CHAPTER XI

CONCERNING THE MAXIMA AND MINIMA OF MULTIVALUED FUNCTIONS, INCLUDING SEVERAL VARIABLES

273. If y were a multivalued function of x ['many–formed' in the original text], thus so that for some one value of x from that there may be obtained several real values, then for that x variation several values of y thus may be connected between themselves, so that they may represent several series of successive values. For if we may consider y as the applied line [*i.e.* the y coordinate] of a curve, with the abscissa x present, however many different real values y may have, just as many different branches of the same curve will correspond to the same abscissa x; and hence those successive values of y, which constitute the same branch, are agreed to group together; but the values for different roots are to be related separate to each other. Therefore we will have whole series of values of y taken together, as many times as different real values may be taken for some value of x; and in any series it is assumed the values of y, while x is assumed increasing, will either increase or decrease or, after increasing, they will decrease again, or vice versa. From which it is evident maxima or minima are to be given equally in each series of values grouped together into uniform [*i.e.* continuous] functions.

274. The same method will prevail too, towards determining these maxima or minima, as we have made use of in the previous chapter for uniform functions. Indeed since, if we may increase the variable x by the increment ω, the function y may accept this form continually

$$y+\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}+\frac{\omega^3 d^3y}{6dx^3}+\text{etc.},$$

it is necessary, in the case of a maximum or minimum, that the term $\frac{\omega dy}{dx}$ may vanish and there is made $\frac{dy}{dx}=0$. Therefore the roots of this equation $\frac{dy}{dx}=0$ will indicate those values of x, to which maxima or minima correspond to individual values of y of the coherent series. Nor truly will there be any ambiguity, to which of the values the maximum or minimum of the coherent series may be given. Since indeed in the equation $\frac{dy}{dx}=0$ both the variable x and y shall be present, the values of x are unable to be defined, unless with the aid of an equation, by which the relation of the function y will be given in terms of x, and the variable y may be eliminated; but before that can be done, an equation is come upon, from which the value of y may be expressed by a rational or uniform function of x. Hence with the values of x found, and for that the corresponding value of y may be found, which will be the maximum or minimum in the series of successive cohering values, to which it relates.

275. Moreover the decision, whether these values of y shall be maxima or minima, may be put in place in the same manner, which we have indicated before. Clearly a value of the finite term $\frac{ddy}{dx^2}$ may be sought , with each one of the x values found successively expressed there in place of x may be substituted; but likewise the value for y may be put, which agrees with any value of x that pleases; with which done it may be considered, whether the expression $\frac{ddy}{dx^2}$ may arrive at a positive or negative value, and in the first place a minimum will be indicated, truly in the second a maximum. But if truly $\frac{ddy}{dx^2}$ may vanish also, then it will be required to proceed to the formula $\frac{d^3y}{dx^3}$ which in the same case may not vanish, neither a maximum nor a minimum will be had; but if $\frac{d^3y}{dx^3}$ may vanish also, a decision will have to be made from the formula $\frac{d^4y}{dx^4}$ in the same manner as we have given an account of the formula $\frac{ddy}{dx^2}$. And if also in a certain case $\frac{d^4y}{dx^4}$ may vanish, it will be required to be progressing to the fifth differential of y; but continually, as far as it should be necessary to progress, the criteria from the differentials of odd orders are similar to these, which we have deduced from the formula $\frac{d^3y}{dx^3}$. Clearly with these cases it will be required to go to that point with the formulas $\frac{ddy}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}$ etc., as long as such may be reached, which in the proposed case may not vanish ; which if it were a differential of odd order, neither a maximum nor a minimum will be indicated ; but it were of even order, a positive value of this indicates a minimum, and truly a negative value a maximum.

276. We may put the function y to be determined from x by some equation ; if which equation may be differentiated, it may adopt a form of this kind $Pdx + Qdy = 0$. Therefore on making $\frac{dy}{dx} = 0$ there will be $\frac{P}{Q} = 0$ and thus either $P = 0$ or $Q = \infty$. Indeed in the latter equation, if the relation between x and y may be expressed by a rational integral equation, it will be unable to be considered, because either x or y or each may be required to become infinite. Whereby the decision will be left to the equation $P = 0$, the roots of which or the values of x, which it arrives at, after the variable y has been eliminated completely with the aid of the proposed equation, they will indicate cases, in which the values of y become a maxima or minima. Truly towards a judgement, whether a maximum or minimum may be produced, the formula to be resolved $\frac{ddy}{dx^2}$ may be examined. Indeed with the differential equation $Pdx + Qdy = 0$ differentiated anew, if there may be put

$$dP = Rdx + Sdy \text{ and } dQ = Tdx + Vdy ,$$

will give on putting dx constant

$$Rdx^2 + Sdxdy + Tdxdy + Vdy^2 + Qddy = 0.$$

But since now there shall be $\frac{dy}{dx} = 0$, with the equation divided by dx^2 , there becomes

$$R + \frac{Qddy}{dx^2} = 0 \quad \text{and thus} \quad \frac{ddy}{dx^2} = -\frac{R}{Q}.$$

Hence in the differential equation $Pdx + Qdy = 0$ only the quantity *P* may be differentiated on putting *y* constant and there will be produced *Rdx;* then the value of the fraction $\frac{R}{Q}$ may be investigated, which if it were positive will indicate a maximum, but if negative then a minimum.

277. Let *y* be a two–valued function of *x*, which may be determined by this equation $yy + py + q = 0$, with p and q denoting some uniform functions of *x*. Therefore there will be on differentiation $2ydy + pdy + ydp + dq = 0$ and thus $Pdx = ydp + dq$. Therefore on putting $P = 0$ there will be $ydp + dq = 0$ and there will be produced $y = -\frac{dq}{dp}$, and thus *y* is expressed by some uniform function of *x*, thus in order that, whatever value were found for *x*, from that and *y* it may acquire a single determined value. Now truly it will be easier with the elimination of *y* ; for if in the proposed equation $yy + py + q = 0$ in place of *y* the value $-\frac{dq}{dp}$ may be substituted, $dq^2 - pdpdq + qdp^2 = 0$ will be had, which equation divided by dx^2 and resolved, will give all the values of *x*, to which the maxima or minima correspond ; which becomes clearer from the following examples.

EXAMPLE 1

With the proposed equation $yy + mxy + aa + bx + nxx = 0$,
to define the maxima or minima of the function y.

With the equation differentiated we will have

$$2ydy + mxdy + mydx + bdx + 2nxdx = 0,$$

from which there becomes

$$P = my + b + 2nx \text{ and } Q = 2y + mx.$$

Therefore on putting $P = 0$ there is made $y = -\frac{b+2nx}{m}$; which value substituted in the equation itself gives

$$\frac{4nn}{mm}xx+\frac{4nb}{mm}x+\frac{bb}{mm}-2nxx-bx+aa+nxx+bx=0$$

or

$$xx=\frac{4nbx+bb+mmaa}{mmn-4nn},$$

from which there is made

$$x=\frac{2nb\pm\sqrt{(mmnbb+mmn(mm-4n)aa)}}{mmn-4nn}$$

or

$$x=\frac{2nb\pm m\sqrt{(nbb+n(mm-4n)aa)}}{mmn-4nn}\quad\text{and}\quad y=\frac{-mb\mp2\sqrt{(nbb+n(mm-4n)aa)}}{mm-4n}$$

Then on putting *x* alone to be variable there becomes $dP=2ndx$ and thus $R=2n$. But there is

$$Q=2y+mx=\pm\frac{\sqrt{(nbb+n(mm-4n)aa)}}{n},$$

from which

$$\frac{R}{Q}=\frac{\pm2nn}{\sqrt{(nbb+n(mm-4n)aa)}};$$

the numerator of which *2nn* since it shall be positive always, if the upper sign may prevail, will produce a maximum value for *y*, if the lower, then it will produce a minimum. Where the following must be noted.

I. If there were $m=0$, $x=-\frac{b}{2n}$ follows at once from the equation $P=0$, so that there shall be no need for elimination. Twin values of *y* correspond to this value on account of $y=\pm\frac{1}{2n}\sqrt{(nbb-4nnaa)}$, of which the positive one is a maximum, the other negative one a minimum.

II. If there shall be $n=0$, there becomes $y=-\frac{b}{m}$ and *x* extends to infinity and *y* retains the same value in the same infinite interval, thus so that neither shall there be a maximum nor a minimum.

III. If there shall be $mm=4n$, there will be $4nbx+bb+mmaa=0$ or $x=\frac{bb+mmaa}{-mmb}$ and there becomes

$$y=-\frac{b+2nx}{m}=-\frac{2b+mmx}{2m}=-\frac{2b}{2m}+\frac{bb+mmaa}{2mb}=\frac{mmaa-bb}{2mb}.$$

Therefore for this value of $x = -\frac{mmaa+bb}{mmb}$ the other value of y which corresponds, $\frac{mmaa-bb}{2mb}$, will be a maximum or a minimum. But because, so that this value of y may be produced, in the expression

$$y = \frac{-mb \mp 2\sqrt{\left(nbb+n\left(mm-4n\right)aa\right)}}{mm-4n}$$

the lower sign must prevail, there will be a minimum value of *y*.

EXAMPLE 2

With the equation $yy - xxy + x - x^3 = 0$ *proposed, to define the maximum or minimum values of* y.

With the equation differentiated there is produced

$$2ydy - xxdy - 2xydx + dx - 3xxdx = 0.$$

And there becomes

$$P = 1 - 3xx - 2xy \quad \text{and} \quad Q = 2y - xx.$$

Whereby on putting $P = 0$ there will be $y = \frac{1-3xx}{2x}$ and thus with this value substituted

$$\frac{1}{4xx} - \frac{3}{2} + \frac{9xx}{4} - \frac{x}{2} + \frac{3}{2}x^3 + x - x^3 = 0$$

or

$$1 - 6xx + 2x^3 + 9x^4 + 2x^5 = 0.$$

One root of which is $x = -1$, to which $y = 1$ corresponds. But on putting y there becomes $R = -6x - 2y$, therefore [recalling $\frac{ddy}{dx^2} = -\frac{R}{Q}$]

$$\frac{ddy}{dx^2} = \frac{2y+6x}{2y-xx};$$

which in the case $x = -1$ and $y = 1$ will change into – 4, thus so that the value of $y = 1$ shall be a maximum. But the twin value if *y* corresponds to $x = -1$ itself from the equation $yy - y = 0$; truly the other value is $y = 0$, which is neither a maximum nor a minimum. But if that fifth order equation be divided by $x + 1$, an equation is produced, the roots of which are unable to be shown simpler.

EXAMPLE 3

Let this equation be proposed $yy+2xxy+4x-3=0$, from which the maxima or minima values of y are required.

Therefore this equation will be produced by differentiation

$$2ydy+2xxdy+4xydx+4dx=0.$$

And on making $\frac{dy}{dx}=0$ there will be $xy+1=0$ and thus $y=-\frac{1}{x}$, which value substituted into the proposed equation there arises

$$\frac{1}{xx}-2x+4x-3=0=2x^3-3xx+1,$$

the roots of which are $x=1$, $x=1$ and $x=-\frac{1}{2}$. Because now there is

$$\frac{dy}{dx}=-\frac{4xy+4}{2y+2xx}=-\frac{2xy+2}{y+xx},$$

there will be on differentiating $\frac{ddy}{dx^2}=-\frac{2y}{y+xx}$ on putting y to be constant on account of $dy=0$ and on making $xy+1=0$. Whereby these values thus themselves will be had :

x	y	$\frac{ddy}{dx^2}$
1	-1	∞
1	-1	∞
$-\frac{1}{2}$	2	$\frac{-16}{9}$ for a maximum.

Because there becomes $\frac{ddy}{dx^2}=\infty$ for the equations, it may not be determined in this case whether a maximum or minimum may be produced. But because likewise there is made $y+xx=0$, for there will not be $\frac{dy}{dx}=0$ in this case on account of $P=0$ and $Q=0$ in the fraction $\frac{dy}{dx}=-\frac{P}{Q}$; whereby since the first property may not be present, neither can there be considered to be a maximum or minimum. But it may be indicated in this case $x=1$ both values of y to become equal to each other. The nature of which I am going to expand on below, since we may arrive at the fundamentals of curved lines. For even if this material also may be concerned here, yet, lest there shall be no need that that be touched on twice, we will reserve all that to be treated in the following.

[See *Institutionum calculi differentialis, Sectio* III, *Opera Postuma* I, 1862 ; see also E812 in Series I, Vol.29 of the *Opera Omnia*; this work had not yet been translated from Latin, I believe.]

278. Truly in addition other kinds of maximas and minimas may be given, which at this stage cannot be found by the method taught, the nature of which can be explained most easily by two–valued functions. For let *y* be some function of *x* of two–valued, thus so that, whatever value may be given to *x*, two values arise for *y*, either both real or both imaginary. We may put in place these values of y to become imaginary, if there is put $x > f$, but to be real, if $x < f$ is established*;* and on putting $x = f$ both values of y merge into one, which shall be $y = g$. Therefore since, if it is supposed $x > f$, the function *y* may have no real value, if it may come about, so that on putting $x < f$ both values of y become either greater than *g* or less than *g*, in the first place the value $y = g$ will be a minimum, in the second, a maximum, because in that case it is less than both the preceding, in the other case truly greater. And nor by the present method will a maximum or minimum be found, because this [equation] on that account does not make $\frac{dy}{dx} = 0$. But there are also maxima or minima of different kinds, since such shall not be from an account of the preceding and subsequent values in a series of grouped values [*i.e.* in a branch of the function], but only from an account of the two disjoined values either of the preceding or of the following parts.

279. This comes about, if the equation proposed were of this kind

$$y = p \pm (f - x)\sqrt{(f - x)q}$$

with the functions *p* and *q* of x present are not divisible by $f - x$; and there may be obtained a positive value *q*, if $x = f$ may be put in place made either greater of less by a small amount. There is made $p = g$ on putting $x = f$ and it is evident in the case $x = f$ both values of y merge into a single $y = g$; but on putting $x > f$ both values of *y* become imaginary. Therefore if we may put *x* a little less than *f*, for example $x = f - \omega$, the function *p* will change into

$$g - \frac{\omega dp}{dx} + \frac{\omega^2 ddp}{2dx^2} - \text{etc.}$$

and *q* into

$$q - \frac{\omega dq}{dx} + \frac{\omega^2 ddq}{2dx^2} - \text{etc.}$$

from which in this case there will be

$$y = g - \frac{\omega dp}{dx} + \frac{\omega^2 ddp}{2dx^2} - \text{etc.} \pm \omega\sqrt{\omega\left(q - \frac{\omega dq}{dx} + \frac{\omega^2 ddq}{2dx^2} - \text{etc.}\right)}.$$

We may make ω minimal; so that besides ω the higher powers of that may vanish, and there will be $y = g - \frac{\omega dp}{dx} \pm \omega\sqrt{\omega q}$; which values of y will be both smaller than g, if $\frac{dp}{dx}$ were positive, but greater, if negative. From which the two–fold value of $y = g$ in the former case will be a maximum, in the latter truly a minimum.

