CHAPTER IV

CONCERNING THE NATURE OF DIFFERENTIALS OF EACH ORDER

112. In the first chapter we saw, if the variable x should take an increase $=\omega$, then the form of any function arising for such an increase of x may be expressed thence $P\omega + Q\omega^2 + R\omega^3 + \text{etc.}$, and this expression shall be either finite or extend to infinity. Therefore the function y, if in that in place of x there may be written $x+\omega$, adopts the following value

$$y^{\text{I}} = y + P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \text{etc.};$$

from which if the former value of y may be subtracted, the difference of the function y will remain, which may be expressed thus

$$\Delta y = P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \text{etc.},$$

and since the value of x following shall be $x^{\text{I}} = x + \omega$, the difference of x will be certainly $\Delta x = \omega$. Moreover the letters P, Q, R etc. depending on y denote functions of x, which we have shown how to find in the first chapter.

113. Therefore hence, in whatever manner the size of the variable x may be augmented by the increase ω, likewise the increase will be able to be defined, which is added to any function y of x, provided that we are able to define the functions P, Q, R, S etc. for any value of y. Moreover in this chapter and in the whole of infinitesimal analysis that increase ω, by which we have decided the variable quantity x to increase, we may set up infinitely small and thus vanishing or $=0$. From which it is evident the increment or difference of the function y to be infinitely small also. But since in this hypothesis the individual terms of the expression

$$P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \text{etc.}$$

from above before vanish(§ 88 et seq.), only the first $P\omega$ will remain and therefore in this case, because ω is infinitely small, the difference of y itself, truly Δy,$= P\omega$.

114. Therefore the infinitesimal analysis, that we begin to explain here, will be nothing other than a particular case of the methods of differences explained in the first chapter, which arise while the differences, which before were assumed finite, are now put in place very small. Therefore so that this case, in which the whole infinitesimal analysis may be contained, may be distinguished from the method of differences, since it may be appropriate to use both special names as well as signs denoting these infinitely small amounts. Therefore here we will call the infinitely small differences *differentials* following Leibniz; and since we were establishing different orders of the differences in the first chapter, now from these too it may be understood easily, what the first, second, third, etc. differentials may signify. But in place of the symbol Δ, by which before we indicated the differences, now we may use the symbol *d*, thus so that *dy* may indicate the first differential of *y*, *ddy* the second differential, d^3y the third and thus henceforth.

115. Because the infinitely small differences, which we discuss here we may call *differentials*, hence the whole calculus, in which the differentials are investigated and adapted according to that use, is accustomed to be called *Differential Calculus.* The English mathematicians among whom Newton first, and equally Leibniz among the Germans began to develop this new part of analysis, employ both other names as well as signs. Indeed the infinitely small differences, which we ourselves call differentials, are accustomed to call chiefly *fluxions*, sometimes also *increments* [by the English mathematicians]; which names may be more convenient to use in Latin speech, thus also they express well enough the thing which they denote. For the magnitude of the variable, by increasing continually takes other and still other values so that it may be considered as a flow, and hence the name fluxion: the designation of fluxion [flowing] which first was used by Newton, was carried over by analogy to an infinitely small increment in the increase of the speed, which quantity he took as if flowing.

116. But although we may easily have been surpassed in considering the use and definition of names, and we ourselves personally by observing the most appropriate expressions by judging the purity of the Latin language, it may be discordant to dispute with the English, yet there is no doubt why we may not first seize the glory for an account of the signs from the English. For the differentials, which they call fluxions themselves, are accustomed to denote by points which they write above letters, thus so that $\dot{y}$ with them may indicate the first fluxion of *y*, $\ddot{y}$ the second fluxion, $\dddot{y}$ the third fluxion and so on thus. Whereby the manner of notation, as it depends on choice, although it cannot be faulted if the number of points were small, as can easily be perceived on counting, yet if several points must be inscribed, great confusion and much inconvenience may arise. Indeed the tenth differential or fluxion may be represented extremely inconveniently in this manner $\overset{\vdots\vdots}{\dot{y}}$, since with our manner of signs $d^{10}y$ may be most easily understood. Moreover there may arise cases, at some point in which much higher orders of differentials and indeed indefinite orders must be expressed, for which the English way becomes completely unsuitable.

117. Therefore we will use both our names and signs, naturally these are accustomed to be used in our region now and are familiar to most, and truly these are more suitable. Yet meanwhile in spite of this, the signs and names of the English will be born in mind here, as these too are required to be understood by any who read their books. Nor indeed do the English adhere so to their own customs, in order that they neither completely reject nor deem it unworthy to read the books that have been written according to our custom. Indeed we have ourselves read through their greatest works with a keen desire and from these gained the greatest pleasure [perhaps Euler refers here in particular to Maclaurin's *Theory of Fluxions,* which was first published in 1745]; but also truly on numerous occasions we have observed that the writings of our own country cannot be read without usefulness. On account of which, although the same thought must certainly be chosen and expressed everywhere and in an equitable manner, yet it is not very difficult that we may become accustomed to each, indeed however much is required for an understanding of the books written in the foreign custom.

118. Therefore since the letter ω up to this point will have specified for us the difference or increment, by which the magnitude of the variable x is considered to increase, but now ω may be set up infinitely small, ω will be the differential of x, and on this account it will be required to be designated in this manner, and there will be $\omega = dx$; and hence dx will be the infinitely small difference, by which x itself is considered to increase. In a similar manner the differential of y thus may be expressed dy; and if y were some function of x, the differential dy will denote the increment, which the function y takes, while x will become $x + dx$. Whereby if in the function y there may be substituted everywhere $x + dx$ in place of x and the resulting quantity may be put $= y^{\mathrm{I}}$, there will be $dy = y^{\mathrm{I}} - y$ and in this manner the differential of each function will be found ; which indeed is required to be understood concerning the first differential or of the first order ; concerning the rest indeed we will consider later.

119. Therefore the letter d here is not to be understood properly to denote a quantity, but only to be used in place of a sign, to which there is required to be expressed the name *differential*, in the same manner, so that we are accustomed to use the letter l for the sign of the logarithm in the instruction of logarithms and in algebra the character $\sqrt{\ }$ for the sign of the square root. Hence dy does not signify, as is customary in common use in analysis, the product of the quantity d by the quantity y, but must be proposed thus, so that it is said to be the differential of y. In a like manner, if there is written $d^2 y$, neither the exponent two nor d^2 signifies the power of d, but may be used only to express the name of the *second differential* more shortly and suitably. Therefore since the letter d in the differential calculus is not a quantity, but may show a sign only, to avoid confusion in calculations, where several constant quantities occur, the letter d cannot be used to designate these, and in the same way we are accustomed to avoid the letter l as a quantity to introduce into a calculation, where logarithms likewise occur. But there is required to choose, so that the letters d and l themselves may be expressed by characters changed a little bit, lest they are not confused with the letters of the alphabet, to which these quantities are accustomed to designate; evidently in a like manner, where in place of the letter r, by which at first the name of the root may be indicated, now it is customary to use that misshapen character $\sqrt{\ }$.

[The original root sign did not have the extended bar on the top, and may be seen to resemble a script letter *r*.]

120. Therefore since we have seen the first differential of y, if y were some function of *x*, to be considered a form of this kind $P\omega$, on account of $\omega = dx$ it will be $dy = Pdx$. Evidently whatever the function y were of x, the differential *dy* may be expressed by a certain function of *x*, for which here we have put *P*, by the differential of x, obviously multiplied by *dx*. Therefore even if the differentials themselves of x and *y* actually are infinitely small and thus equal to nothing, yet they will have a finite ration between each other; evidently it will be $dy{:}dx = P{:}1$. Therefore from that function *P* the ratio between the differential *dx* and the differential *dy* will be known. Therefore since the differential calculus rests on finding differentials, in that not as differentials themselves, which are zero and therefore they may be found without effort, but the mutual geometric ratio of these is investigated.

121. Therefore the differentials may be found much more easily than finite differences. Indeed according to the finite difference Δy, by which the function y increases, while the increment of the quantity *x* takes ω, it is not sufficient to know the function *P*, but it may be required to investigate in addition the functions *Q, R, S* etc., which enter into the finite difference, which we have put

$$= P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \text{etc.};$$

moreover the differential of *y* can be found in a satisfactory manner, if we only know the function *P.* On account of which, the differential of this is defined from the known finite difference of each function of *x* ; truly on the contrary, from the differential of this function it is not yet possible to elicit the finite difference of this. Yet meanwhile below we will instruct, how any finite differences of each proposed function are able to be found from the differentials of all the orders likewise known. Moreover from these it is evident the first differential $dy = Pdx$ gives the first term of the finite difference, which naturally is $= P\omega$.

122. Therefore if the increment ω, which the magnitude of the variable x is considered to take, were exceedingly small, thus so that in the expression $P\omega + Q\omega^2 + R\omega^3 + \text{etc.}$ the terms $Q\omega^2$ and $R\omega^3$ and the more small remaining become so small, that in the computation, in which we assume that rigor is not observed, they are able to be neglected in comparison with the first $P\omega$, then with the differential *Pdx* known, from that the closest finite difference will be known, naturally which will be $= P\omega$; from which on numerous occasions, in which the calculus may be used in practice, some pleasure is derived. And hence some differentials as able to be considered as exceedingly small and these actually deny equality to nothing and are to be considered only as infinitely small. And this idea offers the occasion for others to make accusations about infinitesimal calculus, so that true quantities may not be elicited, but only truly approximations ; which objection always carries a little weight, unless we shall put the infinitely small completely equal to zero.

