CHAPTER VI

CONCERNING THE DIFFERENTIATION OF TRANSCENDING FUNCTIONS

178. Besides the infinitude of transcending functions, or those not of the algebraic kind, which the integral calculus will make available when required, it has become clear from the *Introduction to the Infinitesimal Analysis* for us to become more knowledgeable of the uses of quantities of this kind, which the principles of logarithms and circular arcs may have suggested. Therefore since we have explained the nature of these quantities so clearly, so that they are able to be treated in a calculation with almost the same ease as algebraic quantities, we will investigate the differentials of these also in this chapter, so that the nature and properties of these may be examined more clearly and with this agreed upon the approach to the integral calculus becomes revealed, which is a special source of transcending quantities.

179. Therefore in the first place logarithmic quantities come to mind or functions of x of this kind, which besides algebraic expressions also involve the logarithm of x or of any function of this. Towards which being differentiated since algebraic quantities may be done without further labour, all the difficulty will be put into finding the differential of the logarithm of each function of x. Because truly several different kinds of logarithms are given, which yet hold constant ratios between each other, here chiefly we will be considering hyperbolic logarithms, since from these all the remaining logarithms may be formed easily. For if the hyperbolic logarithm of a function p were $= lp$, then the logarithm of the same function p chosen from another table will be $= mlp$ with m denoting a number, from which the relation of this table of logarithms to the hyperbolic canon is expressed. On account of this proviso here lp will always designate the hyperbolic logarithm of the quantity p.

180. Therefore we may seek the differential of the hyperbolic logarithm of the magnitude x and there is put $y = lx$, thus so that the value of the differential dy must be defined. There may be put $x + dx$ in place of x and thus y will go to $y^{\mathrm{I}} = y + dy$; whereby there may be considered

$$y + dy = l(x + dx) \text{ and } dy = l(x + dx) - lx = l\left(1 + \tfrac{dx}{x}\right).$$

But now above [See the *Introductio* Ch. VII] we may express the hyperbolic logarithm of this kind of expression $1 + z$ by an infinite series, so that there may be

$$l(1 + z) = z - \tfrac{z^2}{2} + \tfrac{z^3}{3} - \tfrac{z^4}{4} + \text{etc.}$$

Therefore on putting $\frac{dx}{z}$ for *z* we will obtain

$$dy = \frac{dx}{x} - \frac{dx^2}{2x^2} + \frac{dx^3}{3x^3} - \text{etc}.$$

Therefore since all the terms of this series except the first vanish, there will be

$$d.lx = dy = \frac{dx}{x}.$$

From which the differential of any other logarithm, of which the ratio to the hyperbolic logarithm is as $n:1$, will be $= \frac{ndx}{x}$.

181. Therefore if the logarithm *lp* may be proposed if each function *p* of *x* function, by the same reasoning the differential of that will be found to be $= \frac{dp}{p}$, from which this rule may be had about finding the differentials of logarithms:

The differential is taken of some quantity p, of which the logarithm is proposed, and this divided by the quantity p will give the differential, of which the logarithm is sought.

This follows by the same rule also from the form $\frac{p^0 - 1^0}{0}$, to which we have reduced the logarithm of p in the above book. There shall be $\omega = 0$, and since there shall $lp = \frac{p^\omega - 1}{\omega}$, there becomes

$$d.lp = d.\frac{1}{\omega} p^\omega = p^{\omega-1} dp = \frac{dp}{p}$$

on account of $\omega = 0$. But it is to be noted that $\frac{dp}{p}$ is the differential of the hyperbolic logarithm of p, thus so that, if the usual logarithm of p may be proposed, that differential $\frac{dp}{p}$ has to be multiplied by this 0,43429448 etc.

182. Therefore with the aid of this rule, the logarithm may be proposed of any function of *x*, and the differential of that will be able to be found easily, just as may be seen from the following examples.

I. If there shall be $y = lx$, there will be

$$dy = \frac{dx}{x}.$$

II. If there shall be $y = lx^n$, there may be put $x^n = p$, so that there shall be $y = lp$, and there will be $dy = \frac{dp}{p}$. But there is $dp = nx^{n-1}dx$, from which there becomes

$$dy=\frac{ndx}{x}.$$

The same also may be deduced from the nature of logarithms ; since indeed there shall be $lx^n=nlx$, then there becomes $d.lx^n=nd.lx=\frac{ndx}{x}$.

III. If there shall be $y=l(1+xx)$, then

$$dy=\frac{2xdx}{1+xx}.$$

IV. If there shall be $y=l\frac{1}{\sqrt{(1-xx)}}$, because there shall be $y=-l\sqrt{(1-xx)}=-\frac{1}{2}l(1-xx)$, there may be found

$$dy=\frac{xdx}{1-xx}.$$

V. If there shall be $y=l\frac{x}{\sqrt{(1+xx)}}$, an account of $y=lx-\frac{1}{2}l(1+xx)$ there becomes

$$dy=\frac{dx}{x}-\frac{xdx}{1+xx}=\frac{dx}{x(1+xx)}.$$

VI. If there shall be $y=l\left(x+\sqrt{(1+xx)}\right)$, there becomes

$$dy=\frac{dx+xdx:\sqrt{(1+xx)}}{x+\sqrt{(1+xx)}}=\frac{xdx+dx\sqrt{(1+xx)}}{\left(x+\sqrt{(1+xx)}\right)\sqrt{(1+xx)}};$$

since the numerator and denominator of which fraction shall be divisible by $x+\sqrt{(1+xx)}$, there becomes

$$dy=\frac{dx}{\sqrt{(1+xx)}}.$$

VII. If there shall be $y=\frac{1}{\sqrt{-1}}l\left(x\sqrt{-1}+\sqrt{(1-xx)}\right)$, there may be put $x\sqrt{-1}=z$. And on account of $y=\frac{1}{\sqrt{-1}}l\left(z+\sqrt{(1+zz)}\right)$ by the preceding there will be $dy=\frac{1}{\sqrt{-1}}dz:\sqrt{(1+zz)}$. Whereby on account of $dz=dx\sqrt{-1}$ there becomes

$$dy=\frac{dx}{\sqrt{(1-xx)}}.$$

Therefore although the proposed logarithm may be involved with imaginary numbers, yet the differential of this shall be real.

183. If a quantity, the logarithm of which is produced, may have factors, then the logarithm itself may be resolved into several others in this manner. If there may be proposed $y = lpqrs$, because there will be $y = lp + lq + lr + ls$, there will be

$$dy = \frac{dp}{p} + \frac{dq}{q} + \frac{dr}{r} + \frac{ds}{s}.$$

This resolution equally has a place, if that quantity, the logarithm of which must be differentiated, were a fraction. Indeed let there be $y = l\frac{pq}{rs}$; on account of $y = lp + lq - lr - ls$ there will be

$$dy = \frac{dp}{p} + \frac{dq}{q} - \frac{dr}{r} - \frac{ds}{s}.$$

Nor also will powers be difficult to move ; if indeed there were $y = l\frac{p^m q^n}{r^\mu s^\nu}$, on account of $y = mlp + nlq - \mu lr - \nu ls$ there will be

$$dy = \frac{mdp}{p} + \frac{ndq}{q} - \frac{\mu dr}{r} - \frac{\nu ds}{s}.$$

I. If there were $y = l(a + x)(b + x)(c + x)$, because there will be

$$y = l(a + x) + l(b + x) + l(c + x),$$

the differential sought becomes

$$dy = \frac{dx}{a+x} + \frac{dx}{b+x} + \frac{dx}{c+x}.$$

II. If there were $y = \frac{1}{2}l\frac{1+x}{1-x}$, there will be

$$y = \frac{1}{2}l(1 + x) - \frac{1}{2}l(1 - x)$$

and hence

$$dy = \frac{\frac{1}{2}dx}{1+x} + \frac{\frac{1}{2}dx}{1-x} = \frac{dx}{1-xx}.$$

III. Let there be $y = \frac{1}{2}l\frac{\sqrt{(1+xx)}+x}{\sqrt{(1+xx)}-x}$, on account of

$$y = \frac{1}{2}l\left(\sqrt{(1 + xx)} + x\right) - \frac{1}{2}l\left(\sqrt{(1 + xx)} - x\right)$$

there will be

$$dy = \frac{\frac{1}{2}dx}{\sqrt{(1+xx)}} + \frac{\frac{1}{2}dx}{\sqrt{(1+xx)}} = \frac{dx}{\sqrt{(1+xx)}}.$$

The same is found more easily from this, if in the fraction $\frac{\sqrt{(1+xx)}+x}{\sqrt{(1+xx)}-x}$ the irrationality in the denominator may be removed by multiplying the numerator and the denominator by $\sqrt{(1+xx)}+x$; for there will be produced

$$y = \tfrac{1}{2} l\left(\sqrt{(1+xx)}+x\right)^2 = l\left(\sqrt{(1+xx)}+x\right),$$

the differential of which we found before to be $dy = \frac{dx}{\sqrt{(1+xx)}}$

IV. If there shall be $y = l\frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}}$, the numerator of this fraction may be put

$$\sqrt{(1+x)}+\sqrt{(1-x)} = p$$

and the denominator

$$\sqrt{(1+x)}-\sqrt{(1-x)} = q$$

there will be $y = l\frac{p}{q} = lp - lq$ and $dy = \frac{dp}{p} - \frac{dq}{q}$. Truly there is

$$dp = \frac{dx}{2\sqrt{(1+x)}} - \frac{dx}{2\sqrt{(1-x)}} = \frac{-dx}{2\sqrt{(1-xx)}}\left(\sqrt{(1+x)}-\sqrt{(1-x)}\right) = \frac{-qdx}{2\sqrt{(1-xx)}}$$

and

$$dq = \frac{dx}{2\sqrt{(1+x)}} + \frac{dx}{2\sqrt{(1-x)}} = \frac{pdx}{2\sqrt{(1-xx)}}.$$

Hence there becomes

$$\frac{dp}{p} - \frac{dq}{q} = \frac{-qdx}{2p\sqrt{(1-xx)}} - \frac{pdx}{2q\sqrt{(1-xx)}} = \frac{-(pp+qq)dx}{2pq\sqrt{(1-xx)}}.$$

But there is $pp+qq = 4$ and $pq = 2x$, from which there will be

$$dy = -\frac{dx}{x\sqrt{(1-xx)}}.$$

But this differential may be found more easily, if the proposed logarithm may be transformed thus

$$y = l\tfrac{1+\sqrt{(1-xx)}}{x} = l\left(\tfrac{1}{x} + \sqrt{\left(\tfrac{1}{xx} - 1\right)}\right).$$

For on putting $\tfrac{1}{x} + \sqrt{\left(\tfrac{1}{xx} - 1\right)} = p$ there will be

$$dp = \tfrac{-dx}{xx} - \tfrac{dx}{x^3\sqrt{\left(\tfrac{1}{xx}-1\right)}} = \tfrac{-dx}{xx} - \tfrac{dx}{xx\sqrt{(1-xx)}} = \tfrac{-dx\left(1+\sqrt{(1-xx)}\right)}{xx\sqrt{(1-xx)}},$$

and thus on account of $p = \tfrac{1+\sqrt{(1-xx)}}{x}$ there will be $dy = \tfrac{dp}{p} = \tfrac{-dx}{x\sqrt{(1-xx)}}$ as before.

184. Therefore since the first differentials of the logarithms, if they may be divided by *dx*, shall be algebraic quantities, the second differentials and of the following orders may be found easily by the precepts of the preceding chapters, if indeed the differential *dx* may be assumed constant. Thus on putting $y = lx$ there will be

$$dy = \tfrac{dx}{x} \quad \text{and} \quad \tfrac{dy}{dx} = \tfrac{1}{x}$$
$$ddy = \tfrac{-dx^2}{x^2} \quad \text{and} \quad \tfrac{ddy}{dx^2} = \tfrac{-1}{x^2}$$
$$d^3y = \tfrac{2dx^3}{x^3} \quad \text{and} \quad \tfrac{d^3y}{dx^3} = \tfrac{2}{x^3}$$
$$d^4y = \tfrac{-6dx^4}{x^4} \quad \text{and} \quad \tfrac{d^4y}{dx^4} = \tfrac{-6}{x^4}$$
etc.

