CHAPTER IX

CONCERNING DIFFERENTIAL EQUATIONS

281. In this chapter it is proposed to explain initially the differentiation of these functions of x, which are not defined explicitly but implicitly by an equation, from which a relation of the function y to x may be satisfied; with which done we may consider the nature of the differential equation in general and we show how they may arise, just as from finite equations. For since in the integral calculus the whole operation consists in the finding of finite equations of this kind in the integration of differential equations, which since they shall agree with the differentials, it is necessary that in this situation we scrutinize more carefully the nature and properties of differential equations which follow from the origin of these equations, and thus we may prepare the way to calculate the integral.

282. Therefore so that we may resolve this matter, let y be a function of this kind x, which may be defined by this quadratic equation

$$yy+Py+Q=0.$$

Therefore since this expression $yy+Py+Q$ shall be $=0$, what ever x may signify, it will be equal to nothing also, if there may be written $x+dx$ in place of x, in which case y will change into $y+dy$. But with this substitution made if from the resulting quantity there may be subtracted the former $yy+Py+Q$, there will remain the differential of this, which also will be completely $=0$. Hence it is apparent, if some expression were $=0$, also the differential of this shall become 0, and if any two expressions should be equal to each other, the differentials of these also shall be equal. Therefore since there shall be

$$yy+Py+Q=0,$$

there will also be

$$2ydy+Pdy+ydP+dQ=0;$$

because indeed P and Q are functions of x, the differentials of these will have a form of this kind

$$dP=pdx \text{ and } dQ=qdx;$$

from which there comes about

$$2ydy+Pdy+ypdx+qdx=0;$$

from which there arises

$$\frac{dy}{dx} = -\frac{yp+q}{2y+P}.$$

283. Therefore just as the finite equation $yy + Py + Q = 0$ sets out the relation between y and x, the differential equation expressed the relation or the ratio, which dy bears to dx. Now because there is $\frac{dy}{dx} = -\frac{yp+q}{2y+P}$, this ratio $dy : dx$ cannot be known, unless the function y itself shall be known; nor indeed can the matter be established otherwise ; for since from the twin finite equation y may obtain a value, each will have its own special differential, and differential of each may be found, as here even that value substituted in place of y in the expression $-\frac{yp+q}{2y+P}$. In a similar manner the function y may be defined by a cubic equation ; the value of the function $\frac{dy}{dx}$ will be three fold, evidently corresponding to the triple value of y. If in the proposed equation with finite y there may be considered four or more dimensions, it is necessary that $\frac{dy}{dx}$ shall be just as many meanings to choose from.

284. Yet meanwhile the function y itself can be eliminated from the equation, since there are two equation containing y, evidently the finite and the differential ; but then just as many differential dimensions of this dy may arise, as y may have had before, and thus this equation likewise may include all the different ratios of dy to dx. We may select the preceding example of the equation $yy + Py + Q = 0$, of which the differential is $2ydy + Pdy + ydP + dQ = 0$, from which there arises

$$y = -\frac{Pdy+dQ}{2dy+dP},$$

which value substituted in place of y in the former equation will give

$$(4Q - PP)dy^2 + (4Q - PP)dPdy + QdP^2 - PdPdQ + dQ^2 = 0,$$

the roots of which are

$$dy = -\frac{1}{2}dP \pm \frac{\frac{1}{2}PdP - dQ}{\sqrt{(PP - 4Q)}},$$

which are two differentials of the two values of y arising from the finite equation

$$y = -\frac{1}{2}P \pm \frac{1}{2}\sqrt{(PP - 4Q)}.$$

285. The value of *ddy* may be found from the value of *dy*, and again of d^3y, d^4y themselves etc. by repeated differentiation. But which since they shall not be determined unless a certain first differential of convenience shall be put in place, therefore we may put *dx* constant and to show this we may consider this example

$$y^3 + x^3 = 3axy\,,$$

from which by differentiation there arises

$$3yydy + 3xxdx = 3axdy + 3aydx$$

and hence

$$\frac{dy}{dx} = \frac{ay-xx}{yy-ax}\,;$$

the differentials may be taken anew on putting *dx* constant, and there will be found

$$\frac{ddy}{dx} = \frac{-ayydy-aaxdy+2xxydy-2xyydx+aaydx+axxdx}{(yy-ax)^2}\,;$$

in place of *dy* the value of this in the manner found $\frac{aydx-xxdx}{yy-ax}$, may be substituted and on division by *dx* there will be made

$$\frac{ddy}{dx^2} = \frac{(ay-xx)(2xxy-ayy-aax)}{(yy-ax)^3} + \frac{axx+aay-2xyy}{(yy-ax)^2}$$

or

$$\frac{ddy}{dx^2} = \frac{6axxyy-2x^4y-2xy^4-2a^3xy}{(yy-ax)^3} = -\frac{2a^3xy}{(yy-ax)^3}$$

since from the finite equation there shall be $2x^4y + 2xy^4 = 6axxyy$; and in this manner with the aid of the finite equation these values are able to be changed into innumerable forms.

286. But the first differential equation can be varied in an infinite number of ways, while it may be combined with the finite equation. Thus since in the preceding example the differential equation may be found

$$yydy + xxdx = axdy + aydx\,,$$

if that be multiplied by *y*, there will become

$$y^3dy + xxydx = axydy + ayydx\,;$$

in which if in place of y^3, the value of this $3axy-x^3$ may be substituted, there will arise this new equation

$$2axydy-x^3dy+xxydx=ayydx;$$

which multiplied by y anew, after the value of this was substituted in place of y^3, there will be given

$$2axy^2dy-x^3ydy+xxyydx=3aaxydx-ax^3dx.$$

Moreover generally, if P, Q, R may denote some functions of x and y, if the differential equation may be multiplied by P, there will be

$$Pyydy+Pxxdx=aPxdy+aPydx.$$

Then, since there shall be $x^3+y^3-3axy=0$, there will be also

$$\left(x^3+y^3-3axy\right)\left(Qdx+Rdy\right)=0;$$

which equations in turn added will give a general differential equation, arising from the proposed finite equation

$$\begin{aligned}&Pyydy-aPxdy+Rx^3dy+Ry^3dy-3aRxydy\\&+Pxxdx-aPydx+Qx^3dx+Qy^3dx-3a\ Qxydx=0.\end{aligned}$$

287. Now also boundless differential equations can be found through the same differentiation from the same finite equation, while those may be multiplied or divided by some quantity before being differentiated. Thus if P were some function of x and y, so that there shall be $dP=pdx+qdy$, if the finite equation be multiplied by P and then finally differentiated, an equation of the general differential will be obtained, which may adopt boundless diverse forms, as other and still other functions may be assumed for P. Then indeed at this point the multiplications will be increased indefinitely, if to this differential equation found there may be added the finite equation itself multiplied by a formula of this kind $Qdx+Rdy$, where it may be possible to assume some functions of x and y for Q and R. But though in all these equations a relation may hold between dy and dx, as the differential of the function y determined by a finite equation through x to dx, it is understood, they extend still much more generally, and the differential of y will be expressed by other finite equations to be determined; an account of which matter will be explained chiefly in the integral calculus.

288. But not only from the same finite equation can innumerable differential equations be deduced, but also indeed infinitely many finite equations can be shown which may lead to the same differential equations. Thus these two equations

$$yy = ax + ab \text{ and } yy = ax$$

are entirely separate, as long as some constant quantity is arranged in place of *b* in the former. Yet meanwhile these two equations differentiated give the differential equation

$$2ydy = adx;$$

why not also take all the equations contained in this form $yy = ax$, whichever the value of *a* that may be attributed to one differential equation in which *a* shall not be present. For that equation may be divided by *x*, so that there shall be $\frac{yy}{x} = a$, and this differentiated will give

$$2xdy - ydx = 0.$$

Also transcending and algebraic equations are able to produce the same differential equation, as happens in these equations

$$yy - ax = 0 \text{ and } yy - ax = bbe^{\frac{x}{a}};$$

if indeed each may be divided by $e^{\frac{x}{a}}$, so that these equations may be considered

$$e^{-\frac{x}{a}}(yy - ax) = 0 \text{ et } e^{-\frac{x}{a}}(yy - ax) = bb,$$

and from the differentiation of each the same differential equation arises

$$2ydy - adx - \frac{yydx}{a} + xdx = 0.$$

289. The account of this difference depends on this, because the differential of the constant quantity shall be $= 0$. But if therefore the finite equation may be reduced to the same form, so that some constant quantity only shall be present neither multiplied nor divided by variables, then by differentiation an equation may be elicited, in which that constant quantity shall not be present at all. In this manner any constant quantity, which may be present in the finite equation, can be removed by differentiation. Thus if the proposed equation were

$$x^3 + y^3 = 3axy$$

if that may be divided by *xy*, so that there may be had $\frac{x^3+y^3}{xy} = 3a$, this equation differentiated will give

$$2x^3ydx + 2xy^3dy - x^4dy - y^4dx = 0,$$

as the constant *a* will be present no further.

290. If several constant quantities must be removed, which are present in the finite equation, that may be done by two or more repeated differentiations and thus only differential equations of higher order will be obtained from these, being completely without the constants. Let this equation be proposed

$$yy = maa - nxx,$$

from which by differentiation the constants *maa* and *n* must be removed. Indeed the first may be removed by the first differentiation, from which there becomes

$$ydy + nxdx = 0;$$

hence again the equation may be formed $\frac{ydy}{xdx} + n = 0$, which on assuming *dx* constant by differentiation will give

$$xyddy + xdy^2 - ydxdy = 0 \;;$$

which even if it contains no constant, yet all the equations contained in this form $yy = maa - nxx$, whatever values may be attributed to the letters *m, n* and *aa*, will in this themselves be understood equally.

291. Not only true constant quantities, which are present in a finite equation, can be removed by differentiation, but also other variables, evidently these, of which the differential may be assumed constant, will be eliminated by differentiation. Indeed from the proposed equation between *x* and *y* the value *x* may be sought, so that there shall be $x = Y$ with *Y* denoting some function of *y*, and there will be $dx = dY$ and on taking *dx* constant on being differentiated $0 = ddY$. But if there were

$$xx + ax + b = Y,$$

there becomes on differentiating three times $0 = d^3Y$ and the equation

$$x^4 + axx + bx + c = Y$$

differentiated four times gives $0 = d^4Y$. But though in these equations only one variable may be considered to be present, which therefore may cease to be variable, yet because the differential *dx* has been assumed constant and the ratio of this ought to be had in the equation, actually it must be thought to enter into the equation. Hence it is not to be wondered at, if more often differential equations of the second or of higher orders occur, in which only a single variable may be considered to be present.

292. But is required to be noted especially that by differentiation irrational and transcending quantities can be removed from an equation. Because indeed concerning irrationals, because it is possible to eliminate irrationals by known reductions, with this done by differentiation an equation is obtained free from irrationality. Now this can be done conveniently on many occasions without that reduction, while by the comparison of differential equations with the finite formulas of irrationals, if only one shall be present, it is possible to be eliminated. But if two or more irrational parts are contained in the finite equation, then the differential equation of this may be differentiated again and thus as many differential equations of higher orders may be sought, as there are required for the elimination of the individual irrational parts. Also in this manner equally indefinite exponents and fractions can be removed. As if there should be

$$y^m = (aa - xx)^n,$$

after differentiation there will be had

$$my^{m-1} = -2n(aa - xx)^{n-1} xdx,$$

which divided by the finite equation gives

$$\frac{mdy}{y} = -\frac{2nxdx}{aa-xx},$$

in which no further indefinite exponent occurs. Hence therefore it is apparent the differential equation is able to arise free from all irrationality from the finite equation and so far from involving transcending functions.