123. But they who do not wish the infinitely small to go to nothing, so that they would seem to destroy the strength of the objection, compare differentials with exceedingly small dust particles to the ratio of the whole earth, the quantity of which nobody may be thought not to have related correctly, who deviated by a single particle from the truth. Therefore they wish such a ratio to be between a finite quantity and the infinitely small, of such a kind as there is between the whole earth and the smallest particle of dust; and if for which this distinction at this point may be considered not to be large enough, they would increase that ratio a thousand times and more, so that altogether it cannot be perceived any smaller. Yet meanwhile they are forced to recognise that the great geometrical rigor to be broken a little; whereby so that they may be opposed to this objection, they flee to examples of this kind, of which both by geometry as well as by infinitesimal analysis they are able to find solutions, and the goodness of the latter method they conclude from the congruence of these. But nevertheless this work does not make a proof, since many times it is elicited by erroneous methods, yet, because by this fault it does not work, rather it overcomes these quantities which shall be ignored in the calculation, not only to be incomprehensibly small , but plainly to be nothing, as we have assumed. From which generally we infer no strength from geometric rigor.

124. We may progress to the nature of the second order differentials requiring to be explained, which arise from second order differences set out in the first chapter by putting in place the quantity ω infinitely small $= dx$. Therefore since, if we may put the variable quantity x to increase by equal increments, thus so that , if the value following x^{I} were $= x + dx$, the following shall become $x^{\mathrm{II}} = x + 2dx$, $x^{\mathrm{III}} = x + 3dx$ etc., on account of the first constant differences $= dx$ the second differences vanish, therefore also the second differential of x, certainly $ddx, = 0$ and on account of this ratio also the differentials beyond will be $= 0$, evidently $d^3x = 0, \quad d^4x = 0, \quad d^5x = 0$ etc. Indeed it is possible to object to these differentials, since they shall be infinitely small, by themselves to be $= 0$ neither this is a property of the variable quantity x, the increments of which shall be considered equal ; but truly this vanishing is required to be interpreted thus, so that the differentials *did*, d^3x etc. shall be seen to be zero not only between themselves, but also on account of the powers of dx, they may vanish since with which others may be compared.

125. So that which may be understood more clearly, the second difference of each function of x, which shall be y, to expressed forms of this kind $P\omega^2 + Q\omega^3 + R\omega^4 + \text{etc.}$
But if therefore ω shall be infinitely small, the terms $Q\omega^3, R\omega^4$ etc. will vanish before the first $P\omega^2$, from which on putting $\omega = dx$ the second differential of y will be $= Pdx^2$ by denoting dx^2 the square of the differential dx. Whereby although the second differential of

y, certainly ddy, by itself shall be $= 0$, yet, since there shall be $ddy = Pdx^2$, to dx^2 there will be had a finite ratio as P to 1; but if there shall be $y = x$, then there is made $P = 0,\ Q = 0,\ R = 0$ etc. and thus in this case the second differential of x also vanishes with respect to dx^2 and to higher powers of dx. And these are to be understood in this manner, which we have said before, evidently to be $ddx = 0,\quad d^3x = 0$ etc.

126. Since the second differences shall be nothing other but the difference of the first difference, the second differential too or, as it is often accustomed to be called the differentio – differential is nothing other than the differential of the first differential. Because then a constant quantity can take no increments either added or subtracted, and no differences are allowed , certainly they are characteristic of variable quantities only, we say in the same sense the differential of constant quantities of each order is to be = 0, that is before to such an extent with all the powers of dx to vanish. Therefore since the differential of dx, that is ddx, shall be $= 0$, the differential dx can be considered as a constant quantity, and as often as the differential of some quantity is said to be constant, as many times this quantity is understood continually to take equal increments. But here we take x for that quantity, the differential of which shall be constant, and thus we will consider the variability of individual functions, on which the differentials of these depend.

127. We may put the first differential of y to be $= pdx$ and towards finding the second differential of which the differential of pdx itself must be differentiated anew. But since dx shall be constant and may not be changed, even if in place of x there may be written $x + dx$, yet it is required, so that the differential of the finite quantity p may be found; therefore if $dp = qdx$, because we have observed the differentials of all functions of x to be returned to a form of this kind ; and since there shall be, as we have shown concerning finite differences, the differential of $np = nqdx$, if n shall be a constant quantity, dx is put in place of n and there will be the differential of $pdx = qdx^2$. On account of this, if there shall be $dy = pdx$ and $dp = qdx$, the second differential will be $ddy = qdx^2$ and thus it is agreed, because now before we have acknowledged that the second differential of y to dx^2 has a finite ratio.

128. Now in the first chapter we have observed that the second and following differences cannot be constructed, unless the successive values of x are assumed to progress by a certain sure law ; which law since it shall be arbitrary, we have attributed an arithmetic progression with these values as most easy and likewise most suitable. Therefore because of the same reason concerning the following differentials nothing of certainty can be put in place, unless the first differentials, by which the magnitude of the variable x is considered to increase continually, are progressing following a given law; and thus we have put the first differentials of x, certainly $dx,\ dx^{\mathrm{I}},\ dx^{\mathrm{II}}$ etc., all equal to each other amongst themselves, from which the second differentials

$$ddx = dx^{\mathrm{I}} - dx = 0, \quad ddx^{\mathrm{I}} = dx^{\mathrm{II}} - dx^{\mathrm{I}} = 0 \text{ etc.}$$

Therefore because the second and further differentials depend on the order, which the differentials of the variable quantity *x* hold between each other, and this order shall be arbitrary, which condition has not affected the first differentials, hence a huge distinction intercedes between the first differentials and found by the following reason.

129. But if moreover the successive values of x : $x, x^{\mathrm{I}}, x^{\mathrm{II}}, x^{\mathrm{III}}, x^{\mathrm{IV}}$ etc. may not be put in place following an arithmetical progression, but may be put to progress by some other law, then also the first differentials of these dx, dx^{I}, $\mathrm{d}x^{\mathrm{II}}$ etc. will not be equal to each other and therefore neither will $ddx = 0$. On this account the second differentials of any functions of *x* may adopt another form ; for if the first differential of this kind of the function *y* were $= pdx$, towards finding the second differential of this it does not suffice for the differential of *p* to be multiplies by *dx*, but in addition the ratio of the differential of *dx*, which is *ddx*, must be considered. Because indeed the second differential arises, if pdx is subtracted from the value of this following which arises, while there is put in place $x + dx$ in place of *x* and $dx + ddx$ in place of *dx*, we may put the value of *p* following to be $= p + qdx$ and there will be the following value of *pdx*

$$= (p + qdx)(dx + ddx) = pdx + pddx + qdx^2 + qdxddx \,;$$

from which there may be taken *pdx* and there will be the second differential

$$ddy = pddx + qdx^2 + qdxddx = pddx + qdx^2,$$

because *qdxddx* will vanish before *pddx*.

130. But nevertheless the ratio of equality is the most simple and the most suitable, which continually is attributing to the increments of *x*, yet often it is accustomed to come about, so that the equal increments may be assumed, not of the variable quantity *x*, of which *y* is a function, but of some other quantity, of which *x* itself shall be a function. So that also often the equal first differentials of a function of this kind may be put in place, of which indeed no relation to *x* may be agreed upon. In the first case the differentials may depend on the second and following differentials of *x* by a ratio, which *x* holds to that quantity, which is put to increase equally, and must be defined from that pair only, from which we have shown how to define here the second differentials of y from the differentials of x . But in the second case the second and following differentials of *x* are to be regarded as unknown and in place of these $ddx,\ d^3x,\ d^4x$ etc. with the signs will have to be taken.

131. But since, just as it may be necessary to resolve the individual differentiations in these cases, that we are to show presently below, here we proceed to assume the variable *x* as increasing uniformly, therefore so that thus the first differentials of this dx, dx^{I}, dx^{II} etc. are all equal to each other and therefore the second differentials and the following are set equal to nothing; which condition thus it may be usual to propose, so that the differential of x, truly *dx*, is said to be assumed constant. Then let *y* be some function of *x;* which since it may be defined by *x* and constants, also the individual first, second, third, fourth, etc., differentials of this also, which are indicated by these signs dy, ddy, d^3y, d^4y etc., will be able to be expressed by *x* and *dx*. Evidently if in *y* in place of *x* there may be written $x+dx$ and from this first value the first is taken away, there will remain the first differential *dy;* in which if again in place of *x* there is put $x+dx$, there will be produced dy^{I} and there will be $ddy = dy^{\mathrm{I}} - dy$; in a similar manner on putting $x+dx$ in place of x from *ddy* there may arise ddy^{I} and $ddy^{\mathrm{I}} - ddy$ will give d^3y and so on thus; in which operations the differential *dx* is always regarded as a constant quantity, which may accept no differential.

132. From the ratio, by which a function *y* may be determined by *x*, as with the help of the method of finite differences which is much more readily from these, which we are about to treat later, the value of the function *p* may be defined, which multiplied by *dx* gives the first differential *dy.* Therefore on putting $dy = pdx$ the differential of this pdx will give the second differential *ddy;* from which, if there should be $dp = qdx$, on account of constant *dx* there may arise $ddy = qdx^2$, as now we have shown before. Therefore by progressing further, since the differential of the second differential will give rise to the third differential, we may put to be $dq = rdx$ and there will be $d^3y = rdx^3$; in a similar manner, if the differential function *r* of this is sought and there should be $dr = sdx$, there will be had the fourth differential $d^4y = sdx^4$ and thus so forth, as long as we may know how to find the first differential of each function we will be able to assign the differential of each order.