And if p were an algebraic quantity and there shall be $y = lp$, even if *y* shall not be an algebraic quantity, yet $\tfrac{dy}{dx}, \tfrac{ddy}{dx^2}, \tfrac{d^3y}{dx^3}$ etc. will be algebraic functions of *x*.

185. The functions from the differentiation of logarithms set forth, which have been mixed together from algebraic and logarithm functions, will be easily differentiated, and likewise these, which are composed from logarithms alone, as will be made evident from the following examples.

I. If there shall be $y = (lx)^2$, there may be put $lx = p$ and on account of $y = p^2$ there will be $dy = 2pdp$, truly $dp = \tfrac{dx}{x}$; and thus there will be $dy = \tfrac{2dx}{x}lx$.

II. In a similar manner, if there shall be $y=(lx)^n$, there will be $dy=\frac{ndx}{x}(lx)^{n-1}$, from which if there shall be $y=\sqrt{lx}$, on account of $n=\frac{1}{2}$ there will be $dy=\frac{dx}{2x\sqrt{lx}}$.

III. And if p were some function of x and there may be put $y=(lp)^n$, there will be

$$dy=\frac{ndp}{lp}(lp)^{n-1}$$

Whereby, since the differential dp by the preceding will be able to be assigned, the differential of y will also be known.

IV. If there shall be $y=lp.lq$ and p and q were some functions of x, by the rule of factors given above there will be

$$dy=\frac{dp}{p}lq+\frac{dq}{q}lp.$$

V. If there shall be $y=xlx$, by the same rule there will be $dy=dxlx+\frac{xdx}{x}=dxlx+dx.$

VI. If there shall be $y=x^m lx-\frac{1}{m}x^m$, there will be found from the differentiation following the parts put in place $d.x^m lx=mx^{m-1}dxlx+x^{m-1}dx$ and $d.\frac{1}{m}x^m=x^{m-1}dx$, from which there will be $dy=mx^{m-1}dxlx$.

VII. If there shall be $y=x^m(lx)^n$, there will become

$$dy=mx^{m-1}dx(lx)^n+nx^{m-1}dx(lx)^{n-1}.$$

VIII. If the logarithms of logarithms occur, as if there should be $y=llx$, there is put $lx=p$; there will be $y=lp$ and $dy=\frac{dp}{p}$; but there is $dp=\frac{dx}{x}$, from which there will become $dy=\frac{dx}{xlx}$.

IX. And if there were $y=lllx$, if there is put $lx=p$, there comes about $y=llp$ and there will be by the preceding example $dy=\frac{dp}{plp}$; but there is $dp=\frac{dx}{x}$, from which values substituted there will be had $dy=\frac{dx}{xlx\cdot llx}$.

186. From the differentiation of logarithms explained we may proceed to exponential quantities or powers of this kind, the exponents of which shall be variable. But the differentials of this kind of function of *x* can be found by logarithmic differentiation in this manner. The differential of a^x may be sought; for which to be found there may be put $y = a^x$ and there will be on taking logarithms $ly = xla$. Now the differentials are taken and there will be obtained $\frac{dy}{y} = dx\, la$, from which there will be $dy = ydx\, la$; but since there shall be $y = a^x$, there will be $dy = a^x dxla$, which is the differential of a^x. In a similar manner, if there shall be some *p* function of *x*, the differential of this exponential quantity a^p will be $= a^p dpla$.

187. But this same differential can be deduced immediately from the nature of the exponential quantity set out in the *Introductione* [Book I, Ch. VII]. Indeed there shall be with a^p proposed with *p* denoting some function of *x*, which on putting $x + dx$ in place of *x* may be changed into $p + dp$. From which if there is put $y = a^p$, if *x* may be changed into $x + dx$, there will be $y + dy = a^{p+dp}$ and thus

$$dy = a^{p+dp} - a^p = a^p \left(a^{dp} - 1 \right)$$

But we have shown above any exponential quantity a^z can be expressed by a series of this kind

$$1 + zla + \frac{z^2 (la)^2}{2} + \frac{z^3 (la)^3}{6} + \text{etc.};$$

from which there will be

$$a^{dp} = 1 + dpla + \frac{dp^2 (la)^2}{2} + \text{etc.}$$

and $a^{dp} - 1 = dpla$, because the following terms all vanish before *dpla*. Consequently there will be

$$dy = d.a^p = a^p dpla.$$

Whereby the differential of the exponential quantity a^p *will be the product from the magnitude of the exponential itself, from the differential of the exponent dp and from the logarithm of the constant quantity a, to which the exponent variable has been raised.*

188. Therefore if e shall be the number, of which the hyperbolic logarithm is $=1$, so that there shall be $le = 1$, the differential of the quantity e^x will be $= e^x dx$. And if dx is assumed constant, the differential of this will be $= e^x dx^2$, which is the second differential of e^x. In as similar manner the third differential will be $= e^x dx^3$. Whereby if there shall be $y = e^{nx}$, there will be

$$\frac{dy}{dx} = ne^{nx} \quad \text{and} \quad \frac{ddy}{dx^2} = n^2 e^{nx} \quad \text{and again} \quad \frac{d^3y}{dx^3} = n^3 e^{nx} \quad \frac{d^3y}{dx^3} = n^4 e^{nx} \text{ etc.}$$

From which it may be apparent that the first, second, and the remaining differentials of e^{nx} constitute a geometric progression and therefore the differential of order *m* of $e^{nx} = y$, evidently will be $d^m y, = n^m e^{nx} dx^m$ and hence therefore $\frac{d^my}{ydx^m}$ is the constant magnitude n^m.

189. If the quantity itself, which is raised, were of a variable, the differential of this will be found in a similar manner. Let *p* and *q* be some functions of *x* and there is proposed the quantity of the exponential $y = p^q$. With the logarithms taken there will be $ly = qlp$, from which differentiation there will be

$$\frac{dy}{y} = dqlp + \frac{qdp}{p},$$

from which there becomes

$$dy = ydqlp + \frac{yqdp}{p} = p^q dqlp + qp^{q-1}dp$$

on account of $y = p^q$. This differential therefore depends on two parts, the first of which $p^q dqlp$ arises, if the proposed quantity p^q thus may be differentiated, as if *p* were a constant quantity and only the exponent *q* variable ; now the other part $qp^{q-1}dp$ arises, if in the proposed quantity p^q the exponent *q* may be regarded as constant and the quantity *p* treated as if it were variable. And this differential therefore may be found by the above general rule for differentiation treated above [§ 170].

190. Now the differential of the same expression p^q can also be elicited from the nature of the exponential quantity in this manner. Let there be $y = p^q$ and there will be on putting $x + dx$ in place of x and certainly $y + dy = (p + dp)^{q+dq}$; which expression if it may be resolved in the usual customary series, will become

$$y + dy = p^{q+dq} + (q + dq)\, p^{q+dq-1}dp + \frac{(q+dq)(q+dq-1)}{1\cdot 2}\, p^{q+dq-2}dp^2 + \text{etc.}$$

and thus

$$dy = p^{q+dq} - p^q + (q + dq)\, p^{q+dq-1}dp \;;$$

for the following terms, which involve higher powers of *dp*, vanish before

$(q+dq)\,p^{q+dq-1}dp$. But there shall be

$$p^{q+dq}-p^{q}=p^{q}\left(p^{dq}-1\right)=p^{q}\left(1+dqlp+\tfrac{dq^{2}(lp)^{2}}{2}+\text{etc.}-1\right)=p^{q}dqlp\,.$$

Now in the other term $(q+dq)\,p^{q+dq-1}dp$ if in place of $q+dq$ we may write q, there may come about $qp^{q-1}dp$ and thus the differential will be as before
$dy=p^{q}dqlp+qp^{q-1}dp$.

191. Truly in this manner this same differential will be found more easily from the nature of the exponential quantities. Since on taking e for the number, of which the hyperbolic logarithm is = 1, there shall be $p^{q}=e^{qlp}$, for the logarithm of each is the same qlp, there will be $y=e^{qlp}$. Whereby, since now the raised quantity e shall be constant, there will be

$$dy=e^{qlp}\left(dqlp+\tfrac{qdp}{p}\right)$$

as we have shown before in the rule given §187. Therefore there may be put in place p^{q} for e^{qlp} and there will be made

$$dy=p^{q}dqlp+p^{q}qdp{:}p=p^{q}dqlp+qp^{q-1}dp$$

Therefore if there should be $y=x^{x}$, there will be $dy=x^{x}dxlx+x^{x}dx$; and hence also the higher differentials of this may be defined ; indeed there may be found

$$\frac{ddy}{dx^{2}}=x^{x}\left(\tfrac{1}{x}+(1+lx)^{2}\right)$$
$$\frac{d^{3}y}{dx^{3}}=x^{x}\left((1+lx)^{3}+\tfrac{3(1+lx)}{x}-\tfrac{1}{xx}\right)$$
etc.

192. Among the differentials of functions of this kind, which include exponential functions, the following are to be noted in the first place, which originate from the differentiation of the formula $e^{x}p$; moreover there shall be

$$d.e^{x}p=e^{x}dp+e^{x}pdx=e^{x}\left(dp+pdx\right).$$

I. If there shall be $y=e^{x}x^{n}$, there will be

$$dy = e^x nx^{n-1} + e^x x^n dx \quad \text{or} \quad dy = e^x dx\left(nx^{n-1} + x^n\right).$$

II. If there shall be $y = e^x(x-1)$, there will be $dy = e^x x dx$.

III. If there shall be $y = e^x\left(x^2 - 2x + 2\right)$, there will be

$$dy = e^x xxdx.$$

IV. If there shall be $y = e^x\left(x^3 - 3x^2 + 6x - 6\right)$, there will be $dy = e^x x^3 dx$.

193. If the exponents were again exponential functions , the differentiation may again be put in place according to the same precepts. Thus, if this quantity e^{e^x} must be differentiated, there may be put in place $e^x = p$, so that there shall be

$$y = e^{e^x} = e^p$$

then there will be $dy = e^p dp$; but there is $dp = e^x dx$, from which, if there were $y = e^{e^x}$, there will be

$$dy = e^{e^x} e^x dx$$

and if there should be $y = e^{e^{e^x}}$, there will be

$$dy = e^{e^{e^x}} e^{e^x} e^x dx$$

But if now there were $y = p^{q^r}$, there may be put in place $q^r = z$; and there will be

$$dy = p^z dzlp + zp^{z-1}dp, \quad \text{but} \quad dz = q^r drlq + rq^{r-1}dq\,,$$

from which

$$dy = p^z q^r drlp \cdot lq + p^z rq^{r-1}dqlp + p^z q^r dp{:}p$$

Whereby if there shall be $y = p^{q^r}$, there will be

$$dy = p^{q^r} q^r\left(drlp \cdot lq + \frac{rdqlp}{q} + \frac{dp}{p}\right).$$

Therefore in this manner, whatever quantity of the exponential occurs, the differential of this can be found.

