CHAPTER V

CONCERNING THE DIFFERENTIATION OF ALGEBRAIC FUNCTIONS INVOLVING ONE VARIABLE

152. Because the differential of the variable quantity *x* is equal to $= dx$, there will be on moving *x* into the nearest position $x^{\mathrm{I}} = x + dx$. Whereby if *y* were some function of *x*, if in place of *x* there may be put $x + dx$, that will change into y^{I} and the difference $\mathrm{y}^{\mathrm{I}} - y$ will give the differential of *y*. Therefore if we may put $y = x^n$, there will be made

$$y^{\mathrm{I}} = (x + dx)^n = x^n + nx^{n-1}dx + \tfrac{n(n-1)}{1\cdot 2}x^{n-2}dx^2 + \text{etc.}$$

and therefore there becomes

$$dy = y^{\mathrm{I}} - y = nx^{n-1}dx + \tfrac{n(n-1)}{1\cdot 2}x^{n-2}dx^2 + \text{etc.}$$

But in this expression the second with the remaining terms vanish before the first and on this account the differential of x^n will be $nx^{n-1}dx$ or

$$d.x^n = nx^{n-1}dx.$$

From which, if a shall be a number or constant quantity, there will be also $d.\,ax^n = nax^{n-1}.dx$. Therefore the differential of any power of *x* is found by multiplying that by the exponent, dividing by *x* and on multiplying the remainder by *dx*, which rule may be retained easily in the memory.

153. With the first differential of x^n known from that the second differential is found easily, provided, as here we may assume constantly, the differential *dx* may be put in place constant. For since in the differential $nx^{n-1}dx$ the factor *ndx* shall be constant, the differential of the other factor x^{n-1} can be assumed, because hence it will be $(n-1)x^{n-2}dx$. Therefore by multiplying this by *ndx* the second differential will be given

$$dd.x^n = n(n-1)x^{n-2}dx^2.$$

In a similar manner, if the differential of x^{n-2}, because it is $= (n-2)x^{n-3}dx$, may be multiplied by $n(n-1)dx^2$, the third differential will be produced

$$d^3.x^n = n(n-1)(n-2)x^{n-3}dx^3$$

And thus again there will be the fourth differential

$$d^4.x^n = n(n-1)(n-2)(n-3)x^{n-4}dx^4$$

and the fifth differential

$$d^5.x^n = n(n-1)(n-2)(n-3)(n-4)x^{n-5}dx^5;$$

from which likewise the form of the following differentials is deduced easily.

154. Therefore as often as *n* is a positive whole number, so also vanishing differentials finally are arrived at ; thus which clearly shall be $=0$, as before they vanish with all the powers of *dx*. Moreover simpler cases of these are to be noted.

$$d.x = dx,\ dd.x = 0,\ d.^3x = 0 \text{ etc.}$$
$$d.x^2 = 2xdx,\ dd.x^2 = 2dx^2,\ d.^3x^2 = 0,\ d.^4x^2 = 0 \text{ etc.}$$
$$d.x^3 = 3x^2dx,\ dd.x^3 = 6xdx^2,\ d.^3x^3 = 6dx^3,\ d.^4\ x^3 = 0 \text{ etc.}$$
$$d.x^4 = 4x^3dx,\ dd.x^4 = 12x^2dx^2,\ d.^3x^4 = 24xdx^3,\ d.^4x^4 = 24dx^4,\ d.^5x^4 = 0 \text{ etc.}$$
$$d.x^5 = 5x^4dx,\ dd.x^5 = 20x^3dx^2,\ d.^3x^5 = 60x^2dx^3,\ d.^4x^5 = 120xdx^4,$$
$$d.^5x^5 = 120dx^5,\ d.^6x^5 = 0 \text{ etc.}$$

Therefore it is clear, if *n* were a positive whole number, the differential of the power x^n of order *n* is a constant, evidently $=1\cdot2\cdot3\cdots n\,dx^n$, and thus the differentials of higher orders are all $=0$.

155. If *n* shall be a negative whole number, the differentials of the negative powers of *x* of this kind $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$ etc. can be taken, since there shall be $\frac{1}{x} = x^{-1}$, $\frac{1}{xx} = x^{-2}$ and generally $\frac{1}{x^m} = x^{-m}$. Therefore if in the preceding formula there may be put $n=-m$, the first differential of $\frac{1}{x^m}$ will be $=\frac{-mdx}{x^{m+1}}$, the second $=\frac{m(m+1)dx^2}{x^{m+2}}$, the third differential $=\frac{-m(m+1)(m+2)dx^3}{x^{m+3}}$ etc., from which the following simpler cases deserve to be observed especially.

$$d.\frac{1}{x} = \frac{-dx}{x^2},\ \ dd.\frac{1}{x} = \frac{2dx^2}{x^3},\ \ d.^3\frac{1}{x} = \frac{-6dx^3}{x^4}\ \text{etc.}$$
$$d.\frac{1}{x^2} = \frac{-2dx}{x^3},\ dd.\frac{1}{x^2} = \frac{6dx^2}{x^4},\ d.^3\frac{1}{x^2} = \frac{-24dx^3}{x^5}\ \text{etc.}$$
$$d.\frac{1}{x^3} = \frac{-3dx}{x^4},\ dd.\frac{1}{x^3} = \frac{12dx^2}{x^5},\ d.^3\frac{1}{x^3} = \frac{-60dx^3}{x^6}dx^3\ \text{etc.}$$
$$d.\frac{1}{x^4} = \frac{-4dx}{x^5},\ dd.\frac{1}{x^4} = \frac{20dx^2}{x^6},\ d.^3\frac{1}{x^4} = \frac{-120dx^3}{x^7}\ \text{etc.}$$
$$d.\frac{1}{x^5} = \frac{-5dx}{x^6},\ dd.\frac{1}{x^5} = \frac{30dx^2}{x^7},\ d.^3\frac{1}{x^5} = \frac{-210dx^3}{x^8}\ \text{etc.}$$
etc.

156. Then on putting fractions for n we will obtain the differentials of irrational formulas. For if there shall be $n = \frac{\mu}{\nu}$; the first differential of the formula $x^{\frac{\mu}{\nu}}$ or $\sqrt[\nu]{x^{\mu}}$ will be

$$= \frac{\mu}{\nu} x^{\frac{\mu-\nu}{\nu}} dx = \frac{\mu}{\nu} dx \sqrt[\nu]{x^{\mu-\nu}}\ ,$$

the second

$$= \frac{\mu(\mu-\nu)}{\nu^2} x^{\frac{\mu-2\nu}{\nu}} dx^2 = \frac{\mu(\mu-\nu)}{\nu\nu} dx^2 \sqrt[\nu]{x^{\mu-2\nu}}\ \text{etc.}$$

Hence there will be

$$d.\sqrt{x} = \frac{dx}{2\sqrt{x}},\ \ dd.\sqrt{x} = \frac{-dx^2}{4x\sqrt{x}},\ \ d.^3\sqrt{x} = \frac{1\cdot 3dx^3}{8x^2\sqrt{x}}\quad \text{etc.}$$
$$d.\sqrt[3]{x} = \frac{dx}{3\sqrt[3]{x^2}},\ dd.\sqrt[3]{x} = \frac{-2dx^2}{9x\sqrt[3]{x^2}},\ d.^3\sqrt[3]{x} = \frac{2\cdot 5dx^3}{27x^2\sqrt[3]{x^2}}\ \text{etc.}$$
$$d.\sqrt[4]{x} = \frac{dx}{4\sqrt[4]{x^3}},\ dd.\sqrt[4]{x} = \frac{-3dx^2}{16x\sqrt[4]{x^3}},\ d.^3\sqrt[4]{x} = \frac{3\cdot 7dx^3}{64x^2\sqrt[4]{x^3}}\ \text{etc.}$$

Which expressions if they may be examined for a short time, the differentials of this kind of expression may be acquired easily, even if without reduction to the form of powers.

157. If μ were not 1, but another number either a positive or negative whole number , the differentials may be defined equally easily. But since the second order differentials and of higher orders may be derived from the first, as these may be derived by the same law from the powers themselves, we will put in place only simpler examples of the first differentials.

$$d.x\sqrt{x} = \tfrac{3}{2}dx\sqrt{x},\ \ d.x^2\sqrt{x} = \tfrac{5}{2}xdx\sqrt{x},\ \ d.x^3\sqrt{x} = \tfrac{7}{2}x^2dx\sqrt{x} \quad \text{etc.}$$

$$d.\frac{1}{\sqrt{x}} = \frac{-dx}{2x\sqrt{x}},\ d.\frac{1}{x\sqrt{x}} = \frac{-3dx}{2xx\sqrt{x}},\ d.\frac{1}{xx\sqrt{x}} = \frac{-5dx}{2x^3\sqrt{x}} \ \text{etc.}$$

$$d.\sqrt[3]{x^2} = \frac{2}{3}\frac{dx}{\sqrt[3]{x}},\ d.x\sqrt[3]{x^2} = \frac{4}{3}dx\sqrt[3]{x},\ d.x\sqrt[3]{x^2} = \frac{5}{3}dx\sqrt[3]{x^2},$$

$$d.xx\sqrt[3]{x} = \tfrac{7}{3}xdx\sqrt[3]{x},\ \ d.xx\sqrt[3]{x^2} = \tfrac{8}{3}xdx\sqrt[3]{x^2} \quad \text{etc.}$$

$$d.\frac{1}{\sqrt[3]{x}} = \frac{-dx}{3x\sqrt[3]{x}},\ d.\frac{1}{\sqrt[3]{x^2}} = \frac{-2dx}{3x\sqrt[3]{x^2}},\ d.\frac{1}{x\sqrt[3]{x}} = \frac{-4dx}{3x^2\sqrt[3]{x}}$$

$$d.\frac{1}{x\sqrt[3]{x^2}} = \frac{-5dx}{3x^2\sqrt[3]{x^2}},\ d.\frac{1}{x^2\sqrt[3]{x}} = \frac{-7dx}{3x^3\sqrt[3]{x}} \quad \text{etc.}$$

158. Now the differentials of all algebraic rational integral can be found, because therefore the individual terms of each are powers of *x*, which we know how to differentiate. Since indeed quantities of this kind

$$p + q + r + s + \text{etc}.$$

on putting $x + dx$ in place of *x* change into

$$p + dp + q + dq + r + dr + s + ds + \text{etc.},$$

the differential of this will be

$$= dp + dq + dr + ds + \text{etc.}$$

Whereby if we are able to assign the differentials of the individual quantities *p, q, r, s,* likewise also the differential of the sum will be known . And since the differential of a multiple of *p* shall equal a multiple of *dp*, that is $d.ap = adp$, the differential of the quantity $ap + bq + cr$ the differential $= adp + bdq + cdr$. And since then the differentials of constant quantities shall be zero, the differential of this quantity also $ap + bq + cr + f$ $= adp + bdq + cdr$.