133. Therefore so that the forms of the individual differentials of these and likewise that ratio required to be found may be shown clearer to the mind, that is seen to be included in the following table. If *y* were some function of *x*,

there will be	and on putting
$dy = pdx$	$dp = qdx$
$ddy = qdx^2$	$dq = rdx$
$d^3y = rdx^3$	$dr = sdx$
$d^4y = sdx^4$	$ds = tdx$
$d^5y = tdx^5$	etc.

Therefore since the function p is known from the function y by differentiation and from p in a like manner q may be found and hence again r and from that further s etc., the differentials of any order of y may easily be found, provided the differential dx is assumed constant.

134. Since p, q, r, s, t etc. shall be finite quantities, without doubt functions of x, the first differential of y will have a finite ratio to the first differential of x, clearly as p to 1, and on account of this reason the differentials dx and dy may be called homogeneous. Then since ddy to dx^2 may have the finite ratio as q to 1, ddy and dx^2 will be homogeneous; in a similar manner d^3y and dx^3 will be homogeneous and likewise d^4y and dx^4 and thus henceforth. From which as the first differentials are homogeneous between themselves or maintaining a finite ratio, thus the second differentials with the squares of the first differentials, moreover the third differentials with the cubes of the first differentials and thus so on will be homogeneous. And generally the differential of y of order n, which is expressed thus d^ny, will be homogeneous with dx^n, that is with the power of the differential dx, the exponent of which is n.

135. Therefore since before dx all the powers of this vanish, of which the exponents are greater than unity, before dy also dx^2, dx^3, dx^4 etc. vanish, and which the differentials ddy, d^3y, d^4y, etc. of higher orders maintain a finite ratio to these powers. In a similar manner before ddy, because it is homogeneous with dx^2 all the powers of dx superior to the square dx^3, dx^4 etc. vanish; therefore also d^3y, d^4y etc. vanish. And before d^3y, dx^4, d^4y, dx^5, d^5y etc. vanish. And hence easily, if some expressions involving differentials of this kind were proposed, they will be able to be distinguished, whether or not they shall be homogeneous. Indeed they must be considered only with the differentials from finite quantities disregarded, clearly which do not disturb the homogeneity ; and for the differentials of the second and higher orders the homogeneous powers of dx itself may be written ; which if they present a number of the same dimension everywhere, the expressions will be homogeneous.

136. Thus it will be apparent these expressions $Pddy^2$ and $Qdyd^3y$ are homogeneous between themselves. For ddy^2 denotes the square of ddy, and because ddy is homogeneous with dx^2, ddy^2 will be homogeneous with dx^4. Then because dx is homogeneous with dy and d^3y with dx^3, there will be produced dyd^3y homogeneous with dx^4; from which it follows that the expressions $Pddy^2$ and $Qdyd^3y$ are homogeneous between themselves and thus have a finite ratio between themselves. In a similar manner it may be deduced these expressions

$$\frac{Pd^3y^2}{dxddy} \text{ and } \frac{Qd^5y}{dy^2}$$

are homogenous; indeed with the substitution for dy, ddy, d^3y and d^5y by these powers of *dx* homogeneous to themselves dx, dx^2, dx^3 and dx^5 these expressions Pdx^3 and Qdx^3 arise, which everywhere will be homogeneous between themselves.

137. But if with this reduction made the proposed expressions may not contain equal powers of *dx*, then they will not be homogeneous nor therefore will they maintain a finite ration between each other. Therefore one will be either infinitely greater or less than the other and hence one vanishes with respect to the other. Thus $\frac{Pd^3y}{dx^2}$ to $\frac{Qddy^2}{dy}$ will have an infinitely large ratio ; for the firs expression is reduced to *Pdx* and the other to Qdx^3, from which the latter vanishes before the former. On account of which if in some calculation a sum of two formulas of this kind may occur

$$\frac{Pd^3y}{dx^2}+\frac{Qddy^2}{dy},$$

the latter term before the former can be rejected with care and only the first term $\frac{Pd^3y}{dx^2}$ retained in the calculation; indeed a perfect ratio of equality remains between the expressions

$$\frac{Pd^3y}{dx^2}+\frac{Qddy^2}{dy} \text{ and } \frac{Pd^3y}{dx^2},$$

because the exponent of the ratio is

$$1+\frac{Qdx^2ddy^2}{Pdyd^3y}=1 \text{ on account of } \frac{Qdx^2ddy^2}{Pdyd^3y}=0.$$

And whenever with this agreed upon, differential expressions are able to be wonderfully contracted.

138. In differential calculus the precepts are taught, with the aid of which the first differential of any proposed quantity can be found ; and because the second differentials from the differentiation of the first, the third by the same operation from the second, and thus again the following are found from the preceding ones, the differential calculus may contain the method of finding all the differentials of each order. But from the name *differential,* by which the infinitely small differences are denoted, other names may be derived, which are customary in use. Thus the words may be had *to differentiate,* which means *to find the differential,* and the quantity is said *to be differentiated*, when the differential of this is elicited. But *Differentiation* denotes the operation, by which differentials are found. Hence the differential calculus is also called the method of *differentiating,* since it may contain the manner of finding differentials.

139. Just as in differential calculus the differential of any quantity is found, thus in turn a kind of calculus is established also into the finding of the quantity, of which the differential is proposed, which is called the *integral calculus*. For if some differential were proposed, with respect of which that quantity, of which it is the differential, may be customarily called the *integral.* An account of which derivation is, because, since the differential may be considered as the infinitely small part by which it increases by some quantity, that quantity itself can be regarded as the total or integral in respect of this part and on account of this reason is called the integral of that. Thus, since *dy* shall be the differential of *y,* in turn *y* will be the integral of *dy*, and since *ddy* shall be the differential of *dy*, *dy* will be the integral of *ddy.* In a similar manner *ddy* will be the integral of d^3y and d^3y of d^4y and thus henceforth ; from which any differentiation, if regarded inversely, shows an example of integration.

140. The origin and nature of integration and differentiation equally can be explained most clearly from the principles of finite differences presented in the first chapter. Indeed after it may be shown, how the difference of each quantity ought to be found, on working backwards we have shown also how, if the difference were proposed, that quantity may be found, of which that will be the difference; as the quantity with regard to its own difference we will call the sum of that. Therefore as the preceding small differences in the differentials go off to infinity, thus the sums, which they have been called there, are allotted the name of integral and on this account the integrals also often are accustomed to be called sums. The English, who call differentials fluxions, call integrals fluent quantities with the custom of talking about finding the fluent of a given fluxion of those, which is the same as by our habit, we say to find the integral of a given differential.

141. As we designate differentiation by the character *d*, thus we use this letter $\int$ to indicate integration, which therefore prefixed to differential quantities will denote these quantities, of which those are the differentials. Thus, if the differential of *y* were pdx or $dy = pdx$, *y* will be the integral of pdx, which will be written in this manner $y = \int pdx$, since there shall be $y = \int dy$. Therefore the integral of *pdx,* which is indicated by $\int pdx$, denotes a quantity, the differential of which is *pdx.* In a similar manner, since there shall be $ddy = qdx^2$ with $dp = qdx$ present, the integral of *ddy* itself, that is there will be $dy, = pdx$ and on account of $p = \int qdx$ there will be $dy = dx\int qdx$ and therefore $y = \int dx\int qdx$. If further there shall be $dq = rdx$, there will be $q = \int rdx$ and $dp = dx\int rdx$, from which , if the character $\int$ may be set in front anew, there will come about

$p=\int dx\int rdx$ and again $dy=dx\int dx\int rdx$ and $y=\int dx\int dx\int rdx$.

142. Because the differential *dy* is a very small quantity, but the integral of this *y* is a finite quantity and in an equal manner the second differential *ddy* is infinitely less than the integral *dy* of this, it is evident the differentials vanish before its integrals. So that this condition may be better understood, the infinitely small are accustomed to be divided into orders and there is said to be an infinitely small of the first order, to which the first differentials *dx, dy* are referred; the infinitely small of the second order includes the differentials of the second order, which are homogeneous with dx^2 ; and in a similar manner, which are homogeneous with dx^3, are called of the third order, to which all the third order differentials pertain, and thus in turn again. From which as the infinitely small of the first order vanish before finite quantities, thus the infinitely small of the second order before the infinitely small of the first order and generally the infinitely small of each higher order vanish before the infinitely small of inferior orders.

143. Therefore with these infinitely small orders in place as the differential of a finite quantity is an infinitely small quantity and the differential of the second order is an infinitely small quantity of the second order and thus so on, thus in turn it is evident the integral of an infinitely small quantity of the first order is a finite quantity, moreover the integral of an infinitely small quantity of the second order is an infinitely small quantity of the first order and thus henceforth. Whereby if the proposed differential were an infinitely small amount of the order *n*, the integral of this will be an infinitely small amount of order $n-1$; and hence as by differentiating the order of an infinitely small quantity may be augmented, thus on integration we are progressing to lower orders, while we may arrive at finite quantities. But id we may integrate the finite quantities anew, then following this rule we may arrive at infinitely large quantities and from these with the integration put in place at this stage to infinitely greater quantities and thus on progression we will obtain similar infinite orders, for each of which to surpass the preceding infinities.

