CHAPTER X

CONCERNING MAXIMA AND MINIMA

250. If a function of *x* were prepared thus, so that with increasing values for *x* it may either increase of decrease continually, then that function will have no maximum or minimum value. Indeed some value of this function may be considered, for which the following values are greater, the preceding truly smaller : x^3+x is such a function of this kind, the value of which will increase continually with increasing *x*, truly with decreasing *x* it will decrease continually; therefore this function is unable to adopt some other maximum value maximum, unless the maximum value for *x* itself may be given, and this is infinite ; and in a similar manner it will obtain the smallest value, if there is put $x=-\infty$. But unless the function were prepared thus, so that with increasing *x* it may increase or decrease continually, it will have a maximum or minimum value somewhere, that is a value of such a kind, which shall be either greater or less than the preceding and following values. Thus that function $xx-2x+3$ adopts a minimum value, if there is put $x=1$; for whatever other value may be attributed of *x*, the function always will come upon a greater value.

251. But so that the nature of the maximas and minimas may be seen more clearly, we may put *y* to be a function of *x* this kind, which may obtain a maximum value, if there is put $x=f$, and it is understood if *x* may be put either greater or smaller than *f*, then the value of *y* thence arising to be less than that arising from that, which it adopts if there is put $x=f$. In a similar manner, if there is put $x=f$ the function *y* may obtain a minimum value, it is necessary that if either *x* may be put either greater or less than *f*, a greater value of *y* may always result ; and this is the definition of absolute maxima and minima. But in addition the function *y* may be said also to have a maximum value on putting for argument's sake $x=f$, provided this value were greater than nearby values, either following or previous, which arise if *x* may be put in place either a little greater or less than *f*, even if for other values substituted in place of *x* the function y perhaps may receive greater values. Similarly the function *y* is said to have a minimum value on putting $x=f$, provided that value were less than these which it adopts, if in place of *x* nearby values either greater or smaller than *f* may be substituted. And in the following we will make use of maxima and minima according to this meaning.

252. But before we may show the manner of finding these maxima and minima, it may be agreed to note particularly this investigation into these functions of *x* only which have a place, which we have called *uniform* above and which have been prepared thus, so that for the individual values of *x* equally they may receive values. But we have called functions *biform* and *multiform*, which introduce two or more values for the individual values of *x*, roots of quadratic and of equations of more dimensions are functions of this kind. Therefore if *y* were a function of this kind of *x* either of a biform or multiform kind, then properly it cannot be said to adopt either a maximum or minimum value on putting $x = f$; for because on putting $x = f$ either two or more values may be obtained likewise and the preceding and following shall be greater by some number, the judgement of the maxima or minima is not so easy, unless perhaps all the values of the function *y*, which correspond to the individual values of *x*, shall be imaginary except one ; in which case functions of this kind pretend to be a special kind of uniform functions. Therefore in the first place we will consider uniform functions and a fictitious kind of these ; then truly, we will indicate how the manner of judgement may be adapted to multiforms.

253. Therefore let *y* be a uniform function of *x*, which therefore, whatever value for *x* may be substituted, always may receive a single real value, and let *x* denote the value of that, which may produce a maximum or minimum value of the function y. Therefore in the first case, if either $x+\alpha$ or $x-\alpha$ be substituted in place of *x*, the value of y will be less than if $\alpha = 0$, and truly greater in the latter case. Therefore since on putting $x+\alpha$ in place of x the function y may change into

$$y+\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}+\frac{\alpha^3 d^3 y}{6dx^3}+\text{etc.},$$

but on putting $x-\alpha$ in place of x into

$$y-\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}-\frac{\alpha^3 d^3 y}{6dx^3}+\text{etc.},$$

it is necessary, that in the case of the maximum there shall be

$$y > y+\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}+\frac{\alpha^3 d^3 y}{6dx^3}+\text{etc.}$$

and

$$y > y-\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}-\frac{\alpha^3 d^3 y}{6dx^3}+\text{etc.}$$

But in the case, in which the value of *y* shall be a minimum, there will be

$$y < y+\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}+\frac{\alpha^3 d^3 y}{6dx^3}+\text{etc.}$$

$$y < y-\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}-\frac{\alpha^3 d^3 y}{6dx^3}+\text{etc.}$$

254. Because these must come about, if α may denote a very small quantity, we may put α to be so small a quantity, that the higher powers of this may be able to be rejected, and there ought to be $\alpha\frac{dy}{dx}=0$ both for the case of the maxima as well as for the minima. For unless $\alpha\frac{dy}{dx}$ should be $=0$, neither a maximum nor minimum value may be possible for *y*.

Hence this common rule is had both for the maxima as well as for the minima, so that nothing may be put equal to the proposed differential of y, and that will be the value of x, the root of this equation, which returns a maximum or minimum function. Truly uncertainty remains whether the value of y to be found in this way will be a maximum or a minimum ; why may it not be possible that y shall become neither a maximum nor a minimum ; for in each case we have found only that $\frac{dy}{dx}=0$ nor also have we affirmed that as often as there shall be $\frac{dy}{dx}=0$, so also the value for y to be produced shall be a maximum or a minimum.

255. Yet meanwhile for the cases, in which the value of y prevails as a maximum or a minimum, this first operation is required to be investigated, so that the differential of the proposed function may be equal to zero and from the equation $\frac{dy}{dx}=0$ all the values of x may be elicited. With which found then it will be required to be discerned, whether from these the function y may adopt a maximum or minimum value or neither. For we may show that it may be possible that it may be treated neither as maximum nor a minimum, even if there shall be $\frac{dy}{dx}=0$.

Let f be the value or one from the values of x, which may be obtained from the equation

$$\frac{dy}{dx}=0,$$

and this value may be substituted into the expressions $\frac{ddy}{dx^2}, \frac{d^3y}{dx^3}$ etc. and there comes from this substitution

$$\frac{ddy}{dx^2}=p, \quad \frac{d^3y}{dx^3}=q, \quad \frac{d^4y}{dx^4}=r \quad \text{etc.}$$

But the function y itself changes into F on putting f in place of x, and if in place of x there may be put $f+\alpha$, that function will change into

$$F+\tfrac{1}{2}\alpha^2p+\tfrac{1}{6}\alpha^3q+\tfrac{1}{24}\alpha^4r+\text{etc.};$$

[Recall the lack of the modern function notation in the preceding; here each distinct value of y corresponding to a different value of x must be named by a different letter.
We may note also in modern terms that $\frac{dy}{dx}=0$ is a necessary condition for a maximum or minimum of a continuous, rather than uniform, function, but that this may or may not be a sufficient condition for a turning point to occur at the point in question; hence all the bother Euler goes to in the following concerning this matter, looking for the extra condition from higher order differentials that prior and post values of the function are both either greater or less than its value at this point, for a minimum or maximum respectively.]

but if in place of x there may be put $f-\alpha$, there will be put

$$F+\tfrac{1}{2}\alpha^2 p-\tfrac{1}{6}\alpha^3 q+\tfrac{1}{24}\alpha^4 r-\text{etc.};$$

from which it may be apparent, if p were a finite quantity, and whether the value may become greater than F, at least if α may denote a very small quantity, and therefore the value F, which the function y adopts on putting $x=f$, is a minimum. But if p shall be a negative quantity, then the value $x=f$ of the function y may lead to a maximum value.

256. But if there should be $p=0$, then the value of q must be considered ; which if it were not $=0$, the value of y will be neither a maximum nor a minimum ; for on putting $x=f+\alpha$ there will be $F+\tfrac{1}{6}\alpha^3 q>F$ and on putting $x=f-\alpha$ there will be $F-\tfrac{1}{6}\alpha^3 q<F$. But if also there were $q=0$, the quantity r will be required to be considered ; which if there were a positive value, the value of the function F, which it receives on putting $x=f$, will be a minimum; but if r may have a negative value, then F will be a maximum. But if r also may vanish, a judgement will depend on the value s from the following letters, because there will be in a like manner from that, as we have formed from the letter q. Obviously if s were not $=0$, the function F will be neither a maximum nor a minimum; but if also there should be $s=0$, then the following letter t, if it should have a positive value, will indicate a minimum ; but if it should have a negative value, it will indicate a maximum. Truly if also this letter t may vanish, then in judging it is required to be progressing further in precisely the same manner, as we have used in the preceding cases. And thus it will be investigated from any root of the equation $\frac{dy}{dx}=0$, whether y introduces a maximum or minimum value, or neither ; and in this manner all the maxima and minima will be found, which a certain function y is able to receive.

257. Therefore if the equation $\frac{dy}{dx}=0$ may have two equal roots, thus so that it may have the square factor $(x-f)^2$, then on putting $x=f$ likewise $\frac{ddy}{dx^2}$ will vanish and there will be $p=0$, but not q. Therefore in this case the function y neither adopts a maximum nor a minimum value. But if the equation $\frac{dy}{dx}=0$ may have three equal roots or $\frac{dy}{dx}$ had the cubic factor $(x-f)^3$, then on putting $x=f$ there becomes $\frac{ddy}{dx^2}=0$ and $\frac{d^3y}{dx^3}=0$, but not $\frac{d^4y}{dx^4}$. Therefore if the value of this term were positive, it will indicate a minimum, if negative, a maximum. Therefore the judgement explained before is reduced to this, so that if the expression $\frac{dy}{dx}$ had a factor $(x-f)^n$ with an odd number n present, then the function y, if there may be put $x=f$ in that, there shall be taken either a maximum or a minimum, but if the exponent *n* were an even number, then on substituting $x = f$ neither a maximum nor a minimum value will be produced.

258. From which the finding of the maxima and minima will be greatly helped many times by the following considerations. Evidently in the cases the function *y* becomes a maximum or minimum, in the same cases some multiple of this may be made *ay*, if *a* were a certain positive quantity, and likewise y^3, y^5, y^7 etc. and generally ay^n if indeed n were a positive odd number, equally will have a maximum or minimum, because formulas of this kind have been prepared thus, so that they increase with *y* increasing and decrease with *y* decreasing. But in which cases *y* is made a maximum or minimum, in the same cases $-y,\ -ay,\ b-ay$ and generally $b-ay^n$, with the number *n* positive and odd, become odd with the order inverted, either minimum or maximum. Similarly in which cases y is made a maximum or minimum, in the same cases these formulas $\frac{a}{y}$, $\frac{a}{y^3}$, $\frac{a}{y^5}$ and generally $\frac{a}{y^n} \pm b$ with *a* denoting a positive quantity and *n* an odd positive number become inverted with the order becoming either minimum or maximum; but if a were a negative quantity, then these formulas will achieve the maximum value, if *y* were a maximum, and the minimum, if *y* shall be a minimum.

259. But these cannot be shown likewise for even powers; because indeed, if y takes negative values, the even powers of this y^2, y^4 etc. adopt positive values, it can arise that while *y* takes a minimum value, evidently negative, the even powers become maxima. Therefore we will be able to affirm in the account had of this condition, if y were a maximum or minimum with a positive value of this present, then the even powers y^2, y^4 etc. also become maxima or minima, but if the negative value of this *y* were a maximum, then the square of this *yy* will be taking a minimum value, and on the contrary, if the negative value of this *y* shall be a minimum, then y^2, y^4 etc. become maximum. But truly if the even exponents of y were negative, then the contrary will come about. Moreover these things which we have noted here about even and odd exponents, prevail not only for integral numbers, but also for fractions, the denominators of which are odd numbers ; for in this matter the fractions $\frac{1}{3}$, $\frac{5}{3}$, $\frac{7}{3}$, $\frac{1}{5}$, $\frac{3}{5}$ etc. with odd numbers are equivalent, and also $\frac{2}{3}$, $\frac{4}{3}$, $\frac{2}{5}$, $\frac{4}{5}$, $\frac{6}{7}$ etc. with even numbers,.

260. But if the denominators were even numbers, then because, if y has a negative value, the powers of that $y^{\frac{1}{2}}, y^{\frac{3}{4}}$ etc. are made imaginary, this will only be possible with these to be positive: If the positive value of *y* were a maximum or minimum, then also $y^{\frac{1}{2}}, y^{\frac{3}{2}}\ y^{\frac{1}{4}}$ etc. become equally either maxima or minima, but on the other hand $y^{-\frac{1}{2}}, y^{-\frac{3}{2}}\ y^{-\frac{1}{4}}$ etc. become minima or maxima. But because these irrational values likewise have twin values, the one positive and the other negative, concerning the negatives the opposite will be required to be understood, to what we have said here concerning the positives. But if the negative value of y becomes a maximum or minimum, then because all the powers of this kind become imaginary, they will not be able to be enumerated by maximas or minimas. Therefore by these aids, often the investigation of a maxima and minima certainly will be returned easily, which otherwise would become exceedingly difficult.

261. Because these pertain particularly to rational functions, clearly which only are uniform, in the first place we may set out integral functions and we will investigate the maxima and minima which occur in these. Therefore since functions of this kind may be referred to this form

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \text{etc.}$$

in the first place it is apparent that the value of these cannot become greater, than if there may be put $x = \infty$; then truly, if $x = -\infty$, the value of this formula produced $= \infty^n$, if n shall be an even number, but $= -\infty^n$, if n shall be an odd number, which therefore the minimum of all. But in addition often other maxima or minima are given by considering that, which we have assigned from these discussed, which we will illustrate by the following examples.

EXAMPLE 1

To find the values of x, for which this function $(x-a)^n$ becomes a maximum or minimum.

On putting $(x-a)^n = y$ there will be

$$\frac{dy}{dx} = n(x-a)^{n-1};$$

from which on putting $= 0$ there becomes $x = a$. Therefore since $\frac{dy}{dx}$ may have the factor $(x-a)^{n-1}$, from § 257 it is understood that y cannot become a maximum or a minimum, unless the number $n-1$ shall be odd or n shall be an even number. But because then there becomes

$$\frac{d^n y}{dx^n} = n(n-1)(n-2)\cdots 1,$$

this is a positive number, it follows that the value of y on putting $x = a$ will produce a minimum. Which indeed is readily apparent ; for on putting $x = a$ there arises $y = 0$, and if x may be put either greater or less than a, on account of the even number n, y takes a positive value, this is greater than zero; but if n were an odd number, then the function $y = (x-a)^n$ permits neither a maximum or minimum. But again it is evident likewise to prevail, if n were a fractional number, either odd or even. Evidently $(x-a)^{\frac{\mu}{\nu}}$ becomes a minimum on putting $x=a$, if μ were an even number and v odd; but if each were odd, it will give neither a maximum nor a minimum.

EXAMPLE 2

To find the cases, in which the value of this formula $xx+3x+2$ becomes a maximum or minimum.

There may be put $xx+3x+2=y$; there will be

$$\frac{dy}{dx}=2x+3,\quad \frac{ddy}{2dx^2}=1.$$

Therefore there may be put in place $2x+3=0$; there is made $x=-\frac{3}{2}$. Which case whether a maximum or a minimum may be produced, will be known from the value $\frac{ddy}{2dx^2}=1$; which since it shall be positive, whatever x shall be, indicates a minimum. But on putting $x=-\frac{3}{2}$ there comes about $y=-\frac{1}{4}$, and if some other values may be given to x itself, the value of y thence arising always will be greater than $-\frac{1}{4}$. Also from the nature of the formula $xx+3x+2$ it is evident that must have a minimum value; for since it may increase to infinity, whether there may be put $x=+\infty$ or $x=-\infty$, it is necessary that the value of a certain x adopts the minimum quantity of all y.

EXAMPLE 3

To find the case, in which this expression $x^3-axx+bx-c$ takes a maximum or minimum value.

On putting $y=x^3-axx+bx-c$ there will be

$$\frac{dy}{dx}=3xx-2ax+b \quad \text{and} \quad \frac{ddy}{2dx^2}=3x-a,\quad \frac{d^3y}{6dx^3}=1,$$

Therefore there is put in place $\frac{dy}{dx}=3xx-2ax+b=0$; there will be

$$x=\frac{a\pm\sqrt{(aa-3b)}}{3},$$

from which it is understood, unless there shall be $aa>3b$, the proposed formula will not be having a maximum or minimum. But if there shall be $aa>3b$, a maximum or minimum happens in two case. Hence indeed there arises

$$\frac{ddy}{2dx^2} = \pm\sqrt{(aa-3b)},$$

from which it is understood, unless there shall be $aa = 3b$, the value $x = \frac{a+\sqrt{(aa-3b)}}{3}$ returns the formula $x^3 - axx + bx - c$ a minimum, truly the other $x = \frac{a-\sqrt{(aa-3b)}}{3}$ a maximum. Moreover the quantities themselves become the values of y, since there shall be $3xx - 2ax + b = 0$ or $x^3 - \frac{2}{3}axx + \frac{1}{3}bx = 0$, there will be

$$y = -\tfrac{1}{3}axx + \tfrac{2}{3}bx - c$$

and on account of $\frac{1}{3}axx - \frac{2aa}{9}x + \frac{ab}{9} = 0$ there is made

$$y = \tfrac{2}{9}(3b-aa)x + \tfrac{ab}{9} - c = -\frac{2a(aa-3b)}{27} \mp \frac{2(aa-3b)\sqrt{(aa-3b)}}{27} + \frac{ab}{9} - c$$

or

$$y = -\frac{2a^3}{27} + \frac{ab}{3} - c \mp \frac{2}{27}(aa-3b)^{\frac{3}{2}},$$

where the upper sign prevails for the minimum, but the lower for the maximum.