159. Therefore in rational integral functions since the individual terms shall be either constants or powers of *x*, the differentiation may be readily resolved by the following given precepts. Thus there will be

$$d\left(a+x\right)=dx,\ d\left(a+bx\right)=bdx;$$
$$d(a+xx)=2xdx,\ d\left(aa-xx\right)=-2xdx$$
$$d\left(a+bx+cxx\right)=bdx+2cxdx$$
$$d\left(a+bx+cxx+ex^3\right)=bdx+2cxdx+3ex^2dx$$
$$d\left(a+bx+cxx+ex^3+fx^4\right)=bdx+2cxdx+3ex^2dx+4fx^3dx.$$

And if the exponents were to be indefinite, there will be

$$d\left(1-x^n\right)=-nx^{n-1}dx,\ d\left(1+x^m\right)=mx^{m-1}dx$$
$$d\left(a+bx^m+cx^n\right)=mbx^{m-1}dx+ncx^{n-1}dx.$$

160. Therefore since rational functions may be distinguished according to the maximum power of x in degree, it is evident, if the differentials of functions of this kind may be taken continually, these finally become constant and afterwards become nothing, if indeed the differential *dx* may be assumed constant. Thus the first differential *bdx* of the function of the first degree $a+bx$ is constant, the second with the following zero. Let $a+bx+cxx=y$ be a function of the second degree ; there will be

$$dy=bdx+2cxdx,\ ddy=2cdx^2,\ d^3y=0.$$

In a similar manner, if a function of the third degree is put in place $a+bx+cxx+ex^3=y$, there will be

$$dy=bdx+2cxdx+3ex^2dx,\quad ddy=2cdx^2+6exdx^2\quad \text{and}\quad d^3y=6edx^3\ \text{and}\quad d^4y=0.$$

Whereby generally, if a function of this kind shall be of degree *n*, the differential of order n will be constant, and all the following zero.

161. Nor also will the differentiation be disturbed, if between the powers of *x*, which make up a function of this kind, such occurs, the exponents of which shall be either negative numbers or fractions. Thus

I. If there shall be $y=a+b\sqrt{x}-\frac{c}{x}$, then there becomes

$$dy=\frac{bdx}{2\sqrt{x}}+\frac{cdx}{xx}.$$

II. If there shall be $y=\frac{a}{\sqrt{x}}+b+c\sqrt{x}-ex,$ then there becomes

$$dy = \frac{-adx}{2x\sqrt{x}} + \frac{cdx}{2\sqrt{x}} - edx \quad \text{and} \quad ddy = \frac{3adx^2}{4xx\sqrt{x}} - \frac{cdx^2}{4x\sqrt{x}}.$$

III. If there shall be $y = a + \frac{b}{\sqrt[3]{xx}} - \frac{c}{x\sqrt[3]{x}} + \frac{f}{xx}$, then there becomes

$$dy = \frac{-2bdx}{3x\sqrt[3]{xx}} + \frac{4cdx}{3xx\sqrt[3]{x}} - \frac{2fdx}{x^3} \quad \text{and} \quad ddy = \frac{10bdx^2}{9x^2\sqrt[3]{xx}} - \frac{28cdx^2}{9x^3\sqrt[3]{x}} + \frac{6fdx^2}{x^4}.$$

Examples of this kind are most easily resolved according to the given precepts.

162. If a proposed quantity to be differentiated were a power of a function of this kind, the differential of which we are able to show, the preceding precepts suffice to define the first differential of this. Indeed let *p* be some function of *x*, the differential of which *dp* is in a power ; the first differential of this power $p^n = np^{n-1}dp$. Hence the following examples are solved.

1. If there shall be $y = (a+x)^n$, then there will be

$$dy = n(a+x)^{n-1}dx.$$

II. If $y = (aa-xx)^2$, then

$$dy = -4xdx(aa-xx).$$

III. If $y = \frac{1}{aa+xx}$ or $y = (aa+xx)^{-1}$, then

$$dy = \frac{-2xdx}{(aa+xx)^2}$$

IV. If $y = \sqrt{(a+bx+cxx)}$, then

$$dy = \frac{bdx+2cxdx}{2\sqrt{(a+bx+cxx)}}.$$

V. If $y = \sqrt[3]{(a^4-x^4)^2}$ or $y = (a^4-x^4)^{\frac{2}{3}}$, then

$$dy = -\frac{8}{3}x^3dx(a^4-x^4)^{-\frac{1}{3}} = -\frac{8x^3dx}{3\sqrt[3]{(a^4-x^4)}}.$$

VI. If $y = \frac{1}{\sqrt{(1-xx)}}$ or $y = (1-xx)^{-\frac{1}{2}}$, then

$$y = xdx(1-xx)^{-\frac{3}{2}} = \frac{xdx}{(1-xx)\sqrt{(1-xx)}}$$

VII. If $y = \sqrt[3]{\left(a+\sqrt{bx}+x\right)}$, then

$$dy = \frac{dx\sqrt{b}:2\sqrt{x}+dx}{3\sqrt[3]{\left(a+\sqrt{bx}+x\right)^2}} = \frac{dx\sqrt{b}+2dx\sqrt{x}}{6\sqrt{x}\sqrt[3]{\left(a+\sqrt{bx}+x\right)^2}}.$$

VIII. If $y = \frac{1}{x+\sqrt{(aa-xx)}}$, on account of $d.\sqrt{(aa-xx)} = \frac{-xdx}{\sqrt{(aa-xx)}}$ then

$$dy = \frac{-dx+xdx:\sqrt{(aa-xx)}}{\left(x+\sqrt{(aa-xx)}\right)^2}, = \frac{xdx-dx\sqrt{(aa-xx)}}{\left(x+\sqrt{(aa-xx)}\right)^2\sqrt{(aa-xx)}}$$

or

$$dy = \frac{dx\left(x-\sqrt{(aa-xx)}\right)^3}{(2xx-aa)^2\sqrt{(aa-xx)}}.$$

IX. If $y = \sqrt[4]{\left(1-\frac{1}{\sqrt{x}}+\sqrt[3]{(1-xx)^2}\right)^3}$, there may be put

$$\frac{1}{\sqrt{x}} = p \quad \text{and} \quad \sqrt[3]{(1-xx)^2} = q;$$

on account of $y = \sqrt[4]{(1-p+q)^3}$ there will be

$$dy = \frac{-3dp+3dq}{4\sqrt[4]{(1-p+q)}}$$

Now by the preceding there is

$$dp = \frac{-dx}{2x\sqrt{x}} \text{ and } dq = \frac{-4xdx}{3\sqrt[3]{(1-xx)}},$$

from which with the values substituted there becomes

$$dy = \frac{3dx:2x\sqrt{x} - 4xdx:\sqrt[3]{(1-xx)}}{4\sqrt[4]{\left(1-\frac{1}{\sqrt{x}}+\sqrt[3]{(1-xx)^2}\right)}}.$$

Moreover in a similar manner the differentials of all functions of this kind are elicited easily by the substitution of some arrangement of terms in place of the individual letters.

163. If the quantity to be differentiated were a product from two or more functions of *x*, upon which the differentials depend, the differential of this may be found most conveniently in the following manner. Let *p* and *q* be functions of *x*, the differentials of which *dp* and *dq* now have been found; because, on putting $x + dx$ in place of *x*, *p* becomes $p + dp$ and *q* changes into $q + dq$, the product *pq* will change into

$$(p + dp)(q + dq) = pq + pdq + qdp + dpdq,$$

from which the differential of the product *pq* will be $= pdq + qdp + dpdq$; where since *pdq* and *qdp* shall be of infinitely small orders, but *dpdq* of the second order, the final term vanishes and there will be

$$d.pq = pdq + qdp.$$

Therefore the differential of the product *pq* depends upon the two members, which may be obtained, if each factor may be multiplied by the differential of other the factor. Hence the differential of the product *pqr* is easily produced, depending on three factors ; for there may be put $qr = z$; there becomes $pqr = pz$ and $d.pqr = pdz + zdp$; now on account of $z = qr$ there will be $dz = qdr + rdq$, with which values substituted in place of *z* and *dz* there will be

$$d.pqr = pqdr + prdq + qrdp.$$

In a similar manner, if the quantity to be differentiated should have four factors, there will be

$$d.pqrs = pqrds + pqsdr + prsdq + qrsdp,$$

from which any differentiation of more factors will be understood easily.

I. Therefore if there were $y = (a + x)(b - x)$, there will be

$$dy = -dx(a + x) + dx(b - x) = -adx + bdx - 2xdx,$$

because the same differential is also found, if the proposed quantity should be expanded out; for there becomes $y = ab - ax + bx - xx$ and thus by the above precepts

$$dy = -adx + bdx - 2xdx .$$

II. If $y = \frac{1}{x}\sqrt{(aa - xx)}$, there may be put

$$\tfrac{1}{x} = p \text{ and } \sqrt{(aa - xx)} = q$$

because there is

$$dp = \tfrac{-dx}{xx} \text{ and } dq = \tfrac{-xdx}{\sqrt{(aa-xx)}},$$

there will be

$$dy = pdq + qdp = \tfrac{-dx}{\sqrt{(aa-xx)}} - \tfrac{dx}{xx}\sqrt{(aa - xx)};$$

which reduced to the same denominator will give

$$\tfrac{-xxdx-aadx+xxdx}{xx\sqrt{(aa-xx)}} = \tfrac{-aadx}{xx\sqrt{(aa-xx)}}.$$

Hence the differential sought will be

$$dy = \tfrac{-aadx}{xx\sqrt{(aa-xx)}}.$$

III. If $y = \frac{xx}{\sqrt{(a^4+x^4)}}$, there may be put

$$xx = p \text{ and } \tfrac{1}{\sqrt{(a^4+x^4)}} = q;$$

because we have found

$$dp = 2xdx \text{ and } dq = \tfrac{-2x^3dx}{(a^4+x^4)^{\frac{3}{2}}},$$

there will be

$$pdq + qdp = \tfrac{-2x^5dx}{(a^4+x^4)^{\frac{3}{2}}} + \tfrac{2xdx}{\sqrt{(a^4+x^4)}} = \tfrac{2a^4xdx}{(a^4+x^4)^{\frac{3}{2}}}.$$

Hence therefore the differential sought will be

$$dy = \tfrac{2a^4xdx}{(a^4+x^4)\sqrt{(a^4+x^4)}}.$$

IV. If $y = \frac{x}{x+\sqrt{(1+xx)}}$, on putting

$$x = p \text{ and } q = \frac{1}{x+\sqrt{(1+xx)}}$$

on account of

$$dp = dx \text{ and } dq = \frac{-dx - xdx:\sqrt{(1+xx)}}{\left(x+\sqrt{(1+xx)}\right)^2} = \frac{-dx\left(x+\sqrt{(1+xx)}\right)}{\left(x+\sqrt{(1+xx)}\right)^2\sqrt{(1+xx)}}$$

$$= \frac{-dx}{\left(x+\sqrt{(1+xx)}\right)\sqrt{(1+xx)}}$$

there will be

$$pdq + qdp = \frac{-xdx}{\left(x+\sqrt{(1+xx)}\right)\sqrt{(1+xx)}} + \frac{dx}{x+\sqrt{(1+xx)}}$$

$$= \frac{dx\left(\sqrt{(1+xx)}-x\right)}{\left(x+\sqrt{(1+xx)}\right)\sqrt{(1+xx)}}.$$

Therefore the differential sought will be made

$$dy = \frac{dx\left(\sqrt{(1+xx)}-x\right)}{\left(x+\sqrt{(1+xx)}\right)\sqrt{(1+xx)}};$$

of which fraction if the numerator and the denominator may be multiplied by $\sqrt{(1+xx)} - x$, there comes about

$$dy = \frac{dx\left(1+2xx-2x\sqrt{(1+xx)}\right)}{\sqrt{(1+xx)}} = \frac{dx+2xxdx}{\sqrt{(1+xx)}} - 2xdx.$$

The same differential can be found more conveniently in another manner ; for since there shall be

$$y = \frac{x}{x+\sqrt{(1+xx)}},$$

the numerator and denominator may be multiplied by $\sqrt{(1+xx)} - x$ and there comes about

$$y = x\sqrt{(1+xx)} - xx = \sqrt{\left(x^2+x^4\right)} - xx,$$

the differential of which by the first rule is

$$dy = \frac{xdx+2x^3dx}{\sqrt{(xx+x^4)}} - 2xdx = \frac{dx+2xxdx}{\sqrt{(1+xx)}} - 2xdx\,.$$

V. If $y=(a+x)(b-x)(x-c)$, then

$$dy=(a+x)(b-x)dx-(a+x)(x-c)dx+(b-x)(x-c)dx\,.$$

VI. If there should be $y = x(aa+xx)\sqrt{(aa-xx)}$, on account of the three factors therefore there may be found

$$dy = dx(aa+xx)\sqrt{(aa-xx)} + 2xxdx\sqrt{(aa-xx)} - \frac{xxdx(aa+xx)}{\sqrt{(aa-xx)}} = \frac{dx(a^4+aaxx-\ 4x^4)}{\sqrt{(aa-xx)}}.$$

164. Though fractions also can be understood from factors, yet more conveniently we will make use of the rule devoted to the differentiation of fractions. Therefore let $\frac{p}{q}$ be this proposed fraction, of which it is required to find the differential. Because on putting $x+dx$ in place of x that fraction will become

$$\frac{p+dp}{q+dq} = (p+dp)\left(\frac{1}{q} - \frac{dq}{qq}\right) = \frac{p}{q} - \frac{pdq}{qq} + \frac{dp}{q} - \frac{dpdq}{qq},$$

from which if the fraction $\frac{p}{q}$ may be taken away, there will remain the differential of this $d.\frac{p}{q} = \frac{dp}{q} - \frac{pdq}{qq}$ on account of the vanishing term $\frac{dpdq}{qq}$; hence therefore there will be

$$d.\frac{p}{q} = \frac{qdp-pdq}{qq},$$

from which this rule for the differentiation of each fraction arises:

From the differential of the numerator multiplied by the denominator the differential of the denominator multiplied by the numerator is subtracted, the remainder may be divided by the square of the denominator and the quotient will be the differential of the fraction sought.