293. But so that it may be understood, how transcending quantities may be eliminated by differentiation, we may begin from logarithms; since the differentials of which shall be algebraic, the matter may be resolved without difficulty. Indeed let there be

$$y = xlx;$$

there will be $\frac{y}{x} = lx$, from which by differentiation there is mad $\frac{xdy-ydx}{xx} = \frac{dx}{x}$ and thus

$$xdy - ydx = xdx\,.$$

If two logarithms shall be present, a twofold integration shall be needed ; for let there be

$$ylx = xly\ ;$$

there will be $\frac{ylx}{x} = ly$ and on differentiation $\frac{dxylx + ydx - ydxlx}{xx} = \frac{dy}{y}$, from which it is concluded to be

$$lx = \frac{xxdy - yydx}{yxdy - yydx}\,.$$

This equation again may be differentiated on putting *dx* constant and there will be produced

$$\frac{dx}{x} = \frac{xxddy + 2xdxdy - 2ydxdy}{yxdy - yydx} + \frac{(yydx - xxdy)(yxddy + xdy^2 - ydxdy)}{(yxdy - yydx)^2}$$

or

$$\frac{dx}{x} = \frac{y^3xdxddy - yyxxdxddy + 3yxxdxdy^2 - y^2xdxdy^2 + y^3dx^2dy - 2xyydx^2dy - x^3dy^3}{(yxdy - yydx)^2}$$

which reduced will give

$$y^3xdxddy - yyxxdxddy + 3yxxdxdy^2 - 2xyydxdy^2 + 3y^3dx^2dy - 2xyydx^2dy - x^3dy^3 - \frac{y^4dx^3}{x} = 0$$

or

$$yyxx(y - x)dxddy + 3yxdxdy(xxdy + yydx) - 2yyxxdxdy(dx + dy) = x^4dy^3 + y^4dx^3.$$

294. Exponential quantities are removed from an equation in the same way as logarithms by differentiation. If indeed it were proposed of this kind there would be

$$P = e^Q\ ,$$

where *P* and *Q* may denote some functions of *x* and *y*, that equation can be transformed into this logarithm $lP = Q$, the differential of which is $\frac{dP}{P} = dQ$ or $dP = PdQ$. Nor is it opposed, if the exponential quantities were more complicated; for then, if a single differentiation is not sufficient, the matter may be resolved in two or more

I. Let $y=\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$; the numerator and denominator of this fraction may be multiplied by e^{x} and there will be $y=\frac{e^{2x}+1}{e^{2x}-1}$; from which there is made $e^{2x}=\frac{y+1}{y-1}$ and $2x=l\frac{y+1}{y-1}$, the differential of which is

$$dx=-\frac{dy}{yy-1}=\frac{dy}{1-yy}$$

II. Let $y=l\frac{e^{x}+e^{-x}}{2}$; by the first differentiation it becomes $dy=\frac{\left(e^{x}-e^{-x}\right)dx}{e^{x}+e^{-x}}$ or $\frac{dy}{dx}=\frac{e^{2x}-1}{e^{2x}+1}$ and $e^{2x}=\frac{dy+dx}{dy-dx}$. Therefore $2x=l\frac{dy+dx}{dy-dx}$. Therefore on assuming *dx* constant there will be $dx=\frac{dxddy}{dx^{2}-dy^{2}}$ or

$$dx^{2}=ddy+dy^{2}.$$

295. In a similar manner transcending quantities depending on the arc of a circle may be removed with the aid of differentiation, as may be understood from these examples.

1. Let $y=a\text{Asin}\frac{x}{a}$; there will be

$$dy=\frac{adx}{\sqrt{(aa-xx)}}.$$

II. Let $y=a\cos\frac{y}{x}$; there will be $\frac{y}{a}=\cos\frac{y}{x}$ and $\frac{dy}{a}=\frac{-xdy+ydx}{xx}\sin\frac{y}{x}$. But since $\cos\frac{y}{x}=\frac{y}{a}$, there will be $\sin\frac{y}{x}=\frac{\sqrt{(aa-yy)}}{a}$, with which value substituted there will be had $\frac{dy}{a}=\frac{(ydx-xdy)\sqrt{(aa-yy)}}{axx}$ or

$$xxdy=\left(ydx-xdy\right)\sqrt{\left(aa-yy\right)}.$$

Ill. Let $y=m\sin x+n\cos x$; after the first differentiation there will be $dy=mdx\cos x-ndx\sin x$; which differentiated anew on putting *dx* constant will give $ddy=-mdx^{2}\sin x-ndx^{2}\cos x$; but this divided by the first gives $\frac{ddy}{y}=-dx^{2}$ or

$$ddy+ydx^{2}=0,$$

from which not only the sine and cosine but also the constants *m* and *n* have vanished.

IV. Let $y = \sin lx$; there will be $\mathrm{Asin}\, y = lx$, from which by differentiation there is made $\frac{dy}{\sqrt{(1-yy)}} = \frac{dx}{x}$; which taken with the square gives $xxdy^2 = dx^2 - yydx^2$, and this on putting dx constant gives differentiated further $2xxdyddy + 2xdxdy^2 = -2ydx^2dy$ or

$$xxddy + xdxdy + ydx^2 = 0 .$$

V. Let there be $y = ae^{mx} \sin nx$; there will be on differentiation

$$dy = mae^{mx} dx \sin nx + nae^{mx} dx \cos nx,$$

which divided by the proposed gives

$$\frac{dy}{y} = mdx + \frac{ndx \cos nx}{\sin nx} = mdx + ndx \cot nx .$$

Hence there will be

$$\mathrm{A} \cot \left(\frac{dy}{nydx} - \frac{m}{n} \right) = nx.$$

Which equation differentiated on putting dx constant gives [the editor of the O.O. edition has corrected the original expression here in Euler's work.]

$$ndx = \frac{ndxdy^2 - nydxddy}{m^2y^2dx^2 + n^2y^2dx^2 - 2mydxdy + dy^2}$$

or

$$\left(m^2 + n^2\right) y^2dx^2 - 2mydxdy = -yddy.$$

Therefore it is evident, even if in no transcending quantities may be present in the differential equation, yet that may be able to arise from the finite equation, which shall be affected by transcending quantities in some manner.

296. Therefore because differential equations either of the first or of higher orders, which contain two variables *x* and *y*, arise from finite equations, from these also a relation between these two variables is expressed. Evidently with any proposed differential equation including any two variables *x* and *y*, from that some relation between *x* and *y* is signified with certainty, from which *y* becomes a certain function of *x*. Hence the nature of the differential equation is evident, if in place of *y* it will be possible to assign that function of *x*, which is indicated by that equation or which shall be prepared thus, so that if those are substituted everywhere in place of *y*, each differential in place of *dy* and higher orders of this in place of *ddy*, d^3y etc. , an identical equation may result. But the integral calculus is about an investigation of functions of this kind, of which the object there aimed at, that with some proposed differential equation that function of *x*, to which the other variable *y* is equal may be defined, so that it returns the same as the finite equation that may be found, by which the relation between *x* and *y* may be contained.

297. If this equation may be proposed as an example

$$2ydy - adx - \frac{yydx}{a} + xdx = 0,$$

which we have come upon above (§ 288), there a relation of this kind is defined between *x* and *y*, which likewise may include this finite equation

$$yy - ax = bbe^{\frac{x}{a}}.$$

Therefore since hence there shall be $yy = ax + bbe^{\frac{x}{a}}$, it appears that $\sqrt{\left(ax + bbe^{\frac{x}{a}}\right)} = y$ is to be a function of *x*, to which the quantity of the variable *y* of the proposed differential equation shall be equal. For if in place of *yy* in the equation we may substitute this value $ax + bbe^{\frac{x}{a}}$ and in place of $2ydy$ the differential of this $adx + \frac{bb}{a}e^{\frac{x}{a}}dx$, an identical equation will arise

$$adx + \frac{bb}{a}e^{\frac{x}{a}}dx - adx - xdx - \frac{bb}{a}e^{\frac{x}{a}}dx + xdx = 0.$$

And thus it is apparent that each differential equation and equally a finite equation show a certain relation between the variables *x* and *y*, but which cannot be found without the aid of the integral calculus.

298. So that this may be understood easier, we may put to be known that function of *x*, which may agree with the quantity of *y* of any differential equation, either of the first or of higher order, and there shall be

$$dy = pdx,\ dp = qdx,\ dq = rdx \text{ etc.},$$

and if in the differential equation *dx* shall be assumed constant, there will be $ddy = qdx^2$, $d^3y = rdx^3$ etc.; which values will be substituted into the equation afterwards, on account of this all the homogeneous differential terms *dx* will vanish by division and there will arise a finite equation containing only the variables *x, y, p, q, r* etc. Therefore since the quantities *p, q, r* etc. shall depend on the nature of the function *y*, an equation will remain actually only between the two variables *x* and *y*; and thus in turn with every differential equation to be consistent with a certain relation to be determined between the variables x and y. On account of which, if in the solution of any problem a differential equation between x and y may be arrived at, through the same equation a relation is required to be considered between x and y, and [from this] if a finite equation might be reached.

299. Therefore in this manner any differential equation can be reduced to a finite form, so that in that not unless finite quantities may be included, moreover the differentials or infinitely small parts completely disappear. For since y shall be a certain function of x, if there may be put $dy = pdx,\ dp = qdx,\ dq = rdx$ etc., whichever differential were taken constant, the second and higher differentials may be expressed by powers of dx, which hence by division may be completely removed. As if this equation may be proposed

$$xyd^3y + xxdyddy + yydxddy - xydx^3 = 0,$$

in which dx is put constant, on making $dy = pdx,\ dp = qdx$ and $dq = rdx$ that will change into $xyr + xxpq + yyq - xy = 0$, evidently after the whole equation has been divided by dx^3. And this equation determines a finite relation between x and y.

300. Therefore all differential equations, of whatever order they shall be, from these substitutions

$$dy = pdx,\ dp = qdx,\ dq = rdx \text{ etc.}$$

are reduced to pure finite quantities. And if the differential equation were of the first order, thus so that the first differentials may enter that, by the same reduction besides the above variables y and x the above quantity p may be introduced. But if the differential equation were of the second order containing second differentials, in addition the quantity q, and if it were of the third order, in addition there may be introduced again the above quantity r and thus henceforth. Therefore because in this manner the differentials may be completely eliminated from the calculation, that ratio of the constant differential may cease completely nor be of a greater amount, even if the quantities q, r arise from second differentials, there will be a need to indicate, or some constant differential shall be assumed. For likewise it is the case, whether in the working some differential may be placed constant or not as it pleases.

301. Therefore if a differential of the second or higher order may be proposed, in which no first differential may be present to be assumed constant, it will be able to be investigated at once in this manner, whether that relation determined between the variables x and y may be satisfied or not. Because indeed no constant differential is assumed, it is left to our choice which differential we may wish to put constant, and hence it will only be required to be considered, whether with the different differentials put constant, the equation may show the same relation between x and y. Because if it may not happen, it is a certain sign the equation expresses no determined relation and thus cannot have a place in the solution of any problem. But the safest and likewise the easiest way of exploring this will be that itself, which we have treated above [§ 277] in a similar argument for differential expressions of higher orders, by discerning whether they may have a fixed meaning. [Recall there that one or other or neither differential may be assumed constant, and transformations are given to achieve this state.]

302. Therefore with a proposed differential equation of this kind of the second or higher order, in which no constant differential shall be put in place, the differential *dx* may be placed constant; then this equation, as we have shown above [§ 276] from the differential expressions, again may be reduced to a form of this kind, which supposes no constant differential, evidently there must be established

$$ddy - \frac{dyddx}{dx} \text{ in place of } ddy \text{ and } d^3y - \frac{3ddxddy}{dx} + \frac{3dyddx^2}{dx^2} - \frac{dydx^3}{dx} \text{ in place of } d^3y \text{ etc.}$$

With which done it may be considered, whether the equation resulting in this manner may agree with the proposed equation; because if it happens, the proposed equation will include the relation determined between x and y ; but if it should happen otherwise, the equation will be undefined nor will it express a definite relation between x and y, just as this has now been shown before.

303. Let there be, so that this may be explained more fully, this proposed equation, which may be present with no constant differential to be found,

$$Pddx + Qddy + Rdx^2 + Sdxdy + Tdy^2 = 0.$$

There is put *dx* constant and that will go over into this

$$Qddy + Rdx^2 + Sdxdy + Tdy^2 = 0\,.$$

Now again this consideration of the constant differential may be removed as in the previous prescribed manner, and there will be obtained

$$-\frac{Qdyddx}{dx} + Qddy + Rdx^2 + Sdxdy + Tdy^2 = 0\,;$$

which because it disagrees from the proposed only on account of the first term, it may be considered, whether there shall be $P = -\frac{Qdy}{dx}$. Because if it may be observed, the proposed equation will show a fixed relation between x and y , which will be found by the rules treated in the integral calculus, whatever first constant first differential may be accepted. But if it should happen that there cannot become $P = -\frac{Qdy}{dx}$, the proposed equation will be impossible.

304. Therefore unless this proposed equation

$$Pddx + Qddy + Rdx^2 + Sdxdy + Tdy^2 = 0.$$

shall be absurd, it is necessary that there shall be $Pdx + Qdy = 0$, which can come about in a twofold manner; either indeed actually there shall be $P = -\frac{Qdy}{dx}$ or the equation $Pdx + Qdy = 0$ identically, or there shall be $Pdx + Qdy = 0$ that differential equation of the first order, from which by differentiation the proposed equation has arisen ; because in the latter case the equation $Pdx + Qdy = 0$ will agree with the proposed and the same relation will be satisfied between *x* and *y* and thus this relation will be able to be elicited without the aid of the integral calculus. Since indeed there shall be $Pdx + Qdy = 0$, there will be on differentiation

$$Pddx + Qddy + dPdx + dQdy = 0,$$

which subtracted from the above equation will leave

$$Rdx^2 + Sdxdy + Tdy^2 = dPdx + dQdy .$$

But since there shall be $dy = -\frac{Pdx}{Q}$, the differentials will be able to cancel each other completely and there will arise a finite equation between *x* and *y* indicating the relation of these.