144. It remains, that in this chapter we may advise about the customary usage of signs, lest a place is left for any ambiguity. And indeed in the first place the sign of the differential *d* only affects the letter immediately following; thus *dxy* does not denote the differential of the product *xy*, but the differential of *x* itself multiplied by the quantity *y*. But it is customary, so that less confusion arises, for the quantity *y* to be written before the sign *d* in this manner *ydx*, by which the product from *y* into *dx* may be indicated. But yet if *y* shall be a quantity having either the square root sign $\sqrt{\ }$ or the logarithm prefix, then it is usual to be put after the differential ; without doubt $dx\sqrt{(aa-xx)}$ signifies the product from the finite quantity $\sqrt{(aa-xx)}$ into the differential *dx* and in a similar manner $dx\, l(1+x)$ is the product from the logarithm of the quantity $1+x$ multiplied by *dx*. On account of the same reason $ddy\sqrt{x}$ expresses the product of the second differential *ddy* and of the finite quantity $\sqrt{x}$.

145. Nor truly does the sign *d* only affect the letter immediately following, but also indeed the exponent, if which it has, thus on considering dx^2 not to express the differential of x^2, but the square of the differential of *x*, thus so that the exponent 2 must not refer to *x*, but to *dx*. It may also be possible to write *dxdx*, just as the product of the two differentials *dx* and *dy* may be set out in this manner *dxdy*; truly the first manner dx^2, as it is shorter, thus is more useful. Especially if higher powers of *dx* might be indicated, it becomes exceedingly long for *dx* to be repeated so many times; thus dx^3 specifies the cube of *dx* and in the differentials of higher orders a like account is observed. Evidently ddy^4 signifies the fourth power of the second order differential *ddy* and $d^3y^2\sqrt{x}$ signifies the square of the third order differential of *y* to be multiplied by $\sqrt{x}$*;* but if it must be multiplies by a rational quantity *x*, that is prefixed in this manner xd^3y^2.

146. But if we may wish, that the sign *d* should affect more than the following letter only, that must be indicated in a special manner. In this case we use a special small bracket, within which that quantity is included, the differential of which has to be indicated ; in order that $d(xx+yy)$ specifies the differential of the quantity $xx+yy$. Truly if we may wish to designate the differential of a power of this kind of quantity, we are scarcely able to avoid ambiguity; indeed if we write $d(xx+yy)^2$, it may be able to understand the square of $d(xx+yy)$. But we may be able in this case to call upon the assistance of a point, thus so that $d.(xx+yy)^2$ will signify the differential of $(xx+yy)^2$, but with the point omitted it becomes $d(xx+yy)^2$ the square of $d(xx+yy)$. Evidently with the point it is possible conveniently to indicate the sign *d* to pertain to the whole quantity after the following point; thus $d.xdy$ expresses the differential of *xdy* and $d.^3xdy\sqrt{(aa+xx)}$ the differential of the third order expression $xdy\sqrt{(aa+xx)}$, which is produced from the finite quantities *x* and $\sqrt{(aa+xx)}$ and from the differential *dy.*

147. But just as the sign of the differentiation *d* only affects the quantity immediately following, unless by a point interposed the strength of this is extended to the whole expression following, thus on the other hand the sign of the integration $\int$ always includes the whole expression, to which it is prefixed. Thus $\int ydx(aa-xx)^n$ signifies the integral or that quantity, the differential of which is $ydx(aa-xx)^n$, and this expression $\int xdx\int dxlx$ signifies the quantity, the differential of which is $xdx\int dxlx$. Hence if we should wish to express the product of the two integrals, namely $\int ydx$ and $\int zdx$, this may wrongly be made in this manner $\int ydx\int zdx$; for it may be understood to be the integral of the quantity $ydx\int zdx$. On account of this cause a point may be accustomed to remove this ambiguity, thus so that $\int ydx\cdot\int zdx$ signifies the product of the integrals $\int ydx$ and $\int zdx$.

148. The infinitesimal analysis therefore is concerned with finding both differentials and integrals and hence on this account is divided into two particular parts, one of which is called differential calculus and the other integral calculus. In the former the differential precepts of any quantities to be found are set out; in the latter indeed a way may be shown to investigate the integrals of proposed differentials ; but likewise in each great use may be indicated, which these calculi offer both to analysis itself as well as to higher geometry. As it is by this reason that the analysis part now has taken so many advancements, that in a little volume it may not be able to be understood completely. Therefore in the first place in the integral calculus over time both new artifices of integration as well as aids of this in solving problems of various kinds are uncovered, so that on account of these new discoveries, which are added to continually, and at no time to become exhausted, will be possible to be explained and described much less perfectly. But I will present the work, so that these things which have been found at this time, either I will explain completely in these books or perhaps I will set out the method, from which they are able to be deduced easily.

149. Generally several parts of infinitesimal analysis are usually considered ; for in addition to the differential and integral calculus here the second order calculus [differentio-differential] and the calculus of the exponential are found. In the second order calculus the method usually examines differentials of the second order, and of higher orders required to be found; but because I am about to set out the manner of finding each order in the differential calculus, we will refrain from this subdivision, which rather seems to have been made from the merit of the discovery than from the making the thing itself. Because then it touches on the exponential calculus, by which the celebrated Johan Bernoulli, to whom we must eternally be grateful on account of innumerable and outstanding advancements of the infinitesimal calculus, transferred the methods of differentiation and integration to exponential quantities, because each calculus may be put in place applied to quantities of all kinds, both algebraic as well as transcending, hence a particular part is made superfluous and the opposite may be put in place.

150. Therefore at first in this book I am to explain and to put in place the manner of dealing with the differential calculus , with the help of which the differentials of all variable quantities not only the first, but also the second and higher orders are able to be found readily. Therefore in the first place I will contemplate algebraic quantities, whether they shall be of one variable or of several, whether at last they may be given explicitly or by equations. Then also I will apply the discovery of differentials to non algebraic quantities, the acquaintance of which indeed it is possible to arrive at without the aid of the integral calculus ; logarithms and exponential quantities are of this kind, then also the arc of the circle and in turn the sines and tangents of the circular arcs. And then also I will show how to differentiate whatever quantities composed and mixed together from these, and thus the first part of the differential calculus, clearly the method of differentiating, will be resolved in full.

151. The other part has been resolved to explaining the use, which the method of differentiating brings to analysis as well as to higher geometry. But in ordinary algebra thence the advantages are too numerous, the part about finding the roots of equations, the part about treating and summing series, about eliciting the maxima and minima, about the values of expressions, which may be considered in certain indeterminate cases, which are to be defines and which are different. But the higher geometry takes the greatest advance from the differential calculus, while with the help of that the tangents of curved lines and the curvature of these themselves are defined with a wonderful facility and many other problems concerning rays reflected or refracted from curved lines are able to be resolved. From which also the largest treaty could be filled, yet I will try, however much it is permitted to be made, to explain everything briefly and clearly.

CAPUT IV

DE DIFFERENTIALIUM CUIUSQUE ORDINIS NATURA

112. In capite primo vidimus, si quantitas variabilis *x* accipiat augmentum $=\omega$, tum cuiusvis functionis ipsius *x* augmentum inde oriundum tali forma exprimi $P\omega+Q\omega^2+R\omega^3+\text{etc.}$, sive haec expressio sit finita sive in infinitum excurrat. Functio ergo *y*, si in ea loco *x* scribatur $x+\omega$, valorem sequentem induet

$$y^{\mathrm{I}}=y+P\omega+Q\omega^2+R\omega^3+S\omega^4+\text{etc.};$$

a quo si valor prior *y* subtrahatur, remanebit differentia functionis *y*, quae ita exprimetur

$$\Delta y=P\omega+Q\omega^2+R\omega^3+S\omega^4+\text{etc.},$$

atque cum valor ipsius *x* sequens sit $x^{\mathrm{I}}=x+\omega$, erit differentia ipsius *x*, nempe $\Delta x=\omega$. Litterae autem *P, Q, R* etc. denotant functiones ipsius *x* pendentes ab *y*, quas capite primo invenire docuimus.

113. Hinc ergo, quicunque augmento ω augeatur quantitas variabilis *x*, simul definiri poterit augmentum, quod cuique ipsius *x* functioni *y* accedit, dummodo pro quovis ipsius *y* valore functiones *P, Q, R, S* etc. definire valeamus. In hoc autem capite atque in universa analysi infinitorum augmentum illud ω, quo quantitatem variabilem *x* crescere sumsimus, statuemus infinite parvum atque adeo evanescens seu $=0$. Unde manifestum est incrementum seu differentiam functionis *y* quoque fore infinite parvam. Cum autem in hac hypothesi singuli termini expressionis

$$P\omega+Q\omega^2+R\omega^3+S\omega^4+\text{etc.}$$

prae antecedentibus evanescant (§ 88 et seqq.), solus primus $P\omega$ remanebit eritque propterea hoc casu, quo ω est infinite parvum, differentia ipsius *y*, nempe $\Delta y,=P\omega$.

114. Erit ergo analysis infinitorum, quam hic tractare coepimus, nil aliud nisi casus particularis methodi differentiarum in capite primo expositae, qui oritur, dum differentiae, quae ante finitae erant assumtae, statuantur infinite parvae. Quo igitur iste casus, quo universa analysis infinitorum continetur, a methodo differentiarum distinguatur, cum peculiaribus nominibus tum etiam signis ad differentias istas infinite parvas denotandas uti conveniet. Differentias igitur infinite parvas hic cum LEIBNIZIO *differentialia* vocabimus;

atque cum differentiarum in primo capite diversos ordines constituissemus, ex iis nunc facile quoque intelligetur, quid differentialia prima, secunda, tertia etc. cuiusque functionis significent. Loco characteris autem Δ, quo ante differentias indicaveramus, nunc utemur charactere *d,* ita ut *dy* significet differentiale primum ipsius *y*, *ddy* differentiale secundum, d^3y tertium et ita porro.

