## CHAPTER VIII

# CONCERNING THE DIFFERENTIATION OF FORMULAS OF HIGHER DIFFERENTIALS

**242.** If a single variable may be present, and the first differential of this assumed constant, the above method now has discussed finding the differential of any order. Clearly, if the differential of any function may be differentiated again, the second differential of this arises, and this differentiated again gives the third differential of the function, and thus henceforth. Now the same rule may be considered also, if the function may involve several variables, or only a single variable of which the first differential is not placed constant. Therefore let $V$ be some function of $x$, nor indeed let $dx$ be constant but some variable, thus so that the differential of $dx$ itself shall be $= ddx$, and the differential of this $= d^3x$ and so on thus, and we shall investigate the second and following differentials of the function $V$.

**243.** We may put the first differential of the function $V$ to be $= Pdx$, where $P$ shall be a certain function of $x$ depending on $V$. If now we may wish to find the second differential of the function $V$, it requires the first differential $Pdx$ to be differentiated anew; which since it shall be produced from the two variable quantities $P$ and $dx$, of which the differential of the first one shall be $dP = pdx$, truly of that $dx$ the differential $ddx$, by the rule given concerning factors, the second differential will be

$$ddV = Pddx + pdx^2$$

Then, if there may be put $dp = qdx$, since the differential of $dx^2$ shall be $2dxddx$, there will be on differentiating again,

$$d^3V = Pd^3x + dPddx + 2pdxddx + dpdx^2;$$

now on account of $dP = pdx$ and $dp = qdx$ there will be

$$d^3V = Pd^3x + 3pdxddx + qdx^3,$$

and the higher differentials may be found in a similar manner.

**244.** We may apply these to powers of $x$, the individual differentials of which we may investigate, if $dx$ shall not be placed constant.

I. Therefore let $V = x$; there will be

$$dV = dx,\ d^2V = d^2x,\ d^3V = d^3x,\ d^4V = d^4x \text{ etc.}$$

II. Let $V = x^2$ ; there will be

$$dV = 2xdx$$

and

$$ddV = 2xddx + 2dx^2$$
$$d^3V = 2xd^3x + 6dxddx$$
$$d^4V = 2xd^4x + 8dxd^3x + 6ddx^2$$
$$d^5V = 2xd^5x + 10dxd^4x + 20ddxd^3x$$

etc.

III. If in general there were put $V = x^n$, there will be

$$dV = nx^{n-1}dx$$
$$ddV = nx^{n-1}ddx + n(n-1)x^{n-2}dx^2$$
$$d^3V = nx^{n-1}d^3x + 3n(n-1)x^{n-2}dxddx + n(n-1)(n-2)x^{n-3}dx^3$$
$$d^4V = nx^{n-1}d^4x + 4n(n-1)x^{n-2}dxd^3x + 3n(n\ \ -1)x^{n-2}ddx^2$$
$$+6n(n-1)(n-2)x^{n-3}dx^2ddx + n(n-1)(n-2)(n-3)x^{n-4}dx^4$$

etc.

Therefore if $dx$ were put constant and therefore $ddx = 0,\ d^3x = 0,\ d^4x = 0$ etc., the same differentials may arise which now have been found above by this hypothesis.

**245.** Therefore because the differentials of each order of $x$ may be differentiated by the same rule, from which finite quantities occur whatever the expressions, into which besides the differentials of this finite quantity are able to be differentiated following the precepts given above . Which operation, since it may happen occasionally, we will illustrate here by some examples.

I. If there were $V = \frac{xddx}{dx^2}$ , by differentiating there will be produced

$$dV = \frac{xd^3x}{dx} + \frac{ddx}{dx} - \frac{2xddx^2}{dx^3}.$$

II. If there were $V = \frac{x}{dx}$ there will be

$$dV = 1 - \frac{xddx}{dx^2},$$

where nothing hinders, that for $V$ we have put an infinitely great quantity.

III. If there were $V = xxl\frac{ddx}{dx^2}$, because $V$ is changed into $xxlddx - 2xxldx$, there will be following the customary rules on differentiation,

$$dV = 2xdxlddx + \frac{xxd^3x}{ddx} - 4xdxldx - \frac{2xxddx}{dx}.$$

Moreover higher differentials of *V* may be found in a like manner.

**246.** If the expression proposed may involve two variables, surely *x* and *y*, either the differential of one is put constant, or of neither; indeed the choice is the differential of either one or the other to be assumed constant, because from our choice may depend just how we may wish to put in place successive values of one variable. Nor indeed are the differentials of each variable to be placed constant at the same time; for from this a relation may be assumed between the variables *x* and *y*, which yet is either zero or put unknown. If indeed, while we have put *x* to increase equally, *y* also may be put in place to increase in equal increments, then from that itself there may be indicated to be $y = ax + b$ and thus *y* depends on *x*, which still cannot be assumed. On this account either only one differential variable can be assumed constant, or none. But if we may know how to absolve the differentials with no differential assumed constant, likewise also the differentials will be agreed upon, if either differential may be assumed constant ; for there is a need still, that if $dx$ be put in place constant, the terms everywhere containing $ddx,\ d^3x,\ d^4x$ etc. may be deleted.

**247.** Therefore *V* may specify any finite function of *x* and *y* and let $dV = Pdx + Qdy$. Towards finding the second differential of *V* we may assume each differential *dx* and *dy* to be variable, and since *P* and *Q* shall be functions of *x* and *y*, we may put in place

$$dP = pdx + rdy$$
$$dQ = rdx + qdy;$$

for we have seen from above that $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right) = r$.

With these in place, $dV = Pdx + Qdy$ may be differentiated and there is found

$$ddV = Pddx + pdx^2 + 2rdxdy + Qddy + qdy^2.$$

Therefore if the differential *dx* be put constant, there will be

$$ddV = pdx^2 + 2rdxdy + Qddy + qdy^2;$$

but if the differential *dy* be placed constant, there becomes

$$ddV = Pddx + pdx^2 + 2rdxdy + qdy^2 .$$

**248.** Therefore if some function of $x$ and $y$ may be differentiated twice with no differential put constant, the second differential of this will have a form of this kind always

$$ddV = Pddx + Qddy + Rdx^2 + Sdy^2 + Tdxdy \;;$$

but the quantities *P*, *Q*, *R*, *S* and *T* thus will depend in turn on each other, so that by the method used in the preceding chapter there shall be specified

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right),\ R = \left(\frac{dP}{dx}\right),\ S = \left(\frac{dQ}{dy}\right) \text{ and } T = 2\left(\frac{dQ}{dx}\right) = 2\left(\frac{dP}{dy}\right);$$

of which conditions, if either one may be absent, certainly we are able to affirm that the formula proposed not be the second differential of any function. Therefore at once it can be discerned, whether a formula of this kind shall be the second differential of some quantity or not.

**249.** In a similar manner the differentials of the third and following orders may be found, which it shall be more expedient to show by particular examples than by using general formulas.
Therefore let $V = xy$ ; there will be

$$\begin{aligned} dV &= ydx + xdy \\ ddV &= yddx + 2dxdy + xddy \\ d^3V &= yd^3x + 3dyddx + 3ddydx + xd^3y \\ d^4V &= yd^4x + 4dyd^3x + 6ddxddy + 4dxd^3y + xd^4y \end{aligned}$$

etc.,

in which example the coefficients follow the rule of binomial numerical powers, and thence they are able to continue as far as desired.

But if there should be $V = \frac{y}{x}$, there will be

$$\begin{aligned} dV &= \frac{dy}{x} - \frac{ydx}{xx} \\ ddV &= \frac{ddy}{x} - \frac{2dxdy}{xx} + \frac{2ydx^2}{x^3} - \frac{yddx}{x^2} \\ d^3V &= \frac{d^3y}{x} - \frac{3dxddy}{xx} + \frac{6dx^2dy}{x^3} - \frac{3dyddx}{x^2} + \frac{6ydxddx}{x^3} - \frac{6ydx^3}{x^4} - \frac{yd^3x}{x^2} \end{aligned}$$

etc.,

In which example the progression of the differentials may not be so easily apparent as in the preceding.

**250.** Nor indeed is this method of differentiation restricted only to finite functions, but also by the same procedure the differential can be found for any expression which now may contain differentials within itself; either a certain single differential may be assumed constant, or none. For since both the individual differentials may be differentiated equally by the same rule as finite quantities, the rules treated in the preceding chapters also prevail here and must be observed. Therefore *V* may specify that expression, which is required to be differentiated, whether it shall be finite, or infinitely large or small ; and an account of the differentiation is observed from these examples.

1. Let $V=\sqrt{\left(dx^2+dy^2\right)}$ ; there will be

$$dV=\frac{dxddx+dyddy}{\sqrt{\left(dx^2+dy^2\right)}}.$$

II. Let $V=\frac{ydx}{dy}$ ; there will be

$$dV=dx+\frac{yddx}{dy}-\frac{ydxddy}{dy^2}.$$

III Let $V=\frac{\left(dx^2+dy^2\right)^{\frac{3}{2}}}{dxddy-dyddx}$ ; there will be

$$dV=\frac{\left(3dxddx+3dyddy\right)\sqrt{\left(dx^2+dy^2\right)}}{dxddy-dyddx}-\frac{\left(dx^2+dy^2\right)^{\frac{3}{2}}\left(dxd^3y-dyd^3x\right)}{\left(dxddy-dyddx\right)^2}.$$

Which differentials, since they shall be the most general with no differential taken considered constant, hence these differentials will be easy to derive, which arise, if either *dx* or *dy* may be considered constant.

**251.** Because also with no constant differentials assumed no law will be prescribed, following which the successive values of the variables may progress, the second differentials and of the following orders will not be determined nor will they have any clear meaning [*i.e*. the differentials may lack meaning or purpose, as later ones to be examined will have; Euler talks of these as being 'vague']. Hence a formula, in which the second and higher order differentials may be contained, will have no determined value, unless a certain differential shall be assumed constant ; but the meaning of this may be vague and it will be varied, as one after another differential is made constant. Yet meanwhile expressions containing second order differentials are given also, which, even if no differential put in place shall be constant, yet they include a meaningful determination, which may remain the same always, whichever differential may be decided to be constant. But we will examine the nature of formulas of this kind with more care below and we will explain the manner in which these are to be distinguished from others, which determine values not to be included.

**252.** So that this account of the formulas, in which the second or higher order differentials are present, may be understood more easily, we may consider in the first place formulas containing a single variable, and it shall be readily apparent that, if in a certain formula the second differential *ddx* of this variable *x* may be present and no constant differential may be in place, then no formula can have a fixed value. If indeed the differential of *x* may be placed constant, there becomes $ddx = 0$; but if the differential of *xx* or *2xdx* or *xdx* may be considered constant, since the differential of *xdx* , $xddx + dx^2$, shall be = 0, there will be made $ddx = \frac{-dx^2}{x}$. Indeed if the differential of any power $x^n$, $nx^{n-1}dx$ or $x^{n-1}dx$ must be considered constant, the second differential of this will be :

$$x^{n-1}ddx + (n-1)x^{n-2}dx^2 = 0 \text{ and thus } ddx = -\tfrac{(n-1)dx^2}{x}.$$

Other values for *ddx* will be produced, if constant differentials of other functions of *x* may be put in place. But it is evident that the formula, in which *ddx* may occur, adopt the most diverse values, as in place of *ddx* there may be written [on selecting $n = 1$ or $2$ ,] either 0, $-\frac{dx^2}{x}$, or $-\frac{(n-1)dx^2}{x}$, or by another expression of this kind. Evidently if there is put in place the formula $\frac{xxddx}{dx^2}$, which must have a finite value on account of the infinitely small homogeneous terms *ddx* and $dx^2$, which on putting *dx* constant will become 0; if there shall be $d.x^2$ constant, that will change into $-x$; if there shall be constant $d.x^3$, that will change into $-2x$ ; if $d.x^4$ shall be constant, that will change into $-3x$ and thus henceforth. Therefore nor is it able to have a determined value, unless a constant differential may be defined, and a constant differential of this kind shall be assumed.

**253.** That inconstancy of meaning is shown in the similar manner, if a differential of the third order be present in a certain formula. We consider this formula $\frac{x^3d^3x}{dxddx}$ which equally brings forward a finite value. If the differential *dx* shall be constant, that formula will turn into $\frac{0}{0}$, the value of which will soon become apparent. Let $d.x^2$ be constant ; there will be $ddx = -\frac{dx^2}{x}$ and on differentiating anew, $d^3x = -\frac{2dxddx}{x} + \frac{dx^3}{x^2} = \frac{3d^3x}{x^2}$, on account of $ddx = -\frac{dx^2}{x}$; therefore in this case the proposed formula $\frac{x^3d^3x}{dxddx}$ will change into $-3x^2$. But if $d.x^n$ were constant, there will be $ddx = -\frac{(n-1)dx^2}{x}$ and hence

$$d^3x = -\frac{2(n-1)dxddx}{x} + \frac{(n-1)dx^3}{x^2} = \frac{2(n-1)^2dx^3}{xx} + \frac{(n-1)dx^3}{xx} = \frac{(2n-1)(n-1)dx^3}{xx}.$$

Therefore in this case there shall be

$$\frac{d^3x}{ddx} = -\frac{(2n-1)dx}{x} \text{ and } \frac{x^3d^3x}{dxddx} = -(2n-1)x^2,$$

from which it may be apparent, if $n=1$ or $dx$ constant, the value of the formula becomes $=-x^2$. From which it is evident that if the third or higher orders occur in a certain differential formula, neither may it be indicated likewise that a differential of this kind be assumed constant, as that formula has no certain value and thus signifies completely nothing; on account of which such expressions cannot occur in the calculation.