280. Therefore these maxima and minima thence have their own origin, because in the first place on putting $x = f$ both values of y become equal, but on putting $x > f$ they become imaginary, and moreover real on putting $x < f$; because then on putting $x = f - \omega$ the other irrational part may give higher powers of ω than the rational part. Hence this happens also, if there were $y = p \pm (f - x)^n \sqrt{(f - x)q}$, provided n shall be a whole number > 0 . But since not only the square root, but also some other root of the even power may introduce the same ambiguity of the signs, it will happen likewise, if there were $y = p \pm (f - x)^{\frac{2n+1}{2m}} q$, provided there shall be $2n + 1 > 2m$; therefore there will be $(y - p)^{2m} = (f - x)^{2n+1} q^{2m}$ or $(y - p)^{2m} = (f - x)^{2n+1} Q$. Therefore whenever the function y is expressed by an equation of this kind, thus so that there shall be $2n + 1 > 2m$, so also, on putting $x = f$, the value of y becomes a maximum or minimum ; indeed in the first case, if $\frac{dp}{dx}$ were a positive quantity, truly in the latter case, if $\frac{dp}{dx}$ shall be a negative quantity on putting $x = f$. [Recall from above : $y = g - \frac{\omega dp}{dx} \pm \omega\sqrt{\omega q}$, etc.]. But if in this case there becomes $\frac{dp}{dx} = 0$, then there will be

$$y = g + \frac{\omega^2 ddp}{2dx^2} \pm \omega^{\frac{2n+1}{2m}} q.$$

Therefore unless there shall be $\frac{2n+1}{2m} > 2$, neither a maximum nor a minimum will be present ; but if $\frac{2n+1}{2m} > 2$, then $y = g$ will be a maxima, if $\frac{ddp}{dx^2}$ had a negative value, and truly a minimum, if it were positive; and thus further, if also $\frac{ddp}{dx^2}$ may vanish, a judgement [about the max. or min.] will be put in place.

281. Therefore if y were a function of this kind of x, it can happen, so that in addition to the maxima and minima, which the first method shows, also maxima or minima of this other kind may be present, which will be able to be examined in the manner set out. That which will be shown in the following examples.

EXAMPLE 1

To determine the maxima and minima of the function y, which is defined by this equation

$$yy - 2xy - 2xx - 1 + 3x + x^3 = 0.$$

Towards investigating the maxima or minima of the first kind, the equation may be differentiated and there will be

$$2ydy - 2xdy - 2ydx - 4xdx + 3dx + 3xxdx = 0$$

and on putting $\frac{dy}{dx} = 0$ there will be

$$y = \frac{3}{2} - 2x + \frac{3}{2}xx$$

which value substituted in the first equation gives

$$9x^4 - 32x^3 + 42xx - 24x + 5 = 0,$$

which is resolved into

$$9xx - 14x + 5 = 0 \text{ and } xx - 2x + 1 = 0.$$

The latter gives $x = 1$ twice and there becomes $y = 1$, from which in this case in the fraction

$$\frac{dy}{dx} = \frac{2y-3+4x-3xx}{2y-2x},$$

the denominator also vanishes and thus a maximum or minimum of the first kind is not given ; truly the first equation $9xx - 14x + 5 = 0$ will give $x = 1$ and $x = \frac{5}{9}$ of which values the former is troubled by the same inconvenience as the preceding. But on putting

$x = \frac{5}{9}$ there is made $y = \frac{3}{2} - \frac{10}{9} + \frac{25}{54} = \frac{23}{27}$. And since there shall be $\frac{dy}{dx} = \frac{2y-3+4x-3xx}{2y-2x}$, there becomes

$$\frac{ddy}{dx^2} = \frac{4-6x}{2y-2x} = \frac{-3x+2}{y-x}$$

on account of $dy = 0$ and the numerator $= 0$. Therefore there will be $\frac{ddy}{dx^2} = \frac{9}{8}$, from which here the value $x = \frac{5}{9}$ gives a minimum of the first kind. Then since there shall be $(y-x)^2 = (1-x)^3$, there will be

$$y = x \pm (1-x)\sqrt{(1-x)}$$

and thus on putting $x=1$ a maximum of the second kind will be produced; for on making $x=1-\omega$ there will be $y=1-\omega\pm\omega\sqrt{\omega}$, either of which is less than unity, if indeed ω may be assumed small.

EXAMPLE 2

To find the maxima and minima of the function $y=2x-xx\pm(1-x)^2\sqrt{(1-x)}$.

The equation may be differentiated for the maxima and minima of the first kind and there will be

$$\frac{dy}{dx}=2-2x\mp\tfrac{5}{2}(1-x)\sqrt{(1-x)},$$

which value put in place $=0$ initially will produce $x=1$, and since there shall be

$$\frac{ddy}{dx^2}=-2\pm\tfrac{15}{2}\sqrt{(1-x)},$$

in this case y shall be a maximum of the first kind and there becomes $y=1$. Indeed with the equation $\frac{dy}{dx}=0$ divided by $1-x$ there will be $4\mp5\sqrt{(1-x)}=0$ or $16=25-25x$, from which there becomes $x=\frac{9}{25}$ and $\frac{ddy}{dx^2}=-2\pm3$. Whereby if the upper sign prevails, there will be a minimum $y=\frac{2869}{3125}$; but if the lower sign may prevail, there will be $y=\frac{821}{3125}$ which may be seen to be a maximum; but truly only the upper sign can be considered, because $4\mp5\sqrt{(1-x)}$ cannot be $=0$, unless there shall be $\sqrt{(1-x)}=+\frac{4}{5}$. Therefore we have found the maximum of the first kind in the case $x=1$ and $y=1$ and the minimum in the case $x=\frac{9}{25}$ and $y=\frac{2869}{3125}$. Truly from the other kind a maximum will be produced also, if $x=1$, in which case there becomes $y=1$. For on putting $x=1-\omega$, $y=1-\omega\omega\pm\omega^2\sqrt{\omega}$ will be <1 in each case. And thus here, if $x=1$, the two maxima of the first and of the higher kind come together and constitute as if a mixed maximum.

282. From these examples not only can the nature of this other kind of maxima and minima be elicited, but also functions of this kind can be formed as it pleases, which permit maxima and minima of the second kind. But just as, if any function were proposed, it will be possible to find, whether or not it shall be provided with maximas or minimas of this kind, that we will show in the following section, because on that account the nature of curved lines is made clear especially in this investigation. Indeed the rest is understood easily, if y were a function of x of this kind, which takes a maximum or minimum of the second kind, then also in turn x becomes a function of y of this kind. Because from the equation $(y-x)^2=(1-x)^3$ on making $x=1$, y adopts a maximum value of the second kind, if the variables y and x may be interchanged, this equation $(y-x)^2=(1-y)^3$ also shows for y a function of x of this kind, which may have a maximum of the second kind. For on making $x=1$ there becomes $(1-y)^2=(1-y)^3$ and hence there is twice $y=1$ and $y=0$ once. But if there may be put $x=1+\omega$, there will be $(1+\omega-y)^2=(1-y)^3$; from which if we may put $y=1+\varphi$, there will be $(\omega-\varphi)^2=(-\varphi)^3=-\varphi^3$ and thus φ must become negative. Therefore let $y=1-\varphi$; there will be $(\omega+\varphi)^2=\varphi^3$, and thus on assuming φ minimal φ^3 may vanish before φ^2, by necessity ω must become negative ; hence no real values of y correspond to the value $x=1+\omega$. But on putting $x=1-\omega$ and $y=1-\varphi$ on account of $(\varphi-\omega)^2=\varphi^3$ there will be $\varphi=\omega\pm\omega\sqrt{\omega}$ [on interchanging φ and ω] and thus $y=1-\omega\mp\omega\sqrt{\omega}$, from which each value of y corresponding to $x=1-\omega$ is less than the value $y=1$, which corresponds to the value $x=1$; and consequently that value of y will be a maximum.

283. Up to the present we have considered only bi–valued functions, the maxima or minima of which, because both values can be expressed easily by the resolution of the quadratic equation, can be recalled for examination. But if the function y may be expressed by a higher equation, the method related before, by which we have investigated the maxima and minima of the first kind, will be able to be used with the same success. Truly we may reserve the finding of maxima and minima of the second kind to the following section. Therefore just as it may be required to treat tri–valued and multivalued functions, we may show by some examples.

EXAMPLE 1

The function y, of which the maxima or minima are to be found, may be defined by this equation

$$y^3+x^3=3axy.$$

By differentiation this equation becomes

$$3y^2dy+3xxdx=3axdy+3aydx$$

and thus

$$\frac{dy}{dx}=\frac{ay-xx}{yy-ax}.$$

Therefore the maximum or minimum will be given, if there were $ay=xx$ or $y=\frac{xx}{a}$, which value substituted into the proposed equation gives

$$\frac{x^6}{a^3} + x^3 = 3x^3 \quad \text{or} \quad x^6 = 2a^3x^3.$$

Therefore there will be $x = 0$ three times, in which case also the denominator becomes $yy - ax = 0$ on account of $y = \frac{xx}{a} = 0$. Therefore whether in this case a maximum or minimum may be produced, will become apparent, if we may give *x* a value disagreeing minimally from 0. Therefore let $x = \omega$ and $y = \varphi$ on account of $\varphi^3 + \omega^3 = 3a\omega\varphi$ there becomes either $\varphi = \alpha\sqrt{\omega}$ or $\varphi = \beta\omega^2$. In the first case there will be $\alpha^3\omega\sqrt{\omega} = 3\alpha a\omega\sqrt{\omega}$ and thus $\alpha = \sqrt{3a}$. Hence on putting $x = \omega$ there will be $y = +\sqrt{3a\omega}$. From which even if ω may be unable to accept a negative value, yet one of the two other values of *y* will be greater than 0, the other less and hence $y = 0$ will become neither a maximum nor a minimum. But if there may be put in place $\varphi = \beta\omega^2$, there will be $\omega^3 = 3a\beta\omega^3$ and thus $\beta = \frac{1}{3a}$ and $\varphi = \frac{\omega^2}{3a}$. Therefore in this case, either there is taken $x = +\omega$ or $= -\omega$, the value of $y = \varphi$ will be greater than zero and thus in this case $y = 0$ will be minimum. Therefore there remains the third case to be examined from the equation $x^3 = 2a^3$, which gives $x = a\sqrt[3]{2}$ and $y = a\sqrt[3]{4}$. Which whether each shall be a maximum or minimum, the second differential may be sought from the equation $\frac{dy}{dx} = \frac{ay - xx}{yy - ax}$, which on account of $dy = 0$ and $ay - xx = 0$ will be $\frac{ddy}{dx^2} = \frac{-2x}{yy - ax}$, the value of which for the present case is $-\frac{2a\sqrt[3]{2}}{2a^2\sqrt[3]{2} - aa\sqrt[3]{2}} = -\frac{2}{a}$, which indicates the value of *y* to be a maximum.

EXAMPLE 2

If the function y may be defined by this equation, $y^4 + x^4 + ay^3 + ax^3 = b^3x + b^3y$, *to find the maximum or minimum values of this.*

Since by differentiation there may arise

$$4y^3dy + 3ayydy - b^3dy = b^3dx - 3axxdx - 4x^3dx,$$

there will be

$$\frac{dy}{dx} = \frac{b^3 - 3axx - 4x^3}{4y^3 + 3ayy - b^3}$$

and it will be required to put $b^3 = 3axx + 4x^3$. The question thus is reduced to this, so that the maxima and minima of the uniform function $b^3 - 3axx - 4x^3$ may be investigated, which likewise shall be maxima or minima of the function *y*. Let $a = 2$ and $b = 3$ or this equation may be proposed $y^4+x^4+2y^3+2x^3=27x+27y$; there will be $\frac{dy}{dx}=\frac{27-6xx-4x^3}{4y^3+6yy-27}$ and $4x^3+6xx-27=0$, which divided by $2x-3=0$ gives $2xx+6x+9=0$; since the roots of which latter equation shall be imaginary, there will be $x=\frac{3}{2}$ and $y^4+2y^3-27y=\frac{459}{16}$, of which the individual roots will be either the maxima or minima. But since there shall be $\frac{dy}{dx}=\frac{27-6xx-4x^3}{4y^3+6yy-27}$, there will be $\frac{ddy}{dx^2}=\frac{-12x-12xx}{4y^3+6yy-27}$, which on putting $x=\frac{3}{2}$, if positive, will indicate a minimum, on the contrary truly a maximum.

EXAMPLE 3

If there were $y^m+ax^n=by^px^q$, to determine the maxima and minima of y.

By differentiation there is made

$$\frac{dy}{dx}=\frac{qby^px^{q-1}-nax^{n-1}}{my^{m-1}-pby^{p-1}x^q},$$

from which on putting $=0$ there will be initially $x=0$, if indeed n and q were greater than unity, and likewise $y=0$. In which case either a maximum or minimum may be given, the approximate values are to be investigated, since also the denominator becomes $=0$; which investigation will depend chiefly on the exponents. Therefore truly the equation $\frac{dy}{dx}=0$ will give $y^p=\frac{na}{qb}x^{n-q}$, which value substituted into the equation on putting $\frac{na}{qb}=g$ will give

$$g^{\frac{m}{p}}x^{\frac{mn-mq}{q}}+ax^n=\frac{na}{q}x^n \quad \text{or} \quad g^{\frac{m}{p}}x^{\frac{mn-mq-np}{p}}=\frac{(n-q)a}{q},$$

from which there becomes

$$x=\left(\frac{(n-q)a}{q}\right)^{p:(mn-mq-np)}:g^{m:(mn-mq-np)}$$

and likewise the value of y will become known. Next it is required to discern, whether the second order differential equation

$$\frac{ddy}{dx^2}=\frac{q(q-1)by^px^{q-2}-n(n-1)ax^{n-2}}{my^{m-1}-pby^{p-1}x^q}$$

may take a positive or negative value, so that from the former a minimum, truly from the latter a maximum, may be pronounced.

EXAMPLE 4

If there were $y^4 + x^4 = 4xy - 2$, *to assign the maxima and minima of the function y.*

With the differentiation put in place there becomes

$$\frac{dx}{dy} = \frac{y-x^3}{y^3-x}$$

and hence there arises $y = x^3$; therefore there will be $x^{12} = 3x^4 - 2$ or $x^{12} - 3x^4 + 2 = 0$, which equation is resolved into these $x^4 - 1 = 0$ and $x^8 + x^4 - 2 = 0$ and the latter into $x^4 - 1 = 0$ and $x^4 + 2 = 0$. Hence there will be twice either $x = +1$ or $x = -1$; and truly in each case the denominator of the fraction $\frac{dy}{dx}$ also vanishes. Therefore towards finding, whether in these cases a maximum or a minimum should be considered, we may put $x = 1 - \omega$ and $y = 1 - \varphi$; there will be

$$1 - 4\varphi + 6\varphi^2 - 4\varphi^3 + \varphi^4 + 1 - 4\omega + 6\omega^2 - 4\omega^3 + \omega^4$$
$$= 4 - 4\omega - 4\varphi + 4\omega\varphi - 2$$

and thus

$$4\omega\varphi = 6\varphi^2 + 6\omega^2 - 4\varphi^3 - 4\omega^3 + \omega^4$$

and on account of the minimal ω and φ, $4\omega\varphi = 6\varphi^2 + 6\omega^2$. Therefore the value of φ will be imaginary, if ω may be taken either positive or negative. Or if *y* and *x* may designate the coordinates of the curve, in that case $x = 1$ and $y = 1$ will have a common point. Therefore nor can the value be had here for a maximum or minimum. because the preceding and following values, since they must be compared with which, therefore become imaginary.