194. Therefore we may go on to transcending quantities, the consideration of circular arcs above has led us to an understanding of which [*Introductio*, Book I, Ch. VIII]. Therefore, in a circle the radius of which we have always put equal to one, there shall be the proposed arc, the sine of which shall be $= x$, which arc we may express in this manner $y = A\sin x$, [*i.e.* the arc *y* is equal to the arcsine of *x*] and we may investigate the differential of this arc or the increment which it takes if the sine *x* be augmented by its own differential *dx*. But this can be shown from the differentiation of logarithms, because in the *Introductione* [*loc.cit.* §138] we have shown that this expression $y = A\sin x$ can be reduced to this logarithm $\frac{1}{\sqrt{-1}} l\left(\sqrt{(1-xx)} + x\sqrt{-1}\right)$. Therefore on putting $y = A\sin x$ there will be also

$$y = \frac{1}{\sqrt{-1}} l\left(\sqrt{(1-xx)} + x\sqrt{-1}\right)$$

which differentiated gives [§ 182, Ch.VII]

$$dy = \frac{\frac{1}{\sqrt{-1}}\left(\frac{-xdx}{\sqrt{(1-xx)}} + dx\sqrt{-1}\right)}{\sqrt{(1-xx)} + x\sqrt{-1}} = \frac{dx\left(x\sqrt{-1} + \sqrt{(1-xx)}\right)}{\left(\sqrt{(1-xx)} + x\sqrt{-1}\right)\sqrt{(1-xx)}},$$

from which there becomes

$$dy = \frac{dx}{\sqrt{(1-xx)}}.$$

195. That differential of the circular arc also can be found in this manner more easily without the aid of logarithms. For if there shall be $y = A\sin x$, *x* will be the sine of *y* or $x = \sin y$. Since on putting $x + dx$ in place of x, *y* changes into $y + dy$, there becomes $x + dx = \sin(y + dy)$. But because there is

$$\sin(a+b) = \sin a \cdot \cos b + \cos a \cdot \sin b,$$

there will be

$$\sin(y+dy) = \sin y \cdot \cos dy + \cos y \cdot \sin dy;$$

but the sine of the vanishing arc *dy* is equal to that arc *dy* and the cosine of this sine is equal to the whole sine [§ 201]; hence on this account

$$\sin(y+dy) = \sin y + dy\cos y \quad \text{and thus} \quad x + dx = \sin y + dy\cos y.$$

Truly because there is $\sin y = x$, there will be the cosine of *y*, or $\cos y, = \sqrt{(1-xx)}$, with which values substituted there will be $dx = dy(1-xx)$, from which there will be obtained $dy = \frac{dx}{\sqrt{(1-xx)}}$.

Therefore the differential of the arc, of which the sine is proposed, is equal to the differential of the sine divided by the cosine.

196. Therefore since, if p were some function of x and y may denote the arc, the arc of which the sine is $= p$, or $y = A\sin p$, the differential of this arc shall be $dy = \frac{dp}{\sqrt{(1-pp)}}$, where $\sqrt{(1-pp)}$ expresses the cosine of the same arc, also it will be able to find the differential of the arc, the cosine of which is proposed. For let $y = A\cos x$; there will be the sine of the same arc $= \sqrt{(1-xx)}$ and thus $y = A\sin\sqrt{(1-xx)}$. Therefore on making $p = \sqrt{(1-xx)}$ there will be

$$dp = \frac{-xdx}{\sqrt{(1-xx)}} \quad \text{and} \quad \sqrt{(1-pp)} = x;$$

from which there arises

$$dy = \frac{-xdx}{\sqrt{(1-xx)}}.$$

Therefore the differential of the arc, of which the cosine is proposed, is equal to the differential of the cosine taken negatively, and divided by the sine of the same arc.

Because also in this manner it can be shown; if there shall be $y = A\cos x$, there may be put $z = A\sin x$; there will be $dz = \frac{dx}{\sqrt{(1-xx)}}$; but the arcs y and z taken at the same time give the constant arc 90° and there will be $y + z =$ constant and thus $dy + dz = 0$ or $dy = -dz$; from which there becomes $dy = \frac{-xdx}{\sqrt{(1-xx)}}$ as before.

197. If an arc may be proposed to be differentiated, the tangent of which is given, thus so that there shall be $y = A\text{tang}\, x$: but the sine of the arc, of which the tangent is x, will be $= \frac{x}{\sqrt{(1+xx)}}$ and the cosine $= \frac{1}{\sqrt{(1+xx)}}$. Therefore on putting $\frac{x}{\sqrt{(1+xx)}} = p$, so that there shall be $\sqrt{(1-pp)} = \frac{1}{\sqrt{(1+xx)}}$, there becomes $y = A\sin p$; from which by the rule in the manner given there will be $dy = \frac{dp}{\sqrt{(1-pp)}}$. But on account of $p = \frac{x}{\sqrt{(1+xx)}}$ there will be $dp = \frac{dx}{(1+xx)^{\frac{3}{2}}}$, from which with the values substituted there becomes

$$dy = \frac{dx}{1+xx}$$

Therefore the differential of the arc, of which the tangent is proposed, is equal to the differential of the tangent divided by the square of the secant. Indeed $\sqrt{(1+xx)}$ is the secant, if x shall be the tangent.

198. In a similar manner if the arc is proposed, of which the cotangent is given, thus so that there shall be $y = A\cot x$, because the tangent of the same arc is $\frac{1}{x}$ on putting $\frac{1}{x} = p$ there will be $y = A\tan p$ and therefore $dy = \frac{dp}{1+pp}$. Now since there shall be $dp = \frac{-dx}{xx}$, with the substitution made there shall be

$$dy = \frac{-dx}{1+xx},$$

which is *the differential of the cotangent taken negative and divided by the square of the cosecant.*

Again if there is proposed $y = A\sec x$, because there is $y = A\cos\frac{1}{x}$, there becomes

$$dy = \frac{dx}{xx\sqrt{\left(1-\frac{1}{xx}\right)}} = \frac{dx}{x\sqrt{(xx-1)}}$$

And if there shall be $y = A\operatorname{cosec} x$, there will be $y = A\sin\frac{1}{x}$ and thus

$$dy = \frac{-dx}{x\sqrt{(xx-1)}}$$

Often also the versed sine *sv* has occurred [The versed sine of the angle θ is $\operatorname{sv}(\theta) = 1-\cos\theta$, which is the part of the radius left on subtracting the cosine] ; thus if there is put $y = A\operatorname{sv} x$, because there shall be $y = A\cos(1-x)$ and the sine of this arc is $= \sqrt{(2x-xx)}$, there arises

$$dy = \frac{dx}{\sqrt{(2x-xx)}}.$$

199. Though the arc therefore is a transcending quantity, of which either the sine, cosine, tangent, cotangent, secant, cosecant, or finally the versed sine is given, yet the differential of this, if divided by dx will be an algebraic quantity and therefore also the second, third, fourth, etc. differentials of this, if they are divided by agreeing powers of dx. Moreover, so that this differentiation may be better understood, we adjoin the following examples.

1. If there shall be $y = \mathrm{A}\sin 2x\sqrt{(1-xx)}$, there is put $p = 2x\sqrt{(1-xx)}$, so that there shall be $y = A\sin p$, and there will be $dy = \frac{dp}{\sqrt{(1-pp)}}$. But there is

$$dp = 2dx\sqrt{(1-xx)} - \frac{2xxdx}{\sqrt{(1-xx)}} = \frac{2dx(1-2xx)}{\sqrt{(1-xx)}} \quad \text{and} \quad \sqrt{(1-pp)} = 1-2xx,$$

from which values substituted there will be

$$dy = \frac{2dx}{\sqrt{(1-xx)}}.$$

Because also thence it will be apparent, that $2x\sqrt{(1-xx)}$ shall be the sine of twice the arc, while x is the simple sine ; therefore there will be $y = 2A\sin x$ and therefore $dy = \frac{2dx}{(1-xx)}$.

II. If there shall be $y = A\sin\frac{1-xx}{1+xx}$ there may be put $\frac{1-xx}{1+xx} = p$; there will be

$$dp = \frac{-4xdx}{(1+xx)^2} \quad \text{and} \quad \sqrt{(1-pp)} = \frac{2x}{1+xx}.$$

Whereby, since there shall be $dy = \frac{dp}{\sqrt{(1-pp)}}$, there will be

$$dy = \frac{-2dx}{1+xx}.$$

III. If there shall be $y = A\sin\sqrt{\frac{1-x}{2}}$ there may be put $\sqrt{\frac{1-x}{2}} = p$; there will be

$$\sqrt{(1-pp)} = \sqrt{\frac{1+x}{2}} \quad \text{and} \quad dp = \frac{-dx}{4\sqrt{\frac{1-x}{2}}},$$

from which there becomes

$$dy = \frac{dp}{\sqrt{(1-pp)}} = \frac{-dx}{2\sqrt{(1-xx)}}.$$

IV. If there shall be $y = A\tang\frac{2x}{1-xx}$, by putting $p = \frac{2x}{1-xx}$ there will be

$$1+pp = \frac{(1+xx)^2}{(1-xx)^2} \quad \text{and} \quad dp = \frac{2dx(1+xx)}{(1-xx)^2}$$

Whereby, since there shall be $dy = \frac{dp}{1+pp}$ by the rule of tangents (§197), there will be $dy = \frac{2dx}{1+xx}$.

V. If there shall be $y = A\tang\frac{\sqrt{(1+xx)}-1}{x}$, on putting $p = \frac{\sqrt{(1+xx)}-1}{x}$ there becomes

$$pp=\frac{2+xx-2\sqrt{(1+xx)}}{xx}$$

and

$$1+pp=\frac{2+2xx-2\sqrt{(1+xx)}}{xx}=\frac{2\left(\sqrt{(1+xx)}-1\right)\sqrt{(1+xx)}}{xx}$$

and

$$dp=\frac{-dx}{xx\sqrt{(1+xx)}}+\frac{dx}{xx}=\frac{dx\left(\sqrt{(1+xx)}-1\right)}{xx\sqrt{(1+xx)}}.$$

Whereby, since there shall be $dy=\frac{dp}{1+pp}$, there will be made

$$dy=\frac{dx}{2(1+xx)};$$

because also thence it is understood, that there shall be

$$A\tan\!g\frac{\sqrt{(1+xx)}-1}{x}=\frac{1}{2}A\tan\!g\, x.$$

VI. If there shall be $y=e^{A\sin x}$, this formula also by the preceding arguments will be differentiated ; indeed there will be made

$$dy=e^{A\sin x}\frac{dx}{\sqrt{(1-xx)}}.$$

Hence in this manner all the functions of *x* can be differentiated, in which as well as logarithms and exponential quantities also circular arcs enter.

200. Because the differentials of arcs divided by *dx* are algebraic quantities, the second differentials of these and through these the following may be found, which we have established from the differentiation of algebraic functions. Let $y=A\sin x$; because there is $dy=\frac{dx}{\sqrt{(1-xx)}}$, there will be $\frac{dy}{dx}=\frac{1}{\sqrt{(1-xx)}}$, the value of which will give the value of $\frac{ddy}{dx^2}$, if indeed *dx* is assumed constant; from which the differentials of *y* of whatever order thus may be had itself.

