CHAPTER III

CONCERNING INFINITE AND INFINITELY SMALL QUANTITIES

72. Since a whole quantity, however great it shall be, shall be able to be increased further, nor may anything oppose that some other quantity of the same kind be added to that quantity, the whole quantity also will be able to be increased without limit ; nor indeed at any time may it become so great, that nothing further may be possible to be added to it. Therefore no quantity is given so great, which cannot be considered greater, and hence it will be placed beyond doubt that *every quantity can be increased to infinitely*. Anyone who indeed may deny this, is forced to affirm to be given a limit, which when it is reached, the quantity is unable to surpass, and thus a quantity must be put in place, to which nothing further is possible to be added; since which shall be absurd and may be opposed to the notion of quantity, by necessity it is to be agreed every quantity is able to be continually increased without end, that is to infinity.

73. In individual kinds of quantities also this may be examined more clearly. Thus nobody may easily find, who puts in place the series of natural numbers 1, 2, 3, 4, 5, 6 etc. to be determined thus at any time, so that it may not be possible to continue further. Indeed no number may be given, to which one cannot be added above and thus the following number is able to be shown greater ; hence the series of natural numbers is progressing without end nor at any time does it reach a maximum number, so that in short it cannot be made greater. In a similar manner a straight line cannot be produced to that point, so that it cannot be made any further. From which it prevails that both numbers as well as lines are able to be increased indefinitely. Which since they shall be examples of quantities, likewise it is understood for any quantity, however great it shall be, still to be given more and with this more anew and thus by increasing continually without limit, this is able to proceed to infinity.

74. But although so far these are evident, so that anyone who might wish to deny those must contradict himself by these, yet this teaching of the infinite by several people, who have tried to explain that [idea], has been obscured and enveloped with such difficulties and also with such contradictions, that no way may be apparent by which they may extricate themselves. From that idea, because a quantity may be able to increased to infinity, certain have inferred an actual infinite quantity to be given and thus they have described that quantity, so that no further increase will be able to be undertaken. But by this means they destroy this idea of the quantity itself [as Euler considers a finite increasing quantity], while they put in place an infinite quantity of this kind, which cannot be increased further. Besides indeed they dispute with themselves over admitting the idea of infinity itself; for while they make the increments finite, by which a quantity shall be capable of increase, yet at the same time they deny that it is possible for a quantity to increase without limit ; hence likewise they deny that a quantity can be increased to infinity, because the argument has agreed with each idea; and thus, while they put in place an infinite quantity, likewise they take it away. If indeed a quantity cannot be increased without end, that is to infinity, certainly no infinite quantity will be able to exist.

75. Hence therefore from that [argument] itself, because the whole quantity may increase to infinity, it may seem to follow that no infinite quantity be given. For a quantity does not emerge increased to infinity by being increased by continuous increments, unless now it may increase without limit; but because now it must become without limit, that now cannot be considered as done. Yet meanwhile not only a quantity of this kind may be arrived at by increments piled up without end, (to be indicated by a certain character and thus in due course allowed to be lead into the calculation, as we may show further soon), but also in a world of this kind cases exist or at any rate can be considered, in which an infinite number actually may be seen to exist. Thus, if matter may be infinitely divisible, as several philosophers maintain, the number of parts, from which each crumb of matter may consist, will actually be infinite ; for if it may be decided to be finite, the matter certainly cannot be divisible indefinitely. In a similar manner, if the whole universe were infinite, as would satisfy many, the number of bodies composing the universe certainly would not be finite and therefore also to become infinite.

76. Even if these may seem to be in dispute amongst themselves, yet, if they may be examined more carefully, they can be freed from all inconveniences. For anyone who decides matter to be infinitely divisible, denies the continual division of matter at any time to have reached parts so small, that they are unable to be divided further; therefore the matter will have no further individual parts, since the individual particles, to which by continual division now shall have been reached, they may be extended to subdivide themselves further. Therefore one who says in this case that the number of parts to be infinite , he means these final parts which may not be divisible further ; to which since at no time may they be reached and which shall be therefore zero, he tries to count these parts themselves which are zero. For if matter can be subdivided continually without end, it is completely without the indivisible or the simple parts nor indeed do any exist which are able to be counted. On account of this , who puts in place matter to be infinitely divisible, he likewise denies that matter is composed from simple parts.

77. But if now, while we speak about any parts of bodies or matter, not the final or most simple parts, which we may understand obviously are nothing, but these, which actually it has produced on division, then, from this for the hypothesis concerning the infinite division of matter, even each smallest crumb of matter not only is to be cut into many parts, but also no number can be assigned however great, from which no greater number of the cuts is able to be shown from that crumb. Therefore the number of parts indeed not of the final, but which themselves at this time shall be divisible further, which make up a single body, all will be greater than an assignable number. In a similar manner, if the whole universe shall be infinite, the number of bodies making up the universe equally will exceed all to be assigned ; which since it cannot be finite, there follows an infinite number and the whole greater number to be assigned to be synonymous with the name.

78. Therefore anyone who may consider the infinite divisibility of matter in this manner, with no disadvantages, which generally are ascribed to this belief, is himself forced to involve and to affirm nothing which may be against sound reasoning. But those, who argue against the infinite divisibility of matter, move towards the greatest difficulties, from which no one is able to extricate themselves completely in any manner. For they force a single body to be set up not only able to be divided into a certain number of parts, to which if it should come about, no place can be found for further division; which final particles are called by some *atoms*, by others *monads*, and still others *simple entities*. But why these ultimate particles allow no further division, there may be two causes : the one, because they are all without strain ; the other, because indeed they shall be under strain, but yet so hard and prepared thus, so that no force may be sufficient to cut them apart. Whichever the advocates of this opinion may say, they themselves are equally involved with difficulties. [The interested reader may wish to refer in particular to Ch. 2 of Euler's *An Introduction to Natural Science, Establishing the Fundamentals....*, from his *Opera Postuma.* Translated from German by E.Hirsch. (E842), and presented on this website.]

79. Indeed the ultimate particles shall be free from all extension [*i.e.* finite size], thus so that they may be completely without parts ; from which analysis indeed the idea of simple entities may be best held. But, just as a body may be able to be constructed from a finite number of particles of this kind, in no manner can it be conceived. We may put a cubic foot to be composed from a thousand simple entities of these kind and hence actually to be cut into a thousand parts; which if they shall be equal, will be of a cubic digit [literally 'a finger width', and so roughly a cubic inch]; but if they should be unequal, some would will be greater, others less. Therefore one cubic digit becomes the entity [*i.e.* something having existence] and thus a great contradiction may result, unless perhaps in the cubic digit there is only one simple entity and the remainder of the space they [*i.e.* such philosophers] may wish to call a vacuum; but truly in this manner they remove the continuation of the body, apart from the fact that these philosophers clearly overthrow the vacuum from the universe. But if they may object that the number of simple entities which constitute a cubic foot of matter to be greater than a thousand, they gain entirely nothing; indeed the disadvantage which follows from the number one thousand, may remain equally with some other number of whatever greater size. Leibniz who was the sharpest of men, and the first inventor of monads, understood this difficulty correctly, as long as matter is considered to be absolutely divisible indefinitely. Nor therefore before monads were come upon was it allowed that a body actually could be divisible indefinitely. But by monads the existence of simple entities, by which bodies may exist, is completely removed; for anyone who denies bodies to be composed from simple entities, and those who consider bodies to be divisible indefinitely, are in complete agreement.

80. But nor do these philosophers themselves agree more, if they say the final particles of bodies indeed to be extended, but on account of great hardness they cannot be torn into parts. For since they allow the first extension in the ultimate particles, these are to be considered to be composed from parts, which whither or not they are able to be separated in turn from each other, actually matters little, even if they can give no reason, by which such hardness shall have arisen. But now most people, who deny the infinite divisibility of matter, may be seen to have considered this latter inconvenience well enough, because in the first place they adhere chiefly to the idea of the ultimate particles of matter ; and these difficulties they are unable to remove in any other way apart from distinguishing some smaller and lighter metaphysical particle, which part they stretch the most there, so that neither from the consequences, which are formed following mathematical principles, nor we may trust, do they involve dimensions required to be given from the simple parts. But first they ought to demonstrate these ultimate parts themselves, of which the number determined may constitute the body, with no extension to be present at all.

[One may know that around this time Benjamin Franklin had estimated the length of a soap molecule by measuring the area a drop of soapy water made in a smooth fresh water pond, assuming it forms a layer one molecule thick, now a well-known elementary experiment involving oleic acid and lycopodium powder.]

81. Therefore since no exit can be found from this labyrinth nor from the objections arising from the manner in which they can occur, they take refuge in the corresponding distinctions that these objections suffice from the senses and the imagination, but in this matter only the pure intellect is required to be used, but sense and reasoning depending on this most often fail. Clearly from the pure intellect is can be recognised, that the thousandth part of the cubic foot of matter may be without the whole extension, which may seem absurd to the imagination. Then truly, because the senses may fail on many occasions, the matter indeed is, that nothing can be put in place other than by mathematics. For mathematics defends us in the first place from the failings of the senses and teaches that objects, which are perceived by the senses, actually are to be prepared otherwise, so that they appear different; and this most guiding knowledge handles precepts, which those who follow are immune to illusions of the senses. Therefore it is only by being freed from answers of this kind, that the metaphysicians may protect their principles, so that they may prove that which is rather more doubtful.

82. Truly so that we may return to the proposition, even if from which it may be denied infinite numbers actually exist in the universe, yet in mathematical speculations the questions occur, to which, unless the infinite number may be admitted, it may not be able to answer. Thus, if the sum is sought of all the numbers which constitute this series $1+2+3+4+5+$ etc., because these numbers are progressing without end and increasing, the sum of all of these certainly is not able to be finite; from which itself it is proved that it is infinite. Hence, which quantity is so great, so that it shall be greater than all finite quantities, that is unable not to be infinite. To a quantity of this kind the mathematicians are accustomed to designate by this sign ∞, by which there may be denoted or to be assigned a quantity greater than all finite quantities. Thus, since the parabola thus is able to be defined, so that it is said to be an ellipse infinitely long, we are able to correctly affirm the axis of the parabola to be an infinite right line.

