + 9 = 0;$$

quae cum pertineat ad casum tertium, duas radices habebit imaginarias. Tum vero ob $B = 2$, $C = 8$, $D = 9$ erit

$$\sqrt{\left(CC - \tfrac{32}{9}BD\right)} = \sqrt{(64 - 64)} = 0.$$

Comparetur ergo $D = 9$ cum $\frac{3}{16}\cdot 8\sqrt[3]{-8} = -3$. Cum ergo sit $D = 9 > -3$, etiam duae reliquae radices erunt imaginariae.

EXEMPLUM 5

Sit proposita haec aequatio $x^4 - 4x^3 - 7x^2 + 34x - 24 = 0$,
cuius radices constat esse 1, 2, 4 *et* – 3.

Quodsi autem regulas applicemus, sublato secundo termino ponendo $x = y + 1$ fiet

$$y^3 - 13yy + 12y + 0 = 0,$$

quae cum casu secundo comparata dat $B = 13, \quad C = 12, \quad D = 0$. Debet ergo esse
$D > \frac{3}{16} \cdot 24 \sqrt[3]{-24}$ seu $0 > -9\sqrt[3]{3}$ et $D < 0$; cum igitur D non sit maius quam 0, aequatio quatuor radices reales habere indicatur. Si enim sit $D = 0$, altera aequatio abit in
$D < \frac{3}{16}\left(\frac{16BD}{9C}\right) \sqrt[3]{4C}$ ideoque $1 < \frac{B}{3C}\sqrt[3]{4C}$ seu $27CC < 4B^3$;
est vero $27 \cdot 144 < 4.13^3$ seu $36 \cdot 27 < 13^3$.

310. Opus foret maxime difficile, si simile iudicium ad aequationes altiorum graduum transferre vellemus, propterea quod aequationum differentialium radices plerumque exhiberi non possunt; quoties autem has radices assignare licet, ex traditis principiis facile colligitur, quot aequatio proposita habeat radices reales et imaginarias. Hinc omnis aequationis, quae tantum ex tribus terminis constat, radices, utrum sint reales an imaginariae, definiri poterunt. Sit enim proposita haec aequatio generalis

$$x^{m+n} + Ax^n + B = 0 = z.$$

Sumatur eius differentialis

$$\frac{dz}{dx} = (m+n)x^{m+n-1} + nAx^{n-1};$$

qua nihilo aequali posita erit primo $x^{n-1} = 0$; unde si n fuerit impar numerus, nulla radix maximum minimumve exhibens oritur; sin autem sit n numerus par, una radix in computum ducenda erit $x = 0$. Tum vero erit $(m+n)x^m + nA = 0$; quae aequatio, si m sit numerus par et A affirmativa quantitas, nullam habet radicem realem. Hinc sequentes casus erunt expendendi.

I. Sit m numerus par et n numerus impar et radix $x = 0$ non valebit. Si igitur fuerit A quantitas affirmativa, nulla prorsus habebitur radix maximum minimumve exhibens; unde ob $m + n$ numerum imparem aequatio proposita unicam habebit radicem realem. Sin autem fuerit A quantitas negativa, puta $A = -E$, $x = \pm\sqrt[m]{\frac{nE}{m+n}}$ unde

$$\alpha = +\sqrt[m]{\frac{nE}{m+n}} \quad \text{et} \quad \beta = -\sqrt[m]{\frac{nE}{m+n}}.$$

Ex quibus valoribus fit

$$\mathfrak{A} = (x^m - E)x^n + B = -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B$$

atque

$$\mathfrak{B} = +\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} + B$$

Si igitur fuerit $\mathfrak{A}$ quantitas negativa seu

$$\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m} > B$$

aequatio unam habebit radicem realem $> \alpha$. Si insuper fuerit

$$B > -\frac{mE}{m+n}\left(\frac{nE}{m+n}\right)^{n:m}$$

hoc est, ambas conditiones in unam complectendo si fuerit

$$(m+n)^{m+n} B^m < m^m n^n E^{m+n},$$

tum aequatio tres habebit radices reales, et nisi haec conditio locum habeat, aequationis unica radix erit realis. Valent haec de aequatione $x^{m+n} - Ex^n + B = 0$, si fuerit *m* numerus par et *n* numerus impar; ubi si *E* fuerit numerus negativus, aequatio semper unicam radicem habebit realem.