115. Quoniam differentias infinite parvas, quas hic tractamus, *differentialia* vocamus, hinc totus calculus, quo differentialia investigantur atque ad usum accommodantur, appellari solet *Calculus differentialis.* Mathematici Angli, inter quos primum NEUTONUS aeque ac LEIBNIZIUS inter Germanos hanc novam analyseos partem excolere coepit, aliis tam nominibus quam signis utuntur. Differentias enim infinite parvas, quas nos differentialia vocamus, potissimum *fluxiones* nominare solent, interdum quoque *incrementa;* quae voces uti latino sermoni magis conveniunt, ita quoque res, quas denotant, satis commode exprimunt. Quantitas enim variabilis crescendo continuo alios atque alios valores recipiens tanquam fluens considerari potest hincque vox fluxionis, quae primum a NEUTONO ad celeritatem crescendi adhibebatur, ad incrementum infinite parvum, quod quantitas quasi fluendo accipit, designandum analogice est translata.

116. Quamvis autem circa vocum usum atque definitionem cum Anglis disceptare absonum foret nosque coram iudice puritatem latinae linguae atque expressionum commoditatem spectante facile superaremur, tamen nullum est dubium, quin Anglis ratione signorum palmam praeripiamus. Differentialia enim, quae ipsi fluxiones appellant, punctis, quae litteris superscribunt, denotare solent, ita ut $\dot{y}$ iis significet fluxionem primam ipsius *y*, $\ddot{y}$ fluxionem secundam $\ddot{y}$ fluxionem tertiam atque ita porro. Qui notandi modus, uti ab arbitrio pendens, etsi improbari nequit, si punctorum numerus fuerit parvus, ut numerando facile percipi queat, tamen, si plura puncta inscribi debeant, maximam confusionem plurimaque incommoda affert. Differentiale enim seu fluxio decima perquam incommode hoc modo $\overset{\vdots\vdots}{\dot{y}}$ repraesentatur, cum nostro signandi modo $d^{10}y$ facillime comprehendatur. Oriuntur autem casus, quibus multo adhuc superiores differentialium ordines atque adeo indefiniti exprimi debent, ad quos Anglorum modus prorsus fit ineptus.

117. Nostris igitur tam nominibus quam signis utemur, quippe quorum illa in nostris regionibus iam sunt usu recepta atque plerisque familiaria, haec vero commodiora. Interim tamen non abs re erat Anglorum denominationes et signationes hic commemorare, ut, qui eorum libros evolvunt, eos quoque intelligere queant. Neque enim Angli suo mori tam pertinaciter adhaerent, ut, quae nostro more sunt scripta, prorsus repudient nec legere dignentur. Nos quidem ipsorum opera maxima cum aviditate perlegimus ex iisque summum fructum percipimus; saepenumero vero etiam animadvertimus ipsos nostratium scripta non sine utilitate legisse. Quamobrem etsi idem ubique atque aequabilis modus cogitata sua exprimendi maxime esset optandus, tamen non admodum est difficile, ut utrique assuescamus, quantum quidem intelligentia librorum alieno more scriptorum postulate

118. Cum igitur littera ω nobis hactenus denotaverit differentiam seu incrementum, quo quantitas variabilis *x* crescere concipitur, nunc autem ω statuatur infinite parvum, erit ω differentiale ipsius *x* et hancobrem recepto signandi modo erit $\omega = dx$; atque *dx* proinde erit differentia infinite parva, qua ipsa *x* crescere concipitur. Simili modo differentiale ipsius *y* ita exprimetur *dy;* atque si *y* fuerit functio quaecunque ipsius *x*, differentiale *dy* denotabit incrementum, quod functio *y* capit, dum *x* abit in $x + dx$. Quare si in functione *y* ubique loco *x* substituatur $x + dx$ et quantitas resultans ponatur $= y^{\mathrm{I}}$, erit $dy = y^{\mathrm{I}} - y$ hocque modo differentiale cuiusque functionis reperietur; quod quidem intelligendum est de differentiali primo seu primi ordinis; de reliquis enim postea videbimus.

119. Probe ergo tenendum est litteram *d* hic non quantitatem denotare, sed tantum loco signi adhiberi, ad vocem *differentialis* exprimendam, eodem modo, quo in doctrina logarithmorum littera *l* pro signo logarithmi et in algebra charactere $\sqrt{\ }$ pro signo radicis uti consuevimus. Hinc *dy* non significat, uti vulgo in analysi usu est receptum, productum ex quantitate *d* in quantitatem *y*, sed ita enunciari debet, ut dicatur differentiale ipsius *y*. Simili modo, si scribatur d^2y, neque binarius exponentem neque d^2 potestatem ipsius *d* significat, sed adhibetur tantum ad nomen *differentialis secundi* breviter et apte exprimendum. Cum igitur littera *d* in calculo differentiali non quantitatem, sed signum tantum exhibeat, ad confusionem vitandam in calculis, ubi plures quantitates constantes occurrunt, littera *d* ad earum designationem usurpari nequit, perinde atque evitare solemus litteram *l* tanquam quantitatem in calculum inducere, ubi simul logarithmi occurrunt. Optandum autem esset, ut litterae istae *d* et *l* per characteres aliquantulum alteratos exprimerentur, ne cum litteris alphabeti, quibus quantitates designari solent, confundantur; simili scilicet modo, quo loco litterae *r*, qua primum vox radicis indicabatur, nunc character iste distortus $\sqrt{\ }$ in usum est receptus.

120. Quoniam igitur vidimus differentiale primum ipsius *y*, si *y* fuerit functio quaecunque ipsius *x*, habiturum esse huiusmodi formam $P\omega$, ob $\omega = dx$ erit $dy = Pdx$. Qualiscunque scilicet fuerit *y* functio ipsius *x*, eius differentiale *dy* exprimetur certa quadam functione ipsius *x*, pro qua hic ponimus *P*, per differentiale ipsius *x*, nempe per *dx*, multiplicata. Etiamsi ergo differentialia ipsarum *x* et *y* revera sint infinite parva ideoque nihilo aequalia, tamen inter se finitam habebunt rationem; erit scilicet $dy:dx = P:1$. Inventa ergo functione ista *P* innotescit ratio inter differentiale *dx* et differentiale *dy*. Cum igitur calculus differentialis in inventione differentialium consistat, in eo non tam ipsa differentialia, quae sunt nihilo aequalia ac propterea nullo labore invenirentur, quam eorum ratio mutua geometrica investigatur.

121. Differentialia igitur multo facilius inveniuntur quam differentiae finitae. Ad differentiam enim finitam Δy, qua functio *y* crescit, dum quantitas variabilis *x* incrementum ω accipit, non sufficit functionem *P* nosse, sed indagari insuper oportet functiones *Q*, *R*, *S* etc., quae in differentiam finitam, quam posuimus

$$= P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \text{etc.},$$

ingrediuntur; ad differentiale ipsius *y* autem inveniendum satis est, si noverimus solam functionem *P*. Quamobrem ex cognita differentia finita cuiusque functionis ipsius *x* facillime eius differentiale definitur; verum contra ex differentiali eius functionis nondum erui potest eius differentia finita. Interim tamen infra [§ 49 partis posterioris] docebitur, quemadmodum ex differentialibus omnium ordinum simul cognitis differentia quaevis finita cuiusque functionis propositae inveniri queat. Ceterum ex his manifestum est differentiale primum $dy = Pdx$ praebere terminum primum differentiae finitae, quippe qui est $= P\omega$.

122. Si igitur incrementum ω, quod quantitas variabilis *x* accipere concipitur, fuerit vehementer parvum, ita ut in expressione $P\omega + Q\omega^2 + R\omega^3 + \text{etc.}$ termini $Q\omega^2$ et $R\omega^3$ multoque magis reliqui fiant tam parvi, ut in computo, quo summus rigor non observatur, prae primo $P\omega$ negligi queant, tum cognito differentiali *Pdx* ex eo differentia finita vero proxime cognoscetur, quippe quae erit $= P\omega$; unde in pluribus occasionibus, quibus calculus ad praxin adhibetur, non parum fructus hauritur. Atque hinc nonnulli arbitrantur differentialia tanquam incrementa vehementer parva considerari posse eaque nihilo revera aequalia esse negant atque tantum indefinite parva statuunt. Haecque idea aliis occasionem praebuit analysin infinitorum accusandi, quod non veras rerum quantitates eliciat, sed tantum vero proximas; quae obiectio semper aliquam vim retineret, nisi infinite parva prorsus nihilo aequalia statueremus.

123. Qui autem nolunt infinite parva plane in nihilum abire, ii, ut vim obiectionis destruere videantur, differentialia comparant minimis pulvisculis ratione totius terrae, cuius quantitatem nemo non veram tradidisse censeretur, qui unico pulvisculo a veritate aberraverit. Talem igitur rationem inter quantitatem finitam et infinite parvam esse volunt, qualis est inter totam terram minimumque pulvisculum; atque si cui hoc discrimen adhuc non satis magnum videatur, eam rationem millies magisque adaugent, ut parvitas amplius omnino percipi nequeat. Interim tamen agnoscere coguntur summum rigorem geometricum aliquantulum infringi; quare quo huic obiectioni occurrant, ad eiusmodi exempla confugiunt, quorum tam per geometriam quam per analysin infinitorum solutiones inveniri possunt, ex earumque congruentia bonitatem posterioris methodi concludunt. Quanquam autem hoc argumentum negotium non conficit, cum saepenumero per erroneas methodos verum elici queat, tamen, quia hoc vitio non laborat, potius evincit eas quantitates, quae in calculo sint neglectae, non solum non incomprehensibiliter parvas, sed plane nullas esse, uti nos assumimus. Ex quo rigori geometrico nullam omnino vim inferimus.

