THE FUNDAMENTALS OF DIFFERENTIAL CALCULIS

SECOND PART

CONTAINING

THE USE OF THIS CALCULUS IN FINITE ANALYSIS

AND ALSO IN THE PRINCIPLES OF SERIES.

CHAPTER I

CONCERNING THE TRANSFORMATION OF SERIES

1. Since the intention for us shall be to show the use of differential calculus both in the whole of analysis as well as in the instruction of series, several aids from common algebra are to be recalled here, which generally are not accustomed to be treated. Which although we have now included the greatest part of that in the *Introductione*, yet certain parts have been set aside there, either from the enthusiasm which enabled that to be then finally explained, when the need to do that could then be decided, or because all things for which there would be a future need were unable to be foreseen. Here the transformation of series, to which we have designated this chapter, pertains to how some series may be changed into innumerable other series, which all may have the same common sum, thus so that if the sum of a proposed series shall be known, all the remaining series likewise are able to be summed. Moreover in this chapter we will be able to enlarge in a fertile manner the principles of the summation of series through the differential and integral calculus, to be sent forth from that.

2. But we will consider mainly series of this kind, the individual terms of which are to be multiplied by successive powers of a certain indeterminate quantity, because these extend more widely and may bring about a greater use.

Therefore let the following general series be proposed, the sum of which, whether it shall be known or not, we may put = S, and let there be

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc}.$$

Now we may put $x = \frac{y}{1+y}$, and since there shall be these values substituted by the infinite series

$$\begin{aligned}
x &= y - y^2 + y^3 - y^4 + y^5 - y^6 + \text{etc.}\\
x^2 &= y^2 - 2y^3 + 3y^4 - 4y^5 + 5y^6 - 6y^7 + \text{etc.}\\
x^3 &= y^3 - 3y^4 + 6y^5 - 10y^6 + 15y^7 - 21y^8 + \text{etc.}\\
x^4 &= y^4 - 4y^5 + 10y^6 - 20y^7 + 35y^8 - 56y^9 + \text{etc.}\\
&\text{etc.,}
\end{aligned}$$

and with the series arranged following the powers of y, there will be given
[recalling Euler's habit of putting the powers on the first line only]

$$\begin{array}{llllll} S = ay & -ay^2 & +ay^3 & -ay^4 & +ay^5 & \text{etc.} \\ & +b & -2b & +3b & -4b & \\ & & +c & -3c & +6c & \\ & & & +d & -4d & \\ & & & & +e & \end{array}$$

3. Because we have put $x = \frac{y}{1+y}$ there will be $y = \frac{x}{1-x}$; with which value substituted in place of y the proposed series

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc}.$$

will be transformed into this

$$S = a\frac{x}{(1-x)} + (b-a)\frac{x^2}{(1-x)^2} + (c-2b+a)\frac{x^3}{(1-x)^3} + \text{etc}.,$$

in which the coefficient of the second term $b-a$ is the first difference of a from the series a, b, c, d, e etc., which we have set out above by Δa; the coefficient of the third term $c-2b+a$ is the second difference $\Delta^2 a$; the coefficient of the fourth is the third difference $\Delta^3 a$ etc. Hence with the continued differences used of a, which are formed from the series a, b, c, d, e etc., the proposed series will be transformed into this

$$S = \frac{x}{(1-x)}a + \frac{x^2}{(1-x)^2}\Delta a + \frac{x^3}{(1-x)^3}\Delta^2 a + \frac{x^4}{(1-x)^4}\Delta^3 a + \text{etc}.,$$

therefore the sum of this series will be found, if the proposed sum were known.

4. Therefore if the series a, b, c, d etc. were prepared thus, so that it may have constant differences only, which comes about, if the general term [in this series] were a function of a rational whole number, then the following series $\frac{x}{(1-x)}a + \frac{x^2}{(1-x)^2}\Delta a + \text{etc.}$ finally will have vanishing terms, and thus the sum will be able to be shown by a finite expression. Now if the first differences of the series a, b, c, d etc. were constants, then the sum of this series $ax + bx^2 + cx^3 + dx^4 + \text{etc}.$ will be

$$= \frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a.$$

But if the second differences of the coefficients of this series become constant, then the sum of the proposed series will be

$$= \frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a + \frac{x^3}{(1-x)^3}\Delta\Delta a$$

From which the sums of series of this kind may be found easily from the differences of the coefficients.

I. *The sum of this series may be sought*

$$1x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \text{etc.}$$
$$\text{Diff.I} \quad 2, \quad 2, \quad 2, \quad 2 \quad \text{etc.}$$

Therefore since the first differences shall be constant, on account of $a = 1$ and $\Delta a = 2$ the sum of the proposed series will be

$$= \frac{x}{1-x} + \frac{2xx}{(1-x)^2} = \frac{x+x^2}{(1-x)^2}$$

II. *The sum of this series may be sought*

$$1x + 4xx + 9x^3 + 16x^4 + 25x^5 + \text{etc.}$$
$$\text{Diff. I} \quad 3, \quad 5, \quad 7, \quad 9 \quad \text{etc.}$$
$$\text{Diff.II} \quad 2, \quad 2, \quad 2 \quad \text{etc.}$$

And thus because there is $a = 1,\ \Delta a = 3,\ \Delta^2 a = 2$, the sum of the proposed series will be

$$= \frac{x}{1-x} + \frac{3xx}{(1-x)^2} + \frac{2x^3}{(1-x)^3} = \frac{x+xx}{(1-x)^3}.$$

III. *The sum of this series may be sought*

$$S = 4x + 15x^2 + 40x^3 + 85x^4 + 156x^5 + 259x^6 + \quad \text{etc.}$$
$$\text{Diff. I} \quad 11, \quad 25, \quad 45, \quad 71, \quad 103 \quad \text{etc.}$$
$$\text{Diff.II} \quad 14, \quad 20, \quad 26, \quad 32 \quad \text{etc.}$$
$$\text{Diff.III} \quad 6, \quad 6, \quad 6 \quad \text{etc.}$$

Because there is $a = 4,\ \Delta a = 11,\ \Delta^2 a = 14,\ \Delta^3 a = 6$, the sum will be

$$S = \frac{4x}{1-x} + \frac{11xx}{(1-x)^2} + \frac{14x^3}{(1-x)^3} + \frac{6x^4}{(1-x)^4}$$

or

$$S = \frac{4x - xx + 4x^3 - x^4}{(1-x)^4} = \frac{x(1+xx)(4-x)}{(1-x)^4}.$$

5. Though the sums are found of these series to infinity in this manner, yet from the same principles these series also are able to be summed to some given term. For let this series be proposed

$$S = ax + bx^2 + cx^3 + dx^4 + ...+ ox^n$$

and the sum of this is sought, if it may progress to infinity, which will be

$$= \tfrac{x}{(1-x)} a + \tfrac{x^2}{(1-x)^2} \Delta a + \tfrac{x^3}{(1-x)^3} \Delta^2 a + \text{etc.}$$

Now the terms of the same series may be considered following after the final ox^n, which shall be

$$px^{n+1} + qx^{n+2} + rx^{n+3} + sx^{n+4} + \text{etc.}$$

the sum of which series, if it may be divided by x^n, can be found as before ; which again multiplied by x^n, will be

$$\tfrac{x^{n+1}}{1-x} p + \tfrac{x^{n+2}}{(1-x)^2} \Delta p + \tfrac{x^{n+3}}{(1-x)^3} \Delta^2 p + \text{etc.};$$

which sum, if it may be subtracted from the sum of the whole series to infinity, will leave the sum of the portion of the series sought

$$S = \tfrac{x}{1-x}\left(a - x^n p\right) + \tfrac{x^2}{(1-x)^2}\left(a\Delta - x^n \Delta p\right) + \tfrac{x^3}{(1-x)^3}\left(\Delta^2 a - x^n \Delta^2 p\right) + \text{etc.}$$

1. *The sum of this finite series may be sought*

$$S = 1x + 2x^2 + 3x^3 + 4x^4 + ...+ nx^n \ .$$

As the differences of these coefficients as well as of the final terms are sought

1, 2, 3, 4 etc	$n+1$, $n+2$, $n+3$ etc.
1 1 1 etc.	1, 1, etc

and there will be $a = 1, \Delta a = 1, p = n+1, \Delta p = 1$, from which the sum sought is

$$S = \tfrac{x}{1-x}\left(1 - (n+1)x^n\right) + \tfrac{x^2}{(1-x)^2}\left(1 - x^n\right)$$

or

$$S = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}.$$

II. *The sum of this finite series may be sought*

$$S = 1x + 4x^2 + 9x^3 + 16x^4 + \ldots\ldots\ldots + n^2x^n$$

First the differences may be found in this manner

1, 4, 9, 16 etc.	$(n+1)^2$, $(n+2)^2$, $(n+3)^2$ etc.
3, 5, 7 etc.	$2n+3$, $2n+5$ etc.
2, 2 etc.	2 etc.

with which found the sum sought will be

$$S = \frac{x}{1-x}\left(1-(n+1)^2x^n\right) + \frac{x^2}{(1-x)^2}\left(3-(2n+3)x^n\right) + \frac{x^3}{(1-x)^3}\left(2-2x^n\right)$$

or

$$S = \frac{x+xx-(n+1)^2x^{n+1}+(2nn+2n-1)x^{n+2}-nnx^{n+3}}{(1-x)^3}.$$

6. But if moreover the proposed series may not have coefficients of this kind, which finally may lead to constant differences, then the transformation shown here adds nothing towards determining the sum of this series. Nor truly also with the aid of this can the sum [of nearby terms] be defined more conveniently, as the addition of the terms of the proposed series is allowed to become [otherwise]. If indeed in the series $ax + bx^2 + cx^3 + dx^4 +$ etc., there were $x < 1$, in which case alone the summation can be said properly to have a place, there will be $\frac{x}{1-x} > x$ and thus the new series proposed will be less convergent than the proposed series. But if in the proposed series there were put $x = 1$, then plainly all the terms of the new series become infinite, therefore in which case the transformation will be of no use at all.

