CHAPTER II

CONCERNING THE USE OF DIFFERENCES IN THE TEACHING OF SERIES

37. It is known well enough that the nature of series can be illustrated especially from first principles by differences. Indeed the [principle of] an arithmetical progression, which it is customary to consider first, is concerned with this particular property, as the first differences of this are equal to each other; hence the second differences and the rest are all zero. Then series are given, of which the second differences shall be equal at last, which on this account are commonly called of the *second order*, while arithmetical progressions are called series of the *first order*. Therefore again there are series of the *third order,* of which the third differences are constant, and to the *fourth order* and these following sequences refer to series, of which the fourth or higher differences at last are constant.

38. Boundless kinds of series are dealt with in this division, and yet it is not permitted to recall all series to this kind. Indeed innumerable series occur, which with the differences taken never lead to constant terms ; of this kind, besides innumerable others, are the geometrical progressions, which under no circumstances will give constant differences, as it can be seen from this example

$$1, \quad 2, \quad 4, \quad 8, \quad 16, \quad 32, \quad 64, \quad 128 \text{ etc.}$$
$$1, \quad 2, \quad 4, \quad 8, \quad 16, \quad 32, \quad 64 \quad \text{etc.}$$
$$1, \quad 2, \quad 4, \quad 8, \quad 16, \quad 32 \quad \text{etc.}$$

For since the series of the differences of each order shall be equal to the proposed series itself, in short the equality of the differences is excluded. On this account several classes of series must be established; in this chapter we will consider chiefly as the class, only the one which is subdivided into these orders which finally are reduced to constant differences;

39. But initially two things are required to be understood about the nature of series usually, the general term and the sum, or the term expressing the sum. The general term is an indefinite expression, which includes each one of the terms of a series and therefore is a function of this kind of the variable quantity *x*, which on putting $x = 1$ exhibits the first term of the series, the second truly on putting $x = 2$, the third on putting $x = 3$, the fourth on putting $x = 4$ and thus so on. Therefore with the general term known any term of the series may be found, even if the law may not be considered by which the individual terms are summed together. Thus for example on putting $x = 1000$ at once the thousandth term may be known. Thus the general term of this series

$$1, 6, 15, 28, 45, 66, 91, 120 \text{ etc}.$$

is $2xx-x$; for on putting $x=1$ this formula gives the first term 1; on putting $x=2$ the second term 6 arises; if there is put $x=3$, the third arises 15 etc.; from which it is clear that the hundredth term of this series on putting $x=100$ becomes $=2\cdot 10000-100=19900$.

40. The *indices* or *exponents* in any series are called the numbers, which indicate how great each term shall be in the order; thus the index of the first term shall be 1, of the second 2, of the third 3 and so on thus. Hence the indices of the individual terms of any series are accustomed to be written in this manner :

Indices

1, 2, 3, 4, 5, 6, 7 etc.

Terms

A, B, C, D, E, F, G etc.,

from which it is at once clear that *G* is the seventh term of the proposed series, since the index of which shall be 7. Hence the general term is nothing other than the term of the series of which the index or the exponent is the indefinite number *x*. Therefore just as in any order of the series, of which either the first or second or any other following are constant, it is required that first we show how to find the general term; then truly we shall move on to the investigation of the sum.

41. We may start from the first order, which contains arithmetical progressions, of which the first differences are constant; and let *a* be the first term of the series and *b* the first term of the difference, to which all the following are equal; from which the series thus will be prepared :

Indices

1, 2, 3, 4, 5, 6 etc.

Terms

a, $a+b$, $a+2b$, $a+3b$, $a+4b$, $a+5b$ etc.

Differences

b, b, b, b, b etc.

From which it is evident at once that the term, the index of which shall be $=x$, is $a+(x-1)b$ and therefore the general term will be $=bx+a-b$, which is composed of both the first term as well as of the differences of the series. But if the second term of the series $a+b$ may be called a^{I}, on account of $b=a^{\mathrm{I}}-a$ the general term will be

$$=\left(a^{\mathrm{I}}-a\right)x+2a-a^{\mathrm{I}}=a^{\mathrm{I}}(x-1)-a(x-2),$$

from which the general term of this series will be formed from the known first and second terms of the arithmetic progression.