**254.** In a similar manner if a formula contains two or more variables and in that the differentials of second or higher orders occur, it may be understood that it is not possible to determine a value, unless a certain differential may be established constant, only with these cases excepted, which now we may consider carefully. For in the first place with *ddx* present in a certain formula, because for the various differentials which may be put constant, the value of *ddx* itself may change constantly, and it cannot happen that the formula in place may obtain a value ; and the same prevails with this concerning any differential of higher order of *x*, and also concerning the differentials of the remaining variables of second and of higher orders. But if second differentials of two or more variables may be present, it can happen that the inconstancy arising from one may be cancelled by the inconstancy arising from the rest; and hence that case arises which we bear in mind, so that a formula of this kind involving second differentials of two or more variables can be defined without an obstacle, because no constant differential shall be put in place.

**255.** Therefore this formula

$$\frac{yddx+xddy}{dxdy}$$

cannot have an appointed and fixed meaning, unless some differential first may be made constant. For if *dx* be put constant, there will be had $\frac{xddy}{dxdy}$, but if *dy* be put constant, there will be had $\frac{yddx}{dxdy}$; but it is evident these formulas by necessity are not equal to each other. For if by necessity they should be equal, they must remain such, whatever function of *x* may be substituted in place of *y* . We may put such to be $y = xx$, and since on putting *dx* constant there shall be $ddy = 2dx^2$, the formula $\frac{xddy}{dxdy}$ will change into 1; but if *dy* or $2xdx$ may be put constant, there is made $ddy = 2xddx + 2dx^2 = 0$ and thus $ddx = -\frac{dx^2}{x}$ from which the formula $\frac{yddx}{dxdy}$ will change into $-\frac{1}{2}$. Therefore since in one case a discrepancy may be found, much less in general will $\frac{xddy}{dxdy}$, on putting *dx* constant, be equal to $\frac{yddx}{dxdy}$ on putting *dy* constant. Then because the formula $\frac{yddx+xddy}{dxdy}$ itself is not agreed on, while either $dx$ or $dy$ may be put constant, much less will it be in agreement itself, if a constant differential of any function of $x$ or of $y$ or of each is put in place.

**256.** Hence it appears that a formula of this kind cannot have an appointed value, unless it shall be prepared thus, so that some functions of $x$ were substituted in place of the variables $y$, $z$ etc., which are present besides $x$, and disappear completely from the calculation after the second and higher differentials of $x$, surely $ddx$, $d^3x$ etc. If indeed after some such substitution at this point $ddx$, or $d^3x$, or $d^4x$ etc. may be left in the formula, because these differentials, as other and still other constants are assumed, change their meaning, the value of the formula also will be uncertain. Thus the proposed formula has been put in place before: $\frac{yddx+xddy}{dxdy}$; which if it should have an appointed value, whatever $y$ may signify, should also have an appointed value, if $y$ may denote some value of $x$. But if we may put $y = x$ only, the formula will change into $\frac{2xddy}{dx^2}$, which certainly on account of $ddx$ is uncertain in that and adopts other and yet other values, as other and still other constant differentials are put in place, as is evident enough from § 252.

**257.** But here doubt may arise later, whether such formulas may be given containing two or more differentials of the second or higher orders, which enjoy this property so that, if in place of the remaining variables some functions of one may be substituted, the differentials of the second order may completely cancel each other. We come across this doubt in the first place, so that we may propose formulas of this kind, which shall be endowed with this property, so that the strength of the proposal shall be better understood by exploration. Therefore I say that this formula,

$$\frac{dyddx-dxddy}{dx^3},$$

possesses the property mentioned ; for whatever the function of $x$ substituted in place of $y$, the differentials of the second order always vanish completely ; as we may demonstrate in the following examples.

I. Let $y = x^2$ ; there will be

$$dy = 2xdx \text{ and } ddy = 2xddx + 2dx^2,$$

which values substituted into the formula $\frac{dyddx-dxddy}{dx^3}$ will give

$$\frac{2xdxddx-2xdxddx-2dx^3}{dx^3} = -2.$$

II. If there shall be $y = x^n$ ; there will be

$$dy = nx^{n-1}dx \ \text{ and } \ ddy = n(n-1)x^{n-2}dx^2,$$

which values substituted into the formula $\frac{dyddx-dxddy}{dx^3}$ will be changed into this

$$\frac{nx^{n-1}dxddx-nx^{n-1}dxddx-n(n-1)x^{n-2}dx^3}{dx^3} = -n(n-1)x^{n-2}.$$

III. If $y = -\sqrt{(1-xx)}$ ; there will be

$$dy = \frac{xdx}{\sqrt{(1-xx)}} \ \text{ and } \ ddy = \frac{xddx}{\sqrt{(1-xx)}} + \frac{dx^2}{(1-xx)^{\frac{3}{2}}}$$

and the formula $\frac{dyddx-dxddy}{dx^3}$ into

$$\frac{xddx}{dx^2\sqrt{(1-xx)}} - \frac{xddx}{dx^2\sqrt{(1-xx)}} - \frac{1}{(1-xx)^{\frac{3}{2}}} = \frac{-1}{(1-xx)^{\frac{3}{2}}}.$$

Therefore in all these examples the second order differentials *ddx* carry each other away and this thus may come about, whatever other functions may be substituted in place of $y$.

**258.** Since now the truth of our proposition may have been approved from these examples, because the formula $\frac{dyddx-dxddy}{dx^3}$ may have a fixed value, even if no constant differential shall be assumed, we will be able to establish a demonstration for that more easily. Let *y* be some function of *x* and the differential *dy* will be of this kind, so that there shall be $dy = pdx$, and *p* will be some function of *x* and the differential of this will have a form of this kind $dp = qdx$ and again *q* will be a function of *x*. Therefore since there shall be $dy = pdx$, on differentiating there will be $ddy = pddx + qdx^2$ and

$$dyddx - dxddy = pdxddx - pdxddx - qdx^3 = -qdx^3;$$

in which expression since no second differential shall be present, this will have an established value and $\frac{dyddx-dxddy}{dx^3}$ will be $= -q$. Therefore in whatever manner *y* may depend on *x*, the second differentials in this formula always mutually cancel each other and because of this, which otherwise would be undefined, the value of this may become appointed and established.

**259.** Although here we have put *y* to be a function of $x$, yet the truth remains equally, if *y* may not depend at all on *x* as we have assumed. For while for *y* we have substituted some function, nor may it be of a kind such as we have determined, and we attribute no dependence of $y$ on $x$ . Yet meanwhile a demonstration can be formed without the mention of a function; for if $y$ shall be such a quantity, either depending on $x$ or not, the differential of this $dy$ will be homogeneous with $dx$ and thus $\frac{dy}{dx}$ will denote a finite quantity, the differential of which will be established, because it is taken while $x$ changes into $x+dx$ and $y$ into $y+dy$ , nor will it depend on a law from the second differentials. Therefore let there be $\frac{dy}{dx}=p$ ; there will be $dy=pdx$ and $ddy=pddx+dpdx$ , from which there becomes

$$dxddy-dyddx=dpdx^2,$$

the value of which is not unclear, because it may contain only the first differential; and therefore the same remains, whether a certain constant differential be admitted, of whatever kind that shall be at last, or no constant differential shall be put in place.

**260.** Therefore because $dyddx-dxddy$ has a fixed meaning, with no opposing second differentials, which influences can be thought to cancel each other out, any expression in which no other second order differentials are present besides the formula $dyddx-dxddy$ , equally will have an established meaning. Or if there may be put in place $dyddx-dxddy=\omega$ and $V$ were a quantity composed in some manner from $x, y,$ with the first differentials of those $dx, dy$ and from $\omega$, that will have an established value. Indeed since in the first differentials $dx$ and $dy$ no account may be had of this arbitrary rule, by which the values of $x$ are put to increase successively, in $\omega$ the second differentials cancel out, also the quantity $V$ will itself not be undefined, but have an established value. Thus this expression

$$\frac{\left(dx^2+dy^2\right)^{\frac{3}{2}}}{dxddy-dyddx}$$

will maintain a fixed value, although it may seem to be corrupted by those second differentials, and in addition, because the numerator is homogenous to the denominator, it will maintain a finite value, unless this prevails in the case to become infinitely large or small.

**261.** Just as the formula $dxddy-dyddx$ has been shown to have an established value, thus also, if a third variable $z$ be added, these formulas

$$dxddz-dzddx \quad \text{and} \quad dyddz-dzddy$$

will have established values. Hence the expressions, which involve the three variables $x, y$ and $z$ , if in these no other second differentials occur besides these assigned, then likewise they will be established, and if plainly no [other] second differentials may be present. Thus this expression

$$\frac{\left(dx^2+dy^2+dz^2\right)^{\frac{3}{2}}}{(dx+dz)ddy-(dy+dz)ddx+(dx-dy)ddz}$$

may entertain an established meaning, without opposing second differentials. And in a similar manner formulas can be shown containing more variables, in which the second differentials are not a hindrance, by which the meaning of these would be less established.

**262.** Therefore with the exception of this kind of formula, which include second order differentials, all the remaining will have undefined meanings and therefore cannot have a place in a calculation, unless a certain first differential may be defined, which shall be assumed constant. Now immediately some first differential is assumed constant, all expressions, however many variables they may contain and differentials may be present in those of whatever order after the first, will maintain established meanings, nor are they excluded from a calculation. If indeed for argument's sake *dx* shall be assumed constant, the second and following differentials of *x* itself vanish ; and any functions of *x* may be substituted in place of the remaining variables *y*, *z* etc., the second differentials of these by $dx^2$, the third by $dx^3$ etc. and thus will be determined, and thus uncertainties arising from the second differentials have been removed. The same occurs, if the first differential of another variable or of some function may be put constant.

**263.** From these therefore it follows that the second differential and of higher orders do not in any circumstances actually enter into the calculation and on account of the undefined meaning are to be completely unsuitable for analysis. Indeed when the second differentials may be seen to be present, either some first differential is assumed to be constant or zero. In the first case the second differentials vanish completely from the calculation , while they may be determined by the first differentials. But in the latter case, unless they mutually cancel each other, the meaning will be indeterminate and therefore cannot be used in analysis ; but if they mutually cancel, only apparently are they present, and actually only finite quantities with their first differentials are required to be agreed to be present. Yet because still most often they are only apparently used in a calculation, it would be by necessity, that a method of treating these should be explained. But soon we will show the manner, with the help of which differentials of the second and higher orders are able to be dismissed always.

**264.** If an expression may contain the single variable *x* and the higher differentials $ddx,\ d^3x,\ d^4x$ etc. may occur in this, this cannot have a fixed meaning, unless a certain first differential shall be put in place. Therefore let *t* be that variable quantity, the differential of which *dt* shall be placed constant, thus so that there shall be $ddt=0,\ d^3t=0,\ d^4t=0$ etc. There may be put $dx=pdt$ and $p$ will be a finite quantity, the differential of which may not be affected by a differential of the second order with an undefined meaning, and hence also $\frac{dp}{dt}$ will be a finite quantity. Let $dp = qdt$ and in a similar manner further $dq = rdt$, $dr = sdt$ etc.; *q, r, s* etc. will be finite quantities having defined meanings. Therefore since there shall be $dx = pdt$, there will be

$$ddx = dpdt = qdt^2,\ \ d^3x = dqdt^2 = rdt^3,\ d^4x = dr\,d\,t^3 = sdt^4 \text{ etc.};$$

which values if they may be substituted in place of $ddx,\ d^3x,\ d^4x$ etc. , the whole expression will contain separated finite quantities with the first differential *dt* and thus no more will it be undefined without meaning.

**265.** If *x* shall be a function of *t*, in this manner the quantity *x* will be able to be eliminated completely, thus so that only the quantity *t* with its own differential *dt* may remain in expressions; but if *t* were a function of *x*, in turn also *x* will be a function of *t*. Yet meanwhile the quantity *x* itself with its first differential *dx* can be retained in the calculation, provided after the substitutions made before everywhere in place of *t* and *dt* the values of these may be restored to be expressed by *x* and *dx*. So that which may become plainer, we may put *t* to be $= x^n$, thus so that the first differential of $x^n$ shall be put constant. Therefore because there is $dt = nx^{n-1}dx$, there will be

$$p = \frac{1}{nx^{n-1}} \text{ et } dp = \frac{-(n-1)dx}{nx^n} = qdt = nqx^{n-1}dx;$$

from which there will become

$$q = \frac{-(n-1)}{nnx^{2n-1}} \text{ and } dq = \frac{(n-1)(2n-1)dx}{nnx^{2n}} = rdt = nrx^{n-1}dx.$$

Hence again there shall become

$$r = \frac{(n-1)(2n-1)}{n^3x^{3n-1}} \text{ and } s = -\frac{(n-1)(2n-1)(3n-1)}{n^4x^{4n-1}}.$$

Whereby if the differential of $x^n$ may be put constant, there will be

$$ddx = -\frac{(n-1)dx^2}{x}$$
$$d^3x = \frac{(n-1)(2n-1)dx^3}{xx}$$
$$d^4x = -\frac{(n-1)(2n-1)(3n-1)dx^4}{x^3}$$

etc.