284. If the equation, between which the relation between *x* and *y* is expressed, were prepared thus, so that a function of *y* may be equal to a function of *x*, for example $Y = X$, towards finding the maxima or minima it will be required to put $dX = 0$; therefore *y* becomes a maximum or minimum in the same cases, in which *X* becomes a maximum or minimum. In a similar manner, if *x* may be considered as a function of *y*, *x* becomes a maximum or minimum, if $dY = 0$, that is, if *Y* were a maximum or minimum. Yet hence neither does it follow likewise *y* and *x* become maxima or minima. For if there were $2ay - yy = 2bx - xx$, *y* will be a maximum or minimum, if there were $x = b$, and there will be $y = a \pm \sqrt{(aa - bb)}$. Truly on the other hand *x* becomes a maximum or minimum, if there were $y = a$, and there is made $x = b \pm \sqrt{(bb - aa)}$ nor therefore does *y* become a maximum or minimum, if $x = b \pm \sqrt{(bb - aa)}$, in which case still *x* is a maximum or minimum. Moreover in this case, if y may have maxima or minima values, so *x* will be without this innate character; for *y* cannot become a maximum or minimum, unless $a > b$, in which case the maximum or minimum of *x* is made imaginary.

285. Then indeed also it may come about, so that not all the roots of the equation $dX = 0$ may give maxima or minima values for *y;* if indeed that equation had two equal roots, from that neither a maximum nor minimum follows; and this likewise comes about, if some even number of roots were equal to each other. Thus if the equation may be proposed $b(y-a)^2 = (x-b)^3 + c^3$, because with the differentials taken there becomes $2bdy(y-a) = 3dx(x-b)^2$, the function *y* becomes neither a maxima nor a minima on putting $x = b$, because here on that account two equal roots occur. But if *x* may be considered as a function of *y*, that becomes a maximum or minimum, if there may be put $y = a$ and $x = b - c$, there will be a minimum. And finally because in equation of this kind $Y = X$ the variables *x* and y are not allowed to be mixed together, if to *x* itself a value may be given, which shall be a root of the equation $dX = 0$, all the values of *y,* however many shall be real, will be maxima or minima; which does not come about, if both variables were mixed together in the equation.

286. Besides what remains concerning the nature of maxima and minima requiring to be explained, we preserve for the following section [as noted previously], because they will be able to be represented and explained more conveniently with the aid of figures brought to mind. Therefore we may proceed to functions, which are composed from several variables, and we may investigate the values, which it may be necessary to attribute to the individual variables, in order that that function may obtain either a maximum or minimum value. And indeed in the first place it may be apparent, if the variables were not mixed among themselves, thus so that a function proposed shall be of this kind $X + Y$ with X present a function of *x* and Y of y only, then the proposed function $X + Y$ becomes a maximum, if *X* and *Y* likewise may emerge a maximum, and a minimum, if likewise *X* and *Y* become a minimum. Hence towards finding a maximum we may inquire about the values of *x*, for which X becomes a maximum, and in a similar manner the values of *y*, for which *Y* is made a maximum, and these values found for *x* and *y* bring about the maximum of the function $X + Y$, which similarly will be understood concerning the minimum. Hence one must beware, lest two values of *x* and *y* of different natures may be combined, of which the former may return X a maximum, truly the latter *Y* a minimum, or the contrary. For if this should happen, the function $X + Y$ may neither become a maximum nor a minimum. But a function of this kind $X - Y$ is made a maxima, if X were a maximum and likewise *Y* a minimum; truly on the other hand $X - Y$ becomes a minimum, if *X* were a minimum and *Y* a maximum. But if each function *X* and *Y* may be put in place either a maximum or minimum, the difference of these $X - Y$ neither may become a maximum nor a minimum; which all have been explained before both in a clear and transparent manner from the nature of maxima and minima.

287. But if the values of the maxima or minima may be sought of functions of two variables, the investigation is subject to much more caution, than if it were a single variable. For not only the cases for each variable, for which a maximum or minimum is produced, are to be carefully distinguished, but also from these two of the same kind are to be taken together, so that the function proposed is made a maximum or a minimum ; that which will become clearer from examples.

EXAMPLE 1

Let this function of the two variables x and y be proposed
$y^4-8y^3+18y^2-8y+x^3-3xx-3x$ *and the values for y and x to be substituted may be sought, so that this function may obtain a maximum or minimum value.*

Because this expression is resolved into two parts of this kind $Y+X$, of which that one is a function of *y*, and this truly a function of *x* only, the cases may be investigated, in which each becomes either maxima or minima. Therefore since there shall be

$$Y=y^4-8y^3+18y^2-8y$$

there will be

$$\frac{dY}{dy}=4y^3-24y^2+36y-8;$$

from which expression put equal to zero becomes divided by 4

$$y^3-6y^2+9y-2=0,$$

the roots of which are $y=2$ and $y=2\pm\sqrt{3}$. Therefore since there shall be $\frac{ddY}{4dy^2}=3yy-12y+9$, in the case $y=2$ a maximum will be produced. For the remaining two roots $y=2\pm\sqrt{3}$, which arise from the equation $yy-4y+1=0$, there is made $\frac{ddY}{12dx^2}=yy-4y+3=2$, from which each gives a minimum. Moreover there will be in these cases, as follows :

$y=2$	$Y=8$ maximum
$y=2-\sqrt{3}$	$Y=-1$ minimum
$y=2+\sqrt{3}$	$Y=-1$ minimum

In a similar manner since there shall be

$$X=x^3-3xx-3x,$$

there is

$$\frac{dX}{dx}=3xx-6x-3,$$

from which this equation arises

$$xx = 2x + 1$$

and $x = 1 \pm \sqrt{2}$. Now there is $\frac{ddX}{6dx^2} = x - 1 = \pm\sqrt{2}$. Hence the root $x = 1 + \sqrt{2}$ gives a minimum, surely $X = -5 - 4\sqrt{2}$, and $x = 1 - \sqrt{2}$ gives a maximum, surely $X = -5 - 4\sqrt{2}$. On account of which the proposed formula

$$y^4 - 8y^3 + 18y^2 - 8y + x^3 - 3xx - 3x$$

becomes a maximum, if there is put $y = 2$ and $x = 1 - \sqrt{2}$, and there will be produced $X + Y = 3 + 4\sqrt{2}$. But the same formula $X + Y$ makes a minimum, if there is assumed either $y = 2 - \sqrt{3}$ or $y = 2 + \sqrt{3}$ and $x = 1 + \sqrt{2}$; in each case there will be $X + Y = -6 - 4\sqrt{2}$.

EXAMPLE 2

If this function of two variables may be proposed
$y^4 - 8y^3 + 18y^9 - 8y - x^3 + 3xx + 3x$, *which may be investigated in which cases there becomes maxima or minima.*

On placing, as we had in the preceding example,

$$Y = y^4 - 8y^3 + 18y^2 - 8y \text{ and } X = x^3 - 3xx - 3x$$

the proposed formula will be $Y - X$ and thus it becomes a maxima, if Y were a maximum and X a minimum. Therefore since now we have elicited these cases before, it may be apparent that $Y - X$ takes a maximum value, if there is put $y = 2$ and $x = 1 + \sqrt{2}$; and there comes about $Y - X = 13 + 4\sqrt{2}$. Truly the minimum value of $Y - X$ will emerge, if Y shall be a minimum and X a maximum, which will arise on putting $y = 2 + \sqrt{3}$ et $x = 1 - \sqrt{2}$ *;* moreover there becomes $Y - X = 4 - 4\sqrt{2}$. Moreover in each example it is apparent these values, which we have found, neither are the maxima or minima of all ; for if there may be put in turn for argument's sake $y = 100$ and $x = 0$, without doubt a greater value may be produced for that, which we have found ; and in a similar manner on putting $y = 0$ and also $x = -100$ or $x = +100$ a smaller value may be produced, than these are which we have found for the minimum case. Therefore the idea expressed above is to be understood properly, as we have given concerning the nature of maximas and minimas, clearly that value is to be called a maximum, which shall be greater both for the preceding values as for the following nearest neighbouring ones, but the minimum will be that, which should be less than both the preceding values as with the following. Thus in this example the value of $Y - X$, which is produced on putting $y = 2$ and $x = 1 + \sqrt{2}$, is greater than these, which result, if there may be put $y = 2 \pm \omega$ and $x = 1 + \sqrt{2} \pm \varphi$, on taking small enough quantities for ω and φ.

288. From these examples explained there will be an easier way towards finding the general solution. Let *V* denote some function of the two variables *x* and *y* and values for *x* and *y* shall be required to be found, which lead to a maximum or minimum value of the function *V*. Therefore since towards bringing that about for each variable *x* and *y* a determined value must be given, we may put the one *y* now to have that value, which is required returning a maximum or minimum value for the function *V*, and with this put in place there will be a need, so that for the other *x* a suitable value may be found also, which happens, provided the function *V* is differentiated on putting only *x* to be variable and the differential is taken equal to nothing. In a similar manner if we may put in place the variable *x* now to have that value, which shall be suitable making the function *V* either a maximum a minimum, the value of *y* may be found by differentiating *V* on putting only *y* to be variable and by putting this variable equal to zero. Hence if the differential of the function *V* were $= Pdx + Qdy$, there will required to be both $P = 0$ and $Q = 0$, from which two equations the values of each variable *x* and *y* will be able to be elicited.

289. Because truly from this agreement the values may be found for *x* and *y* without any trouble , by which the function *V* is returned either a maximum or minimum, the cases in which either a maximum or minimum arise, are properly distinguished from each other in turn. Indeed in order that the function *V* is made a maximum, it is necessary, that both variables act together towards this; for if the one may show a maximum, the other a minimum, that function may avoid becoming either a maximum or a minimum. On account of which with the values of *x* and *y* found from the equations $P = 0$ and $Q = 0$, it is to be inquired, whether both likewise adopt either maximum or minimum values of the function *V*; and then finally, since the value of each variable was ascertained, hence elicited for a maximum value, we are able to affirm the function in this case to adopt a maximum value. Because the same will be understood concerning the minimum, thus so that the function *V* may be unable to arrive at a minimum value, unless both the variables *x* and *y* likewise may produce a minimum. Hence therefore all these cases will be rejected, in which one of the variable may be take to indicate a maximum value, the other truly a minimum. Yet meanwhile truly it comes about, that the values of one or also of each variable to arise from the equations $P = 0$ and $Q = 0$ may show neither a maximum nor a minimum, which cases hence equally are to be rejected as absolutely without meaning.

290. But whether the values found for x and y may prevail for maxima or minima, will be investigated for each separately in a similar manner, by which we have used above, with a single variable being present. Clearly towards a decision about the variable x being established, the other y may be regarded as constant, and since there shall be $dV = Pdx$ or $\frac{dV}{dx} = P$, P may be differentiated anew on putting y constant, so that there may be produced $\frac{ddV}{dx^2} = \frac{dP}{dx}$ and it may be examined, whether the value of $\frac{dP}{dx}$ after the values found previously were substituted in place of x and y, it is made positive or negative; for in the first case it will indicate a minimum, in the latter truly a maximum. In a similar manner with x placed constant there shall be $dV = Qdy$ or $\frac{dV}{dy} = Q$, Q may be differentiated anew on putting y alone to be variable and with the value $\frac{dQ}{dy}$ substituted in place of x and y with the values, which have been found from the equations $P = 0$ and $Q = 0$; which if it were positive, will declare a minimum, on the contrary truly a maximum. Hence therefore it is deduced, if from the values for x and y found, the formulas $\frac{dP}{dx}$ and $\frac{dQ}{dy}$ may adopt values affected by different signs, clearly the one positive, the other negative, then the function V neither is effecting a maximum nor a minimum; but if each formula $\frac{dP}{dx}$ and $\frac{dQ}{dy}$ becomes positive, a minimum will result, and on the contrary, if each becomes negative, a maximum.

291. But if truly either the formula $\frac{dP}{dx}$ or $\frac{dQ}{dy}$ or also each should vanish, if the values found for x and y may be substituted, then it will be required to progress to the following differentials $\frac{ddP}{dx^2}$ and $\frac{ddQ}{dy^2}$; which unless equally they may vanish, neither a maximum nor a minimum can be considered ; but if they do not vanish, a decision will be required from the following differentials $\frac{d^3P}{dx^3}$ and $\frac{d^3Q}{dy^3}$ and in a similar manner, so that it has been made for the formulas $\frac{dP}{dx}$ and $\frac{dQ}{dy}$. But so that, from which cases it may come about from this use, the value $x = \alpha$ may have appeared which if the formula $\frac{dP}{dx}$ may be returned vanishing, it is necessary that $\frac{dP}{dx}$ may have the factor $x - \alpha$; which value if it were solitary nor likewise should it have another partner equal to itself, would indicate neither a maximum nor a minimum; which likewise comes about, if $\frac{dP}{dx}$ had the factor $(x-\alpha)^3$ or $(x-\alpha)^5$ etc. But if the factor were $(x-\alpha)^2$ or $(x-\alpha)^4$ etc., then indeed a maximum or minimum will be indicated ; but in addition it will be required to be observed, whether it may agree with the case indicated by y.

292. But the labour with these cases on progressing to higher differentials will be lightened wonderfully, if indeed we may put, so that we may include the more general case, $\alpha x+\beta=0$ to be found and the formula $\frac{dP}{dx}$ to have the factor $(\alpha x+\beta)^2$, thus so that there shall be $\frac{dP}{dx}=(ax+\beta)^2 T$, because there is $ax+\beta=0$, there is made $\frac{d^3P}{dx^3}=2\alpha^2 T$ and hence on account of positive $2\alpha^2$, from the quantity T itself the judgement will be able to be resolved ; which it will pronounce a minimum if it may adopt a positive value, truly the opposite for a maximum. And this same help in the investigation of maxima and minima can be given, if a single variable shall be present, so that on no occasion is there a need to ascend to higher differentials. Indeed why also is there a need to precede to differentials of the second order ; for if from the equation $P=0$ there is made $\alpha x+\beta=0$, it is necessary that P may have the factor $\alpha x+\beta$ *;* let there be $P=(ax+\beta)T$, and since there shall be

$$\frac{dP}{dx}=\alpha T+(\alpha x+\beta)\frac{dT}{dx}$$

on account of $\alpha x+\beta=0$ there will be $\frac{dP}{dx}=\alpha T$ and hence now the other factor T itself , exactly as the value of αT were either positive or negative, will indicate at once either a minimum or a maximum.

293. From these precepts discussed it will not be difficult to investigate the cases, if some function involving two variables were proposed, in which this function becomes either maxima or minima. If which in addition were required to be noted, that itself will suggest the working out of examples, on account of which it may be useful to illustrate the given rules by some examples.

EXAMPLE 1

Let this function of two variables be proposed $V=xx+xy+yy-ax-by$ *;*
which may be examined, in which cases it becomes either maxima or minima.