Hence the case remains, for which $aa = 3b$; in which since there comes about $\frac{ddy}{dx^2} = 0$, truly the following term $\frac{d^3y}{6dx^3} = 1$ shall not be $= 0$, and it follows in this case that the proposed formula cannot take either a maximum or a minimum.

EXAMPLE 4

To find the cases, in which this function of x, $x^4 - 8x^3 + 22x^2 - 24x + 12$*, shall be made a maximum or minimum.*

On putting $y = x^4 - 8x^3 + 22x^2 - 24x + 12$ there will be

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 44x - 24,\quad \frac{ddy}{dx^2} = 6x^2 - 24x + 22.$$

Now there is put in place

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 44x - 24 = 0 \quad \text{or} \quad x^3 - 6x^2 + 11x - 6 = 0;$$

three real values will arise for x

I. $x = 1$, II. $x = 2$, III. $x = 3$..

From the first value there becomes $\frac{ddy}{2dx^2} = 4$ and thus on putting $x = 1$ the proposed function becomes a minimum.
From the second value $x = 2$ there is made $\frac{ddy}{2dx^2} = -2$ and thus the proposed function has a maximum. From the third value $x = 3$ there becomes $\frac{ddy}{2dx^2} = +4$ and thus the proposed function again becomes a minimum.

EXAMPLE 5

Let this function be proposed $y = x^5 - 5x^4 + 5x^3 + 1$; *which is sought, in which cases it becomes a maximum or minimum.*

Since there shall be

$$\frac{dy}{dx} = 5x^4 - 20x^3 + 15xx$$

the equation may be formed $x^4 - 4x^3 + 3xx = 0$, the roots of which are

$$\text{I. and II.} \quad x = 0, \text{ III.} \quad x = 1, \text{ IV.} \quad x = 3.$$

Because the first and second roots are equal, neither a maximum nor minimum follows from these ; for there becomes $\frac{ddy}{dx^2} = 0$, but $\frac{d^3y}{dx^3}$ does not vanish. But the third root $x = 1$ on account of $\frac{ddy}{2dx^2} = 10x^3 - 30x^2 + 15x$ gives $\frac{ddy}{2dx^2} = -5$ and therefore in this case the function becomes a maximum. From the fourth root $x = 3$ there becomes $\frac{ddy}{2dx^2} = 45$ and thus the proposed function becomes a minimum.

EXAMPLE 6

To find the cases, in which this formula $y = 10x^6 - 12x^5 + 15x^4 - 20x^3 + 20$ *shall be a maximum or minimum.*

Hence there will be

$$\frac{dy}{dx} = 60x^5 - 60x^4 + 60x^3 - 60x^2 \quad \text{and} \quad \frac{ddy}{60dx^2} = 5x^4 - 4x^3 + 3x^2 - 2x.$$

The equation may be formed $x^5 - x^4 + x^3 - xx = 0$; since which resolved into factors shall be $x^2(x-1)(xx+1) = 0$, it has two equal roots $x = 0$ and besides the root $x = 1$ and the two imaginaries in addition from $xx + 1 = 0$. Therefore since the two equal roots $x = 0$ may show neither a maximum nor a minimum, only the root $x = 1$ remains to be considered, from which there comes about $\frac{ddy}{60dx^2} = 2$, the positive value of which indicates a minimum.

262. Therefore the determination of the maxima or minima depends on the roots of the differential equation $\frac{dy}{dx}=0$; the greatest power of which since it shall be less by one degree than the proposed function itself *y*, if indeed this were a rational integral function, it is clear, if this function were proposed in general

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \text{etc.} = y,$$

the maxima and minima of this are to be determined by the roots of this equation

$$nx^{n-1} + (n-1)Ax^{n-2} + (n-2)Bx^{n-3} + (n-3)Cx^{n-4} + \text{etc.} = 0.$$

We may put the real roots of this equation set out following the order of the magnitude to be $\alpha, \beta, \gamma, \delta$ etc. thus so that α shall be the greatest, $\beta < \alpha$, $\gamma < \beta$ etc. And indeed in the first place, if these roots were all unequal, each one of the proposed formulas *y* may lead to a maximum or minimum value and on this account as often as the function *y* will have maxima or minima, so also the equation $\frac{dy}{dx}=0$ will have real unequal roots. But if two or more roots should be equal to each other, thus the condition between the roots will be had, so that two equal roots may neither show a maximum or minimum, truly three equal roots may be equivalent to one ; and in general, if the number of equal roots were even, thence no maxima or minima result ; but if it were odd, thence one arises, either a maximum or minimum.

263. But which roots may produce maxima and which minima, can be defined thus without the help of the rule examined before. Since the function *y* on putting $x=\infty$ equally becomes infinite nor may the values of *x* within the bounds ∞ and α produce either a maximum or minimum, it is evident the values of the function *y*, while successively may be put in place of *x* from ∞ as far as α, are required to decrease continually ; and thus the value $x=\alpha+\omega$ induces a greater value to the function *y* than the value $x=\alpha$; from which, since $x=\alpha$ may produce a maximum or minimum, it is necessary, that in this case the function *y* is made a minimum. Hence on decreasing *x* further or on putting $x=\alpha-\omega$ the value of *y* will increase again, then there is made $x=\beta$, which is the second root of the equation $\frac{dy}{dx}=0$ producing a maximum or minimum root ; whereby this second root $x=\beta$ will give a maximum and the value $x=\beta-\omega$ will bring about a smaller value to the function *y* than $x=\beta$, then there is arrived at $x=\gamma$, which consequently again will generate a minimum. From which reasoning it is evident the first, third, fifth, etc. roots of the equation $\frac{dy}{dx}=0$ show a minimum, but the second, fourth, sixth, etc. show a maximum. But hence likewise it is understood in the case of two equal roots the maximum and minimum coalesce and thus neither has a place.

264. Therefore if in the proposed function

$$y = x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \text{etc.}$$

the maximum exponent *n* were an even number, the equation

$$\frac{dy}{dx} = x^{n-1} + (n-1)Ax^{n-2} + \text{etc.} = 0$$

[the coefficients *A*, etc. have been redefined on division by *n*,]
will be of odd degree and thus either it will have one real root or three or five or by an odd number. If a single root were real, that will give a minimum ; if three were real, the largest will become a minimum, the middle one a maximum, and the smallest a minimum again ; and if five roots were real, the function *y* will have three minima and two maxima ; and thus henceforth.

But if the exponent *n* were an odd number, the equation $\frac{dy}{dx} = 0$ thus will pertain to an even degree and thus it will have either no real roots, or two, or four or six, etc. In the first case *y* will have neither a maximum nor a minimum; in the other case, in which two roots are given, the greater of these will indicate a minimum, but the lesser a maximum; but the first of four roots (which is the greatest) and the third produce a minimum, truly the second and the fourth produce a maximum. Moreover however many roots were real, the maxima and minima always are following each other mutually in turn.

265. We may progress to rational fractional functions, from which the other kind of functions may be constructed. Therefore let there be

$$y = \frac{P}{Q}$$

for any [integral] functions *P* and *Q* of *x* present; and in the first place it appears indeed, if a value of such a kind may be given to *x*, so that there becomes $Q = 0$: unless *P* may likewise vanish, the function *y* goes off to infinity, which certainly may be considered as a maximum. Truly that case is unable to take anything less for a maximum, since indeed the inverse fraction $\frac{Q}{P}$ is made a minimum for the same cases, for which the proposed $\frac{P}{Q}$ is made a maximum, the fraction $\frac{Q}{P}$ must become a minimum, if *Q* vanishes; but this does not always come about, because hence at some point it may be able to adopt smaller values, evidently negative. Therefore with this doubt removed likewise the rule given before may be confirmed, because the maxima and minima ought to be elicited from the equation $\frac{dy}{dx} = 0$. Therefore in the case proposed there becomes

$$\frac{dy}{dx} = \frac{QdP - PdQ}{QQdx}$$

and thus

$$QdP - PdQ = 0$$

and the roots of this equation makes the function *y* either a maximum or minimum. And if there shall be doubt, whether a maximum or minimum may be established, it is required to take recourse to the value $\frac{ddy}{dx^2}$; which if it were positive, will indicate a minimum, but if it were negative, a maximum. But if indeed this value $\frac{ddy}{dx^2}$ also may vanish, which will happen, if the equation $\frac{dz}{dx} = 0$ may have two or more equal roots, always it is required to be understood for an equal even number neither a maximum nor a minimum is produced.

EXAMPLE 1

To find the cases, in which the function $\frac{x}{1+xx}$ becomes a maximum or minimum.

Indeed in the first place it is evident this function goes to zero in the three cases $x = \infty, x = 0$ et $x = -\infty$, from which according to two minima it will take either a maximum or minimum. Towards finding which there may be put $y = \frac{x}{1+xx}$ and there will be

$$\frac{dy}{dx} = \frac{1-xx}{(1+xx)^2} \quad \text{and} \quad \frac{ddy}{dx^2} = \frac{-6x+2x^3}{(1+xx)^3}.$$

Now there may be put in place $\frac{dy}{dx} = 0$; there will be $1 - xx = 0$ and either $x = +1$ or $x = -1$. In the first case $x = +1$ there becomes $\frac{ddy}{dx^2} = -\frac{4}{2^3}$ and thus *y* maximum $= \frac{1}{2}$; in the latter case $x = -1$ there becomes $\frac{ddy}{dx^2} = +\frac{4}{2^3}$ and thus *y* minimum $= -\frac{1}{2}$.

These also may be found more easily, if the proposed fraction $\frac{x}{1+xx}$ may be inverted by putting $y = \frac{1+xx}{x}$, while we may record then, which maxima are come upon, and in turn must be changed into minima. But there will be

$$\frac{dy}{dx} = 1 - \frac{1}{xx} \quad \text{and} \quad \frac{ddy}{dx^2} = \frac{2}{x^3}.$$

Therefore on putting $\frac{dy}{dx} = 0$ there is made $xx - 1 = 0$ from thence either $x = +1$ or $x = -1$ as before. But in the case $x = +1$ there is made $\frac{ddy}{dx^2} = 2$ and thus y a minimum and the proposed formula $\frac{1}{y}$ a maximum. But in the case $x=-1$ there becomes $\frac{ddy}{dx^2}=-2$, from which *y* becomes a maximum and $\frac{1}{y}$ a minimum.

EXEMPLUM 2

To find the cases, in which the formula $\frac{2-3x+xx}{2+3x+xx}$ is made a maximum or minimum.

On putting $y=\frac{2-3x+xx}{2+3x+xx}$ there will be

$$\frac{dy}{dx}=\frac{6x^2-12}{(xx+3x+2)^2} \quad \text{and} \quad \frac{ddy}{dx^2}=\frac{-12x^2+72x+72}{(xx+3x+2)^3}.$$

There may be put in place $\frac{dy}{dx}=0$; there is made either $x=+\sqrt{2}$ or $x=-\sqrt{2}$. In the first place $x=+\sqrt{2}$ there will be

$$\frac{ddy}{dx^2}=\frac{48\sqrt{2}+72}{\left(4+3\sqrt{2}\right)^3}$$

and thus positive on account of the positive numerator and denominator ; hence *y* will be a minimum

$$=\frac{4-3\sqrt{2}}{4+3\sqrt{2}}=12\sqrt{2}-17=-0{,}02943725.$$

In the latter case $x=-\sqrt{2}$ there becomes

$$\frac{ddy}{dx^2}=\frac{-48\sqrt{2}+72}{\left(4-3\sqrt{2}\right)^3}=\frac{24\left(3-2\sqrt{2}\right)}{\left(4-3\sqrt{2}\right)^3},$$

the value of which on account of the positive numerator and the negative denominator will be negative will be negative and thus *y* becomes a maximum

$$=\frac{4+3\sqrt{2}}{4-3\sqrt{2}}=-12\sqrt{2}-17=-33{,}97056274.$$

Which value, even if it is less than the first minimum, yet thus is a maximum, because it shall be the greater of the near neighbours, which arise, if in place of *x* either a little greater or smaller values than $-\sqrt{2}$ may be put in place. Therefore since $\sqrt{2}$ may be contained between the bounds $\frac{4}{3}$ and $\frac{3}{2}$, the evidence may be easily put in place in this manner :

if $x = \frac{4}{3}$, there becomes $y = -\frac{2}{70} = -0{,}0285$
if $x = \sqrt{2}$, there becomes $y = 12\sqrt{2} - 17 = -0{,}0294$ minimum
if $x = \frac{3}{2}$, there becomes $y = -\frac{1}{35} = -0{,}0285$

if $x = -\frac{4}{3}$, there becomes $y = -35$
if $x = -\sqrt{2}$, there becomes $y = -33{,}970$ maximum
if $x = -\frac{3}{2}$, there becomes $y = -35$.

EXAMPLE 3

To find the cases, in which the formula $\frac{xx-x+1}{xx+x-1}$ *becomes a maximum or minimum.*

There may be put $y = \frac{xx-x+1}{xx+x-1}$ and there will be

$$\frac{dy}{dx} = \frac{2xx-4x}{(xx+x-1)^2} \quad \text{and} \quad \frac{ddz}{dx^2} = \frac{-4x^3+12xx+4}{(xx+x-1)^3}.$$

There may be put $\frac{dy}{dx} = 0$; there will be either $x = 0$ or $x = 2$; in the first case there becomes $\frac{ddy}{dx^2} = \frac{4}{-1}$ and thus the maximum y will be $= -1$. In the latter case $x = 2$ there is made $\frac{ddy}{dx^2} = \frac{20}{5^3}$ and thus the minimum $y = \frac{3}{5}$, even if that maximum shall be less than this minimum. The evidence will be apparent from these put in place

if $x = -\frac{1}{3}$, there will be $y = -\frac{13}{11}$
if $x = 0$, there will be $y = -1$ maximum
if $x = +\frac{1}{3}$, there will be $y = -\frac{7}{5}$

if $x = 2 - \frac{1}{3}$, there will be $y = \frac{19}{31}$
if $x = 2$, there will be $y = \frac{3}{5}$ minimum
if $x = 2 + \frac{1}{3}$, there will be $y = \frac{37}{61}$.

Because moreover, if there is put $x = 1$, there is made $y = 1$ and thus > -1, the reason being, because between the values of x, 0 and 1, there may be one present, for which there becomes $y = \infty$.

EXAMPLE 4

The cases are sought, in which this function $\frac{x^3+x}{x^4-xx+1}$ *becomes a maximum or minimum.*

On putting $y=\frac{x^3+x}{x^4-xx+1}$ there will be

$$\frac{dy}{dx}=\frac{-x^6-4x^4+4xx+1}{\left(x^4-xx+1\right)^2} \quad \text{and} \quad \frac{ddy}{dx^2}=\frac{2x^9+18x^7-30x^5-16x^3+12xx}{\left(x^4-xx+1\right)^3}$$

Hence we will have this equation

$$x^6+4x^4-4xx-1=0,$$

which is resolved into these two equations

$$xx-1=0 \text{ and } x^4+5x^2+1=0;$$

the roots of the former are $x=+1$ and $x=-1$, the latter resolved gives indeed

$$xx=-\frac{5\pm\sqrt{21}}{2},$$

from which no real root emerges. Therefore of the two roots found initially, $x=+1$ makes $\frac{ddy}{dx^2}=-14$ and therefore the maximum y = 2; the other root $x=-1$ makes $\frac{ddy}{dx^2}=+14$ and therefore the minimum $y=-2$.
[Some minor changes have been made in the *O.O.* edition to correct errors in the first edition.]