83. But this principle of the infinite will be illustrated more, if we may establish what shall be the infinitely small of the mathematicians. Moreover there is no doubt, why the whole of a quantity cannot be diminished to that point, as long as it may vanish completely and become nothing. But an infinitely small quantity is nothing other than a vanishing quantity and thus may be returned $=0$. Also that agrees with the definition of the infinitely small, which are said to be assigned to all quantities smaller than a given quantity; if indeed the quantity were so small that all assignable quantities shall be less, that certainly cannot avoid being nothing ; if indeed unless it should be $=0$, a quantity might be able to be assigned equal to that, which is contrary to the hypothesis. Therefore if it is required to find a quantity which shall be infinitely small in mathematics, we may respond that actually it is $=0$; nor therefore do as many mysteries lie hidden in this idea, as many commonly are considered to be and which in several return a very doubtful calculation of the infinitely small, yet meanwhile the doubts if such remain will be taken away completely in the following, where we are about to examine this calculation.

84. Therefore since we have shown an infinitely small quantity actually to be zero, at first it is required to meet the objection, why we may not designate infinitely small quantities always be the same character 0, but we use peculiar notations in designating these. Because indeed all zero quantities are equal to each other, it may seem superfluous to denote these by different signs. Truly and any two ciphers thus are equal to each other, so that the difference of these shall be zero, since there shall be two ways of comparison, the one arithmetic, the other geometric, the difference of which from that, truly this amount, we consider to arise from a comparison of the quantities, indeed the arithmetical account between any two ciphers is equality, but not truly the geometrical account. This may be seen easily from this geometric proportion $2:1=0:0$, in which the fourth term is $=0$ as of the third. But from the nature of proportion, since the first term shall be twice as large as the second, it is necessary also that the third shall be twice as large as the fourth.

85. But these are most obvious in common arithmetic also ; for it is observed in general a cipher multiplied by any number gives a cipher and to be $n\cdot 0=0$ and thus to become $n:1=0:0$. From which it may appear to happen, that any two ciphers may hold the same geometric ratio between each other, even if the matter to be considered arithmetically always gives the same ratio of this which shall be of equality. Therefore since any ratio will be possible to exist between the ciphers, to indicate this diversity in a purposeful manner various characters are used, especially then, since the geometric ratio, which the ciphers hold between each other is to be investigated. But in the calculation of the infinitely small nothing other is done, so that unless the geometric ratio between the various infinitely small parts may be investigated, so that therefore the matter, unless we may use it with these different signs to be indicated, might slide into great confusion nor be able to be extricated in any manner.

86. If therefore, as the manner of indicating quantities has been adopted in the infinitesimal analysis, dx may denote an infinitesimally small quantity, certainly there will be both $dx = 0$ as well as $adx = 0$ with a denoting some finite quantity. Yet from this without opposition there will be the finite geometric ratio $adx : dx$, certainly as $a:1$, and that on account of the matter these two infinitely small parts dx and adx, even if each shall be $= 0$, cannot be combined together, if indeed the ratio of these may be investigated. In a similar manner, if different infinitely small parts are present dx and dy, even if each shall be $= 0$, yet the ratio of these may not be exist. And the strength of the differential calculus is about the investigation of the ratio between any two infinitely small parts of this kind . But the uses of this comparison, even if from first consideration it may seem very small, yet it may be seized upon most fully and may be elucidated more at this point in time.

87. Therefore since the infinitely small actually shall be zero, it may be apparent that a finite quantity neither be increased or diminished, if to that we may either add or subtract an infinitely small part. Let a be a finite quantity and dx be infinitely small; then both $a + dx$ as well as $a - dx$ and generally $a \pm ndx = a$. For if we may consider the relation between $a + ndx$ and a arithmetically or geometrically, in each case a ratio of equality can be seized upon. Indeed the arithmetic ratio is evidently equal; since indeed there shall be $ndx = 0$, therefore

$$a \pm ndx - a = 0;$$

now the ratio of the geometric equality may be apparent thus, which shall be

$$\frac{a \pm ndx}{a} = 1.$$

Hence the rule taken from that especially follows, so that *infinitely small parts vanish before finite parts and thus are able to be rejected with respect to these.* Whereby that objection falls at once, from which it is argued that the rigor of geometry is neglected in the infinitesimal analysis, since nothing other is rejected, unless what shall actually be zero. And therefore it is allowed to affirm justly that the highest rigor of geometry in this higher branch of knowledge, which may be grasped in the old books, is to be observed equally diligently.

88. Because the infinitely small quantity dx actually is $= 0$, the square of this too dx^2, the cube dx^3 and any other power having a positive exponent will be $= 0$ and thus equally before finite quantities well vanish. But truly also the infinitely small quantity dx^2 will vanish before dx itself; for there will be $dx \pm dx^2$ to dx in the ratio of equality, whether the arithmetic or the geometric comparison is put in place. Concerning the first indeed there is no doubt ; but on being compared geometrically there will be

$$dx \pm dx^2 : dx = \frac{dx \pm dx^2}{dx} = 1 \pm dx = 1.$$

In a like manner there will be $dx \pm dx^3 = dx$ and generally $dx \pm dx^{n+1} = dx$, provided *n* shall be a number greater than nothing; indeed there will be the geometric ratio $dx \pm dx^{n+1} : dx = 1 \pm dx^n$ and thus on account of ob $dx^n = 0$ the ratio of equality. If therefore, as in the powers made, *dx* may be called an infinitely small amount of the first order, dx^2 of the second order, dx^3 of the third order and thus henceforth, it is evident that the infinitely small of a higher order vanish before the infinitely small of the first order.

89. In a similar manner it may be shown an infinitely small quantity of the third order and of the above orders vanished before infinitely small lower orders and in general infinitely small quantities of each higher order vanish before infinitely small quantities of lower orders. Thus if *m* were a number less than *n*, there will be

$$adx^m + bdx^n = adx^m,$$

because dx^n will vanish before dx^m, as we have shown. And in this also exponents with a fraction may have a place ; thus *dx* may vanish before $\sqrt{dx}$ or $dx^{\frac{1}{2}}$ and there will be

$$a\sqrt{dx} + bdx = a\sqrt{dx}.$$

But if the exponent of *dx* shall be $= 0$, then there will be $dx^0 = 1$, although there shall be $dx = 0$; hence the power dx^n, since it is made = 1, if there shall be *n* = 0, from a finite quantity suddenly becomes infinitely small, and the exponent *n* shall be greater than nothing. Hence therefore an infinitude of infinitely small orders exist, which, although all shall be equal to zero $= 0$, yet they must be distinguished between themselves, if we attend to the mutual relation of these, which is explained by the geometric ratio.

90. We will be able to explain from the idea established of the infinitely small the nature of the infinite or of the infinitely large. The value of the fraction $\frac{1}{z}$ is to be noted with that to become greater there, so that the denominator *z* may be diminished more ; whereby if *z* is made a quantity with any assignable magnitude either smaller or infinitely small, it is necessary that the value of the fraction $\frac{1}{z}$ becomes greater than all assignable quantity and thus infinite. On account of which , if one or any other finite quantity may be divided by an infinitely small amount or zero 0, the quotient will be infinitely large and thus an infinite quantity. Therefore since this sign ∞ may denote an infinitely large quantity, this will be the equation $\frac{a}{dx} = \infty$ the truth of which also hence will be apparent, because there shall be on finding $\frac{a}{\infty} = dx = 0.$ For by how much greater the denominator *z* of the fraction $\frac{a}{z}$ is put in place, from that the value of the fraction will be less, and if z is made an infinite quantity or $z=\infty$, it is necessary that the value of the fraction $\frac{a}{\infty}$ is made infinitely small.

91. Anyone who will deny any of these accounts, moves him into a state of maximum inconvenience and most certainly it is necessary to overturn the foundations of analysis. Indeed anyone who has decided that the value of the fraction $\frac{a}{0}$ is to be that finite quantity *b,* on multiplying both sides by the denominator there is produced $a=0\cdot b$ and thus a finite quantity *b* multiplied by nothing 0 gives rise to the finite quantity *a,* which would be absurd. The value of the fraction *b* and much less of the fraction $\frac{a}{0}$ will be able to be $=0$; for 0 multiplied by 0 will not be able to produce a finite quantity *a* in any manner. Anyone who denies $\frac{a}{\infty}=0$ falls into the same absurdity ; indeed for that it will be said to be $\frac{a}{\infty}=$ to the finite quantity *b;* whereby since from the equation $\frac{a}{\infty}=b$ it may follow properly this equation $\infty=\frac{a}{b}$ may become the infinitely great value of the fraction $\frac{a}{b}$ of which the numerator and the denominator are finite quantities. Nor indeed also are the values of the fractions $\frac{a}{0}$ and $\frac{a}{\infty}$ able to be considered imaginary because the value of the fraction, of which the numerator is finite, the denominator truly imaginary, can neither be infinitely large or small.

92. Therefore an infinitely large quantity, to which this consideration has led us and which may only have place in infinitesimal analysis, is defined most conveniently by saying an infinitely large quantity to be the quotient, which arises from the division of a finite quantity by an infinitely small quantity. Therefore in turn there will be an infinitely small quotient, which arises from the division of a finite quantity by an infinitely large quantity. Whereby since a geometrical proportion of this kind shall remain, so that there shall be an infinitely small quantity to a finite quantity as a finite quantity to an infinite quantity : thus as the infinite quantity is infinitely greater than the finite, thus so the finite quantity will be infinitely greater than the infinitely small. Therefore talking about things in this manner, by which some people are offended, are not to be rejected, since they depend on the most reliable principles. Then also from the equation $\frac{a}{0}=\infty$ it follows to become possible, that nothing multiplied by an infinitely large amount may produce a finite quantity, because the contrary may be able to be considered, unless it can be deduced [otherwise] most clearly by a legitimate consequence.