124. Progrediamur ad differentialium secundi ordinis naturam explicandam, quae oriuntur ex differentiis secundis in capite primo expositis ponendo quantitatem ω infinite parvam $= dx$. Cum igitur, si ponamus quantitatem variabilem *x* aequalibus incrementis crescere, ita ut, si valor secundus x^{I} fuerit $= x + dx$, sequentes futuri sint $x^{\mathrm{II}} = x + 2dx$, $x^{\mathrm{III}} = x + 3dx$ etc., ob differentias primas constantes $= dx$ differentiae secundae evanescant, erit ergo quoque differentiale secundum ipsius *x*, nempe $ddx,= 0$ atque ob hanc rationem quoque differentialia ulteriora erunt $= 0$, scilicet $d^3x = 0, \quad d^4x = 0, \quad d^5x = 0$ etc. Obiici quidem potest haec differentialia, cum sint infinite parva, per se esse $= 0$ neque hoc proprium esse eius quantitatis variabilis *x*, cuius incrementa aequalia concipiantur; at vero hanc evanescentiam ita interpretari oportet, ut differentialia *ddx*, d^3x etc. non solum in se spectata sint nulla, sed etiam ratione potestatum ipsius *dx*, cum quibus alias comparari possent, evanescant.

125. Quae quo clarius intelligantur, recordandum est differentiam secundam cuiusque functionis ipsius *x*, quae sit *y*, huiusmodi forma exprimi $P\omega^2 + Q\omega^3 + R\omega^4 + \text{etc.}$ Quodsi ergo ω sit infinite parvum, termini $Q\omega^3, R\omega^4$ etc. prae primo $P\omega^2$ evanescent, unde posito $\omega = dx$ differentiale secundum ipsius *y* erit $= Pdx^2$ denotante dx^2 quadratum differentialis *dx*. Quare etsi differentiale secundum ipsius *y*, nempe *ddy*, per se sit $= 0$, tamen, cum sit $ddy = Pdx^2$, ad dx^2 habebit rationem finitam uti *P* ad 1; sin autem sit $y = x$, tum fit $P = 0, Q = 0, R = 0$ etc. ideoque hoc casu differentiale secundum ipsius *x* etiam respectu dx^2 altiorumque ipsius *dx* potestatum evanescit. Hocque modo intelligenda sunt ea, quae ante diximus, esse scilicet $ddx = 0, d^3x = 0$ etc.

126. Cum differentia secunda nil aliud sit nisi differentia differentiae primae, differentiale quoque secundum seu, uti saepe vocari solet, differentio differentiale nil aliud erit praeter differentiale differentialis primi. Quia deinde quantitas constans nulla neque augmenta neque decrementa accipit nullasque admittit differentias, quippe quae solis quantitatibus variabilibus sunt propriae, dicimus eodem sensu quantitatum constantium differentialia omnia cuiusque ordinis esse = 0, hoc est prae omnibus adeo potestatibus ipsius *dx* evanescere. Cum igitur differentiale ipsius *dx*, hoc est *ddx*, sit $= 0$, differentiale *dx* tanquam quantitas constans considerari potest, et quoties differentiale cuiuspiam quantitatis dicitur constans, toties ea quantitas intelligenda est continuo aequalia incrementa accipere. Sumimus hic autem *x* pro ea quantitate, cuius differentiale sit constans, sicque singularum eius functionum variabilitatem, cui earum differentialia sunt obnoxia, aestimabimus.

127. Ponamus differentiale primum ipsius *y* esse $= pdx$ atque ad eius differentiale secundum inveniendum ipsius *pdx* denuo differentiale quaeri debet. Cum autem *dx* sit constans neque varietur, etiamsi loco *x* scribatur $x + dx$, tantum opus est, ut quantitatis finitae *p* differentiale quaeratur; sit igitur $dp = qdx$, quoniam vidimus omnium functionum ipsius *x* differentialia ad huiusmodi formam revocari; et cum sit, uti de differentiis finitis ostendimus, differentiale ipsius $np = nqdx$, si *n* sit quantitas constans, ponatur *dx* loco *n* eritque differentiale ipsius $pdx = qdx^2$. Hanc ob rem, si sit $dy = pdx$ et $dp = qdx$, erit differentiale secundum $ddy = qdx^2$ sicque constat, quod iam ante innuimus, differentiale secundum ipsius y ad dx^2 habere rationem finitam.

128. In capite primo iam notavimus differentias secundas atque sequentes constitui non posse, nisi valores successivi ipsius x certa quadam lege progredi assumantur; quae lex cum sit arbitraria, his valoribus progressionem arithmeticam tanquam facillimam simulque aptissimam tribuimus. Ob eandem ergo rationem de differentialibus secundis nihil certi statui poterit, nisi differentialia prima, quibus quantitas variabilis x continuo crescere concipitur, secundum datam legem progrediantur; ponimus itaque differentialia prima ipsius x, nempe dx, dx^{I}, dx^{II} etc., omnia inter se aequalia, unde fiunt differentialia secunda

$$ddx = dx^{\mathrm{I}} - dx = 0, \quad ddx^{\mathrm{I}} = dx^{\mathrm{II}} - dx^{\mathrm{I}} = 0 \text{ etc.}$$

Quoniam ergo differentialia secunda et ulteriora ab ordine, quem differentialia quantitatis variabilis x inter se tenent, pendent hicque ordo sit arbitrarius, quae conditio differentialia prima non afficit, hinc ingens discrimen inter differentialia prima ac sequentia ratione inventionis intercedit.

129. Quodsi autem successivi ipsius x valores $x, x^{\mathrm{I}}, x^{\mathrm{II}}, x^{\mathrm{III}}, x^{\mathrm{IV}}$ etc. non secundum arithmeticam progressionem statuantur, sed alia quacunque lege progredi ponantur, tum eorum quoque differentialia prima $dx, dx^{\mathrm{I}}, dx^{\mathrm{II}}$ etc. non erunt inter se aequalia neque propterea erit $ddx = 0$. Hanc ob rem differentialia secunda quarumvis functionum ipsius x aliam formam induent; si enim huiusmodi functionis y differentiale primum fuerit $= pdx$, ad eius differentiale secundum inveniendum non sufficit differentiale ipsius p per dx multiplicasse, sed insuper ratio differentialis ipsius dx, quod est ddx, haberi debet. Quoniam enim differentiale secundum oritur, si pdx a valore eius sequente, qui oritur, dum $x + dx$ loco x et $dx + ddx$ loco dx ponitur, subtrahatur, ponamus valorem ipsius p sequentem esse $= p + qdx$ eritque ipsius pdx valor sequens

$$= (p + qdx)(dx + ddx) = pdx + pddx + qdx^2 + qdxddx;$$

a quo subtrahatur pdx eritque differentiale secundum

$$ddy = pddx + qdx^2 + qdxddx = pddx + qdx^2,$$

quia $qdxddx$ prae $pddx$ evanescit.

130. Quanquam autem ratio aequalitatis est simplicissima atque aptissima, quae continuo ipsius x incrementis tribuatur, tamen frequenter evenire solet, ut non eius quantitatis variabilis x, cuius y est functio, incrementa aequalia assumantur, sed alius cuiuspiam quantitatis, cuius ipsa x sit functio quaedam. Quin etiam saepe eiusmodi alius quantitatis differentialia prima statuuntur aequalia, cuius nequidem relatio ad x constet. Priori casu pendebunt differentialia secunda et sequentia ipsius x a ratione, quam x tenet ad illam quantitatem, quae aequabiliter crescere ponitur, ex eaque pari modo definiri debent, quo hic differentialia secunda ipsius y ex differentialibus ipsius x definire docuimus. Posteriori autem casu differentialia secunda et sequentia ipsius x tanquam incognita spectari eorumque loco signa ddx, d^3x, d^4x etc. usurpari debebunt.

131. Cum autem, quemadmodum his casibus differentiationes singulas absolvi oporteat, infra fusius simus ostensuri, hic pergamus quantitatem variabilem x tanquam uniformiter crescentem assumere, ita ut eius differentialia prima dx, dx^{I}, dx^{II} etc. inter se omnia aequalia ac propterea differentialia secunda ac sequentia nihilo aequalia statuantur; quae conditio ita enunciari solet, ut differentiale ipsius x, nempe dx, constans assumi dicatur. Sit deinde y functio quaecunque ipsius x; quae cum per x et constantes definiatur, singula quoque eius differentialia prima, secunda, tertia, quarta etc., quae his signis indicantur dy, ddy, d^3y, d^4y etc., per x et dx exprimi poterunt. Scilicet si in y loco x scribatur $x+dx$ ab hocque valore prior subtrahatur, remanebit differentiale primum dy; in quo si porro loco x ponatur $x+dx$, prodibit dy^{I} eritque $ddy = dy^{\mathrm{I}} - dy$; simili modo ponendo $x+dx$ loco x ex ddy nascetur ddy^{I} atque $ddy^{\mathrm{I}} - ddy$ dabit d^3y et ita porro; in quibus operationibus differentiale dx perpetuo tanquam quantitas constans spectatur, quae nullum differentiale recipiat.