EXAMPLE 5

To find the cases, in which this fraction $\frac{x^3-x}{x^4-xx+1}$ *becomes a maximum or a minimum.*

On putting $y=\frac{x^3-x}{x^4-xx+1}$ there will be

$$\frac{dy}{dx}=\frac{-x^6+2x^4+2x^2-1}{\left(x^4-x^2+1\right)^2} \quad \text{and} \quad \frac{ddy}{dx^2}=\frac{2x^9-6x^7-18x^5+20x^3}{\left(x^4-x^2+1\right)^3}.$$

But on putting $\frac{dy}{dx}=0$ there will be

$$x^6-2x^4-2x^2+1=0$$

which divided by $xx+1$ gives

$$x^4 - 3x^2 + 1 = 0,$$

and this may be resolved further into

$$xx - x - 1 = 0 \quad \text{and} \quad xx + x - 1 = 0$$

from which the following four real roots arise

$$\text{I.} \quad x = \tfrac{1+\sqrt{5}}{2}, \qquad \text{II.} \quad x = \tfrac{1-\sqrt{5}}{2},$$
$$\text{III.} \quad x = -\tfrac{1+\sqrt{5}}{2}, \quad \text{IV.} \quad x = -\tfrac{1-\sqrt{5}}{2}.$$

Which since all may be contained in the equation $x^4 - 3xx + 1 = 0$, on putting $x^4 = 3xx - 1$ there becomes for all

$$\frac{ddy}{dx^2} = \frac{2x(10-20xx)}{8x^6} = \frac{5(1-2xx)}{2x^5} = \frac{5(1-2xx)}{2x(3xx-1)} \quad \text{and} \quad y = \frac{x^3-x}{2xx} = \frac{xx-1}{2x}.$$

But for the first two roots coming from the equation $xx = x + 1$ there will be

$$\frac{ddy}{dx^2} = -\frac{5(2x+1)}{2x(3x+2)} = -\frac{5(2x+1)}{2(5x+3)} \quad \text{and} \quad y = \tfrac{1}{2}.$$

Therefore the first root $x = \frac{1+\sqrt{5}}{2}$ gives

$$\frac{ddy}{dx^2} = -\frac{5(2+\sqrt{5})}{11+5\sqrt{5}}$$

and thus y is a maximum. The second root $x = \frac{1-\sqrt{5}}{2}$ gives

$$\frac{ddy}{dx^2} = -\frac{5(2-\sqrt{5})}{11-5\sqrt{5}} = -\frac{5(\sqrt{5}-2)}{5\sqrt{5}-11}$$

and thus $y = \frac{1}{2}$ will be a maximum also. The two remaining roots $y = -\frac{1}{2}$ give a minimum.

266. Therefore in these examples investigated, whether some value found will produce a maximum or minimum, will be able to be put in place more easily; for since there shall be $\frac{dy}{dx} = 0$, the value of the term $\frac{ddy}{dx^2}$ of which equation will be able to be expressed more simply. For let the proposed fraction be $y = \frac{P}{Q}$; since there shall be

$$dy = \frac{QdP-PdQ}{QQ} \text{ and } QdP - PdQ = 0,$$

there will be

$$ddy = \frac{d\left(QdP-PdQ\right)}{Q^2} - \frac{2dQ\left(QdP-PdQ\right)}{Q^3}.$$

But truly on account of $QdP - PdQ = 0$ here the latter term vanishes and there will be

$$ddy = \frac{d\left(QdP-PdQ\right)}{QQ} = \frac{QddP-PddQ}{Q^2}.$$

Truly because the judgement is demanded from the value of this term, either positive or negative, but the denominator Q^2 shall be positive constantly, the matter will be able to be completed from the numerator alone, so that as often as $QddP - PddQ$ or $\frac{d\left(Qd-PdQ\right)}{dx^2}$ should be positive, it may be proclaimed a minimum, but if it shall be negative, a maximum. Or if after $\frac{dy}{dx}$ had been found, of which the form shall be of this kind $\frac{R}{QQ}$, only $\frac{dR}{dx}$ may be sought and the root for which expression adopts a positive value, from that there may arise a minimum and with the contrary a maximum.

267. If the denominator of the fraction were a quadratic or some higher power, thus so that there shall be $y = \frac{P}{Q^n}$, there is made

$$dy = \frac{QdP-nPdQ}{Q^{n+1}}$$

and on putting $\frac{QdP-nPdQ}{dx} = R$ there will be

$$\frac{dy}{dx} = \frac{R}{Q^{n+1}}$$

and the maxima and minima will be determined from the roots of the equation $R = 0$. Since then there shall be

$$\frac{ddy}{dx} = \frac{QdR-\left(n+1\right)RdQ}{Q^{n+2}},$$

on account of $R = 0$ there becomes

$$\frac{ddy}{dx} = \frac{dR}{Q^{n+1}};$$

a positive value of which will indicate a minimum, but a negative value a maximum. Moreover it is understood, if n were an odd number, on account of Q^{n+1} positive a judgement can be brought about from $\frac{dR}{dx}$ alone; but if *n* shall be an even number, the formula $\frac{QdR}{dx}$ may be used.

But again we may put in place a fraction to be proposed of this kind $\frac{P^m}{Q^n} = y$; there will be

$$dy = \frac{(mQdP - nPdQ)P^{m-1}}{Q^{n+1}};$$

And thus if there is put in place $\frac{mQdP - nPdQ}{dx} = R$, the roots of the equation $R = 0$ will indicate the cases, in which the function *y* becomes either a maximum or minimum. Therefore since there shall be

$$\frac{dy}{dx} = \frac{P^{m-1}R}{Q^{n+1}}$$

there will be

$$\frac{ddy}{dx} = \frac{P^{m-2}R((m-1)QdP - (n+1)PdQ)}{Q^{n+2}} + \frac{P^{m-1}dR}{Q^{n+1}},$$

and on account of $R = 0$ there becomes

$$\frac{ddy}{dx^2} = \frac{P^{m-1}dR}{Q^{n+1}dx};$$

which can be divided in addition by some quadratic $\frac{P^{2\mu}}{Q^{2\nu}}$ according to the judgement to be resolved. Therefore indeed the equation $P = 0$ will give a maximum or minimum also, if *m* were an even number ; and in a like manner by considering the inverse formula $\frac{Q^n}{P^m}$ a maximum or minimum will be produced on putting $Q = 0$, if *n* were an even number, as we have considered above (§ 257) ; but here we will not look at the maxima or minima hence arising, but consider those only requiring to be explained by the use of the method, which may arise from the equation $R = 0$.

EXAMPLE 1

The fraction may be proposed $\frac{(\alpha+\beta x)^m}{(\gamma+\delta x)^n}$; *in which the case is sought which makes a maximum or minimum.*

On putting $y=\frac{(\alpha+\beta x)^m}{(\gamma+\delta x)^n}$ indeed in the first place there appears to be $y=0$, if $x=-\frac{\alpha}{\beta}$, and $y=\infty$, if $x=-\frac{\gamma}{\delta}$; of which cases the former will give a minimum, the latter truly a minimum, if m and n were even numbers. Therefore indeed there will be

$$\frac{dy}{dx}=\frac{(\alpha+\beta x)^{m-1}}{(\gamma+\delta x)^{n+1}}\left((m-n)\beta\delta x+m\beta\gamma-n\alpha\delta\right)$$

and thus

$$R=(m-n)\beta\delta x+m\beta\gamma-n\alpha\delta.$$

Whereby on putting $R = 0$ there will be

$$x=\frac{n\alpha\delta-m\beta\gamma}{(m-n)\beta\delta}.$$

Then on account of $\frac{dR}{dx}=(m-n)\beta\delta$ it is to be considered, whether

$$\frac{P^{m-1}dR}{Q^{n+1}dx}=\frac{m^{m-1}\beta^{n+1}}{n^{n+1}\delta^{m-1}}\left(\frac{\alpha\delta-\beta\gamma}{m-n}\right)^{m-n-2}\frac{dR}{dx}$$

shall be a positive or negative quantity. In the first case the proposed formula will be a minimum, in the latter a maximum.

Thus if there were $y=\frac{(x+3)^3}{(x+2)^2}$, there becomes $\frac{P^{m-1}dR}{Q^{n+1}dx}=\frac{9}{8}$ and thus the formula $\frac{(x+3)^3}{(x+2)^2}$ becomes a minimum, if there may be put $x=0$.

But if there shall be $y=\frac{(x-1)^m}{(x+1)^n}$, there will be

$$\frac{P^{m-1}dR}{Q^{n+1}dx}=\frac{m^{m-1}}{n^{n+1}}\left(\frac{n-m}{2}\right)^{n-m+2}(m-n)$$

and $x=\frac{n+m}{n-m}$. But since m and n may be put positive numbers, the judgement will be desired from the $(n-m)^{n-m+2}(m-n)$ or $(n-m)^{n-m}(m-n)$. Therefore if there were

$n > m$, the value elicited $x = \frac{n+m}{n-m}$ will give a maximum always; but if $n < m$, the even number $m - n$ will give a minimum, but odd a maximum; thus $\frac{(x-1)^3}{(x+1)^2}$ becomes a maximum on putting $x = -5$; indeed there is made $y = -\frac{6^3}{4^2} = -\frac{27}{2}$.

EXAMPLE 2

The formula may be proposed $y = \frac{(1+x)^3}{(1+xx)^2}$.

There will be

$$\frac{dy}{dx} = \frac{(1+x)^2}{(1+xx)^3}(3-4x-xx) \quad \text{and} \quad \frac{P^{m-1}dR}{Q^{n+1}dx} = -\frac{(1+x)^2}{(1+xx)^3}(2x+4).$$

where since $(1+x)^2$ and $(1+xx)^3$ may always have a positive value, the judgement may be left to the formula $-x-2$; which, if it were positive, may indicate a minimum, but if negative, a maximum. But truly from the equation $3-4x-xx=0$ there follows either

$$x = -2+\sqrt{7} \quad \text{or} \quad x = -2-\sqrt{7}.$$

In the first case there becomes $-x-2 = -\sqrt{7}$ and thus the fraction proposed will be a maximum, truly in the latter case a minimum on account of $-x-2 = +\sqrt{7}$. But on putting $x = -2+\sqrt{7}$ there will be $1+x = -1+\sqrt{7}$ and $1+xx = 12-4\sqrt{7}$, from which

$$y = \left(\frac{-1+\sqrt{7}}{12-4\sqrt{7}}\right)^2\left(\sqrt{7}-1\right) = \frac{\left(2+\sqrt{7}\right)^2\left(\sqrt{7}-1\right)}{16} = \frac{17+7\sqrt{7}}{16} = 2{,}220.$$

But on putting $x = -2-\sqrt{7}$ there becomes

$$y = \frac{17-7\sqrt{7}}{16} = -0{,}0950.$$

268. Also irrational and transcending functions are given, which have the property of uniform functions, and for this reason the maxima and minima can be found in the same manner. Indeed the roots of the cube and of all the odd powers actually are uniform, since they may show a single real root only ; but the roots of the quadratic and of all even powers although in fact, as often as they are real, indicate a twin value, the one positive, the other negative, yet each one can be regarded separately and in this sense also the maxima and minima are able to be investigated. Thus if y were some function of x, although $\sqrt{y}$ has a twin value, yet each can be treated separately. Clearly $+\sqrt{y}$ will have a maximum or minimum value, if y should have such a value, provided it were positive, because otherwise

$\sqrt{y}$ may emerge imaginary. But on the other hand $-\sqrt{y}$ becomes a minimum or a maximum in the same cases, in which $+\sqrt{y}$ is made a maximum or minimum. But some power $y^{\frac{m}{n}}$ in the same cases will become a maximum or minimum, if indeed n were an odd number; but if n were an even number, only these cases prevail, in which y adopts a positive value, and from these cases on account of the double value, the twins will produce maxima or minima.

269. Because the differential equation, which arises from the power of the function y^m, is $\frac{y^{m-1}dy}{dx}=0$, likewise the roots of which indicate the cases, in which the powers from the irrational $y^{\frac{m}{n}}$ becomes a maximum or minimum, towards finding this a two-fold equating is had, the one $y^{m-1}=0$, the other $\frac{dy}{dx}=0$, of which the former will become $y=0$ and then only it may show a maxima and minima, if $m-1$ were an odd number or if m were an even number, on account of the reasons put forward in § 257. Whereby since n shall be an odd number, if m were an even number, if we may indicate the even number by 2μ and the odd number by $2\nu-1$, the function $y^{2\mu:(2\nu-1)}$ will come out either maximum or minimum with the values of x attributed, which arise from this equation $y=0$ as well as from $\frac{dy}{dx}=0$. But if m shall be an odd number, the function $y^{(2\mu-1):(2\nu-1)}$ or $y^{(2\mu-1):2\nu}$ then only will become a maximum or minimum, when the value is substituted in place of x from this equation $\frac{dy}{dx}=0$. And indeed in the second case $y^{(2\mu-1):2\nu}$ maxima and minima only are arrived at, if y may take positive values from the values found from the equation $\frac{dy}{dx}=0$.

270. Thus this formula $x^{\frac{2}{3}}$ becomes a minimum on putting $x=0$, therefore since in this case x^2 becomes a minimum. But unless we may reduce the formula $x^{\frac{2}{3}}$ to the form x^2, the method treated before may be indicated very briefly, therefore because in the case $x=0$ the terms of the series

$$y+\frac{\omega dy}{dx}+\frac{\omega^2 ddz}{2dx^2}+\frac{\omega^3 d^3z}{6dx^3}+\text{etc.},$$

from which a judgement must be sought, besides the first all become infinite. For on making $y=x^{\frac{2}{3}}$ there will be

$$\frac{dy}{dx}=\frac{2}{3x^{\frac{1}{3}}},\quad \frac{ddy}{dx^2}=\frac{-2}{9x^{\frac{4}{3}}},\quad \frac{d^3y}{dx^3}=\frac{2\cdot 4}{27x^{\frac{7}{3}}}\quad \text{etc.}$$

Hence neither the equation $\frac{dy}{dx} = \frac{2}{3x^{\frac{1}{3}}} = 0$ will show the value $x = 0$ nor the following terms will show the ratio of the maximum or minimum. Therefore since we have assumed the series

$$y + \frac{\omega dy}{dx} + \frac{\omega^2 ddz}{2dx^2} + \frac{\omega^3 d^3 z}{6dx^3} + \text{etc.}$$

to be made converging, if a very small quantity ω were put in place, these cases certainly avoid the general method, in which this series becomes diverging, which happens in the example reported here $y = x^{\frac{2}{3}}$, if there may be put $x = 0$. On account of which in these cases from that reduction, which we have used before, there will be a need, by which the proposed expression may be recalled to another form, which shall not be subjected to this inconvenience. But this only arises in use in a very small number of cases, which may be contained in the formula $y^{\frac{2\mu}{2\nu-1}}$ or may be reduced easily to that. Thus if the maxima or minima of the formula $y^{\frac{2\mu}{2\nu-1}}z$ may be required for some function *z* of *x*, this form $y^{2\mu}\, z^{2\nu-1}$ may be considered, clearly which in the same cases shall be a maximum or minimum, with that itself proposed.

271. With this case excepted, which now may be set out easily, functions which contain irrational quantities, can be treated in the same manner in which rationals are to be treated and the maxima and minima of these determined, that which we will illustrate in the following examples.

EXAMPLE 1

The formula shall be proposed $\sqrt{(aa + xx)} - x$ *; which shall be sought in which cases it becomes a maximum or minimum.*

On putting $y = \sqrt{(aa + xx)} - x$ there will be

$$\frac{dy}{dx} = \frac{x}{\sqrt{(aa+xx)}} - 1 \quad \text{and} \quad \frac{ddy}{dx^2} = \frac{aa}{(aa+xx)^{3:2}}.$$

Hence with $\frac{dy}{dx} = 0$ put in place there will be $x = \sqrt{(aa + xx)}$ and thus $x = \infty$ and there is made $\frac{ddy}{dx^2} = 0$. In a similar manner indeed all the following terms $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ etc. become $= 0$; from which an uncertain judgement is left, whether it shall be a maximum or minimum. The reason is, because actually there becomes $x = -\infty$ as well as $x = +\infty$. Meanwhile of putting $x = +\infty$ on putting

$$\sqrt{(aa+xx)}=x+\tfrac{aa}{2x}$$

there is made $y=0$, which value is a minimum of all.