93. Because among the infinitely small, if the following geometrical account may be prepared between them, the greatest distinction must be taken between them, thus also between infinitely large quantities much greater differences lie between, since the preparations may disagree not only geometrically but also arithmetically. Thus there may be put an infinite quantity = *A* thus, which arises from the division of a finite quantity *a* by an infinitely small quantity *dx,* thus so that there shall be $\frac{a}{dx}=A$; certainly there will be

$$\frac{2a}{dx} = 2A \quad \text{and} \quad \frac{na}{dx} = nA;$$

therefore since indeed nA shall be an infinite quantity, it follows that between infinitely great quantities it is possible to have some ratio in place. And hence, if an infinite quantity may either be multiplied by finite number, there will be produced infinite quantities. Nor therefore can it be denied concerning infinite quantities that these are able to be increased further. Moreover it may be seen easily, if the geometric ratio which the two infinite quantities maintain between each other, were not of equality, much less is it possible for the arithmetic ratio of those to be equal, when the difference of these rather shall be infinitely great always.

94. But however great some idea of infinity, which we use in mathematics, may appear suspect, whereby on account of this reason it may decide to overthrow the infinitesimal analysis, yet we are not able to be without this mathematical idea indeed even in the trivial parts. For in arithmetic, where the instruction of logarithms is usually treated, the logarithm of zero and of both negative as well as infinitely large numbers is to be decided, nor by any so considered in the mind, that this logarithm may be dared to be said either equal to a finite number or indeed equal to zero. But in geometry and trigonometry this may appear clearer; by whom indeed at any time will it be denied that the tangent or secant of a right angle shall not be infinitely large ? And since the rectangle from the tangent into the cotangent shall be equal to the square of the radius, but the contingent of the right angle shall be $= 0$, thus in geometry it must be conceded the product from nothing and infinity can be equal to a finite number.

95. Since therefore $\frac{a}{dx}$ shall be the infinite quantity A, it hence appears this quantity $\frac{A}{dx}$ to be an infinite quantity infinitely more great than A; for there is $\frac{a}{dx} : \frac{A}{dx} = a : A$, that is as a finite number to an infinite magnitude. Therefore there are given between infinitely greater quantities relations of this kind, so that there are able to be other greater infinities from these. Thus $\frac{a}{dx^2}$ will be an infinite quantity infinitely greater than $\frac{a}{dx}$; for on putting $\frac{a}{dx} = A$ there will be $\frac{a}{dx^2} = \frac{A}{dx}$. In a similar manner the infinite quantity $\frac{a}{dx^3}$ will be infinitely greater than $\frac{a}{dx^2}$ and thus infinitely greater than $\frac{a}{dx}$. Therefore there given infinite orders of infinity, whichever of which is infinitely greater than the preceding; and thus, if m shall be a number even so little greater than n, $\frac{a}{dx^m}$ will be an infinite quantity infinitely greater than the infinite quantity $\frac{a}{dx^n}$.

96. Just as with infinitely small quantities unequal geometric ratios are given, since yet all the arithmetic ratios shall be equal, thus with quantities of infinite magnitude there are given equal geometric ratios, while yet the arithmetic quantities shall be unequal by some great amount. For if a and b denote finite quantities, these two infinite quantities $\frac{a}{dx}+b$ and $\frac{a}{dx}$ have the geometric ratio of equality; for there will be the quotient arising from their division $=1+\frac{bdx}{a}=1$ on account of $dx=0$; yet meanwhile, if they may be compared arithmetically, on account of the difference $=b$ the ratio will be of inequality [unless b is zero]. In a similar manner $\frac{a}{dx^2}+\frac{a}{dx}$ to $\frac{a}{dx^2}$ has the ratio of equality ; indeed the exponent of the is $=1+dx=1$; yet truly the difference is $\frac{a}{dx}$ and thus infinite. Hence, if we should consider the geometric ratio, the infinitely great quantities of lesser order vanish before the infinitely great quantities of higher orders.

97. With these orders of infinities forewarned soon it will appear to become possible to happen, that not only may a finite quantity be the product from an infinitely large quantity into an infinitely small quantity, because we have seen above that it may eventuate, but also a product of this kind will be able to be either infinitely large or infinitely small. Thus the infinite quantity $\frac{a}{dx}$ if it may be multiplied by the infinitely small quantity dx, gives the finite product $=a$; but if dx may be multiplied either by the infinitely small quantity dx^2 or dx^3 or of some other higher order, there will be produced either adx, adx^2 or adx^3 etc. and thus infinitely small. In the same manner it may be understood, if the infinite quantity $\frac{a}{dx^2}$ may be multiplied by the infinitely small dx, the product will be infinitely large ; and generally, if $\frac{a}{dx^n}$ may be multiplied by bdx^m, the product $abdx^{m-n}$ will be infinitely small, if m surpasses n, finite, if m equals n, and infinitely great, is m is surpassed by n.

98. Both infinitely small as well as infinitely large quantities most often occur in series of numbers; in which since they shall be mixed together with finite numbers, from these it will be splendidly apparent, just as the transition is made following the rules of continuity from finite to infinitely great and infinitely small quantities. In the first place we may consider the series of natural numbers, which likewise continued backwards and forwards will be

$$\text{etc. } -4,\ -3,\ -2,\ -1,\ +0,\ +1,\ +2,\ +3,\ +4 \text{ etc.}$$

Therefore the numbers by continually decreasing reach 0 only or become infinitely small, from which they avoid continuing further to become negative. Hence on which account it is understood with positive numbers decreasing to pass through 0 to increasing negative numbers. But if we may consider the squares of these numbers, because all shall be positive,

$$\text{etc. } +16,\ +9,\ +4,\ +1,\ +0,\ +1,\ +4,\ +9,\ +16 \text{ etc.,}$$

0 will also be the transition of positive decreasing numbers to positive increasing numbers; and if the signs may be changed, also 0 will be the transition of negative decreasing numbers to negative increasing numbers.

99. If the series may be considered, the general term of which is $\sqrt{x}$, which also continued backwards will be of this kind

$$\text{etc.} + \sqrt{-3}, + \sqrt{-2}, + \sqrt{-1}, + 0, + \sqrt{1}, + \sqrt{2}, + \sqrt{3}, + \sqrt{4} \ \text{ etc.},$$

from that it is apparent that 0 also can be considered as the limit, through which real quantities may be cross over into imaginary ones. If these terms may be considered as the applied points of curves [*i.e.* the *y*-coordinates], it may be seen, if these were positive and had decreased to that point, so that finally they vanish, then these continued further either become negative, or positive again, or thus imaginary. Likewise it may come about, if at first the applied lines were negative ; then indeed equally, after they had vanished , if they were continued further, they become either positive, negative, or imaginary ; of which phenomena several examples give the principles of curves lines treated in the previous book [The *Introductio* again].

100. In the same manner with series there often occur infinite terms; thus in the harmonic series, the general term of which is $\frac{1}{x}$ to indicate $x = 0$ will correspond to the term infinitely great $\frac{1}{0}$ and the whole series thus itself will be considered

$$\text{etc.} - \tfrac{1}{4}, - \tfrac{1}{3}, - \tfrac{1}{2}, - \tfrac{1}{1}, + \tfrac{1}{0}, + \tfrac{1}{1}, + \tfrac{1}{2}, + \tfrac{1}{3} \,\text{etc.}$$

Therefore on progressing from the right to the left the terms increase, so that now $\frac{1}{0}$ shall be infinitely great; which when they will have passed thorough, they are made negative decreasing. Hence an infinitely great quantity can be seen as the limit [*i.e.* boundary], through which positive numbers progress and in turn are made negative; from which with many it is seen that negative numbers can be considered as infinitely greater, therefore because in that series the terms continually increase, after they will have reached infinity, they may change into negative numbers. But truly if we may attend to the series the general term of which is $\frac{1}{xx}$, after the transition through infinity again they may be produced positive terms

$$\text{etc.} + \tfrac{1}{9}, + \tfrac{1}{4}, + \tfrac{1}{1}, + \tfrac{1}{0}, + \tfrac{1}{1}, + \tfrac{1}{4}, + \tfrac{1}{9} \,\text{etc.}$$

which still no one will have said to be greater than infinity.

101. Repeatedly also in series an infinite term establishes a limit separating real terms from imaginary terms, as happens in this series, the general term of which is $\frac{1}{\sqrt{x}}$,

$$\text{etc.} + \frac{1}{\sqrt{-3}}, + \frac{1}{\sqrt{-2}}, + \frac{1}{\sqrt{-1}}, + \frac{1}{0}, + \frac{1}{\sqrt{1}}, + \frac{1}{\sqrt{2}}, + \frac{1}{\sqrt{3}} \text{ etc.}$$

nor yet hence does it follow that the imaginary terms are to be infinitely greater, because from the series mentioned before

$$\text{etc.} + \sqrt{-3}, + \sqrt{-2}, + \sqrt{-1}, + 0, + \sqrt{1}, + \sqrt{2}, + \sqrt{3} \text{ etc.}$$

equally it may follow that the imaginary terms to be less than zero. Then in truth also a transition can be shown from real terms to imaginary terms, of which the limit shall be neither 0 nor ∞, as will happen, if the general term were $1+\sqrt{x}$. From these cases, since on account of the irrationality any term may have a double value, in the limit between the real and imaginary terms these two values always are made equal to each other. But as often as terms which before were positive, change into negative terms, the transition always is made through a limit either infinitely small or infinitely large, which all may be manifest clearly from the law of continuity, as in the lines of curves we have considered.

102. Also from the summation of series extending to infinity many can be reported here, which since this principle of the infinite requires to be illustrated more, then truly they serve to remove many doubts which are accustomed to arise in this business. And in the first place, if the series may be agreed to be from equal terms so that

$$1+1+1+1+1+1+\text{etc}.$$

and that without end, that is it may be continued to infinity, there is no doubt, why the sum of all of these terms may not be greater than any number to be assigned ; and that therefore by necessity shall be infinite. This also may confirm the origin of this, while it arises from the setting out of the fraction

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \text{etc.}$$

on putting $x=1$; there will therefore be

$$\frac{1}{1-1} = 1+1+1+1+\text{etc.}$$

and thus the sum $= \frac{1}{1-1} = \frac{1}{0} =$ to infinity.

