CHAPTER VII

CONCERNING THE DIFFERENTIATION OF FUNCTIONS INVOLVING TWO OR MORE VARIABLES

208. If two or more variable quantities x, y, z may not in turn depend on each other in any manner, it can happen that even if all shall be variables, yet, while one increases or decreases, the remaining may remain unchanged ; because indeed they have no connection between each other, a change of one does not affect the others. Therefore nor will the differentials of the quantities y and z thus depend on the differential of x and thus, while x may be increased by its differential dx, the quantities y and z either remain the same or are able to be varied in whatever manner it pleases. Therefore if the differential dx of the quantity x may be put in place, the differentials of the remaining quantities dy and dz stay undetermined and by our choice either may be denoted to be completely zero or maintaining any infinitely small ratio to dx.

209. But generally the letters y and z are accustomed to specify functions of x either unknown, or the relation of which to x is not seen, and in this latter case the differentials dy and dz of these will have a certain relation to dx. But either y and z may depend on x , or otherwise an account of the differentiation reverts to the same as we consider here. For we will seek the differential of a function, which shall be formed in some manner from several variables x, y and z, which it assumes while the individual variables x, y and z increase by their own differentials dx, dy et dz. Therefore according to this being found, everywhere in place of the variable quantities x, y, z in the proposed function there is written respectively $x+dx,\ y+dy,\ z+dz$ and from the expression resulting in this manner the proposed function itself may be taken away ; the remainder will give the differential itself which is sought, just as may be agreed upon completely from the nature of the differentiation.

210. Let X be a function of x and the differential of this or the increase, while x increases by its differential dx, shall be $=Pdx$. Then let Y be a function of y with the differential of this $=Qdy$, which increase Y takes, while y changes into $y+dy$, and Z shall be a function of z the differential of this shall be $=Rdz$; which differentials Pdx, Qdy, Rdz can be found from the nature of the functions X, Y and Z with the aid of the precepts given above. But if this quantity were proposed $X+Y+Z$, which everywhere will be a function of the three variables x, y and z, the differential of this will be $=Pdx+Qdy+Rdz$. But likewise it is the case that these three differentials may or may not be homogeneous to each other. Indeed the terms which contain the powers of dx, equally vanish before Pdx, and if the remaining members Qdy and Rdz should be missing, the ratio of the terms is of a like kind to that in which the differentiation of the functions Y and Z have been ignored.

211. *X*, *Y* and *Z* may retain the same meanings and let the function *XYZ* be proposed of these *x*, *y* and *z*, the differential of which it is required to investigate. Because, if there may be written $x+dx$ in place of *x*, $y+dy$ in place of *y* and $z+dz$ in place of *z*, *X* changes into $X+Pdx$, Y into $Y+Qdy$ and Z into $Z+Rdz$, the proposed function itself XYZ will change into

$$(X+Pdx)(Y+Qdy)(Z+Rdz)=XYZ+YZPdx+XZQdy+XYRdz$$
$$+ZPQdxdy+YPRdxdz+XQRdydz+PQRdxdydz.$$

But because *dx*, *dy* et *dz* are infinitely small, whether they are homogeneous or not, the final term vanishes before the preceding one. Then the term *ZPQdxdy* vanishes before *YZPdx* as well as before *XZQdy*; and on that same account the terms *YPRdxdz* and *XQRdydz* will vanish. Therefore with the proposed function *XYZ* taken away the differential of this

$$=YZPdx+XZQdy+XYRdz.$$

212. These examples of functions of three variables *x*, *y* and *z*, to each of which many more can be added as it pleases, are sufficient to show, if a function of the three variables *x*, *y* and *z* may be proposed, also these variables can be interchanged between themselves in any manner, the differential of this will always have a form of this kind $pdx+qdy+rdz$, where *p*, *q* and *r* shall be the future individual functions either of all the three variables *x*, *y* and *z* or of two or of one only, just as an account of the composition may be prepared, from which the proposed function may be formed from the variables *x*, *y* et *z* and constants. In a similar manner, if a function is proposed with four or more variables *x*, *y*, *z* and *v*, the differential will have a form of this kind always

$$pdx+qdy+rdz+sdv.$$

213. In the first place we may consider a function of two variables only *x* and *y*, which shall be $=V$ of which therefore the differential thus will be had, so that there shall be

$$dV=pdx+qdy.$$

Therefore if the quantity *y* may be assumed constant, there shall be $dy=0$ and thus the differential of the function *V* becomes *pdx;* but if *x* may be put in place constant, so that there shall be $dx=0$ and only *y* may remain variable, then the differential of *V* may emerge $=qdy$. Therefore since with each quantity *x* and *y* placed to be variable there shall be $dV=pdx+qdy$, this rule for differentiating will result in a function *V* involving two variables *x* and *y*:
In the first place only x may be put variable, truly the other y may be treated as a constant and the differential of V may be sought, which shall be $=pdx$. *Then only the quantity y is*

put variable, the other x taken to be constant, and the differential of V is sought, which shall be $=qdy$. *With which made, on putting each quantity x and y variable, there will be produce* $dV = pdx + qdy$.

214. In a similar manner, since the differential of a function of three variables *x, y* and *z,* which shall be $=V$ may have a form of this kind

$$dV = pdx + qdy + rdz\ ,$$

it is evident, if only the quantity *x* were put variable, truly the others *y* and *z* should have remained constant, on account of $dy = 0$ and $dz = 0$ the differential of *V* shall be produced $= pdx$. In a like manner the differential of $V = qdy$ may be found, if *x* and *z* were constant and only *y* may be made variable ; and if *x* and *y* may be treated as constants and only *z* may be put in place variable, the differential of *V* may be produced $= rdz$. Whereby towards differentiating a function of three or more variables, some quantity must be considered separately and the function for this must be differentiated, as if all the remaining variables are to be constants; then these individual differentials which have been found from the individual variable quantities are to be gathered together and the sum of the proposed functions will be the differential sought.

215. In this rule, which we have found for the differentiation of a function of any number of variables, the demonstration of the general rule given above (§ 170) may be contained, with the aid of which any function involving a single variable can be differentiated. If indeed there for the individual parts, the different letters may be assembled with just as many kept in mind, the function adopts the form of a function of all the different variables and thus here will be differentiated successively in the prescribed manner one part at a time, as if it should be the only variable; and with all the variables being treated the differentials, which originate from the individual parts, may be put together into a single sum; which sum will be the differential sought, after the values of the individual letters were restored. Therefore this rule may appear the widest and is extended even to functions of several variables, in whatever manner they were prepared. From which there is the fullest use of this in all the differential calculus.

216. Therefore with the general rule found, with the aid of this, functions of any number of variables can be differentiated, and it will help to show the use of this in several example.

I. If there should be $V = xy$, there will be $dV = xdy + ydx$.

II. If there should be $V = \frac{x}{y}$, there will be

$$dV = \frac{dx}{y} - \frac{xdy}{yy}.$$

III. If there should be $V=\frac{y}{\sqrt{(aa-xx)}}$, there will be

$$dV=\frac{dy}{\sqrt{(aa-xx)}}+\frac{yxdx}{(aa-xx)^{\frac{3}{2}}}.$$

IV. If there should be $V=(\alpha x+\beta y+\gamma)^m(\delta x+\varepsilon y+\zeta)^n$, there will be

$$dV=m(\alpha x+\beta y+\gamma)^{m-1}(\delta x+\varepsilon y+\zeta)^n(\alpha dx+\beta dy)$$
$$+n(\alpha x+\beta y+\gamma)^m(\delta x+\varepsilon y+\zeta)^{n-1}(\delta dx+\varepsilon dy)$$

Or

$$dV=(\alpha x+\beta y+\gamma)^{m-1}(\delta x+\varepsilon y+\zeta)^{n-1}\quad \text{into}$$
$$\left(\begin{matrix}+m\alpha\delta\\+n\alpha\delta\end{matrix}\right\}xdx\begin{matrix}+m\beta\delta\\+n\alpha\varepsilon\end{matrix}\Big\}xdy\begin{matrix}+m\alpha\varepsilon\\+n\beta\delta\end{matrix}\Big\}ydx\begin{matrix}+m\beta\varepsilon\\+n\beta\varepsilon\end{matrix}\Big\}ydy\begin{matrix}+m\alpha\zeta\\+n\gamma\delta\end{matrix}\Big\}dx\left.\begin{matrix}+m\beta\zeta\\+n\gamma\varepsilon\end{matrix}\right\}dy\Bigg).$$

V. If there should be $V=ylx$, there will be

$$dV=dylx+\frac{ydx}{x}.$$

VI. If there should be $V=x^y$, there will be

$$dV=yx^{y-1}dx+x^y dylx.$$

VII. If there should be $V=\text{Atang}\frac{y}{x}$, there will be $dV=\frac{xdy-ydx}{xx+yy}$.

VIII. If there should be $V=\sin x\cos y$, there will be

$$dV=dx\cos x\cos y-dy\sin x\sin y.$$

IX. If there should be $V=\frac{e^z y}{\sqrt{(xx+yy)}}$, there will be

$$dV=\frac{e^z ydz}{\sqrt{(xx+yy)}}+\frac{e^z(xxdy-yxdx)}{(xx+yy)\sqrt{(xx+yy)}}.$$

X. If there should be $V = e^{z}\text{Asin}\frac{x-\sqrt{(xx-yy)}}{x+\sqrt{(xx-yy)}}$, there may be found

$$dV = e^{z}dz\text{Asin}\frac{x-\sqrt{(xx-yy)}}{x+\sqrt{(xx-yy)}} + e^{z}\cdot\frac{xydy-yydx}{\left(x+\sqrt{(xx-yy)}\right)(xx-yy)^{\frac{3}{4}}\sqrt{x}}$$

217. Because we have seen, if *V* were some function of the two variables *x* and *y*, the differential of this is accustomed to be a form of this kind $dV = Pdx + Qdy$, in which there shall be the functions *P* and *Q* depending on the function *V* determined by that [equation], it follows that these two quantities *P* and *Q* depend on each other in a certain manner, because therefore each may depend on the same function *V*. Therefore whatever this connection shall be between the finite quantities *P* and *Q*, which we will investigate henceforth, it is evident that not all differential formulas of this kind $Pdx + Qdy$, in which P and Q for argument's sake shall be formed from *x* and *y*, are able to be the differentials of any finite function *V* of *x* and *y*. For unless that relation intercedes between the functions P and *Q*, as the nature of the differentiation requires, a differential of this kind $Pdx + Qdy$ plainly cannot arise by differentiation, and thus in turn the integral will not be obtained.