**266.** If the expression may contain two variables *x* and *y* and the differential of one of these *x* shall be constant, on account of $ddx = 0$ the second and higher differentials will not be present besides $ddy$, $d^3y$ etc. But these in the same manner, by which before we have used, can be removed on putting

$$dy = pdx,\ dp = qdx,\ dq = rdx,\ dr = sdx \quad \text{etc.}$$

for there is made

$$ddy = qdx^2,\ d^3y = rdx^3,\ d^4y = sdx^4 \quad \text{etc.}$$

from which substituted an expression may arise, which besides the finite quantities *x*, *y*, *p*, *q*, *r*, *s* etc., will only contain the first differential *dx*. Thus if this expression were proposed

$$\frac{ydx^4 + xdyd^3y + xd^4y}{(xx+yy)ddy},$$

In which *dx* had been taken constant, there may be put

$$dy = pdx,\ dp = qdx,\ dq = rdx \ \text{ and } \ dr = sdx \quad \text{etc.,}$$

with which values substituted the expression proposed will be changed into this :

$$\frac{(y+xpr+xs)dx^2}{(xx+yy)q},$$

which contains no further second or higher differentials.

**267.** In a similar manner the second and higher differentials may be removed, if *dy* were assumed constant. Now if some other first differential *dt* may be set up constant, then the first and higher differentials of *x* in the manner indicated before may be removed from the calculation, by putting in place

$$dx = pdt,\ dp = qdt,\ dq = rdt,\ dr = sdt \quad \text{etc.,}$$

from which there comes about

$$ddx = pdt^2,\ d^3x = rdt^3,\ d^4x = sdt^4 \quad \text{etc.,}$$

then in a similar manner, the higher differentials of *y* , on putting

$$dy = Pdt,\ dP = Qdt,\ dQ = Rdt,\ dR = Sdt \quad \text{etc.,}$$

from which there will become,

$$ddy = Qdt^2,\ d^3y = Rdt^3,\ d^4y = Sdt^4 \quad \text{etc.};$$

from which substitutions an expression will be obtained, which besides the finite quantities *x, p, q, r, s* etc., *y, P, Q, R, S* etc. may include only the differential *dt* , nor therefore will it have an undefined meaning.

**268.** If the first differential, which is put constant, may depend on either *x* or *y* or likewise on both, then there is no need, that the twofold series of finite quantities *p, q, r* etc. be introduced. If indeed *dt* depends on *x* only, then the letters *p, q, r* etc. may become functions of *x* and only the letters *P, Q, R* etc. enter ; it comes about likewise, if the constant differential *dt* may depend on *y* only. But if *dt* may depend on both, the operation must be changed a little. For the sake of an example we may put this differential *ydx* assumed to be constant and there will be $yddx + dxdy = 0$, from which there will be

$$ddx = -\frac{dxdy}{y}.$$

Now since

$$dy = pdx,\ dp = qdx,\ dq = rdx \quad \text{etc.}$$

and there will be

$$ddx = -\frac{pdx^2}{y}$$

and by differentiating further

$$d^3x = -\frac{qdx^3}{y} + \frac{ppdx^3}{yy} - \frac{2pdxddx}{y}\ ;$$

here in place of *ddx* the value $-\frac{pdx^2}{y}$ of this may be substituted; there arises

$$d^3x = -\frac{qdx^3}{y} + \frac{3ppdx^3}{yy}$$

and again

$$d^4x = -\frac{rdx^4}{y} + \frac{pqdx^4}{yy} + \frac{6pqdx^4}{yy} - \frac{6p^3dx^4}{y^3} + \left(\frac{3pp}{yy} - \frac{q}{y}\right)3dx^2ddx$$

and on substituting the value $-\frac{pdx^2}{y}$ for *ddx* there comes about

$$d^4x = \left(\frac{-r}{y} + \frac{10pq}{yy} - \frac{15p^3}{y^3}\right)dx^4$$

etc. Then since there shall be $dy = pdx$, there will be

$$ddy = qdx^2 + pddx = \left(q - \frac{pp}{y}\right)dx^2$$

and by continually substituting the value $-\frac{pdx^2}{y}$ for *ddx* there will come about

$$d^3y = \left(r - \frac{4pq}{y} + \frac{3p^3}{yy}\right)dx^3$$

and

$$d^4y = \left(s - \frac{7pr}{y} - \frac{4qq}{y} + \frac{25ppq}{yy} - \frac{15p^4}{y^3}\right)dx^4$$

etc. Which values substituted in place of the higher differentials of $x$ and $y$ will change the proposed expression into a form of this kind, which will contain no further higher differentials and hence by a consideration of some kind it will be deprived of constant differentials. For with this transformation made because the second differentials are not present, it may be kept in mind lest indeed there is the need, that such a differential assumed may be constant.

**269.** But most often in a calculation it is accustomed to come upon an application to curved lines, so that this first differential $\sqrt{\left(dx^2 + dy^2\right)}$ may be assumed constant ; whereby we may show in what way in this case the second and higher differentials must be eliminated. Thus indeed likewise the way will be apparent for resolving the same matter, if some other differential shall be assumed constant. Again there is put

$$dy = pdx,\ dp = qdx,\ dq = rdx,\ dr = sdx \ \text{ etc}.$$

and the differential $\sqrt{\left(dx^2 + dy^2\right)}$ may adopt this form $dx\sqrt{(1 + pp)}$ *;* which since it shall be constant, there arises

$$ddx\sqrt{(1 + pp)} + \frac{pqdx^2}{\sqrt{(1+pp)}} = 0$$

and thus

$$ddx = -\frac{pqdx^2}{1+pp},$$

from which now the value of *ddx* itself will be had ; hence again there will be

$$d^3x = -\frac{prdx^3}{1+pp} - \frac{qqdx^3}{1+pp} + \frac{2ppqqdx^3}{(1+pp)^2} - \frac{2pqdxddx}{1+pp}$$

$$= -\frac{prdx^3}{1+pp} - \frac{qqdx^3}{1+pp} + \frac{4ppqqdx^3}{(1+pp)^2} = -\frac{prdx^3}{1+pp} + \frac{(3pp-1)qqdx^3}{(1+pp)^2}.$$

Then there is made

$$d^4x = -\frac{psdx^4}{1+pp} + \frac{(10pp-3)qrdx^4}{(1+pp)^2} - \frac{(15pp-13)pq^3dx^4}{(1+pp)^3}.$$

But because we have assumed $dy = pdx$, there becomes on differentiation

$$ddy = qdx^2 + pddx = qdx^2 - \frac{ppqdx^2}{1+pp} = \frac{qdx^2}{1+pp}$$

$$d^3y = \frac{rdx^3}{1+pp} - \frac{2pqqdx^3}{(1+pp)^2} + \frac{2qdxddx}{1+pp}$$

and thus

$$d^3y = \frac{rdx^3}{1+pp} - \frac{4pqqdx^3}{(1+pp)^2}$$

and again on differentiation

$$d^4y = \frac{sdx^4}{1+pp} - \frac{13pqrdx^4}{(1+pp)^2} + \frac{4(6pp-1)q^3dx^4}{(1+pp)^3}.$$

Therefore all the higher differentials of each of the variables *x* and *y* and the powers of *dx* may be expressed by finite quantities, and after making these substitutions, an expression will result absolutely free of second differentials.

**270.** Therefore with the manner explained of setting aside second and higher differentials, it may be appropriate for this matter to be illustrated by some examples.

I. Let this expression be proposed $\frac{xddy}{dx^2}$, in which *dx* is put constant.

Therefore there is put $dy = pdx$ and $dp = qdx$, on account of $ddy = qdx^2$ the proposed expression will change into this finite form *xq*.

II. Let this expression be proposed $\frac{dx^2+dy^2}{ddx}$, in which *dy* shall be put constant.

There is put $dx = pdy$, $dp = qdy$; on account of $ddx = qdy^2$ there will arise $\frac{1+pp}{q}$. But if as before we may wish $dy = pdx$, $dp = qdx$, on account of constant *dy* there will be

$0 = pddx + dpdx$ and $ddx = -\frac{qdx^2}{p}$ from which the proposed expression will change into $\frac{-p(1+pp)}{q}$.

III. Let this expression be proposed $\frac{ydddx - dxddy}{dxdy}$, in which $ydx$ shall be put constant. There is put $dy = pdx$ and $dp = qdx$ and from § 268 there will be $ddx = -\frac{pdx^2}{y}$, $ddy = qdx^2 - \frac{ppdx^2}{y}$, with which substituted the proposed expression is changed into this : $-1 - \frac{xq}{p} + \frac{xp}{y}$.

IV. Let this expression be proposed $\frac{dx^2 + dy^2}{ddy}$, in which $\sqrt{\left(dx^2 + dy^2\right)}$ is put constant. Again there is put $dy = pdx$, $dp = qdx$ and from the preceding paragraph there will be $ddy = \frac{qdx^2}{1+pp}$ ; from which the proposed expression will change into $\frac{(1+pp)^2}{q}$.

Moreover from these examples it is understood well enough, how in whatever case presented, whatever first differential shall be assumed constant, the second and higher differentials may be eliminated.

**271.** Therefore since in this manner with the finite quantities *p, q, r,* s etc. introduced the second and higher differentials thus are able to be eliminated, so that the whole expression besides the finite quantities *x, y, p, q, r,* s etc. may include only the differential *dx*, in turn if a reduced expression of this kind may be proposed, that again will be transformed into the first form, in place of the letters *p, q, r, s* etc. with second and higher differentials introduced. But now likewise there will be, whatever first differential may be assumed constant, and either that itself, which was assumed before can be put in place constant, or some other. Also why not assume that no differential at all can be constant and in this manner expressions will be produced containing second or higher order differentials, which, even if no constant differential shall be assumed, yet they may maintain a definite meaning, the expressions of this kind we have shown to be given above.

**272.** Therefore let there be a proposed expression containing the finite letters *x, y, p, q, r* etc. together with the differential *dx*, in which there shall be

$$p = \frac{dy}{dx}, \quad q = \frac{dp}{dx}, \quad r = \frac{dq}{dx} \quad \text{etc.}$$

For if we should wish to eliminate these letters *p, q, r* etc., so that in place of these we may introduce second and higher differentials with no differential of *x* and *y* assumed constant, there comes about

$$dp = \frac{dxddy - dyddx}{dx^2} \quad \text{and hence} \quad q = \frac{dxddy - dyddx}{dx^3}$$

which formula differentiated will give

$$dq = \frac{dx^2 d^3y - 3dxddxddy + 3dyddx^2 - dxdyd^3x}{dx^4},$$

from which there will be made

$$r = \frac{dx^2 d^3y - 3dxddxddy \ + 3dyddx^2 - dxdyd^3x}{dx^5}.$$

But if in addition letter *s* shall be present, which denotes the value $\frac{dr}{dx}$, for that to be substituted there will be this value

$$s = \frac{dx^3 d^4y - 6dx^2 ddxd^3y - 4dx^2 ddyd^3x \ + 15dxddx^2 ddy + 10dxdyddxd^3x - 15dyddx^3 - dx^2 dyd^4x}{dx^7}.$$

Therefore with these values substituted in place of the quantities *p, q, r, s* etc. the proposed expression will be changed into another higher differential containing *x* and *y* themselves, which, even if no first differential shall be assumed constant, yet it is not undefined, but has an established meaning.

**273.** Therefore in this manner any formula of a higher grade differential, in which some first order differential is assumed constant, can be transformed into another form, in which no differential is put constant, which may have the same established value without this opposing. Clearly in the first place with the aid of the method treated before with the values assumed $dy = pdx,\ dp = qdx,\ dy = rdx, dr = sdx$ etc. the higher differentials may be eliminated ; then in place of *p, q, r, s* etc. the values now found may be substituted and an expression may arise equal to the first involving no constant differential ; as the examples in the following transformation will illustrate.

I. Let this expression be proposed $\frac{xddy}{dx^2}$, in which *dx* may be placed constant, which must be transformed into another form involving no constant differential. There may be put $dy = pdx,\ dp = qdx$ and, as we have seen before (§ 270) , the proposed expression will be changed into this *qx*. Now in place of *q* the value may be substituted, which it maintains with no differential assumed constant, $q = \frac{dxddy - dyddx}{dx^3}$ and this expression may be found

$$\frac{xdxddy - ddyddx}{dx^3}$$

equal to the proposed, and no longer involving a constant differential.

II. Let this expression be proposed $\frac{dx^2+dy^2}{ddx}$, in which *dy* is assumed constant. There may be put $dy = pdx$ and $dp = qdx$ and that will be transformed into this $-\frac{p(1+pp)}{q}$ [§ 270]; now there may be put in place $p = \frac{dy}{dx}$ and $q = \frac{dxddy-dyddx}{dx^3}$, and there may be found

$$\frac{dy\left(dx^2+dy^2\right)}{dyddx-dxddy},$$

which, with no constant differential assumed, has the same established value which was proposed.

III. Let this expression be proposed $\frac{yddx-xddy}{dxdy}$, in which the differential *ydx* is assumed constant. There may be put $dy = pdx$, $dp = qdx$ and, as we have seen above (§ 270), this expression may be transformed into this : $-1-\frac{xq}{p}+\frac{xp}{y}$, which with no differential assumed constant will be transformed into that :

$$-1-\frac{xdxddy-xdyddx}{dx^2dy}+\frac{xdy}{ydx}=\frac{xdxdy^2-ydx^2dy-yxdxddy+yxdyddx}{ydx^2dy}$$

IV. Let this expression be proposed $\frac{dx^2+dy^2}{ddx}$, in which the differential $\sqrt{\left(dx^2+dy^2\right)}$ is assumed constant. On putting $dy = pdx$ and $dp = qdx$ this expression may arise $\frac{(1+pp)^2}{q}$ (loc. cit.). Now there may be put $p = \frac{dy}{dx}$ and $q = \frac{dxddy-dyddx}{dx^3}$ and with no constant differential assumed we will obtain that expression :

$$\frac{\left(dx^2+dy^2\right)^2}{dx^2ddy-dxdyddx}$$

equivalent to the proposed.