Since there shall be $dV=2xdx+ydx+xdy+2ydy-adx-bdy$, if it may be compared with the general form $dV=Pdx+Qdy$, there will be

$$P=2x+y-a \text{ and } Q=2y+x-b,$$

from which there equations will be formed

$$2x+y-a=0 \text{ and } 2y+x-b=0,$$

from which taken together with y being eliminated there becomes $x-b=4x-2a$ and thus

$$x = \frac{2a-b}{3} \text{ and } y = a - 2x = \frac{2b-a}{3}$$

Therefore since there shall be

$$\frac{dP}{dx} = 2 \quad \text{and} \quad \frac{dQ}{dy} = 2,$$

each will show a minimum; from which we conclude the formula $xx + xy + yy - ax - by$ becomes a minimum, if there is put $x = \frac{2a-b}{3}$ et $y = \frac{2b-a}{3}$, and there will be produced in this manner

$$V = \frac{-3aa+3ab-3bb}{9} = \frac{-aa+ab-bb}{3};$$

which since it shall be single, it will be the minimum of everything. Therefore in a single way it may become

$$xx + xy + yy - ax - by = \frac{-aa+ab-bb}{3}$$

and because it cannot become less, this equation will be impossible

$$xx + xy + yy - ax - by = \frac{-aa+ab-bb}{3} - cc.$$

EXAMPLE 2

If this formula $V = x^3 + y^3 - 3axy$ may be proposed, the cases may be sought, in which the maximum or minimum values of V may be arrived at.

On account of $dV = 3xxdx + 3yydy - 3aydx - 3axdy$ there will be

$$P = 3xx - 3ay \text{ and } Q = 3yy - 3ax,$$

from which there is made

$$ay = xx \text{ and } ax = yy.$$

Therefore since there shall be $yy = x^4 : aa = ax$, there will be $x^4 - a^3x = 0$ and thus either $x = 0$ or $x = a$. In the first case there becomes $y = 0$, truly in the latter $y = a$. Hence because there is

$$\frac{dP}{dx} = 6x, \quad \frac{ddP}{dx^2} = 6 \quad \text{and} \quad \frac{dQ}{dy} = 6y, \quad \frac{ddQ}{dy^2} = 6,$$

therefore in the first case, from which $x = 0$ and $y = 0$, neither a maximum nor a minimum may result. Truly in the second case, from which both $x = a$ and $y = a$, a minimum will be produced, if indeed a were a positive quantity, and there becomes

$V = -a^3$, but which value is only less with the nearby preceding and following values ; for without doubt V can adopt a much value, if to each variable x and y negative values may be given.

EXAMPLE 3

This function shall be proposed $V = x^3 + ayy - bxy + cx$, the maxima and minima values of which are sought.

Because there is $dV = 3xxdx + 2aydy - bydx - bxdy + cdx$, there will be

$$P = 3xx - by + c \text{ and } Q = 2ay - bx,$$

from which values with the equations put equal to nothing there will be $y = \frac{bx}{2a}$ and thus

$$3xx - \frac{bbx}{2a} + c = 0 \text{ or } xx = \frac{2bbx - 4ac}{12a}$$

from which there becomes

$$x = \frac{bb \pm \sqrt{(b^4 - 48aac)}}{12a}$$

Therefore unless there shall be $b^4 - 48aac > 0$, neither a maximum nor a minimum may be considered.
Therefore we may put to be $b^4 - 48aac = bbff$, so that there shall be $c = \frac{bb(bb - ff)}{48aa}$; there will be

$$x = \frac{bb \pm bf}{12a} \quad \text{and} \quad y = \frac{bb(b \pm f)}{24aa}$$

Again, because there is

$$\frac{dP}{dx} = 6x \text{ and } \frac{dQ}{dy} = 2a,$$

there becomes

$$\frac{dP}{dx} = \frac{b(b \pm f)}{2a}.$$

Therefore unless $2a$ and $\frac{b(b \pm f)}{2a}$ shall be quantities of the same sign, neither a maximum nor a minimum can be considered. But if both shall be either positive or negative, which comes about, if the product of these $b(b \pm f)$ were positive, then the function V will come out a minimum, if a shall be a positive quantity, on the contrary truly a maximum, if a shall be a negative quantity. Hence if there were $f = 0$ or $c = \frac{b^4}{48aa}$, on account of the positive quantity bb the function V becomes a minimum, if a shall be a positive quantity and there may be put $x = \frac{bb}{12a}$ and $y = \frac{b^3}{24aa}$; truly on the other hand if a shall be negative, these substitutions produce a maximum. If there shall be $f < b$, for the two cases there arises a maximum or a minimum; but if $f > b$, then only the case $x = \frac{b(b+f)}{12a}$ and $y = \frac{bb(b+f)}{24aa}$ will give a maximum or a minimum, exactly as a were either negative or positive. Let there be $a = 1$, $b = 3$ and $f = 1$, so that this formula may be had $V = x^3 + yy - 3xy + \frac{3}{2}x$; this makes a minima on account of positive a, if there is put either $x = 1$ and $y = \frac{3}{2}$ or $x = \frac{1}{2}$ and $y = \frac{3}{4}$. In the first case there arises $V = \frac{1}{4}$ in the latter truly $V = \frac{5}{16}$. Yet meanwhile it is apparent with negative numbers in place of x much smaller values can arise for V. Therefore it must be understood thus the value of $V = \frac{1}{4}$ to be smaller, than if there is put $x = 1 + \omega$ and $y = \frac{3}{2} + \varphi$, provided ω and φ shall be small numbers, either positive or negative ; but the limit, that ω must not cross, is $-\frac{15}{4}$; for if $\omega < -\frac{15}{4}$, it will happen that V is made less than $\frac{1}{4}$.

EXAMPLE 4

To find the maxima or minima of this function

$$V = x^4 + y^4 - axxy - axyy + ccxx + ccyy.$$

With the differentials taken there will be

$$P = 4x^3 - 2axy - ayy + 2ccx \text{ and } Q = 4y^3 - axx - 2axy + 2ccy,$$

with which values put equal to nothing, if they may be taken from each other, there will be

$$4x^3 - 4y^3 + axx - ayy + 2ccx - 2ccy = 0;$$

which since it may be divided by $x - y$, in the first place there shall be $y = x$ and $4x^3 - 3axx + 2ccx = 0$, which gives

$$x = 0 \text{ and } 4xx = 3ax - 2cc \text{ or } x = \frac{3a \pm \sqrt{(9aa - 32cc)}}{8}.$$

If we may assume $x = 0$, there will be also $y = 0$ and on account of

$$\frac{dP}{dx} = 12xx - 2ay + 2cc \text{ and } \frac{dQ}{dy} = 12yy - 2ax + 2cc$$

the function V becomes a minimum $=0$. But if we may put $x=y=\frac{3a\pm\sqrt{(9aa-32cc)}}{8}$, if indeed there were $9aa>32cc$, on account of $4xx=3ax-2cc$ there will be

$$\frac{dP}{dx}=\frac{dQ}{dy}=12xx-2ax+2cc=7ax-4cc$$
$$=\frac{21aa-32cc\pm7a\sqrt{(9aa-32cc)}}{8};$$

which value since it shall always positive on account of $32cc<9aa$, the value V in this case always shall be a minimum and there will be

$$V=\tfrac{-27}{256}a^4+\tfrac{9}{16}aacc-\tfrac{1}{2}c^4\mp\tfrac{a}{256}(9aa-32cc)^{\frac{3}{2}}.$$

But we may divide the equation $4x^3-4y^3+axx-ayy+2ccx-2ccy=0$ by $x-y$ and there becomes $4xx+4xy+4yy+ax+ay+2cc=0$. But from the equation $P=0$ there will be $yy=-2xy+\frac{4}{a}x^3+\frac{2ccx}{a}$, with which value substituted in that there becomes ,

$$y=\frac{16x^3+4axx+aax+8ccx+2acc}{4ax-aa}$$

Truly with that given

$$y=-x\pm\sqrt{\frac{4x^3+axx+2accx}{a}},$$

from which there is effected

$$16x^3+8axx+8ccx+2acc=(4x-a)\sqrt{\left(4ax^3+aaxx+2accx\right)},$$

[a small correct had been made here to the original equation in the First Ed. and henceforth] which reduced to rationality gives

$$\left.\begin{aligned}&256x^6+192ax^5+80aax^4+4a^3x^3-a^4x^2\\&-2a^3ccx+4a^2c^4+256cc+160acc\\&+48aacc+32ac^4+64c^4\end{aligned}\right\}=0;$$

the roots of which, if which it may have real, indicate the maxima or minima of the function V, if indeed $\frac{dP}{dx}$ and $\frac{dQ}{dy}$ become quantities affected with the same sign.

EXAMPLE 5

To find the maxima and minima of this expression

$$x^4 + mxxyy + y^4 + aaxx + naaxy + aayy = V .$$

With the differentiation performed there will be

$$P = 4x^3 + 2mxyy + 2aax + naay = 0,$$
$$Q = 4y^3 + 2mxxy + 2aay + naax = 0,$$

which equations in turn either subtracted or added give

$$(4xx + 4xy + 4yy - 2mxy + 2aa - naa)(x - y) = 0,$$
$$(4xx - 4xy + 4yy + 2mxy + 2aa + naa)(x + y) = 0,$$

which divided by $x - y$ and $x + y$ and again added or subtracted give

$$4xx + 4yy + 2aa = 0 \text{ and } 4xy - 2mxy - naa = 0 .$$

From the latter of which there is made $y = \frac{naa}{2(2-m)x}$; but the former does not admit real values. Therefore we have the three cases.

I. There shall be $y = x$ and there shall be $4x^3 + 2mx^3 + 2aax + naax = 0$, from which there becomes either $x = 0$ or $2(2 + m)xx + (2 + n)aa = 0$. Let $x = 0$; there will be also $y = 0$ and on account of

$$\frac{dP}{dx} = 12xx + 2myy + 2aa \text{ and } \frac{dQ}{dy} = 12yy + 2mxx + 2aa$$

in this case $V = 0$ is made a minimum, if indeed the coefficient aa were positive. The other case gives $xx = -\frac{(n+2)aa}{2(m+2)}$, which cannot be real, unless $\frac{n+2}{m+2}$ shall be a negative number. Let there be $\frac{n+2}{m+2} = -2kk$ or $n = -2kkm - 4kk - 2$; there will be $x = \pm ka$ and $y = \pm ka$. But

$$\frac{dP}{dx} = 12kkaa + 2mkkaa + 2aa \quad \text{and} \quad \frac{dQ}{dy} = 12kkaa + 2mkkaa + 2aa ;$$

which since they shall be equal, V will be either a minimum or a maximum, exactly as these quantities shall be either positive or negative.

II. Let $y=-x$ and there will be $2(m+2)x^3=(n-2)aax$, hence either $x=0$ or $xx=\frac{(n-2)aa}{2(m+2)}$. The first root $x=0$ has been dealt with in the preceding. Truly the latter will be real, if $\frac{(n-2)aa}{2(m+2)}$ were a positive quantity, and since there is made $\frac{dP}{dx}=\frac{dQ}{dy}$, it will produce either a maximum or minimum.

III. Let there be $y=\frac{naa}{2(2-m)x}$; there will be

$$4x^3+\frac{mn^2a^4}{2(2-m)^2x}+2aax+\frac{nna^4}{2(2-m)x}=0 \quad \text{or} \quad 4x^4+2aaxx+\frac{nna^4}{(2-m)^2}=0,$$

no root of which equation is real, unless *aa* shall be a negative quantity.

EXAMPLE 6

The proposed equation shall be this function determined $V=x^4+y^4-xx+xy-yy$, the maxima or minima values of which may be investigated.

Hence since there becomes $P=4x^3-2x+y=0$ and $Q=4y^3-2y+x=0$, from the first there will be $y=2x-4x^3$, which substituted in the other gives

$$256x^9-384x^7+192x^5-40x^3+3x=0.$$

One root of which is $x=0$, from which there is made also $y=0$. Therefore in this case on account of

$$\frac{dP}{dx}=12xx-2 \text{ and } \frac{dQ}{dy}=12yy-2$$

a maximum $V=0$ will be produced. But with the equation found divided by *x* there will be

$$256x^8-384x^6+192x^4-40x^3+3=0,$$

which has the factor $4xx-1$, from which there arises
$4xx=1$ and $x=\pm\frac{1}{2}$ and also $y=\pm\frac{1}{2}$; then truly there will be $\frac{dP}{dx}=\frac{dQ}{dy}=1$; therefore in each case the minimum $V=-\frac{1}{8}$ arises. That equation may be divided by $4xx-1$ and there will be obtained

$$64x^6 - 80x^4 + 28xx - 3 = 0,$$

which anew contains $4xx - 1 = 0$ twice, thus so that the preceding case may arise. In addition truly thence there becomes $4xx - 3 = 0$ and $x = \frac{\pm\sqrt{3}}{2}$; to which there corresponds $y = \frac{\mp\sqrt{3}}{2}$. There will be also $\frac{dP}{dx} = \frac{dQ}{dy} = 7$ and thus V becomes the minimum $= -\frac{9}{8}$; which is the value of all the minima, which indeed the function V can receive, and on this account this equation $V = -\frac{9}{8} - cc$ is impossible always. But hence the way is apparent to determining the maxima and minima of functions which involve three or four variables.

CAPUT XI

DE MAXIMIS ET MINIMIS FUNCTIONUM MULTIFORMIUM PLURESQUE VARIABILES COMPLECTENTIUM

273. Si y fuerit functio multiformis ipsius x, ita ut pro unoquoque valore ipsius x ea plures obtineat valores reales, tum variato x plures illi ipsius y valores ita inter se connectentur, ut plures series valorum successivorum repraesentent. Si enim y tanquam applicatam lineae curvae consideremus, x existente abscissa, quot y habuerit valores reales diversos, totidem diversi eiusdem curvae rami eidem abscissae x respondebunt; atque hinc illi ipsius y valores successivi, qui eundem ramum constituunt, cohaerere censendi sunt; valores autem ad diversos ramos relati erunt inter se disiuncti. Tot igitur series valorum cohaerentium ipsius y habebimus, quot diversos valores reales pro quovis ipsius x valore receperit; atque in qualibet serie valores ipsius y, dum x crescens assumitur, vel crescent vel decrescent vel, postquam creverint, iterum decrescent vel vice versa. Ex quo perspicuum est in unaquaque valorum cohaerentium serie aeque dari maxima minimave atque in functionibus uniformibus.

274. Ad haec maxima minimave determinanda eadem quoque methodus valebit, quam capite praecedente pro functionibus uniformibus tradidimus. Cum enim, si variabilis x incremento ω augeatur, functio y perpetuo recipiat hanc formam

$$y+\frac{\omega dy}{dx}+\frac{\omega^2 ddy}{2dx^2}+\frac{\omega^3 d^3y}{6dx^3}+\text{etc.},$$

necesse est, ut casu maximi minimive terminus $\frac{\omega dy}{dx}$ evanescat fiatque $\frac{dy}{dx}=0$. Radices ergo huius aequationis $\frac{dy}{dx}=0$ eos ipsius x valores indicabunt, quibus in singulis valorum ipsius y cohaerentium seriebus maxima minimave respondeant. Neque vera ambiguum erit, in quanam valorum cohaerentium serie detur maximum minimumve. Cum enim in aequatione $\frac{dy}{dx}=0$ ambae insint variabiles x et y, valores ipsius x definiri nequeunt, nisi ope equationis, qua relatio functionis y ab x continetur, variabilis y eliminetur; antequam autem hoc fit, pervenitur ad aequationem, qua valor ipsius y per functionem rationalem seu uniformem ipsius x exprimitur. Hinc inventis valoribus ipsius x cuique respondens valor ipsius y reperietur, qui erit maximus vel minimus in serie valorum successivorum cohaerentium, ad quam pertinet.