218. Therefore in integration it is of the greatest concern to know this relation between the quantities *P* and *Q*, so that the differentials, which actually have arisen from the differentiation of some finite functions, are able to be discerned from these, which have been formed freely and cannot be integrated. But although we cannot yet undertake the business of integration, still it may be agreed that the relation be investigated by looking deeper into the nature of the actual differential ; clearly a knowledge of this is of the greatest necessity not only towards calculating the integral, for which here we prepare the way, but also sheds a conspicuous light on the process of differentiation itself. Therefore in the first place, *V* shall be a function of the two variables *x* and *y*, in the differential of this $Pdx + Qdy$ the differential of each *dx* et *dy* is required to be present. Therefore neither can it be that $P = 0$ or $Q = 0$. Hence if P were a function of *x* and *y* themselves, the formula *Pdx* cannot be the differential of a finite quantity or no finite quantity exists, of which the differential shall be *Pdx*.

219. Thus no finite quantity *V* is given, either algebraic or transcending, the differential of which shall be *yxdx*, if indeed *y* shall be a variable quantity not depending on *x*. For if we may put a finite quantity *V* of this kind to be given, because *y* enters into the differential, it is necessary that *y* also be present in the quantity *V* itself; truly if *V* should contain *y*, on account of the variability of *y* by necessity also *dy* must be present in the differential of *V*. Yet since which is not present, it cannot happen, that the differential *yxdx* shall have arisen from the differentiation of some finite quantity. Therefore since it may be apparent that the formula $Pdx + Qdy$, if *Q* shall be 0 and *P* may contain *y*, the actual differential cannot be possible, and likewise it is understood for the quantity *Q* not to be attributed any value desired, but with a value depending on the value of *P*.

220. Therefore so that we may investigate this relation between the differentials P and Q in the differential $dV = Pdx + Qdy$, in the first place we may put V to be a function of no dimensions of x and y; indeed we may rise from particular cases to the general relation. Therefore if we may put $y = tx$, the quantity x vanishes at once from the function V and there will be produced a function of t only, which shall be $= T$, the differential of this shall be $= \Theta dt$ for some function of Θ of t present. Therefore we may put everywhere in the differential $Pdx + Qdx$ also $y = tx$ and $dy = tdx + xdt$, with which done there will be produced $Pdx + Qtdx + Qxdt$, in which since dx may not be present, it is necessary, so that there shall be $P + Qt = 0$ and thus $Q = -\frac{P}{t} = -\frac{Px}{y}$, or there shall be

$$Px + Qy = 0,$$

from which the relation between P and Q will become known for this case. Then there must be $\Theta = Qx$ and thus Qx = to a function of t, that is to a function of zero dimensions of x and y themselves. And on account of $Q = \frac{\Theta}{x}$ there will be made $P = -\frac{\Theta y}{xx}$ and as Px as well as Qy will be functions of no dimension of x and y.

221. Therefore if a function of no dimensions of x and y which shall be $= V$, may be differentiated, the differential of this $dV = Pdx + Qdy$ thus always will be prepared, so that there shall be $Px + Qy = 0$. That is: if in the differential in place of the differentials dx and dy there are written x and y, there will result a quantity $= 0$, as it may be apparent to come upon from the use in these examples.

I. Let there be $V = \frac{x}{y}$; there shall be

$$dV = \frac{ydx - xdy}{yy}$$

and on putting x in place of dx and y in place of dy there shall be $\frac{yx - xy}{yy} = 0.$.

II. Let there be $V = \frac{x}{\sqrt{(xx - yy)}}$; there shall be

$$dV = \frac{-yydx + yxdy}{(xx - yy)^{\frac{3}{2}}},$$

from which there will become $\frac{-yyx + yyx}{(xx - yy)^{\frac{3}{2}}} = 0.$

III. Let there be $V = \frac{y + \sqrt{(xx + yy)}}{-y + \sqrt{(xx + yy)}}$, which is a function of zero dimensions of x and y; there shall be

$$dV = \frac{2xxdy - 2xydx}{\left(\sqrt{(xx+yy)} - y\right)^2 \sqrt{(xx+yy)}},$$

which form on putting x and y in place of dx and dy becomes $= 0$.

IV. Let there be $V = l\frac{x+y}{x-y}$; there shall be

$$dV = \frac{2xdy - 2ydx}{xx - yy}$$

And

$$\frac{2xy - 2yx}{xx - yy} = 0.$$

V. Let there be $V = \mathrm{Asin}\frac{\sqrt{(x-y)}}{\sqrt{(x+y)}}$; there shall be

$$dV = \frac{ydx - xdy}{(x+y)\sqrt{2y(x-y)}},$$

which formula enjoys the same property.

222. Now we may contemplate other homogeneous functions and let V be a function of n dimensions of x and y. Whereby if there is put $y = tx$, V will adopt a form of this kind Tx^n with T proving to be a function of t and let $dT = \Theta dt$; there will be

$$dV = x^n \Theta dt + nTx^{n-1}dx.$$

But if therefore we may put in place $dV = Pdx + Qdy$, on account of $dy = tdx + xdt$ there will become

$$dV = Pdx + Qtdx + Qxdt;$$

which form because it must agree with that, will be

$$P + Qt = nTx^{n-1} = \frac{nV}{x}$$

on account of $V = Tx^n$. On account of this because $t = \frac{y}{x}$ will become

$$Px + Qy = nV;$$

which equation thus defines the relation between P and Q, so that if the one shall be known, the other can be found easily. Because again there is $Qx = x^n\Theta$, Qx will be a function of n dimensions of x and y and thus also Qy and Px.

223. Therefore if in the differential of any homogeneous function of *x* and *y* in place of *dx* and *dy* there may be put *x* and *y*, the quantity arising will be equal to the function itself, of which the differential was proposed, multiplied by the number of dimensions.

1. If there shall be $V=\sqrt{(xx+yy)}$, there will be $n=1$ and on account of

$$dV=\frac{xdx+ydy}{\sqrt{(xx+yy)}}$$

there becomes

$$\frac{xx+yy}{\sqrt{(xx+yy)}}=V=\sqrt{(xx+yy)}$$

II. If there shall be $V=\frac{y^3+x^3}{y-x}$, there will be n = 2 and

$$dV=\frac{2y^3dy-3yyxdy+3yxxdx-2x^3dx+y^3dx-x^3dy}{(y-x)^2}.$$

There may be put *x* for *dx* and *y* for *dy;* there will arise

$$\frac{2y^4-2y^3x+2yx^3-2x^4}{(y-x)^2}=\frac{2y^3+2x^3}{y-x}=2V.$$

III. If there shall be $V=\frac{1}{(yy+xx)^2}$, there will be $n=-4$ and

$$dV=-\frac{4ydy+4xdx}{(yy+xx)^3}.$$

Which formula with *x* and *y* put in place of *dx* and *dy* will change into

$$-\frac{4yy+4xx}{(yy+xx)^3}=-4V.$$

IV. If there shall be $V=xxl\frac{y+x}{y-x}$, there will be $n=2$ and

$$dV=2xdxl\frac{y+x}{y-x}+\frac{2xx(ydx-xdy)}{yy-xx};$$

but with the substitution recalled made there arises

$$2xxl\frac{y+x}{y-x}=2V.$$

224. A similar property will be observed, if V were a homogeneous function of several variables ; therefore let V be a function of the quantities $x, y, z,$ which taken together everywhere complete n dimensions, and the differential of this kind will have the form $Pdx + Qdy + Rdz$. Now there is put $y = tx$ and $z = sx$, so that there shall be

$$dy = tdx + xdt \text{ and } dz = sdx + xds,$$

and the function V will adopt this form Ux^n with U proving to be a function of the two variables t and s; hence therefore, if there is put in place $dU = pdt + qds$, there will become

$$dV = x^n pdt + x^n qds + nUx^{n-1}dx.$$

But the first form will give

$$dV = Pdx + Qtdx + Qxdt + Rsdx + Rxds;$$

which combined with that will give

$$P + Qt + Rs = nUx^{n-1} = \tfrac{nV}{x},$$

from which there will be obtained

$$Px + Qy + Rz = nV;$$

which same property may be extended to any number of extra variables.

225. Therefore if there were a proposed homogeneous function of some number of variables x, y, z, v etc., the differential of this will always have this property, so that, if in place of the differentials dx, dy, ds, dv etc. there may be written the finite quantities $x, y, z,$ v etc., the proposed function itself may be produced multiplied by the number of variables. And this rule prevails also, if V were a homogeneous function of only a single variable x. For in this case V will be a power of x, for example $V = ax^n$, which is a homogeneous function of n dimensions; clearly no other function of x may be given, in which x may be established with n dimensions everywhere besides the power x^n. Therefore since there shall be $dV = nax^{n-1}dx$, there may be put x in place of dx and there will be produced nax^n, that is nV. Therefore this conspicuous property of homogeneous functions merits to be noted carefully, since it will bring great usefulness to the integral calculus.

226. So that now we may inquire into a general relation between the quantities P and Q, which constitute the differential $Pdx + Qdy$ of some function V of the two variables x and y, that it will be necessary to attend to in the following. Therefore let V be some function of *x* and *y* and we may put *V* to change into *R*, if in place of *x* there is put $x+dx$ *;* but on putting $y+dy$ in place of *y*, *V* may become *S;* but if at the same time there may be written $x+dx$ in place of *x* and $y+dy$ in place of *y*, *V* may be changed into V^{I}. And thus since *R* may arise from *V* on putting $x+dx$ in place of *x*, it is evident, if further in *R* there may be put $y+dy$ in place of *y*, then V^{I} is produced; indeed it is the same, as if in *V* at once there may be put $x+dx$ in place of *x* and $y+dy$ in place of *y*. In a similar manner, if in *S* there is put $x+dx$ in place of *x*, because *S* now has come from *V* on putting $y+dy$ in place of *y*, and V^{I} will be produced anew, as may be seen more clearly from this table.