V. Let this expression be proposed $\frac{dxd^3y}{ddy}$, in which the differential *dx* shall be assumed constant. There may be put $dy = pdx$, $dp = qdx$ and $dq = rdx$ and on account of $ddy = qdx^2$ and $d^3y = rdx^3$ the proposed formula will change into this $\frac{rdx^2}{q}$. Now in place of $q$ and $r$ the values may be substituted, which they recover with no constant differential assumed, evidently

$$q=\frac{dxddy-dyddx}{dx^3} \text{ and } r=\frac{dx^2d^3y-3dxddxddy+3dyddx^2-dxdyd^3x}{dx^5}$$

and the following expression will be obtained equivalent to the proposed :

$$\frac{dx^2d^3y-3dxddxddy+3dyddx^2-dxdyd^3x}{dxddy-dyddx}=\frac{dx\left(dxd^3y-dyd^3x\right)}{dxddy-dyddx}-3ddx.$$

**274.** If we may consider these transformations more carefully, we will be able to deduce a method of perfecting these more easily, thus so that there shall be no need to introduce the letters *p*, *q*, *r* etc. Moreover various ways of resolving this need occur, as another and yet another differential should be assumed constant in the proposed formula. In the first place in the proposed formula the differential $dx$ is assumed to be constant, and because in place of $dy$ we have put $pdx$ and again $\frac{dy}{dx}$ in place of *p*, the first differentials $dx$ and $dy$, where ever they occur in the expression, may be left without alteration. But where $ddy$ has occurred, because in place of this there will be written $qdx^2$ and in place of $q$ the value $\frac{dxddy-dyddx}{dx^3}$, the transformation will be completed, if everywhere in place of $ddy$ at once there may be put $\frac{dxddy-dyddx}{dx}$ or $ddy-\frac{dyddx}{dx}$. If in addition in the proposed expression there may occur $d^3y$, which in place of this there is put $rdx^3$, on account of the value of *r* found before everywhere in place of $d^3y$ there will have to be written

$$d^3y-\frac{3ddxddy}{dx}+\frac{3dyddx^2}{dx^2}-\frac{dyd^3x}{dx},$$

with which made the proposed expression will be changed into another, which will involve no constant differential. Thus if this expression may be proposed $\frac{\left(dx^2+dy^2\right)^{\frac{3}{2}}}{dxddy}$, in which $dx$ has been put constant, to that the equal will be by putting $ddy-\frac{dyddx}{dx}$ in place of $ddy$, this involving no constant differential

$$\frac{\left(dx^2+dy^2\right)^{\frac{3}{2}}}{dxddy-dyddx}.$$

**275.** Hence it is easily deduced, if in some proposed expression the differential $dy$ were assumed constant, then everywhere in place of $ddx$ there must be written $ddx-\frac{dxddy}{dy}$ and with this in place of $d^3x$,

$$d^3x-\frac{3ddxddy}{dy}+\frac{3dxddy^2}{dy^2}-\frac{dxd^3y}{dy},$$

so that an equivalent expression may be obtained, in which no differential may be put constant. But if in the proposed expression there were assumed *ydx* constant, because there will become [§ 268]

$$ddx = -\frac{pdx^2}{y} \text{ and } ddy = qdx^2 - \frac{ppdx^2}{y},$$

in place of *ddx* there will be written everywhere $-\frac{dxdy}{y}$ and in place of *ddy* everywhere $ddy - \frac{dyddx}{dx} - \frac{dy^2}{y}$; I will not progress to higher differentials, because in this matter they are accustomed to occur very rarely. But if indeed in the proposed expression this differential $\sqrt{\left(dx^2 + dy^2\right)}$ were assumed constant, because we have found [§ 269]

$$ddx = -\frac{pqdx^2}{1+pp} \text{ and } ddy = \frac{qdx^2}{1+pp}$$

for *ddx* everywhere there ought to be written $\frac{dy^2 ddx - dxdyddy}{dx^2+dy^2}$ and in place of *ddy* everywhere $\frac{dx^2 ddy - dxdyddx}{dx^2+dy^2}$. Thus if the expression were proposed $\frac{dy\sqrt{\left(dx^2+dy^2\right)}}{ddx}$, in which $\sqrt{\left(dx^2 + dy^2\right)}$ may be assumed constant, that will be transformed into this

$$\frac{\left(dx^2+dy^2\right)^{\frac{3}{2}}}{dyddx-dxddy}$$

In which no constant differential is assumed.

**276.** So that these reductions can be applied in use more easily, it has been considered to include these in the following table. Therefore the formula of a differential of a higher order will be transformed into another involving no constant differential with the aid of the following substitution,

1. If the differential *dx* were assumed constant,

| in place of | there may be written |
|---|---|
| $ddy$ | $ddy - \frac{dyddx}{dx}$ |
| $d^3y$ | $d^3y - \frac{3ddxddy}{dx} + \frac{3dyddx^2}{dx^2} - \frac{dyd^3x}{dx}$ |

II. If the differential *dy* were assumed constant,

| in place of | there may be written |
|---|---|
| $ddx$ | $ddx - \frac{dxddy}{dy}$ |
| $d^3x$ | $d^3x - \frac{3ddxddy}{dy} + \frac{3dxddy^2}{dy^2} - \frac{dxd^3y}{dy}$ |

III. If the differential *ydx* were assumed constant,

| in place of | there may be written |
|---|---|
| $ddx$ | $-\frac{dxdy}{y}$ |
| $ddy$ | $ddy - \frac{dyddx}{dx} - \frac{dy^2}{y}$ |
| $d^3x$ | $\frac{dyddx}{y} - \frac{dxddy}{y} + \frac{3dxdy^2}{yy}$ |
| $d^3y$ | $d^3y - \frac{3ddxddy}{dx} + \frac{3dyddx^2}{dx^2} - \frac{dyd^3x}{dx} - \frac{4dyddy}{y} + \frac{4dy^2ddx}{ydx} + \frac{3dy^3}{yy}$ |

IV. If the differential $\sqrt{\left(dx^2 + dy^2\right)}$ were assumed constant,

| in place of | there may be written |
|---|---|
| $ddx$ | $\frac{dy^2ddx - dxdyddy}{dx^2 + dy^2}$ |
| $ddy$ | $\frac{dx^2ddy - dxdyddx}{dx^2 + dy^2}$ |
| $d^3x$ | $\frac{dy^2d^3x - dxdyd^3y}{dx^2 + dy^2} + \frac{\left(dxddy - dyddx\right)\left(3dy^2ddy - dx^2ddy + 4dxdyddx\right)}{\left(dx^2 + dy^2\right)^2}$ |
| $d^3y$ | $\frac{dx^2d^3y - dxdyd^3x}{dx^2 + dy^2} + \frac{\left(dyddx - dxddy\right)\left(3dx^2ddx - dy^2ddx + 4dxdyddy\right)}{\left(dx^2 + dy^2\right)^2}.$ |

**277.** Therefore these expressions, which include no constant differential, thus will be prepared, so that any differential can be assumed constant as it pleases. And hence the differential expressions of higher grades, in which no constant differential is assumed present, are able to be considered, whether or not the meaning of these shall be defined. Indeed there may be put constant some differential as it pleases, for example *dx*, then by the rule of the preceding paragraph the expression may be reduced again to the previous form, in which no differential shall be assumed constant; which if it may agree with the proposed, that will be established, nor will it depend on inconsistencies of the second differentials ; but if the expression may emerge different, then the proposed [expression] may have no definite meaning. Thus if this expression $yddx - xddy$ is put in place in which no differential shall be constant, requiring to be investigated, whether it shall have an established meaning or not, *dx* may be put constant and that will change into $-xddy$ ; now by the preceding rule there may be put $ddy-\frac{dyddx}{dx}$ in place of *ddy* and there will be produced $-xddy+\frac{xdyddx}{dx}$, of which a discrepancy from the proposed form may indicate that the proposed expression may not have an established and appointed meaning.

**278.** In a similar manner if there may be proposed an expression of this general kind

$$Pddx+Qdxdy+Rddy\ ,$$

a condition will be able to be defined, under which that may have a defined value with no differential assumed constant. For *dx* may be put constant and the proposed expression will change into this $Qdxdy+Rddy$ ; now this may be transformed again into another form, so that the meaning of this may remain the same, even if no constant differential may be imagined, and thus there will be produced $Qdxdy+Rddy-\frac{Rdyddx}{dx}$, which form will agree with the proposed, if there should be $Pdx+Rdy=0$; and with this case only the value will be defined. Indeed if there should not be $P=-\frac{Rdy}{dx}$ or $R=-\frac{Pdx}{dy}$, then the proposed expression $Pddx+Qdxdy+Rddy$ will not have a defined value, but the meaning of this shall be vague and differing, according as one or another constant differential is assumed.

**279.** Also from these principles a differential expression, in which a certain differential has been put constant, will be easy to transform into another form, in which another differential may be assumed constant. For it may be reduced to a form of the same kind, which may involve no constant differential, with which made there may be put in place that other constant differential. Thus if in the proposed expression the differential *dx* shall be assumed constant and that shall be changed into another, which may involve the constant differential *dy*, in the above formulas in place of *ddy* and $d^3y$ on substituting on account of constant *dy* there may be put in place $ddy=0,\ d^3y=0$ and the question may be satisfied, if in place of *ddy* there may be substituted $\frac{-dyddx}{dx}$ and $\frac{3dyddx^2}{dx^2}-\frac{dyd^3x}{dx}$ in place of $d^3y$. In this manner that formula $-\frac{\left(dx^2+dy^2\right)^{\frac{3}{2}}}{dxddy}$, in which *dx* has been placed constant, will be changed into this $\frac{\left(dx^2+dy^2\right)^{\frac{3}{2}}}{dyddx}$, in which *dy* is put constant.

**280.** If the formula in the opposite direction, in which *dy* has been put constant, must be changed into another, in which *dx* shall be constant, then in place of *ddx* there must be substituted $\frac{-dxddy}{dy}$, and this expression $\frac{3dxddy^2}{dy^2}-\frac{dxdd^3y}{dy}$ in place of $d^3x$. In a similar manner if the formula, in which $\sqrt{\left(dx^2+dy^2\right)}$ has been made constant, must be transformed in to another, in which *dx* shall be constant, then in place of *ddx* there may be written $-\frac{dxdyddy}{dx^2+dy^2}$ and $\frac{dx^2ddy}{dx^2+dy^2}$ in place of *ddy*. But if the formula, in which *dx* has been assumed constant, must be transformed into another, in which $\sqrt{\left(dx^2+dy^2\right)}$ shall be constant, because on account of constant $dx^2+dy^2$ there will be made

$$dxddx+dyddy=0 \text{ and } ddx=-\frac{dyddy}{dx},$$

with this value in place of *ddx*, the assumed value for *ddy* will be written

$$ddy+\frac{dy^2ddy}{dx^2}=\frac{\left(dx^2+dy^2\right)ddy}{dx^2}.$$

Thus this $-\frac{\left(dx^2+dy^2\right)^{\frac{3}{2}}}{dxddy}$, in which *dx* is constant, will be transformed into another, in which $\sqrt{\left(dx^2+dy^2\right)}$ is put constant, which shall be $-\frac{dx\sqrt{\left(dx^2+dy^2\right)}}{ddy}$.

**CAPUT VllI**

# DE FORMULARUM DIFFERENTIALIUM ULTERIORI DIFFERENTIATIONE

**242.** Si unica variabilis adsit eiusque differentiale primum constans assumatur, supra iam methodus est tradita differentialia cuiusque gradus inveniendi. Scilicet si functionis cuiusvis differentiale denuo differentietur, oritur eius differentiale secundum hocque iterum differentiatum dat functionis differentiale tertium atque ita porro. Haec vero eadem regula locum quoque habet, sive functio plures involvat variabiles sive unicam tantum, cuius differentiale primum non ponitur constans. Sit igitur $V$ functio quaecunque ipsius $x$ neque vero $dx$ sit constans, sed utcunque variabile, ita ut ipsius $dx$ differentiale sit $=ddx$ huiusque differentiale $=d^3x$ et ita porro, atque investigemus differentialia secundum et sequentia functionis $V$.

**243.** Ponamus differentiale primum functionis $V$ esse $=Pdx$, ubi erit $P$ functio quaepiam ipsius $x$ pendens ab $V$. Si iam functionis $V$ differentiale secundum invenire velimus, eius differentiale primum $Pdx$ denuo differentiari oportet; quod cum sit productum ex duabus quantitatibus variabilibus $P$ et $dx$, quarum illius differentiale sit $dP = pdx$, huius vero $dx$ differentiale $ddx$, per regulam de factoribus datam erit differentiale secundum

$$ddV = Pddx + pdx^2$$

Deinde si ponatur $dp = qdx$, cum differentiale ipsius $dx^2$ sit $2dxddx$, erit iterum differentiando

$$d^3V = Pd^3x + dPddx + 2pdxddx + dpdx^2;$$

iam ob $dP = pdx$ et $dp = qdx$ erit

$$d^3V = Pd^3x + 3pdxddx + qdx^3$$

similique modo ulteriora differentialia invenientur.

**244.** Applicemus haec ad potestates ipsius $x$, quarum singula differentialia investigemus, si $dx$ non ponatur constans.

I. Sit igitur $V = x$; erit

$$dV = dx,\ d^2V = d^2x,\ d^3V = d^3x,\ d^4V = d^4x \text{ etc.}$$

II. Sit $V = x^2$; erit

$$dV = 2xdx$$

et

$$ddV = 2xddx + 2dx^2$$
$$d^3V = 2xd^3x + 6dxddx$$
$$d^4V = 2xd^4x + 8dxd^3x + 6ddx^2$$
$$d^5V = 2xd^5x + 10dxd^4x + 20ddxd^3x$$

etc.