275. Iudicium autem, utrum isti valores ipsius *y* sint maximi an minimi, instituetur eodem modo, quem ante indicavimus. Scilicet quaeratur valor ipsius $\frac{ddy}{dx^2}$ finitis terminis expressus in eoque loco *x* substituatur unusquisque ipsius *x* valor inventus successive; simul autem pro *y* ponatur valor, qui ipsi pro quolibet ipsius *x* valore convenit; quo facto dispiciatur, utrum expressio $\frac{ddy}{dx^2}$ adeptura sit valorem affirmativum an negativum, priorique casu minimum, posteriori vero maximum indicabitur. Quodsi vera et $\frac{ddy}{dx^2}$ evanescat, tum procedendum erit ad formulam $\frac{d^3y}{dx^3}$ quae si eodem casu non evanescat, neque maximum habebitur neque minimum; sin autem quoque $\frac{d^3y}{dx^3}$ evanescat, iudicium formari oportebit ex formula $\frac{d^4y}{dx^4}$ eodem modo, quo ratione formulae $\frac{ddy}{dx^2}$ praecepimus. Atque si quoque $\frac{d^4y}{dx^4}$ quopiam casu evanescat, ad differentiale quintum ipsius *y* erit progrediendum; perpetuo autem, quousque progredi necesse fuerit, iudicia ex differentialibus ordinum imparium similia sunt illi, quod de formula $\frac{d^3y}{dx^3}$ dedimus. His scilicet casibus in formulis $\frac{ddy}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}$ etc. eousque erit pergendum, quoad perveniatur ad talem, quae proposito casu non evanescat; quae si fuerit differentialis ordinis imparis, neque maximum neque minimum indicabitur; sin autem fuerit ordinis paris, eius valor affirmativus minimum, negativus vero maximum innuet.

276. Ponamus functionem *y* determinari ex *x* per aequationem quamcunque; quae aequatio si differentietur, induet huiusmodi formam $Pdx + Qdy = 0$. Facto ergo $\frac{dy}{dx} = 0$ erit $\frac{P}{Q} = 0$ ideoque vel $P = 0$ vel $Q = \infty$. Posterior quidem aequatio, si relatio inter *x* et *y* exprimatur per aequationem rationalem integram, locum habere nequit, quia vel *x* vel *y* vel utramque fieri oporteret infinitam. Quare iudicium relinquetur aequationi $P = 0$, cuius radices seu valores ipsius *x*, quos adipiscitur, postquam ope aequationis propositae variabilis *y* penitus fuerit eliminata, indicabunt casus, quibus valores ipsius *y* fiunt maximi vel minimi. Ad iudicium vero, utrum prodeat maximum an minimum, absolvendum examinetur formula $\frac{ddy}{dx^2}$. Aequatio vero differentialis $Pdx + Qdy = 0$ denuo differentiata, si ponamus

$$dP = Rdx + Sdy \quad \text{et} \quad dQ = Tdx + Vdy\,,$$

dabit posito *dx* constante

$$Rdx^2 + Sdxdy + Tdxdy + Vdy^2 + Qddy = 0\,.$$

Cum autem iam sit $\frac{dy}{dx} = 0$, aequatione per dx^2 divisa fiet

$$R + \frac{Qddy}{dx^2} = 0 \quad \text{ideoque} \quad \frac{ddy}{dx^2} = -\frac{R}{Q}\,.$$

Hinc in aequatione differentiali $Pdx + Qdy = 0$ differentietur tantum quantitas P ponendo y constans prodibitque Rdx; tum indagetur valor fractionis $\frac{R}{Q}$, qui, si fuerit affirmativus, maximum, sin negativus, minimum indicabit.

277. Sit y functio biformis ipsius x, quae determinetur per hanc aequationem $yy + py + q = 0$ denotantibus p et q functiones quascunque ipsius x uniformes. Erit ergo differentiando $2ydy + pdy + ydp + dq = 0$ ideoque $Pdx = ydp + dq$. Posito igitur $P = 0$ erit $ydp + dq = 0$ prodibitque $y = -\frac{dq}{dp}$ sicque y per functionem ipsius x uniformem exprimitur, ita ut, quicunque valor pro x fuerit inventus, ex eo et y valorem determinatum unicum acquirat. Eliminatio vero nunc ipsius y erit facilis; nam si in aequatione proposita $yy + py + q = 0$ loco y valor $-\frac{dq}{dp}$ substituatur, habebitur $dq^2 - pdpdq + qdp^2 = 0$, quae aequatio divisa per dx^2 et resoluta praebebit valores ipsius x omnes, quibus maxima vel minima respondent; quod clarius, fiet sequentibus exemplis.

EXEMPLUM 1

Proposita aequatione $yy + mxy + aa + bx + nxx = 0$
definire maxima vel minima functionis y.

Differentiata aequatione habebimus

$$2ydy + mxdy + mydx + bdx + 2nxdx = 0,$$

unde fit

$$P = my + b + 2nx \quad \text{et} \quad Q = 2y + mx.$$

Posito ergo $P = 0$ fiet $y = -\frac{b+2nx}{m}$; qui valor in ipsa aequatione substitutus dat

$$\frac{4nn}{mm}xx + \frac{4nb}{mm}x + \frac{bb}{mm} - 2nxx - bx + aa + nxx + bx = 0$$

seu

$$xx = \frac{4nbx + bb + mmaa}{mmn - 4nn},$$

unde fit

$$x = \frac{2nb \pm \sqrt{(mmnbb + mmn(mm - 4n)aa)}}{mmn - 4nn}$$

seu

$$x = \frac{2nb \pm m\sqrt{(nbb + n(mm-4n)aa)}}{mmn-4nn} \quad \text{et} \quad y = \frac{-mb \mp 2\sqrt{(nbb + n(mm-4n)aa)}}{mm-4n}$$

Tum posito solo x variabili fit $dP = 2ndx$ ideoque $R = 2n$. At est

$$Q = 2y + mx = \pm\frac{\sqrt{(nbb + n(mm-4n)aa)}}{n},$$

unde

$$\frac{R}{Q} = \frac{\pm 2nn}{\sqrt{(nbb + n(mm-4n)aa)}};$$

cuius numerator *2nn* cum sit perpetuo affirmativus, si signum superius valeat, prodibit pro y valor maximus, sin inferius, prodibit minimus. Ubi sequentia annotari debent.

I. Si fuerit $m = 0$, ex aequatione $P = 0$ statim sequitur $x = -\frac{b}{2n}$, ut nulla eliminatione opus sit. Huicque valori geminus ipsius y respondet ob $y = \pm\frac{1}{2n}\sqrt{(nbb - 4nnaa)}$, quorum alter affirmativus est maximus, alter negativus minimus.

II. Si sit $n = 0$, fit $y = -\frac{b}{m}$ et x in infinitum excrescit atque y per spatium infinitum eundem valorem retinet, ita ut neque maximus sit neque minimus.

III. Si sit $mm = 4n$, erit $4nbx + bb + mmaa = 0$ seu $x = \frac{bb+mmaa}{-mmb}$ fietque seu

$$y = -\frac{b+2nx}{m} = -\frac{2b+mmx}{2m} = -\frac{2b}{2m} + \frac{bb+mmaa}{2mb} = \frac{mmaa-bb}{2mb}.$$

Huic ergo valori ipsius $x = -\frac{mmaa+bb}{mmb}$ alter ipsius y valor qui respondet, $\frac{mmaa-bb}{2mb}$, erit maximus vel minimus. Quia autem, ut iste ipsius y valor prodeat, in expressione

$$y = \frac{-mb \mp 2\sqrt{(nbb + n(mm-4n)aa)}}{mm-4n}$$

signum inferius valere debet, erit valor ipsius y minimus.

EXEMPLUM 2

Proposita aequatione $yy - xxy + x - x^3 = 0$ *definire valores ipsius y maximos vel minimos.*

Differentiata aequatione prodit

$$2ydy - xxdy - 2xydx + dx - 3xxdx = 0.$$

Fitque

$$P = 1 - 3xx - 2xy \text{ et } Q = 2y - xx.$$

Quare posito $P = 0$ erit $y = \frac{1-3xx}{2x}$ ideoque hoc valore substuto

$$\frac{1}{4xx} - \frac{3}{2} + \frac{9xx}{4} - \frac{x}{2} + \frac{3}{2}x^3 + x - x^3 = 0$$

seu

$$1 - 6xx + 2x^3 + 9x^4 + 2x^5 = 0.$$

Cuius una radix est $x = -1$, cui respondet $y = 1$. At posito y constante fit $R = -6x - 2y$, ergo

$$\frac{ddy}{dx^2} = \frac{2y+6x}{2y-xx};$$

quod casu $x = -1$ et $y = 1$ abit in -4, ita ut valor ipsius $y = 1$ sit maximus. Ipsi $x = -1$ autem geminus valor ipsius y respondet ex aequatione $yy - y = 0$; alter ergo est $y = 0$, qui neque maximus est neque minimus. Quodsi aequatio illa quinti gradus per $x + 1$ dividatur, prodit aequatio, cuius radices simpliciter exhiberi nequeunt.

EXEMPLUM 3

Sit proposita haec aequatio $yy + 2xxy + 4x - 3 = 0$, ex qua maximi minimive valores ipsius y requiruntur.

Per differentiationem ergo prodibit haec aequatio

$$2ydy + 2xxdy + 4xydx + 4dx = 0.$$

Factoque $\frac{dy}{dx} = 0$ erit $xy + 1 = 0$ ideoque $y = -\frac{1}{x}$, qui valor substitutus in ipsa aequatione proposita oritur

$$\frac{1}{xx} - 2x + 4x - 3 = 0 = 2x^3 - 3xx + 1,$$

cuius radices sunt $x = 1$, $x = 1$ et $x = -\frac{1}{2}$. Quia nunc est

$$\frac{dy}{dx} = -\frac{4xy+4}{2y+2xx} = -\frac{2xy+2}{y+xx},$$

erit differentiando $\frac{ddy}{dx^2} = -\frac{2y}{y+xx}$ posito *y* constanti ob $dy = 0$ et facto $xy + 1 = 0$. Quare isti valores ita se habebunt:

x	y	$\frac{ddy}{dx^2}$
1	–1	∞
1	–1	∞
$-\frac{1}{2}$	2	$\frac{-16}{9}$ pro maximo.

Quoniam pro radicibus aequalibus fit $\frac{ddy}{dx^2} = \infty$, utrum hoc casu maximum an minimum prodeat, non determinatur. Quia autem simul fit $y + xx = 0$, nequidem hoc casu erit $\frac{dy}{dx} = 0$ ob $P = 0$ et $Q = 0$ in fractione $\frac{dy}{dx} = -\frac{P}{Q}$; quare cum primaria proprietas desit, neque maximum nec minimum habet locum. Indicatur autem hoc casu $x = 1$ ambos ipsius *y* valores inter se fieri aequales. Quam indolem infra fusius sumus expositurі, cum ad usum calculi differentialis in doctrina de lineis curvis perveniemus. Etiamsi enim haec materia et huc pertineat, tamen, ne eam bis attingere opus sit, eam totam sequenti tractationi reservamus.

278. Datur vero insuper in functionibus multiformibus alia species maximorum ac minimorum, quae methodo hactenus tradita non invenitur, cuius natura ex functionibus biformibus facillime explicari potest. Sit enim *y* functio quaecunque biformis ipsius *x*, ita ut, quicunque valor ipsi *x* tribuatur, pro *y* oriantur bini valores, vel ambo reales vel ambo imaginarii. Ponamus hos ipsius *y* valores fieri imaginarios, si ponatur $x > f$, reales autem esse, si statuatur $x < f$; atque posito $x = f$ ambo ipsius *y* valores in unum coalescent, qui sit $y = g$. Cum igitur, si sumatur $x > f$, functio *y* nullum habeat valorem realem, si eveniat, ut posito $x < f$ ambo ipsius *y* valores fiant vel maiores quam *g* vel minores quam *g*, priori casu valor $y = g$ erit minimus, posteriori maximus, quoniam illo casu minor est quam ambo praecedentes, hoc vero maior. Neque hoc maximum minimumve methodo hactenus tradita reperietur, propterea quod hic non fit $\frac{dy}{dx} = 0$. Sunt autem quoque haec maxima vel minima generis diversi, cum talia non sint ratione valorum antecedentium et consequentium in serie cohaerentium, sed ratione binorum valorum disiunctorum vel antecedentium vel sequentium tantum.

279. Evenit hoc, si aequatio proposita fuerit huiusmodi

$$y = p \pm (f - x)\sqrt{(f - x)q}$$

existentibus *p* et *q* functionibus ipsius *x* per $f - x$ non divisibilibus; obtineatque

q valorem affirmativum, si ponatur vel $x = f$ vel aliquanto maius minusve. Fiat $p = g$ posito $x = f$ et manifestum est casu $x = f$ ambos ipsius y valores in unum $y = g$ coalescere; posito autem $x > f$ ambo valores ipsius y fient imaginarii. Si igitur ponamus x aliquanto minus quam f, puta $x = f - \omega$, functio p abibit in

$$g - \frac{\omega dp}{dx} + \frac{\omega^2 ddp}{2dx^2} - \text{etc.}$$

et q in

$$q - \frac{\omega dq}{dx} + \frac{\omega^2 ddq}{2dx^2} - \text{etc.}$$

unde hoc casu erit

$$y = g - \frac{\omega dp}{dx} + \frac{\omega^2 ddp}{2dx^2} - \text{etc.} \pm \omega\sqrt{\omega\left(q - \frac{\omega dq}{dx} + \frac{\omega^2 ddq}{2dx^2} - \text{etc.}\right)}.$$

Ponamus ω minimum; ut prae ω altiores eius potestates evanescant, eritque $y = g - \frac{\omega dp}{dx} \pm \omega\sqrt{\omega q}$; qui valores ambo ipsius y minores erunt quam g, si $\frac{dp}{dx}$ fuerit affirmativum, maiores autem, si negativum. Unde valor duplex ipsius $y = g$ illo casu erit maximus, hoc vero minimus.

280. Haec igitur maxima atque minima inde ortum suum habent, quod primo posito $x = f$ ambo ipsius y valores fiant aequales, posito autem $x > f$ imaginarii, at posito $x < f$ reales; deinde, quod posito $x = f - \omega$ alterum membrum irrationale praebeat altiores potestates ipsius ω quam membrum rationale. Hoc ergo evenit quoque, si fuerit $y = p \pm (f - x)^n \sqrt{(f - x)q}$, dummodo sit n numerus integer > 0. Cum autem non solum radix quadrata, sed etiam quaecunque alia radix potestatis paris eandem ambiguitatem signorum introducat, idem eveniet, si fuerit $y = p \pm (f - x)^{\frac{2n+1}{2m}} q$, dummodo sit $2n + 1 > 2m$; erit ergo $(y - p)^{2m} = (f - x)^{2n+1} q^{2m}$ seu $(y - p)^{2m} = (f - x)^{2n+1} Q$. Quoties ergo functio y per huiusmodi aequationem exprimitur, ita ut sit $2n + 1 > 2m$, toties posito $x = f$ valor ipsius y fiet maximus vel minimus; prius quidem, si fuerit $\frac{dp}{dx}$ quantitas affirmativa, posterius vero, si sit $\frac{dp}{dx}$ quantitas negativa posito $x = f$. Sin autem fiat hoc casu $\frac{dp}{dx} = 0$, tum erit

$$y = g + \frac{\omega^2 ddp}{2dx^2} \pm \omega^{\frac{2n+1}{2m}} q.$$

Nisi ergo sit $\frac{2n+1}{2m} > 2$, neque maximum neque minimum locum habebit; at si $\frac{2n+1}{2m} > 2$, turn $y = g$ erit maxima, si $\frac{ddp}{dx^2}$ habuerit valorem negativum, minimum vero, si affirmativum; sicque ulterius, si quoque $\frac{ddp}{dx^2}$ evanescat, iudicium erit instituendum.