EXAMPLE 2

The cases are sought, in which this form $\sqrt{(aa+2bx+mxx)}-nx$ *becomes a maximum or minimum.*

On putting $y=\sqrt{(aa+2bx+mxx)}-nx$ there will be

$$\tfrac{dy}{dx}=\tfrac{b+mx}{\sqrt{(aa+2bx+mxx)}}-n\,;$$

so that on making $=0$ there will be

$$bb+2mbx+mmxx=nnaa+2nnbx+mnnxx$$

or

$$xx=\tfrac{2bx(nn-m)+nnaa-bb}{mm-mnn}$$

and thus

$$x=\tfrac{(nn-m)b\pm\sqrt{(mnn(m-nn)aa-nn(m-nn)bb)}}{m(m-nn)}$$

or

$$x=-\tfrac{b}{m}\pm\tfrac{n}{m}\sqrt{\tfrac{maa-bb}{m-nn}}$$

from which there comes about

$$\sqrt{(aa+2bx+mxx)}=\tfrac{b+mx}{n}=\pm\sqrt{\tfrac{maa-bb}{m-nn}}.$$

Therefore since there shall be

$$\tfrac{ddy}{dx^2}=\tfrac{maa-bb}{(aa+2bx+mxx)^{\frac{3}{2}}},$$

there will be

$$\tfrac{ddy}{dx^2}=\tfrac{maa-bb}{\pm\left(\frac{maa-bb}{m-nn}\right)^{\frac{3}{2}}}=\tfrac{\pm(m-nn)\sqrt{(m-nn)}}{\sqrt{(maa-bb)}}$$

Therefore unless $\tfrac{m-nn}{maa-bb}$ becomes a positive quantity, plainly neither a maximum nor minimum is given. But if it shall be a positive quantity, the upper sign will give a minimum, if $m > nn$, truly a maximum, if $m < nn$; the opposite comes about, if the lower sign may prevail. Therefore if there shall be $m = 2,\ n = 1$ and $b = 0$, the formula $\sqrt{(aa + 2xx)} - x$ is made a minimum on putting $x = +\frac{1}{2}\sqrt{2aa} = \frac{a}{\sqrt{2}}$, but a maximum on putting $x = -\frac{a}{\sqrt{2}}$. Therefore the minimum will be $= a\sqrt{2} - \frac{a}{\sqrt{2}} = \frac{a}{\sqrt{2}}$ and the maximum $= a\sqrt{2} + \frac{a}{\sqrt{2}} = \frac{3a}{\sqrt{2}}$.

EXAMPLE 3

The cases are sought, in which this expression $\sqrt[4]{(1+mx^4)} + \sqrt[4]{(1-nx^4)}$ *is made a maximum or minimum.*

Since there shall be $\frac{dy}{dx} = \frac{mx^3}{(1+mx^4)^{\frac{3}{4}}} - \frac{nx^3}{(1-nx^4)^{\frac{3}{4}}}$, there becomes

$mx^3(1-nx^4)^{\frac{3}{4}} = nx^3(1+mx^4)^{\frac{3}{4}}$ and thus $m^4(1-nx^4)^3 = n^4(1+mx^4)^3$

or

$$n^4 - m^4 + 3mn\left(n^3 + m^3\right)x^4 + 3m^2n^2\left(n^2 - m^2\right)x^8 + m^3n^3(n+m)x^{12} = 0.$$

Therefore unless this equation may have a positive root for x^4, thus a maximum or minimum is not given. Because this equation generally cannot be resolved conveniently, for there becomes

$$x^4 = \frac{m^{\frac{4}{3}} - n^{\frac{4}{3}}}{mn\left(\sqrt[3]{m} + \sqrt[3]{n}\right)} \quad \text{or} \quad x^4 = \frac{m - \sqrt[3]{m^2n} + \sqrt[3]{mn^2} - n}{mn},$$

we may put for a special case $m = 8n$ and there will be

$$-4095 + 24 \cdot 513nx^4 - 3 \cdot 63 \cdot 64n^2x^8 + 9 \cdot 512n^3x^{12} = 0$$

or

$$512n^3x^{12} - 1344n^2x^8 + 1368nx^4 - 455 = 0;$$

there may be put $8nx^4 = z$; there will be

$$z^3 - 21z^2 + 171z - 455 = 0,$$

which has the divisor $z-5$, and the other factor will $zz-16z+91=0$ containing imaginary roots. Therefore there will be only $z=8nx^4=5$ and thus $x=\sqrt[4]{\frac{5}{8n}}$, which value returns the expression $\sqrt[4]{\left(1+mx^4\right)}+\sqrt[4]{\left(1-nx^4\right)}$ a maximum or minimum.

Whether of which may eventuate, there may be sought

$$\frac{ddy}{dx^2}=\frac{3mxx}{\left(1+mx^4\right)^{\frac{7}{4}}}-\frac{3nxx}{\left(1-nx^4\right)^{\frac{7}{4}}}.$$

But on account of $m=8n$ on putting $x^4=\frac{5}{8n}$ there will be

$$\frac{ddy}{dx^2}=\left(\frac{24n}{6^{\frac{7}{4}}}-\frac{3n}{(3:8)^{\frac{7}{4}}}\right)xx=-\frac{360nxx}{6^{\frac{7}{4}}}$$

and thus negative; therefore $\sqrt[4]{\left(1+8nx^4\right)}+\sqrt[4]{\left(1-nx^4\right)}$ becomes a maximum on putting $x=\sqrt[4]{\frac{5}{8n}}$. Truly this maximum will be $\sqrt[4]{6}+\sqrt[4]{\frac{3}{8}}=\frac{3\sqrt[4]{6}}{2}$. If in place of nx^4 we may put *u*, it is apparent this expression $\sqrt[4]{(1+8u)}+\sqrt[4]{(1-u)}$ becomes a maximum on putting $u=\frac{5}{8}$ and this maximum value will be $=\frac{3\sqrt[4]{6}}{2}=2{,}347627$. Therefore whichever value may be written besides $\frac{5}{8}$ or *u*, the expression will take a smaller value.

272. The maxima and minima will be determined in a similar manner, if transcending quantities also may be present in the proposed expression. For unless the function proposed were of several forms and some significance of that likewise must be considered, the roots from the differential equations will show maxima or minima, unless equal roots were present, the number of which shall be even. Therefore we will indicate this investigation by some examples.

EXAMPLE 1

To find the number which may hold the minimum ratio to its logarithm.

Thence it is apparent $\frac{x}{lx}$ be given the minimum ratio of this kind, because this ratio is made infinite both on putting $x=1$ as well as $x=\infty$. Therefore in turn the fraction $\frac{lx}{x}$ somewhere will have a maximum value, evidently in the same case, by which $\frac{x}{lx}$ becomes a minimum. Towards finding this case there is put $y=\frac{lx}{x}$; and there becomes

$$\frac{dy}{dx} = \frac{1}{xx} - \frac{lx}{xx}.$$

So that on putting equal to zero, there will be $lx = 1$, and because here we have assumed the hyperbolic logarithm, if there may be put the number *e*, the hyperbolic logarithm of which shall be $= 1$, there will be $x = e$. Therefore since all the logarithms shall be in a given ratio to the hyperbolic ones, in any table of logarithms $\frac{e}{le}$ will be a minimum or $\frac{le}{e}$ a maximum. Because there is $le = 0{,}4342944819$ from tables of logarithms, the fraction $\frac{lx}{x}$ always will be less than $\frac{4842944819}{27182818284}$ or approximately as $\frac{47}{305}$, nor may any other number be given which holds a ratio to its logarithm smaller than 305 to 47. But in this case $\frac{lx}{x}$ thence may be shown to be a maximum, because on account of $\frac{ddy}{dx} = \frac{1-lx}{xx}$ there is made

$$\frac{ddy}{dx^2} = -\frac{1}{x^3} - \frac{2(1-lx)}{x^3} = -\frac{1}{x^3}$$

on account of $1 - lx = 0$, and thus is negative.

EXAMPLE 2

To find the number x, so that these powers $x^{1:x}$ becomes a maximum.

The maximum value of this formula thence appears to be given, because in place of the number *x* on being substituted there shall be

$$\begin{aligned} 1^{1:1} &= 1{,}000000 \\ 2^{1:2} &= 1{,}414213 \\ 3^{1:3} &= 1{,}442250 \\ 4^{1:4} &= 1{,}414213. \end{aligned}$$

Therefore there may be put $x^{1:x} = y$ and there will be

$$\frac{dy}{dx} = x^{1:x}\left(\frac{1}{xx} - \frac{lx}{xx}\right).$$

With which value placed equal to zero there will be $lx = 1$ and $x = e$ with $e = 2{,}718281828$ present.
And since there shall be $\frac{dy}{dx} = (1 - lx)\frac{x^{1:x}}{xx}$, there will be

$$\frac{ddy}{dx^2} = -\frac{x^{1:x}}{x^3} + (1 - lx)\frac{d}{dx}.\frac{x^{1:x}}{xx} = -\frac{x^{1:x}}{x^3}$$

on account of $1 - lx = 0$. Whereby since $\frac{ddy}{dx^2}$ shall be a negative quantity, $x^{1:x}$ is made a maximum in the case $x = e$. But since there shall be $e = 2{,}718281828$, there is found to be $e^{\frac{1}{e}} = 1{,}444667861009764$, which value may be obtained easily from the series

$$e^{\frac{1}{e}} = 1 + \frac{1}{e} + \frac{1}{2e^2} + \frac{1}{6e^3} + \frac{1}{24e^4} + \text{etc.}$$

This example is resolved also from the preceding; if indeed there shall be a maximum $x^{1:x}$, also the logarithm of that, which is $\frac{lx}{x}$, will be a maximum; because with that made, there must be $x = e$, as we have found.

EXAMPLE 3

To find the arc x, so that the sine of that shall be a maximum or minimum.

On putting $\sin x = y$ there will be $\frac{dy}{dx} = \cos x$ and thus $\cos x = 0$, from which the following values will be produced for x: $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, $\pm\frac{5\pi}{2}$, etc. But there becomes $\frac{ddy}{dx^2} = -\sin x$. Therefore since these values substituted for x will give for $\sin x$ either $+1$ or -1, those former will be maxima, these latter truly minima, as agreed.

EXAMPLE 4

To find the arc x, so that the rectangle x sin *x becomes a maximum.*

A maximum to be given thence is apparent, because on putting either $x = 0°$ or $x = 180°$ in each case the proposed rectangle may vanish. Therefore let $y = x\sin x$; there will be

$$\frac{dy}{dx} = \sin x + x\cos x$$

and thus [the condition for a max. or min. becomes]

$$\operatorname{tg} x = -x .$$

Let $x = 90° + u$; there will be $\operatorname{tg} x = -\cot u$, therefore $\cot u = 90° + u$. But towards resolving that equation treated above there is put $z = 90° + u - \cot u$ and let f be the value of the arc u. Since there shall be $dz = du + \frac{du}{\sin^2 u}$, there will be

$$p = \frac{du}{dz} = \frac{\sin^2 u}{1+\sin^2 u}, \quad dp = \frac{2du \sin u \cos u}{(1+\sin u)^2}$$

and thus

$$\frac{dp}{dz} = q = \frac{2\sin^3 u \cos u}{\left(1+\sin^2 u\right)^3}, \quad dq = \frac{6du \sin^2 u \cos^2 u - 2du \sin^4 u}{\left(1+\sin^2 u\right)^3} - \frac{12du \sin^4 u \cos^2 u}{\left(1+\sin^2 u\right)^4}.$$

Hence

$$\frac{dq}{dz} = r = \frac{6\sin^4 u \cos^2 u - 2\sin^6 u}{\left(1+\sin^2 u\right)^4} - \frac{12\sin^6 u \cos^2 u}{\left(1+\sin^2 u\right)^5} = \frac{6\sin^4 u - 14\sin^6 + 4\sin^8 u}{\left(1+\sin^2 u\right)^5}.$$

From which there will be

$$f = u - pz + \tfrac{1}{2} qzz - \tfrac{1}{6} rz^3 + \text{etc.}$$

There may be put, after the nearest value of f has been uncovered from some attempts, $u = 26°15'$; there will be $90° + u = 116°15'$ and the arc equal to the cotangent u thus may be defined. From

$$l \cot u = 10{,}3070250$$

there may be subtracted $\quad 4{,}6855749$

$$5{,}6214501$$

Hence

$$\cot u = 418263, 7''$$

or

$$\cot u = 116^0 11' 3\tfrac{7}{10}''$$

from which

$$z = 3' 56\tfrac{3}{10}'' = 236{,}3''.$$

Now for the value of the term pz required to be found the calculation may be put in place:

$$l \sin u = 9{,}6457058$$
$$l \sin^2 u = 9{,}2914116$$
$$1 + \sin^2 u = 1{,}19561$$
$$l\left(1+\sin^2 u\right) = 0{,}0775895$$
$$lp = 9{,}2138221$$
$$lz = 2{,}3734637$$
$$lpz = 1{,}5872858$$

Hence

$$pz = 38{,}6621 \text{ seconds of arc}$$

or

$$pz = 38'' \, 39''' \, 43''''$$

from

$$u = 26°15'$$

there becomes

$$f = 26° \, 14' \, 21'' \, 20''' \, 17''''$$

and the arc sought

$$x = 116°14'21''20'''17''''.$$

Truly the third term must be added to the above $\frac{1}{2}qzz = \frac{\sin^3 u \cos u}{\left(1+\sin^2 u\right)^3} zz$. In order that the value of which may be found, one z must be expressed in parts of the radius in this manner:

$$lz = 2{,}3734637$$

there may be added

$$\underline{4{,}6855749}$$
$$7{,}0590386$$

there may be added

$$l\frac{\sin^2 u}{1+\sin^2 u} z = \underline{1{,}5872858}$$
$$8{,}6463244$$

there may be added

$$l\sin u = 9{,}6457058$$
$$l\cos u = 9{,}9527308$$
$$\underline{8{,}2447600}$$

taking

$$l\left(1+\sin^2 u\right)^2 = \underline{0{,}1551790}$$
$$\tfrac{1}{2} lqzz = 8{,}08958$$

Therefore

$$\tfrac{1}{2} qzz = 0{,}012291$$

or

$$\tfrac{1}{2} qzz = 44''''15'''''.$$

And hence from using this term the arc sought becomes

$$x = 116°14'21''21'''0'''' ;$$

but with more logarithms being used there is found

$$x = 116^\circ 14' 21'' 20''' 35'''' 47''''' .$$

[Thus a method for finding the approximate root of a transcending equation is set out from an initial approximation refined by a Taylor series expansion.]

CAPUT X

DE MAXIMIS ET MINIMIS

250. Si functio ipsius *x* ita fuerit comparata, ut crescentibus valoribus *x* ipsa continuo crescat vel decrescat, tum ista functio nullum habebit valorem maximum minimumve. Quicunque enim huius functionis valor consideretur, sequentes erunt maiores, praecedentes vero minores. Huiusmodi functio est $x^3 + x$, cuius valor crescentibus *x* continuo crescit, decrescentibus vero *x* continuo decrescit; maximum ergo haec functio valorem alium induere nequit, nisi ipsi *x* valor maximus, hoc est infinitus, tribuatur; similique modo minimum obtinebit valorem, si ponatur $x = -\infty$. Nisi autem functio ita fuerit comparata, ut crescente *x* continuo crescat decrescatve, maximum vel minimum alicubi habebit valorem, hoc est eiusmodi valorem, qui sit vel maior vel minor quam antecedentes et sequentes. Sic ista functio $xx - 2x + 3$ minimum valorem induit, si ponatur $x = 1$; quicunque enim alius valor ipsi *x* tribuatur, perpetuo functio maiorem adipiscetur valorem.

251. Quo autem natura maximorum ac minimorum clarius perspiciatur, ponamus *y* eiusmodi esse functionem ipsius *x*, quae maximum obtineat valorem, si ponatur $x = f$, atque intelligitur, si *x* ponatur sive maius sive minus quam *f*, tum valorem ipsius *y* inde oriundum minorem fore ino, quem induit, si ponatur $x = f$. Simili modo, si posito $x = f$ functio *y* minimum obtineat valorem, necesse est, ut, sive *x* ponatur maius quam *f* sive minus, semper maior ipsius *y* valor resultet; haecque est definitio maximorum et minimorum absolutorum. Praeterea autem quoque functio *y* maximum valorem recipere dicitur posito verbi gratia $x = f$, dummodo iste valor maior fuerit quam proximi, sive sequentes sive antecedentes, qui oriuntur, si *x* aliquantillum sive maius sive minus quam *f* statuatur, etiamsi aliis valoribus loco *x* substituendis functio y maiores forte valores recipiat. Similiter functio *y* minimum valorem recipere dicitur posito $x = f$, dummodo ille valor minor fuerit iis, quos induit, si loco *x* valores proxime sive maiores sive minores quam *f* substituantur. Atque in hac posteriori significatione istis maximorum et minimorum vocabulis utemur.