7. But we may examine series, in which the signs + and – may follow each other in turn, which may be deduced from the preceding on putting x negative. And if thus there were

$$S = ax - bx^2 + cx^3 - dx^4 + ex^5 - \text{etc.},$$

the negative of which series may arise if x be put negative in the preceding, therefore so that the differences Δa, $\Delta^2 a$, $\Delta^3 a$ etc. are taken from the series of coefficients a, b, c, d, e

etc., with the signs related only to the powers of x, and the proposed series will be transformed into this

$$S = \tfrac{x}{1+x}a - \tfrac{x^2}{(1+x)^2}\Delta a + \tfrac{x^3}{(1+x)^3}\Delta^2 a - \tfrac{x^4}{(1+x)^4}\Delta^3 a + \text{etc.},$$

from which it is evident the proposed equation can be summed in the same cases with the preceding, clearly if the series *a, b, c, d* etc. finally leads to constant differences.

8. But in this case that transformation provides a suitable approximation to the value of the proposed series $ax - bx^2 + cx^3 - dx^4 + ex^5 - fx^6 +$ etc. ; for however large the number x should be, the fraction $\tfrac{x}{1+x}$ shall become less than one, the other series of which progresses following the powers; and if there shall be $x = 1$, then there will be $\tfrac{x}{1+x} = \tfrac{1}{2}$. But if $x < 1$, for example $x = \tfrac{1}{n}$, there will become $\tfrac{x}{1+x} = \tfrac{1}{n+1}$ and thus the series by the transformation found always will converge more than the proposed. In the first place we may consider the case, in which $x = 1$, which brings some help to the huge series requiring to be summed, and there shall be

$$S = a - b + c - d + e - f + \text{etc}.$$

and the first, second and following differences of *a*, which the progression *a, b, c, d, e* etc. gives, may be denoted by $\Delta a,\ \Delta^2 a,\ \Delta^3 a$ etc. ; with which found there shall be

$$S = \tfrac{1}{2}a - \tfrac{1}{4}\Delta a + \tfrac{1}{8}\Delta^2 a - \tfrac{1}{16}\Delta^3 a + \text{etc.},$$

which, unless it may actually terminate, the sum may be shown approximately well enough.

9. Therefore we may show the use of this final transformation, in which we have assumed $x = 1$, in some examples and indeed in the first of this kind, in which a true sum can be expressed. Such series are diverging, in which the numbers *a, b, c, d* etc. may be reduced to constant differences finally; the sums of which are unable to be shown properly with the acceptance of this name, here we accept the name of the sum in that sense, which we have attributed above [see §§111 of Ch.3, Part I], thus so that it may denote the value of a finite expression, from the working out of which the proposed series may arise.

1. *Therefore let the proposed series be this of LEIBNIZ*

$$S = 1 - 1 + 1 - 1 + 1 - 1 + \text{etc.};$$

in which since all the terms shall be equal, all the differences become $= 0$ and thus on account of $a = 1$ there will be $S = \frac{1}{2}$.

II. *Let that series be proposed*

$$\begin{array}{ll} S = 1 - 2 + 3 - 4 + 5 - 6 + \text{etc.} \\ \text{Diff. I} = 1, \ 1, \ 1, \ 1, \ 1 \ \text{etc.} \end{array}$$

Therefore since there shall be $a = 1$, $\Delta a = 1$, there will be $S = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

[Recall that $S = \frac{1}{2}a - \frac{1}{4}\Delta a + \frac{1}{8}\Delta^2 a - \frac{1}{16}\Delta^3 a + \text{etc.}$ from above , with $x = 1 \rightarrow \frac{1}{1+1} = \frac{1}{2}$.]

III. *Let this series be proposed*

$$\begin{array}{ll} S = 1 - 3 + 5 - 7 + 9 - \text{etc.} \\ \text{Diff. I} = 2, \ 2, \ 2, \ 2 \ \text{etc.} \end{array}$$

On account of $a = 1$, and $\Delta a = 1$, there becomes $S = \frac{1}{2} - \frac{2}{4} = 0$.

IV. *Let this series of triangular numbers be proposed*

$$\begin{array}{ll} S = 1 - 3 + 6 - 10 + 15 - 21 + \text{etc.} \\ \text{Diff. I} = 2, \ 3, \ 4, \ 5, \ 6 \ \text{etc.} \\ \text{Diff. II} = 1, \ 1, \ 1, \ 1 \ \text{etc.} \end{array}$$

Therefore here on account of $a = 1$, $\Delta a = 2$ and $\Delta\Delta a = 1$ there will be $S = \frac{1}{2} - \frac{2}{4} + \frac{1}{8} = \frac{1}{8}$.

V. *Let this series of squares be proposed*

$$\begin{array}{ll} S = 1 - 4 + 9 - 16 + 25 - 36 + \text{etc.} \\ \text{Diff. I} = 3, \ 5, \ 7, \ 9, \ 11 \ \text{etc.} \\ \text{Diff. II} = 2, \ 2, \ 2, \ 2 \ \text{etc.} \end{array}$$

On account of $a = 1$, $\Delta a = 3$, $\Delta\Delta a = 2$, there will be $S = \frac{1}{2} - \frac{3}{4} + \frac{2}{8} = 0$.

VI. *Let this series of biquadratics be proposed*

$$\begin{array}{ll} & S = 1 - 16 + 81 - 256 + 625 - 1296 + \text{ etc} \\ \text{Diff. I} = & 15,\ 65,\ 175,\ 369,\ 671 \quad \text{etc.} \\ \text{Diff. II} = & 50,\ 110,\ 194,\ 302 \quad \text{etc.} \\ \text{Diff. III} = & 60,\ 84,\ 108 \ \text{etc.} \\ \text{Diff. IV} = & 24,\ 24 \ \text{etc.} \end{array}$$

Therefore there will be $S = \frac{1}{2} - \frac{15}{4} + \frac{50}{8} - \frac{60}{16} + \frac{24}{32} = 0.$

10. If the series may be more divergent, as geometric and other similar series are, these at once are transformed in this manner into more converging series, which at this stage unless it may converge well enough, in the same manner [the new series] may be converted into another series converging more.

1. *Let this geometric series be proposed*

$$\begin{array}{ll} & S = 1 - 2 + 4 - 8 + 16 - 32 + \text{ etc.} \\ \text{Diff. I} = & 1,\ 2,\ 4,\ 8,\ 16 \quad \text{etc.} \\ \text{Diff. II} = & 1,\ 2,\ 4,\ 8 \quad \text{etc.} \\ \text{Diff. III} = & 1,\ 2,\ 4 \ \text{etc.} \end{array}$$

Therefore since in all the differences the first term shall be $=1$, the sum of the series may be expressed in this manner

$$S = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \text{etc.},$$

of which the sum is $= \frac{1}{3}$; for this arises from the expansion of the fraction $\frac{1}{2+1}$, while the proposed arises from $\frac{1}{1+2}$.

II. *Let this recurring series be proposed*

$$[\, a_{n+1} = 2a_n + a_{n-1} \ \ for\ \ n > 2 \ \ with\ \ a_1 = 1 \ \ and\ \ a_2 = 2 \,]$$

$$S = 1 - 2 + 5 - 12 + 29 - 70 + 169 - \text{etc.}$$

$$\begin{array}{ll} \text{Diff. I} = & 1,\ 3,\ 7,\ 17,\ 41,\ 99 \ \text{etc.} \\ \text{Diff. II} = & 2,\ 4,\ 10,\ 24,\ 58 \ \text{etc.} \\ \text{Diff. III} = & 2,\ 6,\ 14,\ 34 \ \text{etc.} \\ \text{Diff. IV} = & 4,\ 8,\ 20 \ \text{etc.} \\ \text{Diff. V} = & 4,\ 12 \ \text{etc.} \\ \text{Diff. VI} = & 8 \ \text{etc.} \\ & \text{etc.} \end{array}$$

Therefore the first terms of the continued differences constitute this twin geometric progression 1, 1, 2, 2, 4, 4, 8, 8, 16, 16 etc., from which there shall be

$$S = \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{2}{8} - \tfrac{2}{16} + \tfrac{4}{32} - \tfrac{4}{64} + \tfrac{8}{128} - \ \text{etc.};$$

therefore since the remaining first terms besides continually cancel each other , there will be $S = \frac{1}{2}$. But the proposed series arises from the expansion of the fraction $\frac{1}{1+2-1} = \frac{1}{2}$, as we have shown in the expression of the nature of the recurring series.

III. *Let the hypergeometric series be proposed* $[\, a_{n+1} = (n+1)a_n$; with $a_1 = 1\,]$

$$S = 1 - 2 + 6 - 24 + 120 - 720 + 5040 - \ \text{etc.},$$

the continued differences of which we will investigate in this more convenient manner :

	Diff. I	Diff. II	Diff. III	
1	1	3	11	
2	4	14	64	
6	18	78	426	
24	96	504	3216	
120	600	3720	27240	etc.
720	4320	30960	256320	
5040	35280	287280	2656080	
40320	322560	2943360		
362880	3265920			
3628800				

[We may add the relevant part of the remainder of Euler's table used in the summation :

Diff.4	Diff.5	Diff.6
53	309	2119
362	2428	18806
2790	21234	183822
24024	205056	1965624
229080	2170680	22852200
2399760	25022880	287250480
27422640	312273360	
339696000		

Diff.7	Diff.8	Diff.9
16687	148329	1468457
165016	1616786	17487988
1781802	19104774	224406930
20886576	243511704	
264398280		

Diff.10	Diff.11
16019531	190899411
206918942	

]

From which differences continued further there will be :
[recalling $S=\frac{1}{2}a-\frac{1}{4}\Delta a+\frac{1}{8}\Delta^2 a-\frac{1}{16}\Delta^3 a+\text{etc.}$]

$$S=\frac{1}{2}-\frac{1}{4}+\frac{3}{8}-\frac{11}{16}+\frac{53}{32}-\frac{309}{64}+\frac{2119}{128}-\frac{16687}{256}+\frac{148329}{512}+\frac{1468457}{1024}$$
$$+\frac{16019531}{2048}-\frac{190899411}{4096}+\text{etc.}$$