Quantity	will become	if in place of	there may be put
V	R	x	$x+dx$
V	S	y	$y+dy$
V	V^{I}	x y	$x+dx$ $y+dy$
R	V^{I}	y	$y+dy$
S	V^{I}	x	$x+dx$

227. Therefore if *V* may be differentiated thus, so that only *x* may be treated as the variable, *y* indeed may be treated as a constant, because on putting $x+dx$ in place of *x* the function *V* will change into *R*, the differential of this will be $=R-V$ *;* but from the form $dV=Pdx+Qdy$ the same differential follows to be $=Pdx$, from which there will be $R-V=Pdx$. But if now in place of *y* there may be put $y+dy$, *x* truly may be treated as constant, because *R* will change into V^{I} and *V* into *S*, the quantity $R-V$ will change into $V^{\mathrm{I}}-S$ and thus the differential of this $R-V=Pdx$, which arises, if only *y* may be assumed to be variable, will be

$$=V^{\mathrm{I}}-R-S+V\,.$$

In a similar manner, since on putting $y+dy$ in place of *y*, *V* may change into *S*, $S-V$ will be the differential of *V* itself on putting *y* alone to be variable and therefore there will be $S-V=Qdy$; now since on putting $x+dx$ in place of x, *S* crossed into V^{I} and *V* into *R*, the quantity $S-V$ will change into $V^{\mathrm{I}}-R$ and the differential of $S-V=Qdy$, which arises, if only the variable *x* may be put in place, will be

$$=V^{\mathrm{I}}-R-S+V$$

which agrees completely with the differential found before.

228. From this agreement the following conclusion is deduced. If the differential of any function *V* of the two variables *x* and *y* were $dV = Pdx + Qdy$, then the differential of *Pdx*, which arises, if only the quantity *y* may be treated as a variable, *x* truly as a constant, will be equal to the differential of *Qdy*, which arises, if only *x* may be treated as a variable, *y* truly as a constant. Clearly if by putting only *y* to be variable there were $dP = Zdy$, the differential of *Pdx* taken in the prescribed manner will be $= Zdxdy$; and on putting only *x* to be variable there will be also $dQ = Zdx$; for thus the differential of *Qdy* taken in the prescribed manner also will become $= Zdxdy$. And by this reasoning the relation is understood, which exists between the quantities *P* and *Q* and consists of this in a few words, that the differential of *Pdx* on placing *x* constant shall be equal to the differential of *Qdy* on placing *y* constant.

229. That conspicuous property may be understood more clearly, if we illustrate it with some examples
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1. Therefore let there be $V = yx$; there will be

$$dV = ydx + xdy$$

and thus

$$P = y \text{ and } Q = x;$$

from which on putting *x* constant there will be

$$d.Pdx = dxdy$$

and on putting *y* constant there will be

$$d.Qdy = dxdy$$

and thus these two differentials are equal to each other.

II. Let there be $V = \sqrt{(xx + 2xy)}$; there will be

$$dV = \frac{xdx + ydx + xdy}{\sqrt{(xx + 2xy)}}$$

and thus

$$P = \frac{x + y}{\sqrt{(xx + 2xy)}} \text{ and } Q = \frac{x}{\sqrt{(xx + 2xy)}},$$

from which on putting *x* constant there will be

$$d.Pdx = \frac{xydxdy}{(xx + 2xy)^{\frac{3}{2}}}$$

and on putting y constant there will be

$$d.Qdy = \frac{xydxdy}{(xx+2xy)^{\frac{3}{2}}}.$$

III. Let there be $V = x\sin Ay + y\sin Ax$ and there will be [Ay means 'the arc y' , etc.]

$$dV = dx\sin Ay + x\,dy\cos y + dy\sin Ax + y\,dx\cos x.$$

Whereby there will be

$$Pdx = dx\sin Ay + ydx\cos x \text{ and } Qdy = dy\sin Ax + xdy\cos y.$$

Therefore on putting x constant there will be

$$d.Pdx = dxdy\cos y + dxdy\cos x$$

and on putting y constant there will be

$$d.Qdy = dxdy\cos y + dxdy\cos x.$$

IV. Let there be $V = x^y$; there will be

$$dV = x^y dylx + yx^{y-1}dx$$

And

$$Pdx = yx^{y-1}dx \text{ and } Qdy = x^y dylx.$$

On account of which on putting x constant there will be had

$$d.Pdx = x^{y-1}dxdy + yx^{y-1}dxdylx$$

and on putting y constant there will be

$$d.Qdy = yx^{y-1}dxdylx + x^{y-1}dxdy.$$

230. That property also may be stated in this manner, by which the nature of all functions selection may become known, which involve two variables. If some function V of the two variables x and y may be differentiated by putting x alone for the variable and this differential again be differentiated on putting y alone for the variable, then after this twofold differentiation the same will be produced, as if the in the inverse order the function V may be differentiated first on putting y alone to be variable and this differential may be differentiated again on putting x alone to be variable; clearly in each case the same expression will be produced of this form $Zdxdy$. The reason of this identity evidently follows from the preceding property ; for if V may be differentiated on putting x alone to be variable, Pdx will be produced, and if V may be differentiated on putting y alone to be variable, Qdy will be produced ; now we have shown before, that the differentials of these differentials in the manner taken between these , are equal to each other. Moreover this characteristic follows at once from the reasoning advanced in § 227.

231. The relation between P and Q, if $Pdx+Qdy$ were the differential of the function V can be shown in the following manner also. Because P and Q are functions of x and y, they may be differentiated both on putting each of x and y to be variable. Clearly if there should be $dV = Pdx+Qdy$, there shall be

$$dP = pdx+rdy \text{ and } dQ = qdx+sdy .$$

Therefore on putting x constant there will be

$$dP = rdy \text{ and } d.Pdx = rdxdy .$$

Then on putting y constant there will be

$$dQ = qdx \text{ and } d.Qdy = qdxdy .$$

Therefore since the two differentials $rdxdy$ and $qdxdy$ shall be equal to each other, it follows that there shall be

$$q = r .$$

Therefore the functions P and Q thus in turn are connected to each other, so that, if both may be differentiated, as we have done, the quantities q and r will be made equal to each other. But for the sake of brevity at least in this chapter thus r and q may be denoted in the usual manner, so that r may be indicated by $\left(\frac{dP}{dy}\right)$ by which writing P may thus be designated to be differentiated, so that y alone may be treated as the variable and that differential may be divided by dy ; for thus there will be produced the finite quantity r. In a similar manner $\left(\frac{dQ}{dx}\right)$ will signify the finite quantity q, because by this ratio it may be indicated that the function Q to be differentiated on putting x to be variable and then the differential must be divided by dx.

232. Therefore we may use this manner of writing, even if it may bring another ambiguity, which yet here may be avoided by the conclusions, so that we may avoid long winded ways of describing the conditions of differentiation, and thus we may express briefly in words the relation between P and Q, so that we may say

$$\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$$

Evidently the denominator in fractions of this kind besides its own significance, by which the numerator must be divided, may indicate that the differential of the numerator is required to be taken, so that quantity alone, of which the differential constitutes the denominator, may be seen as the variable. For in this manner by division the differentials at once emerge from the calculation and these fractions $\left(\frac{dP}{dy}\right)$ and $\left(\frac{dQ}{dx}\right)$ will show finite quantities, which in the present case are equal to each other. And thus in this manner in turn the quantities p and s thus will be allowed to be denoted, so that there shall be

$$p=\left(\frac{dP}{dx}\right) \text{ and } s=\left(\frac{dQ}{dy}\right),$$

if indeed, as has been warned, the differentiation of the numerator by the denominator may be restricted.

233. This property agrees wonderfully with the property, which before we have shown to be present in homogenous functions. For let V be a homogeneous function of n dimensions of x and y and there may be put $dV = Pdx + Qdy$ and we have shown to be $nV = Px + Qy$ and thus

$$Q=\frac{nV}{y}-\frac{Px}{y}$$

Let $dP = pdx + rdy$ and there will be

$$\left(\frac{dP}{dy}\right)=r,$$

to which thus there may be shown $\left(\frac{dQ}{dx}\right)$ to be equal. Q may be differentiated on putting x alone to be variable, and because by this hypothesis there is

$$dQ=\frac{nPdx}{y}-\frac{Pdx}{y}-\frac{xpdx}{y},$$

there will become

$$\left(\frac{dQ}{dx}\right)=\frac{(n-1)P}{y}-\frac{Px}{y}$$

and there must be

$$\frac{(n-1)P}{y}-\frac{px}{y}=r \text{ or } (n-1)P=px+ry.$$

Which equality thence will be made transparent, because *P* shall be a homogeneous function of dimensions $n-1$ in *x* and *y*, from which the differential $dP = pdx + rdy$ on account of the property of homogeneous functions thus must be prepared, so that there shall be

$$(n-1)P = px + ry.$$

234. That property, which shall be $\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$, as we have shown to be common for all functions of two variables *x* and *y*, also may reveal the nature of functions of three or more variables for us. Let *V* be some function of the three variables *x, y* and *z* and there may be put

$$dV = Pdx + Qdy + Rdz.$$

But if therefore in this differentiation *z* may be treated as a constant, there becomes everywhere $dV = Pdx + Qdy$; but in this case from the preceding there must become $\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$. Then if the quantity *y* may be assumed constant, there becomes $dV = Pdx + Rdz$; and therefore $\left(\frac{dP}{dz}\right)=\left(\frac{dR}{dx}\right)$. And then on putting *x* constant there may be found $\left(\frac{dQ}{dz}\right)=\left(\frac{dR}{dy}\right)$. Therefore in the differential $Pdx + Qdy + Rdz$ of the function *V* the quantities *P, Q* and *R* thus depend on each other, so that there shall be

$$\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right),\quad \left(\frac{dP}{dz}\right)=\left(\frac{dR}{dx}\right),\quad \text{and}\quad \left(\frac{dQ}{dz}\right)=\left(\frac{dR}{dy}\right).$$

235. Hence an analogous property follows for that of functions which involve three or more variables, which we have shown above with functions of two variables (§ 230). If *V* were some function of the three variables *x, y* and *z* and that may be continually differentiated three times, thus so that in the first place a single quantity, for example *x,* alone may be put variable, in the second differentiation only *y* and in the third only *z* may be assumed variable, an expression of this form will be produced *Zdxdydz,* which may be found the same , in whatever order the quantities *x, y* et *z* may be arranged. Therefore the same expression *Zdxdydz* may be come upon in six different ways after the triple differentiation, because the order of the quantities *x, y* et *z* can be varied six times. Therefore whatever order may be selected, if the function *V* may be differentiated on putting only the first quantity variable and this differential may be differentiated anew with only the second variable and this differential again may be differentiated with only the third variable, the same expression will be produced, in whatever the order of the quantities *x, y* and *z* may be varied.