103. But here although no doubt can arise, since the same finite number taken an infinite number of times must go off to infinity, yet the origin itself from the general series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \text{ etc.}$$

may be considered to be produced most inconveniently and heavily; if indeed for x successively there are put the numbers 1, 2, 3, 4 etc., the following series with the sums of these will emerge

$$A\ldots\ 1+1+1+1+1+\text{ etc. } = \frac{1}{1-1} = \text{ to infinity}$$
$$B\ldots\ 1+2+4+8+16+\text{ etc. } = \frac{1}{1-2} = -1$$
$$C\ldots\ 1+3+9+27+81+\text{ etc. } = \frac{1}{1-3} = -\frac{1}{2}$$
$$D\ldots\ 1+4+16+64+256+\text{ etc. } = \frac{1}{1-4} = -\frac{1}{3}$$
etc.

Therefore since the series B besides the first may have greater terms than the series A, the sum of the series B by necessity must be much greater than the sum of the series A; meanwhile yet this calculation will show an infinite sum of the series A, and indeed a negative sum of the series B, that is less than nothing, which cannot be conceived. Much less can it be reconciled with the usual ideas, how the sum of this and of the following series C, D etc. are made negative, since yet all the terms shall be positive.

104. On account of this reason the opinion usually considered from the above reported most probably, whenever evidently negative quantities can be considered as infinitely greater or more than infinite quantities; and since the numbers also beyond zero on being diminished may reach negative numbers, a distinction may be established between negative numbers of this kind $-1, -2, -3$ etc. and of this kind $\frac{+1}{-1}, \frac{+2}{-1}, \frac{+3}{-1}$ etc., by saying those are less than zero, these indeed infinitely greater. But yet in this understanding they do not remove the difficulty, as this series suggests

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \text{etc.} = \frac{1}{(1-x)^2},$$

from which the following series arise

$$A\ldots\ 1+2+3+4+5+\text{etc.} = \frac{1}{(1-1)^2} = \frac{1}{0} = \text{to infinito}$$
$$B\ldots\ 1+4+12+32+80+\text{etc.} = \frac{1}{(1-2)^2} = 1;$$

where since the individual terms of the second series B shall be greater than the individual terms of the first series A with only the first term excepted, just as the sum of the series A

shall be infinite, truly the sum of the series B is equal to 1, that is only with the first term, from which principle it cannot be explained entirely.

105. But since, if we should wish to deny that there is $-1 \;=\frac{+1}{-1}$ and $\frac{+a}{-b}=\frac{-a}{+b}$,
the most substantial analytical fundaments would collapse, from that the explanation mentioned before cannot be admitted at all. Why not rather that we ought to deny in these equations that the sums are true, upon which the general formulas depend,. Since indeed these series arise from continued division, while the remainder is continually divided further, but the remainder always becomes greater, so that the longer we may progress, we cannot disregard it at any time; and with a final smallest remainder, that is, because by division it remains infinitesimal, it can be omitted, [but here] evidently because it is made infinitely great. But because this is not observed in the above series, while an account of no smaller residue may be had, it is no wonder that these sums are reduced to absurdity. And this reply, as it is demanded from the origin of the series itself, thus also is the most true and all doubt may be removed.

106. So that this may be made to appear more evident, we will consider the evolution of the fraction $\frac{1}{1-x}$ as at first it is resolved into finite terms only. Therefore there will be

$$\frac{1}{1-x}=1+\frac{x}{1-x}$$
$$\frac{1}{1-x}=1+x+\frac{x^2}{1-x}$$
$$\frac{1}{1-x}=1+x+x^2+\frac{x^3}{1-x}$$
$$\frac{1}{1-x}=1+x+x^2+x^3+\frac{x^4}{1-x}$$
etc.

Whereby therefore it may be wished to say the sum of this finite series $1+x+x^2+x^3$ is $\frac{1}{1-x}$ that will err from the true quantity by $\frac{x^4}{1-x}$, and whereby the sum of this series

$$1+x+x^2+x^3+.........+x^{1000}$$

it may be wished to put in place $=\frac{1}{1-x}$ it will err by the quantity $\frac{x^{1001}}{1-x}$ which error, if x shall be a number greater than unity, will be a maximum.

107. From these it is evident, whereby the sum it may be wished to be put in place of this series continued to infinity

$$1+x+x^2+x^3+\cdots\cdots+x^{\infty}$$

of this $=\frac{1}{1-x}$ is to differ from the truth by the quantity $\frac{x^{\infty+1}}{1-x}$, which, if there shall be $x > 1$, certainly will be infinitely large. Likewise truly hence an account may be apparent, why the sum of the series continued to infinity $1+x+x^2+x^3+x^4+$ etc. actually will be $=\frac{1}{1-x}$, if x were a fraction less than unity ; then indeed the error $\frac{x^{\infty+1}}{1-x}$ is made infinitely small and thus zero, therefore an account of this may be ignored with safety. Thus on putting $x=\frac{1}{2}$ there will be actually

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\text{etc.}=\frac{1}{1-\frac{1}{2}}=2$$

and similarly for the rest of the series, if x shall be a fraction less than unity, the true sum may be indicated in this manner.

108. This same answer prevails concerning the sum of diverging series, in which the signs + and – alternate, which generally from the same formula are accustomed to be shown on putting negative numbers for x. Since indeed there shall be

$$\frac{1}{1+x}=1-x+x^2-x^3+x^4-x^5+\text{etc.},$$

unless an account of the final remainder may be considered, there shall be

$$\begin{aligned} &A...1-1+1-1+1-1+\text{etc.}=\tfrac{1}{2}\\ &B...1-2+4-8+16-32+\text{etc.}=\tfrac{1}{3}\\ &C...1-3+9-27+81-243+\text{etc.}=\tfrac{1}{4}\end{aligned}$$

etc.

But it may be apparent that the sum of the second series B thus cannot be $=\frac{1}{3}$, since, when more terms actually may be added, there the aggregates recede more from $\frac{1}{3}$. But always the sum of the series must be the limit, to which it may approach closer there, when more terms actually are added.

109. Indeed from these series of this kind are inferred, which are called diverging, in short they do not have a fixed sum, therefore because actually by gathering together the terms no approach to fixed limit is made, which may be considered for the sum of a series continued to infinity; which opinion is especially agreeable to the truth, since these sums now may be shown erroneous on account of ignoring the final remainders. Yet meanwhile against that I can add justly, [though] it is possible to object to these sums mentioned, however far from the truth they may seem to differ, yet at no time do they lead to errors, so that rather with these admitted, many outstanding series may be elicited, if we may wish to reject these summations completely. Nor truly are these sums, if they may be false, always able to lead us to the truth, so that since they may disagree not by a little, but rather infinitely from the truth, we too must be led away from the truth to infinity. Because still while it may not come about, the more difficult for us the knot remains requiring to be loosened.

110. Therefore I say in using the name of *sum* a whole difficulty lies hidden; if indeed the *sum* of the series, as general use considers, is taken for the aggregate of all the terms of this acting gathered together, then there is no doubt, so that only the sums of these shown extending to infinity are able to be shown, which shall be converging and they continually lead closer to a certain appointed value, to which more terms actually are gathered together. But diverging series, the terms of which are not decreasing, either with the signs alternate + and – or otherwise, evidently will have no fixed sum, if indeed the use of the word sum is taken in this sense for the aggregate of all terms. But truly in these cases, of which we keep in mind, by which from erroneous sums of this kind yet the true value may be elicited, that does not happen, in as much as the finite expression, for example $\frac{1}{1-x}$ is the sum of the series $1+x+x^2+x^3+$ etc., but rather that expression expanded out may give this series ; and thus in this matter the name sums may be omitted completely.

111. Therefore we will shun completely these inconveniences and these apparent contradictions, if we may attribute another meaning to what is called *sum*, and commonly accustomed to be made. Therefore we may say of each series the *sum* to infinity to be a finite expression and that series is generated from the expansion of this. And in this sense the sum of the infinite series $1+x+x^2+x^3+$ etc. actually will be $=\frac{1}{1-x}$ because that series arises from the expansion of this fraction, whatever number may be substituted in place of *x*. With this understood, if the series were converging, this new definition of referring to a sum will agree with the customary definition, and because divergent series thus are said to have no proper sums, hence no inconvenience may arise from this new name. Finally with the aid of this definition the usefulness of diverging series is looked after and we will be able to free from all injustices.

CAPUT III

DE INFINITIS ATQUE INFINITE PARVIS

72. Cum omnis quantitas, quantumvis sit magna, ulterius augeri possit neque quicquam obstet, quominus ad datam quantitatem quamcunque alia quantitas eiusdem generis addi queat, omnis quoque quantitas sine fine augeri poterit; neque enim unquam tam magna fiet, ut ipsi nihil amplius adiici posset. Nulla igitur datur quantitas tam magna, qua maior concipi nequeat, hincque extra dubium erit positum *omnem quantitatem in infinitum augeri posse.* Qui enim hoc negaverit, is affirmare cogitur dari limitem, quem quantitas, cum attigerit, superare nequeat, atque ideo statuere debebit quantitatem, cui nihil amplius adiici posset; quod cum sit absurdum atque quantitatis notioni adversetur, necessario concedendum est omnem quantitatem sine fine continuo magis, hoc est in infinitum, augeri posse.

73. In singulis quantitatum speciebus hoc etiam clarius perspicietur. Sic nemo facile reperietur, qui statuerit seriem numerorum naturalium 1, 2, 3, 4, 5, 6 etc. ita usquam esse determinatam, ut ulterius continuari non possit. Nullus enim datur numerus, ad quem non insuper unitas addi sicque numerus sequens maior exhiberi queat; hinc series numerorum naturalium sine fine progreditur neque unquam pervenitur ad numerum maximum, quo maior prorsus non detur. Simili modo linea recta nunquam eousque produci potest, ut insuper ulterius prolongari non posset. Quibus evincitur tam numeros in infinitum augeri quam lineas in infinitum produci posse. Quae cum sint species quantitatum simul intelligitur omni quantitati, quantumvis sit magna, adhuc dari maiorem hacque denuo maiorem sicque augendo continuo ulterius sine fine, hoc est in infinitum, procedi posse.