305. We may put in the solution of a problem with no constant differential assumed to be reached, for this equation

$$x^3ddx + xxyddy - yydx^2 + xxdy^2 + aadx^2 = 0 .$$

Therefore there will be, since it may be agreed that it does not contain an absurd equation,

$$x^3dx + xxydy = 0 \text{ or } xdx + ydy = 0,$$

the differential of which will be

$$x^3ddx + xxyddy + 3xxdx^2 + 2xydxdy + xxdy^2 = 0,$$

which subtracted from the proposed equation leaves

$$aadx^2 - yydx^2 - 3xxdx^2 - 2xydxdy = 0$$

or

$$aadx - yydx - 3xxdx - 2xydy = 0.$$

But since there shall be $xdx + ydy = 0$, there will be

$$2xydy = -2xxdx$$

and thus

$$aadx - yydx - xxdx = 0 \text{ or } yy + xx = aa;$$

which equation expresses the true relation between *x* and *y*, if indeed that agrees with the differential first found $xdx + ydy = 0$. Which proposed equation unless the agreement should reveal itself, has to be considered as impossible ; but since in this case it shall have a place, the finite equation $xx + yy = aa$ has been allowed to be elicited without integral calculus.

306. Indeed so that now we may offer an example of an impossible equation, this equation shall be proposed

$$yyddx - xxddy + ydx^2 - xdy^2 + adxdy = 0,$$

in which no constant differential shall be assumed. Therefore there shall be $yydx - xxdy = 0$ and thus on differentiation

$$yyddx - xxddy + 2ydxdy - 2xdxdy = 0,$$

which put equal to the proposed will give

$$ydx^2 - xdy^2 + adxdy = 2ydxdy - 2xdxdy.$$

Now since there shall be $dy = \frac{yydx}{xx}$, with the differentials removed there will be obtained

$$y - \frac{y^4}{x^3} + \frac{ayy}{xx} = \frac{2y^3}{xx} - \frac{2yy}{x}$$

or

$$x^3 - y^3 + axy = 2xyy - 2xxy;$$

which whether it may agree with the differential $yydx - xxdy = 0$ or not, will easily become evident on differentiation ; indeed there is produced

$$3xxdx - 3yydy + axdy + aydx = 2yydx + 4xydy - 2xxdy - 4xydx$$

or

$$\frac{dy}{dx} = \frac{3xx + ay - 2yy + 4xy}{3yy - ax + 4xy - 2xx};$$

but from that there is $\frac{dy}{dx} = \frac{yy}{xx}$ and therefore the equation becomes

$$3x^4 + 4x^3y + axxy = 3y^4 + 4xy^3 - axyy$$

or

$$axy = \frac{3y^4 + 4x^3y - 4x^3y - 3x^4}{x + y} = 3y^3 + xyy - xxy - 3x^3$$

Truly from the finite equation first found there is

$$axy = y^3 + 2xyy - 2xxy - x^3,$$

which subtracted from that equation leaves

$$0 = 2y^3 - xyy + xxy - 2x^3,$$

which is resolved into these parts

$$0 = y - x \text{ and } 2yy + yx + 2xx = 0.$$

Of which that part $y = x$ indeed can agree with the differential $dy = \frac{yydx}{xx}$, but truly disagrees with the first equation found; unless there may be put $a = 0$ or unless each of the variables *x* and *y* may be made constant, indeed in which case on account of $dx = 0$ and $dy = 0$ with all the differential equations satisfied, the proposed equation is unable to remain.

307. Now we may consider also differential equations involving three variables *x*, *y* and *z*, which will be either of first, or of second, or of higher order. A careful examination of the nature of which requires a finite equation to be noted including three variables to determine the relation, which each one may hold to the two remaining ; therefore such a function *z* of *x* and *y* is defined. Therefore just as a finite equation of this kind is resolved, if it may be found, such a function of *x* and *y* must be substituted in place of *z*, so that for the equation to be satisfied, thus also a differential equation involving three variables will be determined, such a function shall be one of the remaining [variables] ; and that it is understood that an equation of this kind shall be resolved, which will have indicated that function of the two variables x and y, which substituted in place of the third z satisfies the equation or returns that identical. Therefore the differential equation is resolved, either if a function of x and y of z may be defined showing the value, or a finite equation may be assigned, by which the same value of z owed may be expressed.

308. But just as every differential equation including only two variables will always express a relation determined between these, yet this does not always come about in differential equations of three variables. For there are given equations of this kind, in which plainly in no manner will it be possible to be satisfied, whatever function of these x and y may be substituted in place of z. As if this equation were proposed $zdy = ydx$, it is readily apparent that completely no function of x and y can be given, which substituted in place of z may return $zdy = ydx$; for the differentials dx and dy in no way may be removed. In a similar manner it may be apparent that no function of x and z can be given, which substituted in place of y may satisfy the same equation. For just as for y, there may be considered a function of x and z, in the differential of this for dy, dz is present, which, because it is not present in the equation [on the r.h.s.], cannot be removed. Hence on this account no finite equation can be given between x, y and z, which may be appropriate for the differential equation $zdy = ydx$.

309. Hence it is required that differential equations containing three variables be distributed into imaginary [i.e. in the sense that they are not actual equations, but meaningless] and real ones. But an equation of this kind will be imaginary or absurd, if it cannot be satisfied by a finite equation, of this kind was that equation $zdy = ydx$, as we have just considered. But an equation will be actual, for which a finite equation can be shown, which comes about, if one of the variables shall be made equal to a certain function of the two remaining. This equation is of which kind

$$zdy + ydz = xdz + zdx + xdy + ydx;$$

for this agrees with that finite equation $yz = xz + xy$ and there is made $z = \frac{xy}{y-x}$.
Therefore that distinction between real and not actual equations of this kind has to be observed most carefully, especially in the integral calculus, because it would be silly to wish to integrate a differential equation of this kind, this is sought requiring to satisfy a finite equation, which plainly may have none.

310. Therefore it is apparent in the first place that all differential equations of three variables, in which only differentials of two occur, are absurd and not actual equations. Indeed we may put in an equation, which may contain the variable z, only the differentials dx and dy to be present, but with the differential dz to be completely absent, and it will be evident that no function of x and y, which on substitution in place of z may produce and identified equation; for the differentials dx and dy in no manner will be removed. Therefore from these cases generally no finite equation is given satisfying the equation ; unless perhaps a relation of this kind may be able to be assigned between *x* and *y*, which, whatever *z* shall be, may be able to remain, as comes about in this equation

$$zdy - zdx = ydy - xdx,$$

which is satisfied by the equation $y = x$. But it is investigated easily, in which cases this may come about, by seeking the relation between *x* and *y*, in the first case if $z = 0$, and then also, whether that relation may be satisfied for some value of *z*.

311. Nor indeed only is an equation involving three variables absurd, if only two may contain differentials, but also if in these all three differentials occur, it can be of such a kind. So that we may set out which cases, we may put *P* and *Q* to be functions of *x* and *y* only, and to satisfy this equation $dz = Pdx + Qdy$; which if it shall not be absurd, *z* will be some function of *x* and *y*, the differential of which shall be sit $dz = pdx + qdy$, and there will be $P = p$ and $Q = q$. But we have shown above [§ 232] that $pdx + qdy$ cannot be the differential of any function of *x* and *y*, unless there shall be $\left(\frac{dp}{dy}\right) = \left(\frac{dq}{dx}\right)$ with $\left(\frac{dp}{dy}\right)$ denoting, as we have assumed before, the differential of *p*, on putting *y* alone to be variable, divided by *dy* and $\left(\frac{dq}{dx}\right)$ the differential of *q*, on putting *x* alone to be variable, divided by *dx*. On account of which $dz = Pdx + Qdy$ is unable to be real, unless there shall be $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$.

312. An account of this equation will be entirely similar

$$dZ = Pdx + Qdy,$$

if *Z* may denote some function of *z*, now *P* and *Q* shall be functions of *x* and *y* not including the third variable *z*. Indeed so that *Z* shall become equal to a function of *x* and *y*, it is necessary that there shall be $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$. Therefore from this criterion the differential equation proposed with each, which indeed may be contained in this general form, is able to be decided, whether it shall be an actual one or absurd. Thus it will be apparent that the equation $zdz = ydx + xdy$ is an actual equation ; for on account of $P = y$ and $Q = x$ there comes about

$$\left(\frac{dP}{dy}\right) = 1 = \left(\frac{dQ}{dx}\right) = 1.$$

Now this equation $azdz = yydx + xxdy$ is absurd; for there becomes

$$\left(\frac{dP}{dy}\right) = 2y \quad \text{and} \quad \left(\frac{dQ}{dx}\right) = 2x$$

which values are unequal.

313. But in order that we may examine the criterion extending the widest, let *P*, *Q* and *R* be some functions of *x*, *y* and *z;* and every differential equation of three variables, if indeed it shall be of the first order, will be satisfied by this form

$$Pdx + Qdy + Rdz = 0.$$

Therefore as often as this equation is real [or actual], *z* will be equal to a certain function of *x* and *y* and the differential of this thus will be of this form $dz = pdx + qdy$. Whereby if in the proposed equation that function of *x* and *y* may be substituted in place of *z* and $pdx + qdy$ in place of *dz*, it is necessary that the identical equation may be produced $0 = 0$. And since from the proposed equation there may be made

$$dz = -\frac{Pdx}{R} - \frac{Qdy}{R},$$

if in *P*, *Q* and *R* that value may be substituted in place of *z*, it is necessary that there becomes

$$p = -\frac{P}{R} \text{ and } q = -\frac{Q}{R}.$$

314. Now because there shall be $dz = pdx + qdy$, there will be $\left(\frac{dp}{dy}\right) = \left(\frac{dq}{dx}\right)$ as shown before. Therefore since by substituting the value in terms of *x* and *y* in place of *z* there shall be

$$p = -\frac{P}{R} \text{ and } q = -\frac{Q}{R},$$

there will be

$$\left(\frac{dp}{dy}\right) = \left(\frac{-RdP + PdR}{RRdy}\right) \text{ and } \left(\frac{dq}{dx}\right) = \left(\frac{-RdQ + QdR}{RRdx}\right)$$

and thus this equation will be satisfied on multiplying by *RR*

$$P\left(\frac{dR}{dy}\right) - R\left(\frac{dP}{dy}\right) = Q\left(\frac{dR}{dx}\right) - R\left(\frac{dQ}{dx}\right),$$

where the denominators *dy* and *dx* again indicate that quantity alone must be assumed variable in the differentials of the numerators, the differential of which constitutes the denominator. But these differentials *dP*, *dQ*, *dR* cannot be known before, as in these quantities *P*, *Q* and *R* the value owed were substituted in place of *z* ; but which since it shall be unknown, it will be required to proceed in the following manner.

315. Because P, Q and R are functions of x, y and z, we may put

$$dP = \alpha dx + \beta dy + \gamma dz$$
$$dQ = \delta dx + \varepsilon dy + \zeta dz\,,$$
$$dR = \eta dx + \theta dy + \iota dz,$$

where $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. denote these functions, which arise from the differentials. Now we may consider the value of this expressed in terms of x and y to be substituted in place of z everywhere, and in place of dz we may put the value $pdx + qdy$ and there will be made

$$dP = (\alpha + \gamma p)dx + (\beta + \gamma q)dy$$
$$dQ = (\delta + \zeta p)dx + (\varepsilon + \zeta q)dy$$
$$dR = (\eta + \iota p)dx + (\theta + \iota q)dy.$$

Therefore from these values there will be

$$\left(\frac{dR}{dy}\right) = \theta + \iota q, \quad \left(\frac{dR}{dx}\right) = \eta + \iota p$$
$$\left(\frac{dP}{dy}\right) = \beta + \gamma q, \quad \left(\frac{dQ}{dx}\right) = \delta + \zeta p.$$

316. Therefore since there is required for the reality of the equation [§ 314], so that there shall be

$$P\left(\frac{dR}{dy}\right) - R\left(\frac{dP}{dy}\right) = Q\left(\frac{dR}{dx}\right) - R\left(\frac{dQ}{dx}\right)$$

there will become, if the values found may be substituted

$$P(\theta + \iota q) - R(\beta + \gamma q) = Q(\eta + \iota p) - R(\delta + \zeta p).$$

But previously we have found to be

$$p = -\tfrac{P}{R} \text{ and } q = -\tfrac{Q}{R}$$

which values, since the differentials may come no further into the calculation, will be able to be used, even if the value in terms of x and y may not be substituted in place of z. Therefore there will be

$$P\theta - \tfrac{PQ\iota}{R} - R\beta + Q\gamma = Q\eta - \tfrac{PQ\iota}{R} - R\delta + P\zeta$$

or

$$0 = P(\zeta - \theta) + Q(\eta - \gamma) + R(\beta - \delta).$$

But because the quantities $\beta, \delta, \gamma, \eta, \zeta, \theta$ are found by differentiation, there will be required to be noted from the manner used above [note that these are partial derivatives in this notation]

$$0 = P\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) + Q\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + R\left(\frac{dP}{dy} - \frac{dQ}{dx}\right).$$

Which property unless it may have a place in an equation, the equation will not be real but will be imaginary and absurd.

317. Though we have elicited this rule from the consideration of the variable *z*, yet, since all the quantities enter equally, it is evident, all of the remaining may be produced by considering the same expression. Therefore with the proposed differential equation of the first order which may involve three variables, from whichever it is possible to decide at once, whether it shall be real or imaginary. For it may be compared with this general form

$$Pdx + Qdy + Rdz = 0$$

and the value may be sought of this formula

$$P\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) + Q\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + R\left(\frac{dP}{dy} - \frac{dQ}{dx}\right);$$

which if it shall be $= 0$, the equation is a real one ; but if it were not $= 0$, here there is a certain sign that the equation is imaginary or absurd.