236. So that the account of this property may be understood more clearly, we may put in place

$$dV = Pdx + Qdy + Rdz \text{ ;}$$

then also we may differentiate the quantities P, Q and R and they will be the differentials shown before, prepared thus

$$dP = pdx + sdy + tdz$$
$$dQ = sdx + qdy + udz$$
$$dR = tdx + udy + rdz.$$

Now V may be differentiated on putting x alone variable, and Pdx will be produced ; which differential may again be differentiated on putting y alone variable and there will be had $sdxdy$; which if it may be differentiated on putting z alone variable, after it were divided by $dxdydz$, there would be obtained $\left(\frac{ds}{dz}\right)$. Now the variables may be rearranged in this order y, x, z and the first differentiation will give Qdy, the second $sdxdy$ and the third (with the division performed by $dxdydz$) will given $\left(\frac{ds}{dz}\right)$ as before. The variables may be arranged in this order z, y; x and the first differentiation will give Rdz, the second $udydz$, the third now on division by $dxdydz$ provides $\left(\frac{du}{dx}\right)$. But since on putting y constant there shall be $dQ = sdx + udz$, there will be $\left(\frac{ds}{dz}\right) = \left(\frac{du}{dx}\right)$, as equally has been shown.

237. We may put to be $V = \frac{xxy}{aa - zz}$ and we may differentiate this function three times in as many ways, as the order of the variables can be varied x, y, z.

	I. Differential.	*II. Differential.*	*III. Differential.*
on placing as variable	only x $\frac{2xydx}{aa-zz}$,	only y $\frac{2xdxdy}{aa-zz}$,	only z $\frac{4xzdxdydz}{(aa-zz)^2}$.
on placing as variable	only x $\frac{2xydx}{aa-zz}$,	only z $\frac{4xyzdxdz}{(aa-zz)^2}$,	only y $\frac{4xzdxdydz}{(aa-zz)^2}$.
on placing as variable	only y $\frac{xxdy}{aa-zz}$,	only x $\frac{2xdxdy}{aa-zz}$,	only z $\frac{4xyzdxdz}{(aa-zz)^2}$.
on placing as variable	only y	only z	only x

	$\frac{xxdy}{aa-zz}$,	$\frac{2xxzdydz}{(aa-zz)^2}$,	$\frac{4xzdxdydz}{(aa-zz)^2}$.
on placing as variable	only z,	only x	only y
	$\frac{2xxyzdz}{(aa-zz)^2}$,	$\frac{4xyzdxdz}{(aa-zz)^2}$,	$\frac{4xzdxdydz}{(aa-zz)^2}$.
on placing as variable	only z	only y	only x
	$\frac{2xxyzdz}{(aa-zz)^2}$,	$\frac{2xxzdydz}{(aa-zz)^2}$,	$\frac{4xzdxdydz}{(aa-zz)^2}$.

From which example it may be apparent, in whatever order the three variables have been assumed, the same expression is produced always after the triple differentiation

$$\frac{4xzdxdydz}{(aa-zz)^2}.$$

238. But as after the threefold differentiation the same expression has been arrived at, thus also an agreement may be reached with differentials, which the second differentiation will provide. Clearly in these any expression occurs twice ; from which it is apparent, which formulas may be affected by the same differentials, also the same are to be equal to each other and the third differentials therefore are all equal to each other, because they have been affected by the same differentials $dxdydz$. Therefore hence we conclude, if V were a function of some number of variables x, y, z, v, u etc. and it may be differentiated a number of times, so that always only with a single quantity assumed variable, then, as often as they may arrive at the same expression, which have been affected by the same differentials, these too become equal to each other. Thus in a twofold differentiation an expression of this kind $Zdxdy$ may be arrived at, while in the one only x, in the other only y has been assumed variable, and likewise it is, either the first or the latter shall be assumed variable. In a similar manner the same expression $Zdxdydz$ may arise in six different ways from a threefold differentiation, and by twenty four various ways there may be come upon the same expression of this form $Zdxdydzdv$ after a fourfold differentiation and thus henceforth.

239. Anyone can acknowledge the truth of these theorems from the principles explained before, by applying a little attention and may be examined more easily from their own meditation than by so many circuitous explanations, without which the demonstrations may not be brought forwards. Because truly the recognition of these properties is of the greatest importance in the integral calculus, beginners are to be reminded, that not only these properties themselves are to be carefully considered and the truth of these scrutinised, but also they may be approved by many examples, so that by this agreement this material itself becomes more familiar and the reward thence produced after they are able to be understood more easily. Nor truly only beginners, but also these who have been imbued with the principles of calculus, to this they are to be encouraged, because in almost all guides to this part of analysis, this argument is accustomed to be omitted completely. For several authors have been content to show only the rules of differentiation to be prescribed and the use of these in higher geometry, nor have they inquired into the nature and properties of the differential, yet from which there are numerous aids in the integral calculus. Which nearly new argument on this account has been seen to be pursued further in this chapter, so that likewise a way may be prepared for the other more difficult integrations, and the work to be undertaken later may be lightened.

240. Therefore from these known properties, in which the differentials of functions have the use of involving two or more variables, we will be able to discern easily, each proposed formula of the differential, in which there occurs two or more variables, shall arise from the differentiation of some finite functions or otherwise. For if in the formula $Pdx+Qdy$ there should not be $\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$, certainly we will be able to affirm that no function of x and y may arise, the differential of which shall be $=Pdx+Qdy$, nor therefore below in the integral calculus will integral formulas of this kind be able to be found. Thus since in $yxdx+xxdy$ the required condition shall not be present, no function is given, the differential of which is $=yxdx+xxdy$. But whether always, as often as there is $\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$, a formula sought may arise from the differentiation of some function, which certainly will be able to be confirmed at last from the principles of integration.

241. If in the differential formula proposed there shall be three or more variables present, such as $Pdx+Qdy+Rdx$, then that arising from the differentiation generally cannot be treated, unless these three conditions have been accommodated in that, so that there shall be

$$\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right),\quad \left(\frac{dP}{dz}\right)=\left(\frac{dR}{dx}\right),\quad \text{and}\quad \left(\frac{dQ}{dz}\right)=\left(\frac{dR}{dy}\right).$$

Of which conditions if only one is present, certainly we must affirm that there is no obvious function of *x*, *y* et *z*, the differential of which shall be $Pdx+Qdy+Rdz$; therefore the integrals of differential formulas of this kind cannot be required and hence they are said to be completely unreceptive to integration [according to this rule, at least, by not being a total derivative]. But it is understood easily that it is required to distinguish the differential formulas before the calculation of the integral, whether they shall be capable of integration, as the investigation of the integral actually is undertaken.

CAPUT VII

DE DIFFERENTIATIONE FUNCTIONUM DUAS PLURESVE VARIABILES INVOLVENTIUM

208. Si duae pluresve quantitates variabiles x, y, z a se invicem prorsus non pendeant, fieri potest, ut, etiamsi omnes sint variabiles, tamen, dum una crescit decrescitve, reliquae maneant invariatae; quia enim nullum nexum inter se habere ponuntur, immutatio unius reliquas non afficit. Neque ergo differentialia quantitatum y et z pendebunt a differentiali ipsius x ideoque, dum x differentiali suo dx augetur, quantitates y et z vel eaedem manere vel quomodocunque pro lubitu variari possunt. Quodsi igitur differentiale quantitatis x statuatur dx, reliquarum quantitatum differentialia dy et dz manent indeterminata atque pro arbitrio nostro vel prorsus nihil vel infinite parva ad dx quamvis rationem tenentia denotabunt.

209. Plerumque autem litterae y et z functiones ipsius x vel incognitas, vel quarum relatio ad x non spectatur, significare solent hocque casu earum differentialia dy et dz certam ad dx relationem habebunt. Sive autem y et z pendeant ab x sive secus, ratio differentiationis, quam hic spectamus, eodem redit. Quaerimus enim functionis, quae ex pluribus variabilibus x, y et z utcunque sit formata, differentiale, quod accipit, dum singulae variabiles x, y et z suis differentialibus dx, dy et dz crescunt. Ad hoc ergo inveniendum in functione proposita ubique loco variabilium quantitatum x, y, z scribatur respective $x+dx$, $y+dy$, $z+dz$ et ab expressione hoc modo resultante auferatur ipsa functio proposita; residuum dabit ipsum differentiale, quod quaeritur, quemadmodum ex natura differentialium luculenter constat.