252. Antequam autem modum ostendamus haec maxima et minima inveniendi, notari convenit hanc investigationem proprie in iis tantum ipsius *x* functionibus locum habere, quas supra *uniformes* vocavimus et quae sunt ita comparatae, ut pro singulis ipsius *x* valoribus singulos pariter valores recipiant. *Biformes* autem et *multiformes* functiones vocavimus, quae pro singulis valoribus ipsius *x* binos pluresve valores inducunt, cuiusmodi functiones sunt radices aequationum quadraticarum et plurium dimensionum. Si igitur *y* huiusmodi fuerit functio ipsius *x* vel biformis vel multiformis, tum proprie dici nequit eam posito $x = f$ valorem sive maximum sive minimum induere; quoniam enim posito $x = f$ vel duos pluresve valores simul obtinet atque praecedentes aeque ac sequentes sint numero plures, diiudicatio maximi minimive non tam facile instituitur, nisi forte omnes functionis y valores, qui singulis ipsius x valoribus respondent, sint imaginarii praeter unum; quo casu huiusmodi functiones speciem functionum uniformium mentiuntur. Primum ergo functiones uniformes harumque speciem mentientes contemplabimur; tum vero, quomodo iudicium ad multiformes accommodari debeat, indicabimus.

253. Sit igitur y functio ipsius x uniformis, quae propterea, quicunque valor pro x substituatur, semper unum recipiat valorem realem, denotetque x eum valorem, qui functioni y maximum minimumve valorem inducat. Priori ergo casu, sive loco x substituatur $x+\alpha$ sive $x-\alpha$, valor ipsius y minor erit, quam si $\alpha=0$, posteriori vero casu maior. Cum igitur posito $x+\alpha$ loco x functio y abeat in

$$y+\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}+\frac{\alpha^3 d^3 y}{6dx^3}+\text{etc.},$$

at posito $x-\alpha$ loco x in

$$y-\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}-\frac{\alpha^3 d^3 y}{6dx^3}+\text{etc.},$$

necesse est, ut casu maximi sit

$$y>y+\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}+\frac{\alpha^3 d^3 y}{6dx^3}+\text{etc.}$$

et

$$y>y-\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}-\frac{\alpha^3 d^3 y}{6dx^3}+\text{etc.}$$

In casu autem, quo valor ipsius y fit minimus, erit

$$y<y+\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}+\frac{\alpha^3 d^3 y}{6dx^3}+\text{etc.}$$
$$y<y-\frac{\alpha dy}{dx}+\frac{\alpha^2 ddy}{2dx^2}-\frac{\alpha^3 d^3 y}{6dx^3}+\text{etc.}$$

254. Quoniam haec evenire debent, si α denotet quantitatem minimam, statuamus α tam parvum, ut eius potestates altiores reiici queant, debebitque tam pro casu maximi quam minimi esse $\alpha\frac{dy}{dx}=0$. Nisi enim $\alpha\frac{dy}{dx}$ esset $=0$, neque valor ipsius y maximus neque minimus esse posset. Hinc tam pro maximis quam pro minimis investigandis haec habetur regula communis, ut differentiale propositae y nihilo aequale ponatur, eritque ille ipsius x valor, qui functionem reddit vel maximam vel minimam, radix istius aequationis. Utrum vero valor hoc modo inventus ipsius y futurus sit maximus an minimus, incertum relinquitur; quin etiam fieri potest, ut y neque maximum neque minimum sit futurum;

tantum enim invenimus utroque casu fore $\frac{dy}{dx}=0$ neque vicissim affirmavimus, quoties sit $\frac{dy}{dx}=0$, toties quoque valorem pro *y* prodire vel maximum vel minimum.

255. Interim tamen ad casus, quibus valor ipsius *y* vel maximus vel minimus evadat, investigandos haec prima operatio instituenda est, ut differentiale functionis propositae nihilo aequetur atque ex aequatione $\frac{dy}{dx}=0$ omnes ipsius *x* valores eliciantur. Quibus inventis deinceps dispiciendum erit, utrum iis functio *y* maximum induat valorem an minimum an neutrum. Ostendemus enim fieri posse, ut neque maximum neque minimum locum habeat, etiamsi sit $\frac{dy}{dx}=0$.

Sit f valor seu unus ex valoribus ipsius *x*, quem obtinet ex aequatione

$$\frac{dy}{dx}=0,$$

hicque valor substituatur in expressionibus $\frac{ddy}{dx^2}, \frac{d^3y}{dx^3}$ etc. fiatque hac substitutione

$$\frac{ddy}{dx^2}=p, \quad \frac{d^3y}{dx^3}=q, \quad \frac{d^4y}{dx^4}=r \quad \text{etc.}$$

Abeat autem functio ipsa *y* posito f loco *x* in *F*, atque si loco *x* ponatur $f+\alpha$, ista functio abibit in

$$F+\tfrac{1}{2}\alpha^2 p+\tfrac{1}{6}\alpha^3 q+\tfrac{1}{24}\alpha^4 r+\text{etc.};$$

sin autem loco *x* ponatur $f-\alpha$, prodibit

$$F+\tfrac{1}{2}\alpha^2 p-\tfrac{1}{6}\alpha^3 q+\tfrac{1}{24}\alpha^4 r-\text{etc.};$$

unde patet, si *p* fuerit quantitas affirmativa, utrumque valorem maiorem fore quam *F*, saltem si α quantitatem valde parvam denotet, ac propterea valorem *F*, quem functio *y* induit posito $x=f$, fore minimum. Sin autem *p* sit quantitas negativa, tum valor $x=f$ functioni *y* inducet valorem maximum.

256. Quodsi autem fuerit $p=0$, tum spectari debet valor ipsius *q;* qui si non fuerit $=0$, valor ipsius *y* neque maximus erit neque minimus; nam posito $x=f+\alpha$ erit $F+\frac{1}{6}\alpha^3 q>F$ et posito $x=f-\alpha$ erit $F-\frac{1}{6}\alpha^3 q<F$. Sin autem quoque fuerit $q=0$, ad quantitatem *r* erit respiciendum; quae si habuerit valorem affirmativum, valor functionis *F*,

quem recipit posito $x = f$, erit minimum; sin autem *r* habeat valorem negativum, erit *F* maximum. At si quoque *r* evanescat, iudicium ex sequentis litterae *s* valore erit petendum, quod simile erit illi, quod ex littera *q* formavimus. Scilicet si *s* non fuerit = 0, functio *F* neque maximum erit neque minimum; sin autem sit quoque $s = 0$, tum sequens littera *t*, si habeat valorem affirmativum, indicabit minimum; sin autem habeat valorem negativum, indicabit maximum. Verum si et haec littera *t* evanescat, tum in iudicando ulterius est procedendum eodem prorsus modo, quo in casibus praecedentibus sumus usi. Sicque de qualibet radice aequationis $\frac{dy}{dx} = 0$ indagabitur, utrum functioni *y* inducat valorem maximum an minimum an neutrum; atque hoc modo omnia maxima et minima, quae quidem functio *y* recipere potest, invenientur.

257. Si ergo aequatio $\frac{dy}{dx} = 0$ duas radices habeat aequales, ita ut factorem habeat quadratum $(x - f)^2$, tum posito $x = f$ simul $\frac{ddy}{dx^2}$ evanescet eritque $p = 0$, non autem *q*. Hoc ergo casu functio *y* neque maximum neque minimum valorem induet. Sin autem aequatio $\frac{dy}{dx} = 0$ tres radices habeat aequales seu $\frac{dy}{dx}$ factorem cubicum $(x - f)^3$, tum posito $x = f$ fiet $\frac{ddy}{dx^2} = 0$ et $\frac{d^3y}{dx^3} = 0$, non autem $\frac{d^4y}{dx^4}$. Huius ergo termini valor si fuerit affirmativus, indicabit minimum, sin negativus, maximum. Iudicium ergo ante explicatum huc redit, ut, si expressio $\frac{dy}{dx}$ factorem habuerit $(x - f)^n$ existente *n* numero impari, functio *y*, si in ea ponatur $x = f$, valorem sit acceptura vel maximum vel minimum, sin autem exponens *n* fuerit numerus par, tum substitutio $x = f$ neque maximum neque minimum valorem producat.

258. Deinde inventio maximi ac minimi saepenumero non mediocriter adiuvabitur sequentibus considerationibus. Quibus scilicet casibus functio *y* fit maximum vel minimum, iisdem casibus fiet quodvis eius multiplum *ay*, si quidem *a* fuerit quantitas affirmativa, itemque y^3, y^5, y^7 etc. atque generaliter ay^n siquidem *n* fuerit numerus affirmativus impar, pariter maximum vel minimum, quoniam huiusmodi formulae ita sunt comparatae, ut crescente *y* crescant et decrescente *y* decrescant. Quibus autem casibus fit *y* maximum vel minimum, iisdem casibus $-y, -ay$, $b - ay$ et generaliter $b - ay^n$ existente *n* numero affirmativo impari fiet ordine inverso vel minimum vel maximum. Similiter quibus casibus *y* fit maximum vel minimum, iisdem casibus formulae hae $\frac{a}{y}$, $\frac{a}{y^3}$, $\frac{a}{y^5}$ et generaliter $\frac{a}{y^n} \pm b$ denotante *a* quantitatem affirmativam et *n* numerum affirmativum imparem fient inverso ordine vel minimum vel maximum; sin autem *a* fuerit quantitas negativa, tum istae formulae maximum impetrabunt valorem, si *y* fuerit maximum, et minimum, si *y* sit minimum.

259. Ad potestates autem pares haec non item traduci possunt; quoniam enim, si *y* valores recipit negativos, eius potestates pares y^2, y^4 etc. valores affirmativos inducunt, fieri potest, ut, dum *y* minimum valorem, negativum scilicet, recipit, eius potestates pares fiant maxima. Huius igitur conditionis ratione habita affirmare poterimus, si *y* fuerit maximum vel minimum existente eius valore affirmativo, tum eius potestates pares y^2, y^4 etc. quoque fore maxima vel minima, sin autem valor ipsius *y* negativus fuerit maximum, tum eius quadratum *yy* accepturum esse valorem minimum, et contra, si valor ipsius *y* negativus sit minimum, tum y^2, y^4 etc. fore maximum. Quodsi vero exponentes ipsius *y* pares fuerint negativi, tum contrarium eveniet. Ceterum quae hic de exponentibus paribus et imparibus annotavimus, ea non solum pro numeris integris valent, sed etiam pro fractis, quorum denominatores sunt numeri impares; in hoc enim negotio fractiones $\frac{1}{3}, \frac{5}{3}, \frac{7}{3}, \frac{1}{5}, \frac{3}{5}$ etc. numeris imparibus, at $\frac{2}{3}, \frac{4}{3}, \frac{2}{5}, \frac{4}{5}, \frac{6}{7}$ etc. numeris paribus aequivalent.

260. Sin autem denominatores fuerint numeri pares, tum, quoniam, si *y* negativum habet valorem, eius potestates $y^{\frac{1}{2}}, y^{\frac{3}{4}}$ etc. fiunt imaginariae, hoc tantum de iis affirmari poterit: Si valor ipsius *y* affirmativus fuerit maximum vel minimum, tum quoque $y^{\frac{1}{2}}, y^{\frac{3}{2}}\ y^{\frac{1}{4}}$ etc. fore pariter vel maxima vel minima, contra autem $y^{-\frac{1}{2}}, y^{-\frac{3}{2}}\ y^{-\frac{1}{4}}$ etc. minima vel maxima. Quia autem haec irrationalia simul geminos valores habent, alterum affirmativum, alterum negativum, de negativis contrarium erit tenendum, quod hic de affirmativis diximus. Sin autem valor ipsius *y* negativus evadat maximum vel minimum, tum, quia huiusmodi potestates omnes fiunt imaginariae, neque maximis neque minimis annumerari poterunt. His igitur subsidiis investigatio maximi et minimi saepe admodum reddetur facilis, quae alias futura esset vehementer difficilis.

261. Quoniam haec proprie ad functiones rationales, quippe quae sunt solae uniformes, pertinent, primum functiones integras evolvamus atque maxima minimaque, quae in ipsis occurrunt, indagemus. Cum igitur huiusmodi functiones ad hanc formam referantur

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \text{etc.}$$

primum patet earum valorem maiorem fieri non posse, quam si ponatur $x = \infty$; tum vero, si $x = -\infty$, valor huius formulae prodit $= \infty^n$, si *n* sit numerus par, at $= -\infty^n$, si *n* sit numerus impar, qui propterea valor erit omnium minimus. Dantur antem praeterea saepe alia maxima et minima eo sensu, quem his vocibus attribuimus, quae sequentibus exemplis illustrabimus.

EXEMPLUM 1

Invenire valores ipsius x, quibus haec funciio $(x-a)^n$ *fit maximum vel minimum.*

Posito $(x-a)^n = y$ erit

$$\frac{dy}{dx} = n(x-a)^{n-1};$$

quo posito $=0$ fiet $x=a$. Cum igitur $\frac{dy}{dx}$ factorem habeat $(x-a)^{n-1}$, ex § 257 intelligitur y maximum minimumve esse non posse, nisi sit $n-1$ numerus impar seu n numerus par. Quia autem tum fit

$$\frac{d^n y}{dx^n} = n(n-1)(n-2)\cdots 1,$$

hoc est numerus affirmativus, sequitur valorem ipsius y posito $x=a$ prodititurum esse minimum. Quod quidem facile patet; nam posito $x=a$ fit $y=0$, et si x ponatur vel maius vel minus quam a, ob n numerum parem accipiet y valorem positivum, hoc est nihilo maiorem; sin autem n fuerit numerus impar, tum functio $y=(x-a)^n$ neque maximum neque minimum admittit. Perspicuum autem porro est hoc idem valere, si n fuerit numerus fractus, sive impar sive par. Scilicet $(x-a)^{\frac{\mu}{\nu}}$ fiet posito $x=a$ minimum, si μ fuerit numerus par et ν impar; sin autem uterque fuerit impar, neque maximum dabitur neque minimum.

EXEMPLUM 2

Invenire casus, quibus valor huius formulae $xx+3x+2$ *fit maximum vel minimum.*

Ponatur $xx+3x+2=y$; erit

$$\frac{dy}{dx} = 2x+3, \quad \frac{ddy}{2dx^2} = 1.$$

Statuatur ergo $2x+3=0$; fiet $x=-\frac{3}{2}$. Qui casus utrum maximum an minimum producat, cognoscetur ex valore $\frac{ddy}{2dx^2}=1$; qui cum sit affirmativus, quicquid sit x, indicat minimum. Posito autem $x=-\frac{3}{2}$ fit $y=-\frac{1}{4}$, et si alii quicunque valores ipsi x tribuantur, valor ipsius y inde oriundus perpetuo maior erit quam $-\frac{1}{4}$. Ex natura quoque ipsius formulae $xx+3x+2$ perspicitur eam minimum valorem habere debere; nam cum in infinitum excrescat, sive ponatur $x=+\infty$ sive $x=-\infty$, necesse est, ut quispiam valor ipsius x ipsi y omnium minimam quantitatem inducat.

EXEMPLUM 3

Invenire casus, quibus expressio haec $x^3 - axx + bx - c$ maximum minimumve valorem accipit.

Posito $y = x^3 - axx + bx - c$ erit

$$\frac{dy}{dx} = 3xx - 2ax + b \quad \text{et} \quad \frac{ddy}{2dx^2} = 3x - a, \quad \frac{d^3y}{6dx^3} = 1,$$

Statuatur ergo $\frac{dy}{dx} = 3xx - 2ax + b = 0$; erit

$$x = \frac{a \pm \sqrt{(aa - 3b)}}{3},$$

ex quo intelligitur, nisi sit $aa > 3b$, formulam propositam neque maximum neque minimum esse habituram. Sin autem sit $aa > 3b$, duobus casibus fit maximum vel minimum. Hinc vero oritur

$$\frac{ddy}{2dx^2} = \pm\sqrt{(aa - 3b)},$$

unde intelligitur, nisi sit $aa = 3b$, valorem $x = \frac{a + \sqrt{(aa-3b)}}{3}$ reddere formulam $x^3 - axx + bx - c$ minimam, alterum vero $x = \frac{a - \sqrt{(aa-3b)}}{3}$ maximam.