318. A proposed equation by division also can be reduced to a form of this kind always

$$Pdx + Qdy + dz = 0;$$

since it changed into which form before if there were made $R = 1$, a simpler criterion may be expressed in this manner

$$P\left(\frac{dQ}{dz}\right) - Q\left(\frac{dP}{dz}\right) + \left(\frac{dP}{dy}\right) - \left(\frac{dQ}{dx}\right) = 0.$$

For as often as this expression actually is found to be zero, the proposed equation will be real as many times ; but if the opposite may arise, the equation will be imaginary. Indeed after these we have shown, it is certain; but from the start at this point it may be possible to doubt, whether an equation shall be a real one always, indeed as often as this criterion may indicate that. Since which may not be able to be demonstrated fully in this place, but in the integral calculus at last it will be possible to be confirmed by a demonstration, yet we may affirm that here nor thence moreover is danger to be feared, if with which for the present there might be some doubt concerning the truth of this.

319. Therefore from this criterion it is apparent, if in the equation

$$Pdx + Qdy + Rdz = 0$$

P were a function of *x*, *Q* a function of *y* and *R* a function of *z* only, the equation will always be a real one. For there shall become

$$\left(\frac{dP}{dy}\right) = 0,\ \left(\frac{dP}{dz}\right) = 0,\ \left(\frac{dQ}{dz}\right) = 0,\ \left(\frac{dQ}{dx}\right) = 0,\ \left(\frac{dR}{dx}\right) = 0 \text{ and } \left(\frac{dR}{dy}\right) = 0$$

and thus the whole expression of the criterion will vanish at once.

320. If there were as before *P* a function of *x* and *Q* a function of *y* only, but *R* some function of *x*, *y* and *z*, the equation will be of an actual formula, if there were

$$P\left(\frac{dR}{dy}\right) = Q\left(\frac{dR}{dx}\right) \text{ or } \left(\frac{dR}{dx}\right):\left(\frac{dR}{dy}\right) = P:Q.$$

Thus if this equation were proposed

$$\frac{2dx}{x} + \frac{3dy}{y} + \frac{x^2 y^3 dz}{z^6} = 0,$$

because here there is

$$P = \frac{2}{x},\quad Q = \frac{3}{y},\quad R = \frac{x^2 y^3}{z^6},$$

hence

$$\left(\frac{dR}{dx}\right) = \frac{2xy^3}{z^6} \text{ and } \left(\frac{dR}{dy}\right) = \frac{3xxyy}{z^6},$$

there will be

$$P\left(\frac{dR}{dy}\right) = Q\left(\frac{dR}{dx}\right) = \frac{6xyy}{z^6}$$

and thus the proposed equation will be real.

321. If P and Q were functions of x and y, but R a function of z only, on account of

$$\left(\frac{dP}{dz}\right)=0,\quad \left(\frac{dQ}{dz}\right)=0,\quad \left(\frac{dR}{dx}\right)=0,\quad \text{and}\quad \left(\frac{dR}{dy}\right)=0$$

the equation will be real, if there were $\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$. Truly this same condition is required, if $Pdx+Qdy$ should be a determined differential or arising from the differentiation of a certain finite function of x and y themselves. And this will be returned, which we have now observed above (§312), the equation $dZ=Pdx+Qdy$, if Z shall be a function of z only, but the functions P and Q of x and y themselves cannot be real, unless there shall be $\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$. But both these cases completely agree amongst themselves ; for in place of *Rdz*, if R shall be a function of z only, there can be put dZ with Z some function of z present.

322. So that we may illustrate by an example this criterion found, we may consider this equation

$$\left(6xy^2z-5yz^3\right)dx+\left(5x^2yz-4xz^3\right)dy+\left(4x^2y^2-6xyz^2\right)dz=0.$$

which since compared with the general form there gives

$$P=6xy^2z-5yz^3,\quad \left(\frac{dP}{dy}\right)=12xyz-5z^3,\quad \left(\frac{dP}{dz}\right)=6xy^2-15yz^2,$$
$$Q=5x^2yz-4xz^3,\quad \left(\frac{dQ}{dx}\right)=10xyz-\ 4z^3,\quad \left(\frac{dQ}{dz}\right)=5x^2y-12xzz,$$
$$R=4x^2y^2-6xyz^2,\quad \left(\frac{dR}{dx}\right)=8xy^2-6yz^2,\quad \left(\frac{dR}{dy}\right)=8x^2y-6xz^2.$$

With these equations found the equation containing the judgement will be this
[from $P\left(\frac{dQ}{dz}-\frac{dR}{dy}\right)+Q\left(\frac{dR}{dx}-\frac{dP}{dz}\right)+R\left(\frac{dP}{dy}-\frac{dQ}{dx}\right)$:]

$$\begin{aligned}&\left(6xy^2z-5yz^3\right)\left(-3xxy-6xzz\right)\\+&\left(5x^2yz-4xz^3\right)\left(2xyy+9yzz\right)\\+&\left(4x^2y^2-6xyz^2\right)\left(2xyz-z^3\right)=0.\end{aligned}$$

But this equation if it may be expanded out, all the terms actually cancel each other and there is made $0=0$, which indicates the proposed equation to be real.

323. But when the expression elicited from the criterion in this manner has not vanished, then that has indicated that the proposed equation is not real. Because truly with this agreed on from the criterion a finite equation is found, that, if indeed it may agree with a differential equation, likewise it will indicate a relation which the variables maintain between themselves. And in this manner these cases are explained, of which we remember above (§310). For let this equation be proposed

$$(z-x)dx+(y-z)dy=0;$$

there becomes

$$P=z-x,\ Q=y-z \text{ et } R=0,$$

again

$$\left(\tfrac{dP}{dz}\right)=1 \text{ and } \left(\tfrac{dQ}{dz}\right)=-1.$$

The equation showing the judgement shall become

$$P\left(\tfrac{dQ}{dz}\right)=Q\left(\tfrac{dP}{dz}\right)$$

or

$$z-x=z-y;$$

from which $y=x$.
Therefore because here in this case it comes about, that the equation $y=x$ likewise satisfies the differential equation, it is required to be said that the proposed equation signifies nothing else, except to be $y=x$.

324. Therefore with the proposed differential equation containing three variables

$$Pdx+Qdy+Rdz=0$$

the three following cases will be required to be considered, to which this equation may be reduced

$$P\left(\tfrac{dQ}{dz}-\tfrac{dR}{dy}\right)+Q\left(\tfrac{dR}{dx}-\tfrac{dP}{dz}\right)+R\left(\tfrac{dP}{dy}-\tfrac{dQ}{dx}\right)=0.$$

The first is, if this expression actually becomes $=0$, and then the proposed equation will be real. But if this finite equation shall not be identical, then it must be considered, whether that will satisfy the proposed equation ; which if it happens, the finite equation will be established, which is the second case. But the third case may have to be considered, if the finite equation is unable to remain with the proposed differential equation, and then the proposed equation will be imaginary ; nor indeed will it be able to show any finite equation which it satisfies.

325. The first and third cases are self evident, but the second, even if it rarely happens, yet merits to be noted properly ; and since we have shown an example of this now above in an equation, which contains only two differentials, we may also bring forwards an equation, in which all three differentials may be present,

$$(z-y)dx + xdy + (y-z)dz = 0.$$

Therefore there will be

$$P = z-y,\quad \left(\frac{dQ}{dz}\right) = 0,\quad \left(\frac{dR}{dy}\right) = 1,$$
$$Q = x,\quad \left(\frac{dR}{dx}\right) = 0,\quad \left(\frac{dP}{dz}\right) = 1,$$
$$R = y-z,\quad \left(\frac{dP}{dy}\right) = -1,\quad \left(\frac{dQ}{dx}\right) = 1,$$

from which equation the equation will arise containing the finite equation

$$z-x-y=0 \text{ or } z=x+y;$$

here the value may be substituted for z in the differential equation and there becomes

$$xdx + xdy - x(dx+dy) = 0;$$

which equation since it shall be identical to zero, it follows that the differential equation can have no other significance apart from $z = x + y$.

326. Because we have said that all differential equations of the first order, in which three variables are present, are to be contained in this form

$$Pdx + Qdy + Rdz = 0,$$

here a doubt may arise about these equations, in which the first differentials constitute two or more dimensions, this equation is of this kind

$$Pdx^2 + Qdy^2 + Rdz^2 = 2Sdxdy + 2Tdxdz + 2Vdydz.$$

The truth concerning equations of this kind to be noted is that in no manner are they able to be real, unless they may have divisors of the prior form, which therefore constitute simple equations. For since there may arise from this equation

$$dz = \frac{Tdx + Vdy \pm \sqrt{\left(dx^2\left(T^2 - PR\right) + 2dxdy\left(TV + RS\right) + dy^2\left(V^2 - QR\right)\right)}}{R},$$

it is readily apparent z cannot become equal to some function of x and y or dz to become equal to an expression of this kind $pdx + qdy$, unless an irrational quantities prevails as rational, which may come about, if there should be

$$\left(T^2 - PR\right)\left(V^2 - QR\right) = \left(TV + RS\right)^2$$

or

$$R = \frac{PVV + 2STV + QTT}{PQ - SS}.$$

Therefore unless this finite equation satisfies the proposed equation, this equation will be imaginary.

327. It may remain, that in this chapter we consider carefully differential equations of higher order also, which include three variables, and define the cases into which these emerge either real or imaginary; truly because the criteria become exceedingly intricate, here we pass over this labour, especially since they may follow from the same sources, which we have uncovered here. Moreover if in the integral calculus there shall be a need for these criteria, then these will be elicited easily. For the same reason also we do not contemplate equations here which include more variables , since hardly ever do they occur and, if at some time they do occur, they may be able to be examined from the principles treated here without bother. Whereby from these expositions of the Principles of Differential Calculus here we impose an end to be progressing soon to the conspicuous uses requiring to be shown, which that calculus offers to analysis itself as well as higher geometry.

CAPUT IX
DE AEQUATIONIBUS DIFFERENTIALIBUS

281. In hoc capite imprimis est propositum earum functionum ipsius *x*, quae non explicite, sed implicite per aequationem, qua relatio functionis istius *y* ad *x* continetur, definiuntur, differentiationem explicare; quo facto naturam aequationum differentialium in genere perpendemus et, quemadmodum ex aequationibus finitis oriantur, ostendemus. Cum enim in calculo integrali summum negotium consistat in integratione aequationum differentialium seu in inventione eiusmodi aequationum finitarum, quae cum differentialibus conveniant, necesse est, ut hoc loco indolem ac proprietates aequationum differentialium, quae ex earum origine sequuntur, diligentius scrutemur sicque viam ad calculum integralem praeparemus.

282. Ut igitur hoc negotium absolvamus, sit *y* functio eiusmodi ipsius *x*, quae per hanc aequationem quadratam

$$yy+Py+Q=0$$

definiatur. Cum ergo haec expressio $yy+Py+Q$ sit $=0$, quicquid *x* significet, nihilo quoque aequalis erit, si loco *x* scribatur $x+dx$, quo casu *y* abit in $y+dy$. Facta autem hac substitutione si a quantitate resultante subtrahatur prior $yy+Py+Q$, remanebit eius differentiale, quod propterea quoque erit $=0$. Hinc patet, si expressio quaecunque fuerit $=0$, eius etiam differentiale fore aequale 0, atque si duae quaecunque expressiones inter se fuerint aequales, earum quoque differentialia fore aequalia. Cum igitur sit $yy+Py+Q=0$, erit quoque

$$2ydy+Pdy+ydP+dQ=0;$$

quia vero *P* et *Q* sunt functiones ipsius *x*, earum differentialia huiusmodi formam habebunt

$$dP=pdx \quad \text{et} \quad dQ=qdx;$$

unde fiet

$$2ydy+Pdy+ypdx+qdx=0;$$

ex qua oritur

$$\frac{dy}{dx}=-\frac{yp+q}{2y+P}.$$

283. Quemadmodum ergo aequatio finita $yy+Py+Q=0$ exponit relationem inter *y* et *x*, ita aequatio differentialis exprimit relationem seu rationem, quam *dy* tenet ad *dx*. Quoniam vero est $\frac{dy}{dx} = -\frac{yp+q}{2y+P}$, haec ratio $dy:dx$ cognosci non potest, nisi ipsa functio *y* sit cognita; neque vero res aliter se habere potest; cum enim ex aequatione finita *y* geminum obtineat valorem, uterque suum peculiare habebit differentiale et utriusque differentiale reperietur, prouti hic vel ille valor in expressione $-\frac{yp+q}{2y+P}$; loco *y* substituatur. Simili modo functio *y* per aequationem cubicam definiatur; valor functionis $\frac{dy}{dx}$ erit triplex, triplici scilicet ipsius *y* valori respondens. Si in aequatione proposita finita *y* quatuor pluresve habeat dimensiones, necesse est, ut $\frac{dy}{dx}$ totidem significationes sortiatur.