74. Quanquam autem haec sunt adeo perspicua, ut, qui ea negare vellet, sibi ipse contradicere deberet, tamen ista infiniti doctrina a pluribus, qui eam explicare sunt conati, tantopere est offuscata tantisque difficultatibus atque etiam contradictionibus obvoluta, ut, qua se extricarent, nulla via pateret. Ex eo, quod quantitas in infinitum augeri possit, quidam concluserunt dari revera quantitatem infinitam eamque ita descripserunt, ut nullum amplius augmentum suscipere possit. Hoc autem ipso ideam quantitatis evertunt, dum eiusmodi quantitatem statuunt, quae ulterius augeri nequeat. Praeterea vero secum ipsi infinitum admittentes pugnant; dum enim incrementi, quo quantitas sit capax, finem faciunt, simul negant quantitatem sine fine augeri posse; negant ergo quoque quantitatem in infinitum augeri posse, quoniam utraque locutio congruit; sicque, dum quantitatem infinitam statuunt, eam simul tollunt. Si enim quantitas sine fine, hoc est in infinitum, augeri nequeat, certe nulla quantitas infinita existere poterit.

75. Hinc igitur ex eo ipso, quod omnis quantitas in infinitum augeri possit, sequi videatur nullam dari quantitatem infinitam. Quantitas enim continuis incrementis aucta infinita non evadet, nisi iam sine fine increverit; quod autem sine fine fieri debet, id non tanquam iam factum concipi potest. Interim tamen non solum huiusmodi quantitatem, ad quam incrementis sine fine congestis pervenitur, certo charactere indicare sicque debito modo in calculum inducere licet, uti mox fusius ostendemus, sed etiam in mundo eiusmodi casus existere vel saltem concipi possunt, quibus numerus infinitus actu existere videatur. Sic, si materia in infinitum sit divisibilis, uti plures philosophi statuerunt, numerus partium, quibus datum quodque materiae frustum constat, revera erit infinitus; si enim statueretur finitus, materia certe non in infinitum foret divisibilis. Simili modo, si universus mundus esset infinitus, uti pluribus placuit, numerus corporum mundum componentium finitus certe esse non posset foretque ideo quoque infinitus.

76. Haec etiamsi inter se pugnare videantur, tamen, si attentius perpendantur, a cunctis incommodis liberari poterunt. Qui enim statuit materiam in infinitum esse divisibilem, is negat in divisione materiae continua unquam ad partes tam parvas perveniri, quae ulterius dividi nequeant; nullas ergo materia habebit partes ulterius individuas, cum singulae particulae, ad quas per continuam divisionem iam sit perventum, ulterius se subdividi patiantur. Qui igitur dicit hoc casu numerum partium fore infinitum, is partes ultimas, quae ulterius sint individuae, intelligit; ad quas cum nunquam perveniatur et quae propterea nullae sint, is has ipsas partes, quae nullae sunt, numerare conatur. Si enim materia sine fine continuo ulterius subdividi potest, partibus individuis seu simplicibus prorsus caret neque adeo quicquam superest, quod numerari queat. Hanc ob rem, qui materiam in infinitum divisibilem statuit, is simul negat materiam ex partibus simplicibus esse compositam.

77. Quodsi autem, dum de partibus alicuius corporis seu materiae loquimur, non ultimas seu simplices, quippe quae nullae sunt, intelligamus, sed eas, quas divisis revera produxit, tum, admissa hac hypothesi de divisibilitate materiae in infinitum, unumquodque vel minimum materiae frustum non solum in plurimas partes dissecari, sed etiam nulIus numerus tam magnus assignari poterit, quo non maior partium ex illo frusto sectarum numerus exhiberi queat. Numerus ergo partium non quidem ultimarum, sed quae ipsae adhuc sint ulterius divisibiles, quae unumquodque corpus componunt, omni numero assignabili erit maior. Simili modo, si universus mundus sit infinitus, numerus corporum mundum constituentium pariter omni assignabili erit maior; qui cum finitus esse nequeat, sequitur numerum infinitum et numerum omni assignabili maiorem esse nomina synonyma.

78. Qui ergo hoc modo divisibilitatem materiae in infinitum intuetur, nullis incommodis, quae vulgo huic opinioni imputantur, se implicat nihilque affirmare cogitur, quod sanae rationi adversetur. Qui autem contra materiam in infinitum divisibilem esse negant, ii in maximas difficultates prolabuntur, ex quibus se nullo prorsus modo extrahere possunt.

Statuere enim coguntur unumquodque corpus nonnisi in certum partium numerum dissecari posse, ad quas si fuerit perventum, nulla divisio ulterior locum inveniat; quas ultimas particulas alii *atomos,* alii *monades* atque *entia simplicia* vocant. Cur autem istae ultimae particulae nullam amplius divisionem admittant, duplex esse potest causa: altera, quod omni extensione careant; altera, quod quidem sint extensae, sed tamen tam durae atque ita comparatae, ut nulla vis ad eas dissecandas sufficiat. Utrumvis patroni huius opinionis dicant, sese aeque difficultatibus implicant.

79. Sint enim ultimae particulae omnis extensionis expertes, ita ut partibus prorsus careant; qua explicatione quidem ideam entium simplicium optime tuentur. At, quemadmodum corpus ex finito huiusmodi particularum numero constare queat, concipi nullo modo potest. Ponamus pedem cubicum materiae ex mille huiusmodi entibus simplicibus esse compositum huncque actu in mille partes secari; quae si sint aequales, erunt digiti cubici; sin autem sint inaequales, aliae erunt maiores, aliae minores. Unus igitur digitus cubicus foret ens simplex sicque maxima resultaret contradictio, nisi forte in digito cubico inesse tantum unum ens simplex reliquumque spatium vacuum esse dicere velint; at vero hoc modo continuitatem corporum tollerent, praeterquam quod isti philosophi vacuum plane ex mundo profligant. Quodsi obiiciant numerum entium simplicium, quae pedem cubicum materiae constituunt, millenario longe esse maiorem, nihil omnino lucrantur; incommodum enim, quod ex numero millenario sequitur, ex quovis alia numero quantumvis magno aeque manat. Hanc difficultatem acutissimus Leibnizius, primus monadum inventor, probe perspexit, dum materiam absolute in infinitum divisibilem esse statuit. Neque ergo ante ad monades pervenire licet, quam corpus actu in infinitum sit divisum. Hoc ipso autem existentiam entium simplicium, ex quibus corpora constent, penitus tollit; nam qui negat corpora ex entibus simplicibus esse composita, et ille, qui statuit corpora in infinitum esse divisibilia, in eadem prorsus sunt sententia.

80. Neque magis autem sibi constant, si dicunt ultimas corporum particulas extensas quidem esse, sed ob summam duritiem in partes divelli non posse. Cum primum enim in ultimis particulis extensionem admittunt, eas ex partibus compositas esse statuunt, quae utrum revera a se invicem separari queant necne, parum refert, etiamsi nullam causam assignare possint, unde tanta durities sit orta. Nunc autem plerique, qui divisibilitatem materiae in infinitum negant, hoc posterius incommodum satis sensisse videntur, quia priori ideae partium ultimarum potissimum inhaerent; hasque difficultates aliter diluere non possunt nisi aliquot leviusculis metaphysicis distinctionibus, quae maximam partem eo tendunt, ut ne consequentiis, quae secundum mathematica principia formantur, fidamus, neque dimensiones in partibus simplicibus adhiberi oportere regerunt. At primum demonstrare debuissent istas suas partes ultimas, quarum determinatus numerus corpus constituat, extensas prorsus non esse.

81. Cum igitur ex hoc labyrintho exitum nullum invenire neque obiectionibus debito modo occurrere queant, ad distinctiones confugiunt respondentes has obiectiones a sensibus atque imaginatione suppeditari, in hoc autem negotio solum intellectum purum adhiberi oportere, sensus autem ac ratiocinia inde pendentia saepissime fallere. Intellectus scilicet purus agnoscet fieri posse, ut pars millesima pedis cubici materiae omni extensione careat, quod imaginationi absurdum videatur. Tum vero, quod sensus saepenumero fallant, res vera quidem est, at nemini minus quam mathematicis opponi potest. Mathesis enim nos imprimis a fallacia sensuum defendit atque docet obiecta, quae sensibus percipiuntur, aliter revera esse comparata, aliter vero apparere; haecque scientia tutissima tradit praecepta, quae qui sequuntur, ab illusione sensuum immunes sunt. Huiusmodi ergo responsionibus tantum abest, ut metaphysici suam doctrinam tueantur, ut eam potius magis suspectam efficiant.

82. Verum ut ad propositum revertamur, etiamsi quis neget in mundo numerum infinitum revera existere, tamen in speculationibus mathematicis saepissime occurrunt quaestiones, ad quas, nisi numerus infinitus admittatur, responderi non posset. Sic, si quaeratur summa omnium numerorum, qui hanc seriem $1+2+3+4+5+$ etc. constituunt, quia isti numeri sine fine progrediuntur atque crescunt, eorum omnium summa certe finita esse non poterit; quo ipso efficitur eam esse infinitam. Hinc, quae quantitas tanta est, ut omni quantitate finita sit maior, ea non infinita esse nequit. Ad huiusmodi quantitatem designandam mathematici utuntur hoc signo ∞, quo denotatur quantitas omni quantitate finita seu assignabili major. Sic, cum parabola ita definiri queat, ut dicatur esse ellipsis infinite longa, recte affirmare poterimus axem parabolae esse lineam rectam infinitam.

83. Haec autem infiniti doctrina magis illustrabitur, si, quid sit infinite parvum mathematicorum, exposuerimus. Nullum autem est dubium, quin omnis quantitas eousque diminui queat, quoad penitus evanescat atque in nihilum abeat. Sed quantitas infinite parva nil aliud est nisi quantitas evanescens ideoque revera erit $=0$. Consentit quoque ea infinite parvorum definitio, qua dicuntur omni quantitate assignabili minora; si enim quantitas tam fuerit parva, ut omni quantitate assignabili sit minor, ea certe non poterit non esse nulla; namque nisi esset $=0$, quantitas assignari posset ipsi aequalis, quod est contra hypothesin. Quaerenti ergo, quid sit quantitas infinite parva in mathesi, respondemus eam esse revera $=0$; neque ergo in hac idea tanta mysteria latent, quanta vulgo putantur et quae pluribus calculum infinite parvorum admodum suspectum reddiderunt. Interim tamen dubia, si quae supererunt; in sequentibus, ubi hunc calculum sumus traditurі, funditus tollentur.