Let there be $y=A\sin x$, then

$$\frac{dy}{dx}=\frac{1}{\sqrt{(1-xx)}}$$

and on taking *dx* constant

$$\frac{ddy}{dx^2}=\frac{x}{(1-xx)^{\frac{3}{2}}}$$

$$\frac{d^3y}{dx^3}=\frac{1+2xx}{(1-xx)^{\frac{5}{2}}}$$

$$\frac{d^4y}{dx^4}=\frac{9x+6x^3}{(1-xx)^{\frac{7}{2}}}$$

$$\frac{d^5y}{dx^5}=\frac{9+72x^2+24x^4}{(1-xx)^{\frac{9}{2}}}$$

$$\frac{d^6y}{dx^6}=\frac{225x+600x^3+120x^5}{(1-xx)^{\frac{11}{2}}}$$

etc.,

from which we infer as above (§177) to be generally

$$\frac{d^{n+1}y}{dx^{n+1}}=\frac{1\cdot 2\cdot 3\cdots n}{(1-xx)^{n+\frac{1}{2}}}\left\{\begin{array}{l} x^n+\frac{1}{2}\cdot\frac{n(n-1)}{1\cdot 2}x^{n-2}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^{n-2} \\ +\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}x^{n-6}+\text{etc.}\end{array}\right\}$$

201. The quantities remain, which arise from the inversion of these, evidently the sines and tangents of the given arcs, which it is necessary that we may show how to differentiate. Therefore let x be the arc of a circle and $\sin x$ may denote the sine of this, the differential of which we may want to find. We may put $y=\sin x$ and on putting $x+dx$ in place of x, because y will change into $y+dy$, there will be $y+dy=\sin(x+dx)$ and

$$dy=\sin(x+dx)-\sin x.$$

But there is

$$\sin(x+dx)=\sin x\cdot\cos dx+\cos x\cdot\sin dx,$$

and since there shall be, as we have shown in the *Introductione*,

$$\sin z=\frac{z}{1}-\frac{z^3}{1\cdot 2\cdot 3}+\frac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\text{etc.}$$

$$\cos z=1-\frac{z^2}{1\cdot 2}+\frac{z^4}{1\cdot 2\cdot 3\cdot 4}-\text{etc.},$$

there will be with the vanishing terms rejected $\cos dx=1$ and $\sin dx=dx$, from which there will be made

$$\sin(x+dx)=\sin x+dx\cos x.$$

Whereby on putting $y=\sin x$ there will be

$$dy=dx\cos x.$$

Therefore the differential of the sine of any arc is equal to the differential of the arc multiplied by the cosine.

Therefore if p were some function of x, in a similar manner there will be $d.\sin p=dp\cos p$.

202. Similarly, if there is put $\cos x$ or the cosine of the arc x, it may be required to investigate the differential of which , there may be put $y=\cos x$ and on putting $x+dx$ in place of x there comes about $y+dy=\cos(x+dx)$. Now there is

$$\cos(x+dx)=\cos x\cdot\cos dx-\sin x\cdot\sin dx,$$

and because, as we have seen just now, there is $\cos dx=1$ and $\sin dx=dx$, there will be

$$y+dy=\cos x-dx\sin x$$

and thus

$$dy=-dx\sin x.$$

Whereby the differential of the cosine of each arc is equal to the differential of the negative arc multiplied by the sine of the same arc taken.

Thus, if p were some function of x, there will be

$$d.\cos p=-dp\sin p.$$

These differentiations also can be elicited from the preceding in this manner If there were $y=\sin p$, there will be $p=A\sin y$ and

$$dp=\frac{dy}{\sqrt{(1-yy)}};$$

but on account of $y=\sin p$ there will be $\cos p=\sqrt{(1-yy)}$, with which value substituted there will be $dp=\frac{dy}{\cos p}$

and

$$dy=dp\cos p$$

as before. In a similar manner, if there shall be
$y = \cos p$, there will be $\sqrt{(1-yy)} =$ sin p and $p = A\cos y$
and thus

$$dp = \frac{-dy}{\sqrt{(1-yy)}} = \frac{-dy}{\sin p},$$

from which there becomes as before

$$dy = -dp \sin p.$$

203. If there were $y = \text{tang}\, x$, there will be

$$dy = \text{tang}(x+dx) - \text{tang}\, x;$$

but there is

$$\text{tang}(x+dx) = \frac{\text{tang}\, x + \text{tang}\, dx}{1 - \text{tang}\, x \cdot \text{tang}\, dx};$$

from which fraction if the tangent *x* may be subtracted, there will remain

$$dy = \frac{\text{tang} dx (\text{tang}\, x + \text{tang}\, x)}{1 - \text{tang}\, x \cdot \text{tang}\, dx}.$$

Indeed the tangent of the vanishing arc *dx* is equal to the arc itself and thus $\text{tang}\, dx = dx$ and the denominator $1 - dx\, \text{tang}\, x$ will change into unity; on which account there becomes

$$dy = dx\left(1 + \text{tang}^2 x\right).$$

Truly there is

$$1 + \text{tang}^2 x = \sec^2 x = \frac{1}{\cos^2 x}$$

with $\cos^2 x$ denoting the square of the cosine of *x;* consequently, if there were $y = \text{tang}\, x$, there will be

$$dy = dx \sec^2 x = \frac{dx}{\cos^2 x}.$$

Because the differential can also be found by the differentiation of the sines and cosines , for since there shall be $\text{tang}\, x = \frac{\sin x}{\cos x}$ there will be [§164]

$$dy = \frac{dx \cos x \cdot \cos x + dx \cdot \sin x \cdot \sin x}{\cos^2 x} = \frac{dx}{\cos^2 x}$$

on account of $\sin^2 x + \cos^2 x = 1$.

204. Also this differential may be found in another way. Since there shall be $y = \tan g\, x$, there will be $x = A\tan g\, y$ and by the above precepts there becomes

$$dx = \frac{dy}{1+yy}.$$

But since there shall be $y = \tan g\, x$, there will be $\sqrt{(1+yy)} = \sec x = \frac{1}{\cos x}$ and thus $dx = dy \cos^2 x$ and

$$dy = \frac{dx}{\cos^2 x}$$

as before.

Therefore the differential of the tangent of any arc is equal to the differential of the arc divided by the square of the cosine of the same arc.

In a similar manner if there may be put $y = \cot x$, there becomes $x = A\cot y$ and

$$dx = \frac{-dy}{1+yy}.$$

But now there will be $\sqrt{(1+yy)} = \operatorname{cosec} x = \frac{1}{\sin x}$, from which there will be had $dx = -dy \sin^2 x$ and

$$dy = \frac{-dx}{\sin^2 x}.$$

Therefore the differential of the cotangent of any arc is equal to the negative differential of the arc taken and divided by the square of the sine of the same arc.

Or because there is $\cot x = \frac{\cos x}{\sin x}$, there becomes this fraction by differentiation

$$dy = \frac{-dx \sin^2 x - dx \cos^2 x}{\sin^2 x} = \frac{-dx}{\sin^2 x},$$

just as we have found.

205. If the secant of the arc may be proposed, so that there shall be $y = \sec x$, because there will be $y = \frac{1}{\cos x}$, there will be

$$dy = \frac{dx \sin x}{\cos^2 x} = dx \tan g\, x \sec x.$$

In a similar manner if there were $y = \operatorname{cosec} x$, on account of $y = \frac{1}{\sin x}$ there will be

$$dy = \frac{-dx \cos x}{\sin^2 x} = -dx \cot x \operatorname{cosec} x,$$

for which singular cases the rules of forming may appear superfluous. If the versed sine of the arc is proposed $y = \text{vs}\, x$, because there shall be $y = 1 - \cos x$, there will be $dy = dx \sin x$. Therefore all cases, in which it is proposed to relate some right line to an arc, because always it can be expressed by the sine or cosine, they can be differentiated without difficulty. Nor indeed only the first differentials, but also the second and the following may be found by the rules given. We may put to be $y = \sin x$ and $z = \cos x$ and dx to be constant; there will be as follows :

$y = \sin x$	$z = \cos x$
$dy = dx \cos x$	$dz = -dx \sin x$
$ddy = -dx^2 \sin x$	$ddz = -dx^2 \cos x$
$d^3y = -dx^3 \cos x$	$d^3z = dx^3 \sin x$
$d^4y = dx^4 \sin x$	$d^4z = dx^4 \cos x$
etc.	etc.

206. In a similar manner the differentials may be found of all orders of the tangents of the arc *x*. For let there be $y = \text{tang}\, x = \frac{\sin x}{\cos x}$ and on putting *dx* constant, there will be :

$$y = \frac{\sin x}{\cos x}$$

$$\frac{dy}{dx} = \frac{1}{\cos^2 x}$$

$$\frac{ddy}{dx^2} = \frac{2\sin x}{\cos^3 x}$$

$$\frac{d^3y}{dx^3} = \frac{6}{\cos^4 x} - \frac{4}{\cos^2 x}$$

$$\frac{d^4y}{dx^4} = \frac{24\sin x}{\cos^5 x} - \frac{8\sin x}{\cos^3 x}$$

$$\frac{d^5y}{dx^5} = \frac{120}{\cos^6 x} - \frac{120}{\cos^4 x} + \frac{16}{\cos^2 x}$$

$$\frac{d^6y}{dx^6} = \frac{720\sin x}{\cos^7 x} - \frac{480\sin x}{\cos^5 x} + \frac{32\sin x}{\cos^3 x}$$

$$\frac{d^7y}{dx^7} = \frac{540}{\cos^8 x} - \frac{6720}{\cos^6 x} + \frac{2016}{\cos^4 x} - \frac{64}{\cos^2 x}$$

etc.

207. Therefore any functions, in which the sine or the cosine of the arc are present, can be differentiated by these precepts, as it is possible to see in the following examples.

I. If there shall be $y = 2\sin x \cdot \cos x = \sin 2x$, there will be

$$dy = 2dx\cos^2 x - 2dx\sin^2 x = 2dx\cos 2x.$$

II If there shall be $y = \sqrt{\frac{1-\cos x}{2}}$ or $y = \sin\frac{1}{2}x$, there will be

$$dy = \frac{dx\sin x}{2\sqrt{2(1-\cos x)}}.$$

But since there shall be

$$2\sqrt{2(1-\cos x)} = 2\sin\tfrac{1}{2}x \quad \text{and} \quad \sin x = 2\sin\tfrac{1}{2}x\cos\tfrac{1}{2}x,$$

there becomes

$$dy = \tfrac{1}{2}dx\cos\tfrac{1}{2}x$$

as follows at once from the form $y = \sin\frac{1}{2}x$.

III. If there shall be $y = \cos l\frac{1}{x}$ there will be on putting

$l\frac{1}{x} = p$, $y = \cos p$ and $dy = -dp\sin p$.

But on account of $p = l1 - lx$ there will be $dp = \frac{-dx}{x}$ and thus $dy = \frac{dx}{x}\sin l\frac{1}{x}$.

IV. If there shall be $y = e^{\sin x}$, there will be

$$dy = e^{\sin x}dx\cos x.$$

V. If there shall be $y = e^{\frac{-n}{\cos x}}$, there will be

$$dy = -\frac{e^{\frac{-n}{\cos x}}ndx\sin x}{\cos^2 x}.$$

VI. If there shall be $y = l\left(1 - \sqrt{\left(1 - e^{\frac{-n}{\sin x}}\right)}\right)$, there may be put $e^{\frac{-n}{\sin x}} = p$ and on account of

$$y = l\left(1 - \sqrt{(1-p)}\right)$$

there will be

$$dy=\frac{dp}{2\left(1-\sqrt{(1-p)}\right)\sqrt{(1-p)}}.$$

But there is

$$dp=\frac{e^{\frac{-n}{\sin x}}ndx\cos x}{\sin^2 x}$$

With which value substituted there will be

$$dy=\frac{ne^{\frac{-n}{\sin x}}dx\cos x}{2\sin^2 x\left(1-\sqrt{\left(1-e^{\frac{-n}{\sin x}}\right)}\right)\sqrt{\left(1-e^{\frac{-n}{\sin x}}\right)}}.$$

CAPUT VI
DE DIFFERENTIATIONE FUNCTIONUM TRANSCENDENTIUM

178. Praeter infinita quantitatum transcendentium seu non algebraicarum genera, quae calculus integralis suppeditabit, in *Introductione in analysin infinitorum* ad cognitionem aliquot huiusmodi quantitatum magis usitatarum nobis pervenire licuit, quas doctrina de logarithmis et arcubus circularibus suggesserat. Quoniam igitur harum quantitatum naturam tam dilucide exposuimus, ut fere eadem facilitate atque quantitates algebraicae in calculo tractari queant, earum quoque differentialia in hoc capite investigabimus, quo earum indoles ac proprietates clarius perspiciantur hocque pacta aditus ad calculum integralem, qui quantitatum transcendentium est fons proprius, patefiat.