132. Ex ratione, qua functio y per x determinatur, tam ope methodi differentiarum finitarum quam multo expeditius ex iis, quae postea sumus tradituri, definietur valor functionis p, quae per dx multiplicata praebeat differentiale primum dy. Posito ergo $dy = pdx$ differentiale ipsius pdx dabit differentiale secundum ddy; unde, si fuerit $dp = qdx$, ob dx constans orietur $ddy = qdx^2$, uti iam ante ostendimus. Ulterius igitur progrediendo, cum differentialis secundi differentiale praebeat differentiale tertium, ponamus esse $dq = rdx$ eritque $d^3y = rdx^3$; simili modo, si huius functionis r differentiale quaeratur fueritque $dr = sdx$, habebitur differentiale quartum $d^4y = sdx^4$ sicque porro, dummodo noverimus differentiale primum cuiusque functionis invenire, differentiale cuiusque ordinis assignare poterimus.

133. Quo igitur formae singulorum horum differentialium simulque ratio ea inveniendi clarius menti repraesentetur, ea sequenti tabella complecti visum est. Si y fuerit functio quaecunque ipsius x,

erit	atque posito
$dy = pdx$	$dp = qdx$
$ddy = qdx^2$	$dq = rdx$
$d^3y = rdx^3$	$dr = sdx$
$d^4\ y = sdx^4$	$ds = tdx$
$d^5y = tdx^5$	etc.

Cum igitur functio p ex functione y per differentiationem cognoscatur similique modo ex p inveniatur q hincque porro r et ex eo ulterius s etc., differentialia cuiusvis ordinis ipsius y facile reperientur, dummodo differentiale dx assumatur constans.

134. Cum p, q, r, s, t etc. sint quantitates finitae, functiones nimirum ipsius x, differentiale primum ipsius y rationem finitam habebit ad differentiale primum ipsius x, scilicet ut p ad 1, hancque ob causam differentialia dx et dy vocantur homogenea. Deinde cum ddy ad dx^2 habeat rationem finitam ut q ad 1, erunt ddy et dx^2 homogenea; simili modo homogenea erunt d^3y et dx^3 itemque $d^4\ y$ et dx^4 et ita porro. Unde uti differentialia prima sunt inter se homogenea seu rationem finitam tenentia, sic differentialia secunda cum quadratis differentialium primorum, differentialia autem tertia cum cubis differentialium primorum atque ita porro erunt homogenea. Atque generatim differentiale ipsius y ordinis n, quod ita exprimitur $d^n y$, homogeneum erit cum dx^n, hoc est cum potestate differentialis dx, cuius exponens est n.

135. Cum igitur prae dx evanescant omnes eius potestates, quarum exponentes sunt unitate maiores, prae dy quoque evanescent dx^2, dx^3, dx^4 etc., et quae ad has potestates rationem finitam tenent differentialia altiorum ordinum ddy, d^3y, d^4y etc. Simili modo prae ddy, quia est homogeneum cum dx^2 omnes ipsius dx potestates quadrato superiores dx^3, dx^4 etc. evanescent; evanescent ergo quoque d^3y, d^4y etc. Atque prae d^3y evanescent dx^4, d^4y, dx^5, d^5y etc. Hincque facile, si propositae fuerint quaecunque expressiones huiusmodi differentialia involventes, dignosci poterunt, utrum sint homogeneae necne. Respici enim debebunt tantum differentialia omissis quantitatibus finitis, quippe quae homogeneitatem non turbant; atque pro differentialibus secundi altiorumque ordinum scribantur potestates ipsius dx ipsis homogeneae; quae si praebeant ubique eundem dimensionum numerum, expressiones erunt homogeneae.

136. Ita patebit has expressiones $Pddy^2$ et $Qdyd^3y$ esse inter se homogeneas. Nam ddy^2 denotat quadratum ipsius ddy, et quia ddy homogeneum est cum dx^2, erit ddy^2 homogeneum cum dx^4. Deinde quia dy cum dx et d^3y cum dx^3 homogeneum est, erit productum dyd^3y cum dx^4 homogeneum; ex quo sequitur expressiones $Pddy^2$ et $Qdyd^3y$ inter se esse homogeneas ideoque rationem inter se finitam habere. Simili modo colligetur has expressiones

$$\frac{Pd^3y^2}{dxddy} \text{ et } \frac{Qd^5y}{dy^2}$$

esse homogeneas; substitutis enim pro dy, ddy, d^3y et d^5y his ipsius dx potestatibus ipsis homogeneis dx, dx^2, dx^3 et dx^5 orientur hae expressiones Pdx^3 et Qdx^3, quae utique erunt inter se homogeneae.

137. Quodsi facta hac reductione expressiones propositae non contineant aequales ipsius dx potestates, tum non erunt homogeneae neque propterea inter se rationem finitam tenebunt. Erit ergo altera infinities sive maior sive minor altera hincque una respectu alterius evanescet. Sic $\frac{Pd^3y}{dx^2}$ ad $\frac{Qddy^2}{dy}$ rationem habebit infinite magnam; prior enim expressio reducitur ad Pdx et altera ad Qdx^3, unde haec prae illa evanescet. Quamobrem si in quopiam calculo aggregatum huiusmodi binarum formularum occurrat

$$\frac{Pd^3y}{dx^2}+\frac{Qddy^2}{dy},$$

posterior terminus prae priori tuto reiici solusque primus $\frac{Pd^3y}{dx^2}$ in calculo retineri poterit; subsistet enim perfecta ratio aequalitatis inter expressiones

$$\frac{Pd^3y}{dx^2}+\frac{Qddy^2}{dy} \text{ et } \frac{Pd^3y}{dx^2},$$

quia exponens rationis est

$$1+\frac{Qdx^2ddy^2}{Pdyd^3y}=1 \text{ ob } \frac{Qdx^2ddy^2}{Pdyd^3y}=0.$$

Hocque pacto expressiones differentiales quandoque mirifice contrahi possunt.

138. In calculo differentiali praecepta traduntur, quorum ope cuiusvis quantitatis propositae differentiale primum inveniri potest; et quoniam differentialia secunda ex differentiatione primorum, tertia per eandem operationem ex secundis et ita porro sequentia ex praecedentibus reperiuntur, calculus differentialis continet methodum omnia cuiusque ordinis differentialia inveniendi. Ex voce autem *differentialis*, qua differentia infinite parva denotatur, alia nomina derivantur, quae usu sunt recepta. Sic verbum habetur *differentiare*, quod significat *differentiale invenire*, quantitasque *differentiari* dicitur, quando eius differentlale elicitur. *Differentiatio* autem denotat operationem, qua differentialia inveniuntur. Hinc calculus differentialis quoque vocatur methodus *differentiandi*, cum modum differentialia inveniendi contineat.

139. Quemadmodum in calculo differentiali cuiusvis quantitatis differentiale investigatur, ita vicissim calculi species constituitur quoque in inventione eius quantitatis, cuius differentiale proponitur, qui *calculus integralis* vocatur. Si enim propositum fuerit differentiale quodcunque, eius respectu ea quantitas, cuius est differentiale, vocari solet *integrale*. Cuius denominationis ratio est, quod, cum differentiale considerari possit tanquam pars infinite parva, qua quantitas quaepiam crescit, ipsa illa quantitas respectu huius partis tanquam totum seu integrum spectari potest hancque ob causam eius vocatur integrale. Sic, cum dy sit differentiale ipsius y, vicissim y erit integrale ipsius dy, et cum ddy sit differentiale ipsius dy, erit dy integrale ipsius ddy. Similique modo erit ddy integrale ipsius d^3y et d^3y ipsius d^4y et ita porro; unde quaelibet differentiatio, si inverse spectatur, integrationis exemplum exhibet.

140. Origo et natura integralium pariter ac differentialium clarissime ex differentiarum finitarum doctrina in capite primo exposita explicari potest. Postquam enim esset ostensum, quomodo cuiusque quantitatis differentiam inveniri oporteat, retrogrediendo quoque monstravimus, quomodo, si proposita fuerit differentia, ea quantitas inveniri queat, cuius illa sit differentia; quam quantitatem respectu suae differentiae vocavimus eius summam. Uti igitur ad infinite parva procedendo differentiae in differentialia abierunt, ita summae, quae ibi erant vocatae, integralium nomen sortiuntur et hanc ob causam integralia quoque non raro summae appellari solent. Angli, qui differentialia fluxiones nominant, integralia vocant quantitates fluentes eorumque loquendi more datae fluxionis fluentem invenire idem est, quod nostro more dati differentialis integrale invenire dicimus.

141. Uti differentialia charactere d designamus, ita ad integralia indicanda hac littera $\int$ utimur, quae ergo quantitatibus differentialibus praefixa eas denotabit quantitates, quarum illa sunt differentialia. Sic, si differentiale ipsius y fuerit pdx seu $dy = pdx$, erit y integrale ipsius pax, quod hoc modo scribitur $y = pdx$, cum sit $y = \int dy$. Integrale ergo ipsius pdx, quod per $\int pdx$ indicatur, denotat quantitatem, cuius differentiale est pdx. Simili modo, cum sit $ddy = qdx^2$ existente $dp = qdx$, erit integrale ipsius ddy, hoc est dy,$= pdx$ atque ob $p = \int qdx$ erit $dy = dx\int qdx$ ac propterea $y = \int dx\int qdx$. Si ulterius sit $dq = rdx$, erit $q = \int rdx$ et $dp = \int dx\int rdx$, unde, si character $\int$ denuo praefigatur, fiet $p = \int dx\int rdx$ porroque $dy = dx\int dx\int rdx$ atque $y = \int dx\int dx\int rdx$.