84. Cum igitur ostenderimus quantitatem infinite parvam revera esse cyphram, primum occurrendum est obiectioni, cur quantitates infinite parvas non perpetuo eodem charactere 0 designemus, sed peculiares notas ad eas designandas adhibeamus. Quia enim omnia nihila sunt inter se aequalia, superfluum videtur variis signis ea denotare. Verum quamquam duae quaevis cyphrae ita inter se sunt aequales, ut earum differentia sit nihil, tamen, cum duo sint modi comparationis, alter arithmeticus, alter geometricus, quorum illo differentiam, hoc vero quotum ex quantitatibus comparandis ortum spectamus, ratio quidem arithmetica inter binas quasque cyphras est aequalitatis, non vero ratio geometrica. Facillime hoc perspicietur ex hac proportione geometrica $2:1=0:0$, in qua terminus quartus est $=0$ uti tertius. Ex natura autem proportionis, cum terminus primus duplo sit maior quam secundus, necesse est, ut et tertius duplo maior sit quam quartus.

85. Haec autem etiam in vulgari arithmetica sunt planissima; cuilibet enim notum est cyphram per quemvis numerum multiplicatam dare cyphram esseque $n \cdot 0 = 0$ sicque fore $n : 1 = 0 : 0$. Unde patet fieri posse, ut duae cyphrae quamcunque inter se rationem geometricam teneant, etiamsi rem arithmetice spectando earum ratio semper sit aequalitatis. Cum igitur inter cyphras ratio quaecunque intercedere possit, ad hanc diversitatem indicandam consulto varii characteres usurpantur, praesertim tum, cum ratio geometriea, quam cyphrae variae inter se tenent, est investiganda. In calculo autem infinite parvorum nil aliud agitur, nisi ut ratio geometrica inter varia infinite parva indagetur, quod negotium propterea, nisi diversis signis ad ea indicanda uteremur, in maximam confusionem illaberetur neque ullo modo expediri posset.

86. Si ergo, prouti in analysi infinitorum modus signandi est receptus, denotet dx quantitatem infinite parvam, erit utique tam $dx = 0$ quam $adx = 0$ denotante a quantitatem quamcunque finitam. Hoc tamen non obstante erit ratio geometrica $adx : dx$ finita, nempe ut $a : 1$, et hanc ob rem haec duo infinite parva dx et adx, etiamsi utrumque sit $= 0$, inter se confundi non possunt; si quidem eorum ratio investigetur, Simili modo, si diversa occurrunt infinite parva dx et dy, etiamsi utrumque sit $= 0$, tamen eorum ratio non constat. Atque in investigatione rationis inter duo quaeque huiusmodi infinite parva omnis vis calculi differentialis versatur. Usus autem huius comparationis, etiamsi primo intuitu admodum exiguus videatur, tamen amplissimus deprehenditur atque adhuc in dies magis elucet.

87. Cum igitur infinite parvum sit revera nihil, patet quantitatem finitam neque augeri neque diminui, si ad eam infinite parvum vel addamus vel ab ea subtrahamus. Sit a quantitas finita atque dx infinite parva; erit tam $a + dx$ quam $a - dx$ et generaliter $a \pm ndx = a$. Sive enim relationem inter $a + ndx$ et a arithmetice intueamur sive geometrice, utroque casu ratio aequalitatis deprehendetur. Arithmetica quidem ratio aequalitatis manifesta est; cum enim sit $ndx = 0$, erit

$$a \pm ndx - a = 0;$$

geometrica vero ratio aequalitatis inde patet, quod sit

$$\frac{a \pm ndx}{a} = 1.$$

Hinc sequitur canon ille maxime receptus, quod *infinite parva prae finitis evanescant atque adeo horum respectu reiici queant.* Quare illa obiectio, qua analysis infinitorum rigorem geometricum negligere arguitur, sponte cadit, cum nil aliud reiiciatur, nisi quod revera sit nihil. Ac propterea iure affirmare licet in hac sublimiori scientia rigorem geometricum summum, qui in veterum libris deprehenditur, aeque diligenter observari.

88. Quoniam quantitas infinite parva dx revera est $=0$, eius quoque quadratum dx^2, cubus dx^3 et quaevis alia potestas affirmativum habens exponentem erit $=0$ ideoque aeque prae quantitatibus finitis evanescent. At vero etiam quantitas infinite parva dx^2 prae ipsa dx evanescit; erit enim $dx \pm dx^2$ ad dx in ratione aequalitatis, sive comparatio arithmetice sive geometrice instituatur. De priori quidem dubium est nullum; at geometrice comparando erit

$$dx \pm dx^2 : dx = \tfrac{dx \pm dx^2}{dx} = 1 \pm dx = 1.$$

Pari modo erit $dx \pm dx^3 = dx$ et generaliter $dx \pm dx^{n+1} = dx$, dummodo sit n numerus nihilo maior; erit enim ratio geometrica $dx \pm dx^{n+1} = 1 \pm dx^n$ ideoque ob $dx^n = 0$ ratio aequalitatis. Si igitur, uti in potestatibus fit, vocetur dx infinite parvum primi ordinis, dx^2 secundi ordinis, dx^3 tertii ordinis et ita porro, manifestum est prae infinite parvis primi ordinis evanescere infinite parva altiorum ordinum.

89. Simili modo ostendetur infinite parva tertii ac superiorum ordinum evanescere prae infinite parvis ordinis secundi atque in genere infinite parva cuiusque ordinis superioris evanescere prae infinite parvis ordinis inferioris. Ita si m fuerit numerus minor quam n, erit

$$adx^m + bdx^n = adx^m,$$

quia dx^n evanescit prae dx^m, uti ostendimus. Hocque etiam in exponentibus fractis habet locum; ita dx evanescet prae $\sqrt{dx}$ seu $dx^{\frac{1}{2}}$ eritque

$$a\sqrt{dx} + bdx = a\sqrt{dx}.$$

Quodsi autem exponens ipsius dx sit $=0$, erit $dx^0 = 1$, quamvis sit $dx = 0$; hinc potestas dx^n, cum fiat = 1, si sit n = 0, ex finita statim fit quantitas infinite parva, atque exponens n nihilo fit maior. Hinc ergo infiniti ordines infinite parvorum existunt, quae, etsi omnia sunt $=0$, tamen inter se probe distingui debent, si ad earum relationem mutuam, quae per rationem geometricam explicatur, attendamus.

90. Stabilita notione infinite parvorum facilius indolem infinitorum seu infinite magnorum exponere poterimus. Notum est valorem fractionis $\frac{1}{x}$ eo maiorem evadere, quo magis diminuatur denominator *z;* quare si *z* fiat quantitas omni assignabili quantitate minor seu infinite parva, necesse est, ut valor fractionis $\frac{1}{x}$ fiat omni assignabili quantitate maior ideoque infinitus. Quamobrem, si unitas seu quaevis alia quantitas finita dividatur per infinite parvum seu 0, quotus erit infinite magnus ideoque quantitas infinita. Cum igitur hoc signum ∞ denotet quantitatem infinite magnam, ista habebitur aequatio

$\frac{a}{dx}=\infty$ cuius veritas quoque hinc patet, quod sit invenendo $\frac{a}{\infty}=dx=0.$ Namque quo maior statuitur fractionis $\frac{a}{z}$ denominator *z*, eo minor fit fractionis valor, atque si *z* fiat quantitas infinite magna seu $z=\infty$, necesse est, ut fractionis valor $\frac{a}{\infty}$ fiat infinite parvus.

91. Qui utrumvis horum ratiociniorum negaverit, eum in maxima incommoda prolabi atque adeo certissima analyseos fundamenta evertere necesse est. Qui enim statuit valorem fractionis $\frac{a}{0}$ esse finitum uti *b*, utrinque per denominatorem multiplicando prodiret $a=0\cdot b$ atque ideo quantitas finita *b* per nihil 0 multiplicata praeberet quantitatem finitam *a*, quod esset absurdum. Multo minus valor ille *b* fractionis $\frac{a}{0}$ poterit esse $=0$; nam 0 per 0 multiplicata quantitatem *a* producere nullo modo poterit. In idem absurdum incidit, qui negat esse $\frac{a}{\infty}=0$; ei enim dicendum erit esse $\frac{a}{\infty}=$ quantitati finitae *b;* quare cum ex aequatione $\frac{a}{\infty}=b$ legitime sequatur haec $\infty=\frac{a}{b}$ foret valor fractionis $\frac{a}{b}$ cuius numerator ac denominator sunt quantitates finitae, infinite magnus, quod perinde foret absurdum. Neque vero etiam valores fractionum $\frac{a}{0}$ et $\frac{a}{\infty}$ imaginarii statui possunt, propterea quod valor fractionis, cuius numerator est finitus, denominator vero imaginarius, neque infinite magnus neque infinite parvus esse potest.

92. Quantitas ergo infinite magna, ad quam nos haec consideratio perduxit et quae sola in analysi infinitorum locum habet, commodissime definitur dicendo quantitatem infinite magnam esse quotum, qui ex divisione quantitatis finitae per infinite parvam oritur. Vicissim ergo erit quantitas infinite parva quotus, qui oritur ex divisione quantitatis finitae per infinite magnam. Quare cum eiusmodi proportio geometrica subsistat, ut sit quantitas infinite parva ad finitam uti finita ad infinite magnam: uti quantitas infinita infinities maior est quam finita, ita quantitas finita infinities maior erit quam infinite parva. Huiusmodi igitur locutiones, quibus plures offenduntur, non sunt improbandae, cum certissimis innitantur principiis. Deinde etiam ex aequatione $\frac{a}{0}=\infty$ sequitur fieri posse, ut nihil per quantitatem infinite magnam multiplicatum producat quantitatem finitam, quod alienum videri posset, nisi planissime per legitimam consequentiam esset deductum

93. Quoniam inter infinite parva, si secundum rationem geometricam inter se comparantur, maximum deprehenditur discrimen, ita quoque inter quantitates infinite magnas multo maior differentia intercedit, cum non solum geometrice, sed etiam arithmetice comparatae discrepent. Ponatur quantitas ita infinita, quae ex divisione quantitatis finitae *a* per infinite parvam *dx* oritur, = *A*, ita ut sit $\frac{a}{dx}=A$*;* erit utique

$$\frac{2a}{dx}=2A \quad \text{et} \quad \frac{na}{dx}=nA;$$

cum igitur et *nA* sit quantitas infinita, sequitur inter quantitates infinite magnas rationem quamcunque locum habere posse. Hincque, si quantitas infinita per numerum finitum sive muItiplicetur sive dividatur, prodibit quantitas infinita. Neque ergo de quantitatibus infinitis negari potest eas ulterius augeri posse. Facile autem perspicitur, si ratio geometrica, quam duae quantitates infinitae inter se tenent, non fuerit aequalitatis, multo minus earum rationem arithmeticam aequalitatis esse posse, cum potius earum differentia semper sit infinite magna.