The use of this rule will be illustrated by the following examples.

I. If there were $y=\frac{x}{aa+xx}$, by this rule there will be

$$dy = \frac{(aa+xx)dx-2xxdx}{(aa+xx)^2} = \frac{(aa-xx)dx}{(aa+xx)^2}.$$

II. If $y = \frac{\sqrt{(aa+xx)}}{aa-xx}$, there is found

$$dy = \frac{(aa-xx)xdx:\sqrt{(aa+xx)}+2xdx\sqrt{(aa+xx)}}{(aa-xx)^2}$$

and from the reduction made

$$dy = \frac{(3aa+xx)xdx}{(aa-xx)^2\sqrt{(aa+xx)}}.$$

Many times it will be expedient to use that rule, which follows from the first formula

$$d.\frac{p}{q} = \frac{dp}{q} - \frac{pdq}{qq},$$

from which the differential of the fraction is found equally with the differential of the numerator divided by the denominator, with the differential of the denominator multiplied by the numerator taken away, but divided by the square of the denominator. Thus

III. If there were $y = \frac{aa-xx}{a^4+aaxx+x^4}$, there will be

$$dy = \frac{-2xdx}{a^4+aaxx+x^4} - \frac{(aa-xx)(2aaxdx+4x^3dx)}{(a^4+aaxx+x^4)^2},$$

which recalled to the same denominator gives

$$dy = \frac{-2xdx(2a^4+2aaxx-x^4)}{(a^4+aaxx+x^4)^2},$$

165. Now these are adequate for the investigation of the differential of each rational function of *x* proposed; if indeed it were integral, the manner of differentiation above now has been set out. Therefore let the function be the proposed fraction, which may always be reduced to this form

$$y = \frac{A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\text{etc.}}{\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4+\zeta x^5+\text{etc.}}$$

The numerator may be put $= p$ and the denominator $= q$, so that there is made $y = \frac{p}{q}$, and there will be

$$dy = \frac{qdp - pdq}{qq}.$$

But since there will be

$$p = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

and

$$q = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \zeta x^5 + \text{etc.}$$

there will be

$$dp = Bdx + 2Cxdx + 3Dx^2dx + 4Ex^3dx + \text{etc.}.$$

and

$$dq = \beta dx + 2\gamma xdx + 3\delta x^2dx + 4\varepsilon x^3dx + \text{etc.}$$

from which there will be obtained by multiplication

$$\begin{aligned} qdp = \alpha Bdx + 2\alpha Cxdx &+ 3\alpha Dx^2dx + 4\alpha Ex^3dx + \text{etc.} \\ &+ \beta Bxdx + 2\beta Cx^2dx + 3\beta Dx^3dx + \text{etc.} \\ &+ \gamma Bx^2dx + 2\gamma Cx^3dx + \text{etc.} \\ &+ \delta Bx^3dx + \text{etc.} \\ pdq = \beta Adx + \beta Bxdx &+ \beta Cx^2dx + \beta Dx^3dx + \text{etc.} \\ &+ 2\gamma Axdx + 2\gamma Bx^2dx + 2\gamma Cx^3dx + \text{etc.} \\ &+ 3\delta .Ax^2dx + 3\delta Bx^3dx + \text{etc.} \\ &+ 4\varepsilon Ax^3dx + \text{etc.} \end{aligned}$$

And thus from these the differential sought may be obtained

$$dy = \frac{\left.\begin{matrix}+\alpha B\\-\beta A\end{matrix}\right\}dx \quad \left.\begin{matrix}+2\alpha C\\-2\gamma A\end{matrix}\right\}xdx \quad \left.\begin{matrix}+3\alpha D\\+\beta C\\-\gamma B\\-3\delta A\end{matrix}\right\}x^2dx \quad \left.\begin{matrix}+4\alpha E\\+2\beta D\\-2\delta B\\-4\varepsilon A\end{matrix}\right\}x^3dx \quad \left.\begin{matrix}+5\alpha F\\+3\beta E\\+\gamma D\\-\delta C\\-3\varepsilon B\\-5\zeta A\end{matrix}\right\}x^4dx \text{ etc.}}{\left(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \zeta x^5 + \text{etc.}\right)^2}$$

Which expression has been especially adapted for finding readily the differential of any rational function. For just as the numerator of the differential may be put together from the coefficients of the numerator and from the proposed function of the denominator is soon understood by inspection. Indeed the denominator of the differential is the square of the denominator of the proposed function.

166. If either the numerator or the denominator or both parts of the fraction proposed may depend on factors, with the multiplication actually set up a form indeed may arise, but of such a kind we have differentiated ; and yet a rule for these special cases will be formed more easily. Therefore let the proposed fraction be of this kind $y = \frac{pr}{q}$. The numerator may be put in place $pr = P$, so that there shall be

$$dP = pdr + rdp.$$

And on account of $y = \frac{P}{q}$, there will be

$$dy = \frac{qdP - Pdq}{qq} \;;$$

but on substituting the values in place of P et dP there will be had

I. If there shall be $y = \frac{pr}{q}$, the differential of this

$$dy = \frac{pqdr + qrdp - prdq}{qq}.$$

If there shall be $y = \frac{p}{qs}$, on putting the denominator $qs = Q$, there will be

$$dQ = qds + sdq,$$

and

$$dy = \frac{Qdp - pdQ}{qqss}.$$

Whereby :

II. If there should be $y = \frac{p}{qs}$, there will be

$$dy = \frac{qsdp - pqds - psdq}{qqss}.$$

If there should be $y = \frac{pr}{qs}$, there may be put $pr = P$ and $qs = Q$, so that there may be considered $y = \frac{P}{Q}$ and $dy = \frac{QdP - PdQ}{QQ}$.

But since there shall be

$$dP = pdr + rdp \;\text{ et }\; dQ = qds + sdq,$$

the following differential will be produced :

III. If there should be $y = \frac{pr}{qs}$, there will be $dy = \frac{pqsdr + qrsdp - pqrds - prsdq}{qqss}$

or

$$dy = \frac{rdp}{qs} + \frac{pdr}{qs} - \frac{prdq}{qqs} - \frac{prds}{qss}$$

In a similar manner, if the numerator and denominator of the proposed fraction should have several factors, the differentials will be investigated in the same manner nor will there be a need for setting this out. On account of which also I will omit some examples pertaining to this, since soon a general method including all these special methods will be advanced.

167. But cases are given of products as well as of fractions, in which the differential can be expressed more conveniently than by the general rules set out here. This comes about, if the factors were powers, which either constitute the function itself, or the numerator and denominator of the function. We may put the function to be differentiated to be $y = p^m q^n$, according to the differential of which requiring to be found there shall be
$p^m = P$ and $q^n = Q$, so that there may become

$$y = PQ \text{ and } dy = PdQ + QdP.$$

But since there shall be

$$dP = mp^{m-1}dp \text{ and } dQ = nq^{n-1}dq,$$

with these values in place there becomes

$$dy = np^m q^{n-1}dq + mp^{m-1}q^n dp = p^{m-1}q^{n-1}\left(npdq + mqdp\right),$$

from which the following rule emerges :

I. If there were $y = p^m q^n$, there will be

$$dy = p^{m-1}q^{n-1}\left(npdq + mqdp\right).$$

In a similar manner, if there were three factors, the differential may be come upon and found expressed in this manner:

II. If $y = p^m q^n r^k$, then

$$dy = p^{m-1}q^{n-1}r^{k-1}\left(mqrdp + nprdq + kpqdr\right).$$

168. But if a fraction were proposed, of which either the numerator or the denominator may have a factor, which is a power, also the particular rules can be treated.

In the first place let the proposed fraction be of this kind $y = \frac{p^m}{q}$; by the rule of fractions serving :

$$dy = \frac{mp^{m-1}qdp - p^m dq}{qq},$$

which differential may be expressed more conveniently thus

I. If $y = \frac{p^m}{q}$, then

$$dy = \frac{p^{m-1}(mqdp - pdq)}{qq}$$

Now let there be $y = \frac{p}{q^n}$ by the same rule above there will be made

$$dy = \frac{q^n dp - npq^{n-1}dq}{q^{2n}};$$

of which expression, if the numerator and denominator may be divided by q^{n-1}, there will be

$$dy = \frac{qdp - npdq}{q^{n+1}}.$$

On account of which :

II. If there were $y = \frac{p}{q^n}$, there will be

$$dy = \frac{qdp - npdq}{q^{n+1}}.$$

But if now there may be proposed $y = \frac{p^m}{q^n}$, there may be found

$$dy = \frac{mp^{m-1}q^n dp - np^m q^{n-1}dq}{q^{2n}},$$

which is reduced to

$$dy = \frac{mp^{m-1}q^n dp - np^m dq}{q^{n+1}},$$

On which account :

III. If $y = \frac{p^m}{q^n}$, then q

$$dy = \frac{p^{m-1}(mqdp - npdq)}{q^{n+1}}$$

Finally if a fraction of this kind were proposed $y=\frac{r}{p^m q^n}$, there will be had by the general rule of fractions

$$dy=\frac{p^m q^n dr-mp^{m-1}q^n rdp-np^m q^{n-1}rdq}{p^{2m}q^{2n}}.$$

Of which expression since the numerator and denominator shall be divisible by $p^{m-1}q^{n-1}$:

IV. If there were $y=\frac{r}{p^m q^n}$, then

$$dy=\frac{pqdr-mqrdp-nprdq}{p^{m+1}q^{n+1}}$$

If more factors occur, special rules of this kind, which it would be superfluous to express in words, it is an easy matter for whatever case they may be elicited.

169. The rules required for differentiation, which we have explained up to this stage, extend so widely, so that no algebraic function of x may be devised, which is unable to be differentiated with the aid of these. For if a function of x were rational, either it will be integral or a fraction ; in the first case in §159 we have given the manner functions of this kind are to be differentiated, now we have resolved the matter for the latter case in §165. Likewise indeed we have shown also short cuts of the differentiation if factors are involved. Then truly we have shown how to differentiate irrational quantities of any kind, which, in whatever way the affect the proposed function, whether they shall be involved in addition, subtraction, multiplication, or division, they can always be recalled to the cases now treated. But it is required to be understood these are about explicit functions; for concerning implicit functions, the nature of which is given by an equation, below, after we have treated the differentiation of functions of two or more variables, will be the place to be treated.