281. Si igitur *y* fuerit huiusmodi functio ipsius *x*, fieri potest, ut praeter maxima et minima, quae prior methodus exhibet, etiam maxima minimave huius alterius speciei adsint, quae modo hic exposito explorari poterunt. Id quod sequentibus exemplis declarabimus.

EXEMPLUM 1

Determinare maxima ac minima functionis y, quae definitur hac aequatione

$$yy - 2xy - 2xx - 1 + 3x + x^3 = 0.$$

Ad maxima minimave primae speciei investiganda differentietur aequatio eritque

$$2ydy - 2xdy - 2ydx - 4xdx + 3dx + 3xxdx = 0$$

positoque $\frac{dy}{dx} = 0$ erit

$$y = \frac{3}{2} - 2x + \frac{3}{2}xx$$

qui valor in prima aequatione substitutus dat

$$9x^4 - 32x^3 + 42xx - 24x + 5 = 0,$$

quae resolvitur in

$$9xx - 14x + 5 = 0 \text{ et } xx - 2x + 1 = 0.$$

Posterior bis dat $x = 1$ fitque $y = 1$, unde hoc casu in fractione

$$\frac{dy}{dx} = \frac{2y - 3 + 4x - 3xx}{2y - 2x}$$

denominator quoque evanescit sicque maximum minimumve primi generis non datur; prior vero aequatio $9xx - 14x + 5 = 0$ dabit $x = 1$ et $x = \frac{5}{9}$ quorum valorum ille eodem incommodo laborat quo praecedentes. Posito autem

$x = \frac{5}{9}$ fit $y = \frac{3}{2} - \frac{10}{9} + \frac{25}{54} = \frac{23}{27}$. Et cum sit $\frac{dy}{dx} = \frac{2y - 3 + 4x - 3xx}{2y - 2x}$, fiet

$$\frac{ddy}{dx^2} = \frac{4-6x}{2y-2x} = \frac{-3x+2}{y-x}$$

ob $dy = 0$ et numeratorem $= 0$. Erit ergo $\frac{ddy}{dx^2} = \frac{9}{8}$, unde hic valor $x = \frac{5}{9}$ dat minimum primi generis. Deinde cum sit $(y-x)^2 = (1-x)^3$, erit

$$y = x \pm (1-x)\sqrt{(1-x)}$$

ideoque posito $x = 1$ prodit maximum secundae speciei; facto enim $x = 1-\omega$ erit $y = 1-\omega \pm \omega\sqrt{\omega}$, quorum uterque minor est quam unitas, siquidem ω sumatur minimum.

EXEMPLUM 2

Invenire maxima ac minima functionis $y = 2x - xx \pm (1-x)^2\sqrt{(1-x)}$.

Pro primi generis maximis et minimis differentietur aequatio eritque

$$\frac{dy}{dx} = 2 - 2x \mp \frac{5}{2}(1-x)\sqrt{(1-x)},$$

qui valor positus $= 0$ prodit primo $x = 1$, et cum sit

$$\frac{ddy}{dx^2} = -2 \pm \frac{15}{2}\sqrt{(1-x)},$$

erit *y* hoc casu maximum primi generis fitque $y = 1$. Aequatione vero $\frac{dy}{dx} = 0$ per $1-x$ divisa erit $4 \mp 5\sqrt{(1-x)} = 0$ seu $16 = 25 - 25x$, unde fit $x = \frac{9}{25}$ et $\frac{ddy}{dx^2} = -2 \pm 3$. Quare si signum superius valet,erit $y = \frac{2869}{3125}$ minimum; sin autem signum inferius valeat, erit $y = \frac{821}{3125}$ quod maximum videatur; at vero tantum signum superius locum habere potest, quoniam $4 \mp 5\sqrt{(1-x)}$ nequit esse $= 0$, nisi sit $\sqrt{(1-x)} = +\frac{4}{5}$. Primi ergo generis invenimus maximum casu $x = 1$ et $y = 1$ atque minimum casu $x = \frac{9}{25}$ et $y = \frac{2869}{3125}$. Ex genere vero altero maximum quoque prodit, si $x = 1$, quo casu fit $y = 1$. Nam posito $x = 1-\omega$ erit $y = 1 - \omega\omega \pm \omega^2\sqrt{\omega}$ utroque casu < 1. Hic itaque, si $x = 1$, maxima duo primae et alterius speciei coalescunt maximumque quasi mixtum constituunt.

282. Ex his exemplis non solum natura huius alterius speciei maximorum et minimorum elucet, sed etiam pro lubitu istiusmodi functiones formari possunt, quae maxima vel minima secundae speciei admittant. Quemadmodum autem, si proposita fuerit functio quaecunque, explorari possit, utrum eiusmodi maximis minimisve sit praedita necne, id in sequenti sectione ostendemus, propterea quod natura linearum curvarum hac investigatione maxime illustratur. Ceterum vero facile intelligitur, si fuerit y eiusmodi functio ipsius x, quae maximum minimumve secundae speciei recipiat, tum quoque vicissim x eiusmodi fore functionem ipsius y. Nam quia ex hac aequatione $(y-x)^2=(1-x)^3$ facto $x=1$ obtinet y valorem maximum secundae speciei, si variabiles y et x permutentur, haec aequatio $(y-x)^2=(1-y)^3$ exhibet pro y quoque eiusmodi functionem ipsius x, quae habeat maximum secundae speciei. Facto enim $x=1$ fiet $(1-y)^2=(1-y)^3$ hincque erit bis $y=1$ et semel $y=0$. Sin autem ponatur $x=1+\omega$, erit $(1+\omega-y)^2=(1-y)^3$; unde si statuamus $y=1+\varphi$, erit $(\omega-\varphi)^2=(-\varphi)^3=-\varphi^3$ ideoque φ debet esse negativum. Sit ergo $y=1-\varphi$; erit $(\omega+\varphi)^2=\varphi^3$, atque cum sumto φ minimo φ^3 prae φ^2 evanescat, debebit necessario ω esse negativum; hinc valori $x=1+\omega$ nulli valores reales ipsius y respondent. At posito $x=1-\omega$ et $y=1-\varphi$ ob $(\varphi-\omega)^2=\varphi^3$ erit $\varphi=\omega\pm\omega\sqrt{\omega}$ ideoque $y=1-\omega\mp\omega\sqrt{\omega}$, unde uterque valor ipsius y respondens ipsi $x=1-\omega$ minor est valore $y=1$, qui respondet valori $x=1$; eritque consequenter iste ipsius y valor maximus.

283. Hactenus tantum functiones biformes sumus contemplati, quarum maxima vel minima, quia ambo valores facile per resolutionem aequationis quadraticae exprimi possunt, ad examen revocari possunt. Sin autem functio y per aequationem altiorem exprimatur, methodus ante tradita, qua maxima minimaque primae speciei indagavimus, eodem successu adhiberi poterit. Inventionem vero maximorum ac minimorum secundae speciei sequenti sectioni reservamus. Functiones ergo triformes ac multiformes, quemadmodum tractari oporteat, aliquot exemplis ostendamus.

EXEMPLUM 1

Definiatur functio y, cuius maxima vel minima quaeruntur, per hanc aequationem

$$y^3+x^3=3axy.$$

Differentiata hac aequatione fit

$$3y^2dy+3xxdx=3axdy+3aydx$$

ideoque

$$\frac{dy}{dx}=\frac{ay-xx}{yy-ax}.$$

Maximum ergo vel minimum dabitur, si fuerit $ay = xx$ seu $y = \frac{xx}{a}$, qui valor in aequatione proposita substitutus dat

$$\frac{x^6}{a^3} + x^3 = 3x^3 \quad \text{seu} \quad x^6 = 2a^3x^3.$$

Erit ergo ter $x = 0$, quo casu quoque fit denominator $yy - ax = 0$ ob $y = \frac{xx}{a} = 0$. Utrum ergo hoc casu maximum minimumve prodeat, patebit, si ipsi *x* valorem tribuamus minime ab 0 discrepantem. Sit ergo $x = \omega$ et $y = \varphi$ ob $\varphi^3 + \omega^3 = 3a\omega\varphi$ fiet vel $\varphi = \alpha\sqrt{\omega}$ vel $\varphi = \beta\omega^2$. Priori casu erit $\alpha^3\omega\sqrt{\omega} = 3\alpha a\omega\sqrt{\omega}$ ideoque $\alpha = \sqrt{3a}$. Hinc posito $x = \omega$ erit $y = +\sqrt{3a\omega}$. Unde etiamsi ω negative accipi nequeat, tamen binorum ipsius *y* valorum alter maior erit quam 0, alter minor hincque $y = 0$ neque maximum erit neque minimum. Sin autem statuatur $\varphi = \beta\omega^2$, erit $\omega^3 = 3a\beta\omega^3$ ideoque $\beta = \frac{1}{3a}$ et $\varphi = \frac{\omega^2}{3a}$. Ergo hoc casu, sive *x* capiatur $= +\omega$ sive $= -\omega$, valor ipsius $y = \varphi$ nihilo erit maior ideoque hoc casu $y = 0$ erit minimum. Restat ergo tertius casus ex aequatione $x^3 = 2a^3$ examinandus, qui dat $x = a\sqrt[3]{2}$ et $y = a\sqrt[3]{4}$. Qui utrum sit maximus an minimus, ex aequatione $\frac{dy}{dx} = \frac{ay-xx}{yy-ax}$ quaeratur differentiale secundum, quod ob $dy = 0$ et $ay - xx = 0$ erit $\frac{ddy}{dx^2} = \frac{-2x}{yy-ax}$, cuius valor praesenti casu est $-\frac{2a\sqrt[3]{2}}{2a^2\sqrt[3]{2} - aa\sqrt[3]{2}} = -\frac{2}{a}$, qui indicat
valorem ipsius *y* esse maximum.

EXEMPLUM 2

Si functio y definiatur per hanc aequationem, $y^4 + x^4 + ay^3 + ax^3 = b^3x + b^3y$,
invenire eius maximos minimosve valores.

Cum per differentiationem oriatur

$$4y^3dy + 3ayydy - b^3dy = b^3dx - 3axxdx - 4x^3dx,$$

erit

$$\frac{dy}{dx} = \frac{b^3 - 3axx - 4x^3}{4y^3 + 3ayy - b^3}$$

ponique oportet $b^3 = 3axx + 4x^3$. Quaestio ergo huc reducitur, ut functionis uniformis $b^3 - 3axx - 4x^3$ maxima ac minima indagentur, quae simul erunt maxima seu minima functionis *y*. Sit $a = 2$ et $b = 3$ seu proponatur haec aequatio
$y^4 + x^4 + 2y^3 + 2x^3 = 27x + 27y$; erit $\frac{dy}{dx} = \frac{27 - 6xx - 4x^3}{4y^3 + 6yy - 27}$ et $4x^3 + 6xx - 27 = 0$,

quae divisa per $2x-3=0$ dat $2xx+6x+9=0$; cuius posterioris radices cum sint imaginariae, erit $x=\frac{3}{2}$ et $y^4+2y^3-27y=\frac{459}{16}$ cuius singulae radices erunt vel maximae vel minimae. Cum autem sit $\frac{dy}{dx}=\frac{27-6xx-4x^3}{4y^3+6yy-27}$, erit $\frac{ddy}{dx^2}=\frac{-12x-12xx}{4y^3+6yy-27}$, qui posito $x=\frac{3}{2}$, si affirmitivus, indicabit minimum, contra vero maximum.

EXEMPLUM 3

Si fuerit $y^m+ax^n=by^px^q$, definire maxima et minima ipsius y.

Per differentiationem fit

$$\frac{dy}{dx}=\frac{qby^px^{q-1}-nax^{n-1}}{my^{m-1}-pby^{p-1}x^q},$$

quo posito $=0$ erit primo $x=0$, si quidem *n* et *q* fuerint unitate maiores, atque simul $y=0$. Quo casu an detur maximum vel minimum, valores proximi sunt investigandi, quoniam quoque denominator fit $=0$; quae investigatio ab exponentibus potissimum pendebit. Praeterea vero aequatio $\frac{dy}{dx}=0$ dabit $y^p=\frac{na}{qb}x^{n-q}$, qui valor in proposita substitutus ponendo $\frac{na}{qb}=g$ dabit

$$g^{\frac{m}{p}}x^{\frac{mn-mq}{q}}+ax^n=\frac{na}{q}x^n \quad \text{seu} \quad g^{\frac{m}{p}}x^{\frac{mn-mq-np}{p}}=\frac{(n-q)a}{q},$$

unde fit

$$x=\left(\frac{(n-q)a}{q}\right)^{p:(mn-mq-np)}:g^{m:(mn-mq-np)}$$

simulque valor ipsius *y* innotescit. Deinde dispiciendum est, utrum differentio – differentiale

$$\frac{ddy}{dx^2}=\frac{q(q-1)by^px^{q-2}-n(n-1)ax^{n-2}}{my^{m-1}-pby^{p-1}x^q}$$

obtineat valorem affirmativum an negativum, ut ex priori minimum, ex posteriori vero maximum pronuncietur.

EXEMPLUM 4

Si fuerit $y^4+x^4=4xy-2$, maxima et minima functionis y assignare.

Differentiatione instituta fit

$$\frac{dx}{dy}=\frac{y-x^3}{y^3-x}$$

hincque oritur $y = x^3$; erit ergo $x^{12} = 3x^4 - 2$ seu $x^{12} - 3x^4 + 2 = 0$, quae aequatio resolvitur in has $x^4 - 1 = 0$ et $x^8 + x^4 - 2 = 0$ posteriorque in $x^4 - 1 = 0$ et $x^4 + 2 = 0$. Hinc erit bis vel $x = +1$ vel $x = -1$; utroque vero casu et denominator fractionis $\frac{dy}{dx}$ evanescit. Ad investigandum ergo, utrum his casibus maximum minimumve locum habeat, ponamus $x = 1 - \omega$ et $y = 1 - \varphi$; erit

$$\begin{aligned}1 - 4\varphi + 6\varphi^2 - 4\varphi^3 + \varphi^4 + 1 - 4\omega + 6\omega^2 - 4\omega^3 + \omega^4 \\ = 4 - 4\omega - 4\varphi + 4\omega\varphi - 2\end{aligned}$$

ideoque

$$4\omega\varphi = 6\varphi^2 + 6\omega^2 - 4\varphi^3 - 4\omega^3 + \omega^4$$

et ob ω et φ minima $4\omega\varphi = 6\varphi^2 + 6\omega^2$. Valor ergo ipsius φ erit imaginarius, sive ω capiatur affirmative sive negative. Seu si y et x designent coordinatas curvae, ea casu $x = 1$ et $y = 1$ habebit punctum coniugatum. Neque ergo hic valor pro maximo neque pro minimo haberi potest, propterea quod antecedentes et consequentes, cum quibus comparari deberet, fiunt imaginarii.