94. Quantumvis autem nonnullis idea infiniti, qua in mathesi utimur, suspecta videatur, qui hanc ob causam analysin infinitorum profligandam arbitrantur, tamen hac idea ne in partibus quidem matheseos trivialibus carere possumus. In arithmetica enim, ubi doctrina logarithmorum tradi solet, logarithmus cyphrae et negativus et infinite magnus statuitur neque quisquam est tam mente captus, ut hunc logarithmum vel finitum vel adeo nihilo aequalem dicere audeat. In geometria autem et trigonometria hoc clarius apparet; quis enim unquam negabit tangentem secantemve anguli recti non esse infinite magnam? Et cum rectangulum ex tangente in cotangentem sit radii quadrato aequale, cotangens autem anguli recti sit $=0$, in geometria adeo concedi debet productum ex nihilo et infinito esse posse finitum.

95. Cum sit $\frac{a}{dx}$ quantitas infinita *A*, patet hanc quantitatem $\frac{A}{dx}$ fore quantitatem infinities maiorem quam *A;* est enim $\frac{a}{dx}:\frac{A}{dx}=a:A$, hoc est ut numerus finitus ad infinite magnum. Dantur ergo inter quantitates infinite magnas eiusmodi relationes, ut aliae aliis infinities maiores esse queant. Sic $\frac{a}{dx^2}$ erit quantitas infinita infinities maior quam $\frac{a}{dx}$*;* posito enim $\frac{a}{dx}=A$ erit $\frac{a}{dx^2}=\frac{A}{dx}$. Simili modo erit $\frac{a}{dx^3}$ quantitas infinita infinities maior quam $\frac{a}{dx^2}$ ideoque infinities infinities maior quam $\frac{a}{dx}$. Dantur ergo infiniti gradus infinitorum, quorum quisque infinities maior est quam praecedentes; atque adeo, si numerus *m* vel tantillum maior sit quam *n*, erit $\frac{a}{dx^m}$ quantitas infinita infinities maior quam quantitas infinita $\frac{a}{dx^n}$.

96. Quemadmodum in quantitatibus infinite parvis dantur rationes geometricae inaequales, cum tamen rationes arithmeticae omnes sint aequales, ita in quantitatibus infinite magnis dantur rationes geometricae aequales, cum tamen arithmeticae sint quantumvis inaequales. Si enim *a* et *b* denotent quantitates finitas, hae duae quantitates infinitae $\frac{a}{dx}+b$ et $\frac{a}{dx}$ rationem geometricam habent aequalitatis; erit enim quotus ex earum divisione ortus $=1+\frac{bdx}{a}=1$ ob $dx=0$; interim tamen, si arithmetice comparentur, ob differentiam $=b$ ratio erit inaequalitatis. Simili modo $\frac{a}{dx^2}+\frac{a}{dx}$ ad $\frac{a}{dx^2}$ rationem geometricam habet aequalitatis; exponens enim rationis est $=1+dx=1$; verum tamen differentia est $\frac{a}{dx}$ ideoque infinita. Hinc, si ad rationem geometricam spectemus, infinite magna inferiorum graduum prae infinite magnis superiorum graduum evanescunt.

97. His de gradibus infinitorum praemonitis mox apparebit fieri posse, ut productum ex quantitate infinite magna in infinite parvam non solum quantitatem finitam producat, quod supra evenisse vidimus, sed etiam huiusmodi productum esse poterit sive infinite magnum sive infinite parvum. Sic quantitas infinita $\frac{a}{dx}$ si per infinite parvam dx multiplicetur, dat productum finitum $=a$; sin autem dx multiplicetur per infinite parvum dx^2 vel dx^3 vel alius superioris ordinis, productum erit vel adx vel adx^2 vel adx^3 etc. ideoque infinite parvum. Eodem modo intelligetur, si quantitas infinita $\frac{a}{dx^2}$ multiplicetur per infinite parvam dx, productum fore infinite magnum; atque generatim, si $\frac{a}{dx^n}$ multiplicetur per bdx^m, productum $abdx^{m-n}$ erit infinite parvum, si m superat n, finitum, si m aequat n, et infinite magnum, si m superatur ab n.

98. Quantitates tam infinite parvae quam infinite magnae in seriebus numerorum saepissime occurrunt; in quibus cum sint numeris finitis permixtae, ex iis luculenter patebit, quemadmodum secundum leges continuitatis a quantitatibus finitis ad infinite magnas atque infinite parvas transitio fiat. Consideremus primum seriem numerorum naturalium, quae simul retro continuata erit

$$\text{etc.} -4, -3, -2, -1, +0, +1, +2, +3, +4 \text{ etc.}$$

Numeri ergo continuo decrescendo praebent tandem 0 seu infinite parvum, unde ulterius continuati negativi evadunt. Quamobrem hinc intelligitur a numeris finitis affirmativis decrescentibus transiri per 0 ad negativos crescentes. Sin autem eorum numerorum quadrata spectentur, quia omnia sunt affirmativa,

$$\text{etc.} +16, +9, +4, +1, +0, +1, +4, +9, +16 \text{ etc.},$$

erit 0 quoque transitus numerorum affirmativorum decrescentium ad affirmativos crescentes; atque si signa mutentur, erit quoque 0 transitus numerorum negativorum decrescentium ad negativos crescentes.

99. Si series consideretur, cuius terminus generalis est $\sqrt{x}$, quae etiam retro continuata erit huiusmodi

$$\text{etc.} +\sqrt{-3}, +\sqrt{-2}, +\sqrt{-1}, +0, +\sqrt{1}, +\sqrt{2}, +\sqrt{3}, +\sqrt{4} \text{ etc.},$$

ex ea patet 0 quoque tanquam limitem considerari posse, per quem a quantitatibus realibus ad imaginaria transeatur. Si isti termini tanquam applicatae curvarum considerentur, perspicitur, si eae fuerint affirmativae atque eousque decreverint, ut tandem evanescant, tum eas ulterius continuatas vel fieri negativas vel iterum affirmativas vel adeo imaginarias. Idem eveniet, si applicatae primum fuerint negativae; tum enim aeque, postquam evanuerint, si ulterius continuentur, vel affirmativae fient vel negativae vel imaginariae; quorum phaenomenorum plurima exempla praebet doctrina de lineis curvis in libro praecedente tractata.

100. Eodem modo in seriebus occurrunt saepe termini infiniti; sic in serie harmonica, cuius terminus generalis est $\frac{1}{x}$ indici $x=0$ respondebit terminus infinite magnus $\frac{1}{0}$ totaque series ita se habebit

$$\text{etc.} -\frac{1}{4}, -\frac{1}{3}, -\frac{1}{2}, -\frac{1}{1}, +\frac{1}{0}, +\frac{1}{1}, +\frac{1}{2}, +\frac{1}{3} \text{ etc.}$$

A dextra ergo ad sinistram progrediendo termini crescunt, ita ut $\frac{1}{0}$ iam sit infinite magnus; quem cum transierint, fient negativi decrescentes. Hinc quantitas infinite magna spectari potest tanquam limes, per quem numeri affirmativi progressi fiunt negativi et vicissim; unde pluribus visum est numeros negativos considerari posse tanquam infinito maiores, propterea quod in hac serie termini continuo crescentes, postquam infinitum attigerint, abeant in negativos. At vero si ad seriem, cuius terminus generalis est $\frac{1}{xx}$ attendamus, post transitum per infinitum rursus prodeunt termini affirmativi

$$\text{etc.} +\frac{1}{9}, +\frac{1}{4}, +\frac{1}{1}, +\frac{1}{0}, +\frac{1}{1}, +\frac{1}{4}, +\frac{1}{9} \text{ etc.}$$

quos tamen nemo infinito maiores dixerit.

101. Saepenumero quoque in seriebus terminus infinitus constituit limitem terminos reales ab imaginariis segregantem, uti fit in serie hac, cuius terminus generalis est $\frac{1}{\sqrt{x}}$,

$$\text{etc.} +\frac{1}{\sqrt{-3}}, +\frac{1}{\sqrt{-2}}, +\frac{1}{\sqrt{-1}}, +\frac{1}{0}, +\frac{1}{\sqrt{1}}, +\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{3}} \text{ etc.}$$

neque tamen hinc sequitur imaginaria esse infinito maiora, quoniam ex serie ante allata

$$\text{etc.} +\sqrt{-3}, +\sqrt{-2}, +\sqrt{-1}, +0, +\sqrt{1}, +\sqrt{2}, +\sqrt{3} \text{ etc.}$$

aeque sequeretur imaginaria esse nihilo minora. Deinde vera etiam a terminis realibus transitus ad imaginarios exhiberi potest, quorum limes neque sit 0 neque ∞, uti fit, si terminus generalis fuerit $1+\sqrt{x}$. His autem casibus, cum ob irrationalitatem quilibet terminus geminum habeat valorem, in limite inter realia et imaginaria semper bini illi valores fiunt inter se aequales. At quoties termini, qui ante erant affirmativi, abeunt in negativos, transitus semper fit per limitem vel infinite parvum vel infinite magnum, quae omnia ex lege continuitatis, quam in lineis curvis deprehendimus, clarius elucent.