Quanti autem futuri isti sint ipsius *y* valores, cum sit $3xx - 2ax + b = 0$ seu $x^3 - \frac{2}{3}axx + \frac{1}{3}bx = 0$, erit

$$y = -\frac{1}{3}axx + \frac{2}{3}bx - c$$

et ob $\frac{1}{3}axx - \frac{2aa}{9}x + \frac{ab}{9} = 0$ fit

$$y = \frac{2}{9}(3b - aa)x + \frac{ab}{9} - c = -\frac{2a(aa-3b)}{27} \mp \frac{2(aa-3b)\sqrt{(aa-3b)}}{27} + \frac{ab}{9} - c$$

sive

$$y = -\frac{2a^3}{27} + \frac{ab}{3} - c \mp \frac{2}{27}(aa - 3b)^{\frac{3}{2}},$$

ubi signum superius valet pro minimo, inferius autem pro maximo.

Restat ergo casus, quo $aa = 3b$; in quo cum fiat $\frac{ddy}{dx^2} = 0$, sequens vero terminus $\frac{d^3y}{6dx^3} = 1$ non sit $= 0$, sequitur hoc casu formulam propositam neque maximum neque minimum esse recepturam.

EXEMPLUM 4

Invenire casus, quibus haec functio ipsius x, $x^4 - 8x^3 + 22x^2 - 24x + 12$, fit maximum vel minimum.

Posito $y = x^4 - 8x^3 + 22x^2 - 24x + 12$ erit

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 44x - 24,\ \frac{ddy}{dx^2} = 6x^2 - 24x + 22\,.$$

Statuatur nunc

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 44x - 24 = 0 \quad \text{seu} \quad x^3 - 6x^2 + 11x - 6 = 0\,;$$

orientur tres valores reales pro x

I. $x = 1$, II. $x = 2$, III. $x = 3.$.

Ex primo valore fit $\frac{ddy}{2dx^2} = 4$ ideoque posito $x = 1$ functio proposita fit minimum.
Ex secundo valore $x = 2$ fit $\frac{ddy}{2dx^2} = -2$ ideoque functio proposita maximum. Ex tertio valore $x = 3$ fit $\frac{ddy}{2dx^2} = +4$ ideoque functio proposita iterum minimum.

EXEMPLUM 5

Proposita sit haec functio $y = x^5 - 5x^4 + 5x^3 + 1$; *quae quibus casibus fiat maximum minimumve, quaeritur.*
Cum sit

$$\frac{dy}{dx} = 5x^4 - 20x^3 + 15xx$$

formetur aequatio $x^4 - 4x^3 + 3xx = 0$, cuius radices sunt

$$\text{I. et II.} \quad x = 0,\ \text{III.} \quad x = 1,\ \text{IV.} \quad x = 3\,.$$

Quoniam prima et secunda radices sunt aequales, ex iis neque maximum neque minimum sequitur; fit enim $\frac{ddy}{dx^2} = 0$, at $\frac{d^3y}{dx^3}$ non evanescit. Tertia radix autem x = 1 ob $\frac{ddy}{2dx^2} = 10x^3 - 30x^2 + 15x$ praebet $\frac{ddy}{2dx^2} = -5$ hocque ergo casu functio fit maximum. Ex quarta radice x = 3 fit $\frac{ddy}{2dx^2} = 45$ ideoque functio proposita minimum.

EXEMPLUM 6

Invenire casus, quibus haec formula $y = 10x^6 - 12x^5 + 15x^4 - 20x^3 + 20$ *fit maximum vel minimum.*

Erit ergo

$$\frac{dy}{dx} = 60x^5 - 60x^4 + 60x^3 - 60x^2 \quad \text{et} \quad \frac{ddy}{60dx^2} = 5x^4 - 4x^3 + 3x^2 - 2x.$$

Formetur aequatio $x^5 - x^4 + x^3 - xx = 0$; quae cum in factores resoluta sit $x^2(x-1)(xx+1) = 0$, duas habet radices aequales $x = 0$ et praeterea radicem $x = 1$ duasque insuper ex $xx + 1 = 0$ imaginarias. Cum igitur binae radices aequales $x = 0$ neque maximum neque minimum exhibeant, tantum consideranda superest radix $x = 1$, ex qua fit $\frac{ddy}{60dx^2} = 2$, cuius valor affirmativus indicat minimum.

262. Determinatio ergo maximorum et minimorum pendet a radicibus aequationis differentialis $\frac{dy}{dx} = 0$; cuius potestas summa cum sit uno gradu inferior quam in ipsa functione proposita *y*, si quidem haec fuerit functio rationalis integra, manifestum est, si in genere proponatur haec functio

$$x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \text{etc.} = y,$$

eius maxima et minima determinari per radices huius aequationes

$$nx^{n-1} + (n-1)Ax^{n-2} + (n-2)Bx^{n-3} + (n-3)Cx^{n-4} + \text{etc.} = 0.$$

Ponamus huius aequationis radices reales secundum ordinem quantitatis dispositas esse $\alpha, \beta, \gamma, \delta$ etc. ita ut α sit maxima, $\beta < \alpha$, $\gamma < \beta$ etc. Ac primo quidem, si hae radices omnes fuerint inaequales, unaquaeque formulae propositae *y* inducet valorem maximum vel minimum totque idcirco functio *y* habebit maxima vel minima, quot aequatio $\frac{dy}{dx} = 0$ haberit radices reales inaequales. Sin autem duae pluresve radices inter se fuerint aequales, res ita se habebit, ut duae radices aequales neque maximum neque minimum exhibeant, ternae vero radices aequales uni aequivaleant; atque in genere, si numerus radicum aequalium fuerit par, nullum inde resultat maximum minimumve; sin autem sit impar, unum inde oritur sive maximum sive minimum.

263. Quaenam autem radices maxima et quae minima producant, sine subsidio regulae ante traditae ita definiri poterit. Cum functio *y* posito $x = \infty$ fiat pariter infinita neque valores ipsius *x* intra limites ∞ et α ullum producant sive maximum sive minimum, perspicuum est valores functionis *y*, dum loco *x* successive valores ab ∞ usque ad α substituantur, continuo decrescere oportere; ideoque valor $x = \alpha + \omega$ functioni *y* maiorem valorem inducet quam valor $x = \alpha$; unde , cum $x = \alpha$ maximum minimumve producat, necesse est, ut hoc casu functio *y* fiat minimum. Ulterius ergo *x* diminuendo seu ponendo $x = \alpha - \omega$ valor ipsius *y* iterum crescet, donec fiat $x = \beta$, quae est secunda aequationis $\frac{dy}{dx} = 0$ radix maximum minimumve producens; quare haec secunda radix $x = \beta$ maximum praebebit et valor $x=\beta-\omega$ minorem efficiet functionem y quam $x=\beta$, donec perveniatur ad $x=\gamma$, quae consequenter iterum minimum generabit. Ex quo ratiocinio perspicitur radices aequationis $\frac{dy}{dx}=0$ primam, tertiam, quintam etc. minima, secundam autem, quartam, sextam etc. maxima exhibere. Simul autem hinc intelligitur in casu duarum radicum aequalium maximum et minimum coalescere sicque neutrum locum habere.

264. Si ergo in functione proposita

$$y=x^{n}+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\text{etc.}$$

maximus exponens n fuerit numerus par, aequatio

$$\frac{dy}{dx}=nx^{n-1}+(n-1)Ax^{n-2}+\text{etc.}=0$$

erit gradus imparis ideoque vel unam habebit radicem realem vel tres vel quinque vel numero impares. Si unica radix fuerit realis, ea dabit minimum; sin tres fuerint reales, maxima praebebit minimum, media maximum et minima iterum minimum; et si quinque radices fuerint reales, functio y tria habebit minima et duo maxima; sicque porro.

At si exponens n fuerit numerus impar, aequatio $\frac{dy}{dx}=0$ ad gradum parem pertinebit ideoque vel nullam habebit radicem realem vel duas vel quatuor vel sex etc. Primo casu functio y neque maximum habebit neque minimum; altero casu, quo duae dantur radices, earum maior minimum, minor autem maximum indicabit; quatuor autem radicum prima (quae est maxima) et tertia minimum, secunda vero et quarta maximum producunt. Perpetuo autem quotcunque radices fuerint reales, maxima et minima se mutuo alternatim insequuntur.

265. Progrediamur ad functiones rationales fractas, quibus altera species functionum uniformium constituitur. Sit igitur

$$y=\frac{P}{Q}$$

existentibus P et Q functionibus quibuscunque ipsius x; ac primo quidem apparet, si ipsi x eiusmodi valor tribuatur, ut fiat $Q=0$: nisi simul P evanescat, functionem y evadere infinitam, quod utique maximum videatur.
Nihilo vero minus iste casus pro maximo haberi nequit; cum enim fractio inversa $\frac{Q}{P}$ iisdem casibus fiat minimum, quibus proposita $\frac{P}{Q}$ fit maximum, deberet fractio $\frac{Q}{P}$ fieri minimum, si Q evanescit; hoc autem non semper evenit, propterea quod adhuc minores valores,

negativos scilicet, induere posset. Hoc igitur dubio exemto simul regula ante data confirmatur, quod maxima et minima ex aequatione $\frac{dy}{dx} = 0$ elici debeant. Fiet ergo casu proposito

$$\frac{dy}{dx} = \frac{QdP - PdQ}{QQdx}$$

ideoque

$$QdP - PdQ = 0$$

huiusque aequationis radices efficient functionem y vel maximum vel minimum. Atque si dubium sit, utrum maximum an minimum locum habeat, confugiendum est ad valorem $\frac{ddy}{dx^2}$; qui si fuerit affirmativus, minimum indicabit, sin autem sit negativus, maximum. Quodsi vero et hic valor $\frac{ddy}{dx^2}$ evanescat, quod evenit, si aequatio $\frac{dz}{dx} = 0$ habeat duas pluresve radices aequales, perpetuo tenendum est radices aequales numero pares neque maximum neque minimum producere.

EXEMPLUM 1

Invenire casus, quibus functio $\frac{x}{1+xx}$ fit maximum vel minimum.

Primum quidem apparet hanc functionem in nihilum abire casibus tribus $x = \infty, x = 0$ et $x = -\infty$, unde ad minimum duo recipiet sive maxima sive minima. Ad quae invenienda ponatur $y = \frac{x}{1+xx}$ eritque

$$\frac{dy}{dx} = \frac{1-xx}{(1+xx)^2} \quad \text{et} \quad \frac{ddy}{dx^2} = \frac{-6x+2x^3}{(1+xx)^3}.$$

Iam statuatur $\frac{dy}{dx} = 0$; erit $1 - xx = 0$ et vel $x = +1$ vel $x = -1$. Priori casu $x = +1$ fit $\frac{ddy}{dx^2} = -\frac{4}{2^3}$ ideoque y maximum $= \frac{1}{2}$; posteriori $x = -1$ fit $\frac{ddy}{dx^2} = +\frac{4}{2^3}$ ideoque minimum $= -\frac{1}{2}$.

Haec quoque facilius inveniuntur, si fractio proposita $\frac{x}{1+xx}$ invertatur ponendo $y = \frac{1+xx}{x}$, dummodo recordemur tum, quae maxima inveniuntur, in minima et vicissim transmutari debere. Erit autem

$$\frac{dy}{dx} = 1 - \frac{1}{xx} \quad \text{et} \quad \frac{ddy}{dx^2} = \frac{2}{x^3}.$$

Statuto ergo $\frac{dy}{dx}=0$ fit $xx-1=0$ indeque vel $x=+1$ vel $x=-1$ ut ante. Atque casu $x=+1$ fit $\frac{ddy}{dx^2}=2$ ideoque y minimum et formula proposita $\frac{1}{y}$ maximum. Casu autem $x=-1$ fit $\frac{ddy}{dx^2}=-2$, unde y maximum et $\frac{1}{y}$ minimum.

EXEMPLUM 2

Invenire casus, quibus formula $\frac{2-3x+xx}{2+3x+xx}$ fit maximum vel minimum.

Posito $y=\frac{2-3x+xx}{2+3x+xx}$ erit

$$\frac{dy}{dx}=\frac{6x^2-12}{(xx+3x+2)^2} \quad \text{et} \quad \frac{ddy}{dx^2}=\frac{-12x^2+72x+72}{(xx+3x+2)^3}.$$

Statuatur $\frac{dy}{dx}=0$; fiet vel $x=+\sqrt{2}$ vel $x=-\sqrt{2}$. Priori casu $x=+\sqrt{2}$ erit

$$\frac{ddy}{dx^2}=\frac{48\sqrt{2}+72}{\left(4+3\sqrt{2}\right)^3}$$

ideoque affirmativum ob denominatorem affirmativum; hinc erit y minimum

$$=\frac{4-3\sqrt{2}}{4+3\sqrt{2}}=12\sqrt{2}-17=-0{,}02943725.$$

Posteriori casu $x=-\sqrt{2}$ fit

$$\frac{ddy}{dx^2}=\frac{-48\sqrt{2}+72}{\left(4-3\sqrt{2}\right)^3}=\frac{24\left(3-2\sqrt{2}\right)}{\left(4-3\sqrt{2}\right)^3},$$

cuius valor ob numerator affirmativum et denominatorem negativum erit negativus ideoque y fiet maximum

$$=\frac{4+3\sqrt{2}}{4-3\sqrt{2}}=-12\sqrt{2}-17=-33{,}97056274.$$

Qui valor, etsi minor est quam prior minimus, tamen ideo est maximus, quod maior sit contiguis proximis, qui oriuntur, si loco x vel aliquantillum maiores vel minores valores quam $-\sqrt{2}$ substituantur. Cum igitur $\sqrt{2}$ inter limites $\frac{4}{3}$ et $\frac{3}{2}$ contineatur, probatio facile instituetur hoc modo:

si $x=\frac{4}{3}$, fit $y=-\frac{2}{70}=-0{,}0285$
si $x=\sqrt{2}$, fit $y=12\sqrt{2}-17=-0{,}0294$ minimum
si $x=\frac{3}{2}$, fit $y=-\frac{1}{35}=-0{,}0285$

si $x=-\frac{4}{3}$, fit $y=-35$
si $x=-\sqrt{2}$, fit $y=-33{,}970$ maximum
si $x=-\frac{3}{2}$, fit $y=-35$.

EXEMPLUM 3

Invenire casus, quibus formula $\frac{xx-x+1}{xx+x-1}$ fit maximum vel minimum.

Ponatur $y=\frac{xx-x+1}{xx+x-1}$ eritque

$$\frac{dy}{dx}=\frac{2xx-4x}{(xx+x-1)^2} \quad \text{et} \quad \frac{ddz}{dx^2}=\frac{-4x^3+12xx+4}{(xx+x-1)^3}.$$

Statuatur $\frac{dy}{dx}=0$; erit vel $x=0$ vel $x=2$; priori casu fit $\frac{ddy}{dx^2}=\frac{4}{-1}$ ideoque erit y maximum $=-1$. Posteriori casu $x=2$ fit $\frac{ddy}{dx^2}=\frac{20}{5^3}$ ideoque y minimum $=\frac{3}{5}$, etiamsi illud maximum minus sit quam hoc minimum.Probatio patebit ex his positionibus

si $x=-\frac{1}{3}$, erit $y=-\frac{13}{11}$
si $x=0$, erit $y=-1$ maximum
si $x=+\frac{1}{3}$, erit $y=-\frac{7}{5}$
si $x=2-\frac{1}{3}$, erit $y=\frac{19}{31}$
si $x=2$, erit $y=\frac{3}{5}$ minimum
si $x=2+\frac{1}{3}$, erit $y=\frac{37}{61}$.

Quod autem, si ponatur $x=1$, fiat $y=1$ ideoque >-1, causa est, quod inter valores ipsius x 0 et 1 contineatur unus, quo fit $y=\infty$.

EXEMPLUM 4

Quaerantur casus, quibus haec fractio $\frac{x^3+x}{x^4-xx+1}$ *fiat maxima vel minima.*

Posito $y=\frac{x^3+x}{x^4-xx+1}$ erit

$$\frac{dy}{dx}=\frac{-x^6-4x^4+4xx+1}{\left(x^4-xx+1\right)^2} \quad \text{et} \quad \frac{ddy}{dx^2}=\frac{2x^9+18x^7-30x^5-16x^3+12xx}{\left(x^4-xx+1\right)^3}$$

Habebimus ergo hanc aequationem

$$x^6+4x^4-4xx-1=0,$$

quae resolvitur in has duas

$$xx-1=0 \ \text{ et } \ x^4+5x^2+1=0;$$

quarum illius radices sunt $x=+1$ et $x=-1$, haec vero resoluta dat

$$xx=-\frac{5\pm\sqrt{21}}{2},$$

ex qua nulla radix realis emergit. Duarum igitur radicum inventarum prior $x=+1$ facit $\frac{ddy}{dx^2}=-14$ ac propterea *y* maximum = 2; altera radix $x=-1$ facit $\frac{ddy}{dx^2}=+14$ ac propterea *y* minimum $=-2$.