170. If we carefully assess the particular rules treated here and collect them together, we will be able to reduce all to one especially general rule ; as below finally moreover it will be allowed to be strengthened by a rigid demonstration [§ 214]; yet meanwhile and in this place it will not be very difficult to consider attending to the truth of this. Any algebraic function has been composed from the parts, which have included amongst themselves either addition, subtraction, multiplication or division ; and these parts will be either rational or irrational. Therefore we may call these quantities some function of these parts put together:

Then a proposed function thus may be differentiated by any part separately, as if that part alone is to be variable, truly all the remaining parts constant. With which done these individual differentials, which are elicited from the individual parts in the manner described, may be gathered together into one sum and thus the differential of the propose function will be obtained.

With the aid of this rule all functions generally are able to be differentiated lest indeed with the exception of the transcending functions, as we will show below.

171. Towards illustrating this rule we may put in place a function of y to be made from two parts connected either by addition or subtraction, thus so that there shall be

$$y = p \pm q .$$

In the first place there may be put only the part p variable, the other q constant; there will be the differential $= dp$; then there may be put the other part $\pm q$ only variable, truly the other part p constant and there will be the differential $= \pm dq$. And from these differentials the differential sought thus may be composed, so that there shall be

$$dy = dp \pm dq ,$$

altogether the same as we have now found above. Hence indeed likewise it may be clear, if the function should depend on several parts either in turn by addition or subtraction, evidently

$$y = p \pm q \pm r \pm s ,$$

with the help of this rule there is going to be found

$$dy = dp \pm dq \pm dr \pm ds ,$$

plainly as the above rule will instruct.

172. If the parts shall be multiplied in turn between themselves, thus to that there shall be $y = pq$, it is evident with the part p alone to be variable, the differential to be $= qdp$; but if the other part q only may be put in place variable, the differential will be $= pdq$. Therefore these two differentials may be added in turn and the differential sought will be produced

$$dy = qdp + pdq ,$$

just as it is agreed upon from that now reported. If there were several parts connected by multiplication, such as

$$y = pqrs ,$$

if successively each one alone may be put in place variable, there will arise these differentials

$$qrsdp,\ prsdq,\ pqsdr \text{ and } pqrds,$$

the sum of which will give the differential sought, clearly

$$dy = qrsdp + prsdq + pqsdr + pqrds,$$

absolutely as we have found before. Therefore the differential is composed from just as many parts, whether the parts constituting the function shall be in turn added or subtracted, or in turn multiplied into each other.

173. If the parts forming the function shall be connected by division, evidently

$$y = \tfrac{p}{q},$$

there may be put following the first rule the first part p only variable and on account of q constant the differential $= \frac{dp}{q}$; then only the part q may be put variable; on account of $y = p\ q^{-1}$ the differential will be $= -\frac{pdq}{qq}$, which two differentials gathered together will give the differential of the function proposed

$$dy = \tfrac{dp}{q} - \tfrac{pdq}{qq} = \tfrac{qdp - pdq}{qq},$$

thus as now we have found above. In a similar manner, if the proposed function shall be

$$y = \tfrac{pq}{rs},$$

on putting successively the individual parts alone p, q, r and s variable, there will be produced the following differentials

$$\tfrac{qdp}{rs},\ \tfrac{pdq}{rs},\ \tfrac{-pqdr}{rrs} \text{ and } \tfrac{-pqds}{rss},$$

from which there becomes

$$dy = \tfrac{qrsdp + prsdq - pqsdr - pqrds}{rrss}.$$

174. Therefore provided the individual parts, from which the function may be composed, have been prepared thus, so that the differentials of these are able to be shown, likewise too the differential of the whole function can be found. But if therefore the parts were rational functions, then the differentials of these are found not only with the aid of the precepts now given before, but these also will be able to be elicited from this general rule itself; but if the parts were irrational, because the irrational number is reduced to a power, the exponents of which are fractional numbers, these will be differentiated by the differentiation of the powers, from which there is $d.x^n = nx^{n-1}dx$. And from the same source there may be drawn also the differential of this kind of irrational formulas, which in the other expressions above involve surds. From which it is apparent, if with the rule given here generally, truly below there is required to be demonstrated a rule for the differentiation of powers joined together, then generally the differentials of all algebraic functions can be shown.

175. From all these now it follows clearly, if y were some algebraic function of x, the differential of this dy shall be having a form of this kind $dy = pdx$, in which the value of p may always be able to be assigned by the precepts here set out. But p will be an algebraic function of x too, since in the determination of this no other operations are present unless the customary ones, from which algebraic functions are usually constituted. On this account if y were an algebraic function of x, $\frac{dy}{dx}$ also will be an algebraic function of x. And if z were also an algebraic function of x, thus so that there shall be $dz = qdx$, on account of q being an algebraic function of x, $\frac{dz}{dy}$ also will be an algebraic function of x, evidently which is $= \frac{q}{p}$. Whereby if formulas of this kind $\frac{dz}{dy}$ may be present in other algebraic expression, these will not hinder if that expression shall not be algebraic, provided y and z shall be algebraic functions.

176. Moreover we will be able to extend this reasoning to second differentials order and to of higher orders. For if, with y remaining some algebraic function of x there were $dy = pdx$ and $dp = qdx$, there will be on taking the differential dx constant $ddy = qdx^2$, as we have seen above. Therefore since, on account of the reasons alleged before, q shall be also an algebraic function of x, also not only will $\frac{ddy}{dx^2}$ be a finite quantity, but also an algebraic function of x, provided y were a function of this kind. In a similar manner there may be seen to be the algebraic functions of x : $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ etc. , provided y were such a function; and if z also shall be an algebraic function of x, all finite expressions which will be composed from the differentials of any order of y, z and from dx, of which kind there shall be $\frac{ddy}{ddz}$, $\frac{d^3y}{dzddy}$, $\frac{dxd^4y}{dy^3ddz}$ etc., likewise will be algebraic functions of x.

177. Therefore since now a method shall have treated of finding the first differential of each algebraic function of x, likewise by the same method we will be able also to investigate the second order differentials and of higher orders. For if y were a some algebraic function of x, from the differentiation of this $dy = pdx$ the value of p may become known. Which if it may e differentiated again and there is found $dp = qdx$, there will be $ddy = qdx^2$ on putting dx constant, and thus the second differential may be defined.

Again on differentiating q, so that there shall be $dq = rdx$, the third differential $d^3y = rdx^3$ will be had and thus the differentials of higher orders will be investigated, since all the quantities p, q, r etc. are algebraic functions of x, for which differentiations required the given precepts are sufficient. Therefore this differentiation may be effected continually ; for with [the differentiation of] dx omitted the differentiation of y will produce this value of $\frac{dy}{dx} = p$, which differentiated anew and divided by dx, because there becomes, while everywhere the differential dx may be omitted, the value of $q = \frac{ddy}{dx^2}$ will be given. In a similar manner again there is found $r = \frac{d^3y}{dx^3}$ etc.

I. Let there be $y = \frac{aa}{aa+xx}$, the differentials of which both the first as well as of the following orders are required .

Therefore in the first place on differentiating and likewise on dividing by dx there will be

$$\frac{dy}{dx} = \frac{-2aax}{(aa+xx)^2}$$

and hence again

$$\frac{ddy}{dx^2} = \frac{-2a^4+6aaxx}{(aa+xx)^3}$$

$$\frac{d^3y}{dx^3} = \frac{24a^4x-24aax^3}{(aa+xx)^4}$$

$$\frac{d^4y}{dx^4} = \frac{24a^6-240a^4xx+120aax^4}{(aa+xx)^5}$$

$$\frac{d^5y}{dx^5} = \frac{-720a^6x+2400a^4x^3-720aax^5}{(aa+xx)^6}$$

etc.

II. Let there be $y = \frac{1}{\sqrt{(1-xx)}}$ and the differentials of the first and following orders will be

$$\frac{dy}{dx} = \frac{x}{(1-xx)^{\frac{3}{2}}}$$

$$\frac{ddy}{dx^2} = \frac{1+2xx}{(1-xx)^{\frac{5}{2}}}$$

$$\frac{d^3y}{dx^3} = \frac{9x+6x^3}{(1-xx)^{\frac{7}{2}}}$$

$$\frac{d^4y}{dx^4} = \frac{9+72x^2+24x^4}{(1-xx)^{\frac{9}{2}}}$$

$$\frac{d^5y}{dx^5} = \frac{225x+600x^3+120x^5}{(1-xx)^{\frac{11}{2}}}$$

$$\frac{d^6y}{dx^6} = \frac{225+4050x^4+720x^6}{(1-xx)^{\frac{13}{2}}}$$

etc.

These differentials may be continued further easily ; yet meanwhile the law, by which the terms of these are progressing, may not be quickly apparent. Indeed the coefficient of the greatest powers of x always is the product of the natural numbers from one as far as the order of the differential, which is sought. Meanwhile if we may continue these forms further and we consider carefully, we may come upon the general form to be, if $y = \frac{1}{\sqrt{(1-xx)}}$,

$$\frac{d^ny}{dx^n} = \frac{1\cdot 2\cdot 3\cdots n}{(1-xx)^{n+\frac{1}{2}}}\left\{\begin{array}{l} x^n + \frac{1}{2}\cdot\frac{n(n-1)}{1\cdot 2}x^{n-2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{n(n-1)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^{n-4} \\ \quad + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{n(n-1)\cdots(n-5)}{1\cdot 2\cdots 6}x^{n-6} \\ \quad + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{n(n-1)\cdots(n-7)}{1\cdot 2\cdots 8}x^{n-8} \\ \qquad + \text{etc.} \end{array}\right\}$$

Therefore examples of this kind not only serve the purpose of having the task of differentiation to be acquired, but also the laws, which may be observed in the differentials of all orders, which by themselves are worthy of note and which are able to draw out other discoveries.

CAPUT V

DE DIFFERENTIATIONE FUNCTIONUM ALGEBRAICARUM UNICAM VARIABILEM INVOLVENTIUM

152. Quia quantitatis variabilis *x* differentiale est $= dx$, erit *x* in proximum promovendo $x^{\mathrm{I}} = x + dx$. Quare si fuerit *y* quaecunque functio ipsius *x*, si in ea loco *x* ponatur $x + dx$, ea abibit in y^{I} atque differentia $y^{\mathrm{I}} - y$ dabit differentiale ipsius *y*. Si igitur ponamus $y = x^n$, fiet

$$y^{\mathrm{I}} = (x + dx)^n = x^n + nx^{n-1}dx + \frac{n(n-1)}{1\cdot 2}x^{n-2}dx^2 + \text{etc.}$$

eritque ergo

$$dy = y^{\mathrm{I}} - y = nx^{n-1}dx + \frac{n(n-1)}{1\cdot 2}x^{n-2}dx^2 + \text{etc.}$$

At in hac expressione terminus secundus cum reliquis sequentibus prae primo evanescit eritque idcirco $nx^{n-1}dx$ differentiale ipsius x^n seu

$$d.x^n = nx^{n-1}dx.$$

Unde, si *a* sit numerus seu quantitas constans, erit quoque $d.\,ax^n = nax^{n-1}$.
Cuiuscunque ergo ipsius *x* potestatis differentiale invenitur multiplicando eam per exponentem, dividendo per *x* et reliquum per *dx* multiplicando, quae regula facile memoria retinetur.