179. Primum igitur occurrunt quantitates logarithmicae seu eiusmodi functiones ipsius x, quae praeter expressiones algebraicas quoque logarithmum ipsius x seu cuiusvis ipsius functionis involvunt. Ad quas differentiandas cum quantitates algebraicae nullum negotium amplius facessant, omnis difficultas in inveniendo differentiali logarithmi cuiusque ipsius x functionis erit posita. Quia vero logarithmorum plurima dantur genera diversa, quae tamen inter se constantes tenent rationes, hic logarithmos hyperbolicos potissimum contemplabimur, cum ex iis omnes reliqui logarithmi facile formentur. Si enim functionis p logarithmus hyperbolicus fuerit $= lp$, tum eiusdem functionis p logarithmus ex alio canone desumtus erit $= mlp$ denotante m numerum, quo relatio huius logarithmorum canonis ad hyperbolicos exprimitur. Hanc ob causam lp perpetuo hic designabit logarithmum hyperbolicum quantitatis p.

180. Quaeramus ergo differentiale logarithmi hyperbolici quantitatis x ponaturque $y = lx$, ita ut differentialis dy valor definiri debeat. Ponatur $x + dx$ loco x sicque transibit y in $y^{\mathrm{I}} = y + dy$; quare habebitur

$$y + dy = l\left(x + dx\right) \text{ et } dy = l\left(x + dx\right) - lx = l\left(1 + \tfrac{dx}{x}\right).$$

At iam supra logarithmum hyperbolicum huiusmodi expressionis $1 + z$ ita per seriem infinitam expressimus, ut esset

$$l\left(1 + z\right) = z - \tfrac{z^2}{2} + \tfrac{z^3}{3} - \tfrac{z^4}{4} + \text{etc.}$$

Posito ergo $\frac{dx}{z}$ pro z obtinebimus

$$dy = \tfrac{dx}{x} - \tfrac{dx^2}{2x^2} + \tfrac{dx^3}{3x^3} - \text{etc}.$$

Cum igitur huius seriei omnes termini prae primo evanescant, erit

$$d.lx = dy = \frac{dx}{x}.$$

Unde alius cuiuscunque logarithmi, cuius ad hyperbolicum ratio est ut $n:1$, differentiale $= \frac{ndx}{x}$.

181. Si igitur cuiusque ipsius *x* functionis *p* logarithmus *lp* proponatur, eodem ratiocinio reperietur eius differentiale esse $= \frac{dp}{p}$, unde ad logarithmorum differentialia invenienda haec habetur regula:

Quantitatis p, cuius logarithmus proponitur, sumatur differentiale hocque per ipsam quantitatem p divisum dabit differentiale logarithmi quaesitum.

Sequitur haec eadem regula quoque ex forma $\frac{p^0 - 1^0}{0}$, ad quam superiori libro logarithmum ipsius *p* reduximus. Sit $\omega = 0$, et cum sit $lp = \frac{p^\omega - 1}{\omega}$, erit

$$d.lp = d.\frac{1}{\omega} p^\omega = p^{\omega-1} dp = \frac{dp}{p}$$

ob $\omega = 0$. Notandum autem est $\frac{dp}{p}$ esse differentiale logarithmi hyperbolici ipsius *p*, ita ut, si logarithmus vulgaris ipsius *p* proponeretur, differentiale illud $\frac{dp}{p}$ multiplicari deberet per hunc numerum 0,43429448 etc.

182. Ope huius ergo regulae, cuiuscunque functionis ipsius *x* logarithmus proponatur, eius differentiale facillime inveniri poterit, quemadmodum ex sequentibus exemplis perspicietur.

I. Si sit $y = lx$, erit

$$dy = \frac{dx}{x}.$$

II. Si sit $y = lx^n$, ponatur $x^n = p$, ut sit $y = lp$, eritque $dy = \frac{dp}{p}$. At est $dp = nx^{n-1}dx$, unde fit

$$dy = \frac{ndx}{x}.$$

Idem quoque ex logarithmorum natura colligitur; cum enim sit $lx^n = nlx$, erit $d.lx^n = nd.lx = \frac{ndx}{x}$.

III. Si sit $y = l(1+xx)$, erit

$$dy = \frac{2xdx}{1+xx}.$$

IV. Si sit $y = l\frac{1}{\sqrt{(1-xx)}}$, quia erit $y = -l\sqrt{(1-xx)} = -\frac{1}{2}l(1-xx)$, invenitur

$$dy = \frac{xdx}{1-xx}.$$

V. Si sit $y = l\frac{x}{\sqrt{(1+xx)}}$, ob $y = lx - \frac{1}{2}(1+xx)$ fiet

$$dy = \frac{dx}{x} - \frac{xdx}{1+xx} = \frac{dx}{x(1+xx)}.$$

VI. Si sit $y = l\left(x+\sqrt{(1+xx)}\right)$, fiet

$$dy = \frac{dx + xdx:\sqrt{(1+xx)}}{x+\sqrt{(1+xx)}} = \frac{xdx + dx\sqrt{(1+xx)}}{\left(x+\sqrt{(1+xx)}\right)\sqrt{(1+xx)}};$$

cuius fractionis cum numerator ac denominator per $x+\sqrt{(1+xx)}$ sit divisibilis, fiet

$$dy = \frac{dx}{\sqrt{(1+xx)}}.$$

VII. Si sit $y = \frac{1}{\sqrt{-1}}l\left(x\sqrt{-1}+\sqrt{(1-xx)}\right)$, ponatur $x\sqrt{-1} = z$. Atque ob

$y = \frac{1}{\sqrt{-1}}l\left(z+\sqrt{(1+zz)}\right)$ erit per praecedens $dy = \frac{1}{\sqrt{-1}}dz:\sqrt{(1+zz)}$. Quare ob $dz = dx\sqrt{-1}$ fiat

$$dy = \frac{dx}{\sqrt{(1-xx)}}.$$

Quamvis ergo logarithmus propositus imaginaria involvat, tamen eius differentiale fit reale.

183. Si quantitas, cuius logarithmus proponitur, habeat factores, tum ipse logarithmus in plures alios resolvetur hoc modo. Si proponatur $y = lpqrs$, quia erit $y = lp + lq + lr + ls$, erit

$$dy = \frac{dp}{p} + \frac{dq}{q} + \frac{dr}{r} + \frac{ds}{s}.$$

Haec resolutio pariter locum habet, si illa quantitas, cuius logarithmus differentiari debet, fuerit fractio. Sit enim $y = l\frac{pq}{rs}$; ob $y = lp + lq - lr - ls$ erit

$$dy = \frac{dp}{p} + \frac{dq}{q} - \frac{dr}{r} - \frac{ds}{s}.$$

Neque etiam potestates difficultatem movebunt; si etiam fuerit $y = l\frac{p^m q^n}{r^\mu s^\nu}$,ob $y = mlp + nlq - \mu lr - \nu ls$ erit

$$dy = \frac{mdp}{p} + \frac{ndq}{q} - \frac{\mu dr}{r} - \frac{\nu ds}{s}.$$

I. Si fuerit $y = l(a+x)(b+x)(c+x)$, quia erit

$$y = l(a+x) + l(b+x) + l(c+x),$$

fiet differentiale quaesitum

$$dy = \frac{dx}{a+x} + \frac{dx}{b+x} + \frac{dx}{c+x}.$$

II. Si fuerit $y = \frac{1}{2}l\frac{1+x}{1-x}$, erit

$$y = \frac{1}{2}l(1+x) - \frac{1}{2}l(1-x)$$

hincque

$$dy = \frac{\frac{1}{2}dx}{1+x} + \frac{\frac{1}{2}dx}{1-x} = \frac{dx}{1-xx}.$$

III. Sit $y = \frac{1}{2}l\frac{\sqrt{(1+xx)}+x}{\sqrt{(1+xx)}-x}$, ob

$$y = \frac{1}{2}l\left(\sqrt{(1+xx)}+x\right) - \frac{1}{2}l\left(\sqrt{(1+xx)}-x\right)$$

erit

$$dy = \frac{\frac{1}{2}dx}{\sqrt{(1+xx)}} + \frac{\frac{1}{2}dx}{\sqrt{(1+xx)}} = \frac{dx}{\sqrt{(1+xx)}}.$$

Hoc idem facilius invenitur, si in fractione $\frac{\sqrt{(1+xx)}+x}{\sqrt{(1+xx)}-x}$ irrationalitas in denominatore tollatur multiplicando numeratorem ac denominatorem per $\sqrt{(1+xx)}+x$; prodibit enim

$$y=\tfrac{1}{2}l\left(\sqrt{(1+xx)}+x\right)^{2}=l\left(\sqrt{(1+xx)}+x\right),$$

cuius differentiale ante vidimus esse $dy=\frac{dx}{\sqrt{(1+xx)}}$

IV. Si sit $y=l\frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}}$, ponatur huius fractionis numerator

$$\sqrt{(1+x)}+\sqrt{(1-x)}=p$$

et denominator

$$\sqrt{(1+x)}-\sqrt{(1-x)}=q$$

erit $y=l\frac{p}{q}=lp-lq$ et $dy=\frac{dp}{p}-\frac{dq}{q}$. Est vero

$$dp=\frac{dx}{2\sqrt{(1+x)}}-\frac{dx}{2\sqrt{(1-x)}}=\frac{-dx}{2\sqrt{(1-xx)}}\left(\sqrt{(1+x)}-\sqrt{(1-x)}\right)=\frac{-qdx}{2\sqrt{(1-xx)}}$$

et

$$dq=\frac{dx}{2\sqrt{(1+x)}}+\frac{dx}{2\sqrt{(1-x)}}=\frac{pdx}{2\sqrt{(1-xx)}}.$$

Hinc fiet

$$\frac{dp}{p}-\frac{dq}{q}=\frac{-qdx}{2p\sqrt{(1-xx)}}-\frac{pdx}{2q\sqrt{(1-xx)}}=\frac{-(pp+qq)dx}{2pq\sqrt{(1-xx)}}.$$

At est $pp+qq=4$ et $pq=2x$, unde erit

$$dy=-\frac{dx}{x\sqrt{(1-xx)}}.$$

Hoc autem differentiale facilius invenietur, si logarithmus propositus ita transformetur

$$y=l\frac{1+\sqrt{(1-xx)}}{x}=l\left(\frac{1}{x}+\sqrt{\left(\frac{1}{xx}-1\right)}\right).$$

Posito enim $\frac{1}{x}+\sqrt{\left(\frac{1}{xx}-1\right)}=p$ erit

$$\frac{-dx}{xx}-\frac{dx}{x^3\sqrt{\left(\frac{1}{xx}-1\right)}}=\frac{-dx}{xx}-\frac{dx}{xx\sqrt{(1-xx)}}=\frac{-dx\left(1+\sqrt{(1-xx)}\right)}{xx\sqrt{(1-xx)}},$$

ideoque ob $p=\frac{1+\sqrt{(1-xx)}}{x}$ erit $dy=\frac{dp}{p}=\frac{-dx}{x\sqrt{(1-xx)}}$ ut ante.

184. Cum igitur logarithmorum differentialia prima, si per *dx* dividantur, sint quantitates algebraicae, differentialia secunda ac sequentium ordinum per praecepta praecedentis capitis facile inveniuntur, siquidem differentiale *dx* assumatur constans. Sic posito $y=lx$ erit

$$dy=\frac{dx}{x} \quad \text{et} \quad \frac{dy}{dx}=\frac{1}{x}$$
$$ddy=\frac{-dx^2}{x^2} \quad \text{et} \quad \frac{ddy}{dx^2}=\frac{-1}{x^2}$$
$$d^3y=\frac{2dx^3}{x^3} \quad \text{et} \quad \frac{d^3y}{dx^3}=\frac{2}{x^3}$$
$$d^4y=\frac{-6dx^4}{x^4} \quad \text{et} \quad \frac{d^4y}{dx^4}=\frac{-6}{x^4}$$
etc.