III. Si in genere fuerit $V = x^n$, erit

$$dV = nx^{n-1}dx$$
$$ddV = nx^{n-1}ddx + n(n-1)x^{n-2}dx^2$$
$$d^3V = nx^{n-1}d^3x + 3n(n-1)x^{n-2}dxddx + n(n-1)(n-2)x^{n-3}dx^3$$
$$d^4V = nx^{n-1}d^4x + 4n(n-1)x^{n-2}dxd^3x + 3n(n-1)x^{n-2}ddx^2$$
$$+ 6n(n-1)(n-2)x^{n-3}dx^2ddx + n(n-1)(n-2)(n-3)x^{n-4}dx^4$$

etc.

Si igitur fuerit $dx$ constans ac propterea $ddx = 0$, $d^3x = 0$, $d^4x = 0$ etc., orientur eadem differentialia, quae iam supra pro hac hypothesi sunt inventa.

**245.** Quoniam igitur differentialia cuiusque ordinis ipsius $x$ eadem lege differentiantur qua quantitates finitae, expressiones quaecunque, in quibus praeter quantitatem finitam eius differentialia occurrunt, secundum praecepta supra data differentiari poterunt. Quam operationem, cum nonnumquam occurrat, hic aliquot exemplis illustrabimus.

I. Si fuerit $V = \frac{xddx}{dx^2}$, differentiando prodibit

$$dV = \frac{xd^3x}{dx} + \frac{ddx}{dx} - \frac{2xddx^2}{dx^3}.$$

II. Si fuerit $V = \frac{x}{dx}$ erit

$$dV = 1 - \frac{xddx}{dx^2},$$

ubi nihil impedit, quod pro $V$ quantitatem infinite magnam posuimus.

III. Si fuerit $V = xxl\frac{ddx}{dx^2}$, quia transmutatur $V$ in $xxlddx - 2xxldx$, erit secundum regulas consuetas differentiando

$$dV = 2xdxlddx + \frac{xxd^3x}{ddx} - 4xdxldx - \frac{2xxddx}{dx}.$$

Simili autem modo differentialia altiora ipsius *V* reperientur.

**246.** Si expressio proposita duas variabiles involvat, nempe *x* et *y*, vel unius differentiale ponitur constans vel neutrius; arbitrarium enim est alterutrius differentiale constans assumi, quia ab arbitrio nostro pendet, quemadmodum unius valores successivos crescere statuere velimus. Neque vero utriusque variabilis differentialia simul statui possunt constantia; hoc ipso enim relatio inter variabiles *x* et *y* assumeretur, quae tamen vel nulla est vel incognita ponitur. Si enim, dum *x* aequabiliter crescere ponimus, *y* quoque aequalia incrementa capere statueretur, tum eo ipso indicaretur fore $y = ax + b$ sicque *y* ab *x* penderet, quod tamen assumere non licet. Hanc ob rem vel unius tantum variabilis differentiale constans assumi potest vel nullum. Quodsi autem differentiationes absolvere noverimus nullo differentiali assumto constante, simul quoque differentialia constabunt, si alterutrum differentiale ponatur constans; tantum enim opus est, ut, si *dx* constans statuatur, ubique termini continentes $ddx,\ d^3x,\ d^4x$ etc. deleantur.

**247.** Denotet ergo *V* functionem quamcunque finitam ipsarum *x* et *y* sitque $dV = Pdx + Qdy$. Ad differentiale ipsius *V* secundum inveniendum assumamus utrumque differentiale *dx* et *dy* variabile, et cum *P* et *Q* sint functiones ipsarum *x* et *y*, statuamus

$$dP = pdx + rdy$$
$$dQ = rdx + qdy;$$

supra enim vidimus esse $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right) = r$.

His positis differentietur $dV = Pdx + Qdy$ et reperietur

$$ddV = Pddx + pdx^2 + 2rdxdy + Qddy + qdy^2.$$

Si igitur differentiale *dx* statuatur constans, erit

$$ddV = pdx^2 + 2rdxdy + Qddy + qdy^2;$$

sin autem differentiale *dy* statueretur constans, foret

$$ddV = Pddx + pdx^2 + 2rdxdy + qdy^2.$$

**248.** Si igitur functio quaecunque ipsarum *x* et *y* bis differentietur nullo differentiali posito constante, eius differentiale secundum semper huiusmodi formam habebit

$$ddV = Pddx + Qddy + Rdx^2 + Sdy^2 + Tdxdy\ ;$$

pendebunt autem quantitates $P$, $Q$, $R$, $S$ et $T$ ita a se invicem, ut sit signandi modo capite praecedente adhibito

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right),\ R = \left(\frac{dP}{dx}\right),\ S = \left(\frac{dQ}{dy}\right)\ \text{ et }\ T = 2\left(\frac{dQ}{dx}\right) = 2\left(\frac{dP}{dy}\right);$$

quarum conditionum si vel unica desit, certo affirmare poterimus formulam propositam nullius functionis esse differentiale secundum. Statim ergo dignosci poterit, utrum huiusmodi formula sit cuiuspiam quantitatis differentiale secundum an minus.

**249.** Simili modo differentialia tertia ac sequentia invenientur, quod in exemplo particulari ostendisse magis expediet quam formulas generales adhibendo.
Sit igitur $V = xy$ ; erit

$$dV = ydx + xdy$$
$$ddV = yddx + 2dxdy + xddy$$
$$d^3V = yd^3x + 3dyddx + 3ddydx + xd^3y$$
$$d^4V = yd^4x + 4dyd^3x + 6ddxddy + 4dxd^3y + xd^4y$$
etc.,

in quo exemplo coefficientes numerici legem potestatum binomii sequuntur indeque, quousque libuerit, continuari possunt.

At si fuerit $V = \frac{y}{x}$, erit

$$dV = \frac{dy}{x} - \frac{ydx}{xx}$$
$$ddV = \frac{ddy}{x} - \frac{2dxdy}{xx} + \frac{2ydx^2}{x^3} - \frac{yddx}{x^2}$$
$$d^3V = \frac{d^3y}{x} - \frac{3dxddy}{xx} + \frac{6dx^2dy}{x^3} - \frac{3dyddx}{x^2} + \frac{6ydxddx}{x^3} - \frac{6ydx^3}{x^4} - \frac{yd^3x}{x^2}$$
etc.,

In quo exemplo progressio differentialium non tam facile patet quam in praecedente.

**250.** Neque vero tantum haec differentiandi methodus ad functiones finitas adstringitur, sed etiam eodem negotio cuiusvis expressionis , quae iam differentialia in se continet, differentiale inveniri potest, sive unum quoddam differentiale assumitur constans sive minus. Cum enim singula differentialia aeque et eadem lege differentientur ac quantitates finitae, regulae in praecedentibus capitibus traditae etiam hic valent atque observari debent. Denotet igitur $V$ eam expressionem, quam differentiari oportet, sive sit finita sive infinite magna sive infinite parva; atque ratio differentiationis ex his exemplis perspicitur.

1. Sit $V=\sqrt{\left(dx^2+dy^2\right)}$; erit

$$dV=\frac{dxddx+dyddy}{\sqrt{\left(dx^2+dy^2\right)}}.$$

II. Sit $V=\frac{ydx}{dy}$ ; erit

$$dV=dx+\frac{yddx}{dy}-\frac{ydxddy}{dy^2}.$$

III Sit $V=\frac{\left(dx^2+dy^2\right)^{\frac{3}{2}}}{dxddy-dyddx}$; erit

$$dV=\frac{\left(3dxddx+3dyddy\right)\sqrt{\left(dx^2+dy^2\right)}}{dxddy-dyddx}-\frac{\left(dx^2+dy^2\right)^{\frac{3}{2}}\left(dxd^3y-dyd^3x\right)}{\left(dxddy-dyddx\right)^2}.$$

Quae differentialia cum sint generalissime sumta nullo differentiali pro constante habito, hinc facile ea differentialia derivari poterunt, quae oriuntur, si vel $dx$ vel $dy$ statuatur constans.

**251.** Quia nullo differentiali constante assumto nulla etiam lex, secundum quam successivi variabilium valores progrediantur, praescribitur, differentialia secunda et sequentium ordinum non erunt determinata neque quicquam certi significabunt. Hinc formula, in qua differentialia secunda atque altiora continentur, nullum determinatum habebit valorem, nisi quodpiam differentiale constans sit assumtum; sed eius significatio erit vaga atque variabitur, prouti aliud atque aliud differentiale fuerit constans positum. Interim tamen dantur quoque eiusmodi expressiones differentialia secunda continentes, quae, etiamsi nullum differentiale positum sit constans, tamen significatum determinatum complectuntur, qui perpetuo idem maneat, quodcunque differentiale constans statuatur. Huiusmodi autem formularum naturam infra diligentius scrutabimur modumque trademus eas ab aliis, quae valores determinatos non includunt, dignoscendi.

**252.** Quo haec ratio formularum, in quibus differentialia secunda vel altiora insunt, facilius perspiciatur, contemplemur primum formulas unicam variabilem continentes atque facile patet, si in quapiam formula insit eius variabilis $x$ differentiale secundum $ddx$ nullumque differentiale constans statuatur, formulam nullum valorem fixum habere posse. Si enim statuatur differentiale ipsius $x$ constans, fiet $ddx=0$; sin autem ipsius $xx$ differentiale $2xdx$ seu $xdx$ constans ponatur, cum ipsius $xdx$ differentiale $xddx+dx^2$ sit = 0, fiet $ddx=\frac{-dx^2}{x}$. Verum si potestatis cuiuscunque $x^n$ differentiale $nx^{n-1}dx$ seu $x^{n-1}dx$ debeat esse constans, erit eius differentiale secundum ideoque

$$x^{n-1}ddx+\left(n-1\right)x^{n-2}dx^2=0 \text{ ideoque } ddx=-\frac{(n-1)dx^2}{x}.$$

Alii valores pro *ddx* prodibunt, si aliarum ipsius *x* functionum differentialia constantia ponantur. Manifestum autem est formulam, in qua *ddx* occurrat, diversissimos induere valores, prout loco *ddx* scribatur vel 0 vel $-\frac{dx^2}{x}$ vel $-\frac{(n-1)dx^2}{x}$ vel alia huiusmodi expressio. Scilicet si proponatur formula $\frac{xxddx}{dx^2}$ quae ob *ddx* et $dx^2$ infinite parva homogenea finitum valorem habere deberet, ea posito *dx* constante abit in 0; si sit $d.x^2$ constans, ea abit in - *x;* si sit $d.x^3$ constans, ea abit in $-2x$*;* si $d.x^4$ sit constans, ea abit in $-3x$ et ita porro. Neque ergo determinatum valorem habere potest, nisi definiatur, cuiusmodi differentiale constans sit assumtum.

**253.** Ista inconstantia significationis simili ratione ostenditur, si differentiale tertium in quapiam formula insit. Consideremus hanc formulam $\frac{x^3d^3x}{dxddx}$ quae pariter finitum valorem prae se fert. Si differentiale *dx* sit constans, abit ea in $\frac{0}{0}$, cuius valor mox patebit. Sit $d.x^2$ constans ; erit $ddx=-\frac{dx^2}{x}$ et denuo differentiando $d^3x=-\frac{2dxddx}{x}+\frac{dx^3}{x^2}=\frac{3d^3x}{x^2}$ ob $ddx=-\frac{dx^2}{x}$*;* hoc ergo casu formula proposita $\frac{x^3d^3x}{dxddx}$ abit in $-3x^2$. At si fuerit $d.x^n$ constans, erit $ddx=-\frac{(n-1)dx^2}{x}$ hincque

$$d^3x=-\frac{2(n-1)dxddx}{x}+\frac{(n-1)dx^3}{x^2}=\frac{2(n-1)^2dx^3}{xx}+\frac{(n-1)dx^3}{xx}=\frac{(2n-1)(n-1)dx^3}{xx}.$$

Hoc ergo casu erit

$$\frac{d^3x}{ddx}=-\frac{(2n-1)dx}{x} \quad \text{et} \quad \frac{x^3d^3x}{dxddx}=-(2n-1)x^2,$$

unde patet, si sit $n=1$ seu *dx* constans, valorem formulae fore $=-x^2$. Ex quo manifestum est, si in quapiam formula differentialia tertia vel altiora occurrant neque simul indicetur, cuiusmodi differentiale assumtum sit constans, eam formulam nullum certum valorem habere atque adeo nihil prorsus significare; quamobrem tales expressiones in calculo occurrere non possunt.

**254.** Simili modo si formula contineat duas pluresve variabiles in eaque occurrant differentialia secundi altiorisve gradus, intelligetur valorem determinatum locum habere non posse, nisi differentiale quodpiam constans statuatur, iis tantum exceptis casibus, quos mox perpendemus. Quum primum enim *ddx* in quapiam formula inest, quoniam pro variis differentialibus, quae constantia ponuntur, valor ipsius *ddx* perpetuo variatur, fieri nequit, ut formula statum obtineat valorem; hocque idem valet de quovis differentiali altiori ipsius *x* atque etiam de differentialibus reliquarum variabilium secundis et altioribus. Sin autem duarum pluriumve variabilium differentialia secunda insint, fieri potest, ut inconstantia ab uno oriunda per inconstantiam reliquorum destruatur; hincque nascitur ille casus, cuius meminimus, quo formula huiusmodi differentialia secunda duarum pluriumve variabilium involvens valorem definitum habere potest, non obstante, quod nullum differentiale constans sit positum.