153. Cognito differentiali primo ipsius x^n ex eo facile differentiale secundum reperitur, dummodo, ut hic constanter assumemus, differentiale *dx* constans statuatur. Cum enim in differentiali $nx^{n-1}dx$ factor *ndx* sit constans, alterius factoris x^{n-1} differentiale sumi debet, quod proinde erit $(n-1)x^{n-2}dx$. Hoc ergo per *ndx* multiplicatum dabit differentiale secundum

$$dd.x^n = n(n-1)x^{n-2}dx^2.$$

Simili modo, si differentiale ipsius x^{n-2}, quod est $= (n-2)x^{n-3}dx$, multiplicetur per $n(n-1)dx^2$, prodibit differentiale tertium

$$d^3.x^n = n(n-1)(n-2)x^{n-3}dx^3$$

Porro itaque erit differentiale quartum

$$d^4.x^n = n(n-1)(n-2)(n-3)x^{n-4}dx^4$$

et differentiale quintum

$$d^5.x^n = n(n-1)(n-2)(n-3)(n-4)x^{n-5}dx^5;$$

unde simul forma sequentium differentialium facillime colligitur.

154. Quoties ergo n est numerus integer affirmativus, toties ad differentialia tandem pervenitur evanescentia; quae scilicet ita sunt $=0$, ut prae omnibus ipsius dx potestatibus evanescant. Horum autem notandi sunt casus simpliciores.

$$d.x = dx,\ dd.x = 0,\ d.^3 x = 0 \text{ etc.}$$

$$d.x^2 = 2xdx,\ dd.x^2 = 2dx^2,\ d.^3 x^2 = 0,\ d.^4 x^2 = 0 \text{ etc.}$$

$$d.x^3 = 3x^2dx,\ dd.x^3 = 6xdx^2,\ d.^3 x^3 = 6dx^3,\ d.^4\ x^3 = 0 \text{ etc.}$$

$$d.x^4 = 4x^3dx,\ dd.x^4 = 12x^2dx^2,\ d.^3 x^4 = 24xdx^3,\ d.^4 x^4 = 24dx^4,\ d.^5 x^4 = 0 \text{ etc.}$$

$$d.x^5 = 5x^4dx,\ dd.x^5 = 20x^3dx^2,\ d.^3 x^5 = 60x^2dx^3,\ d.^4 x^5 = 120xdx^4,$$

$$d.^5 x^5 = 120dx^5,\ d.^6 x^5 = 0 \text{ etc.}$$

Patet ergo, si n fuerit numerus integer affirmativus, potestatis x^n differentiale ordinis n esse constans, nempe $= 1\cdot 2\cdot 3\cdots ndx^n$, adeoque differentialia superiorum ordinum omnium esse $=0$.

155. Si n sit numerus integer negativus, huiusmodi ipsius x potestatum negativarum $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$ etc. differentialia sumi poterunt, cum sit $\frac{1}{x} = x^{-1}$, $\frac{1}{xx} = x^{-2}$ et generaliter $\frac{1}{x^m} = x^{-m}$. Si ergo in formula antecedente ponatur $n=-m$, erit ipsius $\frac{1}{x^m}$ differentiale primum $= \frac{-mdx}{x^{m+1}}$, differentiale secundum $= \frac{m(m+1)dx^2}{x^{m+2}}$, differentiale tertium $= \frac{-m(m+1)(m+2)dx^3}{x^{m+3}}$ etc., unde sequentes casus simpliciores imprimis notari merentur.

$$d.\frac{1}{x}=\frac{-dx}{x^2},\ dd.\frac{1}{x}=\frac{2dx^2}{x^3},\ d.^3\frac{1}{x}=\frac{-6dx^3}{x^4}\text{ etc.}$$
$$d.\frac{1}{x^2}=\frac{-2dx}{x^3},\ dd.\frac{1}{x^2}=\frac{6dx^2}{x^4},\ d.^3\frac{1}{x^2}=\frac{-24dx^3}{x^5}\text{ etc.}$$
$$d.\frac{1}{x^3}=\frac{-3dx}{x^4},\ dd.\frac{1}{x^3}=\frac{12dx^2}{x^5},\ d.^3\frac{1}{x^3}=\frac{-60dx^3}{x^6}dx^3\text{ etc.}$$
$$d.\frac{1}{x^4}=\frac{-4dx}{x^5},\ dd.\frac{1}{x^4}=\frac{20dx^2}{x^6},\ d.^3\frac{1}{x^4}=\frac{-120dx^3}{x^7}\text{ etc.}$$
$$d.\frac{1}{x^5}=\frac{-5dx}{x^6},\ dd.\frac{1}{x^5}=\frac{30dx^2}{x^7},\ d.^3\frac{1}{x^5}=\frac{-210dx^3}{x^8}\text{ etc.}$$
etc.

156. Ponendis deinde pro *n* numeris fractis differentialia formularum irrationalium obtinebimus. Sit enim $n=\frac{\mu}{\nu}$; erit formulae $x^{\frac{\mu}{\nu}}$ seu $\sqrt[\nu]{x^{\mu}}$ differentiale primum

$$=\frac{\mu}{\nu}x^{\frac{\mu-\nu}{\nu}}dx=\frac{\mu}{\nu}dx\sqrt[\nu]{x^{\mu-\nu}},$$

secundum

$$=\frac{\mu(\mu-\nu)}{\nu^2}x^{\frac{\mu-2\nu}{\nu}}dx^2=\frac{\mu(\mu-\nu)}{\nu\nu}dx^2\sqrt[\nu]{x^{\mu-2\nu}}\text{ etc.}$$

Hinc erit

$$d.\sqrt{x}=\frac{dx}{2\sqrt{x}},\ dd.\sqrt{x}=\frac{-dx^2}{4x\sqrt{x}},\ d.^3\sqrt{x}=\frac{1\cdot 3dx^3}{8x^2\sqrt{x}}\quad\text{etc.}$$
$$d.\sqrt[3]{x}=\frac{dx}{3\sqrt[3]{x^2}},\ dd.\sqrt[3]{x}=\frac{-2dx^2}{9x\sqrt[3]{x^2}},\ d.^3\sqrt[3]{x}=\frac{2\cdot 5dx^3}{27x^2\sqrt[3]{x^2}}\text{ etc.}$$
$$d.\sqrt[4]{x}=\frac{dx}{4\sqrt[4]{x^3}},\ dd.\sqrt[4]{x}=\frac{-3dx^2}{16x\sqrt[4]{x^3}},\ d.^3\sqrt[4]{x}=\frac{3\cdot 7dx^3}{64x^2\sqrt[4]{x^3}}\quad\text{etc.}$$

Quae expressiones si paulisper inspiciantur, facile habitus acquiretur huiusmodi differentialia, etiam sine praevia reductione ad formam potestatis, inveniendi.

157. Si μ non fuerit 1, sed numerus alius sive affirmativus sive negativus integer, differentialia aeque facile definientur. Cum autem differentialia secunda et altiorum ordinum eadem lege ex primis, qua haec ex ipsis potestatibus, deriventur, exempla simpliciora primorum tantum differentialium apponamus.

$$d.x\sqrt{x} = \tfrac{3}{2}dx\sqrt{x},\ \ d.x^2\sqrt{x} = \tfrac{5}{2}xdx\sqrt{x},\ \ d.x^3\sqrt{x} = \tfrac{7}{2}x^2dx\sqrt{x} \quad \text{etc.}$$

$$d.\frac{1}{\sqrt{x}} = \frac{-dx}{2x\sqrt{x}},\ d.\frac{1}{x\sqrt{x}} = \frac{-3dx}{2xx\sqrt{x}},\ d.\frac{1}{xx\sqrt{x}} = \frac{-5dx}{2x^3\sqrt{x}} \ \text{etc.}$$

$$d.\sqrt[3]{x^2} = \frac{2}{3}\frac{dx}{\sqrt[3]{x}},\ d.x\sqrt[3]{x^2} = \tfrac{4}{3}dx\sqrt[3]{x},\ d.x\sqrt[3]{x^2} = \tfrac{5}{3}dx\sqrt[3]{x^2},$$

$$d.xx\sqrt[3]{x} = \tfrac{7}{3}xdx\sqrt[3]{x},\ \ d.xx\sqrt[3]{x^2} = \tfrac{8}{3}xdx\sqrt[3]{x^2} \quad \text{etc.}$$

$$d.\frac{1}{\sqrt[3]{x}} = \frac{-dx}{3x\sqrt[3]{x}},\ d.\frac{1}{\sqrt[3]{x^2}} = \frac{-2dx}{3x\sqrt[3]{x^2}},\ d.\frac{1}{x\sqrt[3]{x}} = \frac{-4dx}{3x^2\sqrt[3]{x}}$$

$$d.\frac{1}{x\sqrt[3]{x^2}} = \frac{-5dx}{3x^2\sqrt[3]{x^2}},\ d.\frac{1}{x^2\sqrt[3]{x}} = \frac{-7dx}{3x^3\sqrt[3]{x}} \ \text{etc.}$$

158. Ex his iam functionum omnium algebraicarum rationalium integrarum differentialia poterunt inveniri, propterea quod earum singuli termini sunt potestates ipsius *x*, quas differentiare novimus. Cum enim quantitas huiusmodi

$$p + q + r + s + \text{etc}.$$

posito $x + dx$ loco *x* abeat in

$$p + dp + q + dq + r + dr + s + ds + \text{etc}.,$$

erit eius differentiale

$$= dp + dq + dr + ds + \text{etc}.$$

Quare si singularum quantitatum *p, q, r, s* differentialia assignare queamus, simul quoque aggregati earum differentiale innotescet. Atque cum multipli ipsius *p* differentiale sit aeque multiplum ipsius *dp*, hoc est $d.ap = adp$, erit quantitatis $ap + bq + cr$ differentiale $= adp + bdq + cdr$. Cum denique quantitatum constantium differentialia sint nulla, erit quoque quantitatis huius $ap + bq + cr + f$ differentiale $= adp + bdq + cdr$.

159. In functionibus ergo rationalibus integris cum singuli termini sint vel constantes vel potestates ipsius *x*, differentiatio secundum praecepta data facile absolvetur. Sic erit

$$d\left(a + x\right) = dx,\ d\left(a + bx\right) = bdx;$$

$$d(a + xx) = 2xdx,\ d\left(aa - xx\right) = -2xdx$$

$$d\left(a + bx + cxx\right) = bdx + 2cxdx$$

$$d\left(a + bx + cxx + ex^3\right) = bdx + 2cxdx + 3ex^2dx$$

$$d\left(a + bx + cxx + ex^3 + fx^4\right) = bdx + 2cxdx + 3ex^2dx + 4fx^3dx.$$

Atque si exponentes fuerint definiti, erit

$$d\left(1-x^n\right)=-nx^{n-1}dx,\ d\left(1+x^m\right)=mx^{m-1}dx$$
$$d\left(a+bx^m+cx^n\right)=mbx^{m-1}dx+ncx^{n-1}dx.$$

160. Cum igitur functiones rationales integrae secundum maximam ipsius x dignitatem in gradus distinguantur, manifestum est, si huiusmodi functionum continuo differentialia capiantur, ea tandem fieri constantia posteaque in nihilum abire, si quidem differentiale dx assumatur constans. Sic functionis primi gradus $a+bx$ differentiale primum bdx est constans, secundum cum sequentibus nullum. Sit functio secundi gradus $a+bx+cxx=y$; erit

$$dy=bdx+2cxdx,\ ddy=2cdx^2,\ d^3y=0.$$

Simili modo, si ponatur functio tertii gradus $a+bx+cxx+ex^3=y$, erit

$$dy=bdx+2cxdx+3ex^2dx,\quad ddy=2cdx^2+6exdx^2 \quad \text{et} \quad d^3y=6edx^3 \quad \text{atque} \quad d^4y=0.$$

Quare generaliter, si huiusmodi functio sit gradus n, eius differentiale ordinis n erit constans et sequentia omnia nulla.