EXEMPLUM 5

Invenire casus, quibus haec fractio $\frac{x^3-x}{x^4-xx+1}$ *fit maximum vel minimum.*

Posito $y=\frac{x^3-x}{x^4-xx+1}$ erit

$$\frac{dy}{dx}=\frac{-x^6+2x^4+2x^2+1}{\left(x^4-x^2+1\right)^2} \quad \text{et} \quad \frac{ddy}{dx^2}=\frac{2x^9-6x^7-18x^5+20x^3}{\left(x^4-x^2+1\right)^3}.$$

Facto autem $\frac{dy}{dx}=0$ erit

$$x^6-2x^4-2x^2+1=0$$

quae divisa per $xx+1$ dat

$$x^4-3x^2+1=0,$$

haecque ulterius resolvitur in

$$xx-x-1=0 \quad \text{et} \quad xx+x-1=0$$

unde sequentes quatuor oriuntur radices reales

$$\text{I.}\quad x=\tfrac{1+\sqrt{5}}{2},\qquad \text{II.}\quad x=\tfrac{1-\sqrt{5}}{2},$$
$$\text{III.}\quad x=-\tfrac{1+\sqrt{5}}{2},\quad \text{IV.}\quad x=-\tfrac{1-\sqrt{5}}{2}.$$

Quae cum omnes in aequatione $x^4-3xx+1=0$ contineantur, posito $x^4=3xx-1$ fiet pro omnibus

$$\frac{ddy}{dx^2}=\frac{2x(10-20xx)}{8x^6}=\frac{5(1-2xx)}{2x^5}=\frac{5(1-2xx)}{2x(3xx-1)}\quad\text{et}\quad y=\frac{x^3-x}{2xx}=\frac{xx-1}{2x}.$$

Pro duabus autem prioribus ex aequatione $xx=x+1$ ortis erit

$$\frac{ddy}{dx^2}=-\frac{5(2x+1)}{2x(3x+2)}=-\frac{5(2x+1)}{2(5x+3)}\quad\text{et}\quad y=\tfrac{1}{2}.$$

Prima igitur radix $x=\frac{1+\sqrt{5}}{2}$ dat

$$\frac{ddy}{dx^2}=-\frac{5\left(2+\sqrt{5}\right)}{11+5\sqrt{5}}$$

ideoque est y maximum. Secunda radix $x=\frac{1-\sqrt{5}}{2}$ dat

$$\frac{ddy}{dx^2}=-\frac{5\left(2-\sqrt{5}\right)}{11-5\sqrt{5}}=-\frac{5\left(\sqrt{5}-2\right)}{5\sqrt{5}-11}$$

ideoque $y=\frac{1}{2}$ erit quoque maximum. Duae reliquae radices dant $y=-\frac{1}{2}$ minimum.

266. In his igitur exemplis exploratio, utrum valor quispiam inventus maximum an minimum producat, facilius institui poterit; cum enim sit $\frac{dy}{dx}=0$, valor termini $\frac{ddy}{dx^2}$ eius aequationis ratione habita simplicius exprimi poterit. Sit enim proposita fractio $y=\frac{P}{Q}$; cum sit

$$dy=\frac{QdP-PdQ}{QQ}\quad\text{et}\quad QdP-PdQ=0,$$

erit

$$ddy=\frac{d(QdP-PdQ)}{Q^2}-\frac{2dQ(QdP-PdQ)}{Q^3}.$$

At vero ob $QdP-PdQ=0$ hic posterior terminus evanescit eritque

$$ddy = \frac{d(QdP - PdQ)}{QQ} = \frac{QddP - PddQ}{Q^2}.$$

Quoniam vero iudicium ex huius termini valore sive affirmativo sive negativo petitur, denominator autem Q^2 perpetuo sit affirmativus, ex solo numeratore negotium ita confici poterit, ut, quoties $QddP - PddQ$ seu $\frac{d(Qd - PdQ)}{dx^2}$ fuerit affirmativum, minimum pronuncietur, sin sit negativum, maximum. Sive postqumn inventum fuerit $\frac{dy}{dx}$, cuius forma erit huiusmodi $\frac{R}{QQ}$, tantum quaeratur $\frac{dR}{dx}$ et quae radix huic expressioni valorem affirmativum inducit, ex ea proveniet minimum et contra maximum.

267. Si denominator fractionis propositae fuerit quadratum seu altior potestas quaecunque, ita ut sit $y = \frac{P}{Q^n}$, fiet

$$dy = \frac{QdP - nPdQ}{Q^{n+1}}$$

et posito $\frac{QdP - nPdQ}{dx} = R$ erit

$$\frac{dy}{dx} = \frac{R}{Q^{n+1}}$$

et maxima minimaque determinabuntur ex radicibus aequationis $R = 0$. Cum deinde sit

$$\frac{ddy}{dx} = \frac{QdR - (n+1)RdQ}{Q^{n+2}},$$

ob $R = 0$ fiet

$$\frac{ddy}{dx} = \frac{dR}{Q^{n+1}};$$

cuius valor affirmativus indicabit minimum, negativus autem maximum. Perspicuum autem est, si n fuerit numerus impar, ob Q^{n+1} semper affirmativum iudicium ex solo $\frac{dR}{dx}$ perfici posse; sin autem n sit numerus par, adhibeatur $\frac{QdR}{dx}$ formula.

Ponamus autem porro proponi huiusmodi fractionem $\frac{P^m}{Q^n} = y$; erit

$$dy = \frac{(mQdP - nPdQ)P^{m-1}}{Q^{n+1}};$$

Si itaque ponatur $\frac{mQdP - nPdQ}{dx} = R$, aequationis $R = 0$ radices indicabunt casus, quibus functio y fit vel maximum vel minimum. Cum igitur sit

$$\frac{dy}{dx}=\frac{P^{m-1}R}{Q^{n+1}}$$

erit

$$\frac{ddy}{dx}=\frac{P^{m-2}R\big((m-1)QdP-(n+1)PdQ\big)}{Q^{n+2}}+\frac{P^{m-1}dR}{Q^{n+1}},$$

et ob $R=0$ fiet

$$\frac{ddy}{dx^2}=\frac{P^{m-1}dR}{Q^{n+1}dx};$$

quae insuper per quodcunque quadratum $\frac{P^{2\mu}}{Q^{2\nu}}$ dividi potest ad iudicium absolvendum. Praeterea vero quoque aequatio $P = 0$ dabit maximum vel minimum, si m fuerit numerus par; atque simili modo formulam inversam $\frac{Q^n}{P^m}$ spectando prodibit maximum vel minimum ponendo $Q=0$, si n fuerit numerus par, uti supra (§ 257) ostendimus; hic autem ad maxima vel minima hinc oriunda non respicimus, sed tantum ad usum methodi explicandum ea indagamus, quae oriuntur ex aequatione $R=0$.

EXEMPLUM 1

Proponatur fractio $\frac{(\alpha+\beta x)^m}{(\gamma+\delta x)^n}$; *quae quo casu fiat maximum vel minimum, quaeritur.*

Posito $y=\frac{(\alpha+\beta x)^m}{(\gamma+\delta x)^n}$ primo quidem patet fore $y=0$, si $x=-\frac{\alpha}{\beta}$, et $y=\infty$, si $x=-\frac{\gamma}{\delta}$; quorum casuum ille dabit minimum, hic vero maximum, si m et n fuerint numeri pares. Praeterea vero erit

$$\frac{dy}{dx}=\frac{(\alpha+\beta x)^{m-1}}{(\gamma+\delta x)^{n+1}}\big((m-n)\beta\delta x+m\beta\gamma-n\alpha\delta\big)$$

ideoque

$$R=(m-n)\beta\delta x+m\beta\gamma-n\alpha\delta.$$

Quare posito $R = 0$ erit

$$x=\frac{n\alpha\delta-m\beta\gamma}{(m-n)\beta\delta}.$$

Deinde ob $\frac{dR}{dx}=(m-n)\beta\delta$ dispiciendum est, utrum

$$\frac{P^{m-1}dR}{Q^{n+1}dx}=\frac{m^{m-1}\beta^{n+1}}{n^{n+1}\delta^{m-1}}\left(\frac{\alpha\delta-\beta\gamma}{m-n}\right)^{m-n-2}\frac{dR}{dx}$$

sit quantitas affirmativa an negativa. Priori casu formula proposita erit minimum, posteriori maximum.

Sic si fuerit $y=\frac{(x+3)^3}{(x+2)^2}$, fiet $\frac{P^{m-1}dR}{Q^{n+1}dx}=\frac{9}{8}$ ideoque formula $\frac{(x+3)^3}{(x+2)^2}$ fiet mimimum, si ponatur $x=0$.

Sin autem sit $y=\frac{(x-1)^m}{(x+1)^n}$, erit

$$\frac{P^{m-1}dR}{Q^{n+1}dx}=\frac{m^{m-1}}{n^{n+1}}\left(\frac{n-m}{2}\right)^{n-m+2}(m-n)$$

et $x=\frac{n+m}{n-m}$. Cum autem *m* et *n* ponantur numeri affirmativi, iudicium petendum erit ex formula $(n-m)^{n-m+2}(m-n)$ seu $(n-m)^{n-m}(m-n)$. Si igitur fuerit $n>m$, valor erutus $x=\frac{n+m}{n-m}$ semper dabit maximum; sin autem $n<m$, numerus $m-n$ par dabit minimum, at impar maximum; sic $\frac{(x-1)^3}{(x+1)^2}$ fiet maximum posito $x=-5$; fit enim $y=-\frac{6^3}{4^2}=-\frac{27}{2}$.

EXEMPLUM 2

Proponatur formula $y=\frac{(1+x)^3}{(1+xx)^2}$.

Erit

$$\frac{dy}{dx}=\frac{(1+x)^2}{(1+xx)^3}(3-4x-xx) \quad \text{et} \quad \frac{P^{m-1}dR}{Q^{n+1}dx}=-\frac{(1+x)^2}{(1+xx)^3}(2x+4).$$

ubi cum $(1+x)^2$ et $(1+xx)^3$ semper habeant valorem affirmativum, iudicium relinquetur formulae $-x-2$; quae, si fuerit affirmativa, minimum, sin negativa, maximum indicat. At vero ex aequatione $3-4x-xx=0$ sequitur vel

$$x=-2+\sqrt{7} \text{ vel } x=-2-\sqrt{7}.$$

Priori casu fit $-x-2=-\sqrt{7}$ ideoque fractio proposita erit maximum, posteriori vero casu minimum ob $-x-2=+\sqrt{7}$. Posito autem $x=-2+\sqrt{7}$ erit $1+x=-1+\sqrt{7}$ et $1+xx=12-4\sqrt{7}$, unde

$$y=\left(\tfrac{-1+\sqrt{7}}{12-4\sqrt{7}}\right)^2\left(\sqrt{7}-1\right)=\frac{\left(2+\sqrt{7}\right)^2\left(\sqrt{7}-1\right)}{16}=\tfrac{17+7\sqrt{7}}{16}=2{,}220.$$

Posito autem $x=-2-\sqrt{7}$ fiet

$$y=\tfrac{17-7\sqrt{7}}{16}=-0{,}0950.$$

268. Dantur etiam functiones irrationales et transcendentes, quae proprietatem functionum uniformium habent, et hanc ob rem maxima et minima eodem modo inveniti possunt. Radices enim cubicae et omnium imparium potestatum revera sunt uniformes, cum nonnisi unicum valorem realem exhibeant; radices autem quadratae atque omnium potestatum parium etsi revera, quoties sunt reales, geminum valorem indicant, alterum affirmativum, alterum negativum, tamen unusquisque seorsim spectari potest hocque sensu etiam maxima et minima investigari possunt. Sic si *y* fuerit functio quaecunque ipsius *x*, etsi $\sqrt{y}$ geminum habet valorem, tamen uterque seorsim tractari poterit. Scilicet $+\sqrt{y}$ maximum vel minimum habebit valorem, si *y* talem habuerit, dummodo fuerit affirmativus, quia alioquin $\sqrt{y}$ evaderet imaginarium. Vice versa autem $-\sqrt{y}$ fiet minimum vel maximum iisdem casibus, quibus $+\sqrt{y}$ fit maximum vel minimum. Potestas autem quaecunque $y^{\frac{m}{n}}$ iisdem casibus fit maximum vel minimum, siquidem *n* fuerit numerus impar; at si *n* fuerit numerus par, ii tantum casus valent, quibus *y* induit valorem affirmativum, hisque casibus ob ancipitem valorem gemina prodibunt maxima vel minima.

269. Quoniam aequatio differentialis, quae ex potestate functionis y^m nascitur, est $\frac{y^{m-1}dy}{dx}=0$, cuius radices simul casus, quibus potestas surda $y^{\frac{m}{n}}$ fit maximum vel minimum, indicant, ad hoc indagandum duplex habetur aequatio, altera $y^{m-1}=0$, altera $\frac{dy}{dx}=0$, quarum illa abit in $y=0$ atque tum solum maxima et minima exhibet, si $m-1$ fuerit numerus impar seu si *m* fuerit numerus par, ob rationes § 257 allegatas. Quare cum *n* sit numerus impar, si *m* fuerit numerus par, si numeros pares per 2μ et impares per $2\nu-1$ indicemus, functio $y^{2\mu:(2\nu-1)}$ evadet maxima vel minima tribuendis ipsi *x* valoribus, quos tam ex hac aequatione $y=0$ quam ex hac $\frac{dy}{dx}=0$ adipiscitur. Sin autem *m* sit numerus impar, functio $y^{(2\mu-1):(2\nu-1)}$ vel $y^{(2\mu-1):2\nu}$ tum solum fit maxima vel minima, cum loco *x* substituitur valor ex hac aequatione $\frac{dy}{dx}=0$. Ac posteriori quidem casu $y^{(2\mu-1):2\nu}$ maxima et minima tantum proveniunt, si *y* ab inventis ex aequatione $\frac{dy}{dx}=0$ valoribus affirmativos recipiat valores.

270. Sic ista formula $x^{\frac{2}{3}}$ fit minimum ponendo $x=0$, propterea quod hoc casu x^2 fit minimum. Nisi autem formulam $x^{\frac{2}{3}}$ ad formam x^2 reducamus, methodus ante tradita hoc minime indicaret, propterea quod casu $x=0$ termini seriei

$$y+\frac{\omega dy}{dx}+\frac{\omega^2 ddz}{2dx^2}+\frac{\omega^3 d^3 z}{6dx^3}+\text{etc.},$$

unde iudicium peti debet, praeter primum omnes fiuut infiniti. Facto enim $y=x^{\frac{2}{3}}$ erit

$$\frac{dy}{dx}=\frac{2}{3x^{\frac{1}{3}}},\quad \frac{ddy}{dx^2}=\frac{-2}{9x^{\frac{4}{3}}},\quad \frac{d^3y}{dx^3}=\frac{2\cdot 4}{27x^{\frac{7}{3}}}\quad \text{etc.}$$

Hinc neque aequatio $\frac{dy}{dx}=\frac{2}{3x^{\frac{1}{3}}}=0$ ostendit valorem $x=0$ neque termini sequentes rationem maximi minimive. Cum igitur assumsimus seriem

$$y+\frac{\omega dy}{dx}+\frac{\omega^2 ddz}{2dx^2}+\frac{\omega^3 d^3 z}{6dx^3}+\text{etc.}$$

fieri convergentem, si ω statuatur quantitas valde parva, ii casus utique methodum generalem effugiunt, quibus haec series fit divergens, quod evenit exemplo hic allato $y=x^{\frac{2}{3}}$, si ponatur $x=0$. Quamobrem his casibus ea reductione, qua ante usi sumus, erit opus, quo expressio proposita ad aliam formam revocetur, quae huic incommodo non sit subiecta. Hoc autem tantum paucissimis casibus usu venit, qui in formula $y^{\frac{2\mu}{2\nu-1}}$ continentur vel ad eam facile reducuntur. Sic si requirantur maxima minimave formulae $y^{\frac{2\mu}{2\nu-1}}z$ existente *z* functione quacunque ipsius *x*, investigetur forma haec $y^{2\mu}\, z^{2\nu-1}$, quippe quae iisdem casibus fit maxima vel minima, quibus ipsa proposita.