**255.** Haec igitur formula

$$\frac{yddx+xddy}{dxdy}$$

statam atque fixam significationem habere nequit, nisi quodpiam differentiale primum constans statuatur. Nam si $dx$ constans ponatur, habebitur $\frac{xddy}{dxdy}$ sin autem $dy$ constans ponatur, habebitur $\frac{yddx}{dxdy}$; manifestum autem est has formulas non necessario inter se esse aequales. Si enim necessario essent aequales, tales manere deberent, quaecunque functio ipsius $x$ loco $y$ substitueretur. Ponamus tantum esse $y = xx$, et cum posito $dx$ constante sit $ddy = 2dx^2$, formula $\frac{xddy}{dxdy}$ abibit in 1; sin autem $dy$ seu $2xdx$ ponatur constans, fiet $ddy = 2xddx + 2dx^2 = 0$ ideoque $ddx = -\frac{dx^2}{x}$ unde formula $\frac{yddx}{dxdy}$ abit in $-\frac{1}{2}$. Cum igitur in unico casu reperiatur discrepantia, multo minus in genere erit $\frac{xddy}{dxdy}$, positio $dx$ constante, aequalis $\frac{yddx}{dxdy}$ posito $dy$ constante. Deinde quia formula $\frac{yddx+xddy}{dxdy}$ sibi non constat, dummodo vel $dx$ vel $dy$ constans ponatur, multo minus sibi constabit, si functionis cuiusvis vel ipsius $x$ vel ipsius $y$ vel utriusque differentiale constans ponatur.

**256.** Hinc apparet huiusmodi formulam statum valorem habere non posse, nisi ita sit comparata, ut, postquam loco variabilium $y$, $z$ etc., quae praeter $x$ insunt, functiones quaecunque ipsius $x$ fuerint substitutae, differentialia secunda et altiora ipsius $x$, nempe $ddx$, $d^3x$ etc., penitus ex calculo excedant. Si enim post talem substitutionem quamcunque in formula adhuc relinqueretur $ddx$ vel $d^3x$ vel $d^4x$ etc., quia haec differentialia, prout alia aliaque constantia assumuntur, significationem suam variant, valor quoque ipsius formulae erit vagus. Sic comparata est formula ante proposita $\frac{yddx+xddy}{dxdy}$; quae si statum haberet valorem, quicquid $y$ significet, statum quoque habere deberet valorem, si $y$ denotaret functionem quampiam ipsius $x$. At si tantum ponamus $y = x$, formula abit in $\frac{2xddy}{dx^2}$, quae utique ob $ddx$ in ea contentum est vaga atque alios aliosque valores induit, prouti alia atque alia differentialia constantia ponuntur, uti ex § 252 satis est manifestum.

**257.** Dubium autem hic subnascetur, utrum dentur tales formulae duo plurave differentialia secundi altiorisve gradus continentia, quae hac proprietate gaudeant, ut, si loco reliquarum variabilium quaecunque functiones unius substituantur, differentialia secundi gradus prorsus se destruant. Huic dubio primum ita occurramus, ut huiusmodi formulam proponamus, quae ista proprietate sit praedita, quo per explorationem vis quaestionis melius percipiatur. Dico igitur hanc formulam

$$\frac{dyddx-dxddy}{dx^3}$$

memoratam proprietatem possidere; quaecunque enim functio ipsius $x$ loco $y$ substituatur, semper differentialia secundi gradus penitus evanescent; quam proprietatem sequentibus exemplis comprobemus.

I. Sit $y=x^2$ ; erit

$$dy=2xdx \text{ et } ddy=2xddx+2dx^2,$$

qui valores in formula $\frac{dyddx-dxddy}{dx^3}$ substituti dabunt

$$\frac{2xdxddx-2xdxddx-2dx^3}{dx^3}=-2$$

II. Sit $y=x^n$ ; erit

$$dy=nx^{n-1}dx \text{ et } ddy=n(n-1)x^{n-2}dx^2,$$

qui valores substituti formulam $\frac{dyddx-dxddy}{dx^3}$ transmutabunt in hanc

$$\frac{nx^{n-1}dxddx-nx^{n-1}dxddx-n(n-1)x^{n-2}dx^3}{dx^3}=-n(n-1)x^{n-2}.$$

III. Sit $y=-\sqrt{(1-xx)}$ ; erit

$$dy=\frac{xdx}{\sqrt{(1-xx)}} \text{ et } ddy=\frac{xddx}{\sqrt{(1-xx)}}+\frac{dx^2}{(1-xx)^{\frac{3}{2}}}$$

atque formula $\frac{dyddx-dxddy}{dx^3}$ abit in

$$\frac{xddx}{dx^2\sqrt{(1-xx)}}-\frac{xddx}{dx^2\sqrt{(1-xx)}}-\frac{1}{(1-xx)^{\frac{3}{2}}}=\frac{-1}{(1-xx)^{\frac{3}{2}}}.$$

In his igitur omnibus exemplis differentialia secunda $ddx$ se mutuo tollunt hocque ita eveniet, quaecunque aliae functiones loco $y$ substituantur.

**258.** Cum ista exempla iam probaverint veritatem nostrae propositionis, quod formula $\frac{dyddx-dxddy}{dx^3}$ fixum habeat valorem, etiamsi nullum differentiale constans sit assumtum, demonstrationem eo facilius adornare poterimus. Sit $y$ functio quaecunque ipsius $x$ eiusque differentiale *dy* huiusmodi erit, ut sit $dy = pdx$, atque *p* erit functio quaepiam ipsius *x* eiusque differentiale propterea huiusmodi formam habebit $dp = qdx$ eritque *q* iterum functio ipsius *x*. Cum igitur sit $dy = pdx$, erit differentiando $ddy = pddx + qdx^2$ et

$$dyddx - dxddy = pdxddx - pdxddx - qdx^3 = -qdx^3;$$

in qua expressione cum nullum insit differentiale secundum, habebit ea valorem fixum atque $\frac{dyddx - dxddy}{dx^3}$ erit $= -q$. Quomodocunque igitur *y* pendeat ab *x*, differentialia secunda in hac formula semper se mutuo tollent hancque ob causam eius valor, qui alioquin esset vagus, fiet status ac fixus.

**259.** Quanquam hic posuimus *y* esse functionem ipsius *x*, tamen veritas aeque subsistit, si *y* ab *x* prorsus non pendeat, uti assumsimus. Dum enim pro *y* functionem quamcunque substituimus neque, qualis sit, determinavimus, nullam pendentiam ab *x* ipsi *y* tribuimus. Interim tamen sine functionis mentione demonstratio formari potest; quaecunque enim *y* sit quantitas, sive pendens ab *x* sive non pendens, eius differentiale *dy* homogeneum erit cum *dx* sicque $\frac{dy}{dx}$ quantitatem finitam denotabit, cuius differentiale, quod capit, dum *x* in $x + dx$ et $y$ in $y + dy$ abit, erit fixum neque a differentialium secundorum lege pendebit. Sit igitur $\frac{dy}{dx} = p$; erit $dy = pdx$ et $ddy = pddx + dpdx$, unde fit

$$dxddy - dyddx = dpdx^2,$$

cuius valor non est vagus, quia tantum differentialia prima continet; ac propterea idem manet, sive quodpiam differentiale constans accipiatur, qualecunque id demum sit, sive nullum differentiale positum sit constans.

**260.** Quia igitur $dyddx - dxddy$ non obstantibus differentialibus secundis, quae potentia se mutuo destruere censeri possunt, significationem habet fixam, expressio quaecunque, in qua nulla alia differentialia secunda praeter formulam $dyddx - dxddy$ insunt, pariter significationem habebit fixam. Seu si ponatur $dyddx - dxddy = \omega$ atque *V* fuerit quantitas ex *x*, *y*, earum differentialibus primis *dx*, *dy* atque ex $\omega$ utcunque composita, ea valorem habebit fixum. Cum enim in differentialibus primis *dx* et *dy* nulla ratio habeatur eius legis arbitrariae, qua valores successivi ipsius *x* crescere ponuntur, in $\omega$ differentialia secunda se mutuo tollunt, etiam ipsa quantitas $V$ non erit vaga, sed fixa. Sic ista expressio

$$\frac{\left(dx^2 + dy^2\right)^{\frac{3}{2}}}{dxddy - dyddx}$$

valorem obtinet fixum, quamvis ea differentialibus secundis inquinata videatur, atque insuper, quia numerator est homogeneus denominatori, valorem obtinet finitum, nisi is casu vel infinite magnus vel infinite parvus evadat.

**261.** Quemadmodum formula $dxddy - dyddx$ valorem fixum habere ostensa est, ita quoque, si tertia variabilis $z$ accedat, hae formulae

$$dxddz - dzddx \quad \text{et} \quad dyddz - dzddy$$

valores fixos habebunt. Hinc expressiones, quae tres variabiles $x$, $y$ et $z$ involvunt, si in eis nulla alia differentialia secunda occurrant praeter haec assignata, tum perinde erunt fixae, ac si nulla plane differentialia secunda inessent. Ita haec expressio

$$\frac{\left(dx^2+dy^2+dz^2\right)^{\frac{3}{2}}}{(dx+dz)ddy-(dy+dz)ddx+(dx-dy)ddz}$$

non obstantibus differentialibus secundis fixa gaudet significatione. Similique modo formulae exhiberi possunt plures variabiles continentes, in quibus differentialia secunda non impediunt, quominus earum significatio sit fixa.

**262.** Exceptis ergo huius generis formulis, quae differentialia secunda complectuntur, reliquae omnes significationes habebunt vagas neque propterea in calculo locum habere possunt, nisi quodpiam differentiale primum definiatur, quod constans sit assumtum. Statim vero atque differentiale quodpiam primum constans assumitur, omnes expressiones, quotcunque variabiles contineant et cuiuscunque ordinis differentialia post primum in eas ingrediantur, fixas obtinebunt significationes neque amplius ex calculo excluduntur. Si enim verbi gratia $dx$ assumtum sit constans, ipsius $x$ differentialia secunda et sequentia evanescunt; et quaecunque functiones ipsius $x$ loco reliquarum variabilium $y$, $z$ etc. substituantur, earum differentialia secunda per $dx^2$, tertia per $dx^3$ etc. determinabuntur sicque inconstantia a differentialibus secundis oriunda tollitur. Idem evenit, si alius variabilis seu functionis cuiuscunque differentiale primum constans ponatur.

**263.** Ex his igitur sequitur differentialia secunda et altiorum ordinum revera nunquam in calculum ingredi atque ob vagam significationem prorsus ad Analysin esse inepta. Quando enim differentialia secunda adesse videntur, vel differentiale quodpiam primum constans assumitur vel nullum. Priori casu differentialia secunda prorsus ex calculo evanescunt, dum per differentialia prima determinantur. Posteriori casu autem, nisi se mutuo destruant, significatio erit vaga et propterea in Analysi locum nullum inveniunt; sin autem se mutuo destruunt, tantum apparenter adsunt et revera solae quantitates finitae cum suis differentialibus primis adesse censendae sunt. Quoniam tamen saepissime apparenter tantum in calculo usurpantur, necesse fuit, ut methodus eas tractandi exponeretur. Modum autem mox ostendemus, cuius ope differentialia secunda et altiora semper exterminari queant.

**264.** Si expressio unicam contineat variabilem *x* eiusque differentialia altiora $ddx,\ d^3x,\ d^4x$ etc. in ea occurrant, ea significatum fixum habere nequit, nisi quodpiam differentiale primum constans sit positum. Sit igitur *t* illa quantitas variabilis, cuius differentiale *dt* sit constans positum, ita ut sit $ddt=0,\ d^3t=0,\ d^4t=0$ etc. Ponatur $dx=pdt$ eritque *p* quantitas finita, cuius differentiale vaga significatione differentialium secundorum non afficietur, hincque etiam $\frac{dp}{dt}$ erit quantitas finita. Sit $dp=qdt$ similique modo ulterius $dq=rdt,\ dr=sdt$ etc.; erunt *q, r, s* etc. quantitates finitae fixos significatus habentes. Cum igitur sit $dx=pdt$, erit

$$ddx=dpdt=qdt^2,\ d^3x=dqdt^2=rdt^3,\ d^4x=dr\,d\,t^3=sdt^4 \text{ etc.};$$

qui valores si loco $ddx,\ d^3x,\ d^4x$ etc. substituantur, tota expressio meras quantitates finitas cum differentiali primo *dt* continebit ideoque non amplius vagam significationem habebit.

**265.** Si *x* sit functio ipsius *t*, poterit hoc modo quantitas *x* prorsus eliminari, ita ut sola quantitas *t* cum suo differentiali *dt* in expressione remaneat; sin autem *t* sit functio ipsius *x*, vicissim quoque *x* erit ipsius *t* functio. Interim tamen ipsa quantitas *x* cum suo differentiali primo *dx* in calculo retineri potest, dummodo post substitutiones ante factas ubique loco *t* et *dt* earum valores per *x* et *dx* expressi restituantur. Quod quo planius fiat, ponamus *t* esse $=x^n$, ita ut differentiale primum ipsius $x^n$ constans sit positum. Quia igitur est $dt=nx^{n-1}dx$, erit

$$p=\frac{1}{nx^{n-1}} \quad\text{et}\quad dp=\frac{-(n-1)dx}{nx^n}=qdt=nqx^{n-1}dx;$$

unde fit

$$q=\frac{-(n-1)}{nnx^{2n-1}} \quad\text{et}\quad dq=\frac{(n-1)(2n-1)dx}{nnx^{2n}}=rdt=nrx^{n-1}dx\,.$$

Hinc porro fit

$$r=\frac{(n-1)(2n-1)}{n^3x^{3n-1}} \quad\text{et}\quad s=-\frac{(n-1)(2n-1)(3n-1)}{n^4x^{4n-1}}.$$

Quare si differentiale ipsius $x^n$ ponatur constans, erit

$$ddx = -\frac{(n-1)dx^2}{x}$$
$$d^3x = \frac{(n-1)(2n-1)dx^3}{xx}$$
$$d^4x = -\frac{(n-1)(2n-1)(3n-1)dx^4}{x^3}$$
etc.