The two initial terms are taken together and there will be $S=\frac{1}{4}+A$, with

$$A=\frac{3}{8}-\frac{11}{16}+\frac{53}{32}-\frac{309}{64}+\frac{2119}{128}-\text{etc.}$$

If now the differences are taken in the same manner, [in consecutive pairs,] there will be

$$A=\frac{3}{2^4}-\frac{5}{2^6}+\frac{21}{2^8}-\frac{99}{2^{10}}+\frac{615}{2^{12}}-\frac{4401}{2^{14}}+\frac{36585}{2^{16}}-\frac{342207}{2^{18}}+\frac{3565323}{2^{20}}-\frac{40866525}{2^{22}}+\text{etc.}$$

[The table corresponding to this is :

:

Diff.1	Diff.2	Diff.3	Diff.4	Diff.5	Diff.6
3	5	21	99	615	4401
11	31	141	813	5631	45387
53	203	1095	7257	56649	506151
309	1501	9447	71163	619449	6092793
2119	12449	90057	761775	7331691	78843213
16687	114955	941889	8855241	93506595	
148329	1171799	10739019	111217077		
1468457	13082617	132695115			
16019531	158860349				
190899411					

Diff.7	Diff.8	Diff.9	Diff.11
36585	342207	3565323	40866525
415377	4249737	47997171	
5080491	56496645		
66657627			

]

The two initial terms, because they converge, are taken together, and there becomes $A=\frac{7}{2^6}+B$, with $B=\frac{21}{2^8}-\frac{99}{2^{10}}+\text{etc.}$;

[The new original series with $x=1$ will be $B=\frac{21}{2^8}-\frac{99}{2^{10}}+\frac{615}{2^{12}}-\frac{4401}{2^{14}}+\frac{36585}{2^{16}}-\frac{342207}{2^{18}}+\frac{3565323}{2^{20}}-\frac{40866525}{2^{22}}+\text{etc.}$; matching this against the series with $y=\frac{1}{2}$, there will be $S=\frac{1}{2}a-\frac{1}{4}\Delta a+\frac{1}{8}\Delta^2 a-\frac{1}{16}\Delta^3 a+\text{etc.}$ we have $\frac{a}{2}=\frac{21}{2^9}$, $b=\frac{99}{2^{10}}$; $\Delta a=b-a=\frac{99}{2^{10}}-\frac{21}{2^8}=\frac{99-84}{2^{10}}=\frac{15}{2^{10}}$ leading to the term $\frac{1}{4}\Delta a=\frac{15}{2^{12}}$;
$c=\frac{615}{2^{12}}$; $\Delta^2 a=c-2b+a=\frac{615}{2^{12}}-2\cdot\frac{99}{2^{10}}+\frac{21}{2^8}=\frac{615-8\cdot 99+16\cdot 21}{2^{12}}=\frac{159}{2^{12}}$ etc.]

of which series, with the differences taken again, there becomes

$$B=\frac{21}{2^9}-\frac{15}{2^{12}}+\frac{159}{2^{15}}-\frac{429}{2^{18}}+\frac{5241}{2^{21}}-\frac{26283}{2^{24}}+\frac{338835}{2^{27}}-\frac{2771097}{2^{30}}+\text{etc.}$$

[The table for this series :

21	15	159	429	5241	26283	338835	2771097
99	219	1065	6957	47247	443967	4126437	
615	1941	11217	75075	632955	5902305		
4401	18981	119943	933255	8434125			
36585	195867	1413027	12167145				
342207	2196495	17819253					
3565323	26605233						
40866525							

]
The first four terms are gathered together into one, and there may be put $B = \frac{153}{2^{12}} + \frac{843}{2^{18}} + C$ with

$$C = \frac{5241}{2^{21}} - \frac{26283}{2^{24}} + \text{etc.}$$

and there becomes with several terms actually taken together, $C = \frac{15645}{2^{24}} - \frac{60417}{2^{30}}$. Therefore from these it may be concluded finally the sum of the proposed series S =0,40082055, which yet scarcely can have three or four figures further for accuracy on account of the exceedingly divergent series; yet certainly the smaller is true. Indeed from elsewhere this sum is = 0,4036524077, where it does not disagree with the true noted value.
[The editor of the *O.O*. edition notes that S is related to the logarithmic integral li(z); Euler had already started to investigate this rather intriguing transcendental function, and §128 of Book I of his Integral Calculus present on this website shows how it can be expanded in an infinite series. The investigation of the history of this function would be a nice project for someone.]

11. But in the first place this transformation brings great usefulness to the series but now converging slowly, now indeed requiring to be transformed into others which may converge more promptly. Because truly the following terms are less than the preceding, the first difference become negative; from which in the following an account of the signs must be attended to properly.

I. *Let this series be proposed*

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{etc.}$$

$$\begin{array}{ll}
\text{Diff.I} & = -\frac{1}{2}, -\frac{1}{2\cdot3}, \ -\frac{1}{3\cdot4}, -\frac{1}{4\cdot5}, -\frac{1}{5\cdot6}, \text{etc.} \\
\text{Diff. II} & = +\frac{1}{3}, \ \frac{2}{2\cdot3\cdot4}, \ \frac{2}{3\cdot4\cdot5}, \ \frac{2}{4\cdot5\cdot6} \text{etc.} \\
\text{Diff. III} & = -\frac{1}{4}, -\frac{2\cdot3}{2\cdot3\cdot4\cdot5}, -\frac{2\cdot3}{3\cdot4\cdot5\cdot6} \text{etc.} \\
\text{Diff. IV} & = \ \ +\frac{1}{5} \ \ \text{etc.} \\
 & \quad \text{etc.}
\end{array}$$

Hence therefore there will be [recalling $S = \frac{1}{2}a - \frac{1}{4}\Delta a + \frac{1}{8}\Delta^2 a - \frac{1}{16}\Delta^3 a + \text{etc.}$]

$$S = \frac{1}{2} + \frac{1}{2.4} + \frac{1}{3.8} + \frac{1}{4.16} + \frac{1}{5.32} + \text{etc.} \ ;$$

but we have shown now in the *Introductione* that each series produces the hyperbolic logarithm of two.
[For in Ch.7 of Part I, the formula is deduced :

$l(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \text{etc.}$, which produces the above equation on setting $x = -\frac{1}{2}$; while setting $x = 1$ gives $l(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{etc.}$]

II. *Let this equation be proposed for the circle*

$$S = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}$$

$$\begin{array}{ll} \text{Diff. I} & = -\frac{2}{1\cdot3},\quad -\frac{2}{3\cdot5},\quad -\frac{2}{5\cdot7},\quad -\frac{2}{7\cdot9}\quad -\frac{2}{9\cdot11}\ \text{etc.} \\ \text{Diff. II} & = +\frac{2\cdot4}{1\cdot3\cdot5},\quad \frac{2\cdot4}{3\cdot5\cdot7},\quad \frac{2\cdot4}{5\cdot7\cdot9},\quad \frac{2\cdot4}{7\cdot9\cdot11}\ \text{etc.} \\ \text{Diff. III} & = -\frac{2\cdot4\cdot6}{1\cdot3\cdot5\cdot7},\quad -\frac{2\cdot4\cdot6}{3\cdot5\cdot7\cdot9}\ \text{etc.} \\ & \qquad \text{etc.} \end{array}$$

Hence it is concluded that the sum of the series is [*i.e.* $S = \frac{1}{2}a - \frac{1}{4}\Delta a + \frac{1}{8}\Delta^2 a - \frac{1}{16}\Delta^3 a + \text{etc.}$]

$$S = \frac{1}{2} + \frac{1}{3\cdot2} + \frac{1\cdot2}{3\cdot5\cdot2} + \frac{1\cdot2\cdot3}{3\cdot5\cdot7\cdot2} + \text{etc}.$$

or

$$2S = 1 + \frac{1}{3} + \frac{1\cdot2}{3\cdot5} + \frac{1\cdot2\cdot3}{3\cdot5\cdot7} + \frac{1\cdot2\cdot3\cdot4}{3\cdot5\cdot7\cdot9} + \text{etc}.$$

III. *The value of this infinite series is sought*

$$S = l2 - l3 + l4 - l5 + l6 - l7 + l8 - l9 + \text{etc}.$$

Which differences from the beginning are made exceedingly unequal, actually the terms are gathered as far as to $l10$ from tables, the value of which may be found $= -0,3911005$, and there will be

$$S = -0,3911005 + l10 - l11 + l12 - l13 + l14 - l15 + \text{etc}.$$

to infinity. These logarithms may be selected from tables and the differences of these sought in this manner:

	Diff. I	Diff. II	Diff. III	Diff. IV	Diff. V
$l10 = 1{,}0000000$	+	–	+	–	+
$l11 = 1{,}0413927$	413927				
		36042			
$l12 = 1{,}0791812$	377885		5779		
		30263		1292	
$l13 = 1{,}1139434$	347622		4487		368
		25776		924	
$l14 = 1{,}1461280$	321846		3563		
		22213			
$l15 = 1{,}1760913$	299633				

From which there may be shown

$$\begin{aligned} & l10 - l11 + l12 - l13 + \text{etc.} \\ & = \tfrac{1{,}0000000}{2} - \tfrac{413927}{4} - \tfrac{36042}{8} - \tfrac{5779}{16} - \tfrac{1292}{32} - \tfrac{368}{64} = 0{,}4891606. \end{aligned}$$

Hence the value of the proposed series will be

$$S = l2 - l3 + l4 - l5 + \text{etc.} = 0{,}0980601,$$

to which logarithm the number 1,253315 corresponds. [Base 10 of course.]

12. Just as we have obtained these transformations on putting this fraction $\frac{y}{1\pm y}$ in place of x in the series, thus innumerable other transformations may arise, if in place of x other functions of y may be substituted. Let this series be proposed again

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + \text{etc.}$$

and there is put $x = y(1-y)$, with which done the following series arises

$$\begin{aligned} S = ay - ayy & \\ + byy - 2by^3 + by^4 & \\ + cy^3 - 3cy^4 + 3cy^5 - cy^6 & \\ + dy^4 - 4dy^5 + 6dy^6 & \\ + ey^5 - 5ey^6 & \\ + fy^6 \quad \text{etc.} & \end{aligned}$$

But therefore if either of these series were summable, likewise the other sum would be known. Thus if there may be put in place

$$S = x + x^2 + x^3 + x^4 + x^5 + x^6 + \text{etc.} = \tfrac{x}{1-x}$$

there will be

$$S = y - y^3 - y^4 + y^6 + y^7 - y^9 - y^{10} + \text{etc.}$$

Therefore the sum of this series will be $= \frac{y-yy}{1-y+yy}$.