42. There shall be in a series of the second order first terms of the series itself $=a$, of the first differences $=b$, of the second differences $=c$ and thus the series itself with its differences will be composed:

Indices

1, 2, 3, 4, 5, 6 7 etc.

Terms

$a,\ a+b,\ a+2b+c,\ a+3b+3c,\ a+4b+6c,\ a+5b+10c,\ a+6b+15c$ etc.

The first differences

$b,\ \ b+c,\ \ b+2c,\ \ b+3c,\ \ b+4c,\ \ b+5c,$ etc.

The second differences

$c,\ \ c,\ \ c,\ \ c,\ \ c,$ etc.

From the inspection of which it will be apparent that the term of which the index $=x$, shall be

$$=a+(x-1)b+\tfrac{(x-1)(x-2)}{1\cdot 2}c,$$

which therefore is the general term of the proposed series. But the second term of this series is put $=a^{\mathrm{I}}$, the third term $=a^{\mathrm{II}}$; since there shall be

$$b=a^{\mathrm{I}}-a \quad \text{and} \quad c=a^{\mathrm{II}}-2a^{\mathrm{I}}+a,$$

as it is understood from the nature of the differences (§10), the general term will be

$$a+(x-1)\left(a^{\mathrm{I}}-a\right)+\tfrac{(x-1)(x-2)}{1\cdot 2}\left(a^{\mathrm{II}}-2a^{\mathrm{I}}+a\right),$$

which is reduced to this form

$$\frac{a^{\mathrm{II}}(x-1)(x-2)}{1\cdot 2}-\frac{2a^{\mathrm{I}}(x-1)(x-3)}{1\cdot 2}+\frac{a(x-2)(x-3)}{1\cdot 2}$$

or also to this

$$\tfrac{a^{\mathrm{II}}}{2}(x-1)(x-2)-\tfrac{2a^{\mathrm{I}}}{2}(x-1)(x-3)+\tfrac{a}{2}(x-2)(x-3)$$

or finally to this

$$\tfrac{1}{2}(x-1)(x-2)(x-3)\left(\tfrac{a^{\mathrm{II}}}{x-3}-\tfrac{2a^{\mathrm{I}}}{x-2}+\tfrac{a}{x-1}\right)$$

and thus it is defined from three terms of the series.

43. Let $a, a^{\mathrm{I}}, a^{\mathrm{II}}, a^{\mathrm{III}}, a^{\mathrm{IV}}$ etc. be a series of the third order, of which the first differences shall be $b, b^{\mathrm{I}}, b^{\mathrm{II}}, b^{\mathrm{III}}$ etc., the second $c, c^{\mathrm{I}}, c^{\mathrm{II}}, c^{\mathrm{III}}$ etc., and the third d, d, d, d etc., which evidently are constants.

Indices

1, 2, 3, 4, 5, 6 etc.

Terms

$a,\ a^{\mathrm{I}}, a^{\mathrm{II}}, a^{\mathrm{III}}, a^{\mathrm{IV}}, a^{\mathrm{V}}$ etc.

The first differences

$b,\ b^{\mathrm{I}}, b^{\mathrm{II}}, b^{\mathrm{III}}, b^{\mathrm{IV}}$ etc.

The second differences

$c,\ c^{\mathrm{I}}, c^{\mathrm{II}}, c^{\mathrm{III}}$, etc.

The third differences

$d,\ d,\ d$ etc.

Because there is

$$a^{\mathrm{I}}=a+b,\ a^{\mathrm{II}}=a+2b+c,\ a^{\mathrm{III}}=a+3b+3c+d,\ a^{\mathrm{IV}}=a+4b+6c+4d \text{ etc.,}$$

there will be the general term or this, the index of which is x,

$$a+\frac{(x-1)}{1}\,b+\frac{(x-1)(x-2)}{1\cdot 2}c+\frac{(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3}d$$

and thus the general term will be formed from the differences. But since again there will be

$$b=a^{\mathrm{I}}-a,\ c=a^{\mathrm{II}}-2a^{\mathrm{I}}+a,\ d=a^{\mathrm{III}}-3a^{\mathrm{II}}+3a^{\mathrm{I}}-a,$$

if these values may be substituted, the general term will be

$$a^{\mathrm{III}}\frac{(x-1)(x-2)(x-3)}{1\cdot2\cdot3}-3a^{\mathrm{II}}\frac{(x-1)(x-2)(x-4)}{1\cdot2\cdot3}$$
$$+3a^{\mathrm{I}}\frac{(x-1)(x-2)(x-4)}{1\cdot2\cdot3}-a\frac{(x-2)(x-3)(x-4)}{1\cdot2\cdot3},$$

which also may be expressed in this manner, so that it shall be

$$=\frac{(x-1)(x-2)(x-3)(x-4)}{1\cdot2\cdot3}\left(\frac{a^{\mathrm{III}}}{x-4}-\frac{3a^{\mathrm{II}}}{x-3}+\frac{3a^{\mathrm{I}}}{x-2}-\frac{a}{x-1}\right).$$