Atque si *p* fuerit quantitas algebraica sitque $y=lp$, etiamsi *y* non sit quantitas algebraica, tamen $\frac{dy}{dx},\frac{ddy}{dx^2},\frac{d^3y}{dx^3}$ etc. erunt functiones algebraicae ipsius *x*.

185. Exposita logarithmorum differentiatione functiones, quae ex algebraicis ac logarithmis sunt permixtae, facile differentiabuntur, perinde atque eae, quae ex logarithmis solis componuntur, uti ex sequentibus exemplis fiet perspicuum.

I. Si sit $y=(lx)^2$, ponatur $lx=p$ atque ob $y=p^2$ erit $dy=2pdp$, verum $dp=\frac{dx}{x}$; ideoque erit $dy=\frac{2dx}{x}lx$.

II. Simili modo, si sit $y=(lx)^n$, erit $dy=\frac{ndx}{x}(lx)^{n-1}$, unde, si sit $y=\sqrt{lx}$, ob $n=\frac{1}{2}$ erit $dy=\frac{dx}{2x\sqrt{lx}}$.

III. Atque si *p* fuerit functio quaecunque ipsius *x* ponaturque $y=(lp)^n$, erit

$$dy=\frac{ndp}{lp}(lp)^{n-1}$$

Quare, cum differentiale *dp* per praecedentia assignari possit, erit quoque differentiale ipsius *y* cognitum.

IV. Si sit $y = lp.lq$ fuerintque *p* et *q* functiones quaecunque ipsius *x*, per regulam factorum supra datam erit

$$dy = \frac{dp}{p} lq + \frac{dq}{q} lp.$$

V. Si sit $y = xlx$, erit per eandem regulam $dy = dxlx + \frac{xdx}{x} = dxlx + dx$.

VI. Si sit $y = x^m lx - \frac{1}{m} x^m$, differentiatione secundum partes instituta reperietur $d.x^m lx = mx^{m-1}dxlx + x^{m-1}dx$ et $d.\frac{1}{m} x^m = x^{m-1}dx$, unde erit $dy = mx^{m-1}dxlx$.

VII. Si sit $y = x^m (lx)^n$, fiet

$$dy = mx^{m-1}dx(lx)^n + nx^{m-1}dx(lx)^{n-1}.$$

VIII. Si logarithmi logarithmorum occurrant, uti si fuerit $y = llx$, ponatur $lx = p$; erit $y = lp$ et $dy = \frac{dp}{p}$; at est $dp = \frac{dx}{x}$, unde fiet $dy = \frac{dx}{xlx}$.

IX. Atque si fuerit $y = lllx$, si statuatur $lx = p$, fiet $y = llp$ eritque per exemplum praecedens $dy = \frac{dp}{plp}$; at est $dp = \frac{dx}{x}$, quibus valoribus substitutis habebitur $dy = \frac{dx}{xlx \cdot llx}$.

186. Exposita logarithmorum differentiatione progrediamur ad quantitates exponentiales seu eiusmodi potestates, quarum exponentes sint variabiles. Huiusmodi autem ipsius *x* functionum differentialia per logarithmorum differentiationem inveniri possunt hoc modo. Quaeratur differentiale ipsius a^x; ad quod investigandum ponatur $y = a^x$ eritque logarithmis sumendis $ly = xla$. Sumantur iam differentialia atque obtinebitur $dy = dxla$, unde fit $\frac{dy}{y} = ydxla$; cum autem sit $y = a^x$, erit $dy = a^x dxla$, quod est differentiale ipsius a^x. Simili modo, si sit *p* functio quaecunque ipsius *x*, huius quantitatis exponentialis a^p differentiale erit $= a^p dpla$.

187. Hoc idem autem differentiale immediate ex natura quantitatum exponentialium in *Introductione* exposita deduci potest. Sit enim proposita a^p denotante p functionem quamcunque ipsius x, quae posito $x+dx$ loco x abeat in $p+dp$. Unde si ponatur $y=a^p$, si x abeat in $x+dx$, erit $y+dy=a^{p+dp}$ ideoque

$$dy=a^{p+dp}-a^p=a^p\left(a^{dp}-1\right)$$

Ostendimus autem supra quamvis quantitatem exponentialem a^z per huiusmodi seriem exprimi

$$1+zla+\frac{z^2(la)^2}{2}+\frac{z^3(la)^3}{6}+\text{etc.};$$

unde erit

$$a^{dp}=1+dpla+\frac{dp^2(la)^2}{2}+\text{etc.}$$

et $a^{dp}-1=dpla$, quia sequentes termini prae $dpla$ omnes evanescunt. Consequenter erit

$$dy=d.a^p=a^p dpla.$$

Quare quantitatis exponentialis a^p differentiale erit productum ex ipsa quantitate exponentiali, exponentis differentiali dp et logarithmo quantitatis constantis a, quae ad exponentem variabilem est evecta.

188. Si igitur e sit numerus, cuius logarithmus hyperbolicus est $=1$, ut sit $le=1$, erit quantitatis e^x differentiale $=e^x dx$. Atque si dx sumatur constans, erit huius differentiale $=e^x dx^2$, quod est differentiale secundum ipsius e^x. Simili modo differentiale tertium erit $=e^x dx^3$. Quare si sit $y=e^{nx}$, erit

$$\frac{dy}{dx}=ne^{nx} \quad \text{et} \quad \frac{ddy}{dx^2}=n^2e^{nx} \quad \text{porroque} \quad \frac{d^3y}{dx^3}=n^3e^{nx} \quad \frac{d^3y}{dx^3}=n^4e^{nx} \text{ etc.}$$

Unde patet ipsius e^{nx} differentialia primum, secundum et reliqua sequentia constituere progressionem geometricam eritque ergo differentiale ordinis m ipsius $e^{nx}=y$, nempe $d^m y,=n^m e^{nx}dx^m$ hincque igitur $\frac{d^{my}}{ydx^m}$ quantitas constans n^m.

189. Si ipsa quantitas, quae elevatur, fuerit variabilis, eius differentiale simili modo investigabitur. Sint p et q functiones quaecunque ipsius x ac proponatur quantitas exponentialis $y=p^q$. Sumtis logarithmis erit $ly=qlp$, quibus differentiatis erit

$$\frac{dy}{y}=dqlp+\frac{qdp}{p},$$

unde fit

$$dy = ydqlp + \frac{yqdp}{p} = p^q dqlp + qp^{q-1}dp$$

ob $y = p^q$. Hoc ergo differentiale constat duobus membris, quorum prius $p^q dqlp$ oritur, si quantitas proposita p^q ita differentietur, quasi *p* esset quantitas constans solusque exponens *q* variabilis; alterum vero membrum $qp^{q-1}dp$ oritur, si in quantitate proposita p^q exponens *q* tanquam constans spectetur solaque quantitas *p*, quasi esset variabilis, tractetur. Hocque ergo differentiale per regulam generalem differentiandi supra [§ 170] traditam inveniri potuisset.

190. Eiusdem vero expressionis p^q differentiale quoque ex natura quantitatum exponentialium erui potest hoc modo. Sit $y = p^q$ eritque loco *x* posito $x + dx$ utique $y + dy = (p + dp)^{q+dq}$; quae expressio si more solito in seriem resolvatur, fiet

$$y + dy = p^{q+dq} + (q + dq)\, p^{q+dq-1}dp + \frac{(q+dq)(q+dq-1)}{1\cdot 2}\, p^{q+dq-2}dp^2 + \text{etc.}$$

ideoque

$$dy = p^{q+dq} - p^q + (q + dq)\, p^{q+dq-1}dp \ ;$$

sequentes enim termini, qui altiores ipsius *dp* potestates involvunt, prae $(q + dq)\, p^{q+dq-1}dp$ evanescunt. At est

$$p^{q+dq} - p^q = p^q\left(p^{dq} - 1\right) = p^q\left(1 + dqlp + \frac{dq^2(lp)^2}{2} + \text{etc.} - 1\right) = p^q dqlp\,.$$

In altero vero termino $(q + dq)\, p^{q+dq-1}dp$ si loco $q + dq$ scribamus *q*, orietur $qp^{q-1}dp$ ideoque differentiale erit ut ante $dy = p^q dqlp + qp^{q-1}dp$.

191. Facilius vero hoc idem differentiale ex natura quantitatum exponentialium investigabitur hoc modo. Cum sumto *e* pro numero, cuius logarithmus hyperbolicus est = 1, sit $p^q = e^{qlp}$, utriusque enim logarithmus est idem *qlp*, erit $y = e^{qlp}$. Quare, cum nunc quantitas elevata *e* sit constans, erit

$$dy = e^{qlp}\left(dqlp + \frac{qdp}{p}\right)$$

uti ante ostendimus in regula §187 data. Restituatur igitur p^q loco e^{qlp} fietque

$$dy = p^q dqlp + p^q qdp{:}p = p^q dqlp + qp^{q-1}dp$$

Si igitur fuerit $y = x^x$, erit $dy = x^x dxlx + x^x dx$; atque hinc quoque eius ulteriora differentialia definientur; reperietur enim

$$\frac{ddy}{dx^2} = x^x\left(\tfrac{1}{x} + (1+lx)^2\right)$$
$$\frac{d^3y}{dx^3} = x^x\left((1+lx)^3 + \frac{3(1+lx)}{x} - \frac{1}{xx}\right)$$

etc.

192. Inter differentialia huiusmodi functionum, quae quantitates exponentiales complectuntur, imprimis sunt notanda sequentia exempla, quae ex differentiatione formulae $e^x p$ originem habent; est autem

$$d.e^x p = e^x dp + e^x pdx = e^x(dp + pdx).$$

I. Si sit $y = e^x x^n$, erit

$$dy = e^x nx^{n-1} + e^x x^n dx \quad \text{seu} \quad dy = e^x dx\left(nx^{n-1} + x^n\right).$$

II. Si sit $y = e^x(x-1)$, erit

$$dy = e^x xdx.$$

III. Si sit $y = e^x\left(x^2 - 2x + 2\right)$, erit

$$dy = e^x xxdx.$$

IV. Si sit $y = e^x\left(x^3 - 3x^2 + 6x - 6\right)$, erit

$$dy = e^x x^3 dx.$$

193. Si ipsi exponentes fuerint denuo quantitates exponentiales, differentiatio secundum eadem praecepta instituetur. Sic, si haec quantitas e^{e^x} differentiari debeat, statuatur $e^x = p$, ut sit

$$y = e^{e^x} = e^p$$

erit $dy = e^p dp$; at est $dp = e^x dx$, unde, si fuerit $y = e^{e^x}$, erit

$$dy = e^{e^x} e^x dx$$

atque si sit $y = e^{e^{e^x}}$, erit

$$dy = e^{e^{e^x}} e^{e^x} e^x dx$$

Quodsi vero fuerit $y = p^{q^r}$, statuatur $q^r = z$; erit

$$dy = p^z dzlp + zp^{z-1}dp, \quad \text{at} \quad dz = q^r drlq + rq^{r-1}dq\,,$$

unde

$$dy = p^z q^r drlp \cdot lq + p^z rq^{r-1}dqlp + p^z q^r dp{:}p$$

Quare si sit $y = p^{q^r}$, erit

$$dy = p^{q^r} q^r \left(drlp \cdot lq + \frac{rdqlp}{q} + \frac{dp}{p} \right).$$

Hoc ergo modo, quaecunque occurrat quantitas exponentialis, eius differentiale inveniri poterit.