13. If the other [*i.e.* second] series may be terminated somewhere, then the sum of the first can be shown completely. We may put $a = 1$ and in the series found all the terms after the first to vanish, so that there shall be $S = y$, and thus on account of $x = y - yy$ the sum of the first will be $= \frac{1}{2} - \sqrt{\left(\frac{1}{4} - x\right)}$. Moreover on account of making $a = 1$, it follows that,

$$\begin{aligned}
b &= 1 = \tfrac{1}{4} \cdot 2^2 \\
c &= 2 = \tfrac{1\cdot 3}{4\cdot 6} \cdot 2^4 \\
d &= 5 = \tfrac{1\cdot 3\cdot 5}{4\cdot 6\cdot 8} \cdot 2^6 \\
e &= 14 = \tfrac{1\cdot 3\cdot 5\cdot 7}{4\cdot 6\cdot 8\cdot 10} \cdot 2^8 \\
f &= 42 = \tfrac{1\cdot 3\cdot 5\cdot 7\cdot 9}{4\cdot 6\cdot 8\cdot 10\cdot 12} \cdot 2^{10} \\
g &= 132 = \tfrac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14} \cdot 2^{12} \\
&\text{etc.,}
\end{aligned}$$

from which the first series will change into this

$$S = \tfrac{1}{2} - \sqrt{\left(\tfrac{1}{4} - x\right)} = x + x^2 + 2x^3 + 5x^4 + 14x^5 + 42x^6 + 132x^7 + \text{etc.},$$

which is found the same, if the surd quantity $\sqrt{\left(\frac{1}{4} - x\right)}$ may be expanded in a series and taken from $\frac{1}{2}$.

14. We may put in place $x = y(1+ny)^r$, so that the transformation may be extended wider, and the proposed series

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc}.$$

will be changed into the following

$$\begin{array}{llllll}
S = ay + \frac{v}{1}nay^2 & + \frac{v(v-1)}{1\cdot 2}n^2ay^3 & + \frac{v(v-1)(v-2)}{1\cdot 2\cdot 3}n^3ay^4 & + \frac{v(v-1)(v-2)(v-3)}{1\cdot 2\cdot 3\cdot 4}n^4ay^5 \\
\quad + by^2 & + \frac{2v}{1}nby^3 & + \frac{2v(2v-1)}{1\cdot 2}n^2by^4 & + \frac{2v(2v-1)(2v-2)}{1\cdot 2\cdot 3}n^3ay^5 \\
 & + cy^3 & + \frac{3v}{1}ncy^4 & + \frac{3v(3v-1)}{1\cdot 2}n^2by^5 \\
 & & + dy^4 & + \frac{4v}{1}ndy^5 \\
 & & & + ey^5 \\
 & \text{etc.} & &
\end{array}$$

Therefore if the sum of that series deserves to be known, and likewise in turn the sum of this will be found. Because truly n and v can be taken as it pleases, hence from one summable series innumerable other summable series can be found.

15. Also transformations of this kind can come about, so that the sum of the series found becomes irrational, in this way. Let this series be proposed

$$S = ax + bx^3 + cx^5 + dx^7 + ex^9 + fx^{11} + \text{etc.} ;$$

there will be

$$Sx = ax^2 + bx^4 + cx^6 + dx^7 + ex^{10} + fx^{12} + \text{etc.}$$

now there may be put in place

$$x = \frac{y}{\sqrt{(1-nyy)}} ;$$

there will be $xx = \frac{y^2}{1-nyy}$ and the proposed series will be changed into this

$$\begin{array}{l}
\frac{Sy}{\sqrt{(1-nyy)}} = ay^2 + nay^4 + n^2ay^6 + n^3ay^8 + n^4ay^{10} + \text{etc.} \\
\qquad + by^4 + 2nby^6 + 3n^2by^8 + 4n^3by^{10} + \text{etc.} \\
\qquad\qquad + cy^6 + 3ncy^8 + 6n^2cy^{10} + \text{etc.} \\
\qquad\qquad\qquad + dy^8 + 4ndy^{10} + \text{etc.} \\
\qquad\qquad\qquad\qquad + ey^{10} + \text{etc.} \\
\qquad\qquad\qquad \text{etc.}
\end{array}$$

Therefore if the sum S from the first series were known, likewise the sum of the following series will be obtained

$$\frac{S}{\sqrt{(1-nyy)}} = ay + (na+b)y^3 + \left(n^2a + 2nb + c\right)y^5 + \left(n^3a + 3n^2b + 3nc + d\right)y^7 + \text{etc.}$$

16. If it is assumed that $n=-1$, the coefficients of this series of continued differences of a will be from the series a, b, c, d etc.; but if the signs in the proposed series may be alternated, then on putting $n=1$ the coefficients will be these differences. Therefore $\Delta a, \Delta^2 a, \Delta^3 a, \Delta^4 a$ etc. may denote the first, second, third differences etc. of a, from the series of numbers a, b, c, d, e, f etc. And if there should be

$$S = ax + bx^3 + cx^5 + dx^7 + ex^9 + \text{etc.}$$

on putting $x = \frac{y}{\sqrt{(1+yy)}}$ there will be

$$\frac{S}{\sqrt{(1+yy)}} = ay + \Delta a \cdot y^3 + \Delta a^2 \cdot y^5 + \Delta a^3 \cdot y^7 + \text{etc.}$$

But if there should be

$$S = ax - bx^3 + cx^5 - dx^7 + ex^9 - \text{etc.}$$

and there may be put $x = \frac{y}{\sqrt{(1-yy)}}$, there will be

$$\frac{S}{\sqrt{(1-yy)}} = ay - \Delta a \cdot y^3 + \Delta a^2 \cdot y^5 - \Delta a^3 \cdot y^7 + \text{etc.}$$

But if the series a, b, c, d, e etc. therefore may lead to constant differences only, then each series will be absolutely summable; moreover which summation follows also from above.

17. We may put the coefficients a, b, c, d etc. in place to constitute this series

$$1,\ \tfrac{1}{3},\ \tfrac{1}{5},\ \tfrac{1}{7},\ \tfrac{1}{9}\ \text{ etc.}$$

and there will be, as we have seen above,

$$a = 1,\ \ \Delta a = -\tfrac{2}{3},\ \ \Delta a^2 = \tfrac{2\cdot 4}{3\cdot 5},\ \ \Delta a^3 = -\tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\ \text{ etc.,}$$

from which we will be able to sum the following two series.

I. Let there be $S = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \text{etc.}$; there will be $S = \frac{1}{2} l \frac{1+x}{1-x}$. Now on putting $x = \frac{y}{\sqrt{(1+yy)}}$ there becomes

$$S = \tfrac{1}{2} l \frac{\sqrt{(1+yy)}+y}{\sqrt{(1+yy)}-y} = l\left(\sqrt{(1+yy)}+y\right),$$

from which there shall be

$$\frac{l\left(\sqrt{(1+yy)}+y\right)}{\sqrt{(1+yy)}} = y - \tfrac{2}{3}y^3 + \tfrac{2\cdot 4}{3\cdot 5}y^5 - \tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}y^7 + \text{etc.}$$

II. Let there be $S = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \text{etc.}$; there will be $S = \text{Atang}\,x$. Now on putting $x = \frac{y}{\sqrt{(1-yy)}}$ there becomes

$$S = \text{Atang}\frac{y}{\sqrt{(1-yy)}} = \text{Asin}\,y = \text{Acos}\sqrt{(1-yy)}.$$

On this account that sum may be obtained

$$\frac{\text{A sin}\,y}{\sqrt{(1-yy)}} = y + \tfrac{2}{3}y^3 + \tfrac{2\cdot 4}{3\cdot 5}y^5 + \tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}y^7 + \text{etc .}$$

18. Also transcending functions of *y* are able to be substituted in place of *x* and thus other more difficult summations are able to be elicited ; but yet lest the new series become exceedingly complicated, functions of this kind must be selected, the powers of which can be shown easily, of such a kind are the exponential functions e^y. Therefore with this series proposed

$$S = ax + bx^3 + cx^5 + dx^7 + ex^9 + \text{etc.}$$

there may be put $x = e^{ny}y$ with *e* denoting the number, the hyperbolic logarithm of which is $=1$; there will be $x^2 = e^{2ny}y^2$, $x^3 = e^{3ny}y^3$ etc. Truly generally there is, as it is agreed,

$$e^z = 1 + z + \tfrac{z^2}{1\cdot 2} + \tfrac{z^3}{1\cdot 2\cdot 3} + \tfrac{z^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

Whereby the proposed series will be changed into this

$$\begin{aligned} S = ay + 1nay^2 &+ \tfrac{1}{2}n^2ay^3 + \tfrac{1}{6}n^3ay^4 + \tfrac{1}{24}n^4ay^5 + \text{etc.} \\ + by^2 &+ \tfrac{2}{1}nby^3 + \tfrac{4}{2}n^2by^4 + \tfrac{8}{6}n^3by^5 + \text{etc.} \\ &+ cy^3 + \tfrac{3}{1}ncy^4 + \tfrac{9}{2}n^2cy^5 + \text{etc.} \\ &+ dy^4 + \tfrac{4}{1}ndy^5 + \text{etc.} \\ &+ ey^5 + \text{etc.} \\ &\text{etc.} \end{aligned}$$

I. Let the proposed be the geometric series $S = x + x^2 + x^3 + x^4 + x^5 + \text{etc.}$; there will be $S = \frac{x}{1-x}$.