210. Sit X functio ipsius x eiusque differentiale seu augmentum, dum x differentiali suo dx crescit, sit $=Pdx$. Deinde sit Y functio ipsius y eiusque differentiale $=Qdy$, quod augmentum Y accipit, dum y abit in $y+dy$, atque Z sit functio ipsius z eiusque differentiale sit $=Rdz$; quae differentialia Pdx, Qdy, Rdz ex natura functionum X, Y et Z ope praeceptorum supra datorum inveniri poterunt. Quodsi ergo proposita fuerit haec quantitas $X+Y+Z$, quae utique erit functio trium variabilium x, y et z, eius differentiale erit $=Pdx+Qdy+Rdz$. Utrum autem haec tria differentialia sint inter se homogenea necne, perinde est. Termini enim, qui continent potestates ipsius dx, prae Pdx aeque evanescunt, ac si reliqua membra Qdy et Rdz abessent, similisque est ratio terminorum, qui in differentiatione functionum Y et Z sunt neglecti.

211. Retineant *X, Y* et *Z* easdem significationes sitque proposita ista functio *XYZ* ipsarum *x, y* et *z,* cuius differentiale investigari oporteat. Quoniam, si $x+dx$ loco *x*, $y+dy$ loco *y* et $z+dz$ loco *z* scribatur, abit *X* in $X+Pdx$, Y in $Y+Qdy$ et Z in $Z+Rdz$, ipsa functio proposita *XYZ* abibit in

$$\left(X+Pdx\right)\left(Y+Qdy\right)\left(Z+Rdz\right)=XYZ+YZPdx+XZQdy+XYRdz$$
$$+\ ZPQdxdy+YPRdxdz+XQRdydz+PQRdxdydz.$$

At quia *dx, dy* et *dz* sunt infinite parva, sive inter se sint homogenea sive non, ultimus terminus prae uno quoque praecedentium evanescit. Deinde terminus *ZPQdxdy* tam prae *YZPdx* quam prae *XZQdy* evanescit; atque ob eandem rationem termini *YPRdxdz* et *XQRdydz* evanescent. Ablata ergo ipsa functione proposita *XYZ* erit eius differentiale

$$=YZPdx+XZQdy+XYRdz.$$

212. Exempla haec functionum trium variabilium *x, y* et *z,* quibus pro lubitu quisque pIura adiicere potest, sufficiunt ad ostendendum, si functio quaecunque trium variabilium *x, y* et *z* proponatur, utcunque etiam hae variabiles inter se fuerint permixtae, eius differentiale semper huiusmodi formam esse habiturum $pdx+qdy+rdz$, ubi *p, q* et *r* futurae sint singulae functiones vel omnium trium variabilium *x, y* et *z* vel binarum vel unius tantum, prout ratio compositionis, qua functio proposita ex variabilibus *x, y* et *z* atque constantibus formatur, fuerit comparata. Simili modo, si proponatur functio quatuor pluriumve variabilium *x, y, z* et *v*, eius differentiale semper huiusmodi formam habebit

$$pdx+qdy+rdz+sdv.$$

213. Consideremus primum functionem duarum tantum variabilium *x* et *y,* quae sit $=V$ cuius ergo differentiale ita se habebit, ut sit

$$dV=pdx+qdy.$$

Si igitur quantitas *y* assumeretur constans, foret $dy=0$ ideoque functionis *V* differentiale esset *pdx;* sin autem *x* statueretur constans, ut esset $dx=0$ solaque *y* maneret variabilis, tum ipsius *V* differentiale prodiret $=qdy$. Cum igitur utraque quantitate *x* et *y* variabili posita sit $dV=pdx+qdy$, ista regula pro differentianda functione *V* duas variabiles *x* et *y* involvente resultabit:
Ponatur primum sola x variabilis, altera vero y tanquam constans tractetur et quaeratur ipsius V differentiale, quod sit $=pdx$. *Deinde ponatur sola quantitas y variabilis, altera x pro constanti habita, et quaeratur ipsius V differentiale, quod sit* $=qdy$. *Quibus factis, posita utraque quantitate x et y variabili, fiet* $dV=pdx+qdy$.

214. Simili modo, cum functionis trium variabilium *x, y* et *z,* quae sit $=V$ differentiale huiusmodi habeat formam

$$dV = pdx + qdy + rdz\ ,$$

manifestum est, si sola quantitas x fuisset variabilis posita, reliquae vero y et z constantes mansissent, ob $dy = 0$ et $dz = 0$ prodiisset ipsius V differentiale $= pdx$. Pari modo inveniretur differentiale ipsius $V = qdy$, si x et z essent constantes solaque y poneretur variabilis; atque si x et y tanquam constantes tractarentur solaque z statueretur variabilis, prodiret differentiale ipsius $V = rdz$. Quare ad functionem trium pluriumve variabilium differentiandam consideretur seorsim quaelibet quantitas variabilis et functio pro qualibet differentietur, quasi reliquae omnes essent constantes; tum singula haec differentialia, quae ex singulis quantitatibus variabilibus sunt inventa, colligantur eritque aggregatum differentiale quaesitum functionis propositae.

215. In hac regula, quam pro differentiatione functionis quotcunque variabilium invenimus, continetur demonstratio regulae supra (§ 170) datae generalis, cuius ope functio quaecunque unicam variabilem complectens differentiari potest. Si enim pro singulis partibus ibi commemoratis totidem litterae diversae collocentur, functio speciem induet functionis totidem diversarum variabilium atque adeo modo hic praescripto differentiabitur successive unamquamque partem, quasi sola esset variabilis, tractando cunctaque differentialia, quae ex singulis partibus oriuntur, in unam summam coniiciendo; quae summa erit differentiale quaesitum, postquam pro singulis litteris valores fuerint restituti. Haec ergo regula latissime patet atque etiam ad functiones plurium variabilium, quomodocunque fuerint comparatae, extenditur. Unde eius usus per universum calculum differentialem est amplissimus.

216. Inventa ergo regula generali, cuius ope functiones quotcunque variabilium differentiari possunt, eius usum in nonnullis exemplis ostendisse iuvabit.

I. Si fuerit $V = xy$, erit $dV = xdy + ydx$.

II. Si fuerit $V = \frac{x}{y}$, erit

$$dV = \frac{dx}{y} - \frac{xdy}{yy}.$$

III. Si fuerit $V = \frac{y}{\sqrt{(aa-xx)}}$, erit

$$dV = \frac{dy}{\sqrt{(aa-xx)}} + \frac{yxdx}{(aa-xx)^{\frac{3}{2}}}.$$

IV. Si fuerit $V = (\alpha x + \beta y + \gamma)^m (\delta x + \varepsilon y + \zeta)^n$, erit

$$dV = m(\alpha x+\beta y+\gamma)^{m-1}(\delta x+\varepsilon y+\zeta)^{n}(\alpha dx+\beta dy)$$
$$+n(\alpha x+\beta y+\gamma)^{m}(\delta x+\varepsilon y+\zeta)^{n-1}(\delta dx+\varepsilon dy)$$

Sive

$$dV = (\alpha x+\beta y+\gamma)^{m-1}(\delta x+\varepsilon y+\zeta)^{n-1} \quad \text{in}$$
$$\left(\begin{matrix}+m\alpha\delta\\+n\alpha\delta\end{matrix}\right\}xdx\begin{matrix}+m\beta\delta\\+n\alpha\varepsilon\end{matrix}\Bigg\}xdy\begin{matrix}+m\alpha\varepsilon\\+n\beta\delta\end{matrix}\Bigg\}ydx\begin{matrix}+m\beta\varepsilon\\+n\beta\varepsilon\end{matrix}\Bigg\}ydy\begin{matrix}+m\alpha\zeta\\+n\gamma\delta\end{matrix}\Bigg\}dx\begin{matrix}+m\beta\zeta\\+n\gamma\varepsilon\end{matrix}\Bigg\}dy\Bigg).$$

V. Si fuerit $V = ylx$, erit

$$dV = dylx + \tfrac{ydx}{x}.$$

VI. Si fuerit $V = x^{y}$, erit

$$dV = yx^{y-1}dx + x^{y}dylx.$$

VII. Si fuerit $V = \text{Atang}\,\tfrac{y}{x}$, erit

$$dV = \tfrac{xdy-ydx}{xx+yy}.$$

VIII. Si fuerit $V = \sin x\cos y$, erit

$$dV = dx\cos x\cos y - dy\sin x\sin y.$$

IX. Si fuerit $V = \frac{e^{z}y}{\sqrt{(xx+yy)}}$, erit

$$dV = \frac{e^{z}ydz}{\sqrt{(xx+yy)}} + \frac{e^{z}(xxdy-yxdx)}{(xx+yy)\sqrt{(xx+yy)}}.$$

X. Si fuerit $V = e^{z}\text{Asin}\frac{x-\sqrt{(xx-yy)}}{x+\sqrt{(xx-yy)}}$, reperietur

$$dV = e^{z}dz\text{Asin}\frac{x-\sqrt{(xx-yy)}}{x+\sqrt{(xx-yy)}} + e^{z}\cdot\frac{xydy-yydx}{\left(x+\sqrt{(xx-yy)}\right)(xx-yy)^{\frac{3}{4}}\sqrt{x}}$$

217. Quoniam vidimus, si *V* fuerit functio quaecunque binarum variabilium *x* et *y*, eius differentiale huiusmodi habiturum esse formam $dV = Pdx + Qdy$, in qua sint *P* et *Q* functiones a functione *V* pendentes per eamque determinatae, sequitur has duas quantitates *P* et *Q* certo quodam modo etiam a se invicem pendere, propterea quod utraque ab eadem functione *V* pendet. Quicunque igitur sit iste nexus inter quantitates finitas *P* et *Q*, quem deinceps investigabimus, perspicuum est non omnes formulas differentiales huiusmodi $Pdx + Qdy$, in quibus *P* et *Q* pro lubitu sint ex *x* et *y* formatae, posse esse differentialia cuiuspiam functionis finitae *V* ipsarum *x* et *y*. Nisi enim ea relatio inter functiones P et *Q* intercedat, quam natura differentiationis requirit, huiusmodi differentiale $Pdx + Qdy$ oriri plane per differentiationem non potuit ideoque vicissim integrale non habebit.