284. Interim tamen ipsa functio *y* ex aequatione eliminari poterit, cum duae habeantur aequationes *y* continentes, finita scilicet et differentialis; tum autem eius differentiale *dy* totidem dimensiones assurget, quot ante habuerat *y* sicque ista aequatio omnes diversas rationes ipsius *dy* ad *dx* simul complectetur. Sumamus praecedens exemplum aequationis $yy + Py + Q = 0$, cuius differentialis est $2ydy + Pdy + ydP + dQ = 0$, ex qua fit

$$y = -\frac{Pdy+dQ}{2dy+dP},$$

qui valor loco *y* in priori aequatione substitutus dabit

$$(4Q - PP)dy^2 + (4Q - PP)dPdy + QdP^2 - PdPdQ + dQ^2 = 0,$$

cuius radices sunt

$$dy = -\frac{1}{2}dP \pm \frac{\frac{1}{2}PdP - dQ}{\sqrt{(PP - 4Q)}},$$

quae sunt bina differentialia binorum ipsius *y* valorum ex aequatione finita

$$y = -\frac{1}{2}P \pm \frac{1}{2}\sqrt{(PP - 4Q)}.$$

285. Invento valore ipsius *dy* per repetitam differentiationem reperietur valor ipsius *ddy* porroque ipsorum d^3y, d^4y etc. Qui autem cum determinati non sint, nisi aliquod differentiale primum constans statuatur, ponamus commoditatis ergo *dx* constans atque ad hoc ostendendum sumamus hoc exemplum

$$y^3 + x^3 = 3axy$$

unde per differentiationem oritur

$$3yydy + 3xxdx = 3axdy + 3aydx$$

hincque

$$\frac{dy}{dx}=\frac{ay-xx}{yy-ax};$$

sumantur denuo differentialia posito dx constante atque invenietur

$$\frac{ddy}{dx}=\frac{-ayydy-aaxdy+2xxydy-2xyydx+aaydx+axxdx}{(yy-ax)^2}$$

substituatur loco dy eius valor modo inventus $\frac{aydx-xxdx}{yy-ax}$ atque divisione per dx facta habebitur

$$\frac{ddy}{dx^2}=\frac{(ay-xx)(2xxy-ayy-aax)}{(yy-ax)^3}+\frac{axx+aay-2xyy}{(yy-ax)^2}$$

seu

$$\frac{ddy}{dx^2}=\frac{6axxyy-2x^4y-2xy^4-2a^3xy}{(yy-ax)^3}=-\frac{2a^3xy}{(yy-ax)^3}$$

cum ex aequatione finita sit $2x^4y+2xy^4=6axxyy$*;* hocque modo ope aequationis finitae hi valores in innumeras formas transmutari possunt.

286. Aequatio etiam differentialis prima infinitis modis potest variari, dum cum aequatione finita permiscetur. Sic cum exemplo praecedente inventa esset aequatio differentialis

$$yydy+xxdx=axdy+aydx,$$

si ea multiplicetur per y, orietur

$$y^3dy+xxydx=axydy+ayydx;$$

in qua si loco y^3 substituatur eius valor $3axy-x^3$, orietur haec aequatio nova

$$2axydy-x^3dy+xxydx=ayydx;$$

quae denuo per y multiplicata, postquam loco y^3 eius valor fuerit substitutus, praebebit

$$2axy^2dy-x^3ydy+xxyydx=3aaxydx-ax^3dx.$$

Generaliter autem, si P, Q, R denotent functiones quascunque ipsarum x et y, si aequatio differentialis multiplicetur per P, erit

$$Pyydy + Pxxdx = aPxdy + aPydx.$$

Tum, cum sit $x^3 + y^3 - 3axy = 0$, erit quoque

$$\left(x^3 + y^3 - 3axy\right)\left(Qdx + Rdy\right) = 0;$$

quae aequationes invicem additae dabunt aequationem differentialem generalem ex proposita aequatione finita natam

$$\begin{aligned} &Pyydy - aPxdy + Rx^3dy + Ry^3dy - 3aRxydy \\ &+Pxxdx - aPydx + Qx^3dx + Qy^3dx - 3a\ Qxydx = 0. \end{aligned}$$

287. Possunt vero etiam per ipsam differentiationem infinitae aequationes differentiales ex eadem aequatione finita inveniri, dum ea, antequam differentietur, per quantitatem quamcunque aut multiplicatur aut dividitur. Sic si P fuerit functio quaecunque ipsarum x et y, ut sit $dP = pdx + qdy$, si aequatio finita per P multiplicetur atque tum demum differentietur, obtinebitur aequatio differentialis generalis, quae infinitas formas diversas induet, prouti pro P aliae atque aliae functiones assumuntur. Tum vero multiplicitas adhuc in infinitum augebitur, si ad hanc aequationem differentialem inventam addatur ipsa aequatio finita per huiusmodi formulam $Qdx + Rdy$ multiplicata, ubi pro Q et R functiones quascunque ipsarum x et y assumere licet. Quanquam autem in his omnibus aequationibus relatio inter dy et dx, quam differentiale functionis y aequatione finita per x determinatae ad dx tenet, comprehenditur, tamen plerumque multo latius patent et differentiale ipsius y per alias aequationes finitas determinati exprimunt; cuius rei ratio in calculo integrali potissimum explicabitur.

288. Non solum autem ex eadem aequatione finita innumerabiles aequationes differentiales deduci possunt, sed etiam plures, imo infinitae exhiberi possunt aequationes finitae, quae ad easdem aequationes differentiales deducantur. Sic hae duae aequationes

$$yy = ax + ab \text{ et } yy = ax$$

omnino sunt diversae, dum in priori quaecunque quantitas constans in locum ipsius b collocatur. Interim tamen hae ambae aequationes differentiatae eandem dant aequationem differentialem

$$2ydy = adx;$$

quin etiam omnes aequationes in hac forma $yy = ax$ contentae, quicunque valor ipsi a tribuatur, in una aequatione differentiali, in qua a non insit, comprehendi possunt. Dividatur enim aequatio illa per x, ut sit $\frac{yy}{x} = a$, haecque differentiata dabit

$$2xdy - ydx = 0 .$$

Possunt quoque aequationes transcendentes et algebraicae ad eandem aequationem differentialem perduci, uti fit in istis aequationibus

$$yy - ax = 0 \text{ et } yy - ax = bbe^{\frac{x}{a}} ;$$

si enim utraque per $e^{\frac{x}{a}}$ dividatur, ut habeantur istae aequationes

$$e^{-\frac{x}{a}}\left(yy - ax\right) = 0 \text{ et } e^{-\frac{x}{a}}\left(yy - ax\right) = bb ,$$

ex utriusque differentiatione orietur eadem aequatio differentialis

$$2ydy - adx - \frac{yydx}{a} + xdx = 0.$$

289. Ratio huius diversitatis in hoc consistit, quod quantitatis constantis differentiale sit $= 0$. Quodsi ergo aequatio finita ad eiusmodi formam reducatur, ut quantitas quaepiam constans sola adsit neque per variabiles vel multiplicetur vel dividatur, tum per differentiationem eruetur aequatio, in qua illa quantitas constans prorsus non adsit. Hoc modo quaelibet quantitas constans, quae in aequationem finitam ingreditur, per differentiationem tolli potest. Sic si proposita fuerit aequatio

$$x^3 + y^3 = 3axy$$

si ea per xy dividatur, ut habeatur $\frac{x^3+y^3}{xy} = 3a$, haec aequatio differentiata dabit

$$2x^3ydx + 2xy^3dy - x^4dy - y^4dx = 0 ,$$

quam constans a amplius non ingreditur.

290. Si plures quantitates constantes, quae in aequatione finita insunt tolli debeant, id fiet per differentiationem bis pluriesve repetitam sicque tandem obtinebuntur aequationes differentiales altiorum graduum iis constantibus prorsus carentes. Sit proposita haec aequatio

$$yy = maa - nxx ,$$

ex qua per differentiationem constantes *maa* et *n* tolli debeant. Prima quidem tolletur prima differentiatione, unde fit

$$ydy + nxdx = 0;$$

hinc porro formetur aequatio $\frac{ydy}{xdx} + n = 0$, quae sumto *dx* constante per differentiationem dabit

$$xyddy + xdy^2 - ydxdy = 0 \;;$$

quae etsi nullam constantem complectitur, tamen omnes aequationes in hac forma $yy = maa - nxx$ contentas, quicunque valores litteris *m*, *n* et *aa* tribuantur, in se aeque comprehendit.

291. Non solum vera quantitates constantes, quae in aequationem finitam ingrediuntur, per differentiationem tolli possunt, sed etiam altera variabilis, ea scilicet, cuius differentiale constans assumitur, per differentiationem eliminari poterit. Ex aequatione enim inter *x* et *y* proposita quaeratur valor *x*, ut sit $x = Y$ denotante *Y* functionem ipsius *y*, eritque $dx = dY$ et sumto *dx* constante fiet differentiando $0 = ddY$. Sin autem fuerit

$$xx + ax + b = Y \;,$$

fiet ter differentiando $0 = d^3Y$ et aequatio

$$x^4 + axx + bx + c = Y$$

quater differentiata dat $0 = d^4Y$. Quanquam autem in his aequationibus una tantum variabilis inesse videtur, quae propterea variabilis esse cessaret, dum unica variabilis in nulla aequatione adesse potest, tamen, quia differentiale *dx* constans est assumtum eiusque ratio in aequatione haberi debet, revera in aequationem ingredi censendum est. Hinc mirandum non est, si saepius aequationes differentiales secundi altiorisve gradus occurrant, in quibus unica tantum variabilis inesse videatur.

292. Praecipue autem notandum est per differentiationem quantitates irrationales ac transcendentes ex aequatione tolli posse. Quod quidem ad irrationales attinet, quoniam per reductiones cognitas irrationalitas eliminari potest, hoc facto per differentiationem aequatio obtinetur ab irrationalitate libera. Verum hoc saepenumero commodius sine ista reductione fieri potest, dum per comparationem aequationis differentialis cum finita formula irrationalis, si una tantum insit, eliminari potest. Sin autem duae pluresve partes irrationales in aequatione finita contineantur, tum eius aequatio differentialis denuo differentietur sicque aequationes differentiales altiorum graduum tot quaerantur, quot requiruntur ad singulas partes irrationales eliminandas. Hoc modo etiam exponentes indefiniti pariter atque fracti tolli poterunt. Uti si fuerit

$$y^m = (aa - xx)^n,$$

post differentiationem habebitur

$$my^{m-1} = -2n(aa - xx)^{n-1} xdx,$$

quae per finitam divisa dat

$$\frac{mdy}{y} = -\frac{2nxdx}{aa-xx},$$

in qua nullus amplius exponens indefinitus occurrit. Hinc ergo patet aequationem differentialem ab omni irrationalitate liberam ortam esse posse ex aequatione finita irrationali atque adeo quantitates transcendentes involvente.

293. Ut autem intelligatur, quomodo per differentiationem quantitates transcendentes eliminentur, incipiamus a logarithmis; quorum differentialia cum sint algebraica, negotium sine difficultate absolvetur. Sit enim

$$y = xlx;$$

erit $\frac{y}{x} = lx$, unde differentiando fit $\frac{xdy-ydx}{xx} = \frac{dx}{x}$ ideoque

$$xdy - ydx = xdx.$$

Si bini insint logarithmi, duplici differentiatione erit opus; sit enim

$$ylx = xly;$$

erit $\frac{ylx}{x} = ly$ et differentiando $\frac{dxylx+ydx-ydxlx}{xx} = \frac{dy}{y}$, ex qua concluditur fore

$$lx = \frac{xxdy-yydx}{yxdy-yydx}.$$

Haec aequatio iam iterum differentietur posito dx constante atque prodibit

$$\frac{dx}{x} = \frac{xxddy+2xdxdy-2ydxdy}{yxdy-yydx} + \frac{(yydx-xxdy)(yxddy+xdy^2-ydxdy)}{(yxdy-yydx)^2}$$

seu

$$\frac{dx}{x} = \frac{y^3xdxddy-yyxxdxddy+3yxxdxdy^2-y^2xdxdy^2+y^3dx^2dy-2xyydx^2dy-x^3dy^3}{(yxdy-yydx)^2}$$

quae reducta dabit

$$y^3xdxddy - yyxxdxddy + 3yxxdxdy^2 - 2xyydxdy^2 + 3y^3dx^2dy - 2xyydx^2dy - x^3dy^3 - \frac{y^4dx^3}{x} = 0$$

seu

$$yyxx(y-x)dxddy + 3yxdxdy(xxdy + yydx) - 2yyxxdxdy(dx+dy) = x^4dy^3 + y^4dx^3.$$

294. Quantitates exponentiales ex aequatione eodem modo quo logarithmi per differentiationem tolluntur. Si enim huiusmodi proposita fuerit

$$P = e^Q,$$

ubi P et Q functiones quascunqne ipsarum x et y denotent, ea aequatio transmutari poterit in hanc logarithmicam $lP = Q$, cuius differentialis est $\frac{dP}{P} = dQ$ seu $dP = PdQ$. Neque obstat, si quantitates exponentiales magis fuerint complicatae; tum enim, si una differentiatio non sufficit, duabus pluribusve negotium absolvetur.