Now there is put $n = -1$ so that there shall be $x = e^{-y}y$ and $S = \frac{e^{-y}y}{1-e^{-y}y} = \frac{y}{e^y - 1}$;

this sum may be found

$$\frac{y}{e^y-1} = y - \tfrac{1}{2}y^3 - \tfrac{1}{6}y^4 + \tfrac{5}{24}y^5 + \tfrac{19}{120}y^6 - \text{etc.},$$

but the rule of which series is not evident.

II. In the other series, let all the other terms besides the first be $= 0$; there will be

$$b = -na,\ c = \tfrac{3}{2}n^2a,\ d = -\tfrac{8}{3}n^3a,\ e = \tfrac{125}{24}n^4a,\ f = -\tfrac{54}{5}n^5a,\ \text{etc.}$$

Therefore since the sum shall be $S = ay$ and $x = ye^{ny}$, there becomes

$$y = x - nx^2 + \tfrac{3}{2}n^2x^3 - \tfrac{8}{3}n^3x^4 + \tfrac{125}{24}n^4x^5 - \tfrac{54}{5}n^5x^6 + \text{etc.}$$

Now since in these series the law of the progression is not evident, the summations from this substitution may have little use. But it is especially worthy of note that the transformations derived from the substitution $x = \frac{y}{1 \pm y}$, are clearly not only exemplary summations, but also suitable ways of supplying the needs of series requiring to be summed. Therefore from these, which have been extricated without the need of the differential calculus, we may proceed to the premises required to show the use of the calculus itself in the principles of series.

INSTITUTIONUM CALCULI DIFFERENTIALIS

PARS POSTERIOR

CONTINENS

USUM HUIUS CALCULI IN ANALYSI

FINITORUM NEC NON IN DOCTRINA SERIERUM.

CAPUT I

DE TRANSFORMATIONE SERIERUM

1. Cum nobis propositum sit usum Calculi differentialis tam in universa Analysi quam in doctrina de seriebus ostendere, nonnulla subsidia ex Algebra communi, quae vulgo tractari non solent, hic erunt repetenda. Quae quamvis maximam partem iam in *Introductione* sumus complexi, tamen quaedam ibi sunt praetermissa, vel studio, quod expediat ea tum demum explicari, quando necessitas id exigat, vel quia cuncta, quibus opus sit futurum, praevideri non poterant. Huc pertinet transformatio serierum, cui hoc caput destinavimus, qua quaevis series in innumerabiles alias series transmutatur, quae omnes eandem habeant summam communem, ita ut, si seriei propositae summa sit cognita, reliquae series omnes simul summari queant. Hoc autem capite praemisso eo uberius doctrinam serierum per calculum differentialem et integralem amplificare poterimus.

2. Considerabimus autem potissimum eiusmodi series, quarum singuli termini per potestates successivas quantitatis cuiusdam indeterminatae sunt multiplicati, quoniam hae latius patent maioremque utilitatem afferent.

Sit igitur proposita sequens series generalis, cuius summam, sive sit cognita sive secus, ponamus = S, sitque

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc}.$$

Ponatur iam $x = \frac{y}{1+y}$ et cum sit per series infinitas

$$\begin{aligned}
x &= y - y^2 + y^3 - y^4 + y^5 - y^6 + \text{etc.}\\
x^2 &= y^2 - 2y^3 + 3y^4 - 4y^5 + 5y^6 - 6y^7 + \text{etc.}\\
x^3 &= y^3 - 3y^4 + 6y^5 - 10y^6 + 15y^7 - 21y^8 + \text{etc.}\\
x^4 &= y^4 - 4y^5 + 10y^6 - 20y^7 + 35y^8 - 56y^9 + \text{etc.}\\
&\text{etc.,}
\end{aligned}$$

hi valores substituti, serieque secundum potestates ipsius y disposita, dabunt

$$\begin{array}{lllll}
S = ay & -ay^2 & +ay^3 & -ay^4 & +ay^5 \text{ etc.}\\
 & +b & -2b & +3b & -4b\\
 & & +c & -3c & +6c\\
 & & & +d & -4d\\
 & & & & +e
\end{array}$$

3. Quoniam posuimus $x = \frac{y}{1+y}$ erit $y = \frac{x}{1-x}$; quo valore loco y substituto series proposita

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc}.$$

transmutabitur in hanc

$$S = a\frac{x}{(1-x)} + (b-a)\frac{x^2}{(1-x)^2} + (c-2b+a)\frac{x^3}{(1-x)^3} + \text{etc.},$$

in qua coefficiens secundi termini $b-a$ est differentia prima ipsius a ex serie a, b, c, d, e etc., quam supra per Δa exposuimus; coefficiens tertii termini $c-2b+a$ est differentia secunda $\Delta^2 a$; coefficiens quarti est differentia tertia $\Delta^3 a$ etc. Hinc differentiis ipsius a continuis, quae formantur ex serie a, b, c, d, e etc., adhibendis proposita series transmutabitur in hanc

$$S = \frac{x}{(1-x)}a + \frac{x^2}{(1-x)^2}\Delta a + \frac{x^3}{(1-x)^3}\Delta^2 a + \frac{x^4}{(1-x)^4}\Delta^3 a + \text{etc.},$$

cuius ergo seriei summa habebitur, si propositae summa fuerit cognita.

4. Si igitur series a, b, c, d etc. ita fuerit comparata, ut tandem differentias habeat constantes, quod evenit, si eius terminus generalis fuerit functio rationalis integra, tum series posterior $\frac{x}{(1-x)}a + \frac{x^2}{(1-x)^2}\Delta a + \text{etc.}$ tandem habebit terminos evanescentes sicque eius summa per expressionem finitam exhiberi poterit. Ita si seriei a, b, c, d etc. differentiae primae iam fuerint constantes, tum seriei huius $ax + bx^2 + cx^3 + dx^4 + \text{etc}$. summa erit

$$= \frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a.$$

At si illius seriei coefficientium differentiae secundae fiant constantes, tum ipsius seriei propositae summa erit

$$= \frac{x}{1-x}a + \frac{x^2}{(1-x)^2}\Delta a + \frac{x^3}{(1-x)^3}\Delta\Delta a$$

Unde summae huiusmodi serierum ex differentiis coefficientium facile invenientur.

I. *Quaeratur summa huius seriei*

$$1x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \text{etc}.$$
$$\text{Diff.I} \quad 2, \quad 2, \quad 2, \quad 2 \quad \text{etc.}$$

Cum ergo differentiae primae sint constantes, ob $a = 1$ et $\Delta a = 2$ erit seriei propositae summa

$$= \frac{x}{1-x} + \frac{2xx}{(1-x)^2} = \frac{x+x^2}{(1-x)^2}$$

II. *Quaeratur summa huius seriei*

$$1x + 4xx + 9x^3 + 16x^4 + 25x^5 + \text{etc.}$$
$$\begin{array}{llllll} \text{Diff. I} & 3, & 5, & 7, & 9 & \text{etc.} \\ \text{Diff. II} & & 2, & 2, & 2 & \text{etc.} \end{array}$$

Quia itaque est $a = 1,\ \varDelta a = 3,\ \varDelta^2 a = 2$, erit seriei propositae summa

$$= \frac{x}{1-x} + \frac{3xx}{(1-x)^2} + \frac{2x^3}{(1-x)^3} = \frac{x+xx}{(1-x)^3}.$$

III. *Quaeratur summa huius seriei*

$$S = 4x + 15x^2 + 40x^3 + 85x^4 + 156x^5 + 259x^6 + \text{ etc.}$$
$$\begin{array}{lllllll} \text{Diff. I} & 11, & 25, & 45, & 71, & 103 & \text{etc.} \\ \text{Diff. II} & & 14, & 20, & 26, & 32 & \text{etc.} \\ \text{Diff. III} & & & 6, & 6, & 6 & \text{etc.} \end{array}$$

Quia est $a = 4,\ \varDelta a = 11,\ \varDelta^2 a = 14,\ \varDelta^3 a = 6$, erit summa

$$S = \frac{4x}{1-x} + \frac{11xx}{(1-x)^2} + \frac{14x^3}{(1-x)^3} + \frac{6x^4}{(1-x)^4}$$

sive

$$S = \frac{4x - xx + 4x^3 - x^4}{(1-x)^4} = \frac{x(1+xx)(4-x)}{(1-x)^4}.$$

5. Quanquam hoc modo istarum serierum in infinitum progredientium summae inveniuntur, tamen ex iisdem principiis hae series quoque ad datum quemvis terminum summari possunt. Proposita enim sit haec series

$$S = ax + bx^2 + cx^3 + dx^4 + \ldots\ldots\ldots + ox^n$$

et quaeratur eius summa, si in infinitum progrediatur, quae erit

$$= \frac{x}{(1-x)} a + \frac{x^2}{(1-x)^2} \varDelta a + \frac{x^3}{(1-x)^3} \varDelta^2 a + \text{etc.}$$

Nunc considerentur eiusdem seriei termini post ultimum ox^n, sequentes, qui sint

$$px^{n+1} + qx^{n+2} + rx^{n+3} + sx^{n+4} + \text{etc.}$$

cuius seriei, si per x^n dividatur, summa ut ante inveniri poterit; quae rursus per x'', multiplicata erit

$$\tfrac{x^{n+1}}{1-x} p + \tfrac{x^{n+2}}{(1-x)^2} \Delta p + \tfrac{x^{n+3}}{(1-x)^3} \Delta^2 p + \text{etc.};$$

quae summa si a totius seriei in infinitum continuatae summa subtrahatur, remanebit summa portionis propositae quaesita

$$S = \tfrac{x}{1-x}\left(a - x^n p\right) + \tfrac{x^2}{(1-x)^2}\left(a\Delta - x^n \Delta p\right) + \tfrac{x^3}{(1-x)^3}\left(\Delta^2 a - x^n \Delta^2 p\right) + \text{etc.}$$

1. *Quaeratur summa huius seriei finitae*

$$S = 1x + 2x^2 + 3x^3 + 4x^4 + \ldots\ldots\ldots\ldots + nx^n .$$

Tam horum coefficientium quam terminum ultimum sequentium quaerantur differentiae