284. Si aequatio, qua relatio inter x et y exprimitur, ita fuerit comparata, ut functio ipsius y aequetur functioni ipsius x, puta $Y = X$, ad maxima minimave invenienda poni debebit $dX = 0$; fiet ergo y maximum vel minimum iisdem casibus, quibus X fit maximum vel minimum. Simili modo si x tanquam functio ipsius y consideretur, fiet x maximum vel minimum, si $dY = 0$, hoc est si Y fuerit maximum vel minimum. Neque tamen hinc sequitur y et x simul fieri maxima vel minima. Nam si fuerit $2ay - yy = 2bx - xx$, erit y maximum vel minimum, si fuerit $x = b$, eritque $y = a \pm \sqrt{(aa - bb)}$. Contra vero x fit maximum vel minimum, si fuerit $y = a$, fitque $x = b \pm \sqrt{(bb - aa)}$ neque ergo fiet y maximum vel minimum, si $x = b \pm \sqrt{(bb - aa)}$, quo tamen casu x est maximum minimumve. Ceterum hoc casu, si y habeat valores maximos vel minimos, x hac indole prorsus carebit; namque y maximum minimumve fieri nequit, nisi sit $a > b$, quo casu maximum minimumve ipsius x fit imaginarium.

285. Tum vero etiam evenire potest, ut non omnes radices aequationis $dX = 0$ praebeant maximos minimosve valores pro y; si enim illa aequatio duas habuerit radices aequales, exinde neque maximum neque minimum consequitur; hocque idem evenit, si quotcunque radices numero pares fuerint inter se aequales. Sic si proponatur aequatio $b(y-a)^2 = (x-b)^3 + c^3$, quia sumtis differentialibus fit $2bdy(y-a) = 3dx(x-b)^2$, functio y neque maxima fiet neque minima posito $x = b$, propterea quod hic occurrunt duae radices aequales. Sin autem x tanquam functio ipsius y spectetur, ea fiet maxima vel minima, si statuatur $y = a$, eritque $x = b - c$ minimum. Quia denique in huiusmodi aequationibus $Y = X$ variabiles *x* et *y* inter se non permiscentur, si ipsi *x* tribuitur valor, qui sit radix aequationis $dX = 0$, omnes valores ipsius *y*, quotcunque fuerint reales, erunt maximi vel minimi; quod non evenit, si in aequatione ambae variabiles fuerint permixtae.

286. Quae praeterea supersunt de natura maximorum ac minimorum exponenda, ea in sequentem sectionem reservamus, quoniam commodius ope figurarum menti repraesentari atque explicari possunt. Pergamus ergo ad functiones, quae ex pluribus variabilibus sunt compositae, atque investigemus valores, quos singulis variabilibus tribui oportet, ut ipsa functio vel maximum vel minimum valorem obtineat. Ac primo quidem patet, si variabiles non fuerint inter se permixtae, ita ut functio proposita sit huiusmodi $X + Y$ existente *X* functione ipsius *x* et *Y* ipsius *y* tantum, tum functionem propositam $X + Y$ fore maximum, si simul *X* et *Y* maximum evadat, minimumque, si simul *X* et *Y* fiat minimum. Ad maximum ergo inveniendum inquirantur valores ipsius *x*, quibus *X* fiat maximum, similique modo valores ipsius *y*, quibus *Y* fit maximum, hique valores pro *x* et *y* inventi efficient functionem $X + Y$ maximam, quod similiter de minimo erit tenendum. Cavendum ergo est, ne duo valores ipsarum *x* et *y* diversae naturae combinentur, quorum ille reddat *X* maximum, hic vero *Y* minimum, aut contra. Hoc enim si fieret, functio $X + Y$ neque maximum foret neque minimum. At huiusmodi functio $X - Y$ fiet maxima, si *X* fuerit maximum simulque *Y* minimum; contra vero $X - Y$ fiet minimum, si *X* fuerit minimum et *Y* maximum. Sin autem utraque functio *X* et *Y* statueretur vel maxima vel minima, earum differentia $X - Y$ neque foret maxima neque minima; quae omnia sunt ex natura maximorum ac minimorum ante exposita clara ac perspicua.

287. Si ergo quaerantur maximi minimive valores functionis duarum variabilium, quaestio multo magis cautioni obnoxia est, quam si unica fuerit variabilis. Non solum enim pro utraque variabili casus, quibus maximum minimumve producitur, diligenter sunt distinguendi, sed etiam ex his bini eiusmodi sunt coniungendi, ut functio proposita fiat maximum vel minimum; id quod ex exemplis clarius patebit.

EXEMPLUM 1

Sit proposita haec duarum variabilium x et y functio
$y^4 - 8y^3 + 18y^2 - 8y + x^3 - 3xx - 3x$ *et quaerantur valores pro y et x substituendi, ut haec functio maximum vel minimum obtineat valorem.*

Quoniam haec expressio in duas huiusmodi partes $Y + X$ resolvitur, quarum illa est functio ipsius *y*, haec vero ipsius *x* tantum, casus, quibus utraque fit maxima vel minima, investigentur. Cum igitur sit

$$Y = y^4 - 8y^3 + 18y^2 - 8y$$

erit

$$\frac{dY}{dy} = 4y^3 - 24y^2 + 36y - 8;$$

qua expressione nihilo aequali posita fiet per 4 diviso

$$y^3 - 6y^2 + 9y - 2 = 0,$$

cuius radices sunt $y = 2$ et $y = 2 \pm \sqrt{3}$. Cum ergo sit $\frac{ddY}{4dy^2} = 3yy - 12y + 9$, casu $y = 2$ prodibit maximum. Pro reliquis binis radicibus $y = 2 \pm \sqrt{3}$, quae oriuntur ex aequatione $yy - 4y + 1 = 0$, fiet $\frac{ddY}{12dx^2} = yy - 4y + 3 = 2$, unde utraque dat minimum. Erit autem his casibus, ut sequitur:

$y = 2$	$Y = 8$ maximum
$y = 2 - \sqrt{3}$	$Y = -1$ minimum
$y = 2 + \sqrt{3}$	$Y = -1$ minimum

Simili modo cum sit

$$X = x^3 - 3xx - 3x,$$

erit

$$\frac{dX}{dx} = 3xx - 6x - 3,$$

unde oritur haec aequatio

$$xx = 2x + 1$$

et $x = 1 \pm \sqrt{2}$. Est vero $\frac{ddX}{6dx^2} = x - 1 = \pm\sqrt{2}$. Ergo radix $x = 1 + \sqrt{2}$ dat minimum, nempe $X = -5 - 4\sqrt{2}$, et $x = 1 - \sqrt{2}$ dat maximum, nempe $X = -5 - 4\sqrt{2}$. Quocirca formula proposita

$$y^4 - 8y^3 + 18y^2 - 8y + x^3 - 3xx - 3x$$

fiet maxima, si ponatur $y = 2$ et $x = 1 - \sqrt{2}$, prodibitque $X + Y = 3 + 4\sqrt{2}$. Eadem autem formula $X + Y$ fiet minima, si sumatur vel $y = 2 - \sqrt{3}$ vel $y = 2 + \sqrt{3}$ et $x = 1 + \sqrt{2}$; utroque casu erit $X + Y = -6 - 4\sqrt{2}$.

EXEMPLUM 2

Si proponatur haec functio duarum variabilium
$y^4 - 8y^3 + 18y^9 - 8y - x^3 + 3xx + 3x$*, quae quibus casibus fiat maxima vel minima, investigetur.*

Posito, ut in praecedente exemplo habuimus,

$$Y = y^4 - 8y^3 + 18y^2 - 8y \text{ et } X = x^3 - 3xx - 3x$$

formula proposita erit $Y - X$ ideoque fiet maxima, si *Y* fuerit maximum et *X* minimum. Cum igitur hos casus iam ante eruerimus, patet $Y - X$ obtinere valorem maximum, si ponatur $y = 2$ et $x = 1 + \sqrt{2}$; fietque $Y - X = 13 + 4\sqrt{2}$. Minimus vero valor ipsius $Y - X$ evadet, si *Y* sit minimum et *X* maximum, quod evenit ponendo $y = 2 + \sqrt{3}$ et $x = 1 - \sqrt{2}$; fiet autem $Y - X = 4 - 4\sqrt{2}$. Ceterum in utroque exemplo patet hos valores, quos invenimus, neque omnium esse maximos neque minimos; nam si utrinque poneretur verbi gratia $y = 100$ et $x = 0$, sine dubio maior prodiret valor eo, quem invenimus; similique modo ponendo $y = 0$ et vel $x = -100$ vel $x = +100$ minor prodiret valor, quam sunt illi, quos pro casu minimi invenimus. Probe ergo tenenda est idea supra exposita, quam de natura maximorum ac minimorum dedimus, scilicet eum valorem vocari maximum, qui maior sit valoribus tam antecedentibus quam consequentibus contiguis proximis, minimum autem esse eum, qui his valoribus tam antecedentibus quam consequentibus fuerit minor. Sic in hoc exemplo valor ipsius $Y - X$, qui prodit ponendo $y = 2$ et $x = 1 + \sqrt{2}$, maior est iis, qui resultant, si ponatur $y = 2 \pm \omega$ et $x = 1 + \sqrt{2} \pm \varphi$ sumtis pro ω et φ quantitatibus satis exiguis.

288. His exemplis expeditis facilior erit via ad solutionem generalem indagandam. Denotet *V* functionem quamcunque duarum variabilium *x* et *y* sintque pro *x* et *y* valores inveniendi, qui functioni *V* inducant maximum vel minimum valorem. Cum igitur ad hoc efficiendum utrique variabili *x* et *y* determinatus valor tribui debeat, ponamus alteram *y* iam habere eum valorem, qui requiritur ad functionem *V* vel maximam vel minimam reddendam, hocque posito tantum opus erit, ut pro altera *x* idoneus quoque valor investigetur, quod fiet, dum functio *V* differentiatur ponenda sola *x* variabili differentialeque nihilo aequale statuitur. Simili modo si fingamus variabilem *x* iam eum habere valorem, qui aptus sit ad functionem *V* vel maximam vel minimam efficiendam, valor ipsius *y* reperietur differentiando *V* posita sola *y* variabili hocque differentiale nihilo aequali ponendo. Hinc si differentiale functionis *V* fuerit $= Pdx + Qdy$, oportebit esse et $P = 0$ et $Q = 0$, ex quibus duabus aequationibus valores utriusque variabilis *x* et *y* erui poterunt.

289. Quoniam vero hoc pacta sine discrimine reperiuntur valores pro *x* et *y*, quibus functio *V* vel maxima vel minima redditur, casus, quibus vel maximum vel minimum oritur, probe a se invicem sunt distinguendi. Ut enim functio *V* fiat maxima, necesse est, ut ambae variabiles ad hoc conspirent; namque si altera maximum exhiberet, altera minimum, ipsa functio neque maxima neque minima evaderet. Quocirca inventis ex aequationibus $P = 0$ et $Q = 0$ valoribus ipsarum *x* et *y* inquirendum est, utrum ambo simul functioni *V* vel maximum vel minimum valorem inducant; atque tum demum, cum compertum fuerit utriusque variabilis valorem hinc erutum pro maximo valere, affirmare poterimus functionem hoc casu maximum valorem induere. Quod idem de minimo erit tenendum, ita ut functio V minimum valorem adipisci nequeat, nisi simul ambae variabiles x et y minimum producant. Hinc ergo omnes illi casus reiiei debebunt, quibus altera variabilis maximum, altera vero minimum indicare deprehendetur. Interdum vero etiam evenit, ut alterius vel etiam utriusque variabilis valores ex aequationibus $P=0$ et $Q=0$ oriundi neque maximum neque minimum exhibeant, qui casus proinde pariter tanquam prorsus inepti erunt reiiciendi.

290. Utrum autem valores pro x et y reperti valeant pro maximo an minimo, de utroque seorsim simili modo investigabitur, quo supra, cum unica adesset variabilis, sumus usi. Ad iudicium scilicet de variabili x instituendum consideretur altera y tanquam constans, et cum sit $dV=Pdx$ seu $\frac{dV}{dx}=P$, differentietur P denuo posito y constante, ut prodeat $\frac{ddV}{dx^2}=\frac{dP}{dx}$ ac dispiciatur, utrum valor ipsius $\frac{dP}{dx}$ postquam loco x et y valores ante inventi fuerint substituti, fiat affirmativus an negativus; priori enim casu indicabitur minimum, posteriori vero maximum. Simili modo cum posito x constante sit $dV=Qdy$ seu $\frac{dV}{dy}=Q$, differentietur Q denuo posita sola y variabili et examinetur valor $\frac{dQ}{dy}$ substitutis loco x et y valoribus, qui ex aequationibus $P=0$ et $Q=0$ sunt inventi; qui si fuerit affirmativus, declarabit minimum, contra vero maximum. Hinc ergo colligitur, si ex valoribus pro x et y inventis formulae $\frac{dP}{dx}$ et $\frac{dQ}{dy}$ induant valores diversis signis affectos, altera scilicet affirmativum, altera negativum, tum functionem V neque maximam neque minimam effici; sin autem utraque formula $\frac{dP}{dx}$ et $\frac{dQ}{dy}$ fiat affirmativa, minimum resultabit, contraque, si utraque fiat negativa, maximum.

291. Quodsi vero altera formula $\frac{dP}{dx}$ et $\frac{dQ}{dy}$ vel etiam utraque, si pro x et y valores inventi substituantur, evanescat, tum progrediendum erit ad differentialia sequentia $\frac{ddP}{dx^2}$ et $\frac{ddQ}{dy^2}$; quae nisi pariter evanescant, neque maximum neque minimum habebit locum; sin autem evanescant, iudicium ex formulis differentialibus sequentibus $\frac{d^3P}{dx^3}$ et $\frac{d^3Q}{dy^3}$ erit petendum similique modo instituendum, quo pro formulis $\frac{dP}{dx}$ et $\frac{dQ}{dy}$ est factum. Quo autem, quibus casibus hoc usu veniat, clarius exponamus, prodierit valor $x=\alpha$; qui si formulam $\frac{dP}{dx}$ reddat evanescentem, necesse est, ut $\frac{dP}{dx}$ factorem habeat $x-\alpha$; qui factor si fuerit solitarius neque simul alium sibi habeat aequalem socium, neque maximum neque minimum indicabitur; quod idem evenit , si $\frac{dP}{dx}$ factorem habuerit $(x-\alpha)^3$ vel $(x-\alpha)^5$ etc. Sin autem factor fuerit $(x-\alpha)^2$ vel $(x-\alpha)^4$ etc., tum quidem maximum vel minimum indicabitur; at insuper videndum erit, utrum cum casu per y indicato consentiat.