**266.** Si expressio duas contineat variabiles $x$ et $y$ earumque unius $x$ differentiale positum sit constans, ob $ddx = 0$ alia differentialia secunda et altiora non inerunt praeter $ddy$, $d^3y$ etc. Haec autem eodem modo, quo ante usi sumus, tolli poterunt ponendo

$$dy = pdx,\ dp = qdx,\ dq = rdx,\ dr = sdx \quad \text{etc.}$$

fiet enim

$$ddy = qdx^2,\ d^3y = rdx^3,\ d^4y = sdx^4 \quad \text{etc.}$$

quibus substitutis expressio orietur, quae praeter quantitates finitas $x, y, p, q, r, s$ etc. nonnisi differentiale primum $dx$ continebit. Sic si proposita fuerit haec expressio

$$\frac{ydx^4 + xdyd^3y + xd^4y}{(xx+yy)ddy},$$

In qua $dx$ est constans assumtum, ponatur

$$dy = pdx,\ dp = qdx,\ dq = rdx \ \text{ et } \ dr = sdx \quad \text{etc.},$$

quibus valoribus substitutis expressio proposita transmutabitur in hanc

$$\frac{(y+xpr+xs)dx^2}{(xx+yy)q},$$

quae nulla amplius differentialia secunda altiorave continet.

**267.** Simili modo differentialia secunda et altiora tollentur, si $dy$ fuerit constans assumtum. Verum si aliud differentiale primum quodcunque $dt$ statuatur constans, tum primum modo ante indicato differentialia ipsius $x$ altiora ex calculo tollantur ponendo

$$dx = pdt,\ dp = qdt,\ dq = rdt,\ dr = sdt \quad \text{etc.},$$

unde fit

$$ddx = pdt^2,\ d^3x = rdt^3,\ d^4x = sdt^4 \quad \text{etc.},$$

deinde simili modo differentialia altiora ipsius $y$ ponendo

$$dy = Pdt,\ dP = Qdt,\ dQ = Rdt,\ dR = Sdt \text{ etc.},$$

unde fiet

$$ddy = Qdt^2,\ d^3y = Rdt^3,\ d^4y = Sdt^4 \quad \text{etc.};$$

quibus substitutis obtinebitur expressio, quae praeter quantitates finitas *x, p, q, r, s* etc., *y, P, Q, R, S* etc. solum differentiale *dt* complectetur neque propterea vagam habebit significationem.

**268.** Si differentiale primum, quod constans ponitur, vel ab *x* vel ab *y* vel ab utroque simul pendet, tum non opus est, ut duplex quantitatum finitarum *p, q, r* etc. series introducatur. Si enim *dt* ab *x* tantum pendet, tum litterae *p, q, r* etc. fient functiones ipsius *x* solaeque litterae *P, Q, R* etc. ingrediuntur; idemque evenit, si differentiale constans *dt* ab *y* tantum pendeat. At si *dt* ab utraque pendeat, operatio aliquantum immutari debet. Ponamus exempli gratia hoc differentiale *ydx* constans esse assumtum eritque $yddx + dxdy = 0$, unde fit

$$ddx = -\frac{dxdy}{y}.$$

Sit nunc

$$dy = pdx,\ dp = qdx,\ dq = rdx \text{ etc.}$$

eritque

$$ddx = -\frac{pdx^2}{y}$$

ulteriusque differentiando

$$d^3x = -\frac{qdx^3}{y} + \frac{ppdx^3}{yy} - \frac{2pdxddx}{y};$$

substituatur hic loco *ddx* eius valor $-\frac{pdx^2}{y}$; fiet

$$d^3x = -\frac{qdx^3}{y} + \frac{3ppdx^3}{yy}$$

porroque

$$d^4x = -\frac{rdx^4}{y} + \frac{pqdx^4}{yy} + \frac{6pqdx^4}{yy} - \frac{6p^3dx^4}{y^3} + \left(\frac{3pp}{yy} - \frac{q}{y}\right)3dx^2ddx$$

et pro *ddx* substituto valore $-\frac{pdx^2}{y}$ emerget

$$d^4x = \left(\frac{-r}{y} + \frac{10pq}{yy} - \frac{15p^3}{y^3}\right)dx^4$$

etc. Deinde cum sit $dy = pdx$, erit

$$ddy = qdx^2 + pddx = \left(q - \tfrac{pp}{y}\right)dx^2$$

et continuo pro *ddx* valore $-\frac{pdx^2}{y}$ substituendo fiet *y*

$$d^3y = \left(r - \tfrac{4pq}{y} + \tfrac{3p^3}{yy}\right)dx^3$$

et

$$d^4y = \left(s - \tfrac{7pr}{y} - \tfrac{4qq}{y} + \tfrac{25ppq}{yy} - \tfrac{15p^4}{y^3}\right)dx^4$$

etc. Qui valores loco differentialium altiorum ipsarum *x* et *y* substituti mutabunt expressionem propositam in eiusmodi formam, quae nulla amplius differentialia altiora continebit hincque consideratione cuiuspiam differentialis constantis exuetur. Facta enim hac transformatione quia differentialia secunda non insunt, nequidem opus est, ut, quale differentiale sumtum sit constans, commemoretur.

**269.** Saepissime autem in calculo ad lineas curvas applicato evenire solet, ut hoc differentiale primum $\sqrt{\left(dx^2 + dy^2\right)}$ constans assumatur; quare, quemadmodum hoc casu differentialia secunda et altiora eliminari debeant, ostendamus. Sic enim simul via patebit ad idem negotium absolvendum, si aliud quodcunque differentiale assumendum sit constans. Ponatur iterum

$$dy = pdx,\ dp = qdx,\ dq = rdx,\ dr = sdx \ \text{ etc}.$$

atque differentiale $\sqrt{\left(dx^2 + dy^2\right)}$ induet hanc formam $dx\sqrt{(1 + pp)}$; quae cum sit constans, fiet

$$ddx\sqrt{(1 + pp)} + \frac{pqdx^2}{\sqrt{(1+pp)}} = 0$$

ideoque

$$ddx = -\frac{pqdx^2}{1+pp},$$

unde iam ipsius *ddx* valor habebitur; hinc porro erit

$$d^3x = -\frac{prdx^3}{1+pp} - \frac{qqdx^3}{1+pp} + \frac{2ppqqdx^3}{(1+pp)^2} - \frac{2pqdxddx}{1+pp}$$
$$= -\frac{prdx^3}{1+pp} - \frac{qqdx^3}{1+pp} + \frac{4ppqqdx^3}{(1+pp)^2} = -\frac{prdx^3}{1+pp} + \frac{(3pp-1)qqdx^3}{(1+pp)^2}.$$

Deinde fiet

$$d^4x = -\frac{psdx^4}{1+pp} + \frac{(10pp-3)qrdx^4}{(1+pp)^2} - \frac{(15pp-13)pq^3dx^4}{(1+pp)^3}.$$

Quia autem assumsimus $dy = pdx$, fiet differentiando

$$ddy = qdx^2 + pddx = qdx^2 - \frac{ppqdx^2}{1+pp} = \frac{qdx^2}{1+pp}$$

$$d^3y = \frac{rdx^3}{1+pp} - \frac{2pqqdx^3}{(1+pp)^2} + \frac{2qdxddx}{1+pp}$$

ideoque

$$d^3y = \frac{rdx^3}{1+pp} - \frac{4pqqdx^3}{(1+pp)^2}$$

porroque differentiando

$$d^4y = \frac{sdx^4}{1+pp} - \frac{13pqrdx^4}{(1+pp)^2} + \frac{4(6pp-1)q^3dx^4}{(1+pp)^3}.$$

Omnia ergo differentialia altiora utriusque variabilis *x* et *y* per quantitates finitas et potestates ipsius *dx* exprimentur atque post has substitutiones factas resultabit expressio a differentialibus secundis prorsus libera.

**270.** Exposito igitur modo differentialia secunda et altiora exuendi conveniet hoc negotium aliquot exemplis illustrari.

I. Sit proposita haec expressio $\frac{xddy}{dx^2}$, in qua *dx* positum est constans.

Posito ergo $dy = pdx$ et $dp = qdx$, ob $ddy = qdx^2$ expressio proposita abit in hanc finitam *xq*.

II. Sit proposita haec expressio $\frac{dx^2+dy^2}{ddx}$, in qua positum sit *dy* constans.

Ponatur $dx = pdy$, $dp = qdy$; ob $ddx = qdy^2$ orietur $\frac{1+pp}{q}$. Sin autem ut ante statuere velimus $dy = pdx$, $dp = qdx$, ob *dy* constans erit $0 = pddx + dpdx$ et $ddx = -\frac{qdx^2}{p}$; unde expressio proposita transibit in $\frac{-p(1+pp)}{q}$.

III. Sit proposita haec expressio $\frac{ydddx-dxddy}{dxdy}$, in qua $ydx$ positum sit constans constans. Ponatur $dy=pdx$ et $dp=qdx$ eritque ex § 268 $ddx=-\frac{pdx^2}{y}$, $ddy=qdx^2-\frac{ppdx^2}{y}$, quibus substitutis expressio proposita transmutatur in hanc $-1-\frac{xq}{p}+\frac{xp}{y}$.

IV. Sit proposita ista expressio $\frac{dx^2+dy^2}{ddy}$, in qua constans sit positum $\sqrt{\left(dx^2+dy^2\right)}$.. Ponatur iterum $dy=pdx$, $dp=qdx$ et ex paragrapho praecedente erit $ddy=\frac{qdx^2}{1+pp}$ ; unde expressio proposita abibit in $\frac{(1+pp)^2}{q}$.

Ex his autem exemplis satis intelligitur, quemadmodum in quovis casu oblato, quodcunque differentiale primum assumtum sit constans, differentialia secunda atque altiora eliminari debeant.

**271.** Cum igitur hoc modo introducendis quantitatibus finitis $p$, $q$, $r$, s etc. differentialia secunda et altiora ita eliminari queant, ut tota expressio praeter quantitates finitas $x$, $y$, $p$, $q$, $r$, s etc. solum differentiale $dx$ complectatur, vicissim si huiusmodi expressio reducta proponatur, ea iterum in formam priorem transmutari poterit, loco litterarum $p$, $q$, $r$, $s$ etc. introducendis differentialibus secundis et altioribus. Nunc autem perinde erit, quodnam differentiale primum constans assumatur, atque vel id ipsum, quod ante fuit assumtum, constans poni potest vel aliud quodcunque. Quin etiam prorsus nullum differentiale constans assumi poterit hocque modo prodibunt expressiones differentialia secunda altiorave continentes, quae, etiamsi nullum differentiale constans sit assumtum, tamen fixas significationes obtineant, cuiusmodi expressiones dari supra ostendimus.

**272.** Sit ergo proposita expressio quaecunque continens litteras finitas $x$, $y$, $p$, $q$, $r$ etc. una cum differentiali $dx$, in qua sit

$$p=\frac{dy}{dx},\quad q=\frac{dp}{dx},\quad r=\frac{dq}{dx}\quad \text{etc.}$$

Si enim has litteras $p$, $q$, $r$ etc. ita eliminare velimus, ut earum loco introducamus differentialia secunda et altiora ipsarum $x$ et $y$ nullo differentiali constante assumto, fiet

$$dp=\frac{dxddy-dyddx}{dx^2}\quad \text{hincque}\quad q=\frac{dxddy-dyddx}{dx^3}$$

quae formula differentiata dabit

$$dq=\frac{dx^2d^3y-3dxddxddy+3dyddx^2-dxdyd^3x}{dx^4},$$

unde fit

$$r = \frac{dx^2 d^3 y - 3dxddxddy + 3dyddx^2 - dxdyd^3 x}{dx^5}.$$

Quodsi insuper littera *s*, quae denotat valorem $\frac{dr}{dx}$, insit, pro ea substitui debebit hic valor

$$s = \frac{dx^3 d^4 y - 6dx^2 ddxd^3 y - 4dx^2 ddyd^3 x + 15dxddx^2 ddy + 10dxdyddxd^3 x - 15dyddx^3 - dx^2 dyd^4 x}{dx^7}.$$

His igitur valoribus loco quantitatum *p, q, r, s* etc. substitutis expressio proposita transmutabitur in aliam differentialia altiora ipsarum *x* et *y* continentem, quae, etiamsi nullum differentiale primum constans sit assumtum, tamen non vagam, sed fixam habebit significationem.

**273.** Hoc ergo modo quaevis formula differentialis altioris gradus, in qua quodpiam differentiale primum assumtum est constans, transmutari poterit in aliam formam, in qua nullum differentiale constans ponitur, quae hoc non obstante eundem valorem fixum habeat. Primum scilicet ope methodi ante traditae assumtis valoribus $dy = pdx,\ dp = qdx,\ dy = rdx, dr = sdx$ etc. differentialia altiora eliminentur; tum loco *p, q, r, s* etc. valores nunc inventi substituantur atque orietur expressio priori aequalis nullum differentiale constans involvens; quam transformationem exempla sequentia illustrabunt.

I. Sit proposita haec expressio $\frac{xddy}{dx^2}$, in qua *dx* positum sit constans, quae transmutari debeat in aliam formam nullum differentiale constans involventem. Ponatur $dy = pdx,\ dp = qdx$ atque, ut ante (§ 270) vidimus, expressio proposita transibit in hanc *qx*. Nunc loco *q* substituatur valor, quem obtinet nullo differentiali constante assumto, $q = \frac{dxddy - dyddx}{dx^3}$ atque reperietur haec expressio

$$\frac{xdxddy - ddyddx}{dx^3}$$

propositae aequalis et nullum amplius differentiale constans involvens.