II. Sint ambo numeri *m* et *n* impares, ut sit $m+n$ numerus par nullaque radix $x = 0$ in computum veniat. Quia est $(m+n)x^m + nA = 0$, erit $x = -\sqrt[m]{\frac{nA}{m+n}}$; quae unica radix si sit $= \alpha$, fiet

$$\mathfrak{A} = \frac{mA}{m+n}x^n + B = -\frac{mA}{m+n}\left(\frac{nA}{m+n}\right)^{n:m} + B.$$

Qui valor si fuerit negativus, aequatio proposita duas habebit radices reales, contra nullam. Aequatio ergo proposita $x^{m+n} + \mathrm{A}x^n + B = 0$ duas habebit radices reales, si fuerit

$$m^m n^n A^{m+n} > (m+n)^{m+n} B^m;$$

sin fuerit

$$m^m n^n A^{m+n} < (m+n)^{m+n} B^m,$$

nulla prorsus radix erit realis.

III. Sint ambo numeri *m* et *n* pares; erit $m+n$ pariter numerus par unaque radix $x = 0$ maximum minimumve praebebit; quae erit unica, si *A* fuerit quantitas affirmativa, unde facto $\alpha = 0$ erit $\mathfrak{A} = B$. Quare si fuerit *B* quoque quantitas affirmativa, aequatio nullam habebit radicem realem; sin autem *B* sit quantitas negativa, duae habebuntur radices reales neque plures, si quidem *A* fuerit quantitas affirmativa. At ponamus esse *A* quantitatem negativam seu $A = -E$; erit $x = \pm\sqrt[m]{\frac{nE}{m+n}}$ habebimusque tria maxima vel minima, nempe

$$\alpha = +\sqrt[m]{\tfrac{nE}{m+n}}, \quad \beta = 0, \quad \gamma = -\sqrt[m]{\tfrac{nE}{m+n}}.$$

Quibus ipsius $z = x^{m+n} - Ex^n + B$ respondent valores

$$\mathfrak{A} = -\tfrac{mE}{m+n}\left(\tfrac{nE}{m+n}\right)^{n:m} + B, \quad \mathfrak{B} = B, \quad \mathfrak{C} = -\tfrac{mE}{m+n}\left(\tfrac{nE}{m+n}\right)^{n:m} + B$$

Si igitur B sit quantitas negativa, ob $\mathfrak{A}$ et $\mathfrak{B}$ negativas aequatio duas tantum habebit radices reales, propterea quod quoque $\mathfrak{B} = B$ fit negativum. At si B fuerit quantitas affirmativa, aequatio quatuor habebit radices reales, si
sit

$$(m+n)^{m+n} B^m < m^m n^n E^{m+n}.$$

Nullam autem habebit radicem realem, si fuerit

$$(m+n)^{m+n} B^m > m^m n^n E^{m+n}.$$

IV. Sit m numerus impar et n numerus par atque radix $x = 0$ dabit maximum vel minimum. Praeterea vero erit $x = -\sqrt[m]{\tfrac{nA}{m+n}}$. Si ergo A sit numerus affirmativus, fiet $\alpha = 0$ et $\beta = -\sqrt[m]{\tfrac{nA}{m+n}}$ hincque

$$\mathfrak{A} = B, \quad \text{et} \quad \mathfrak{B} = \tfrac{mA}{m+n}\left(\tfrac{nA}{m+n}\right)^{n:m} + B.$$

Quare si sit B quantitas negativa, puta $B = -F$, atque insuper fuerit

$$m^m n^n A^{m+n} > (m+n)^{m+n} F^m,$$

aequatio tres habebit radices reales; contra unica tantum erit realis. Sin autem sit A quantitas negativa, puta $A = -E$, fiet $x = +\sqrt[m]{\tfrac{nE}{m+n}}$ et

$$\alpha = \sqrt[m]{\tfrac{nE}{m+n}} \quad \text{et} \quad \beta = 0,$$

quibus respondent

$$\mathfrak{A} = -\tfrac{mE}{m+n}\left(\tfrac{nE}{m+n}\right)^{n:m} + B \quad \text{et} \quad \mathfrak{B} = B.$$

Quare aequatio tres habebit radices reales, si fuerit B quantitas affirmativa et

$$m^m n^n E^{m+n} > (m+n)^{m+n} B^m;$$

quae proprietas nisi locum inveniat, aequatio unicam habebit radicem realem.