44. Now let a series of any order be proposed;

The indices
$1,\quad 2,\quad 3,\quad 4,\quad 5,\quad 6\quad$ etc.
of the terms
$a,\quad a^{\mathrm{I}},\ a^{\mathrm{II}},a^{\mathrm{III}},a^{\mathrm{IV}},a^{\mathrm{V}}$ etc.
The first differences
$b,\quad b^{\mathrm{I}},\ b^{\mathrm{II}},\ b^{\mathrm{III}},\ b^{\mathrm{IV}}$ etc.
The second differences
$c,\quad c^{\mathrm{I}},\ c^{\mathrm{II}},\ c^{\mathrm{III}},$ etc.
The third differences
$d,\quad d^{\mathrm{I}},\ d^{\mathrm{II}}$ etc.
The fourth differences
$e,\quad e^{\mathrm{I}}$ etc.
The fifth differences
f etc.

Thus the general term may be expressed from the first term of the series itself and from the first terms of the differences, *b, c, d, e, f* etc., so that there shall be

$$a+\frac{(x-1)}{1}b+\frac{(x-1)(x-2)}{1\cdot2}c+\frac{(x-1)(x-2)(x-3)}{1\cdot2\cdot3}d+\frac{(x-1)(x-2)(x-3)(x-4)}{1\cdot2\cdot3\cdot4}e+\text{etc.},$$

until constant differences may be come upon. From which it is evident, if constant differences are not produced in any circumstances, the general term is to be shown by an infinite expression.

45. Because the differences may be formed from the terms of the series itself, if the values of these may be substituted, the expression will produce a general term in a form of this kind, of which kind we have shown for the series of the first, second, and third orders. Evidently for a series of the fourth order there will be a general term

$$=\frac{(x-1)(x-2)(x-3)(x-4)(x-5)}{1\cdot2\cdot3\cdot4}\left(\frac{a^{\mathrm{IV}}}{x-5}-\frac{4a^{\mathrm{III}}}{x-4}+\frac{6a^{\mathrm{II}}}{x-3}-\frac{4a^{\mathrm{I}}}{x-2}+\frac{a}{x-1}\right),$$

from which the general law, from which the law of the following orders of terms are composed, is easily seen. But from these it may be apparent for any order that the general term is to be an integral rational function of *x*, in which the greatest power of *x* shall agree with the order, to which the series may be returned. Thus the general term of a series of the first order will be a function of the first power, of the second order the second power and thus again.

46. But the differences, as we have seen above, thus result themselves from these terms of the series, so that there shall be

$$\begin{array}{lll} b=a^{\mathrm{I}}-a & c=a^{\mathrm{II}}-2a^{\mathrm{I}}+a & d=a^{\mathrm{III}}-3a^{\mathrm{II}}+3a^{\mathrm{I}}-a \\ b^{\mathrm{I}}=a^{\mathrm{II}}-a^{\mathrm{I}} & c^{\mathrm{I}}=a^{\mathrm{III}}-2a^{\mathrm{II}}+a^{\mathrm{I}} & d^{\mathrm{I}}=a^{\mathrm{IV}}-3a^{\mathrm{III}}+3a^{\mathrm{II}}-a^{\mathrm{I}} \\ b^{\mathrm{II}}=a^{\mathrm{III}}-a^{\mathrm{II}} & c^{\mathrm{II}}=a^{\mathrm{IV}}-2a^{\mathrm{III}}+a^{\mathrm{II}} & d^{\mathrm{II}}=a^{\mathrm{V}}-3a^{\mathrm{IV}}+3a^{\mathrm{III}}-a^{\mathrm{II}} \\ \text{etc.} & \text{etc.} & \text{etc.} \end{array}$$