II. Sit proposita haec expressio $\frac{dx^2 + dy^2}{ddx}$, in qua *dy* assumtum est constans.
Ponatur $dy = pdx$ et $dp = qdx$ eaque transibit in hanc $-\frac{p(1+pp)}{q}$ [§ 270]; statuatur nunc $p = \frac{dy}{dx}$ et $q = \frac{dxddy - dyddx}{dx^3}$ atque invenietur

$$\frac{dy(dx^2 + dy^2)}{dyddx - dxddy},$$

quae nullo differentiali constante assumto eundem fixum habet valorem, quem proposita.

III. Sit proposita haec expressio $\frac{yddx-xddy}{dxdy}$, in qua differentiale $ydx$ constans est assumtum. Ponatur $dy = pdx$, $dp = qdx$ atque, uti supra (§ 270) vidimus, haec expressio transmutatur in hanc $-1-\frac{xq}{p}+\frac{xp}{y}$, quae nullo differentiali constante assumto transformabitur in istam

$$-1-\frac{xdxddy-xdyddx}{dx^2dy}+\frac{xdy}{ydx}=\frac{xdxdy^2-ydx^2dy-yxdxddy+yxdyddx}{ydx^2dy}$$

IV. Sit proposita haec expressio $\frac{dx^2+dy^2}{ddx}$, in qua constans assumtum est differentiale $\sqrt{\left(dx^2+dy^2\right)}$. Posito $dy = pdx$ et $dp = qdx$ orietur haec expression $\frac{(1+pp)^2}{q}$ (loco citato). Statuatur nunc $p=\frac{dy}{dx}$ et $q=\frac{dxddy-dyddx}{dx^3}$ atque nullo assumto differentiali constante nanciscemur istam expressionem

$$\frac{\left(dx^2+dy^2\right)^2}{dx^2ddy-dxdyddx}$$

propositae aequivalentem.

V. Sit proposita haec expressio $\frac{dxd^3y}{ddy}$, in qua differentiale $dx$ constans sit assumtum. Ponatur $dy = pdx$, $dp = qdx$ et $dq = rdx$ atque ob $ddy = qdx^2$ et $d^3y = rdx^3$ formula proposita abibit in hanc $\frac{rdx^2}{q}$. Nunc loco $q$ et $r$ substituantur valores, quos nullo differentiali constante assumto recipiunt, scilicet

$$q=\frac{dxddy-dyddx}{dx^3} \text{ et } r=\frac{dx^2d^3y-3dxddxddy+3dyddx^2-dxdyd^3x}{dx^5}$$

atque obtinebitur sequens expressio propositae aequivalens

$$\frac{dx^2d^3y-3dxddxddy+3dyddx^2-dxdyd^3x}{dxddy-dyddx}=\frac{dx\left(dxd^3y-dyd^3x\right)}{dxddy-dyddx}-3ddx.$$

**274.** Si has transformationes diligentius intueamur, methodum eas perficiendi colligere poterimus expeditiorem, ita ut non opus sit litteras *p, q, r* etc. introducere. Varii autem modi hoc opus absolvendi occurrent, prout aliud atque aliud differentiale in formula proposita constans fuerit assumtum. Ponamus primum in formula proposita differentiale $dx$ constans esse assumtum, et quia loco $dy$ posuimus $p\,dx$ rursusque $\frac{dy}{dx}$ loco $p$, differentialia prima $dx$ et $dy$, ubicunque in expressione occurrunt, sine alteratione relinquuntur. Ubi autem occurrit *ddy*, quia eius loco scribitur $qdx^2$ et porro loco *q* valor $\frac{dxddy-dyddx}{dx^3}$, transmutatio absolvetur, si ubique loco *ddy* statim ponatur $\frac{dxddy-dyddx}{dx}$ seu $ddy-\frac{dyddx}{dx}$. Si insuper in expressione proposita occurrat $d^3y$, quia eius loco ponitur $rdx^3$, ob valorem ipsius *r* ante inventum ubique loco $d^3y$ scribi debebit

$$d^3y-\frac{3ddxddy}{dx}+\frac{3dyddx^2}{dx^2}-\frac{dyd^3x}{dx},$$

quo facto expressio proposita transmutabitur in aliam, quae nullum differentiale constans involvit. Sic proponatur itsa expressio $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxddy}$, in qua *dx* positum est constans, ei aequalis erit posito $ddy-\frac{dyddx}{dx}$ loco *ddy* haec nullum differentiale constans involvens

$$\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxddy-dyddx}.$$

**275.** Hinc facile colligitur, si in expressione quapiam proposita assumtum fuerit differentiale *dy* constans, tum ubique loco *ddx* scribi debere $ddx-\frac{dxddy}{dy}$ et loco $d^3x$ hoc

$$d^3x-\frac{3ddxddy}{dy}+\frac{3dxddy^2}{dy^2}-\frac{dxd^3y}{dy},$$

ut obtineatur expressio aequivalens, in qua nullum differentiale constans ponatur. Sin autem in expressione proposita constans fuerit assumtum *ydx*, quoniam fit [§ 268]

$$ddx=-\frac{pdx^2}{y} \quad \text{et} \quad ddy=qdx^2-\frac{ppdx^2}{y},$$

loco *ddx* ubique scribi debebit $-\frac{dxdy}{y}$ et loco *ddy* ubique $ddy-\frac{dyddx}{dx}-\frac{dy^2}{y}$;
ad altiora differentialia, quia in hoc negotio rarissime occurrere solent, non progredior, Quodsi vero in expressione proposita hoc differentiale $\sqrt{(dx^2+dy^2)}$ assumtum fuerit constans, quia invenimus [§ 269]

$$ddx=-\frac{pqdx^2}{1+pp} \quad \text{et} \quad ddy=\frac{qdx^2}{1+pp}$$

pro *ddx* ubique scribi debet $\frac{dy^2ddx-dxdyddy}{dx^2+dy^2}$ et 1oco *ddy* ubique $\frac{dx^2ddy-dxdyddx}{dx^2+dy^2}$.
Sic si proposita fuerit expressio $\frac{dy\sqrt{(dx^2+dy^2)}}{ddx}$, in qua $\sqrt{(dx^2+dy^2)}$ assumtum sit constans, ea transmutabitur in hanc

$$\frac{\left(dx^2+dy^2\right)^{\frac{3}{2}}}{dyddx-dxddy}$$

In qua nullum differentiale constans assumitur.

**276.** Quo istae reductiones facilius ad usum accommodari queant, eas in sequenti tabella complecti visum est. Formula igitur differentialis altioris gradus in aliam nullam differentiale constans involventem transmutabitur ope substitutionum sequentium,

1. Si differentiale $dx$ fuerit constans assumtum,

| loco | scribatur |
|---|---|
| $ddy$ | $ddy-\frac{dyddx}{dx}$ |
| $d^3y$ | $d^3y-\frac{3ddxddy}{dx}+\frac{3dyddx^2}{dx^2}-\frac{dyd^3x}{dx}$ |

II. Si differentiale $dy$ fuerit constans assumtum,

| loco | scribatur |
|---|---|
| $ddx$ | $ddx-\frac{dxddy}{dy}$ |
| $d^3x$ | $d^3x-\frac{3ddxddy}{dy}+\frac{3dxddy^2}{dy^2}-\frac{dxd^3y}{dy}$ |

III. Si differentiale $ydx$ fuerit constans assumtum,

| loco | scribatur |
|---|---|
| $ddx$ | $-\frac{dxdy}{y}$ |
| $ddy$ | $ddy-\frac{dyddx}{dx}-\frac{dy^2}{y}$ |
| $d^3x$ | $\frac{dyddx}{y}-\frac{dxddy}{y}+\frac{3dxdy^2}{yy}$ |
| $d^3y$ | $d^3y-\frac{3ddxddy}{dx}+\frac{3dyddx^2}{dx^2}-\frac{dyd^3x}{dx}-\frac{4dyddy}{y}+\frac{4dy^2ddx}{ydx}+\frac{3dy^3}{yy}$ |

IV. Si differentiale $\sqrt{\left(dx^2+dy^2\right)}$ fuerit constans assumtum,

| loco | scribatur |
|---|---|
| $ddx$ | $\frac{dy^2ddx-dxdyddy}{dx^2+dy^2}$ |
| $ddy$ | $\frac{dx^2ddy-dxdyddx}{dx^2+dy^2}$ |

$$d^3x \;\Big|\; \frac{dy^2d^3x - dxdyd^3y}{dx^2+dy^2} + \frac{(dxddy - dyddx)(3dy^2ddy - dx^2ddy + 4dxdyddx)}{(dx^2+dy^2)^2}$$

$$d^3y \;\Big|\; \frac{dx^2d^3y - dxdyd^3x}{dx^2+dy^2} + \frac{(dyddx - dxddy)(3dx^2ddx - dy^2ddx + 4dxdyddy)}{(dx^2+dy^2)^2}.$$

**277.** Expressiones ergo istae, quae nullum differentiale constans includunt, ita erunt comparatae, ut pro lubitu quodvis differentiale constans assumi queat. Hincque expressiones differentiales altiorum graduum, in quibus nullum differentiale constans assumtum perhibetur, examinari possunt, utrum significatio earum sit vaga an fixa. Ponatur enim pro lubitu quodpiam differentiale, puta *dx,* constans, tum per regulam paragraphi praecedentis priorem reducatur expressio iterum ad formam, in qua nullum differentiale constans sit assumtum; quae si cum proposita conveniat, ea erit fixa neque ab inconstantia differentialium secundorum pendebit; sin autem expressio prodeat diversa, tum proposita vagam habet significationem. Sic si ponatur haec expressio $yddx - xddy$, in qua nullum differentiale positum sit constans, ad investigandum, utrum significationem fixam habeat an vagam, ponatur *dx* constans eaque abibit in $-xddy$; nunc per regulam primam paragraphi praecedentis loco *ddy* ponatur $ddy - \frac{dyddx}{dx}$ ac prodibit $-xddy + \frac{xdyddx}{dx}$, cuius a proposita discrepantia indicat propositam expressionem fixam statamque significationem non habere.

**278.** Simili modo si proponatur expressio generalis huiusmodi

$$Pddx + Qdxdy + Rddy,$$

conditio definiri poterit, sub qua ea nullo differentiali constante assumto valorem fixum habeat. Ponatur enim *dx* constans atque expressio proposita abibit in hanc $Qdxdy + Rddy$; nunc haec iterum transformetur in aliam formam, ut eius significatus idem maneat, etiamsi nullum differentiale constans fingatur, sicque prodibit $Qdxdy + Rddy - \frac{Rdyddx}{dx}$, quae forma cum proposita congruet, si fuerit $Pdx + Rdy = 0$; hocque solo casu valor eius erit fixus. Verum si non fuerit $P = -\frac{Rdy}{dx}$ seu $R = -\frac{Pdx}{dy}$, tum expressio proposita $Pddx + Qdxdy + Rddy$ valorem fixum non habebit, sed eius significatio erit vaga atque diversa, prout aliud atque aliud differentiale constans assumitur.

**279.** Ex his principiis etiam facile erit expressionem differentialem, in qua quodpiam differentiale constans est positum, transmutare in aliam formam, in qua aliud differentiale constans assumatur. Reducatur enim primum ad eiusmodi formam, quae nullum differentiale constans involvat, quo facto illud alterum differentiale constans ponatur. Sic si in expressione proposita differentiale *dx* assumtum sit constans eaque transmutanda sit in aliam, quae differentiale *dy* constans implicet, in formulis supra loco *ddy* et $d^3y$ substituendis ob *dy* constans ponatur $ddy = 0$, $d^3y = 0$ atque quaesito satisfiet, si loco *ddy* substituatur $\frac{-dyddx}{dx}$ et $\frac{3dyddx^2}{dx^2} - \frac{dydd^3x}{dx}$ loco $d^3y$. Hoc modo ista formula $-\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxddy}$, in qua $dx$ positum est constans, transmutabitur in hanc $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dyddx}$, in qua $dy$ ponitur constans.

**280.** Si contra formula, in qua $dy$ constans est positum, transmutari debeat in aliam, in qua $dx$ sit constans, tum loco $ddx$ substitui debet $\frac{-dxddy}{dy}$ et loco $d^3x$ haec expression $\frac{3dxddy^2}{dy^2} - \frac{dxdd^3y}{dy}$. Simili modo si formula, in qua $\sqrt{(dx^2+dy^2)}$ positum est constans, transmutari debeat in aliam, in qua $dx$ sit constans, tum loco $ddx$ scribatur $-\frac{dxdyddy}{dx^2+dy^2}$ et $\frac{dx^2ddy}{dx^2+dy^2}$ loco $ddy$. At si formula, in qua $dx$ constans est assumtum, transmutari debeat in aliam, in qua $\sqrt{(dx^2+dy^2)}$ sit constans, quia ob $dx^2+dy^2$ constans fit

$$dxddx + dyddy = 0 \text{ et } ddx = -\frac{dyddy}{dx},$$

hoc valore loco $ddx$ assumto pro $ddy$ scribi debebit

$$ddy + \frac{dy^2ddy}{dx^2} = \frac{(dx^2+dy^2)ddy}{dx^2}.$$

Sic haec formula $-\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxddy}$, in qua $dx$ est constans, transmutabitur in aliam, in qua $\sqrt{(dx^2+dy^2)}$ ponitur constans, quae erit $-\frac{dx\sqrt{(dx^2+dy^2)}}{ddy}$.