194. Pergamus ergo ad quantitates transcendentes, ad quarum cognitionem consideratio arcuum circularium nos supra deduxit. Sit igitur in circulo, cuius radium constanter ponimus unitati aequalem, propositus arcus, cuius sinus sit $= x$, quem arcum hoc modo exprimamus $y = A\sin x$, huiusque arcus differentiale investigemus seu incrementum, quod accipit, si sinus x differentiali suo dx augeatur. Hoc autem ex differentiatione logarithmorum praestari poterit, quia in *Introductione* [l.c. §138] ostendimus hanc expressionem $y = A\sin x$ reduci posse ad hanc logarithmicam $\frac{1}{\sqrt{-1}} l\left(\sqrt{(1-xx)} + x\sqrt{-1}\right)$. Posito ergo $y = A\sin x$ erit quoque

$$y = \frac{1}{\sqrt{-1}} l\left(\sqrt{(1-xx)} + x\sqrt{-1}\right)$$

quae differentiata dat [§ 182, VII]

$$dy = \frac{\frac{1}{\sqrt{-1}}\left(\frac{-xdx}{\sqrt{(1-xx)}} + dx\sqrt{-1}\right)}{\sqrt{(1-xx)} + x\sqrt{-1}} = \frac{dx\left(x\sqrt{-1} + \sqrt{(1-xx)}\right)}{\left(\sqrt{(1-xx)} + x\sqrt{-1}\right)\sqrt{(1-xx)}}\,,$$

unde fit

$$dy = \frac{dx}{\sqrt{(1-xx)}}.$$

195. Istud arcus circularis differentiale etiam hoc modo facilius sine logarithmorum subsidio inveniri potest. Si enim sit $y = A\sin x$, erit x sinus arcus y seu $x = \sin y$. Cum igitur posito $x + dx$ loco x abeat y in $y + dy$, fiet $x + dx = \sin(y + dy)$. At quia est

$$\sin(a+b) = \sin a \cdot \cos b + \cos a \cdot \sin b,$$

erit

$$\sin(y+dy) = \sin y \cdot \cos dy + \cos y \cdot \sin dy;$$

arcus autem evanescentis dy sinus ipsi illi arcui dy eiusque cosinus sinui toti aequatur [§ 201]; hanc ob rem fiet

$$\sin(y+dy) = \sin y + dy\cos y \quad \text{ideoque} \quad x + dx = \sin y + dy\cos y.$$

Quia vero est $\sin y = x$, erit cosinus ipsius y, seu $\cos y, = \sqrt{(1-xx)}$, quibus valoribus substitutis erit $dx = dy(1-xx)$, ex qua obtinebitur

$$dy = \frac{dx}{\sqrt{(1-xx)}}.$$

Arcus ergo, cuius sinus proponitur, differentiale aequatur differentiali sinus per cosinum diviso.

196. Cum igitur, si p fuerit functio quaecunque ipsius x atque y denotet arcum, cuius sinus est $= p$, seu $y = A\sin p$, sit huius arcus differentiale $dy = \frac{dp}{\sqrt{(1-pp)}}$, ubi $\sqrt{(1-pp)}$ exprimit cosinum eiusdem arcus, inveniri quoque poterit differentiale arcus, cuius cosinus proponitur. Sit enim $y = A\cos x$; erit eiusdem arcus sinus $= \sqrt{(1-xx)}$ ideoque $y = A\sin\sqrt{(1-xx)}$. Facto ergo $p = \sqrt{(1-xx)}$ erit

$$dp = \frac{-xdx}{\sqrt{(1-xx)}} \quad \text{et} \quad \sqrt{(1-pp)} = x;$$

unde fiet

$$dy = \frac{-xdx}{\sqrt{(1-xx)}}.$$

Arcus ergo, cuius cosinus proponitur, differentiale aequatur differentiali cosinus negative sumto atque per sinum eiusdem arcus diviso.

Quod etiam hoc modo ostendi potest. Si sit $y = A\cos x$, ponatur

$z = A\sin x$; erit $dz = \frac{dx}{\sqrt{(1-xx)}}$; at arcus y et z simul sumti dant arcum constantem 90° eritque $y + z =$ constans ideoque $dy + dz = 0$ seu $dy = -dz$; unde fit $dy = \frac{-xdx}{\sqrt{(1-xx)}}$ ut ante.

197. Si arcus proponatur differentiandus, cuius tangens detur, ita ut sit $y = A\operatorname{tang} x$: arcus autem, cuius tangens est x, sinus erit $= \frac{x}{\sqrt{(1+xx)}}$ et cosinus $= \frac{1}{\sqrt{(1+xx)}}$. Posito ergo $\frac{x}{\sqrt{(1+xx)}} = p$, ut sit $\sqrt{(1-pp)} = \frac{1}{\sqrt{(1+xx)}}$, fiat $y = A\sin p$; unde per regulam modo datam erit $dy = \frac{dp}{\sqrt{(1-pp)}}$. At ob $p = \frac{x}{\sqrt{(1+xx)}}$ erit $dp = \frac{dx}{(1+xx)^{\frac{3}{2}}}$, quibus valoribus substitutis fiet

$$dy = \frac{dx}{1+xx}$$

Arcus ergo, cuius tangens proponitur, differentiale aequatur differentiali tangentis per quadratum secantis diviso. Est enim $\sqrt{(1+xx)}$ secans, si x sit tangens.

198. Simili modo si proponatur arcus, cuius cotangens datur, ita ut sit $y = A\cot x$, quia eiusdem arcus tangens est $\frac{1}{x}$ posito $\frac{1}{x} = p$ erit $y = A\operatorname{tang} p$ ac propterea $dy = \frac{dp}{1+pp}$. Cum nunc sit $dp = \frac{-dx}{xx}$, facta substitutione erit

$$dy = \frac{-dx}{1+xx},$$

quod est *differentiale cotangentis negative sumtum atque per quadratum cosecantis divisum.*

Porro si proponatur $y = A\sec x$, quia est $y = A\cos\frac{1}{x}$, fiet

$$dy = \frac{dx}{xx\sqrt{(1-\frac{1}{xx})}} = \frac{dx}{x\sqrt{(xx-1)}}$$

Atque si sit $y = A\operatorname{cosec} x$, erit $y = A\sin\frac{1}{x}$ ideoque

$$dy = \frac{-dx}{x\sqrt{(xx-1)}}$$

Saepe etiam sinus versus occurrit; ita si proponatur $y = A\operatorname{sv} x$, quia est $y = A\cos(1-x)$ huiusque arcus sinus est $= \sqrt{(2x-xx)}$, fiet

$$dy = \frac{dx}{\sqrt{(2x-xx)}}.$$

199. Quanquam ergo arcus, cuius sinus vel cosinus vel tangens vel cotangens vel secans vel cosecans vel denique sinus versus datur, est quantitas transcendens, tamen eius differentiale, si per dx dividatur, erit quantitas algebraica ac propterea quoque eius differentialia secunda, tertia, quarta etc., si per potestates ipsius dx convenientes dividantur. Ceterum, quo haec differentiatio melius percipiatur, adiunximus sequentia exempla.

1. Si sit $y = \mathrm{A}\sin 2x\sqrt{(1-xx)}$, ponatur $p = 2x\sqrt{(1-xx)}$, ut sit $y = A\sin p$, eritque $dy = \frac{dp}{\sqrt{(1-pp)}}$. At est

$$dp = 2dx\sqrt{(1-xx)} - \frac{2xxdx}{\sqrt{(1-xx)}} = \frac{2dx(1-2xx)}{\sqrt{(1-xx)}} \quad \text{et} \quad \sqrt{(1-pp)} = 1-2xx,$$

quibus valoribus substitutis erit

$$dy = \frac{2dx}{\sqrt{(1-xx)}}.$$

Quod etiam inde patet, quod $2x\sqrt{(1-xx)}$ sit sinus arcus dupli, dum x est sinus simpli ; erit ergo $y = 2A\sin x$ ideoque $dy = \frac{2dx}{(1-xx)}$.

II. Si sit $y = A\sin\frac{1-xx}{1+xx}$ ponatur $\frac{1-xx}{1+xx} = p$; erit

$$dp = \frac{-4xdx}{(1+xx)^2} \quad \text{et} \quad \sqrt{(1-pp)} = \frac{2x}{1+xx}.$$

Quare, cum sit $dy = \frac{dp}{\sqrt{(1-pp)}}$, erit

$$dy = \frac{-2dx}{1+xx}.$$

III. Si sit $y = A\sin\sqrt{\frac{1-x}{2}}$ ponatur $\sqrt{\frac{1-x}{2}} = p$; erit

$$\sqrt{(1-pp)} = \sqrt{\frac{1+x}{2}} \quad \text{et} \quad dp = \frac{-dx}{4\sqrt{\frac{1-x}{2}}},$$

unde fit

$$dy = \frac{dp}{\sqrt{(1-pp)}} = \frac{-dx}{2\sqrt{(1-xx)}}$$

IV. Si sit $y = A\tan\!g\frac{2x}{1-xx}$,facto $p = \frac{2x}{1-xx}$ erit

$$1+pp=\frac{(1+xx)^2}{(1-xx)^2} \quad \text{et} \quad dp=\frac{2dx(1+xx)}{(1-xx)^2}$$

Quare, cum sit $dy=\frac{dp}{1+pp}$ per regulam tangentium (§ 197), erit $dy=\frac{2dx}{1+xx}$.

V. Si sit $y=A\operatorname{tang}\frac{\sqrt{(1+xx)}-1}{x}$, posito $p=\frac{\sqrt{(1+xx)}-1}{x}$ fiet

$$pp=\frac{2+xx-2\sqrt{(1+xx)}}{xx}$$

et

$$1+pp=\frac{2+2xx-2\sqrt{(1+xx)}}{xx}=\frac{2\left(\sqrt{(1+xx)}-1\right)\sqrt{(1+xx)}}{xx}$$

atque

$$dp=\frac{-dx}{xx\sqrt{(1+xx)}}+\frac{dx}{xx}=\frac{dx\left(\sqrt{(1+xx)}-1\right)}{xx\sqrt{(1+xx)}}.$$

Quare, cum sit $dy=\frac{dp}{1+pp}$, fiet

$$dy=\frac{dx}{2(1+xx)};$$

quod etiam inde intelligitur, quod sit

$$A\operatorname{tang}\frac{\sqrt{(1+xx)}-1}{x}=\frac{1}{2}A\operatorname{tang}x.$$

VI. Si sit $y=e^{A\sin x}$, haec formula quoque per praecedentia differentiabitur; fiet enim

$$dy=e^{A\sin x}\frac{dx}{\sqrt{(1-xx)}}.$$

Hoc ergo modo omnes functiones ipsius *x*, in quas praeter logarithmos atque exponentiales quantitates etiam arcus circulares ingrediuntur, differentiari poterunt.

200. Quoniam differentialia arcuum per *dx* divisa sunt quantitates algebraicae, eorum differentialia secunda et sequentia per ea, quae de functionum algebraicarum differentiatione exposuimus, invenientur. Sit $y=A\sin x$; quia est $dy=\frac{dx}{\sqrt{(1-xx)}}$, erit

$\frac{dy}{dx} = \frac{1}{\sqrt{(1-xx)}}$, cuius differentiale dabit valorem pro $\frac{ddy}{dx^2}$, si quidem *dx* sumatur constans; unde differentialia ipsius *y* cuiusvis ordinis ita se habebunt.