161. Neque etiam differentiatio turbabitur, si inter potestates ipsius x, quae huiusmodi functionem componunt, occurrant tales, quarum exponentes sint numeri negativi seu fracti. Ita

I. Si sit $y=a+b\sqrt{x}-\frac{c}{x}$, erit

$$dy=\frac{bdx}{2\sqrt{x}}+\frac{cdx}{xx}.$$

II. Si sit $y=\frac{a}{\sqrt{x}}+b+c\sqrt{x}-ex$, erit

$$dy=\frac{-adx}{2x\sqrt{x}}+\frac{cdx}{2\sqrt{x}}-edx \quad \text{et} \quad ddy=\frac{3adx^2}{4xx\sqrt{x}}-\frac{cdx^2}{4x\sqrt{x}}.$$

III. Si sit $y=a+\frac{b}{\sqrt[3]{xx}}-\frac{c}{x\sqrt[3]{x}}+\frac{f}{xx}$, erit

$$dy=\frac{-2bdx}{3x\sqrt[3]{xx}}+\frac{4cdx}{3xx\sqrt[3]{x}}-\frac{2fdx}{x^3} \quad \text{et} \quad ddy=\frac{10bdx^2}{9x^2\sqrt[3]{xx}}-\frac{28cdx^2}{9x^3\sqrt[3]{x}}+\frac{6fdx^2}{x^4}.$$

Cuiusmodi exempla secundum praecepta data facillime absolvuntur.

162. Si quantitas differentianda proposita fuerit potestas eiusmodi functionis, cuius differentiale exhibere valemus, praecedentia praecepta sufficiunt ad eius differentiale primum definiendum. Sit enim *p* functio quaecunque ipsius *x*, cuius differentiale *dp* in potestate est; erit ipsius potestatis p^n differentiale primum $= np^{n-1}dp$. Hinc sequentia exempla solvuntur.

1. Si sit $y = (a + x)^n$, erit

$$dy = n(a + x)^{n-1} dx .$$

II. Si sit $y = (aa - xx)^2$, erit

$$dy = -4xdx(aa - xx).$$

III. Si sit $y = \frac{1}{aa+xx}$ seu $y = (aa + xx)^{-1}$, erit

$$dy = \frac{-2xdx}{(aa+xx)^2}$$

IV. Si sit $y = \sqrt{(a + bx + cxx)}$, erit

$$dy = \frac{bdx+2cxdx}{2\sqrt{(a+bx+cxx)}} .$$

V. Si sit $y = \sqrt[3]{(a^4 - x^4)^2}$ seu $y = (a^4 - x^4)^{\frac{2}{3}}$, erit

$$dy = -\frac{8}{3}x^3dx(a^4 - x^4)^{-\frac{1}{3}} = -\frac{8x^3dx}{3\sqrt[3]{(a^4-x^4)}} .$$

VI. Si sit $y = \frac{1}{\sqrt{(1-xx)}}$ seu $y = (1 - xx)^{-\frac{1}{2}}$, erit

$$y = xdx(1 - xx)^{-\frac{3}{2}} = \frac{xdx}{(1-xx)\sqrt{(1-xx)}}$$

VII. Si sit $y = \sqrt[3]{(a + \sqrt{bx} + x)}$, erit

$$dy = \frac{dx\sqrt{b} : 2\sqrt{x} + dx}{3\sqrt[3]{\left(a+\sqrt{bx}+x\right)^2}} = \frac{dx\sqrt{b} + 2dx\sqrt{x}}{6\sqrt{x}\sqrt[3]{\left(a+\sqrt{bx}+x\right)^2}} .$$

VIII. Si sit $y = \frac{1}{x+\sqrt{(aa-xx)}}$, ob $d.\sqrt{(aa-xx)} = \frac{-xdx}{\sqrt{(aa-xx)}}$ erit

$$dy = \frac{-dx + xdx : \sqrt{(aa-xx)}}{\left(x+\sqrt{(aa-xx)}\right)^2}, = \frac{xdx - dx\sqrt{(aa-xx)}}{\left(x+\sqrt{(aa-xx)}\right)^2\sqrt{(aa-xx)}}$$

seu

$$dy = \frac{dx\left(x-\sqrt{(aa-xx)}\right)^3}{(2xx-aa)^2\sqrt{(aa-xx)}} .$$

IX. Si sit $y = \sqrt[4]{\left(1-\frac{1}{\sqrt{x}}+\sqrt[3]{(1-xx)^2}\right)^3}$, ponatur

$$\frac{1}{\sqrt{x}} = p \quad \text{et} \quad \sqrt[3]{(1-xx)^2} = q;$$

ob $y = \sqrt[4]{(1-p+q)^3}$ erit

$$dy = \frac{-3dp+3dq}{4\sqrt[4]{(1-p+q)}}$$

Iam per antecedentia est

$$dp = \frac{-dx}{2x\sqrt{x}} \text{ et } dq = \frac{-4xdx}{3\sqrt[3]{(1-xx)}} ,$$

quibus valoribus substitutis fiet

$$dy = \frac{3dx : 2x\sqrt{x} - 4xdx : \sqrt[3]{(1-xx)}}{4\sqrt[4]{\left(1-\frac{1}{\sqrt{x}}+\sqrt[3]{(1-xx)^2}\right)}} .$$

Simili autem modo singulares litteras loco terminorum aliquantum compositorum substituendo omnium huiusmodi functionum differentialia facile eruuntur.

163. Si quantitas differentianda fuerit productum ex duabus pluribusve functionibus ipsius x, quarum differentialia constant, eius differentiale sequente modo commodissime invenietur. Sint p et q functiones ipsius x, quarum differentialia dp et dq iam sunt cognita; quia, posito $x+dx$ loco x, p abit in $p+dp$ et q in $q+dq$, productum pq transmutabitur in

$$(p+dp)(q+dq)=pq+pdq+qdp+dpdq,$$

unde producti pq differentiale erit $=pdq+qdp+dpdq$; ubi cum pdq et qdp sint infinite parva primi ordinis, at $dpdq$ secundi ordinis, ultimus terminus evanescet eritque igitur

$$d.pq=pdq+qdp.$$

Differentiale ergo producti pq constat ex duobus membris, quae obtinentur, si uterque factor per differentiale alterius factoris multiplicetur. Hinc facile deducitur differentiatio producti pqr ex tribus factoribus constantis; ponatur enim $qr=z$; fiet $pqr=pz$ et $d.pqr=pdz+zdp$; verum ob $z=qr$ erit $dz=qdr+rdq$, quibus valoribus loco z et dz substitutis erit

$$d.pqr=pqdr+prdq+qrdp.$$

Simili modo, si quantitas differentianda quatuor habeat factores, erit

$$d.pqrs=pqrds+pqsdr+prsdq+qrsdp,$$

unde quilibet differentiationem plurium factorum facile perspiciet.

I. Si ergo fuerit $y=(a+x)(b-x)$, erit

$$dy=-dx(a+x)+dx(b-x)=-adx+bdx-2xdx,$$

quod idem differentiale quoque invenitur, si quantitas proposita evolvatur; fit enim $y=ab-ax+bx-xx$ ideoque per superiora praecepta

$$dy=-adx+bdx-2xdx.$$

II. Si fuerit $y=\frac{1}{x}\sqrt{(aa-xx)}$, ponatur

$$\tfrac{1}{x}=p \quad \text{et} \quad \sqrt{(aa-xx)}=q$$

quia est

$$dp = \frac{-dx}{xx} \text{ et } dq = \frac{-xdx}{\sqrt{(aa-xx)}},$$

erit

$$dy = pdq + qdp = \frac{-dx}{\sqrt{(aa-xx)}} - \frac{dx}{xx}\sqrt{(aa-xx)} \;;$$

quae ad eundem denominatorem reductae dabunt

$$\frac{-xxdx-aadx+xxdx}{xx\sqrt{(aa-xx)}} = \frac{-aadx}{xx\sqrt{(aa-xx)}}.$$

Hinc erit differentiale quaesitum

$$dy = \frac{-aadx}{xx\sqrt{(aa-xx)}}.$$

III. Si fueret $y = \frac{xx}{\sqrt{(a^4+x^4)}}$ ponatur

$$xx = p \text{ et } \frac{1}{\sqrt{(a^4+x^4)}} = q\,;$$

quia invenimus

$$dp = 2xdx \text{ et } dq = \frac{-2x^3dx}{(a^4+x^4)^{\frac{3}{2}}}$$

erit

$$pdq + qdp = \frac{-2x^5dx}{(a^4+x^4)^{\frac{3}{2}}} + \frac{2xdx}{\sqrt{(a^4+x^4)}} = \frac{2a^4xdx}{(a^4+x^4)^{\frac{3}{2}}}.$$

Hinc ergo erit differentiale quaesitum

$$dy = \frac{2a^4xdx}{(a^4+x^4)\sqrt{(a^4+x^4)}}.$$

IV. Si fuerit $y = \frac{x}{x+\sqrt{(1+xx)}}$, ponendo

$$x = p \text{ et } q = \frac{1}{x+\sqrt{(1+xx)}}$$

ob

$$dp = dx \text{ et } dq = \frac{-dx-xdx:\sqrt{(1+xx)}}{\left(x+\sqrt{(1+xx)}\right)^2} = \frac{-dx\left(x+\sqrt{(1+xx)}\right)}{\left(x+\sqrt{(1+xx)}\right)^2\sqrt{(1+xx)}}$$

$$= \frac{-dx}{\left(x+\sqrt{(1+xx)}\right)\sqrt{(1+xx)}}$$

erit

$$pdq+qdp=\frac{-xdx}{\left(x+\sqrt{(1+xx)}\right)\sqrt{(1+xx)}}+\frac{dx}{x+\sqrt{(1+xx)}}$$
$$=\frac{dx\left(\sqrt{(1+xx)}-x\right)}{\left(x+\sqrt{(1+xx)}\right)\sqrt{(1+xx)}}.$$

Fiet ergo differentiale quaesitum

$$dy=\frac{dx\left(\sqrt{(1+xx)}-x\right)}{\left(x+\sqrt{(1+xx)}\right)\sqrt{(1+xx)}};$$

cuius fractionis si numerator ac denominator multiplicetur per $\sqrt{(1+xx)}-x$, fiet

$$dy=\frac{dx\left(1+2xx-2x\sqrt{(1+xx)}\right)}{\sqrt{(1+xx)}}=\frac{dx+2xxdx}{\sqrt{(1+xx)}}-2xdx.$$

Idem differentiale alio modo commodius inveniri potest; cum enim sit

$$y=\frac{x}{x+\sqrt{(1+xx)}},$$

multiplicetur numerator ac denominator per $\sqrt{(1+xx)}-x$ fietque

$$y=x\sqrt{(1+xx)}-xx=\sqrt{\left(x^2+x^4\right)}-xx,$$

cuius differentiale per priorem regulam est

$$dy=\frac{xdx+2x^3dx}{\sqrt{\left(xx+x^4\right)}}-2xdx=\frac{dx+2xxdx}{\sqrt{(1+xx)}}-2xdx.$$

V. Si fuerit $y=(a+x)(b-x)(x-c)$, erit
$$dy=(a+x)(b-x)dx-(a+x)(x-c)dx+(b-x)(x-c)dx.$$

VI. Si fuerit $y=x(aa+xx)\sqrt{(aa-xx)}$, ob tres factores ergo reperietur

$$dy=dx(aa+xx)\sqrt{(aa-xx)}+2xxdx\sqrt{(aa-xx)}-\frac{xxdx(aa+xx)}{\sqrt{(aa-xx)}}=\frac{dx\left(a^4+aaxx-\ 4x^4\right)}{\sqrt{(aa-xx)}}.$$

164. Quanquam etiam fractiones in factoribus comprehendi possunt, tamen commodius utemur regula fractionibus differentiandis inserviente. Sit ergo proposita haec fractio $\frac{p}{q}$, cuius differentiale inveniri oporteat. Quoniam posito $x+dx$ loco x fractio illa abit in

$$\frac{p+dp}{q+dq}=(p+dp)\left(\frac{1}{q}-\frac{dq}{qq}\right)=\frac{p}{q}-\frac{pdq}{qq}+\frac{dp}{q}-\frac{dpdq}{qq},$$

unde si fractio ipsa $\frac{p}{q}$ subtrahatur, remanet eius ditferentiale $d.\frac{p}{q}=\frac{dp}{q}-\frac{pdq}{qq}$ ob evanescentem terminum $\frac{dpdq}{qq}$; hinc ergo erit

$$d.\frac{p}{q}=\frac{qdp-pdq}{qq},$$

unde haec regula pro differentiatione cuiusque fractionis enascitur:

A differentiali numeratoris per denominatorem multiplicato subtrahatur differentiale denominatoris per numeratorem multiplicatum, residuum dividatur per quadratum denuminatoris quotusque erit differentiale fractionis quaesitum.