1,	2,	3,	4	etc	$n+1$,	$n+2$,	$n+3$	etc.
1	1	1	etc.		1,	1,	etc	

eritque $a = 1, \Delta a = 1, p = n+1, \Delta p = 1$, unde summa quaesita est

$$S = \tfrac{x}{1-x}\left(1 - (n+1)x^n\right) + \tfrac{x^2}{(1-x)^2}\left(1 - x^n\right)$$

seu

$$S = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2} .$$

II. Quaeratur summa huius seriei finitae

$$S = 1x + 4x^2 + 9x^3 + 16x^4 + \ldots\ldots\ldots\ldots + n^2 x^n$$

Investigentur primum differentiae hoc modo

1, 4, 9, 16 etc.	$(n+1)^2$, $(n+2)^2$, $(n+3)^2$ etc.
3, 5, 7 etc.	$2n+3$, $2n+5$ etc.
2, 2 etc.	2 etc.

quibus inventis erit summa quaesita

$$S = \frac{x}{1-x}\left(1-(n+1)^2 x^n\right) + \frac{x^2}{(1-x)^2}\left(3-(2n+3)x^n\right) + \frac{x^3}{(1-x)^3}\left(2-2x^n\right)$$

seu

$$S = \frac{x+xx-(n+1)^2 x^{n+1} + (2nn+2n-1)x^{n+2} - nnx^{n+3}}{(1-x)^3}.$$

6. Quodsi autem series proposita non eiusmodi habeat coefficientes, qui tandem ad differentias constantes deducantur, tum transmutatio hic exhibita nihil confert ad eius summam determinandam. Neque vero etiam eius ope summa proxime definiri poterit commodius, quam per ipsam seriei propositae additionem fieri licet. Si enim in serie $ax+bx^2+cx^3+dx^4+$ etc. fuerit $x<1$, quo solo casu summatio proprie sic dicta locum habere potest, erit $\frac{x}{1-x} > x$ ideoque nova series minus convergit quam proposita. Sin autem in serie proposita fuerit $x=1$, tum novae seriei omnes plane termini fiunt infiniti, quo ergo casu ista transmutatio nullius prorsus erit usus.

7. Consideremus autem seriem, in qua signa + et – alternatim se excipiant, quae ex praecedente deducetur ponendo x negativum. Si itaque fuerit

$$S = ax - bx^2 + cx^3 - dx^4 + ex^5 - \text{etc.},$$

cuius seriei negativa oritur, si in praecedente statuatur x negativum, sumantur ergo ut ante differentiae Δa, $\Delta^2 a$, $\Delta^3 a$ etc. ex serie coefficientium a, b, c, d, e etc., signis ad solas ipsius x potestates relatis, atque series proposita transformabitur in hanc

$$S = \frac{x}{1+x}a - \frac{x^2}{(1+x)^2}\Delta a + \frac{x^3}{(1+x)^3}\Delta^2 a - \frac{x^4}{(1+x)^4}\Delta^3 a + \text{etc.},$$

unde perspicitur aequationem propositam iisdem casibus summari posse quibus praecedens scilicet si series a, b, c, d etc. tandem ad differentias constantes deducatur.

8. Hoc autem casu ista transformatio commodam praebet approximationem ad valorem seriei propositae $ax - bx^2 + cx^3 - dx^4 + ex^5 - fx^6 +$ etc.; quantuscunque enim x fuerit

numerus, fractio $\frac{x}{1+x}$, secundum cuius potestates altera series progreditur, fit unitate minor; atque si sit $x=1$, erit $\frac{x}{1+x}=\frac{1}{2}$. Sin autem sit $x < 1$, puta $x=\frac{1}{n}$, fiet $\frac{x}{1+x}=\frac{1}{n+1}$ ideoque series per transformationem inventa semper magis convergit quam proposita.
Consideremus imprimis casum, quo $x=1$, qui ad series summandas ingens affert subsidium, sitque

$$S = a-b+c-d+e-f+\text{etc}.$$

ac denotentur differentiae primae, secundae et sequentes ipsius a, quas progressio a, b, c, d, e etc. praebet, per $\Delta a,\ \Delta^2 a,\ \Delta^3 a$ etc. ; quibus inventis erit

$$S=\tfrac{1}{2}a-\tfrac{1}{4}\Delta a+\tfrac{1}{8}\ \Delta^2 a-\tfrac{1}{16}\Delta^3 a+\text{etc.},$$

quae, nisi actu terminatur, summam vero proximam satis commode exhibet.

9. Usum igitur huius ultimae transmutationis, qua sumsimus $x=1$, in aliquot exemplis ostendamus ac primo quidem in eiusmodi, quibus vera summa finite exprimi potest. Tales sunt series divergentes, quibus numeri a, b, c, d etc. tandem ad differentias constantes deducant; quarum summae cum recepto huius vocis significatu proprie non exhiberi queant, vocem summae hic eo sensu, quem supra tribuimus, accipimus, ita ut denotet valorem expressionis finitae, ex cuius evolutione proposita series nascatur.

1. *Sit igitur proposita haec series LEIBNIZII*

$$S=1-1+1-1+1-1+\text{etc.};$$

in qua cum omnes termini sint aequales, fient omnes differentiae $=0$ ideoque ob $a=1$ erit $S=\frac{1}{2}$.

II. *Sit proposita ista series*

$$S=1-2+3-4+5-6+\text{etc.}$$
$$\text{Diff. I} = 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1\ \ \text{etc.}$$

Cum ergo sit $a=1,\ \Delta a=1$, erit $S=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$

III. *Sit proposita haec series*

$$S=1-3+5-7+9-\text{etc.}$$
$$\text{Diff. I} = 2,\ \ 2,\ \ 2,\ \ 2\ \ \text{etc.}$$

Ob $a=1$, et $\Delta a=1$, fit $S=\frac{1}{2}-\frac{2}{4}=0$.

IV. *Sit proposita haec series trigonalium numerorum*

$$S=1-3+6-10+15-21+\text{ etc.}$$
$$\text{Diff. I} = 2,\ 3,\ 4,\ 5,\ 6 \text{ etc.}$$
$$\text{Diff. II} = 1,\ 1,\ 1,\ 1 \text{ etc.}$$

Hic ergo ob $a=1$, $\Delta a=2$ et $\Delta\Delta a=1$ erit $S=\frac{1}{2}-\frac{2}{4}+\frac{1}{8}=\frac{1}{8}$.

V. *Sit proposita series quadratorum*

$$S=1-4+9-16+25-36+\text{ etc.}$$
$$\text{Diff. I} = 3,\ 5,\ 7,\ 9,\ 11 \text{ etc.}$$
$$\text{Diff. II} = 2,\ 2,\ 2,\ 2 \text{ etc.}$$

Ob $a=1$, $\Delta a=3$, $\Delta\Delta a=2$, erit $S=\frac{1}{2}-\frac{3}{4}+\frac{2}{8}=0$.

VI. *Sit proposita series biquadratorum*

$$S=1-16+81-256+625-1296+\text{ etc}$$
$$\text{Diff. I} = 15,\ 65,\ 175,\ 369,\ 671 \text{ etc.}$$
$$\text{Diff. II} = 50,\ 110,\ 194,\ 302 \text{ etc.}$$
$$\text{Diff. III} = 60,\ 84,\ 108 \text{ etc.}$$
$$\text{Diff. IV} = 24,\ 24 \text{ etc.}$$

Erit ergo $S=\frac{1}{2}-\frac{15}{4}+\frac{50}{8}-\frac{60}{16}+\frac{24}{32}=0.$

10. Si series magis divergant, uti geometricae aliaeque similes, eae hoc modo statim in seriem magis convergentem transmutantur, quae, nisi adhuc satis convergat, eodem modo in aliam magis convergentem convertetur.

1. *Sit proposita haec series geometrica*

$$S = 1 - 2 + 4 - 8 + 16 - 32 + \text{ etc.}$$

$$\begin{array}{llllllll}
\text{Diff. I} & = & 1, & 2, & 4, & 8, & 16 & \text{etc.} \\
\text{Diff. II} & = & & 1, & 2, & 4, & 8 & \text{etc.} \\
\text{Diff. III} & = & & & 1, & 2, & 4 & \text{etc.}
\end{array}$$

Cum igitur in omnibus differentiis primus terminus sit $=1$, summa seriei exprimetur hoc modo

$$S = \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{8} - \tfrac{1}{16} + \tfrac{1}{32} - \tfrac{1}{64} + \text{etc.},$$

cuius summa est $= \frac{1}{3}$; oritur enim ex evolutione fractionis $\frac{1}{2+1}$, dum proposita oritur ex $\frac{1}{1+2}$.

II. *Sit proposita haec series recurrens*

$$S = 1 - 2 + 5 - 12 + 29 - 70 + 169 - \text{etc.}$$

$$\begin{array}{lllllllll}
\text{Diff. I} & = & 1, & 3, & 7, & 17, & 41, & 99 & \text{etc.} \\
\text{Diff. II} & = & & 2, & 4, & 10, & 24, & 58 & \text{etc.} \\
\text{Diff. III} & = & & & 2, & 6, & 14, & 34 & \text{etc.} \\
\text{Diff. IV} & = & & & & 4, & 8, & 20 & \text{etc.} \\
\text{Diff. V} & = & & & & & 4, & 12 & \text{etc.} \\
\text{Diff. VI} & = & & & & & & 8 & \text{etc.} \\
 & & & & & & & \text{etc.} &
\end{array}$$

Continuarum ergo differentiarum termini primi constituunt hanc progressionem geometricam geminatam 1, 1, 2, 2, 4, 4, 8, 8, 16, 16 etc., unde erit

$$S = \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{2}{8} - \tfrac{2}{16} + \tfrac{4}{32} - \tfrac{4}{64} + \tfrac{8}{128} - \text{ etc.};$$

cum igitur praeter primum terminum reliqui bini se continuo destruant, erit $S = \frac{1}{2}$. Oritur autem series proposita ex evolutione fractionis $\frac{1}{1+2-1} = \frac{1}{2}$, uti in expressione naturae serierum recurrentium ostendimus.