Si sit $y = A\sin x$, erit

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1-xx)}}$$

et sumto *dx* constante

$$\frac{ddy}{dx^2} = \frac{x}{(1-xx)^{\frac{3}{2}}}$$

$$\frac{d^3y}{dx^3} = \frac{1+2xx}{(1-xx)^{\frac{5}{2}}}$$

$$\frac{d^4y}{dx^4} = \frac{9x+6x^3}{(1-xx)^{\frac{7}{2}}}$$

$$\frac{d^5y}{dx^5} = \frac{9+72x^2+24x^4}{(1-xx)^{\frac{9}{2}}}$$

$$\frac{d^6y}{dx^6} = \frac{225x+600x^3+120x^5}{(1-xx)^{\frac{11}{2}}}$$

etc.,

unde concludimus ut supra (§ 177) fore generaliter

$$\frac{d^{n+1}y}{dx^{n+1}} = \frac{1\cdot 2\cdot 3\cdots n}{(1-xx)^{n+\frac{1}{2}}}\left\{\begin{array}{l} x^n + \frac{1}{2}\cdot\frac{n(n-1)}{1\cdot 2}x^{n-2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^{n-2} \\ + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}x^{n-6} + \text{etc.}\end{array}\right\}$$

201. Supersunt quantitates, quae ex harum inversione nascuntur, scilicet sinus tangentesve arcuum datorum, quas quomodo differentiare oporteat, ostendamus. Sit igitur *x* arcus circuli et sin *x* denotet eius sinum, cuius differentiale investigemus. Ponamus $y = \sin x$ ac posito $x + dx$ loco *x*, quia *y* abit in $y + dy$, erit $y + dy = \sin(x + dx)$ et

$$dy = \sin(x + dx) - \sin x\,.$$

Est autem

$$\sin(x + dx) = \sin x \cdot \cos dx + \cos x \cdot \sin dx\,,$$

atque cum sit, uti in *Introductione* ostendimus,

$$\sin z = \frac{z}{1} - \frac{z^3}{1\cdot2\cdot3} + \frac{z^5}{1\cdot2\cdot3\cdot4\cdot5} - \text{etc.}$$
$$\cos z = 1 - \frac{z^2}{1\cdot2} + \frac{z^4}{1\cdot2\cdot3\cdot4} - \text{etc.},$$

erit reiectis terminis evanescentibus $\cos dx = 1$ et $\sin dx = dx$, unde fit

$$\sin(x+dx) = \sin x + dx\cos x.$$

Quare posito $y = \sin x$ erit

$$dy = dx\cos x.$$

Differentiale ergo sinus arcus cuiusvis aequatur differentiali arcus per cosinum multiplicato.

Si igitur fuerit *p* functio quaecunque ipsius *x*, erit simili modo $d.\sin p = dp\cos p$.

202. Similiter, si proponatur $\cos x$ seu cosinus arcus *x*, cuius differentiale investigari oporteat, ponatur $y = \cos x$ et posito $x + dx$ loco *x* fiet $y + dy = \cos(x+dx)$. Est vero

$$\cos(x+dx) = \cos x\cdot\cos dx - \sin x\cdot\sin dx,$$

et quia, ut modo vidimus, est $\cos dx = 1$ et $\sin dx = dx$, erit

$$y + dy = \cos x - dx\sin x$$

ideoque

$$dy = -dx\sin x.$$

Quare differentiale cosinus cuiusque arcus aequatur differentiali arcus negative sumto per sinum eiusdem arcus multiplicato.

Sic, si *p* fuerit functio quaecunque ipsius *x*, erit

$$d.\cos p = -dp\sin p.$$

Hae differentiationes quoque ex antecedentibus elici possunt hoc modo.
Si fuerit $y = \sin p$, erit $p = A\sin y$ et

$$dp = \frac{dy}{\sqrt{(1-yy)}};$$

at ob $y = \sin p$ erit $\cos p = \sqrt{(1-yy)}$, quo valore substituto erit $dp = \frac{dy}{\cos p}$

et

$$dy = dp\cos p$$

ut ante. Pari modo, si sit $y = \cos p$, erit $\sqrt{(1-yy)} = \sin p$ et $p = A\cos y$ ideoque

$$dp = \frac{-dy}{\sqrt{(1-yy)}} = \frac{-dy}{\sin p},$$

unde fit ut ante

$$dy = -dp\sin p.$$

203. Si fuerit $y = \operatorname{tang} x$, erit

$$dy = \operatorname{tang}(x+dx) - \operatorname{tang} x;$$

at est

$$\operatorname{tang}(x+dx) = \frac{\operatorname{tang} x + \operatorname{tang} dx}{1 - \operatorname{tang} x \cdot \operatorname{tang} dx};$$

a qua fractione si tangens *x* subtrahatur, remanebit

$$dy = \frac{\operatorname{tang} dx(\operatorname{tang} x + \operatorname{tang} x)}{1 - \operatorname{tang} x \cdot \operatorname{tang} dx}.$$

Verum arcus evanescentis *dx* tangens ipsi arcui est aequalis ideoque $\operatorname{tang} dx = dx$ et denominator $1 - dx\operatorname{tang} x$ abit in unitatem; quocirca fiet

$$dy = dx\left(1 + \operatorname{tang}^2 x\right).$$

Est vero

$$1 + \operatorname{tang}^2 x = \sec^2 x = \frac{1}{\cos^2 x}$$

denotante $\cos^2 x$ quadratum cosinus ipsius *x;* consequenter, si fuerit $y = \operatorname{tang} x$, erit

$$dy = dx\sec^2 x = \frac{dx}{\cos^2 x}.$$

Quod differentiale quoque per differentiationem sinuum et cosinuum inveniri potest; cum enim sit $\operatorname{tang} x = \frac{\sin x}{\cos x}$ erit [§164]

$$dy = \frac{dx\cos x \cdot \cos x + dx \cdot \sin x \cdot \sin x}{\cos^2 x} = \frac{dx}{\cos^2 x}$$

ob $\sin^2 x + \cos^2 x = 1$.

204. Aliter etiam hoc differentiale invenitur. Cum sit $y = \operatorname{tang} x$, erit $x = A\operatorname{tang} y$ et per praecepta superiora fiet

$$dx = \frac{dy}{1+yy}.$$

At cum sit $y = \operatorname{tang} x$, erit $\sqrt{(1+yy)} = \sec x = \frac{1}{\cos x}$ ideoque $dx = dy \cos^2 x$ et

$$dy = \frac{dx}{\cos^2 x}$$

ut ante.

Tangentis ergo cuiusvis arcus differentiale aequatur differentiali arcus diviso per quadratum cosinus eiusdem arcus.

Simili modo si proponatur $y = \cot x$, fiet $x = A\cot y$ et

$$dx = \frac{-dy}{1+yy}.$$

At vero erit $\sqrt{(1+yy)} = \operatorname{cosec} x = \frac{1}{\sin x}$, unde habebitur $dx = -dy \sin^2 x$ et

$$dy = \frac{-dx}{\sin^2 x}.$$

Cotangentis ergo cuiusvis arcus differentiale aequatur differentiali arcus negative sumto ac per quadratum sinus eiusdem arcus diviso.

Vel quia est $\cot x = \frac{\cos x}{\sin x}$, fiet hanc fractionem differentiando

$$dy = \frac{-dx \sin^2 x - dx \cos^2 x}{\sin^2 x} = \frac{-dx}{\sin^2 x},$$

uti modo invenimus.

205. Si proponatur secans arcus, ut sit $y = \sec x$, quia erit $y = \frac{1}{\cos x}$, erit

$$dy = \frac{dx \sin x}{\cos^2 x} = dx \operatorname{tang} x \sec x.$$

Simili modo si fuerit $y = \operatorname{cosec} x$, ob $y = \frac{1}{\sin x}$ erit

$$dy = \frac{-dx \cos x}{\sin^2 x} = -dx \cot x \operatorname{cosec} x,$$

pro quibus casibus peculiares regulas formare superfluum foret. Si sinus versus arcus proponatur $y = \operatorname{sv} x$, quia est $y = 1 - \cos x$, erit $dy = dx \sin x$. Omnes ergo casus, quibus linea quaepiam recta ad arcum relata proponitur, quia semper per sinum cosinumve exprimi

potest, sine difficultate differentiari poterunt. Neque vero tantum differentialia prima, sed etiam secunda et sequentia per regulas datas invenientur. Ponamus esse
$y = \sin x$ et $z = \cos x$ atque dx esse constans; erit, ut sequitur:

$y = \sin x$	$z = \cos x$
$dy = dx \cos x$	$dz = -dx \sin x$
$ddy = -dx^2 \sin x$	$ddz = -dx^2 \cos x$
$d^3 y = -dx^3 \cos x$	$d^3 z = dx^3 \sin x$
$d^4 y = dx^4 \sin x$	$d^4 z = dx^4 \cos x$
etc.	etc.

206. Simili modo inveniri poterunt differentialia omnium ordinum tangentis arcus *x*. Sit enim $y = \tang x = \frac{\sin x}{\cos x}$ et ponatur dx constans; erit

$$y = \frac{\sin x}{\cos x}$$
$$\frac{dy}{dx} = \frac{1}{\cos^2 x}$$
$$\frac{ddy}{dx^2} = \frac{2\sin x}{\cos^3 x}$$
$$\frac{d^3 y}{dx^3} = \frac{6}{\cos^4 x} - \frac{4}{\cos^2 x}$$
$$\frac{d^4 y}{dx^4} = \frac{24\sin x}{\cos^5 x} - \frac{8\sin x}{\cos^3 x}$$
$$\frac{d^5 y}{dx^5} = \frac{120}{\cos^6 x} - \frac{120}{\cos^4 x} + \frac{16}{\cos^2 x}$$
$$\frac{d^6 y}{dx^6} = \frac{720\sin x}{\cos^7 x} - \frac{480\sin x}{\cos^5 x} + \frac{32\sin x}{\cos^3 x}$$
$$\frac{d^7 y}{dx^7} = \frac{540}{\cos^8 x} - \frac{6720}{\cos^6 x} + \frac{2016}{\cos^4 x} - \frac{64}{\cos^2 x}$$
etc.

207. Functiones ergo quaecunque, in quas sinus vel cosinus arcuum ingrediuntur, per haec praecepta differentiari poterunt, uti ex sequentibus exemplis videre licet.

I. Si sit $y = 2\sin x \cdot \cos x = \sin 2x$, erit

$$dy = 2dx \cos^2 x - 2dx \sin^2 x = 2dx \cos 2x.$$

II Si sit $y=\sqrt{\frac{1-\cos x}{2}}$ vel $y=\sin\frac{1}{2}x$, erit

$$dy=\frac{dx\sin x}{2\sqrt{2(1-\cos x)}}.$$

Cum autem sit

$$2\sqrt{2(1-\cos x)}=2\sin\tfrac{1}{2}x \quad \text{et} \quad \sin x=2\sin\tfrac{1}{2}x\cos\tfrac{1}{2}x,$$

fiet

$$dy=\tfrac{1}{2}dx\cos\tfrac{1}{2}x$$

uti ex forma $y=\sin\frac{1}{2}x$ immediate sequitur.

III. Si sit $y=\cos l\frac{1}{x}$ erit posito $l\frac{1}{x}=p$, $y=\cos p$ et $dy=-dp\sin p$. At ob $p=l1-lx$ erit $dp=\frac{-dx}{x}$ ideoque $dy=\frac{dx}{x}\sin l\frac{1}{x}$.

IV. Si sit $y=e^{\sin x}$, erit

$$dy=e^{\sin x}dx\cos x.$$

V. Si sit $y=e^{\frac{-n}{\cos x}}$, erit

$$dy=-\frac{e^{\frac{-n}{\cos x}}ndx\sin x}{\cos^2 x}.$$

VI. Si sit $y=l\left(1-\sqrt{\left(1-e^{\frac{-n}{\sin x}}\right)}\right)$, ponatur $e^{\frac{-n}{\sin x}}=p$ atque ob

$$y=l\left(1-\sqrt{(1-p)}\right)$$

erit

$$dy=\frac{dp}{2\left(1-\sqrt{(1-p)}\right)\sqrt{(1-p)}}.$$

At est

$$dp=\frac{e^{\frac{-n}{\sin x}}ndx\cos x}{\sin^2 x}$$

Quo valore substituto prodibit

$$dy=\frac{ne^{\frac{-n}{\sin x}}dx\cos x}{2\sin^2 x\left(1-\sqrt{\left(1-e^{\frac{-n}{\sin x}}\right)}\right)\sqrt{\left(1-e^{\frac{-n}{\sin x}}\right)}}.$$