311. Sint omnes coefficientes $=1$ atque denotantibus μ et ν numeros integros aequationes sequentes ita diiudicabuntur:

$$x^{2\mu+2\nu-1} + x^{2\nu-1} \pm 1 = 0$$

unicam habebit radicem realem.

$$x^{2\mu+2\nu-1} - x^{2\nu-1} \pm 1 = 0$$

tres habebit radices reales, si fuerit

$$(2\mu+2\nu-1)^{2\mu+2\nu-1} < (2\mu)^{2\mu}(2\nu-1)^{2\nu-1}$$

quod cum nunquam fieri possit, aequatio semper unicam radicem realem habebit.

$$x^{2\mu+2\nu} \pm x^{2\nu-1} - 1 = 0$$

duas habet radices reales.

$$x^{2\mu+2\nu} \pm x^{2\nu-1} + 1 = 0$$

nullam habet radices reales.

$$x^{2\mu+2\nu} \pm x^{2\nu} + 1 = 0$$

nullam habet radicem realem.

$$x^{2\mu+2\nu} \pm x^{2\nu} - 1 = 0$$

duas habet radices reales.

$$x^{2\mu+2\nu+1} + x^{2\nu} \pm 1 = 0$$

unicam habet radicem realem.

$$x^{2\mu+2\nu+1} - x^{2\nu} \pm 1 = 0$$

unicam habet radicem realem.

Ceterum quia in casu tertio ambo exponentes sunt pares, is ponendo $xx = y$ ad formam simpliciorem reduci potest ideoque hic casus praetermitti posset. Quo facto affirmari poterit nullam aequationem tribus terminis constantem plures tribus habere posse radices reales.

EXEMPLUM

Quaerantur casus, quibus aequatio haec $x^5 \pm Ax^2 \pm B = 0$ *tres habeat radices reales.*

Quia haec aequatio pertinet ad casum quartum, patet quantitates A et B esse debere signis contrariis affectas. Quare nisi huiusmodi habeat formam, unicam habebit radicem realem; sin autem aequatio proposita fuerit huiusmodi $x^5 \pm Ax^2 \mp B = 0$, quo ea habeat tres radices reales, necesse est, ut sit $3^3 2^2 A^5 > 5^5 B^3$ seu $A^5 > \frac{3125}{108} B^3$. Quodsi ergo fuerit $B = 1$, oportet esse $A^5 > \frac{3125}{108}$ seu $A > 1,960132$. Si ergo sit $A = 2$, ista aequatio $x^5 - 2x^2 + 1 = 0$ tres habet radices reales; quarum cum una sit $x = 1$, sequitur hanc aequationem biquadratam $x^4 + x^3 + x^2 - x - 1 = 0$ duas habere radices reales. Quod quidem tum ex his datis praeceptis intelligi potest, tum ex iis, quae in libro supereriori sunt demonstrata, manifestum est, ubi ostendimus, quamvis aequationem paris gradus, cuius terminus absolutus sit numerus negativus, habere semper duas radices reales.

312. Ex his principiis quoque aequationes, quae constant quatuor terminis, diiudicari poterunt, dummodo aequationis differentialis radices commode exhiberi queant, quod evenit, si exponentes ipsius x vel in tribus anterioribus vel in tribus posterioribus terminis sint in arithmetica progressione. Cum autem haec diiudicatio in genere suscepta ad plures perducatur casus, eam in nonnullis exemplis absolvamus.