I. Sit $y = \frac{e^x+e^{-x}}{e^x-e^{-x}}$; multiplicetur huius fractionis numerator ac denominator per e^x eritque $y = \frac{e^{2x}+1}{e^{2x}-1}$; unde fit $e^{2x} = \frac{y+1}{y-1}$ et $2x = l\frac{y+1}{y-1}$, cuius differentiale est

$$dx = -\frac{dy}{yy-1} = \frac{dy}{1-yy}$$

II. Sit $y = l\frac{e^x+e^{-x}}{2}$; fiet per primam differentiationem $dy = \frac{(e^x-e^{-x})dx}{e^x+e^{-x}}$ seu $\frac{dy}{dx} = \frac{e^{2x}-1}{e^{2x}+1}$ atque $e^{2x} = \frac{dy+dx}{dy-dx}$. Ergo $2x = l\frac{dy+dx}{dy-dx}$. Sumto ergo dx constante erit $dx = \frac{dxddy}{dx^2-dy^2}$ seu

$$dx^2 = ddy + dy^2.$$

295. Simili modo quantitates transcendentes a circulo pendentes ex aequatione ope differentiationis tollentur, uti ex his exemplis intelligetur.

1. Sit $y = a\text{Asin}\frac{x}{a}$; erit

$$dy = \frac{adx}{\sqrt{(aa-xx)}}.$$

II. Sit $y = a\cos\frac{y}{x}$; erit $\frac{y}{a} = \cos\frac{y}{x}$ et $\frac{dy}{a} = \frac{-xdy+ydx}{xx}\sin\frac{y}{x}$. At cum $\cos\frac{y}{x} = \frac{y}{a}$, erit $\sin\frac{y}{x} = \frac{\sqrt{(aa-yy)}}{a}$, quo valore substituto habebitur $\frac{dy}{a} = \frac{(ydx-xdy)\sqrt{(aa-yy)}}{axx}$ seu

$$xxdy = (ydx - xdy)\sqrt{(aa-yy)}.$$

Ill. Sit $y = m\sin x + n\cos x$; erit post differentiationem primam $dy = mdx\cos x - ndx\sin x$; quae denuo differentiata posito dx constante dabit $ddy = -mdx^2\sin x - ndx^2\cos x$; haec autem per primam divisa dat $\frac{ddy}{y} = -dx^2$ seu $ddy + ydx^2 = 0$, ex qua non solum sinus et cosinus, sed etiam constantes m et n evanuerunt.

IV. Sit $y = \sin lx$; erit $\mathrm{A}\sin y = lx$, unde per differentiationem fit $\frac{dy}{\sqrt{(1-yy)}} = \frac{dx}{x}$; quae sumtis quadratis dat $xxdy^2 = dx^2 - yydx^2$ haecque posito dx constante ulterius differentiata praebet $2xxdyddy + 2xdxdy^2 = -2ydx^2dy$ seu $xxddy + xdxdy + ydx^2 = 0$.

V. Sit $y = ae^{mx}\sin nx$; erit differentiando

$$dy = mae^{mx}dx\sin nx + nae^{mx}dx\cos nx$$

quae per propositam divisa dat

$$\frac{dy}{y} = mdx + \frac{ndx\cos nx}{\sin nx} = mdx + ndx\cot nx.$$

Erit ergo

$$\mathrm{A}\cot\left(\frac{dy}{nydx} - \frac{m}{n}\right) = nx.$$

Quae aequatio posito dx constante differentiata dat

$$ndx = \frac{ndxdy^2 - nydxddy}{m^2y^2dx^2 + n^2y^2dx^2 - 2mydxdy + dy^2}$$

seu

$$\left(m^2 + n^2\right)y^2dx^2 - 2mydxdy = -yddy.$$

Perspicuum igitur est, etiamsi in aequatione differentiali nullae quantitates transcendentes insint, eam tamen ex aequatione finita oriri potuisse, quae a quantitatibus transcenderitibus utcunque sit affecta.

296. Quoniam igitur aequationes differentiales sive primi sive altioris gradus, quae duas variabiles *x* et *y* continent, ex aequationibus finitis oriuntur, iis etiam relatio inter binas istas variabiles exprimitur. Proposita scilicet aequatione differentiali quacunque binas variabiles *x* et *y* complectente, ea significatur certa quaedam relatio inter *x* et *y*, qua *y* fit functio quaedam ipsius *x*. Hinc natura aequationis differentialis perspicitur, si loco *y* ea ipsius *x* functio assignari poterit, quae per aequationem illam indicatur seu quae sit ita comparata, ut, si ea ubique loco *y* eiusque differentiale loco *dy* atque eius altiora differentialia loco *ddy*, d^3y etc. substituantur, aequatio resultet identica. In huius autem functionis investigatione versatur calculus integralis, cuius finis eo tendit, ut proposita aequatione differentiali quacunque functio illa ipsius *x*, cui altera variabilis *y* est aequalis, definiatur seu, quod eodem redit, ut aequatio finita inveniatur, qua relatio inter *x* et *y* contineatur.

297. Si exempli gratia proponatur aequatio haec

$$2ydy - adx - \frac{yydx}{a} + xdx = 0,$$

ad quam supra (§ 288) pervenimus, eiusmodi relatio inter *x* et *y* ea definitur, quae simul hac aequatione finita

$$yy - ax = bbe^{\frac{x}{a}}$$

continetur. Cum igitur hinc sit $yy = ax + bbe^{\frac{x}{a}}$, patet $\sqrt{\left(ax + bbe^{\frac{x}{a}}\right)} = y$ eam esse functionem ipsius *x* cui variabilis *y* vi propositae aequationis differentialis sit aequalis. Namque si in aequatione loco *yy* hunc valorem $ax + bbe^{\frac{x}{a}}$ et loco $2ydy$ eius differentiale $adx + \frac{bb}{a}e^{\frac{x}{a}}dx$ substituamus, orietur aequatio identica

$$adx + \frac{bb}{a}e^{\frac{x}{a}}dx - adx - xdx - \frac{bb}{a}e^{\frac{x}{a}}dx + xdx = 0.$$

Sicque patet omnem aequationem differentialem aeque ac finitam certam relationem inter variabiles *x* et *y* exhibere, quae autem sine subsidio calculi integralis reperiri nequeat.

298. Quo haec facilius intelligantur, ponamus cognitam esse eam functionem ipsius *x*, quae ipsi *y* vi cuiuscunque aequationis differentialis sive primi sive altioris gradus conveniat, sitque

$$dy = pdx,\ dp = qdx,\ dq = rdx \text{ etc.},$$

atque si in aequatione differentiale dx assumtum sit constans, erit $ddy = qdx^2$, $d^3y = rdx^3$ etc.; qui valores postquam in aequatione erunt substituti, ob omnes eius terminos homogeneos differentialia dx per divisionem evanescent orieturque aequatio finitas tantum quantitates x, y, p, q, r etc. complectens. Cum igitur sint p, q, r etc. quantitates a natura functionis y pendentes, aequatio revera tantum inter duas variabiles x et y subsistet; sicque vicissim constat omni aequatione differentiali certam quandam relationem inter variabiles x et y determinari. Quamobrem si in solutione cuiusvis problematis ad aequationem differentialem inter x et y perveniatur, per eam aeque relatio inter x et y exprimi censenda est, ac si ad aequationem finitam esset perventum.

299. Hoc igitur modo aequatio quaevis differentialis ita ad formam finitam reduci potest, ut in ea nonnisi quantitates finitae contineantur, differentialia autem seu infinite parva prorsus excedant. Cum enim sit y certa functio ipsius x, si ponatur $dy = pdx$, $dp = qdx$, $dq = rdx$ etc., quodcunque differentiale fuerit constans acceptum, differentialia secunda et altiora per potestates ipsius dx exprimentur, quae deinceps per divisionem penitus tollentur. Ut si proponeretur haec aequatio

$$xyd^3y + xxdyddy + yydxddy - xydx^3 = 0,$$

in qua dx ponitur constans, facto $dy = pdx$, $dp = qdx$ et $dq = rdx$ ea abibit in $xyr + xxpq + yyq - xy = 0$, postquam scilicet tota aequatio per dx^3 est divisa. Haecque aequatio finita relationem inter x et y determinat.

300. Omnes ergo aequationes differentiales, cuiuscunque sint ordinis, his substitutionibus

$$dy = pdx,\ dp = qdx,\ dq = rdx \text{ etc.}$$

ad meras quantitates finitas reducuntur. Atque si aequatio differentialis fuerit primi ordinis, ita ut differentialia prima eam tantum ingrediantur, per istam reductionem praeter variabiles y et x insuper quantitas p introducetur. Sin autem aequatio differentialis fuerit secundi ordinis continens differentialia secunda, praeterea quantitas q, ac si fuerit differentialis tertii ordinis, introducetur insuper quantitas r sicque porro. Quoniam igitur hoc modo differentialia prorsus ex calculo exterminantur, ratio illa differentialis constantis penitus cessat neque amplius, etiamsi insint quantitates q, r ex differentialibus secundis oriundae, opus erit indicare, an quodpiam differentiale constans sit assumtum. Perinde enim est, utrum in evolutione aliquod differentiale pro lubitu constans statuatur an nullum.

301. Si igitur aequatio differentialis secundi vel altioris gradus proponatur, in qua nullum differentiale primum constans esse assumtum perhibetur, hoc modo statim explorari poterit, utrum ea determinatam relationem inter variabiles x et y contineat necne. Quia enim nullum differentiale constans assumitur, in arbitrio nostro relinquitur, quodnam differentiale constans ponere velimus, hincque tantum erit dispiciendum, utrum diversis differentialibus constantibus positis aequatio eandem relationem inter x et y exhibeat. Quod si non eveniat, certum est signum aequationem nullam determinatam relationem exprimere ideoque in solutione nullius problematis locum habere posse. Tutissimus autem modus simulque facillimus hoc explorandi erit is ipse, quem supra [§ 277] in simili negotio pro expressionibus differentialibus altiorum ordinum, num fixos habeant significatus, dignoscendis tradidimus.

302. Proposita ergo huiusmodi aequatione differentiali secundi altiorisve ordinis, in qua nullum differentiale constans sit positum, statuatur differentiale dx constans; deinde haec aequatio, uti supra [§ 276] de expressionibus differentialibus ostendimus, iterum reducatur ad eiusmodi formam, quae nullum differentiale constans supponat, statuendo scilicet

$$ddy - \frac{dyddx}{dx} \text{ loco } ddy \text{ et } d^3y - \frac{3ddxddy}{dx} + \frac{3dyddx^2}{dx^2} - \frac{dydx^3}{dx} \text{ loco } d^3y \text{ etc.}$$

Quo facto dispiciatur, utrum aequatio hoc modo resultans conveniat cum aequatione proposita; quod si eveniat, aequatio proposita determinatam relationem inter x et y complectetur; sin autem secus accidat, aequatio erit vaga neque definitam rationem inter variabilis x et y exprimet, quemadmodum hoc iam ante fusius est demonstratum.

303. Sit, quo hoc plenius explicetur, haec aequatio proposita, quae nullo differentiali constante posito reperta esse perhibeatur,

$$Pddx + Qddy + Rdx^2 + Sdxdy + Tdy^2 = 0.$$

Ponatur dx constans atque ea transibit in hanc

$$Qddy + Rdx^2 + Sdxdy + Tdy^2 = 0\,.$$

Ex hac nunc iterum consideratio differentialis constantis exuatur modo ante praescripto et obtinebitur

$$-\frac{Qdyddx}{dx} + Qddy + Rdx^2 + Sdxdy + Tdy^2 = 0\,;$$

quae quoniam a proposita tantum ratione primi termini discrepat, videndum est, utrum sit $P = -\frac{Qdy}{dx}$. Quod si deprehendatur, aequatio proposita fixam relationem inter x et y exhibebit, quae per regulas in calculo integrali tradendas reperietur, quodcunque differentiale primum constans accipiatur. At si fieri nequeat $P = -\frac{Qdy}{dx}$, aequatio proposita erit impossibilis.

304. Nisi igitur haec proposita aequatio

$$Pddx + Qddy + Rdx^2 + Sdxdy + Tdy^2 = 0.$$

sit absurda, necesse est, ut sit $Pdx + Qdy = 0$, quod duplici modo evenire potest; vel enim actu erit $P = -\frac{Qdy}{dx}$ seu aequatio $Pdx + Qdy = 0$ identica, vel erit $Pdx + Qdy = 0$ ipsa illa aequatio differentialis primi gradus, ex cuius differentiatione proposita est orta; quo posteriore casu aequatio $Pdx + Qdy = 0$ congruet cum proposita eandemque relationem inter *x* et *y* continebit sicque sine auxilio calculi integralis haec relatio erui poterit. Cum enim sit $Pdx + Qdy = 0$, erit differentiando

$$Pddx + Qddy + dPdx + dQdy = 0,$$

quae ab aequatione proposita subtracta relinquet

$$Rdx^2 + Sdxdy + Tdy^2 = dPdx + dQdy.$$

Cum autem sit $dy = -\frac{Pdx}{Q}$, differentialia prorsus extingui poterunt nasceturque aequatio finita inter *x* et *y* earum relationem indicans.