Whereby, since in series of the first order all the values of $c=0$, there will be

$$a^{\mathrm{II}}=2a^{\mathrm{I}}-a,\quad a^{\mathrm{III}}=2a^{\mathrm{II}}-a^{\mathrm{I}},\quad a^{\mathrm{IV}}=2a^{\mathrm{III}}-a^{\mathrm{II}}\quad \text{etc.},$$

from which it may be evident likewise are to be recurring and the step of the relation to be $2,-1$. Successively, since in series of the second order all the values of $d=0$, there will be

$$a^{\mathrm{III}}=3a^{\mathrm{II}}-3a^{\mathrm{I}}+a,\quad a^{\mathrm{IV}}=3a^{\mathrm{III}}-3a^{\mathrm{II}}+a^{\mathrm{I}}\quad \text{etc.}$$

and thus these are recurring with the steps of the relation present $3,-3,+1$. In a similar manner it will appear all series of this class, of whatever order they shall be, likewise relate to a class of recurring series and thus indeed, so that the steps of the relation may depend on the coefficients of the binomial power one order higher than the order to which the series refers.

47. Because truly for series of the first order also all the values of d itself and e and of all the following differences are $=0$, there will be from these also

$$a^{\mathrm{III}} = 3a^{\mathrm{II}} - 3a^{\mathrm{I}} + a,$$
$$a^{\mathrm{IV}} = 3a^{\mathrm{III}} - 3a^{\mathrm{II}} + a^{\mathrm{I}}$$
etc.

or

$$a^{\mathrm{IV}} = 4a^{\mathrm{III}} - 6a^{\mathrm{II}} + 4a^{\mathrm{I}} - a$$
$$a^{\mathrm{V}} = 4a^{\mathrm{IV}} - 6a^{\mathrm{III}} + 4a^{\mathrm{II}} - a^{\mathrm{I}}$$

Therefore hence they will relate to recurrent series and that in an infinite number of ways, since the steps of the relation are able to be

$$3, -3, +1;\ 4, -6, +4, -1;\ 5, -10, +10, -5, +1 \text{ etc.}$$

In a similar manner it is understood each series of this, that we have discussed, likewise to be a series recurring in classes of innumerable kinds; for the steps of the relation will be

$$\frac{n}{1}, -\frac{n(n-1)}{1\cdot 2}, +\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}, -\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} \text{ etc.,}$$

provided n shall be a whole number greater than the number by which the order is indicated. Therefore this series may arise also from setting out fractions, the denominator of which is $(1-y)^n$, as has been further shown in the above book concerning recurring series.

48. Just as we have seen of all of this class of series, of whatever order they shall be, the general terms are rational whole number functions of *x*, thus in turn it will be apparent all series, of which the general terms shall be functions of *x* of this kind, relate to this class and are led in the end to constant differences. And indeed, if the general term were a function of the first degree $ax+b$, while thence the series arising from that will be of the first order or arithmetic, the first differences will be considered constant. But if the general term were a function of the second degree satisfied by this form $axx+bx+c$, then the series originating from that, while in place of *x* successively the numbers 1, 2, 3, 4, 5 etc. are substituted, will be of the second order and it will have constant second order differences; in a similar manner the general term of the third degree ax^3+bx^2+cx+d will give a series of the third order and thus henceforth.

49. Indeed from the general term not only all the terms of the series may be deduced, but also the series can be deduced both of the first as well as of the following differences. For since, if the first term of the series is subtracted from the second, the first term of the differences of the series may be produced, moreover following, if the second term of the series is taken from the third; thus the term of the differences of the series will be obtained, of which the index is *x,* if the term of this series, of which the index is *x,* is taken from the following, of which the index is $x+1$. Whereby if in place of *x* in the general term of the series there is put $x+1$ and from that value the general term is subtracted, the general term of the differences of the series will remain; therefore if X were a term of a general series, the difference of this would be ΔX (in which manner this can be found has been shown in the previous chapter, if in that place there is put in place $\omega = 1$) the general term of the series of the first differences. In a similar manner therefore there will be $\Delta\Delta X$ the general term of the second order differences, $\Delta^3 X$ of the third and so henceforth.