218. In integratione igitur plurimum interest nosse hanc relationem inter quantitates *P* et *Q*, ut differentialia, quae revera ex differentiatione functionis cuiuspiam finitae sunt orta, dignosci queant ab iis, quae ad libitum sunt formata atque nulla integralia admittunt. Quanquam autem hic nondum integrationis negotium suscipimus, tamen ad naturam differentialium realium penitius inspiciendam conveniet hanc relationem investigari; quippe cuius cognitio non solum ad calculum integralem, ad quem hic viam paramus, est maxime necessaria, sed etiam in ipso calculo differentiali insignem lucem accendit. Primum igitur patet, si *V* sit functio duarum variabilium *x* et *y*, in eius differentiali $Pdx + Qdy$ utriusque differentiale *dx* et *dy* inesse oportere. Neque ergo potest esse $P = 0$ neque $Q = 0$. Hinc si *P* fuerit functio ipsarum *x* et *y*, formula *Pdx* nullius quantitatis finitae poterit esse differentiale seu nulla extat quantitas finita, cuius differentiale sit *Pdx*.

219. Sic nulla datur quantitas finita *V*, sive algebraica sive transcendens, cuius differentiale sit *yxdx*, si quidem sit *y* quantitas variabilis ab *x* non pendens. Si enim ponamus dari eiusmodi quantitatem finitam *V*, quia *y* in eius differentiale ingreditur, necesse est, ut *y* quoque in ipsa quantitate *V* insit; verum si *V* contineret ,*y*, ob variabilitatem ipsius *y* necessario quoque in differentiali ipsius *V* differentiale *dy* inesse deberet. Quod tamen cum non adsit, fieri nequit, ut differentiale *yxdx* ex cuiuspiam quantitatis finitae differentiatione sit ortum. Cum igitur pateat formulam $Pdx + Qdy$, si *Q* sit 0 et *P* contineat *y*, differentiale reale esse non posse, simul intelligitur quantitati *Q* non pro lubitu valorem tribui posse, sed eum a valore ipsius *P* pendere.

220. Quo igitur hanc relationem inter *P* et *Q* in differentiali $dV = Pdx + Qdy$ investigemus, ponamus primo *V* esse functionem nullius dimensionis ipsarum *x* et *y;* a casibus enim particularibus ad relationem generalem ascendamus. Quodsi ergo ponamus $y = tx$, ex functione *V* quantitas *x* prorsus evanescet prodibitque functio ipsius *t* tantum, quae sit $= T$, euius differentiale erit $= \Theta dt$ existente Θ functione ipsius *t*. Ponamus igitur quoque in differentiali $Pdx + Qdx$ ubique $y = tx$ et $dy = tdx + xdt$, quo facto prodibit $Pdx + Qtdx + Qxdt$, in quo cum *dx* non contineatur, necesse est, ut sit $P + Qt = 0$ ideoque $Q = -\frac{P}{t} = -\frac{Px}{y}$, seu erit

$$Px + Qy = 0,$$

unde relatio inter *P* et *Q* pro hoc casu innotescit. Deinde debet esse $\Theta = Qx$ ideoque *Qx* = functioni ipsius *t*, hoc est functioni nullius dimensionis ipsarum *x* et *y*. Atque ob $Q = \frac{\Theta}{x}$ fiet $P = -\frac{\Theta y}{xx}$ et tam *Px* quam *Qy* erunt functiones nullius dimensionis ipsarum *x* et *y*.

221. Si igitur functio nullius dimensionis ipsarum *x* et *y*, quae sit $= V$ differentietur, eius differentiale $dV = Pdx + Qdy$ semper ita erit comparatum, ut sit $Px + Qy = 0$. Hoc est: si in differentiali loco differentialium *dx* et *dy* scribantur *x* et *y*, resultabit quantitas $= 0$, uti in his exemplis usu venire patet.

I. Sit $V = \frac{x}{y}$; erit

$$dV = \frac{ydx - xdy}{yy}$$

atque posito *x* loco *dx* et *y* loco *dy* erit $\frac{yx - xy}{yy} = 0.$.

II. Sit $V = \frac{x}{\sqrt{(xx - yy)}}$; erit

$$dV = \frac{-yydx + yxdy}{(xx - yy)^{\frac{3}{2}}},$$

unde fit $\frac{-yyx + yyx}{(xx - yy)^{\frac{3}{2}}} = 0.$

III. Sit $V = \frac{y + \sqrt{(xx + yy)}}{-y + \sqrt{(xx + yy)}}$, quae est functio nullius dimensionis ipsarum *x* et *y;* erit

$$dV = \frac{2xxdy - 2xydx}{\left(\sqrt{(xx + yy)} - y\right)^2 \sqrt{(xx + yy)}},$$

quae forma positis *x* et *y* loco *dx* et *dy* fit $= 0$.

IV. Sit $V = l\frac{x + y}{x - y}$; erit

$$dV = \frac{2xdy - 2ydx}{xx - yy}$$

Atque

$$\frac{2xy - 2yx}{xx - yy} = 0.$$

V. Sit $V = \text{Asin}\frac{\sqrt{(x - y)}}{\sqrt{(x + y)}}$; erit

$$dV = \frac{ydx - xdy}{(x + y)\sqrt{2y(x - y)}},$$

quae formula eadem proprietate gaudet.

222. Contemplemur nunc alias functiones homogeneas sitque V functio n dimensionum ipsarum x et y. Quare si ponatur $y = tx$, induet V huiusmodi formam Tx^n existente T functione ipsius t sitque $dT = \Theta dt$; erit

$$dV = x^n \Theta dt + nTx^{n-1}dx.$$

Quodsi ergo statuamus $dV = Pdx + Qdy$, ob $dy = tdx + xdt$ fiet

$$dV = Pdx + Qtdx + Qxdt;$$

quae forma quoniam cum illa congruere debet, erit

$$P + Qt = nTx^{n-1} = \frac{nV}{x}$$

ob $V = Tx^n$. Hanc ob rem ob $t = \frac{y}{x}$ fiet

$$Px + Qy = nV;$$

quae aequatio relationem inter P et Q ita definit, ut, si altera sit cognita, altera facile inveniatur. Quia porro est $Qx = x^n \Theta$, erit Qx ideoque etiam Qy et Px functio n dimensionum ipsarum x et y.

223. Si ergo in differentiali cuiusvis functionis homogeneae ipsarum x et y loco dx et dy ponatur x et y, quantitas oriunda aequabitur ipsi functioni, cuius differentiale proponebatur, per numerum dimensionum multiplicatae.

1. Si sit $V = \sqrt{(xx + yy)}$, erit $n = 1$ et ob

$$dV = \frac{xdx + ydy}{\sqrt{(xx + yy)}}$$

fiet

$$\frac{xx + yy}{\sqrt{(xx + yy)}} = V = \sqrt{(xx + yy)}$$

II. Si sit $V = \frac{y^3 + x^3}{y - x}$, erit $n = 2$ et

$$dV = \frac{2y^3dy - 3yyxdy + 3yxxdx - 2x^3dx + y^3dx - x^3dy}{(y - x)^2}.$$

Ponatur x pro dx et y pro dy; orietur

$$\frac{2y^4-2y^3x+2yx^3-2x^4}{(y-x)^2}=\frac{2y^3+2x^3}{y-x}=2V.$$

III. Si sit $V=\frac{1}{(yy+xx)^2}$, erit $n=-4$ atque

$$dV=-\frac{4ydy+4xdx}{(yy+xx)^3}.$$

Quae formula positis x et y loco dx et dy abit in

$$-\frac{4yy+4xx}{(yy+xx)^3}=-4V.$$

IV. Si sit $V=xxl\frac{y+x}{y-x}$, erit $n=2$ atque

$$dV=2xdxl\frac{y+x}{y-x}+\frac{2xx(ydx-xdy)}{yy-xx};$$

facta autem memorata substitutione oritur

$$2xxl\frac{y+x}{y-x}=2V.$$

224. Similis proprietas observabitur, si V fuerit functio homogenea plurium variabilium; sit ergo V functio quantitatum x, y, z, quae coniunctim ubique n dimensiones adimpleant, atque differentiale huiusmodi habebit formam $Pdx+Qdy+Rdz$. Ponatur iam $y=tx$ et $z=sx$, ut sit

$$dy=tdx+xdt \text{ et } dz=sdx+xds,$$

atque functio V induet hanc formam Ux^n existente U functione binarum variabilium t et s; hinc ergo, si statuatur $dU=pdt+qds$, fiet

$$dV=x^npdt+x^nqds+nUx^{n-1}dx.$$

Prior autem forma dabit

$$dV=Pdx+Qtdx+Qxdt+Rsdx+Rxds;$$

quae cum illa collata praebet

$$P+Qt+Rs=nUx^{n-1}=\frac{nV}{x},$$

unde obtinetur

$$Px+Qy+Rz=nV;$$

quae eadem proprietas ad quotcunque plures variabiles extenditur.

225. Si igitur proposita fuerit functio homogenea quotcunque variabilium *x, y, z, v* etc., eius differentiale perpetuo hanc habebit proprietatem, ut, si loco differentialium *dx*, *dy*, *ds*, *dv* etc. scribantur respective quantitates finitae *x, y, z, v* etc., prodeat ipsa functio proposita per numerum dimensionum multiplicata. Haecque regula etiam valet, si *V* fuerit functio homogenea unicae tantum variabilis *x*. Hoc enim casu erit *V* potestas ipsius *x*, puta $V=ax^n$, quae est functio homogenea *n* dimensionum; nulla scilicet alia datur functio ipsius *x*, in qua *x* ubique *n* dimensiones constituat praeter postestatem x^n. Cum igitur sit $dV=nax^{n-1}dx$, ponatur *x* loco *dx* atque prodibit nax^n, hoc est nV. Ista ergo functionum homogenearum insignis proprietas diligenter notari meretur, cum in calculo integrali maximam afferat utilitatem.