271. Hoc casu excepto, qui iam facile expeditur, functiones, quae continent quantitates irrationales, eodem modo quo rationales tractari earumque maxima,et minima determinari possunt, id quod sequentibus exemplis illustrabimus.

EXEMPLUM 1

Proposita sit formula $\sqrt{(aa+xx)}-x$; quae quibus casibus fiat maxima vel minima, quaeritur.

Posito $y=\sqrt{(aa+xx)}-x$ erit

$$\frac{dy}{dx}=\frac{x}{\sqrt{(aa+xx)}}-1 \quad \text{et} \quad \frac{ddy}{dx^2}=\frac{aa}{(aa+xx)^{3:2}}.$$

Facto ergo $\frac{dy}{dx}=0$ erit $x=\sqrt{(aa+xx)}$ ideoque $x=\infty$ ac fit $\frac{ddy}{dx^2}=0$. Simili vero modo fiunt sequentes termini $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ etc. omnes $=0$; ex quo iudicium incertum relinquitur, utrum sit maximum an minimum. Ratio est, quod revera tam fiat $x=-\infty$ quam $x=+\infty$. Interim ponendo $x=+\infty$ ob

$$\sqrt{(aa+xx)}=x+\frac{aa}{2x}$$

fit $y=0$, qui valor omnium est minimus.

EXEMPLUM 2

Quaerantur casus, quibus haec forma $\sqrt{(aa+2bx+mxx)}-nx$ fiat maximum vel minimum.

Posito $y=\sqrt{(aa+2bx+mxx)}-nx$ erit

$$\frac{dy}{dx}=\frac{b+mx}{\sqrt{(aa+2bx+mxx)}}-n;$$

quo facto $=0$ erit

$$bb+2mbx+mmxx=nnaa+2nnbx+mnnxx$$

seu

$$xx=\frac{2bx(nn-m)+nnaa-bb}{mm-mnn}$$

ideoque

$$x=\frac{(nn-m)b\pm\sqrt{(mnn(m-nn)aa-nn(m-nn)bb)}}{m(m-nn)}$$

sive

$$x=-\frac{b}{m}\pm\frac{n}{m}\sqrt{\frac{maa-bb}{m-nn}}$$

unde fit

$$\sqrt{(aa+2bx+mxx)}=\frac{b+mx}{n}=\pm\sqrt{\frac{maa-bb}{m-nn}}.$$

Cum igitur sit

$$\frac{ddy}{dx^2}=\frac{maa-bb}{(aa+2bx+mxx)^{\frac{3}{2}}},$$

erit

$$\frac{ddy}{dx^2} = \frac{maa-bb}{\pm\left(\frac{maa-bb}{m-nn}\right)^{\frac{3}{2}}} = \frac{\pm(m-nn)\sqrt{(m-nn)}}{\sqrt{(maa-bb)}}$$

Nisi ergo fuerit $\frac{m-nn}{maa-bb}$ quantitas affirmativa, maximum minimumve plane non datur. Sin autem sit quantitas affirmativa, signum superius dabit minimum, si $m > nn$, maximum vero, si $m < nn$; contrarium evenit, si signum inferius valeat. Si ergo sit $m = 2, n = 1 \text{ et } b = 0$, formula $\sqrt{(aa+2xx)} - x$ fit minimum ponendo $x = +\frac{1}{2}\sqrt{2aa} = \frac{a}{\sqrt{2}}$, at maximum ponendo $x = -\frac{a}{\sqrt{2}}$. Erit ergo minimum $= a\sqrt{2} - \frac{a}{\sqrt{2}} = \frac{a}{\sqrt{2}}$ et maximum $= a\sqrt{2} + \frac{a}{\sqrt{2}} = \frac{3a}{\sqrt{2}}$.

EXEMPLUM 3

Quaerantur casus, quibus haec expressio $\sqrt[4]{(1+mx^4)} + \sqrt[4]{(1-nx^4)}$ fiat maximum vel minimum.

Cum sit $\frac{dy}{dx} = \frac{mx^3}{(1+mx^4)^{\frac{3}{4}}} - \frac{nx^3}{(1-nx^4)^{\frac{3}{4}}}$, fiet

$$mx^3\left(1-nx^4\right)^{\frac{3}{4}} = nx^3\left(1+mx^4\right)^{\frac{3}{4}} \text{ ideoquae } m^4\left(1-nx^4\right)^3 = n^4\left(1+mx^4\right)^3$$

seu

$$n^4 - m^4 + 3mn\left(n^3+m^3\right)x^4 + 3m^2n^2\left(n^2-m^2\right)x^8 + m^3n^3(n+m)x^{12} = 0.$$

Nisi ergo haec aequatio radicem positivam habeat pro x^4, maximum minimumve prorsus non datur. Quia haec aequatio generaliter commode resolvi nequit, fiet enim

$$x^4 = \frac{m^{\frac{4}{3}} - n^{\frac{4}{3}}}{mn\left(\sqrt[3]{m}+\sqrt[3]{n}\right)} \quad \text{seu} \quad x^4 = \frac{m-\sqrt[3]{m^2n}+\sqrt[3]{mn^2}-n}{mn},$$

ponamus pro casu speciali $m = 8n$ eritque

$$-4095 + 24\cdot 513nx^4 - 3\cdot 63\cdot 64n^2x^8 + 9\cdot 512n^3x^{12} = 0$$

seu

$$512n^3x^{12} - 1344n^2x^8 + 1368nx^4 - 455 = 0$$

ponatur $8nx^4 = z$; erit

$$z^3 - 21z^2 + 171z - 455 = 0,$$

quae divisorem habet $z - 5$, alterque factor erit $zz - 16z + 91 = 0$ radices continens imaginarias. Erit ergo tantum $z = 8nx^4 = 5$ ideoque $x = \sqrt[4]{\frac{5}{8n}}$, qui valor reddet expressionem $\sqrt[4]{(1+mx^4)} + \sqrt[4]{(1-nx^4)}$ maximum vel minimum.

Quorum utrum eveniat, quaeratur

$$\frac{ddy}{dx^2} = \frac{3mxx}{(1+mx^4)^{\frac{7}{4}}} - \frac{3nxx}{(1-nx^4)^{\frac{7}{4}}}.$$

At ob $m = 8n$ posito $x^4 = \frac{5}{8n}$ erit

$$\frac{ddy}{dx^2} = \left(\frac{24n}{6^{\frac{7}{4}}} - \frac{3n}{(3:8)^{\frac{7}{4}}} \right) xx = -\frac{360nxx}{6^{\frac{7}{4}}}$$

ideoque negativum; ergo fiet $\sqrt[4]{(1+8nx^4)} + \sqrt[4]{(1-nx^4)}$ maximum posito $x = \sqrt[4]{\frac{5}{8n}}$. Erit vero hoc maximum $\sqrt[4]{6} + \sqrt[4]{\frac{3}{8}} = \frac{3\sqrt[4]{6}}{2}$. Si loco nx^4 ponamus *u*, patet hanc expression $\sqrt[4]{(1+8u)} + \sqrt[4]{(1-u)}$ fieri maximam posito $u = \frac{5}{8}$ huncque valorem maximum fore $= \frac{3\sqrt[4]{6}}{2} = 2,347627$. Quicunque ergo valor praeter $\frac{5}{8}$ pro *u* scribatur, expressio minorem accipiet valorem.

272. Simili modo maxima ac minima determinabuntur, si quantitates quoque transcendentes in expressione proposita insint. Nisi enim functio proposita fuerit multiformis atque aliquot eius significatus simul considerari debeant, radices aequationis differentialis ostendent maxima vel minima, nisi affuerint radices aequales, quarum numerus sit par. Hanc ergo investigationem in aliquot exemplis declarabimus.

EXEMPLUM 1

Invenire numerum, qui ad suum logarithmum minimam teneat rationem.

Dari huiusmodi rationem minimam $\frac{x}{lx}$ inde patet, quod haec ratio tam posito $x = 1$ quam $x = \infty$ fiat infinita. Vicissim ergo habebit fractio $\frac{lx}{x}$ alicubi maximum valorem, eodem scilicet casu, quo $\frac{x}{lx}$ fit minimum. Ad hunc casum indagandum ponatur $y = \frac{lx}{x}$; fietque

$$\frac{dy}{dx} = \frac{1}{xx} - \frac{lx}{xx}.$$

Quo nihilo aequali posito erit $lx = 1$, et quia hic logarithmum hyperbolicum assumsimus, si e ponatur numerus, cuius logarithmus hyperbolicus sit $= 1$, erit $x = e$. Cum igitur omnes logarithmi ad hyperbolicos in data sint ratione, erit in quocunque logarithmorum canone $\frac{e}{le}$ minimum seu $\frac{le}{e}$ maximum. Quoniam in logarithmis tabularibus est $le = 0{,}4342944819$, fractio $\frac{lx}{x}$ perpetuo erit minor quam $\frac{4842944819}{27182818284}$ seu proxime quam $\frac{47}{305}$ neque ullus datur numerus, qui ad suum logarithmum minorem teneat rationem quam 305 ad 47. Esse autem hoc casu $\frac{lx}{x}$ maximum inde patet, quod ob $\frac{ddy}{dx} = \frac{1-lx}{xx}$ fiat

$$\frac{ddy}{dx^2} = -\frac{1}{x^3} - \frac{2(1-lx)}{x^3} = -\frac{1}{x^3}$$

propter $1 - lx = 0$ ideoque negativum.

EXEMPLUM 2

Invenire numerum x, ut haec potestas $x^{1:x}$ fiat maximum.

Dari huius formulae valorem maximum inde patet, quod numeris loco x substituendis sit

$$\begin{aligned} 1^{1:1} &= 1{,}000000 \\ 2^{1:2} &= 1{,}414213 \\ 3^{1:3} &= 1{,}442250 \\ 4^{1:4} &= 1{,}414213. \end{aligned}$$

Ponatur ergo $x^{1:x} = y$ eritque

$$\frac{dy}{dx} = x^{1:x}\left(\frac{1}{xx} - \frac{lx}{xx}\right).$$

Quo valore nihilo aequali posito erit $lx = 1$ et $x = e$ existente $e = 2{,}718281828$.
Et cum sit $\frac{dy}{dx} = (1 - lx)\frac{x^{1:x}}{xx}$, erit

$$\frac{ddy}{dx^2} = -\frac{x^{1:x}}{x^3} + (1 - lx)\frac{d}{dx}.\frac{x^{1:x}}{xx} = -\frac{x^{1:x}}{x^3}$$

ob $1-lx=0$. Quare cum sit $\frac{ddy}{dx^2}$ quantitas negativa, fiet $x^{1:x}$ maximum casu $x=e$. Cum autem sit $e=2{,}718281828$, reperitur fore $e^{\frac{1}{e}}=1{,}444667861009764$, qui valor obtinetur facile ex serie

$$e^{\frac{1}{e}}=1+\tfrac{1}{e}+\tfrac{1}{2e^2}+\tfrac{1}{6e^3}+\tfrac{1}{24e^4}+\text{etc.}$$

Hoc exemplum quoque ex praecedenti resolvitur; si enim sit $x^{1:x}$ maximum, quoque eius logarithmus, qui est $\frac{lx}{x}$, debebit esse maximum; quod quo fiat, debet esse $x=e$, uti invenimus.

EXEMPLUM 3

Invenire arcum x, ut sit eius sinus maximus vel minimus.

Posito $\sin x=y$ erit $\frac{dy}{dx}=\cos x$ ideoque $\cos x=0$, unde prodeunt sequentes vaIores pro x: $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$, $\pm\frac{5\pi}{2}$, etc. Fit autem $\frac{ddy}{dx^2}=-\sin x$. Cum igitur hi valores pro x substituti dent pro sin x vel +1 vel -1, illi erunt maximi, hi vero minimi, uti constat.

EXEMPLUM 4

Invenire arcum x, ut rectangulum x sin *x fiat maximum.*

Dari maximum inde patet, quod posito vel $x=0°$ vel $x=180°$ utroque casu rectangulum propositum evanescat. Sit igitur $y=x\sin x$; erit

$$\frac{dy}{dx}=\sin x+x\cos x$$

ideoque

$$\operatorname{tg} x=-x\,.$$

Sit $x=90°+u$; erit $\operatorname{tg} x=-\cot u$, ergo $\cot u=90°+u$. Ad quam aequationem modo supra tradito resolvendam ponatur $z=90°+u-\cot u$ sitque f valor arcus u quaesitus. Cum sit $dz=du+\frac{du}{\sin^2 u}$, erit

$$p=\frac{du}{dz}=\frac{\sin^2 u}{1+\sin^2 u},\quad dp=\frac{2du\sin u\cos u}{(1+\sin u)^2}$$

ideoque

$$\frac{dp}{dz}=q=\frac{2\sin^3 u\cos u}{(1+\sin^2 u)^3},\quad dq=\frac{6du\sin^2 u\cos^2 u-2du\sin^4 u}{(1+\sin^2 u)^3}-\frac{12du\sin^4 u\cos^2 u}{(1+\sin^2 u)^4}.$$

Ergo

$$\frac{dq}{dz} = r = \frac{6\sin^4 u\cos^2 u - 2\sin^6 u}{\left(1+\sin^2 u\right)^4} - \frac{12\sin^6 u\cos^2 u}{\left(1+\sin^2 u\right)^5} = \frac{6\sin^4 u - 14\sin^6 + 4\sin^8 u}{\left(1+\sin^2 u\right)^5}.$$

Ex quibus erit

$$f = u - pz + \tfrac{1}{2}qzz - \tfrac{1}{6}rz^3 + \text{etc.}$$

Ponatur, postquam aliquot tentaminibus proximus ipsius *f* valor est detectus, $u = 26°15'$; erit $90° + u = 116°15'$ et arcus cotangenti *u* aequalis ita definiatur. A

$$l\cot u = 10{,}3070250$$

subtrahatur

$$\begin{array}{r} 4{,}6855749 \\ \hline 5{,}6214501 \end{array}$$

Ergo

$$\cot u = 418263{,}\ 7''$$

seu

$$\cot u = 116^0 11' 3\tfrac{7}{10}''$$

unde

$$z = 3' 56\tfrac{3}{10}'' = 236{,}3''.$$

.
Iam ad valorem termini *pz* inveniendum iste instituatur calculus:

$$\begin{array}{rl} l\sin u = & 9{,}6457058 \\ l\sin^2 u = & 9{,}2914116 \\ 1+\sin^2 u = & 1{,}19561 \\ l\left(1+\sin^2 u\right) = & 0{,}0775895 \\ lp = & 9{,}2138221 \\ lz = & 2{,}3734637 \\ lpz = & 1{,}5872858 \end{array}$$

Ergo

$$pz = 38{,}6621 \text{ secundis}$$

seu

$$pz = 38''\ 39'''\ 43''''$$

ab

fiet

$$u = 26°15'$$

$$f = 26°\,14'\,21''\,20'''\,17''''$$

et arcus quaesitus

$$x = 116°14'21''20'''17''''.$$

Tertius vero termInus $\frac{1}{2}qzz = \frac{\sin^3 u \cos u}{\left(1+\sin^2 u\right)^3}zz$ insuper addi debet. Cuius valor ut inveniatur, unum z in partibus radii exprimi debet hoc modo:

$$lz = 2{,}3734637$$

addatur

$$4{,}6855749$$

addatur

$$7{,}0590386$$

addatur

$$l\frac{\sin^2 u}{1+\sin^2 u}z = 1{,}5872858$$

addatur

$$8{,}6463244$$

addatur

$$l\sin u = 9{,}6457058$$

$$l\cos u = 9{,}9527308$$

$$8{,}2447600$$

subtrahatur

$$l\left(1+\sin^2 u\right)^2 = 0{,}1551790$$

$$\tfrac{1}{2}lqzz = 8{,}08958$$

Ergo

$$\tfrac{1}{2}qzz = 0{,}012291$$

seu

$$\tfrac{1}{2}qzz = 44''''15'''''.$$

Unde et hoc termino adhibito fiet arcus quaesitus

$$x = 116°14'\,21''\,21'''\,0'''';$$

maioribus autem logarithmis adhibitis reperitur

$$x = 116°14'\,21''\,20'''\,35''''\,47'''''.$$