50. But if moreover the general term X were a rational integral function, in which the greatest exponent of the power of x shall be n, from the preceding chapter it is deduced that the difference of this ΔX becomes a function less by one degree, clearly degree $n-1$. And hence again $\Delta\Delta X$ will be a function of degree $n-2$ and $\Delta^3 X$ a function of degree $n-3$, and so on in this manner. Whereby, if X were a function of the first degree, as $ax+b$, then the difference of this ΔX will be the constant $= a$; which since it shall be the general term of the series of the first differences, it is seen that the series, of which the general term X shall be a function of the first degree, to be arithmetical or of the first order. In a similar manner, if the general term X were a function of the second degree, on account of the constant $\Delta\Delta X$ the series thence arising will have constant second differences and therefore it will be of the second order; and thus continually; of which degree were a function X constituting the general term, the series arising from that will be of the same order.

51. On account of this matter the series of powers of natural arrive at constant differences, as becomes evident from the following scheme.

First powers
1, 2, 3, 4, 5, 6, 7, 8 etc.

First differences
1, 1, 1, 1, 1, 1, 1 etc.

Second powers
1, 4, 9, 16, 25, 36, 49, 64 etc.

First differences
3, 5, 7, 9, 11, 13, 15 etc.

Second differences
2, 2, 2, 2, 2, 2 etc.

Third powers
1, 8, 27, 64, 125, 216, 343 etc.

First differences
7, 19, 37, 61, 91, 127 etc.

Second differences
12, 18, 24, 30, 36 etc.

Third differences
6, 6, 6, 6 etc.

Fourth powers
1, 16, 81, 256, 625, 1296, 2401 etc.

First differences
15, 65, 175, 369, 671, 1105 etc.

Second differences
50, 110, 194, 302, 434 etc.

Third differences
60, 84, 108, 132 etc.

Fourth differences
24, 24, 24 etc.

Which therefore are the precepts concerning the differences of each order to be found in the preceding chapter, here those account for finding the general terms of any differences which arise from the series in the preceding chapter.

52. If the general term of any series were known, with the help of this not only all the terms of this are to be found indefinitely, but also the series continued backwards, the exponents of which are negative numbers, can be shown by substituting negative numbers in place of x ; thus, if the general term were $\frac{xx+3x}{2}$, on putting both negative as well as positive numbers in place of x a series of this kind will be continued on both sides :

Indices
etc. $-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6$ etc.

Series
etc. $+5, +2, 0, -1, -1, 0, 2, 5, 9, 14, 20, 27$ etc.

First differences
etc. $-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7$ etc.

Second differences
etc. 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 etc.

Therefore since the general term may be formed from the differences, each series can be continued backwards from the differences, thus indeed, so that, if the differences finally become constant, these terms are to be shown within limits, truly contrary to the infinite expression they are able to be assigned. So that also from the general term these terms may be defined, of which the indices are fractions, in which an *interpolation* of the series may put in place.

53. From the admonitions given about the general term of a series we may move on to the sum or the summation term of each order of the series to be found. Moreover for any proposed series the summation term for the series is a function of *x*, which is equal to as many terms of the series, as there are units contained in the number *x*. Therefore the summation term is prepared thus, so that if there is put $x = 1$, the first term of the series may be produced, but if there is put $x = 2$, so that there may be produced the sum of the first and second terms, moreover on making $x = 3$ the sum of the first, second, and of the third terms and thus henceforth. Hence, if from the proposed series a new series may be formed, the first term of which shall be equal to the first term of that, the second equal to the sum of the two, the third equal to the sum of the three and thus henceforth, this new series may be called the *summatory series* of that [*i.e.* the partial sum of the series] and the general term of the summatory series of this series will be the summation term of the proposed series ; from which the finding of the summatory term may be reduced to finding the general term.

54. Therefore let this proposed series be

$$a, \quad a^{\mathrm{I}}, \quad a^{\mathrm{II}}, \quad a^{\mathrm{III}}, \quad a^{\mathrm{IV}}, \quad a^{\mathrm{V}} \text{ etc.}$$

and the summatory series of this series shall be

$$A, \quad A^{\mathrm{I}}, \quad A^{\mathrm{II}}, \quad A^{\mathrm{III}}, \quad A^{\mathrm{IV}}, \quad A^{\mathrm{V}} \text{ etc.};$$