226. Quo nunc in genere in relationem inter quantitates *P* et *Q*, quae differentiale $Pdx+Qdy$ functionis cuiuscunque *V* duarum variabilium *x* et *y* constituunt, inquiramus, ad sequentia attendi oportebit. Sit igitur *V* functio quaecunque ipsarum *x* et *y* atque ponamus *V* abire in *R*, si loco *x* ponatur $x+dx$; posito autem $y+dy$ loco *y* abeat *V* in *S;* quodsi autem simul $x+dx$ loco *x* et $y+dy$ loco *y* scribatur, mutetur *V* in V^{I}. Cum itaque *R* oriatur ex *V* posito $x+dx$ loco *x*, manifestum est, si ulterius in *R* ponatur $y+dy$ loco *y*, tum prodire V^{I}; idem enim est, ac si in *V* statim poneretur $x+dx$ loco *x* et $y+dy$ loco *y*. Simili modo, si in *S* ponatur $x+dx$ loco *x*, quia *S* iam orta est ex *V* posito $y+dy$ loco *y*, denuo prodibit V^{I}, uti ex hoc schematismo clarius perspicitur.

Quantitas	abit in	si loco	ponatur
V	R	x	$x+dx$
V	S	y	$y+dy$
V	V^{I}	x y	$x+dx$ $y+dy$
R	V^{I}	y	$y+dy$
S	V^{I}	x	$x+dx$

227. Si igitur *V* ita differentietur, ut tantum *x* tanquam variabilis, *y* vero tanquam constans tractetur, quia posito $x+dx$ loco *x* functio *V* abit in *R*, eius differentiale erit $=R-V;$ at ex forma $dV=Pdx+Qdy$ sequitur idem differentiale fore $=Pdx$, unde erit $R-V=Pdx$. Quodsi iam loco *y* ponatur $y+dy$, *x* vero tanquam constans tractetur, quia *R* abit in V^{I} et *V* in *S*, quantitas $R-V$ abibit in $V^{\mathrm{I}}-S$ ideoque ipsius $R-V=Pdx$ differentiale, quod oritur, si sola *y* variabilis assumatur, erit

$$=V^{\mathrm{I}}-R-S+V\,.$$

Simili modo, cum posito $y+dy$ loco *y* abeat *V* in *S*, erit $S-V$ differentiale ipsius *V* posita sola *y* variabili eritque propterea $S-V=Qdy$; nunc quia loco *x* posito $x+dx$ transit *S* in V^{I} et *V* in *R*, quantitas $S-V$ abibit in $V^{\mathrm{I}}-R$ atque ipsius $S-V=Qdy$ differentiale, quod oritur, si sola *x* variabilis statuatur, erit

$$=V^{\mathrm{I}}-R-S+V$$

quod prorsus congruit cum differentiali ante invento.

228. Ex hac convenientia deducitur sequens conclusio. Si functionis *V* cuiuscunque binarum variabilium *x* et *y* differentiale fuerit $dV=Pdx+Qdy$, tum differentiale ipsius *Pdx*, quod oritur, si sola quantitas *y* tanquam variabilis, *x* vero tanquam constans tractetur, aequale erit differentiali ipsius *Qdy*, quod oritur, si sola quantitas *x* tanquam variabilis, *y* vero tanquam constans tractetur. Si scilicet posita sola *y* variabili fuerit $dP=Zdy$, erit differentiale ipsius *Pdx* praescripto modo sumtum $=Zdxdy$; atque posita sola *x* variabili erit quoque $dQ=Zdx$; sic enim differentiale ipsius *Qdy* praescripto modo sumtum fiet quoque $=Zdxdy$. Hacque ratione intelligitur relatio, quae inter quantitates *P* et *Q* intercedit atque paucis verbis in hoc consistit, ut differentiale ipsius *Pdx* posito *x* constante aequale sit differentiali ipsius *Qdy* posito *y* constante.

229. Ista insignis proprietas clarius perspicietur, si eam nonnullis exemplis illustremus.

1. Sit igitur $V=yx$; erit

$$dV=ydx+xdy$$

ideoque

$$P=y \text{ et } Q=x;$$

unde posito *x* constante erit

$$d.Pdx=dxdy$$

et posito *y* constante erit

$$d.\,Qdy=dxdy$$

sicque haec duo differentialia inter se aequantur.

II. Sit $V=\sqrt{(xx+2xy)}$; erit

$$dV=\frac{xdx+ydx+xdy}{\sqrt{(xx+2xy)}}$$

ideoque

$$P = \frac{x+y}{\sqrt{(xx+2xy)}} \text{ et } Q = \frac{x}{\sqrt{(xx+2xy)}} ,$$

unde posito x constante erit

$$d.Pdx = \frac{xydxdy}{(xx+2xy)^{\frac{3}{2}}}$$

et posito y constante erit

$$d.Qdy = \frac{xydxdy}{(xx+2xy)^{\frac{3}{2}}} .$$

III. Sit $V = x\sin\text{A}y + y\sin\text{A}x$ eritque

$$dV = dx\sin\text{A}y + x\,dy\cos y + dy\sin\text{A}x + y\,dx\cos x .$$

Quare erit

$$Pdx = dx\sin\text{A}y + ydx\cos x \text{ et } Qdy = dy\sin\text{A}x + xdy\cos y .$$

Posito ergo x constante erit

$$d.Pdx = dxdy\cos y + dxdy\cos x$$

et posito y constante erit

$$d.Qdy = dxdy\cos y + dxdy\cos x .$$

IV. Sit $V = x^{y}$; erit

$$dV = x^{y}dylx + yx^{y-1}dx$$

Atque

$$Pdx = yx^{y-1}dx \text{ et } Qdy = x^{y}dylx.$$

Quamobrem posito x constante habebitur

$$d.Pdx = x^{y-1}dxdy + yx^{y-1}dxdylx$$

et posito y constante erit

$$d.Qdy = yx^{y-1}dxdylx + x^{y-1}dxdy .$$

230. Ista proprietas etiam hoc modo enunciari potest, unde eximia omnium functionum, quae duas variabiles involvunt, indoles cognoscetur. Si functio quaecunque V duarum variabilium x et y differentietur posita sola x variabili hocque differentiale denuo differentietur posita sola y variabili, tum post duplicem hanc differentiationem idem prodibit, ac si ordine inverso functio V primum posita sola y variabili differentiaretur hocque differentiale posita sola x variabili denuo differentiaretur; utroque scilicet casu prodibit eadem expressio huius formae $Zdxdy$. Ratio huius identitatis ex praecedente proprietate manifesto sequitur; si enim V differentietur posita sola x variabili, prodit Pdx, et si V differentietur posita sola y variabili, prodit Qdy;horum differentialium vero differentialia modo indicato sumta inter se aequalia esse ante demonstravimus. Ceterum haec indoles immediate sequitur ex ratiocinio § 227 allato.

231. Relatio inter P et Q, si $Pdx+Qdy$ fuerit differentiale functionis V sequenti etiam modo indicari potest. Quoniam P et Q sunt functiones ipsarum x et y, differentientur ambae posita utraque x et y variabili. Si scilicet fuerit $dV=Pdx+Qdy$, sit

$$dP=pdx+rdy \quad \text{et} \quad dQ=qdx+sdy\,.$$

Posito ergo x constante erit

$$dP=rdy \quad \text{et} \quad d.Pdx=rdxdy\,.$$

Deinde posito y constante erit

$$dQ=qdx \quad \text{et} \quad d.Qdy=qdxdy\,.$$

Cum igitur haec duo differentialia $rdxdy$ et $qdxdy$ sint inter se aequalia, sequitur fore

$$q=r\,.$$

Functiones ergo P et Q ita invicem connectuntur, ut, si ambae differentientur, uti fecimus, quantitates q et r inter se fiant aequales. Brevitatis gratia autem hoc saltem capite quantitates r et q ita commode denotari solent, ut r indicetur per $\left(\frac{dP}{dy}\right)$ qua scriptura designatur P ita differentiari, ut sola y tanquam variabilis tractetur atque differentiale istud per dy dividatur; sic enim prodibit quantitas finita r. Simili modo significabit $\left(\frac{dQ}{dx}\right)$ quantitatem finitam q, quia hac ratione indicatur functionem Q sola x posita variabili differentiari tumque differentiale per dx dividi debere.

232. Utamur ergo hoc scribendi modo, etiamsi alias ambiguitatem afferre possit, quae tamen hic per clausulas evitatur, ut ambages in describendis differentiandi conditionibus evitemus, sicque breviter relationem inter P et Q ita verbis exprimere poterimus, ut dicamus esse

$$\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$$

In huiusmodi scilicet fractionibus denominator praeter propriam significationem, qua numerator per eum dividi debet, indicat numeratoris differentiale ita esse capiendum, ut ea sola quantitas, cuius differentiale denominatorem constituit, tanquam variabilis spectetur. Hoc enim modo per divisionem differentialia prorsus ex calculo egredientur istaeque fractiones $\left(\frac{dP}{dy}\right)$ et $\left(\frac{dQ}{dx}\right)$ exhibebunt quantitates finitas, quae in praesenti casu erunt inter se aequales. Hoc itaque modo recepto quantitates quoque p et s ita denotare licebit, ut sit

$$p=\left(\frac{dP}{dx}\right) \text{ et } s=\left(\frac{dQ}{dy}\right),$$

si quidem, ut monitum est, differentiatio numeratoris per denominatorem restringatur.

233. Consentit haec proprietas mirifice cum proprietate, quam ante in functionibus homogeneis inesse ostendimus. Sit enim V functio homogenea n dimensionum ipsarum x et y ponaturque $dV=Pdx+Qdy$ atque demonstravimus fore $nV=Px+Qy$ ideoque

$$Q=\frac{nV}{y}-\frac{Px}{y}$$

Sit $dP=pdx+rdy$ eritque

$$\left(\frac{dP}{dy}\right)=r,$$

cui aequale esse $\left(\frac{dQ}{dx}\right)$ ita ostendetur. Differentietur Q posita sola x variabili, et quia in hac hypothesi est

$$dQ=\frac{nPdx}{y}-\frac{Pdx}{y}-\frac{xpdx}{y}$$

Fiet

$$\left(\frac{dQ}{dx}\right)=\frac{(n-1)P}{y}-\frac{Px}{y}$$

debebitque esse

$$\frac{(n-1)P}{y}-\frac{px}{y}=r \text{ seu } (n-1)P=px+ry.$$

Quae aequalitas inde fit perspicua, quod P sit functio homogenea $n-1$ dimensionum ipsarum x et y, unde eius differentiale $dP=pdx+rdy$ ob proprietatem functionum homogenearum ita debet esse comparatum, ut sit $(n-1)P=px+ry$.