102. Ex summatione quoque serierum in infinitum excurrentium plura hic afferri possunt, quae cum ad hanc infiniti doctrinam magis illustrandam, tum vera ad plura dubia, quae in hoc negotio suboriri solent, delenda inserviunt. Ac primo quidem, si series constet ex terminis aequalibus ut

$$1+1+1+1+1+1+\text{etc}.$$

eaque sine fine, hoc est in infinitum, continuetur, nullum certe est dubium, quin omnium horum terminorum summa maior sit omni numero assignabili; eaque propterea infinita sit necesse est. Hoc quoque confirmat eius origo, dum oritur ex evolutione fractionis

$$\tfrac{1}{1-x}=1+x+x^2+x^3+\text{etc.}$$

ponendo $x=1$; erit ergo

$$\tfrac{1}{1-1}=1+1+1+1+\text{etc.}$$

ideoque summa

$$=\tfrac{1}{1-1}=\tfrac{1}{0}=\text{infinito.}$$

103. Quamvis autem hic nullum dubium nasci queat, cum idem numerus finitus infinities sumtus in infinitum abire debeat, tamen ipsa origo ex serie generali

$$\tfrac{1}{1-x}=1+x+x^2+x^3+x^4+x^5+\text{etc.}$$

gravissima incommoda afferre videtur; si enim pro *x* successive ponantur numeri 1, 2, 3, 4 etc., sequentes series cum suis summis prodibunt

$$\begin{aligned}
&A\ldots\ 1+1+1+1+1+\text{etc.}=\tfrac{1}{1-1}=\text{infinito}\\
&B\ldots\ 1+2+4+8+16+\text{etc.}=\tfrac{1}{1-2}=-1\\
&C\ldots\ 1+3+9+27+81+\text{etc.}=\tfrac{1}{1-3}=-\tfrac{1}{2}\\
&D\ldots\ 1+4+16+64+256+\text{etc.}=\tfrac{1}{1-4}=-\tfrac{1}{3}\\
&\qquad\qquad\text{etc.}
\end{aligned}$$

Cum igitur series *B* singulos terminos praeter primum habeat maiores quam series *A*, summa seriei *B* necessario multo maior esse deberet quam summa seriei *A;* interim tamen iste calculus ostendit seriei *A* summam infinitam, seriei *B* vero summam negativam, hoc est nihilo minorem, quod concipi non potest. Multo minus cum solitis ideis conciliari potest, quemadmodum huius et sequentium serierum *C, D* etc. summae fiant negativae, cum tamen omnes termini sint affirmativi.

104. Ob hanc rationem opinio supra allata multis probabilis videri solet, quantitates scilicet negativas quandoque considerari posse tanquam infinito maiores seu plus quam infinitas; et cum etiam numeros ultra nihil diminuendo perveniatur ad negativos, discrimen statuunt inter numeros negativos huiusmodi $-1, -2, -3$ etc. et huiusmodi $\frac{+1}{-1}, \frac{+2}{-1}, \frac{+3}{-1}$ etc., illos nihilo minores, hos vero infinito maiores dicendo. Verumtamen hoc pacta difficultatem non tollunt, quam suggerit haec series

$$1+2x+3x^2+4x^3+5x^4+\text{etc.}=\frac{1}{(1-x)^2},$$

unde oriuntur sequentes series

$$A\ldots\ 1+2+\ 3\ +\ 4\ +\ 5+\text{etc.}=\frac{1}{(1-1)^2}=\frac{1}{0}=\text{infinito}$$

$$B\ldots\ 1+4\ +12\ +32\ +\ 80+\text{etc.}=\frac{1}{(1-2)^2}=1;$$

ubi cum singuli termini seriei *B* sint maiores quam singuli termini seriei *A* primis solis exceptis, quemadmodum summa seriei *A* sit infinita, seriei *B* vero summa aequalis 1, hoc est soli termino primo, ex illo principio explicari omnino nequit.

105. Quoniam autem, si vellemus negare esse $-1 = \frac{+1}{-1}$ et $\frac{+a}{-b} = \frac{-a}{+b}$,
firmissima analyseos fundamenta collaberentur, illa ante commemorata explicatio prorsus admitti non potest. Quin potius negare debebimus illas, quas formulae generales suppeditaverant, summas esse veras. Cum enim hae series ex continua divisione oriantur, dum residuum continuo ulterius dividitur, residuum autem perpetuo fiat maius, quo longius progrediamur, id nunquam negligere poterimus; atque minime residuum ultimum, hoc est, quod in divisione infinitesima remanet, omitti potest, quippe quod fit infinite magnum. Quia autem hoc in superioribus seriebus non observatur, dum nullius residui ratio habetur, mirum non est eas summationes ad absurdum deducere. Haecque responsio, uti est ex ipsa serierum genesi petita, ita quoque est verissima atque omnem dubitationem tollit.

106. Quo hoc clarius appareat, contemplemur evolutionem fractionis $\frac{1}{1-x}$ uti in terminis primum finitis tantum absolvitur. Erit ergo

$$\frac{1}{1-x}=1+\frac{x}{1-x}$$
$$\frac{1}{1-x}=1+x+\frac{x^2}{1-x}$$
$$\frac{1}{1-x}=1+x+x^2+\frac{x^3}{1-x}$$
$$\frac{1}{1-x}=1+x+x^2+x^3+\frac{x^4}{1-x}$$
etc.

Qui ergo dicere vellet huius seriei finitae $1+x+x^2+x^3$ summam esse $\frac{1}{1-x}$ is erraret a vero quantitate $\frac{x^4}{1-x}$, et qui summam huius seriei

$$1+x+x^2+x^3+.........+x^{1000}$$

statuere vellet $=\frac{1}{1-x}$ is erraret quantitate $\frac{x^{1001}}{1-x}$ qui error, si x sit numerus unitate maior, foret maximus.

107. Ex his perspicuum est eum, qui eiusdem seriei in infinitum continuatae seu huius

$$1+x+x^2+x^3+\cdots\cdots+x^{\infty}$$

summam statuere velit $=\frac{1}{1-x}$ a veritate esse aberraturum quantitate $\frac{x^{\infty+1}}{1-x}$, quae, si sit $x > 1$, utique erit infinite magna. Simul vero hinc ratio patet, cur seriei in infinitum continuatae $1+x+x^2+x^3+x^4+$ etc. summa revera sit $=\frac{1}{1-x}$, si fuerit x fractio unitate minor; tum enim error $\frac{x^{\infty+1}}{1-x}$ fit infinite parvus ideoque nullus, cuius propterea ratio tuto potest negligi. Sic posito $x=\frac{1}{2}$ erit revera

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\text{etc.}=\frac{1}{1-\frac{1}{2}}=2$$

similiterque reliquarum serierum, si x sit fractio unitate minor, summa vera hoc modo indicatur.

108. Haec eadem responsio valet de summis serierum divergentium, in quibus signa + et – alternantur, quae vulgo ex eadem formula exhiberi solent ponendo pro x numeros negativos. Cum enim sit

$$\frac{1}{1+x}=1-x+x^2-x^3+x^4-x^5+\text{etc.},$$

nisi ultimi residui ratio habeatur, foret

$$A\ldots 1-1+1-1+1-1+\text{etc.}=\tfrac{1}{2}$$
$$B\ldots 1-2+4-8+16-32+\text{etc.}=\tfrac{1}{3}$$
$$C\ldots 1-3+9-27+81-243+\text{etc.}=\tfrac{1}{4}$$
etc.

Patet autem seriei secundae *B* summam ideo non posse esse $=\frac{1}{3}$, cum, quo plures termini actu summentur, aggregata eo magis ab $\frac{1}{3}$ recedant. Perpetuo autem cuiusque seriei summa debet esse limes, ad quem eo propius perveniatur, quo plures termini actu addantur.

109. Ex his quidam concluserunt huiusmodi series, quae vocantur divergentes, prorsus nullas habere summas fixas, propterea quod colligendis actu terminis ad nullum limitem fiat appropinquatio, qui pro summa seriei in infinitum continuatae haberi posset; quae sententia, cum istae summae iam ob neglecta ultima residua erroneae sint ostensae, veritati maxime est consentanea. Interim tamen contra eam summo iure obiici potest has memoratas summas, quantumvis a veritate abhorrere videantur, tamen nunquam in errores inducere, quin potius iis admissis plurima praeclara esse eruta, quibus, si istas summationes prorsus reiicere vellemus, carendum esset. Neque vero hae summae, si essent falsae, perpetuo ad veritatem nos ducere possent, quin potius, cum non parum, sed infinite a veritate discrepent, nos quoque in infinitum a vero seducere deberent. Quod tamen cum non eveniat, difficillimus nobis restat nodus solvendus.

110. Dico igitur in voce *summae* latere totam difficultatem; si enim *summa* seriei, ut vulgo usus fert, sumatur pro aggregato omnium eius terminorum actu collectorum, tum dubium est nullum, quin earum tantum serierum in infinitum excurrentium summae exhiberi queant, quae sint convergentes atque continuo propius ad certum statumque valorem deducant, quo plures termini actu colligantur. Series autem divergentes, quarum termini non decrescunt, sive signa + et – alternentur sive secus, prorsus nullas habebunt summas fixas, si quidem vox summae hoc sensu pro aggregato omnium terminorum accipiatur. At vero in iis casibus, quorum meminimus, quibus ex istiusmodi summis erroneis veritas tamen elicitur, id non fit, quatenus expressio finita, verbi gratia $\frac{1}{1-x}$ est summa seriei $1+x+x^2+x^3+$ etc., sed quatenus ea expressio evoluta hanc seriem praebet; sicque in hoc negotio nomen summae prorsus omitti posset.

111. Haec igitur incommoda hasque apparentes contradictiones penitus evitabimus, si voci *summae* aliam notionem, atque vulgo fieri solet, tribuamus. Dicamus ergo seriei cuiusque infinitae *summam* esse expressionem finitam, ex cuius evolutione illa series nascatur. Hocque sensu seriei infinitae $1+x+x^2+x^3+$ etc. summa revera erit $=\frac{1}{1-x}$ quia illa series ex huius fractionis evolutione oritur, quicunque numerus loco *x* substituatur. Hoc pacto, si series fuerit convergens, ista nova vocis summae definitio cum consueta congruet, et quia divergentes nullas habent summas proprie sic dictas, hinc nullum incommodum ex nova hac appellatione orietur. Denique ope huius definitionis utilitatem serierum divergentium tueri atque ab omnibus iniuriis vindicare poterimus.