292. Labor autem his casibus ad differentialia ulteriora progrediendi mirifice sublevari poterit; si enim ponamus, ut rem generalius complectamur, inventum esse $\alpha x+\beta=0$ atque formulam $\frac{dP}{dx}$ factorem habere $(\alpha x+\beta)^2$, ita ut sit $\frac{dP}{dx}=(ax+\beta)^2 T$, quia est $ax+\beta=0$, fiet $\frac{d^3P}{dx^3}=2\alpha^2 T$ hincque ob $2\alpha^2$ affirmativum ex ipsa quantitate T iudicium absolvi poterit; quae si induet valorem affirmativum, pro minimo, contra vero pro maximo pronunciabit. Hocque idem subsidium in maximorum minimorumque investigatione, si unica insit variabilis, adhiberi poterit, ita ut nunquam opus sit ad altiora differentialia ascendere. Quin etiam nequidem ad differentialia secunda procedere opus erit; si enim ex aequatione $P=0$ fiat $\alpha x+\beta=0$, necesse est, ut P factorem habeat $\alpha x+\beta$; sit $P=(ax+\beta)T$, et cum sit

$$\frac{dP}{dx}=\alpha T+(\alpha x+\beta)\frac{dT}{dx}$$

ob $\alpha x+\beta=0$ erit $\frac{dP}{dx}=\alpha T$ hincque iam ipse alter factor T, prout valor ipsius αT fuerit vel affirmativus vel negativus, statim vel minimum vel maximum indicabit.

293. His igitur traditis praeceptis haud difficile erit, si functio quaecunque duas variabiles involvens fuerit proposita, casus investigare, quibus haec functio fiat vel maxima vel minima. Si quae insuper notanda fuerint, ea ipsa exemplorum evolutio suggeret, quamobrem aliquot exempiis regulas datas illustrari expediet.

EXEMPLUM 1

Sit proposita ista functio duarum variabilium $V=xx+xy+yy-ax-by$; quae quibus casibus fiat vel maxima vel minima, inquiratur.

Cum sit $dV=2xdx+ydx+xdy+2ydy-adx-bdy$, si comparetur cum forma generali $dV=Pdx+Qdy$, erit

$$P=2x+y-a \text{ et } Q=2y+x-b,$$

unde formabuntur istae aequationes

$$2x+y-a=0 \text{ et } 2y+x-b=0,$$

quibus coniunctis eliminando y fiet $x-b=4x-2a$ ideoque

$$x=\frac{2a-b}{3} \text{ et } y=a-2x=\frac{2b-a}{3}$$

Cum igitur sit

$$\frac{dP}{dx} = 2 \quad \text{et} \quad \frac{dQ}{dy} = 2,$$

utraque ostendit minimum; ex quo concludimus formulam $xx + xy + yy - ax - by$ fieri minimum, si ponatur $x = \frac{2a-b}{3}$ et $y = \frac{2b-a}{3}$, prodibit que hoc modo

$$V = \frac{-3aa+3ab-3bb}{9} = \frac{-aa+ab-bb}{3};$$

qui cum sit unicus, omnium erit minimus. Unico ergo modo fieri potest

$$xx + xy + yy - ax - by = \frac{-aa+ab-bb}{3}$$

et quia minor fieri nequit, erit haec aequatio

$$xx + xy + yy - ax - by = \frac{-aa+ab-bb}{3} - cc$$

impossibilis.

EXEMPLUM 2

Si proponatur formula $V = x^3 + y^3 - 3axy$, quaerantur casus, quibus V adipiscatur valorem maximum vel minimum.

Ob $dV = 3xxdx + 3yydy - 3aydx - 3axdy$ erit

$$P = 3xx - 3ay \quad \text{et} \quad Q = 3yy - 3ax,$$

unde fit

$$ay = xx \quad \text{et} \quad ax = yy.$$

Cum ergo sit $yy = x^4 : aa = ax$, erit $x^4 - a^3x = 0$ ideoque vel $x = 0$ vel $x = a$. Priori casu fit $y = 0$, posteriori vero $y = a$. Quoniam ergo est

$$\frac{dP}{dx} = 6x, \text{ et } \frac{ddP}{dx^2} = 6 \quad \text{et} \quad \frac{dQ}{dy} = 6y \text{ atque } \frac{ddQ}{dy^2} = 6,$$

priori ergo casu, quo $x = 0$ et $y = 0$, neque maximum neque minimum resultat. Posteriori vero casu, quo et $x = a$ et $y = a$, minimum prodit, si quidem a fuerit quantitas affirmativa, fietque $V = -a^3$, qui autem valor tantum minor est proximis antecedentibus et consequentibus; nam sine dubio V multo minorem induere potest valorem, si utrique variabili x et y valores negativi tribuantur.

EXEMPLUM 3

Proposita sit haec functio $V = x^3 + ayy - bxy + cx$ *, cuius valores maximi seu minimi inquirantur.*

Quia est $dV = 3xxdx + 2aydy - bydx - bxdy + cdx$, erit

$$P = 3xx - by + c \text{ et } Q = 2ay - bx,$$

quibus valoribus nihilo aequalibus positis erit $y = \frac{bx}{2a}$ ideoque

$$3xx - \frac{bbx}{2a} + c = 0 \text{ seu } xx = \frac{2bbx - 4ac}{12a}$$

unde fit

$$x = \frac{bb \pm \sqrt{(b^4 - 48aac)}}{12a}$$

Nisi ergo sit $b^4 - 48aac > 0$, neque maximum neque minimum habet locum.
Ponamus ergo esse $b^4 - 48aac = bbff$, ut sit $c = \frac{bb(bb - ff)}{48aa}$; erit

$$x = \frac{bb \pm bf}{12a} \quad \text{et} \quad y = \frac{bb(b \pm f)}{24aa}$$

Quoniam porro est

$$\frac{dP}{dx} = 6x \text{ et } \frac{dQ}{dy} = 2a,$$

fiet

$$\frac{dP}{dx} = \frac{b(b \pm f)}{2a}.$$

Nisi ergo *2a* et $\frac{b(b \pm f)}{2a}$ sint quantitates eiusdem signi, neque maximum neque minimum habet locum. At si sint ambae vel affirmativae vel ambae negativae, quod evenit, si earum productum $b(b \pm f)$ fuerit affirmativum, tum functio *V* evadet minimum, si *a* sit quantitas affirmativa, contra vero maximum, si *a* sit quantitas negativa, Hinc si fuerit $f = 0$ seu $c = \frac{b^4}{48aa}$, ob *bb* quantitatem affirmativam functio *V* evadet minima, si *a* sit quantitas positiva ponatque $x = \frac{bb}{12a}$ et $y = \frac{b^3}{24aa}$; contra vero si *a* sit negativum, istae substitutiones producent maximum. Si sit $f < b$, duobus casibus oritur vel maximum vel minimum; at si $f > b$, tum casus tantum $x = \frac{b(b+f)}{12a}$ et $y = \frac{bb(b+f)}{24aa}$ praebebit maximum minimumve, prout a fuerit vel negativum vel affirmativum. Sit $a=1$, $b=3$ et $f=1$, ut habeatur haec formula $V=x^3+yy-3xy+\frac{3}{2}x$; haec fiet minima ob a affirmativum, si ponatur vel $x=1$ et $y=\frac{3}{2}$ vel $x=\frac{1}{2}$ et $y=\frac{3}{4}$. Priori casu oritur $V=\frac{1}{4}$ posteriori vero $V=\frac{5}{16}$. Interim tamen patet loco x numeris negativis ponendis multo minores valores pro V oriri posse. Ita ergo intelligi debet valor ipsius $V=\frac{1}{4}$ minor esse, quam si ponatur $x=1+\omega$ et $y=\frac{3}{2}+\varphi$, dummodo sint ω et φ numeri parvi, sive affirmativi sive negativi; limes autem, quem ω transgredi non debet, est $-\frac{15}{4}$; nam si $\omega<-\frac{15}{4}$, fieri poterit, V fiat minor quam $\frac{1}{4}$.

EXEMPLUM 4

Invenire maxima vel minima huius functionis

$$V=x^4+y^4-axxy-axyy+ccxx+ccyy.$$

Sumto differentiali erit

$$P=4x^3-2axy-ayy+2ccx \quad \text{et} \quad Q=4y^3-axx-2axy+2ccy,$$

quibus valoribus nihilo aequalibus positis, si a se invicem subtrahantur, erit

$$4x^3-4y^3+axx-ayy+2ccx-2ccy=0;$$

quae cum sit divisibilis per $x-y$, erit primo $y=x$ atque $4x^3-3axx+2ccx=0$, quae dat

$$x=0 \quad \text{et} \quad 4xx=3ax-2cc \quad \text{seu} \quad x=\frac{3a\pm\sqrt{(9aa-32cc)}}{8}.$$

Si sumamus $x=0$, erit quoque $y=0$ et ob

$$\frac{dP}{dx}=12xx-2ay+2cc \quad \text{atque} \quad \frac{dQ}{dy}=12yy-2ax+2cc$$

fiet functio V minima $=0$. Sin statuamus $x=y=\frac{3a\pm\sqrt{(9aa-32cc)}}{8}$, si quidem fuerit $9aa>32cc$, ob $4xx=3ax-2cc$ erit

$$\frac{dP}{dx}=\frac{dQ}{dy}=12xx-2ax+2cc=7ax-4cc$$
$$=\frac{21aa-32cc\pm7a\sqrt{(9aa-32cc)}}{8};$$

qui valor cum sit semper affirmativus ob $32cc<9aa$, valor V hoc quoque casu fit minimus eritque

$$V = \tfrac{-27}{256}a^4 + \tfrac{9}{16}aacc - \tfrac{1}{2}c^4 \mp \tfrac{a}{256}\left(9aa - 32cc\right)^{\frac{3}{2}}.$$

Dividamus autem aequationem $4x^3 - 4y^3 + axx - ayy + 2ccx - 2ccy = 0$ per $x - y$ fietque $4xx + 4xy + 4yy + ax + ay + 2cc = 0$. At ex aequatione $P = 0$ erit $yy = -2xy + \frac{4}{a}x^3 + \frac{2ccx}{a}$, quo valore in illa substituto fit ,

$$y = \tfrac{16x^3 + 4axx + aax + 8ccx + 2acc}{4ax - aa}$$

Verum ista dat

$$y = -x \pm \sqrt{\tfrac{4x^3 + axx + 2accx}{a}},$$

unde efficitur

$$16x^3 + 8axx + 8ccx + 2acc = \left(4x - a\right)\sqrt{\left(4ax^3 + aaxx + 2accx\right)},$$

quae ad rationalitatem perducta dat

$$\left.\begin{array}{l} 256x^6 + 192ax^5 + 80aax^4 + 4a^3x^3 - a^4x^2 \\ -2a^3ccx + 4a^2c^4 + 256cc + 160acc \\ +48aacc + 32ac^4 + 64c^4 \end{array}\right\} = 0;$$

cuius radices, si quas habet, reales indicabunt maxima vel minima functionis V, si quidem $\frac{dP}{dx}$ et $\frac{dQ}{dy}$ fiant quantitates eodem signa affectae.

EXEMPLUM 5

Invenire maxima et minima huius expressionis

$$x^4 + mxxyy + y^4 + aaxx + naaxy + aayy = V.$$

Facta differentiatione erit

$$P = 4x^3 + 2mxyy + 2aax + naay = 0,$$
$$Q = 4y^3 + 2mxxy + 2aay + naax = 0,$$

quae aequationes invicem vel subtractae vel additae dant

$$\left(4xx + 4xy + 4yy - 2mxy + 2aa - naa\right)\left(x - y\right) = 0,$$
$$\left(4xx - 4xy + 4yy + 2mxy + 2aa + naa\right)\left(x + y\right) = 0,$$

quae divisae per $x-y$ et $x+y$ et denuo vel additae vel subtractae dant

$$4xx+4yy+2aa=0 \text{ et } 4xy-2mxy-naa=0.$$

Ex quarum posteriori fit $y=\frac{naa}{2(2-m)x}$; prior autem reales valores non admittit.
Tres igitur habemus casus.

I. Sit $y=x$ eritque $4x^3+2mx^3+2aax+naax=0$, unde fit vel $x=0$ vel $2(2+m)xx+(2+n)aa=0$. Sit $x=0$; erit quoque $y=0$ atque ob

$$\frac{dP}{dx}=12xx+2myy+2aa \text{ et } \frac{dQ}{dy}=12yy+2mxx+2aa$$

hoc casu fiet $V=0$ minimum, si quidem coefficiens *aa* fuerit affirmativus. Alter casus dat $xx=-\frac{(n+2)aa}{2(m+2)}$, quae realis esse nequit, nisi sit $\frac{n+2}{m+2}$ numerus negativus. Sit $\frac{n+2}{m+2}=-2kk$ seu $n=-2kkm-4kk-2$; erit $x=\pm ka$ et $y=\pm ka$. At

$$\frac{dP}{dx}=12kkaa+2mkkaa+2aa \quad \text{et} \quad \frac{dQ}{dy}=12kkaa+2mkkaa+2aa;$$

quae cum sint aequales, erit *V* vel minimum vel maximum, prout istae quantitates fuerint vel affirmativae vel negativae.

II. Sit $y=-x$ eritque $2(m+2)x^3=(n-2)aax$, ergo vel $x=0$ vel $xx=\frac{(n-2)aa}{2(m+2)}$. Prior radix $x=0$ recidit in praecedentem. Posterior vero erit realis, si $\frac{(n-2)aa}{2(m+2)}$ fuerit quantitas affirmativa, et cum fiat $\frac{dP}{dx}=\frac{dQ}{dy}$, prodibit vel maximum vel minimum.

III. Sit $y=\frac{naa}{2(2-m)x}$; erit

$$4x^3+\frac{mn^2a^4}{2(2-m)^2x}+2aax+\frac{nna^4}{2(2-m)x}=0 \quad \text{seu} \quad 4x^4+2aaxx+\frac{nna^4}{(2-m)^2}=0,$$

cuius aequationis nulla radix est realis, nisi sit *aa* quantitas negativa.

EXEMPLUM 6

Proposita sit haec functio determinata $V = x^4 + y^4 - xx + xy - yy$, cuius valores maximi vel minimi investigentur.

Cum hinc fiat $P = 4x^3 - 2x + y = 0$ et $Q = 4y^3 - 2y + x = 0$, erit ex priori $y = 2x - 4x^3$, qui in altera substitutus dat

$$256x^9 - 384x^7 + 192x^5 - 40x^3 + 3x = 0.$$

Cuius una radix est $x = 0$, unde fit quoque $y = 0$. Ergo hoc casu ob

$$\frac{dP}{dx} = 12xx - 2 \text{ et } \frac{dQ}{dy} = 12yy - 2$$

prodit maximum $V = 0$. Divisa autem aequatione inventa per x erit

$$256x^8 - 384x^6 + 192x^4 - 40x^3 + 3 = 0,$$

quae factorem habet $4xx - 1$, unde fit $4xx = 1$ et $x = \pm\frac{1}{2}$ atque $y = \pm\frac{1}{2}$; tum vero erit $\frac{dP}{dx} = \frac{dQ}{dy} = 1$; utroque ergo casu oritur minimum $V = -\frac{1}{8}$. Dividatur illa aequatio per $4xx - 1$ atque obtinebitur

$$64x^6 - 80x^4 + 28xx - 3 = 0,$$

quae denuo bis continet $4xx - 1 = 0$, ita ut praecedens casus oriatur. Praeterea vero inde fit $4xx - 3 = 0$ et $x = \frac{\pm\sqrt{3}}{2}$; cui respondet $y = \frac{\mp\sqrt{3}}{2}$. Erit igitur quoque $\frac{dP}{dx} = \frac{dQ}{dy} = 7$ ideoque fiet V minimum $= -\frac{9}{8}$; qui est valor omnium minimus, quos quidem functio V recipere potest, et hanc ob rem ista aequatio $V = -\frac{9}{8} - cc$ semper est impossibilis. Hinc autem patet via determinandi maxima et minima functionum, quae tres pluresve variabiles involvunt.