234. Ista proprietas, quod sit $\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$, quam omnibus functionibus duarum variabilium x et y communem esse ostendimus, nobis quoque patefaciet naturam functionum trium pluriumve variabilium. Sit V functio quaecunque trium variabilium x, y et z ac ponatur

$$dV=Pdx+Qdy+Rdz.$$

Quodsi igitur in hac differentiatione z tanquam constans tractaretur, foret utique $dV = Pdx + Qdy$; hoc autem casu per antecedentia debet esse $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$. Deinde si quantitas y constans assumeretur, foret $dV = Pdx + Rdz$; erit ergo $\left(\frac{dP}{dz}\right) = \left(\frac{dR}{dx}\right)$. Denique posito x constante reperietur $\left(\frac{dQ}{dz}\right) = \left(\frac{dR}{dy}\right)$. In differentiali ergo $Pdx + Qdy + Rdz$ functionis V quantitates P, Q et R ita a se invicem pendent, ut sit

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right), \quad \left(\frac{dP}{dz}\right) = \left(\frac{dR}{dx}\right), \quad \text{et} \quad \left(\frac{dQ}{dz}\right) = \left(\frac{dR}{dy}\right).$$

235. Sequitur hinc ista functionum, quae tres pluresve variabiles involvunt, proprietas analoga ei, quam supra (§ 230) de functionibus duarum variabilium ostendimus. Si fuerit V functio quaecunque trium variabilium x, y et z eaque continuo ter differentietur, ita ut primum una quantitatum, puta x, sola variabilis ponatur, in differentiatione secunda sola y atque in tertia sola z variabilis assumatur, prodibit expressio huius formae $Zdxdydz$, quae eadem reperietur, quocunque alio ordine quantitates x, y et z collocentur.
Sex igitur diversis modis post triplicem differentiationem ad eandem expressionem $Zdxdydz$ pervenietur, quoniam ordo quantitatum x, y et z sexies variari potest. Quicunque ergo ordo eligatur, si functio V differentietur posita sola prima variabili hocque differentiale denuo differentietur posita sola secunda variabili atque differentiale hoc iterum differentietur posita sola tertia variabili, eadem prodibit expressio, utcunque ordo quantitatum x, y et z varietur.

236. Quo ratio huius proprietatis clarius perspiciatur, ponamus esse

$$dV = Pdx + Qdy + Rdz;$$

deinde etiam quantitates P, Q et R differentiemus eruntque earum differentialia per ante demonstrata ita comparata

$$dP = pdx + sdy + tdz$$
$$dQ = sdx + qdy + udz$$
$$dR = tdx + udy + rdz.$$

Differentietur nunc V posito solo x variabili, prodibit Pdx; quod differentiale iterum differentietur posito solo y variabili atque habebitur $sdxdy$; quod si differentietur posito solo z variabili, postquam per $dxdydz$ fuerit divisum, obtinebitur $\left(\frac{ds}{dz}\right)$. Collocentur nunc variabiles hoc ordine y, x, z atque prima differentiatio dabit Qdy, secunda $sdxdy$ et tertia (facta divisione per $dxdydz$) dabit $\left(\frac{ds}{dz}\right)$ ut ante. Disponantur variabiles hoc ordine z, y; x ac prima differentiatio dabit *Rdz*, secunda *udydz*, tertia vero post divisionem per *dxdydz* praebet $\left(\frac{du}{dx}\right)$. At cum posito *y* constante sit $dQ = sdx + udz$, erit $\left(\frac{ds}{dz}\right) = \left(\frac{du}{dx}\right)$, uti pariter est demonstratum.

237. Ponamus esse $V = \frac{xxy}{aa-zz}$ hancque functionem toties ter differentiemus, quoties ordo variabilium *x*, *y*, *z* variari potest.

	I. Differ.	*II. Differ.*	*III. Differ.*
posito variabili	solo *x* $\frac{2xydx}{aa-zz}$,	solo *y* $\frac{2xdxdy}{aa-zz}$,	solo *z* $\frac{4xzdxdydz}{(aa-zz)^2}$.
posito variabili	solo *x* $\frac{2xydx}{aa-zz}$,	solo *z* $\frac{4xyzdxdz}{(aa-zz)^2}$,	solo *y* $\frac{4xzdxdydz}{(aa-zz)^2}$.
posito variabili	solo *y* $\frac{xxdy}{aa-zz}$,	solo *x* $\frac{2xdxdy}{aa-zz}$,	solo *z* $\frac{4xyzdxdz}{(aa-zz)^2}$.
posito variabili	solo *y* $\frac{xxdy}{aa-zz}$,	solo *z* $\frac{2xxzdydz}{(aa-zz)^2}$,	solo *x* $\frac{4xzdxdydz}{(aa-zz)^2}$.
posito variabili	solo *z*, $\frac{2xxyzdz}{(aa-zz)^2}$,	solo *x* $\frac{4xyzdxdz}{(aa-zz)^2}$,	solo *y* $\frac{4xzdxdydz}{(aa-zz)^2}$.
posito variabili	solo *z* $\frac{2xxyzdz}{(aa-zz)^2}$,	solo *y* $\frac{2xxzdydz}{(aa-zz)^2}$,	solo *x* $\frac{4xzdxdydz}{(aa-zz)^2}$.

Ex quo exemplo patet, quocunque ordine tres variabiles fuerint assumtae, post triplicem differentiationem semper eandem prodire expressionem

$$\frac{4xzdxdydz}{(aa-zz)^2}.$$

238. Uti autem post triplicem differentiationem ad eandem expressionem est perventum, ita quoque consensus deprehenditur in differentialibus, quae secunda differentiatio suppeditavit. In iis scilicet expressio quaevis bis occurrit; unde patet, quae formulae iisdem differentialibus sint affectae, easdem quoque inter se esse aequales atque differentialia tertia ideo esse omnia inter se aequalia, quia iisdem differentialibus $dxdydz$ sunt affecta. Hinc igitur concludimus, si V fuerit functio quotcunque variabilium x, y, z, v, u etc. eaque successive aliquoties differentietur, ut semper unica tantum quantitas variabilis assumatur, tum, quoties ad expressiones perveniatur, quae iisdem differentialibus sint affectae, eas quoque inter se aequales fore. Sic duplici differentiatione orietur huiusmodi expressio $Zdxdy$, dum in altera sola x, in altera sola y assumta est variabilis, perindeque est, utra prius posteriusve sit variabilis assumta. Simili modo sex variis modis per triplicem differentiationem eadem exsurget expressio $Zdxdydz$ atque viginti quatuor variis modis pervenietur post quadruplicem differentiationem ad eandem expressionem huius formae $Zdxdydzdv$ atque ita porro.

239. Veritatem horum theorematum quilibet adhibita levi attentione ex ante explicatis principiis facile agnoscet atque propria meditatione facilius intuebitur quam tantis verborum ambagibus, sine quibus demonstrationes proferri non possent. Quia vero harum proprietatum cognitio maximi est momenti in calculo integrali, tirones sunt monendi, ut non solum has proprietates ipsi diligenter meditentur earumque veritatem scrutentur, sed etiam pluribus exemplis comprobent, quo hoc pacto sibi hanc materiam familiariorem reddant fructusque inde natos postmodum facilius percipere queant. Neque vero solum tirones, sed etiam ii, qui principiis calculi differentialis iam sunt imbuti, ad hoc sunt cohortandi, quoniam in omnibus fere manuductionibus ad hanc analyseos partem hoc argumentum penitus praetermitti solet. Plerumque enim auctores solas differentiationis regulas praescripsisse earumque usum in geometria sublimiori ostendisse fuerunt contenti neque in naturam atque proprietates differentialium inquisiverunt, unde tamen maxima subsidia in calculum integralem redundant. Quam ob causam hoc argumentum fere novum in isto capite fusius persequi visum est, quo simul via ad integrationes alias difficiliores pararetur atque negotium postea suscipiendum sublevaretur.

240. Cognitis igitur his proprietatibus, quibus differentialia functionum duas pluresve variabiles involventium gaudent, facile poterimus dignoscere, utrum formula differentialis proposita, in qua occurrunt duae pluresve variabiles, sit orta ex differentiatione cuiuspiam functionis finitae an secus. Si enim in formula $Pdx+Qdy$ non fuerit $\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$, certo poterimus affirmare nullam existere functionem ipsarum x et y, cuius differentiale sit $=Pdx+Qdy$, neque ergo infra in calculo integrali huiusmodi formulae integrale indagari potest. Sic cum in $yxdx+xxdy$ requisita conditio non adsit, nulla datur functio, cuius differentiale est $=yxdx+xxdy$. Utrum autem semper, quoties est $\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$, formula ex differentiatione cuiuspiam functionis sit orta, quaestio est, quae demum ex integrationis principiis solide affirmari poterit.

241. Si in formula differentiali proposita tres pluresve insint variabiles, uti $Pdx+Qdy+Rdx$, tum ea ex differentiatione ortum traxisse omnino nequit, nisi tres istae conditiones in ea locum habeant, ut sit

$$\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right),\quad \left(\frac{dP}{dz}\right)=\left(\frac{dR}{dx}\right),\quad \text{et}\quad \left(\frac{dQ}{dz}\right)=\left(\frac{dR}{dy}\right).$$

Quarum conditionum si una tantum desit, certo affirmare debemus nullam extare functionem ipsarum x, y et $z,$ cuius differentiale sit $Pdx+Qdy+Rdz$; huiusmodi ergo formularum differentialium nequidem requiri possunt integralia hincque integrationem prorsus non recipere dicuntur. Facile autem intelligitur in calculo integrali formulas differentiales ante dignosci oportere, utrum integrationis sint capaces, quam investigatio integralis actu suscipiatur.