III. *Sit proposita series hypergeometrica*

$$S = 1 - 2 + 6 - 24 + 120 - 720 + 5040 - \text{ etc.},$$

cuius differentias continuas hoc modo commodius investigabimus:

	Diff. I	Diff. II	Diff. III	
1	1	3	11	
2	4	14	64	
6	18	78	426	
24	96	504	3216	
120	600	3720	27240	etc.
720	4320	30960	256320	
5040	35280	287280	2656080	
40320	322560	2943360		
362880	3265920			
3628800				

Quibus differentiis ulterius continuatis erit

$$S = \frac{1}{2} - \frac{1}{4} + \frac{3}{8} - \frac{11}{16} + \frac{53}{32} - \frac{309}{64} + \frac{2119}{128} - \frac{1668}{256} + \frac{148329}{512} + \frac{1468457}{1024}$$
$$+ \frac{16019531}{2048} - \frac{190899411}{4096} + \text{ etc.}$$

Colligantur duo termini initiales eritque $S = \frac{1}{4} + A$ existente

$$A = \frac{3}{8} - \frac{11}{16} + \frac{53}{32} - \frac{309}{64} + \frac{2119}{128} - \text{etc.}$$

Si nunc eodem modo differentiae capiantur, erit

$$A = \frac{3}{2^4} - \frac{5}{2^6} + \frac{21}{2^8} - \frac{99}{2^{10}} + \frac{615}{2^{12}} - \frac{4401}{2^{14}} + \frac{36585}{2^{16}} - \frac{342207}{2^{18}} + \frac{3565323}{2^{20}} - \frac{40866525}{2^{22}} + \text{etc.}$$

Colligantur duo termini initiales, quia convergunt, fietque $A = \frac{7}{2^6} + B$ existente $B = \frac{21}{2^8} - \frac{99}{2^{10}} + \text{etc.}$; cuius seriei diffferentlis denuo sumendis fiet

$$B = \frac{21}{2^9} - \frac{15}{2^{12}} + \frac{159}{2^{15}} - \frac{429}{2^{18}} + \frac{5241}{2^{21}} - \frac{26283}{2^{24}} + \frac{338835}{2^{27}} - \frac{2771097}{2^{30}} + \text{ etc.}$$

Colligantur quatuor termini initiales in unum et statuatur $B = \frac{153}{2^{12}} + \frac{843}{2^{18}} + C$ existente

$$C = \frac{5241}{2^{21}} - \frac{26283}{2^{24}} + \text{etc.}$$

fietque aliquot terminis actu colligendis proxime $C=\frac{15645}{2^{24}}-\frac{60417}{2^{30}}$. Ex his ergo tandem concludetur summa seriei propositae S =0,40082055, quae tamen vix ultra tres quatuorve figuras pro accurata haberi potest ob nimiam seriei divergentiam; est tamen certe iusto minor. Aliunde enim inveni hanc summam esse = 0,4036524077, ubi ne ultima quidem nota a vero aberrat.

11. Imprimis autem haec transmutatio ingentem affert utilitatem ad series iam quidem, sed lente convergentes in alias, quae multo promtius convergant, transmutandas. Quoniam vero termini sequentes minores sunt quam praecedentes, differentiae primae fiunt negativae; unde in sequentibus signorum ratio probe est habenda.

1. *Sit proposita haec series*

$$S=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\text{etc.}$$

$$\begin{aligned}
&\text{Diff. I} &&=-\tfrac{1}{2},-\tfrac{1}{2\cdot3},\ -\tfrac{1}{3\cdot4},-\tfrac{1}{4\cdot5},-\tfrac{1}{5\cdot6},\text{etc.}\\
&\text{Diff. II} &&=+\tfrac{1}{3},\ \tfrac{2}{2\cdot3\cdot4},\ \tfrac{2}{3\cdot4\cdot5},\ \tfrac{2}{4\cdot5\cdot6}\,\text{etc.}\\
&\text{Diff. III} &&=-\tfrac{1}{4},-\tfrac{2\cdot3}{2\cdot3\cdot4\cdot5},-\tfrac{2\cdot3}{3\cdot4\cdot5\cdot6}\,\text{etc.}\\
&\text{Diff. IV} &&=\ +\tfrac{1}{5}\ \text{etc.}\\
& &&\text{etc.}
\end{aligned}$$

Hinc ergo erit

$$S=\tfrac{1}{2}+\tfrac{1}{2.4}+\tfrac{1}{3.8}+\tfrac{1}{4.16}+\tfrac{1}{5.32}+\text{etc.}\ ;$$

utramque autem hanc seriem logarithmum hyperbolicum binarii exhibere iam in *Introductione* ostendimus.

II. *Sit proposita ista series pro circulo*

$$S=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{11}+\text{etc.}$$

$$\begin{aligned}
&\text{Diff. I} &&=-\tfrac{2}{1\cdot3},\ -\tfrac{2}{3\cdot5},\ -\tfrac{2}{5\cdot7},\ -\tfrac{2}{7\cdot9}\ \ -\tfrac{2}{9\cdot11}\ \text{etc.}\\
&\text{Diff. II} &&=+\tfrac{2\cdot4}{1\cdot3\cdot5},\ \tfrac{2\cdot4}{3\cdot5\cdot7},\ \tfrac{2\cdot4}{5\cdot7\cdot9},\ \tfrac{2\cdot4}{7\cdot9\cdot11}\ \text{etc.}\\
&\text{Diff. III} &&=-\tfrac{2\cdot4\cdot6}{1\cdot3\cdot5\cdot7},\ -\tfrac{2\cdot4\cdot6}{3\cdot5\cdot7\cdot9}\ \text{etc.}\\
& &&\text{etc.}
\end{aligned}$$

Hinc ergo concluditur fore summam seriei

$$S = \frac{1}{2} + \frac{1}{3\cdot 2} + \frac{1\cdot 2}{3\cdot 5\cdot 2} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7\cdot 2} + \text{etc}.$$

seu

$$2S = 1 + \frac{1}{3} + \frac{1\cdot 2}{3\cdot 5} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \frac{1\cdot 2\cdot 3\cdot 4}{3\cdot 5\cdot 7\cdot 9} + \text{etc}.$$

III. *Quaeratur valor huius seriei infinitae*

$$S = l2 - l3 + l4 - l5 + l6 - l7 + l8 - l9 + \text{etc}.$$

Quia differentiae ab initio nimis fiunt inaequales, colligantur actu termini usque ad $l10$ ex tabulis, quorum valor reperietur $= -0{,}3911005$, eritque

$$S = -0{,}3911005 + l10 - l11 + l12 - l13 + l14 - l15 + \text{etc}.$$

in infinitum. Desumantur hi logarithmi ex tabulis eorumque differentiae quaerantur hoc modo:

	Diff. I	Diff. II	Diff. III	Diff. IV	Diff. V
$l10$ = 1,0000000	+	–	+	–	+
$l11$ = 1,0413927	413927				
		36042			
$l12$ = 1,0791812	377885		5779		
		30263		1292	
$l13$ = 1,1139434	347622		4487		368
		25776		924	
$l14$ = 1,1461280	321846		3563		
		22213			
$l15$ = 1,1760913	299633				

Ex quibus reperitur

$$l10 - l11 + l12 - l13 + \text{etc}.$$
$$= \frac{1{,}0000000}{2} - \frac{413927}{4} - \frac{36042}{8} - \frac{5779}{16} - \frac{1292}{32} - \frac{368}{64} = 0{,}4891606.$$

Hinc valor seriei propositae erit

$$S = l2 - l3 + l4 - l5 + \text{etc.} = 0{,}0980601,$$

cui logarithmo respondet numerus 1,253315.

12. Quemadmodum has transmutationes obtinuimus ponendo in serie loco *x* hanc fractionem $\frac{y}{1\pm y}$, ita innumerabiles aliae transmutationes orientur, si loco *x* aliae functiones ipsius *y* substituantur. Sit iterum proposita ista series

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + fx^6 + \text{etc.}$$

atque ponatur $x = y(1-y)$, quo facto series orietur sequens

$$\begin{aligned} S = ay - ayy & \\ + byy - 2by^3 + by^4 & \\ + cy^3 - 3cy^4 + 3cy^5 - cy^6 & \\ + dy^4 - 4dy^5 + 6dy^6 & \\ + ey^5 - 5ey^6 & \\ + fy^6 \quad \text{etc.} & \end{aligned}$$

Quodsi ergo altera harum serierum fuerit summabilis, simul alterius summa erit cognita. Ita si statuatur

$$S = x + x^2 + x^3 + x^4 + x^5 + x^6 + \text{etc.} = \frac{x}{1-x}$$

erit

$$S = y - y^3 - y^4 + y^6 + y^7 - y^9 - y^{10} + \text{etc.}$$

Cuius ergo serei summa erit $= \frac{y-yy}{1-y+yy}$.

13. Si altera series alicubi abrumpatur, tum summa prioris absolute exhiberi poterit. Ponamus esse $a = 1$ et in serie inventa omnes terminos post primum evanescere, ut sit $S = y$, ideoque ob $x = y - yy$ erit summa prioris $= \frac{1}{2} - \sqrt{\left(\frac{1}{4} - x\right)}$. Fiet autem ob $a = 1$, ut sequitur,

$$\begin{aligned}
b &= \quad 1 = \tfrac{1}{4}\cdot 2^2 \\
c &= \quad 2 = \tfrac{1\cdot 3}{4\cdot 6}\cdot 2^4 \\
d &= \quad 5 = \tfrac{1\cdot 3\cdot 5}{4\cdot 6\cdot 8}\cdot 2^6 \\
e &= \ \ 14 = \tfrac{1\cdot 3\cdot 5\cdot 7}{4\cdot 6\cdot 8\cdot 10}\cdot 2^8 \\
f &= \ \ 42 = \tfrac{1\cdot 3\cdot 5\cdot 7\cdot 9}{4\cdot 6\cdot 8\cdot 10\cdot 12}\cdot 2^{10} \\
g &= 132 = \tfrac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{4\cdot 6\cdot 8\cdot 10\cdot 12\cdot 14}\cdot 2^{12} \\
&\quad \text{etc.,}
\end{aligned}$$

unde prior series abibit in hanc

$$S = \tfrac{1}{2} - \sqrt{\left(\tfrac{1}{4} - x\right)} = x + x^2 + 2x^3 + 5x^4 + 14x^5 + 42x^6 + 132x^7 + \text{etc.},$$

quae eadem invenitur, si quantitas surda $\sqrt{\left(\tfrac{1}{4} - x\right)}$ in seriem evolvatur atque ab $\tfrac{1}{2}$ subtrahatur.