from the nature of this, it can be set out in the manner

$$A = a$$
$$A^{\mathrm{I}} = a + a^{\mathrm{I}}$$
$$A^{\mathrm{II}} = a + a^{\mathrm{I}} + a^{\mathrm{II}}$$
$$A^{\mathrm{III}} = a + a^{\mathrm{I}} + a^{\mathrm{II}} + a^{\mathrm{III}}$$
$$A^{\mathrm{IV}} = a + a^{\mathrm{I}} + a^{\mathrm{II}} + a^{\mathrm{III}} + a^{\mathrm{IV}}$$
$$A^{\mathrm{V}} = a + a^{\mathrm{I}} + a^{\mathrm{II}} + a^{\mathrm{III}} + a^{\mathrm{IV}} + a^{\mathrm{V}}$$

Hence the differences of the summatory series will be

$$A^{\mathrm{I}} - A = a^{\mathrm{I}}, \quad A^{\mathrm{II}} - A^{\mathrm{I}} = a^{\mathrm{II}}, \quad A^{\mathrm{III}} - A^{\mathrm{II}} = a^{\mathrm{III}} \text{ etc.,}$$

from which the proposed series with the small first term will be the series of the first differences of the summatory series. But if therefore there may be set in front of the summatory series a term $= 0$, so that there may be considered

$$0, \quad A, \quad A^{\mathrm{I}}, \quad A^{\mathrm{II}}, \quad A^{\mathrm{III}}, \quad A^{\mathrm{IV}}, \quad A^{\mathrm{V}} \text{ etc.},$$

then the series of the first differences of this series will be the proposed series itself

$$a, \quad a^{\mathrm{I}}, \quad a^{\mathrm{II}}, \quad a^{\mathrm{III}}, \quad a^{\mathrm{IV}}, \quad a^{\mathrm{V}} \text{ etc.}$$

[One may wonder why Euler did not consider a suffix notation for the indices as we now do, which would not have got in the way of powers, etc. ; in any case, the original series is replaced by the first differences of the summatory series, so that the sum of the original series will collapse into a single summatory term;]

55. On account of this matter the first differences of the proposed series are the second differences of the summatory terms and the second differences of that first series are the third differences of this summatory one, moreover the third differences of that first series are the fourth differences of this summatory series, and thus so on. Whereby if the proposed series finally may have constant differences, then also the summatory series of this may be deduced, and therefore it will be a series of the same nature, but higher in order by one. Therefore the summatory term of this kind of series always can be expressed by a finite expression. In as much as the general term of the series

$$0, \quad A, \quad A^{\mathrm{I}}, \quad A^{\mathrm{II}}, \quad A^{\mathrm{III}}, \quad A^{\mathrm{IV}}, \quad \text{etc.}$$

or this, which it is agreed to be indicated by *x*, will show the sum of $x-1$ terms of this series $a, \ a^{\mathrm{I}}, \ a^{\mathrm{II}}, \ a^{\mathrm{III}}, \ a^{\mathrm{IV}}, \ a^{\mathrm{V}}$ etc., and if then in place of x there may be written

$x+1$, there may arise the sum of x terms and the summatory term itself. [Note that in the manner the first series is defined, the term a is the first term, while a^{I} is the second term, corresponding to $x=2$, and thus in general $x+1$ is the summatory term of x elements]

56. Therefore let the series of the first differences of the proposed series

$$a, \quad a^{\mathrm{I}}, \quad a^{\mathrm{II}}, \quad a^{\mathrm{III}}, \quad a^{\mathrm{IV}}, \quad a^{\mathrm{V}}, \quad a^{\mathrm{VI}} \text{ etc.}$$

be

$$b, \quad b^{\mathrm{I}}, \quad b^{\mathrm{II}}, \quad b^{\mathrm{III}}, \quad b^{\mathrm{IV}}, \quad b^{\mathrm{V}}, \quad b^{\mathrm{VI}} \text{ etc.,}$$

the series of the second differences

$$c, \quad c^{\mathrm{I}}, \quad c^{\mathrm{II}}, \quad c^{\mathrm{III}}, \quad c^{\mathrm{IV}}, \quad c^{\mathrm{V}}, \quad c^{\mathrm{VI}} \text{ etc.,}$$

the series of the third differences

$$d, \quad d^{\mathrm{I}}, \quad d^{\mathrm{II}}, \quad d^{\mathrm{III}}, \quad d^{\mathrm{IV}}, \quad d^{\mathrm{V}}, \quad d^{\mathrm{VI}} \text{ etc.,}$$

and thus so on, until constant differences may be reached. Then the summatory series may be formed, which with 0 put in place before the first term with the differences of this considered to be continued in the following manner :

Indices
1, 2, 3, 4, 5, 6, 7 etc.