142. Quia differentiale *dy* est quantitas infinite parva, eius integrale autem *y* quantitas finita parique modo differentiale secundum *ddy* infinities minus est quam eius integrale *dy*, manifestum est differentialia prae suis integralibus evanescere. Quae affectio quo melius percipiatur, infinite parva in ordines dividi solent diciturque infinite parvum primi ordinis, ad quod referuntur differentialia prima *dx*, *dy*; infinite parvum secundi ordinis complectitur differentialia secundi ordinis, quae homogenea sunt cum dx^2 ; similique modo infinite parva, quae cum dx^3 sunt homogenea, vocantur ordinis tertii, ad quem ergo pertinent differentialia tertia omnia, sicque porro. Unde uti infinite parva primi ordinis prae quantitatibus finitis evanescunt, sic infinite parva secundi ordinis prae infinite parvis primi ordinis atque generatim infinite parva cuiusque ordinis altioris prae infinite parvis ordinis inferioris evanescent.

143. His igitur infinite parvorum ordinibus constitutis uti differentiale quantitatis finitae est infinite parvum primi ordinis atque differentiale infinite parvi primi ordinis est infinite parvum secundi ordinis et ita porro, ita vicissim manifestum est integrale infinite parvi primi ordinis esse quantitatem finitam, integrale autem infinite parvi secundi ordinis esse infinite parvum primi ordinis sicque deinceps. Quare si differentiale propositum fuerit infinite parvum ordinis *n*, eius integrale erit infinite parvum ordinis $n-1$; hincque uti differentiando ordo infinite parvorum augetur, ita integratione ad ordines inferiores progredimur, donec ad ipsas quantitates finitas perveniamus. Sin autem quantitates finitas denuo integrare velimus, tum secundum hanc legem perveniemus ad quantitates infinite magnas ab harumque integratione instituta ad quantitates adhuc infinities maiores sicque progrediendo obtinebimus similes infinitorum ordines, quorum quisque praecedentem infinities superat.

144. Superest, ut in hoc capite quaedam de usu signorum recepto moneamus, ne ambiguitati ullus locus relinquatur. Ac primo quidem signum differentiationis *d* tantum afficit litteram immediate sequentem solam; sic *dxy* non denotat differentiale producti *xy*, sed differentiale ipsius *x* per ipsam quantitatem *y* multiplicatum. Solet autem, quo minus confusio nascatur, quantitas *y* ante signum *d* hoc modo scribi *ydx*, quo productum ex *y* in *dx* indicatur. Attamen si *y* sit quantitas vel signum radicale $\sqrt{\ }$ vel logarithmicum habens praefixum, tum post differentiale poni solet; nimirum $dx\sqrt{(aa-xx)}$ significat productum ex quantitate finita $\sqrt{(aa-xx)}$ in differentiale *dx* similique modo $dxl(1+x)$ est productum ex logarithmo quantitatis $1+x$ per *dx* multiplicato. Ob eandem rationem $ddy\sqrt{x}$ exprimit productum differentialis secundi *ddy* et quantitatis finitae $\sqrt{x}$.

145. Neque vero signum *d* litteram immediate sequentem solam afficit, sed etiam nequidem exponentem, si quem habet, spectate ita dx^2 non exprimit differentiale ipsius x^2 , sed quadratum differentialis ipsius *x*, ita ut exponens 2 non ad *x*, sed ad *dx* referri debeat. Posset etiam scribi *dxdx*, quemadmodum productum duorum differentialium *dx* et *dy* hoc modo

$dxdy$ exponitur; verum prior modus dx^2, uti est brevior, ita usitatior. Praesertim si altiores potestates ipsius dx essent indicandae, nimis prolixum foret dx toties repeti; sic dx^3 denotat cubum ipsius dx et in differentialibus altiorum ordinum similis ratio observatur. Scilicet ddy^4 denotat potestatem quartam differentialis secundi ordinis ddy atque $d^3y^2\sqrt{x}$ significat quadratum differentialis tertii ordinis ipsius y multiplicatum esse per $\sqrt{x}$; sin autem per quantitatem rationalem x multiplicari deberet, ea praefigitur hoc modo xd^3y^2.

146. Sin autem velimus, ut signum d plus quam solam litteram subsequentem afficiat, id peculiari modo indicari debet. Utimur hoc casu praecipue uncinulis, quibus ea quantitas includitur, cuius differentiale debet indicari; uti $d(xx+yy)$ denotat differentiale quantitatis $xx+yy$. Verum si velimus differentiale potestatis huiusmodi quantitatis designare, ambiguitatem vix evitare possumus; si enim scribamus $d(xx+yy)^2$, intelligi posset quadratum ipsius $d(xx+yy)$. Poterimus autem hoc casu punctum in auxilium vocare, ita ut $d.(xx+yy)^2$ denotet differentiale ipsius $(xx+yy)^2$, omisso autem puncto $d(xx+yy)^2$ quadratum ipsius $d(xx+yy)$. Puncto scilicet commode indicari potest signum d ad totam quantitatem post punctum sequentem pertinere; sic $d.xdy$ exprimet differentiale ipsius xdy et $d.^3xdy\sqrt{(aa+xx)}$ differentiale tertii ordinis expressionis $xdy\sqrt{(aa+xx)}$, quae est productum ex quantitatibus finitis x et $\sqrt{(aa+xx)}$ atque ex differentiali dy.

147. Quemadmodum autem signum differentiationis d solam quantitatem immediate sequentem afficit, nisi puncto interposito eius vis ad totam expressionem sequentem extendatur, ita contra signum integrationis $\int$ semper totam expressionem, cui est praefixum, complectitur. Ita $\int ydx(aa-xx)^n$ denotat integrale seu eam quantitatem, cuius differentiale est $ydx(aa-xx)^n$, atque haec expressio $\int xdx\int dxlx$ denotat quantitatem, cuius differentiale est $xdx\int dxlx$. Hinc si velimus productum duorum integralium, scilicet $\int ydx$ et $\int zdx$, exprimere, id hoc modo $\int ydx\int zdx$ perperam fiet; intelligeretur enim integrale quantitatis $ydx\int zdx$. Hanc ob causam iterum puncto solet haec ambiguitas tolli, ita ut $\int ydx\cdot\int zdx$ significet productum integralium $\int ydx$ et $\int zdx$.

148. Analysis infinitorum igitur cum in differentialibus tum in integralibus inveniendis versatur et hanc ob rem in duas praecipuas partes dividitur, quarum altera vocatur Calculus differentialis, altera Calculus integralis. In priori praecepta traduntur quantitatum quarumvis differentialia inveniendi; in posteriori vero via monstratur differentialium propositorum integralia investigandi; in utroque autem simul summus usus, quem isti calculi tam ad ipsam analysin quam ad geometriam sublimiorem afferunt, indicatur. Quam ob causam ista analyseos pars iam tanta accepit incrementa, ut modico volumine prorsus comprehendi nequeat. Imprimis vero in calculo integrali in dies tam nova artificia integrandi quam adiumenta eius in solvendis varii generis problematibus deteguntur, ut ob haec nova inventa, quae continuo accedunt, nunquam exhauriri, multo minus perfecte describi atque explicari possit. Dabo autem operam, ut, quae adhuc sunt reperta, vel cuncta in his libris exponam vel saltem methodos explicem, unde ea facile deduci queant.

149. Solent vulgo plures analyseos infinitorum partes numerari; praeter calculos enim differentialem et integralem inveniuntur passim calculi differentio- differentialis atque exponentialis. In calculo differentio-differentiali tradi solet methodus differentialia secundi atque altiorum ordinum inveniendi; quoniam autem modum cuiusque ordinis differentialia inveniendi in ipso calculo differentiali sum expositurus, hac subdivisione, quae potius ex merito inventionis quam ex re ipsa facta esse videtur, supersedebimus. Quod deinde ad calculum exponentialem attinet, quo celeb. JOH. BERNOULLI, cui ob innumera eaque maxima incrementa analyseos infinitorum aeternas debemus gratias, methodos differentiandi atque integrandi ad quantitates exponentiales transtulit, quia utrumque calculum ad omnis generis quantitates tam algebraicas quam transcendentes accommodare constitui, hinc partem peculiarem facere superfluum atque instituto contrarium foret.

150. Primum igitur calculum differentialem in hoc libro pertractare statui modumque sum expositurus, cuius ope omnium quantitatum variabilium differentialia non solum prima, sed etiam secunda et altiorum ordinum expedite inveniri queant. Primum ergo quantitates algebraicas contemplabor, sive sint functiones unius variabilis sive plurium, sive demum explicite dentur sive per aequationes. Deinde inventionem differentialium quoque accommodabo ad quantitates non algebraicas, ad quarum notitiam quidem sine calculi integralis subsidio pervenire licet; cuiusmodi sunt logarithmi atque quantitates exponentiales, deinde etiam arcus circuli vicissimque arcuum circularium sinus et tangentes. Denique etiam quantitates utcunque ex his compositas et permixtas differentiare docebo sicque calculi differentialis pars prior, methodus scilicet differentiandi, absolvetur.

151. Altera pars usui, quem methodus differentiandi tam ad analysin quam geometriam sublimiorem affert, explicando est destinata. In algebram autem communem inde plurima redundant commoda, partim ad radices aequationum inveniendas, partim ad series tractandas atque summandas, partim ad maxima minimaque eruenda, partim ad valores expressionum, quae certis casibus indeterminatae videantur, definiendos et quae sunt alia. Geometria autem sublimior ex calculo differentiali maxima accepit incrementa, dum eius ope tangentes linearum curvarum earumque curvatura ipsa mira facilitate definiri multaque alia problemata circa radios a lineis curvis vel reflexos vel refractos resolvi possunt. Quibus etsi amplissimus tractatus impleri posset, tamen conabor, quantum fieri licet, omnia breviter ac perspicue explicare.