Cuius regulae usus per sequentia exempla illustrabitur.

I. Si fuerit $y=\frac{x}{aa+xx}$, erit per hanc regulam

$$dy=\frac{(aa+xx)dx-2xxdx}{(aa+xx)^2}=\frac{(aa-xx)dx}{(aa+xx)^2}.$$

II. Si fuerit $y=\frac{\sqrt{(aa+xx)}}{aa-xx}$, reperitur

$$dy=\frac{(aa-xx)xdx:\sqrt{(aa+xx)}+2xdx\sqrt{(aa+xx)}}{(aa-xx)^2}$$

et facta reductione

$$dy=\frac{(3aa+xx)xdx}{(aa-xx)^2\sqrt{(aa+xx)}}.$$

Saepenumero expedit ea regula uti, quae sequitur ex formula priori

$$d.\frac{p}{q}=\frac{dp}{q}-\frac{pdq}{qq},$$

qua differentiale fractionis aequale reperitur differentiali numeratoris per denominatorem diviso, demto differentiali denominatoris per numeratorem multiplicato, at per quadratum denominatoris diviso. Ita

III. Si fuerit $y = \frac{aa-xx}{a^4+aaxx+x^4}$,

erit

$$dy = \frac{-2xdx}{a^4+aaxx+x^4} - \frac{(aa-xx)(2aaxdx+4x^3dx)}{(a^4+aaxx+x^4)^2},$$

quae ad eundem denominatorem revocata praebet

$$dy = \frac{-2xdx(2a^4+2aaxx-x^4)}{(a^4+aaxx+x^4)^2},$$

165. Haec iam sufficiunt ad cuiusque functionis rationalis ipsius x propositae differentiale investigandum; si enim fuerit integra, modus differentiandi iam supra est expositus. Sit igitur functio proposita fracta, quae semper ad huiusmodi formam reducetur

$$y = \frac{A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\text{etc.}}{\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4+\zeta x^5+\text{etc.}}$$

Ponatur numerator $= p$ et denominator $= q$, ut fiat $y = \frac{p}{q}$, eritque

$$dy = \frac{qdp-pdq}{qq}.$$

At cum sit

$$p = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{etc.}$$

et

$$q = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \zeta x^5 + \text{etc.}$$

erit

$$dp = Bdx + 2Cxdx + 3Dx^2dx + 4Ex^3dx + \text{etc.}.$$

et

$$dq = \beta dx + 2\gamma xdx + 3\delta x^2dx + 4\varepsilon x^3dx + \text{etc.}$$

unde per multiplicationem obtinebitur

$$\begin{aligned} qdp = {} & \alpha Bdx + 2\alpha Cxdx + 3\alpha Dx^2dx + 4\alpha Ex^3dx + \text{etc.} \\ & + \beta Bxdx + 2\beta Cx^2dx + 3\beta Dx^3dx + \text{etc.} \\ & + \gamma Bx^2dx + 2\gamma Cx^3dx + \text{etc.} \\ & + \delta Bx^3dx + \text{etc.} \end{aligned}$$

$$\begin{aligned} pdq = {} & \beta Adx + \beta Bxdx + \beta Cx^2dx + \beta Dx^3dx + \text{etc.} \\ & + 2\gamma Axdx + 2\gamma Bx^2dx + 2\gamma Cx^3dx + \text{etc.} \\ & + 3\delta.Ax^2dx + 3\delta Bx^3dx + \text{etc.} \\ & + 4\varepsilon Ax^3dx + \text{etc.} \end{aligned}$$

Ex his itaque obtinebitur differentiale quaesitum

$$dy = \frac{\left.\begin{matrix} +\alpha B \\ -\beta A \end{matrix}\right\} dx \;\; \left.\begin{matrix} +2\alpha C \\ -2\gamma A \end{matrix}\right\} xdx \;\; \left.\begin{matrix} +3\alpha D \\ +\beta C \\ -\gamma B \\ -3\delta A \end{matrix}\right\} x^2dx \;\; \left.\begin{matrix} +4\alpha E \\ +2\beta D \\ -2\delta B \\ -4\varepsilon A \end{matrix}\right\} x^3dx \;\; \left.\begin{matrix} +5\alpha F \\ +3\beta E \\ +\gamma D \\ -\delta C \\ -3\varepsilon B \\ -5\zeta A \end{matrix}\right\} x^4dx \text{ etc.}}{\left(\alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \zeta x^5 + \text{etc.}\right)^2}$$

Quae expressio ad cuiusvis functionis rationalis differentiale expedite inveniendum maxime est accommodata. Quemadmodum enim numerator differentialis ex coefficientibus numeratoris ac denominatoris functionis propositae combinatur, ex inspectione mox intelligitur. Denominator vero differentialis est quadratum denominatoris functionis propositae.

166. Si fractionis propositae vel numerator vel denominator vel uterque ex factoribus constet, multiplicatione actu instituta orietur quidem forma, qualem modo differentiavimus; attamen facilius pro his casibus regula peculiaris formabitur. Sit igitur proposita huiusmodi fractio $y = \frac{pr}{q}$. Ponatur numerator $pr = P$, ut sit

$$dP = pdr + rdp.$$

Atque ob $y = \frac{P}{q}$ erit

$$dy = \frac{qdP - Pdq}{qq};$$

substitutis autem loco P et dP valoribus habebitur

I. Si fuerit $y = \frac{pr}{q}$, eius differentiale

$$dy = \frac{pqdr + qrdp - prdq}{qq}.$$

Si sit $y=\frac{p}{qs}$ posito denominatore $qs=Q$ erit

$$dQ=qds+sdq$$

et

$$dy=\frac{Qdp-pdQ}{qqss}.$$

Quare:

II. Si fuerit $y=\frac{p}{qs}$, erit

$$dy=\frac{qsdp-pqds-psdq}{qqss}.$$

Si fuerit $y=\frac{pr}{qs}$, ponatur $pr=P$ et $qs=Q$, ut habeatur $y=\frac{P}{Q}$ et $dy=\frac{QdP-PdQ}{QQ}$.
Cum autem sit

$$dP=pdr+rdp \text{ et } dQ=qds+sdq,$$

prodibit sequens differentiatio:

III. Si fuerit $y=\frac{pr}{qs}$, erit $dy=\frac{pqsdr+qrsdp-pqrds-prsdq}{qqss}$

seu

$$dy=\frac{rdp}{qs}+\frac{pdr}{qs}-\frac{prdq}{qqs}-\frac{prds}{qss}$$

Simili modo, si numerator ac denominator fractionis propositae plures habeant factores, differentialia eadem ratione investigabuntur neque ad hoc ampliori manuductione erit opus. Quamobrem quoque exempla huc pertinentia praetermitto, cum mox modus generalis has omnes differentiandi methodos particulares complectens afferatur.

167. Dantur autem casus tam productorum quam fractionum, quibus differentiale commodius exprimi potest quam per regulas generaliores hic expositas. Evenit hoc, si factores, qui vel functionem ipsam vel functionis numeratorem aut denominatorem constituunt, fuerint potestates. Ponamus functionem differentiandam esse $y=p^m q^n$, ad cuius differentiale inveniendum sit $p^m=P$ et $q^n=Q$, ut fiat

$$y=PQ \text{ et } dy=PdQ+QdP.$$

Cum autem sit

$$dP=mp^{m-1}dp \text{ et } dQ=nq^{n-1}dq,$$

fiet his valoribus substitutis

$$dy = np^m q^{n-1} dq + mp^{m-1} q^n dp = p^{m-1} q^{n-1} (npdq + mqdp),$$

unde sequens oritur regula:

I. Si fuerit $y = p^m q^n$, erit

$$dy = p^{m-1} q^{n-1} (npdq + mqdp).$$

Simili modo, si tres fuerint factores, differentiale invenietur ac reperietur hoc modo expressum:

II. Si fuerit $y = p^m q^n r^k$, erit

$$dy = p^{m-1} q^{n-1} r^{k-1} (mqrdp + nprdq + kpqdr).$$

168. Sin autem fuerit proposita fractio, cuius vel numerator vel denominator habeat factorem, qui est potestas, regulae quoque particulares tradi poterunt.

Sit primum proposita huiusmodi fractio $y = \frac{p^m}{q}$; erit per regulam fractionibus inservientem :

$$dy = \frac{mp^{m-1} qdp - p^m dq}{qq},$$

quod differentiale commodius sic exprimetur

I. Si fuerit $y = \frac{p^m}{q}$ erit

$$dy = \frac{p^{m-1}(mqdp - pdq)}{qq}$$

Sit iam $y = \frac{p}{q^n}$ fiet per eandem superiorem regulam

$$dy = \frac{q^n dp - npq^{n-1} dq}{q^{2n}};$$

cuius expressionis si numerator ac denominator per q^{n-1} dividatur, erit

$$dy = \frac{qdp - npdq}{q^{n+1}}.$$

Quamobrem:

II. Si fuerit $y = \frac{p}{q^n}$, erit

$$dy = \frac{qdp - npdq}{q^{n+1}}.$$

Quodsi vero proponatur $y=\frac{p^m}{q^n}$, invenietur

$$dy=\frac{mp^{m-1}q^n dp-np^m q^{n-1}dq}{q^{2n}},$$

quae reducitur ad

$$dy=\frac{mp^{m-1}q^n dp-np^m dq}{q^{n+1}},$$

Quocirca:

III. Si fuerit $y=\frac{p^m}{q^n}$, erit q

$$dy=\frac{p^{m-1}(mqdp-npdq)}{q^{n+1}}$$

Denique si proposita fuerit huiusmodi fractio $y=\frac{r}{p^m q^n}$, habebitur per regulam fractionum generalem

$$dy=\frac{p^m q^n dr-mp^{m-1}q^n rdp-np^m q^{n-1}rdq}{p^{2m}q^{2n}}.$$

Cuius expressionis cum numerator et denominator sit divisibilis per $p^{m-1}q^{n-1}$:

IV. Si fuerit $y=\frac{r}{p^m q^n}$, erit

$$dy=\frac{pqdr-mqrdp-nprdq}{p^{m+1}q^{n+1}}$$

Si plures occurrant factores, huiusmodi regulae speciales, quas verbis exprimere superfluum foret, facili negotio pro quovis casu erui poterunt.