305. Ponamus in solutione problematis nullo differentiali constante assumto perventum esse ad hanc aequationem

$$x^3ddx + xxyddy - yydx^2 + xxdy^2 + aadx^2 = 0.$$

Erit ergo, cum aequationem absurdum non continere constet,

$$x^3dx + xxydy = 0 \quad \text{seu} \quad xdx + ydy = 0,$$

cuius differentiale erit

$$x^3ddx + xxyddy + 3xxdx^2 + 2xydxdy + xxdy^2 = 0,$$

quae aequatio a proposita subtracta relinquit

$$aadx^2 - yydx^2 - 3xxdx^2 - 2xydxdy = 0$$

seu

$$aadx - yydx - 3xxdx - 2xydy = 0.$$

Cum autem sit $xdx + ydy = 0$, erit

$$2xydy = -2xxdx$$

ideoque

$$aadx - yydx - xxdx = 0 \text{ seu } yy + xx = aa;$$

quae aequatio veram relationem inter x et y exprimit, siquidem ea consentit cum differentiali primum inventa $xdx + ydy = 0$. Qui consensus nisi se manifestasset, aequatio proposita pro impossibili esset habenda; cum autem hoc casu locum habuerit, aequationem finitam $xx + yy = aa$ sine calculo integrali elicere licuit.

306. Ut vero etiam exemplum aequationis impossibilis afferamus, proposita sit haec aequatio

$$yyddx - xxddy + ydx^2 - xdy^2 + adxdy = 0,$$

in qua nullum differentiale constans sit assumtum. Foret ergo $yydx - xxdy = 0$ ideoque differentiando

$$yyddx - xxddy + 2ydxdy - 2xdxdy = 0,$$

quae propositae aequalis posita dabit

$$ydx^2 - xdy^2 + adxdy = 2ydxdy - 2xdxdy.$$

Cum vero sit $dy = \frac{yydx}{xx}$, extinguendis differentialibus obtinebitur

$$y - \frac{y^4}{x^3} + \frac{ayy}{xx} = \frac{2y^3}{xx} - \frac{2yy}{x}$$

seu

$$x^3 - y^3 + axy = 2xyy - 2xxy;$$

quae utrum cum differentiali $yydx - xxdy = 0$ consentiat, eam differentiando facile patebit; fiet enim

$$3xxdx - 3yydy + axdy + aydx = 2yydx + 4xydy - 2xxdy - 4xydx$$

seu

$$\frac{dy}{dx} = \frac{3xx + ay - 2yy + 4xy}{3yy - ax + 4xy - 2xx};$$

at ex illa est $\frac{dy}{dx} = \frac{yy}{xx}$ foretque ergo

$$3x^4 + 4x^3y + axxy = 3y^4 + 4xy^3 - axyy$$

seu

$$axy = \frac{3y^4+4x^3y-4x^3y-3x^4}{x+y} = 3y^3 + xyy - xxy - 3x^3$$

Verum ex aequatione finita primum inventa est

$$axy = y^3 + 2xyy - 2xxy - x^3,$$

quae ab ista subtracta relinquit

$$0 = 2y^3 - xyy + xxy - 2x^3,$$

quae resolvitur in has

$$0 = y - x \text{ et } 2yy + yx + 2xx = 0.$$

Quarum illa $y = x$ quidem cum differentiali $dy = \frac{yydx}{xx}$ constare potest, at vero aequationi finitae primum inventae adversatur; nisi statuatur $a = 0$ vel nisi utraque variabilis x et y constans statuatur, quo quidem casu ob $dx = 0$ et $dy = 0$ omnibus aequationibus differentialibus satisfit, aequatio proposita subsistere nequit.

307. Consideremus nunc etiam aequationes differentiales tres variabiles x, y et z involventes, quae erunt vel primi vel secundi vel altioris gradus. Ad quarum naturam scrutandam notari oportet aequationem finitam tres variabiles complectentem determinare relationem, quam unaquaeque ad binas reliquas teneat; definitur ergo, qualis functio sit z ipsarum x et y. Quemadmodum igitur aequatio huiusmodi finita resolvitur, si reperiatur, qualis functio ipsarum x et y loco z substitui debeat, ut aequationi satisfiat, ita quoque aequatio differentialis tres variabiles complectens determinabit, qualis functio una sit reliquarum; isque huiusmodi aequationem resolvisse censendus est, qui indicaverit eam binarum variabilium x et y functionem, quae loco tertiae z substituta aequationi satisfaciat seu eam identicam reddat. Aequatio ergo differentialis resolvitur, si vel functio ipsarum x et y valorem ipsius z exhibens definiatur vel aequatio finita assignetur, qua idem debitus ipsius z valor exprimatur.

308. Quanquam autem omnis aequatio differentialis duas tantum variabiles complectens semper determinatam relationem inter eas exprimit, tamen hoc non semper evenit in aequationibus differentialibus trium variabilium. Dantur enim eiusmodi aequationes, quibus plane nullo modo satisfieri poterit, quaecunque functio ipsarum x et y in locum ipsius z substituatur. Uti si proposita fuerit haec aequatio $zdy = ydx$, facile patet nullam prorsus dari functionem ipsarum x et y, quae loco z substituta reddat $zdy = ydx$; differentialia enim dx et dy nullo modo extinguentur. Simili modo apparet nullam dari functionem ipsarum x et z, quae loco y substituta eidem aequationi satisfaciat. Quaecunque enim pro y concipiatur functio ipsarum *x* et *z*, in eius differentiali *dy* inest *dz*, quod, quia in aequatione non inest, destrui non poterit. Hanc ob rem nulla aequatio finita inter *x*, *y* et *z* dari potest, quae aequationi differentiali $zdy = ydx$ conveniat.

309. Hinc aequationes differentiales tres variabiles continentes distribui oportet in imaginarias et reales. Huiusmodi autem aequatio erit imaginaria seu absurda, cui per nullam aequationem finitam satisfieri potest, cuiusmodi erat illa $zdy = ydx$, quam modo consideravimus. Aequatio autem erit realis, cui aequivalens aequatio finita exhiberi potest, quod evenit, si una variabilis aequalis fit certae cuipiam functioni binarum reliquarum. Cuiusmodi est haec aequatio

$$zdy + ydz = xdz + zdx + xdy + ydx\,;$$

congruit enim haec cum ista aequatione finita $yz = xz + xy$ fitque $z = \frac{xy}{y-x}$.
Istud ergo discrimen inter huiusmodi aequationes imaginarias et reales diligentissime est observandum, praecipue in calculo integrali, quia ridiculum foret cuiuspiam aequationis differentialis velle integralem, hoc est aequationem finitam satisfacientem quaerere, quae plane nullam habeat.

310. Primum igitur patet omnes aequationes differentiales trium variabilium, in quibus tantum binarum differentialia occurrant, esse imaginarias et absurdas. Ponamus enim in aequatione, quae contineat variabilem *z*, tantum inesse differentialia *dx* et *dy*, differentiale autem *dz* prorsus abesse, atque manifestum erit nullam exhiberi posse functionem ipsarum *x* et *y*, quae loco *z* substituta aequationem identicam producat; differentialia enim *dx* et *dy* nullo modo tollentur. Ilis ergo casibus omnino nulla datur aequatio finita satisfaciens; nisi forte eiusmodi relatio inter *x* et *y* assignari queat, quae, quicquid sit *z*, subsistere possit, uti fit in hac aequatione

$$zdy - zdx = ydy - xdx\,,$$

.cui satisfacit aequatio $y = x$. Facile autem investigatur, quibus casibus hoc, eveniat, quarendo relationem inter *x* et *y*, primo si $z = 0$, et tum, an ista relatio aequationi pro quocunque ipsius *z* valore satisfaciat.

311. Neque vero solum aequatio tres variabiles involvens est absurda, si duo tantum continet differentialia, sed etiam, si in ea omnia tria differentialia occurrant, talis esse poterit. Quos casus ut evolvamus, ponamus *P* et *Q* esse functiones ipsarum *x* et *y* tantum atque haberi hanc aequationem $dz = Pdx + Qdy$; quae si non est absurda, erit *z* functio quaepiam ipsarum *x* et *y*, cuius differentiale sit $dz = pdx + qdy$, eritque $P = p$ et $Q = q$. At supra [§ 232] demonstravimus $pdx + qdy$ non esse posse differentiale cuiusquam functionis ipsarum *x* et *y*, nisi sit $\left(\frac{dp}{dy}\right) = \left(\frac{dq}{dx}\right)$ denotante, uti ante assumsimus, $\left(\frac{dp}{dy}\right)$

differentiale ipsius *p*, posita sola *y* variabili, per *dy* divisum atque $\left(\frac{dq}{dx}\right)$ differentiale ipsius *q*, posita sola *x* variabili, divisum per *dx*. Quocirca aequatio $dz = Pdx + Qdy$ realis esse nequit, nisi sit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$.

312. Similis omnino erit ratio huius aequationis

$$dZ = Pdx + Qdy,$$

si *Z* denotet functionem quamcunque ipsius *z*, *P* vero et *Q* sint functiones ipsarum *x* et *y* tertiam variabilem *z* non complectentes. Ut enim *Z* aequalis fieri possit functioni ipsarum *x* et *y*, necesse est, ut sit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$. Ex hoc ergo criterio aequatio differentialis quaeque proposita, quae quidem in hac forma generali contineatur, diiudicari potest, utrum sit realis an absurda. Sic patebit hanc aequationem $zdz = ydx + xdy$ esse realem; nam ob $P = y$ et $Q = x$ fit

$$\left(\frac{dP}{dy}\right) = 1 = \left(\frac{dQ}{dx}\right) = 1.$$

Haec vero aequatio $azdz = yydx + xxdy$ est absurda; fit enim

$$\left(\frac{dP}{dy}\right) = 2y \text{ et } \left(\frac{dQ}{dx}\right) = 2x$$

qui valores sunt inaequales.

313. Ut autem criterium latissime patens investigemus, sint *P*, *Q* et *R* functiones quaecunque ipsarum *x*, *y* et *z*; atque omnis aequatio differentialis trium variabilium, siquidem sit primi gradus, continebitur in hac forma

$$Pdx + Qdy + Rdz = 0.$$

Quoties ergo haec aequatio est realis, *z* aequabitur functioni cuipiam ipsarum *x* et *y* eiusque adeo differentiale erit huius formae $dz = pdx + qdy$. Quare si in aequatione proposita ista functio ipsarum *x* et *y* loco *z* et $pdx + qdy$ loco *dz* substituatur, necessa est, ut prodeat aequatio identica $0 = 0$. Atque cum ex aequatione proposita fiat

$$dz = -\frac{Pdx}{R} - \frac{Qdy}{R},$$

si in *P*, *Q* et *R* valor ille loco *z* substituatur, necesse est, ut fiat

$$p = -\frac{P}{R} \text{ et } q = -\frac{Q}{R}.$$

314. Quoniam vero est $dz = pdx + qdy$, erit per ante demonstrata $\left(\frac{dp}{dy}\right) = \left(\frac{dq}{dx}\right)$. Cum igitur substituto loco *z* ipsius valore in *x* et *y* sit

$$p = -\tfrac{P}{R} \text{ et } q = -\tfrac{Q}{R},$$

erit

$$\left(\frac{dp}{dy}\right) = \left(\frac{-RdP+PdR}{RRdy}\right) \text{ et } \left(\frac{dq}{dx}\right) = \left(\frac{-RdQ+QdR}{RRdx}\right)$$

ideoque habebitur per *RR* multiplicando haec aequatio

$$P\left(\frac{dR}{dy}\right) - R\left(\frac{dP}{dy}\right) = Q\left(\frac{dR}{dx}\right) - R\left(\frac{dQ}{dx}\right),$$

ubi denominatores *dy* et *dx* iterum indicant in differentialibus numeratorum eam solam quantitatem variabilem assumi debere, cuius differentiale denominatorem constituit. Haec autem differentialia *dP, dQ, dR* ante cognosci non possunt, quam in ipsis quantitatibus *P, Q* et *R* valor debitus loco *z* fuerit substitutus; qui autem cum sit incognitus, sequenti modo erit procedendum.