14. Statuamus, quo transmutatio latius pateat, $x = y(1+ny)^r$ atque series proposita

$$S = ax + bx^2 + cx^3 + dx^4 + ex^5 + \text{etc}.$$

transmutabitur in sequentem

$$\begin{array}{llllll}
S = ay & + \frac{v}{1}nay^2 & + \frac{v(v-1)}{1\cdot 2}n^2ay^3 & + \frac{v(v-1)(v-2)}{1\cdot 2\cdot 3}n^3ay^4 & + \frac{v(v-1)(v-2)(v-3)}{1\cdot 2\cdot 3\cdot 4}n^4ay^5 \\
 & + \ by^2 & + \frac{2v}{1}nby^3 & + \frac{2v(2v-1)}{1\cdot 2}n^2by^4 & + \frac{2v(2v-1)(2v-2)}{1\cdot 2\cdot 3}n^3ay^5 \\
 & & + \ cy^3 & + \frac{3v}{1}ncy^4 & + \frac{3v(3v-1)}{1\cdot 2}n^2by^5 \\
 & & & + \ dy^4 & + \frac{4v}{1}ndy^5 \\
 & & & & + \ ey^5 \\
 & & \text{etc.} & &
\end{array}$$

Si ergo illius seriei summa merit cognita, et huius simul summa habebitur ac vicissim. Quoniam vero n et v pro lubitu accipi possunt, hinc ex una serie summabili innumerae aliae summabiles inveniri possunt.

15. Possunt etiam eiusmodi transmutationes fieri, ut seriei inventae summa fiat irrationalis, hoc modo.

Sit proposita ista series

$$S = ax + bx^3 + cx^5 + dx^7 + ex^9 + fx^{11} + \text{etc. };$$

erit

$$Sx = ax^2 + bx^4 + cx^6 + dx^7 + ex^{10} + fx^{12} + \text{etc.}$$

iam statuatur

$$x = \frac{y}{\sqrt{(1-nyy)}} \;;$$

erit $xx = \frac{y^2}{1-nyy}$ atque series proposita transmutabitur in hanc

$$\begin{aligned} \frac{Sy}{\sqrt{(1-nyy)}} = ay^2 + nay^4 + n^2ay^6 + n^3ay^8 + n^4ay^{10} + \text{etc.} \\ + by^4 + 2nby^6 + 3n^2by^8 + 4n^3by^{10} + \text{etc.} \\ + cy^6 + 3ncy^8 + 6n^2cy^{10} + \text{etc.} \\ + dy^8 + 4ndy^{10} + \text{etc.} \\ + ey^{10} + \text{etc.} \\ \text{etc.} \end{aligned}$$

Si igitur summa S ex priori serie fuerit cognita, habebitur simul summa sequentis seriei

$$\frac{S}{\sqrt{(1-nyy)}} = ay + (na+b)y^3 + \left(n^2a + 2nb + c\right)y^5 + \left(n^3a + 3n^2b + 3nc + d\right)y^7 + \text{etc.}$$

16. Si sumatur $n = -1$, erunt coefficientes huius seriei differentiae continuae ipsius *a* ex serie *a, b, c, d* etc.; sin autem signa in serie proposita alternentur, tum posito $n = 1$ coefficientes erunt istae differentiae. Denotent ergo $\Delta a,\ \Delta^2 a,\ \Delta^3 a,\ \Delta^4 a$ etc. differentias primas, secundas, tertias etc. ipsius *a* ex serie numerorum *a, b, c, d, e, f* etc. Ac si fuerit

$$S = ax + bx^3 + cx^5 + dx^7 + ex^9 + \text{etc.}$$

posito $x = \frac{y}{\sqrt{(1+yy)}}$ erit

$$\frac{S}{\sqrt{(1+yy)}} = ay + \Delta a \cdot y^3 + \Delta a^2 \cdot y^5 + \Delta a^3 \cdot y^7 + \text{etc.}$$

Sin autem fuerit

$$S = ax - bx^3 + cx^5 - dx^7 + ex^9 - \text{etc.}$$

ponaturque $x = \frac{y}{\sqrt{(1-yy)}}$, erit

$$\frac{S}{\sqrt{(1-yy)}} = ay - \Delta a \cdot y^3 + \Delta a^2 \cdot y^5 - \Delta a^3 \cdot y^7 + \text{etc.}$$

Quodsi ergo series *a*, *b*, *c*, *d*, *e* etc. tandem ad differentias constantes deducat, tum utraque series absolute summari poterit; quae summatio autem quoque ex superioribus sequitur.

17. Ponamus coefficientes *a, b, c, d* etc. constituere hanc seriem

$$1,\ \tfrac{1}{3},\ \tfrac{1}{5},\ \tfrac{1}{7},\ \tfrac{1}{9}\ \text{ etc.}$$

eritque, uti supra iam vidimus,

$$a = 1,\ \ \Delta a = -\tfrac{2}{3},\ \ \Delta a^2 = \tfrac{2\cdot 4}{3\cdot 5},\ \ \Delta a^3 = -\tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\ \text{ etc.},$$

unde sequentes duas series summabimus.

I. Sit $S = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \text{etc.}$; erit $S = \frac{1}{2} l \frac{1+x}{1-x}$. Posito iam $x = \frac{y}{\sqrt{(1+yy)}}$
fiet

$$S = \tfrac{1}{2} l \frac{\sqrt{(1+yy)}+y}{\sqrt{(1+yy)}-y} = l\left(\sqrt{(1+yy)} + y\right),$$

unde erit

$$\frac{l\left(\sqrt{(1+yy)}+y\right)}{\sqrt{(1+yy)}} = y - \tfrac{2}{3}y^3 + \tfrac{2\cdot 4}{3\cdot 5}y^5 - \tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}y^7 + \text{etc.}$$

II. Sit $S = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \text{etc.}$; erit $S = \text{Atang}\, x$. Posito iam $x = \frac{y}{\sqrt{(1-yy)}}$
fiet

$$S = \text{Atang}\frac{y}{\sqrt{(1-yy)}} = \text{Asin}\, y = \text{Acos}\sqrt{(1-yy)}.$$

Hanc ob rem obtinebitur ista summatio

$$\frac{\text{A}\sin y}{\sqrt{(1-yy)}} = y + \tfrac{2}{3}y^3 + \tfrac{2\cdot 4}{3\cdot 5}y^5 + \tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}y^7 + \text{etc}\ .$$

18. Possunt quoque loco x functiones transcendentes ipsius y substitui sicque summationes aliae inventu difficiliores erui; verumtamen ne series novae fiant nimis perplexae, eiusmodi functiones eligi debent, quarum potestates facile exhiberi queant, quales sunt quantitates exponentiales e^y. Proposita igitur hac serie

$$S = ax + bx^3 + cx^5 + dx^7 + ex^9 + \text{etc.}$$

ponatur $x = e^{ny}y$ denotante e numerum, cuius logarithmus hyperbolicus $= 1$;
erit $x^2 = e^{2ny}y^2$, $x^3 = e^{3ny}y^3$ etc. Generaliter vero est, uti constat,

$$e^z = 1 + z + \tfrac{z^2}{1\cdot 2} + \tfrac{z^3}{1\cdot 2\cdot 3} + \tfrac{z^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

Quare series proposita in hanc transmutabitur

$$\begin{aligned} S = ay + 1nay^2 + \tfrac{1}{2}n^2ay^3 + {} & \tfrac{1}{6}n^3ay^4 + \tfrac{1}{24}n^4ay^5 + \text{etc.} \\ + \; by^2 + \tfrac{2}{1}nby^3 + {} & \tfrac{4}{2}n^2by^4 + \tfrac{8}{6}n^3by^5 + \text{etc.} \\ + \; cy^3 + {} & \tfrac{3}{1}ncy^4 + \tfrac{9}{2}n^2cy^5 + \text{etc.} \\ + {} & dy^4 + \tfrac{4}{1}ndy^5 + \text{etc.} \\ & + \; ey^5 + \text{etc.} \\ & \text{etc.} \end{aligned}$$

I. Sit series proposita geometrica $S = x + x^2 + x^3 + x^4 + x^5 + \text{etc.}$; erit $S = \frac{x}{1-x}$.
Ponatur iam $n = -1$ ut sit $x = e^{-y}y$ et $S = \frac{e^{-y}y}{1-e^{-y}y} = \frac{y}{e^y - 1}$;
reperietur summa haec

$$\frac{y}{e^y-1} = y - \tfrac{1}{2}y^3 - \tfrac{1}{6}y^4 + \tfrac{5}{24}y^5 + \tfrac{19}{120}y^6 - \text{etc.},$$

cuius autem seriei lex non perspicitur.

II. Sint in altera serie omnes termini praeter primum = 0; erit

$$b = -na,\; c = \tfrac{3}{2}n^2a,\; d = -\tfrac{8}{3}n^3a,\; e = \tfrac{125}{24}n^4a,\; f = -\tfrac{54}{5}n^5a, \text{ etc.}$$

Cum ergo sit summa $S = ay$ et $x = ye^{ny}$, fiet

$$y = x - nx^2 + \frac{3}{2}n^2x^3 - \frac{8}{3}n^3x^4 + \frac{125}{24}n^4x^5 - \frac{54}{5}n^5x^6 + \text{ etc.}$$

Quoniam vero in his seriebus lex progressionis non est manifesta, summationes ex hac substitutione deductae parum habent utilitatis. Praecipue autem notari merentur transformationes ex substitutione $x = \frac{y}{1 \pm y}$ derivatae, quippe quae non solum eximias summationes, sed etiam idoneos modos adsummas serierum appropinquandi suppeditant. His ergo, quae sine calculi differentialis ope sunt expedita, praemissis ad ipsum huius calculi usum in doctrina serierum ostendendum progrediamur.