169. Regulae differentiandi, quas hactenus exposuimus, tam late patent, ut nulla excogitari possit functio ipsius x algebraica, quae non earum ope differentiari queat. Si enim functio ipsius x fuerit rationalis, vel erit integra vel fracta; priori casu § 159 modum dedimus eiusmodi functiones differentiandi, posteriori vero casu in § 165 negotium absolvimus. Simul vero etiam compendia, si factores involvantur, differentiationis exhibuimus. Deinde vero etiam quantitates irrationales cuiusvis generis differentiare docuimus, quae, quomodocunque functionem propositam afficiant, sive ei per additionem sive per subtractionem sive multiplicationem sive divisionem sint implicatae, perpetuo ad casus iam tractatos revocari poterunt. Intelligenda autem haec sunt de functionibus explicitis; nam de implicitis, quarum natura per aequationem datur, infra, postquam functiones duarum pluriumve variabilium differentiare docuerimus, tractandi locus erit.

170. Si regulas hic traditas singulas perpendamus atque inter se conferamus, eas omnes ad unam maxime universalem reducere poterimus; quam autem infra demum rigida demonstratione munire licebit [§ 214]; interim tamen et hoc loco non adeo difficile erit eius veritatem attendenti intueri. Functio quaecunque algebraica composita est ex partibus, quae vel additione vel subtractione vel multiplicatione vel divisione inter se erunt complicatae; haeque partes erunt vel rationales vel irrationales. Vocemus ergo istas quantitates functionem quamvis constituentes eius partes:

Tum pro qualibet parte functio proposita seorsim ita differentietur, quasi ea pars sola esset variabilis, reliquae vero partes omnes constantes. Quo facto singula ista differentialia, quae ex singulis partibus modo descripto eliciuntur, in unam summam colligantur sicque obtinebitur differentiale functionis propositae.

Huiusque regulae ope omnes omnino functiones differentiari poterunt nequidem transcendentibus exceptis, uti infra ostendetur.

171. Ad regulam hanc illustrandam ponamus functionem y duabus constare partibus sive per additionem sive subtractionem connexis, ita ut sit

$$y = p \pm q \,.$$

Ponatur primo sola pars p variabilis, altera q constans; erit differentiale $= dp$; deinde ponatur altera pars $\pm q$ sola variabilis, altera vero p constans eritque differentiale $= \pm dq$. Atque ex his differentialibus differentiale quaesitum ita componetur, ut sit

$$dy = dp \pm dq \,,$$

omnino uti idem iam supra invenimus. Hinc vero simul liquet, si functio pluribus constet partibus sive invicem additis sive subtractis, nempe

$$y = p \pm q \pm r \pm s \,,$$

ope huius regulae inventum iri

$$dy = dp \pm dq \pm dr \pm ds \,,$$

plane uti et superior regula docebat.

172. Si partes sint in se invicem multiplicatae, ita ut sit $y = pq$, manifestum est posita sola parte p variabili fore differentiale $= qdp$; at si altera pars q sola variabilis statuatur, erit differentiale $= pdq$. Addantur ergo haec duo differentialia invicem atque prodibit differentiale quaesitum

$$dy = qdp + pdq \,,$$

quemadmodum ex iam allatis constat. Si plures fuerint partes per multiplicationem connexae, scilicet

$$y = pqrs,$$

si successive unaquaeque sola variabilis statuatur, orientur ista differentialia

$$qrsdp,\ prsdq,\ pqsdr \text{ et } pqrds,$$

quorum summa dabit differentiale quaesitum, nempe

$$dy = qrsdp + prsdq + pqsdr + pqrds,$$

prorsus uti iam ante invenimus. Differentiale ergo ex totidem partibus componitur, sive partes functionem constituentes sint invicem additae subtractaeve sive in se invicem multiplicatae.

173. Si partes functionem formantes per divisionem sint connexae, nempe

$$y = \frac{p}{q},$$

ponatur secundum regulam primum sola pars p variabilis eritque ob q constans differentiale $= \frac{dp}{q}$; deinde ponatur sola pars q variabilis; ob $y = p\, q^{-1}$ erit differentiale $= -\frac{pdq}{qq}$, quae duo differentialia collecta dabunt differentiale functionis propositae

$$dy = \frac{dp}{q} - \frac{pdq}{qq} = \frac{qdp - pdq}{qq},$$

sicut iam supra invenimus. Simili modo, si functio proposita sit

$$y = \frac{pq}{rs},$$

ponendo successive singulas partes solas p, q, r et s variabiles, prodibunt sequentia differentialia

$$\frac{qdp}{rs},\ \frac{pdq}{rs},\ \frac{-pqdr}{rrs} \text{ et } \frac{-pqds}{rss},$$

unde fit

$$dy = \frac{qrsdp + prsdq - pqsdr - pqrds}{rrss}.$$

174. Dummodo ergo singulae partes, ex quibus functio componitur, ita fuerint comparatae, ut earum differentialia exhiberi queant, simul quoque totius functionis differentiale inveniri poterit. Quodsi igitur partes fuerint functiones rationales, tum earum differentialia non solum ope praeceptorum ante iam datorum inveniuntur, sed ea quoque ex hac ipsa regula generali erui poterunt; sin autem partes fuerint irrationales, quia irrationalitas ad potestates, quarum exponentes sunt numeri fracti, reducitur, eae per differentiationem potestatum, qua est $d.x^n = nx^{n-1}dx$, differentiabuntur. Atque ex eodem fonte haurietur quoque differentiatio eiusmodi formularum irrationalium, quae alias insuper expressiones surdas involvunt. Unde patet, si cum regula generali hic data, infra vero demonstranda coniungatur regula differentiandi potestates, tum omnium omnino functionum algebraicarum differentialia exhiberi posse.

175. Ex his omnibus iam dilucide sequitur, si y fuerit functio quaecunque ipsius x, differentiale eius dy huiusmodi habiturum esse formam $dy = pdx$, in qua valor ipsius p per praecepta hic exposita semper assignari queat. Erit autem p functio ipsius x quoque algebraica, cum in eius determinationem nullae aliae operationes ingrediantur nisi consuetae, quibus functiones algebraicae constitui solent. Hanc ob rem si y fuerit functio algebraica ipsius x, erit quoque $\frac{dy}{dx}$ functio algebraica ipsius x. Atque si z fuerit etiam functio algebraica ipsius x, ita ut sit $dz = qdx$, ob q functionem algebraicam ipsius x erit quoque $\frac{dz}{dy}$ functio algebraica ipsius x, quippe quae est $= \frac{p}{q}$. Quare si huiusmodi formulae $\frac{dz}{dy}$ in expressionem cetera algebraicam ingrediantur, eae non impedient, quominus ea expressio sit algebraica, dummodo y et z fuerint functiones algebraicae.

176. Poterimus autem hoc ratiocinium extendere ad differentialia secunda et superiorum ordinum. Si enim manente y functione algebraica ipsius x fuerit $dy = pdx$ atque $dp = qdx$, erit sumto differentiali dx constante $ddy = qdx^2$, uti supra vidimus. Cum igitur ob rationes ante allegatas sit quoque q functio algebraica ipsius x, erit quoque $\frac{ddy}{dx^2}$ non solum quantitas finita, sed etiam functio algebraica ipsius x, dummodo y fuerit eiusmodi functio. Simili modo perspicietur fore $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ etc. functiones algebraicas ipsius x, modo y fuerit talis; atque si z sit quoque functio algebraica ipsius x, omnes expressiones finitae, quae ex differentialibus cuiusvis ordinis ipsarum y, z et ex dx componuntur, cuiusmodi sunt $\frac{ddy}{ddz}$, $\frac{d^3y}{dzddy}$, $\frac{dxd^4y}{dy^3ddz}$ etc., simul erunt functiones algebraicae ipsius x.

177. Cum igitur nunc methodus sit tradita cuiusque functionis ipsius x algebraicae differentiale primum inveniendi, eadem methodo poterimus quoque differentialia secunda altiorumque ordinum investigare. Si enim y fuerit functio quaecunque algebraica ipsius x, ex eius differentiatione $dy = pdx$ innotescet valor ipsius p. Qui si denuo differentietur atque reperiatur $dp = qdx$, erit $ddy = qdx^2$ posito dx constante sicque definietur differentiale secundum. Differentiando porro q, ut sit $dq = rdx$, habebitur differentiale tertium $d^3y = rdx^3$ sicque ulterius differentialia altiorum ordinum indagabuntur, quoniam quantitates p, q, r etc. omnes sunt functiones ipsius x algebraicae, ad quas differentiandas praecepta data sufficiunt. Hoc ergo efficietur continua differentiatione; omissis enim dx in differentiatione ipsius y prodibit valor ipsius $\frac{dy}{dx} = p$, qui denuo differentiatus ac divisus per dx, quod fit, dum ubique differentiale dx omittatur, dabit valorem ipsius $q = \frac{ddy}{dx^2}$. Simili modo porro invenitur $r = \frac{d^3y}{dx^3}$ etc.

I. Sit $y = \frac{aa}{aa+xx}$ cuius differentialia tam primum quam sequentium ordinum requiruntur.

Primum ergo differentiando simulque per dx dividendo erit

$$\frac{dy}{dx} = \frac{-2aax}{(aa+xx)^2}$$

hincque porro

$$\frac{ddy}{dx^2} = \frac{-2a^4+6aaxx}{(aa+xx)^3}$$

$$\frac{d^3y}{dx^3} = \frac{24a^4x-24aax^3}{(aa+xx)^4}$$

$$\frac{d^4y}{dx^4} = \frac{24a^6-240a^4xx+120aax^4}{(aa+xx)^5}$$

$$\frac{d^5y}{dx^5} = \frac{-720a^6x+2400a^4x^3-720aax^5}{(aa+xx)^6}$$

etc.

II. Sit $y = \frac{1}{\sqrt{(1-xx)}}$ eruntque differentialia primum et sequentia

$$\frac{dy}{dx} = \frac{x}{(1-xx)^{\frac{3}{2}}}$$

$$\frac{ddy}{dx^2} = \frac{1+2xx}{(1-xx)^{\frac{5}{2}}}$$

$$\frac{d^3y}{dx^3} = \frac{9x+6x^3}{(1-xx)^{\frac{7}{2}}}$$

$$\frac{d^4y}{dx^4} = \frac{9+72x^2+24x^4}{(1-xx)^{\frac{9}{2}}}$$

$$\frac{d^5y}{dx^5} = \frac{225x+600x^3+120x^5}{(1-xx)^{\frac{11}{2}}}$$

$$\frac{d^6y}{dx^6} = \frac{225+4050x^4+720x^6}{(1-xx)^{\frac{13}{2}}}$$

etc.

Haec differentialia facile ulterius continuantur; interim tamen lex, qua termini eorum progrediuntur, non cito patet. Coefficiens quidem supremarum ipsius *x* potestatum semper est productum numerorum naturalium ab 1 usque ad ordinem differentialis, quod quaeritur. Interim si has formas ulterius continuemus atque perpendamus, deprehendemus fore generaliter, si $y = \frac{1}{\sqrt{(1-xx)}}$,

$$\frac{d^n y}{dx^n} = \frac{1\cdot 2\cdot 3\cdots n}{(1-xx)^{n+\frac{1}{2}}}\left\{\begin{aligned} & x^n + \tfrac{1}{2}\cdot\frac{n(n-1)}{1\cdot 2}x^{n-2} + \tfrac{1\cdot 3}{2\cdot 4}\cdot\frac{n(n-1)(n-3)}{1\cdot 2\cdot 3\cdot 4}x^{n-4} \\ & + \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{n(n-1)\cdots(n-5)}{1\cdot 2\cdots 6}x^{n-6} \\ & + \tfrac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{n(n-1)\cdots(n-7)}{1\cdot 2\cdots 8}x^{n-8} \\ & + \text{etc.} \end{aligned}\right\}$$

Huiusmodi ergo exempla non solum inserviunt ad habitum in differentiationis negotio acquirendum, sed etiam leges, quae in differentialibus omnium ordinum observantur, per se sunt notatu dignissimae atque ad alias inventiones deducere possunt.