315. Quia *P, Q* et *R* sunt functiones ipsarum *x, y* et *z*, ponamus

$$dP = \alpha dx + \beta dy + \gamma dz$$
$$dQ = \delta dx + \varepsilon dy + \zeta dz,$$
$$dR = \eta dx + \theta dy + \iota dz,$$

ubi $\alpha, \beta, \gamma, \delta, \varepsilon$ etc. denotant eas functiones, quae ex differentiatione oriuntur. Concipiamus nunc loco *z* ubique eius valorem in *x* et *y* expressum substitui et loco *dz* ponamus valorem $pdx + qdy$ fietque

$$dP = (\alpha + \gamma p)dx + (\beta + \gamma q)dy$$
$$dQ = (\delta + \zeta p)dx + (\varepsilon + \zeta q)dy$$
$$dR = (\eta + \iota p)dx + (\theta + \iota q)dy.$$

Ex his ergo valoribus erit

$$\left(\frac{dR}{dy}\right) = \theta + \iota q, \quad \left(\frac{dR}{dx}\right) = \eta + \iota p$$
$$\left(\frac{dP}{dy}\right) = \beta + \gamma q, \quad \left(\frac{dQ}{dx}\right) = \delta + \zeta p.$$

316. Cum igitur ad realitatem aequationis requiratur, ut sit

$$P\left(\frac{dR}{dy}\right)-R\left(\frac{dP}{dy}\right)=Q\left(\frac{dR}{dx}\right)-R\left(\frac{dQ}{dx}\right)$$

fiet, si inventi valores substituantur

$$P(\theta+\iota q)-R(\beta+\gamma q)=Q(\eta+\iota p)-R(\delta+\zeta p).$$

At ante invenimus esse

$$p=-\tfrac{P}{R} \text{ et } q=-\tfrac{Q}{R}$$

qui valores, cum differentialia non amplius in computum veniant, adhiberi poterunt, etiamsi loco *z* eius valor in *x* et *y* non substituatur. Eritque ergo

$$P\theta-\tfrac{PQ\iota}{R}-R\beta+Q\gamma=Q\eta-\tfrac{PQ\iota}{R}-R\delta+P\zeta$$

seu

$$0=P(\zeta-\theta)+Q(\eta-\gamma)+R(\beta-\delta).$$

Quia autem quantitates $\beta,\delta,\gamma,\eta,\zeta,\theta$ per differentiationem inveniuntur, erit superiori notandi modo adhibito

$$0=P\left(\tfrac{dQ}{dz}-\tfrac{dR}{dy}\right)+Q\left(\tfrac{dR}{dx}-\tfrac{dP}{dz}\right)+R\left(\tfrac{dP}{dy}-\tfrac{dQ}{dx}\right).$$

Quae proprietas nisi in aequatione locum habeat, aequatio non erit realis, sed imaginaria et absurda.

317. Quanquam hanc regulam ex consideratione variabilis *z* elicuimus, tamen, quia omnes quantitates aeque ingrediuntur, manifestum est et reliquarum consideratione eandem expressionem prodituram fuisse. Proposita ergo aequatione differentiali primi gradus, quae tres variabiles involvat, quacunque statim diiudicari poterit, utrum sit realis an imaginaria. Comparetur enim cum hac forma generali

$$Pdx+Qdy+Rdz=0$$

atque quaeratur valor huius formulae

$$P\left(\tfrac{dQ}{dz}-\tfrac{dR}{dy}\right)+Q\left(\tfrac{dR}{dx}-\tfrac{dP}{dz}\right)+R\left(\tfrac{dP}{dy}-\tfrac{dQ}{dx}\right);$$

qui si fuerit $=0$, aequatio erit realis; sin autem non fuerit $=0$, certum hoc est signum aequationem esse imaginariam seu absurdam.

318. Aequatio proposita per divisionem quoque semper ad huiusmodi formam reduci potest

$$Pdx + Qdy + dz = 0;$$

in quam cum prior abeat, si fiat $R=1$, criterium simplicius exprimetur hoc modo

$$P\left(\frac{dQ}{dz}\right) - Q\left(\frac{dP}{dz}\right) + \left(\frac{dP}{dy}\right) - \left(\frac{dQ}{dx}\right) = 0.$$

Quoties enim haec expressio revera nihilo aequalis reperitur, toties aequatio proposita erit realis; sin autem contrarium eveniat, aequatio erit imaginaria. Posterius quidem ex iis, quae demonstravimus, est certum; de priori autem adhuc dubitari possit, utrum aequatio semper sit realis, quoties quidem hoc criterium id indicat. Quod cum hoc loco plenissime demonstrari nequeat, sed in calculo demum integrali demonstratione confirmari possit, hic tantum id affirmatus neque autem periculum inde est metuendum, si quis tantisper de eius veritate dubitare voluerit.

319. Ex hoc ergo criterio primum patet, si in aequatione

$$Pdx + Qdy + Rdz = 0$$

fuerit *P* functio ipsius *x*, *Q* functio ipsius *y* et *R* functio ipsius *z* tantum, aequationem semper fore realem. Fit enim

$$\left(\frac{dP}{dy}\right) = 0,\ \left(\frac{dP}{dz}\right) = 0,\ \left(\frac{dQ}{dz}\right) = 0,\ \left(\frac{dQ}{dx}\right) = 0,\ \left(\frac{dR}{dx}\right) = 0 \text{ et } \left(\frac{dR}{dy}\right) = 0$$

ideoque tota expressio criterii sponte evanescet.

320. Si fuerit ut ante *P* ipsius *x* et *Q* ipsius *y* functio tantum, *R* autem functio quaecunque ipsarum *x*, *y* et *z*, aequatio erit realis, si fuerit

$$P\left(\frac{dR}{dy}\right) = Q\left(\frac{dR}{dx}\right) \text{ seu } \left(\frac{dR}{dx}\right) : \left(\frac{dR}{dy}\right) = P:Q.$$

Sic si proposita fuerit haec aequatio

$$\frac{2dx}{x} + \frac{3dy}{y} + \frac{x^2y^3dz}{z^6} = 0$$

quia hic est

$$P=\tfrac{2}{x},\quad Q=\tfrac{3}{y},\quad R=\frac{x^2y^3}{z^6},$$

hinc

$$\left(\frac{dR}{dx}\right)=\frac{2xy^3}{z^6}\ \text{ atque }\ \left(\frac{dR}{dy}\right)=\frac{3xxyy}{z^6},$$

erit

$$P\left(\frac{dR}{dy}\right)=Q\left(\frac{dR}{dx}\right)=\frac{6xyy}{z^6}$$

ideoque aequatio proposita erit realis.

321. Si fuerint P et Q functiones ipsarum x et y, at R functio ipsius z tantum, ob

$$\left(\frac{dP}{dz}\right)=0,\quad \left(\frac{dQ}{dz}\right)=0,\quad \left(\frac{dR}{dx}\right)=0,\quad \text{et}\quad \left(\frac{dR}{dy}\right)=0$$

aequatio erit realis, si fuerit $\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$. Haec eadem vero conditio requiritur, si $Pdx+Qdy$ debeat esse differentiale determinatum seu ex differentiatione cuiuspiam functionis finitae ipsarum x et y ortum. Hucque redit, quod supra (§312) iam observavimus, aequationem $dZ=Pdx+Qdy$, si Z sit functio ipsius z tantum, at P et Q functiones ipsarum x et y, realem esse non posse, nisi sit $\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$. Ambo autem isti casus inter se prorsus conveniunt; nam loco Rdz, si R est functio ipsius z tantum, poni potest dZ existente Z functione ipsius z.

322. Ut hoc criterium inventum exemplo illustremus, consideremus hanc aequationem

$$\left(6xy^2z-5yz^3\right)dx+\left(5x^2yz-4xz^3\right)dy+\left(4x^2y^2-6xyz^2\right)dz=0.$$

qua cum forma generali comparata fit

$$P=6xy^2z-5yz^3,\quad \left(\frac{dP}{dy}\right)=12xyz-5z^3,\quad \left(\frac{dP}{dz}\right)=6xy^2-15yz^2,$$
$$Q=5x^2yz-4xz^3,\quad \left(\frac{dQ}{dx}\right)=10xyz-4z^3,\quad \left(\frac{dQ}{dz}\right)=5x^2y-12xzz,$$
$$R=4x^2y^2-6xyz^2,\quad \left(\frac{dR}{dx}\right)=8xy^2-6yz^2,\quad \left(\frac{dR}{dy}\right)=8x^2y-6xz^2.$$

His inventis valoribus aequatio iudicium continens erit haec

$$\left(6xy^2z-5yz^3\right)\left(-3xxy-6xzz\right)$$
$$+\left(5x^2yz-4xz^3\right)\left(2xyy+9yzz\right)$$
$$+\left(4x^2y^2-6xyz^2\right)\left(2xyz-z^3\right)=0.$$

Haec autem expressio si evolvatur, omnes termini actu se mutuo destruunt fitque $0=0$, quod indicat aequationem propositam esse realem.

323. Quando autem expressio hoc modo ex criterio eruta non evanescit, tum id signum est aequationem propositam esse imaginariam. Quoniam vero hoc pacta ex criterio aequatio finita invenitur, ea, si quidem aequationi differentiali conveniat, simul relationem indicabit, quam variabiles inter se tenent. Atque hoc modo ii casus, quorum supra meminimus (§310), evolvuntur. Sit enim proposita ista aequatio

$$\left(z-x\right)dx+\left(y-z\right)dy=0;$$

fiet

$$P=z-x,\ Q=y-z \text{ et } R=0,$$

porro

$$\left(\tfrac{dP}{dz}\right)=1 \text{ et } \left(\tfrac{dQ}{dz}\right)=-1.$$

Aequatio iudicium exhibens fit

$$P\left(\tfrac{dQ}{dz}\right)=Q\left(\tfrac{dP}{dz}\right)$$

seu

$$z-x=z-y;$$

unde fit $y=x$.
Quoniam igitur hic casu evenit, ut aequatio $y=x$ simul aequationi differentiali satisfaciat, dicendum est propositam aequationem nil aliud significare, nisi esse $y=x$.

324. Proposita ergo aequatione differentiali tres variabiles continente

$$Pdx+Qdy+Rdz=0$$

tres considerandi erunt casus sequentes, ad quos haec aequatio deducit

$$P\left(\tfrac{dQ}{dz}-\tfrac{dR}{dy}\right)+Q\left(\tfrac{dR}{dx}-\tfrac{dP}{dz}\right)+R\left(\tfrac{dP}{dy}-\tfrac{dQ}{dx}\right)=0.$$

Primus est, si haec expressio revera fit $=0$, tumque aequatio proposita erit realis. Sin autem haec aequatio finita non sit identica, tum dispiciendum est, utrum ea aequationi propositae satisfaciat; quod si evenit, habebitur aequatio finita, qui est casus secundus. Tertius autem casus locum habet, si aequatio finita cum proposita differentiali subsistere nequeat, atque tum aequatio proposita erit imaginaria; neque enim ulla aequatio finita exhiberi poterit, quae ipsi satisfaciat.

325. Casus primus ac tertius per se sunt perspicui, secundus autem, etsi rarissime occurrit, probe tamen notari meretur; et cum eius exemplum iam supra in aequatione, quae duo tantum continet differentialia, exhibuerimus, etiam aequationem afferamus, in qua omnia tria differentialia insint,

$$(z-y)dx + xdy + (y-z)dz = 0.$$

Erit ergo

$$P = z-y,\quad \left(\frac{dQ}{dz}\right)=0,\quad \left(\frac{dR}{dy}\right)=1,$$
$$Q = x,\quad \left(\frac{dR}{dx}\right)=0,\quad \left(\frac{dP}{dz}\right)=1,$$
$$R = y-z,\quad \left(\frac{dP}{dy}\right)=-1,\left(\frac{dQ}{dx}\right)=1,$$

unde aequatio finita criterium continens evadet

$$z-x-y=0 \text{ seu } z=x+y;$$

substituatur hic valor pro z in aequatione differentiali fietque

$$xdx + xdy - x(dx+dy) = 0;$$

quae aequatio cum sit identica, sequitur aequationem differentialem nil aliud significare nisi $z = x+y$.

326. Quoniam diximus omnes aequationes differentiales primi ordinis, in quibus tres variabiles insunt, contineri in hac forma

$$Pdx + Qdy + Rdz = 0,$$

dubium hic nasci poterit circa eas aequationes, in quibus differentialia prima duas pluresve dimensiones constituunt, cuiusmodi est haec

$$Pdx^2 + Qdy^2 + Rdz^2 = 2Sdxdy + 2Tdxdz + 2Vdydz.$$

Verum de huiusmodi aequationibus notandum est eas nullo modo reales esse posse, nisi habeant divisores prioris formae, qui propterea aequationes simplices constituent. Cum enim ex hac aequatione fiat

$$dz = \frac{Tdx + Vdy \pm \sqrt{\left(dx^2\left(T^2 - PR\right) + 2dxdy\left(TV + RS\right) + dy^2\left(V^2 - QR\right)\right)}}{R},$$

facile patet z functioni cuipiam ipsarum x et y seu dz huiusmodi expressioni $pdx + qdy$ aequale fieri non posse, nisi quantitas irrationalis evadat rationalis, quod eveniet, si fuerit

$$\left(T^2 - PR\right)\left(V^2 - QR\right) = \left(TV + RS\right)^2$$

seu

$$R = \frac{PVV + 2STV + QTT}{PQ - SS}.$$

Nisi ergo haec aequatio finita ipsa aequationi propositae satisfaciat, haec erit imaginaria.

327. Superesset, ut in hoc capite quoque aequationes differentiales altiorum ordinum, quae tres variabiles complectuntur, perpenderemus casusque definiremus, quibus eae vel reales vel imaginariae evadunt; verum quia criteria nimis fierent intricata, hunc laborem hic praetermittimus, praesertim cum ex iisdem fontibus, quos hic aperuimus, sequantur. Ceterum si in calculo integrali his criteriis erit opus, tum ea facile erui poterunt. Ob eandem causam hic quoque aequationes, quae plures variabiles complectuntur, non contemplamur, cum fere nunquam occurrant atque, si unquam occurrerent, ex principiis hic traditis sine negotio examinari possent. Quare his expositis Institutioni Calculi Differentialis hic finem imponimus progressuri ad insignes usus ostendendos, quos iste calculus cum in ipsa Analysi tum in Geometria sublimiori affert.