EXEMPLUM 1

Sit proposita haec aequatio $x^7 - 2x^5 + x^3 - a = 0.$

Facto $z = x^7 - 2x^5 + x^3 - a$ erit

$$\frac{dz}{dx} = 7x^6 - 10x^4 + 3xx$$

quo valore nihilo aequali posito fiet primo $xx = 0$, qui duplex valor pro nullo reputandus. Tum vero erit $7x^4 = 10x^2 - 3$, unde fit $x^2 = \frac{5 \pm 2}{7}$, et quatuor valores pro x emergent, qui secundum magnitudinem ordinati sequentes pro z praebebunt valores:

$$\begin{array}{c|c} \alpha = 1 & \mathfrak{A} = -a \\ \beta = +\sqrt{\frac{3}{7}} & \mathfrak{B} = \frac{48}{343}\sqrt{\frac{3}{7}} - a \\ \gamma = -\sqrt{\frac{3}{7}} & \mathfrak{C} = \frac{-48}{343}\sqrt{\frac{3}{7}} - a \\ \delta = -1 & \mathfrak{D} = -a. \end{array}$$

Si ergo sit a numerus affirmativus, erit vel $a > \frac{48}{343}\sqrt{\frac{3}{7}}$ vel $a < \frac{48}{343}\sqrt{\frac{3}{7}}$; priori casu ob $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ omnes negativas aequatio proposita unicam habebit radicem realem $x > 1$. Posteriori casu, si $a < \frac{48}{343}\sqrt{\frac{3}{7}}$ aequatio tres habebit radices reales, primam $x > 1$, secundam contentam inter limites 1 et $\sqrt{\frac{3}{7}}$ et tertiam intra limites $+\sqrt{\frac{3}{7}}$ et $-\sqrt{\frac{3}{7}}$.

Sin *a* sit quantitas negativa, ponendo $x = -y$ aequatio perducetur ad formam priorem. Quo ergo aequatio proposita tres habeat radices reales, necesse est, ut sit $a < 0{,}0916134$ vel $a < \frac{1}{11}$.

EXEMPLUM 2

Sit proposita haec aequatio $ax^8 - 3x^6 + 10x^3 - 12 = 0.$

Quia hic exponentes trium posteriorum terminorum sunt in arithmetica progressione, ponatur $x = \frac{1}{y}$ atque aequatio transmutabitur in hanc

$$a - 3y^2 + 10y^5 - 12y^8 = 0;$$

ponatur ergo

$$z = 12y^8 - 10y^5 + 3y^2 - a = 0$$

eritque differentiando

$$\frac{dz}{dx} = 96y^7 - 50y^4 + 6y = 0,$$

ex qua aequatione primo fit $y = 0$; tum vero erit

$$y^6 = \frac{50y^3 - 6}{96} \quad \text{et} \quad y^3 = \frac{25 \pm 7}{96}$$

ideoque vel $y = \sqrt[3]{\frac{1}{3}}$ vel $y = \sqrt[3]{\frac{3}{16}}$. His ergo tribus radicibus secundum magnitudinem dispositis respondentes ipsius *z* valores ita se habebunt:

$\alpha = \sqrt[3]{\frac{1}{3}}$	$\mathfrak{A} = \sqrt[3]{\frac{1}{9}} - a$
$\beta = \sqrt[3]{\frac{3}{16}}$	$\mathfrak{B} = \frac{99}{64}\sqrt[3]{\frac{9}{256}} - a = \frac{99}{256}\sqrt[3]{\frac{9}{4}} - a$
$\gamma = 0$	$\mathfrak{C} = -a.$

Quodsi ergo fuerit $a > \sqrt[3]{\frac{1}{9}}$, aequatio proposita duas habebit radices reales, alteram $> \sqrt[3]{\frac{1}{3}}$, alteram < 0; at praeter has insuper habebit duas radices reales, si simul fuerit $\mathfrak{B}$ quantitas affirmativa, hoc est, si fuerit $a < \frac{99}{256}\sqrt[3]{\frac{9}{4}}$. Quamobrem aequatio proposita quatuor habebit radices reales, si quantitas a contineatur intra limites $\sqrt[3]{\frac{1}{9}}$ et $\frac{99}{256}\sqrt[3]{\frac{9}{4}}$; qui limites proxime sunt 0,48075 et 0,50674. Posito ergo $a=\frac{1}{2}$ haec aequatio $x^8-6x^6+20x^3-24=0$ quatuor habet radices reales intra limites $\infty,\ \sqrt[3]{\frac{16}{3}},\ \sqrt[3]{3},\ 0,\ -\infty$; ergo tres erunt affirmativae et una negativa.