Summatory terms

$$0, \quad A, \quad A^{\mathrm{I}}, \quad A^{\mathrm{II}}, \quad A^{\mathrm{III}}, \quad A^{\mathrm{IV}}, \quad \text{etc.}$$

Proposed series

$$a, \quad a^{\mathrm{I}}, \quad a^{\mathrm{II}}, \quad a^{\mathrm{III}}, \quad a^{\mathrm{IV}}, \quad a^{\mathrm{V}}, \quad a^{\mathrm{VI}} \text{ etc.}$$

First differences

$$b, \quad b^{\mathrm{I}}, \quad b^{\mathrm{II}}, \quad b^{\mathrm{III}}, \quad b^{\mathrm{IV}}, \quad b^{\mathrm{V}}, \quad b^{\mathrm{VI}} \text{ etc.}$$

Second differences

$$c, \quad c^{\mathrm{I}}, \quad c^{\mathrm{II}}, \quad c^{\mathrm{III}}, \quad c^{\mathrm{IV}}, \quad c^{\mathrm{V}}, \quad c^{\mathrm{VI}} \text{ etc.}$$

Third differences

$$d, \quad d^{\mathrm{I}}, \quad d^{\mathrm{II}}, \quad d^{\mathrm{III}}, \quad d^{\mathrm{IV}}, \quad d^{\mathrm{V}}, \quad d^{\mathrm{VI}} \text{ etc.}$$

The general term of the summatory series, or which corresponds to the index *x*, will be

$$0+(x-1)a+\frac{(x-1)(x-2)}{1\cdot 2}b+\frac{(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3}c+\text{etc.},$$

which shows likewise the sum of $x-1$ terms of the proposed series

$$a, \quad a^{\text{I}}, \quad a^{\text{II}}, \quad a^{\text{III}}, \quad a^{\text{IV}}, \quad a^{\text{V}}, \quad a^{\text{VI}} \text{ etc.}$$

57. But if therefore in this sum there may be written *x* in place of $x-1$, there will be produced the summatory term of the proposed series in-folding the sum of *x* terms

$$xa+\frac{x(x-1)}{1\cdot 2}b+\frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}c+\frac{x(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3\cdot 4}d+\text{etc.},$$

Hence, if the letters *b*, *c*, *d*, *e* etc. may retain their assigned values, there will be

the general term

$$a+(x-1)b+\frac{(x-1)(x-2)}{1\cdot 2}c+\frac{(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3}d+\frac{(x-1)(x-2)(x-3)(x-4)}{1\cdot 2\cdot 3\cdot 4}e+\text{etc.}$$

of the series

$$a, \quad a^{\text{I}}, \quad a^{\text{II}}, \quad a^{\text{III}}, \quad a^{\text{IV}}, \quad a^{\text{V}}, \quad a^{\text{VI}} \text{ etc.}$$

and

the summatory term

$$xa+\frac{x(x-1)b}{1\cdot 2}+\frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}c+\frac{(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3\cdot 4}d+\text{etc.}$$

Therefore with this found of a series of any order, which we have shown, in the same manner the general term may be found from that summatory term without difficulty, clearly which is constructed from the same differences.

58. Here the manner required for finding the summatory term through the differences of series has been adapted in the first place for series of this kind, which finally may be reduced to constant differences ; for in other cases a finite expression cannot be found. But if that, which before were set out concerning the nature of the summatory terms, we should look at with greater care, another way presents itself at once for finding the summatory term from the general term, which extends much wider and leads to finite expressions in an infinite number of cases it, for which the former way may show infinitudes. For let some proposed series be

$$a,\ b,\ c,\ d,\ e,\ f\quad \text{etc.},$$

of which the general term, or of the corresponding index x, shall be $=X$; but the summatory term shall be $=S$; which since it may show the sum of as many terms from the beginning, as the number x may contain units, the sum of $x-1$ terms $=S-X$ and clearly X will be the difference of the expression $S-X$, since it may be left, if this is subtracted from the following S.

59. Therefore since there shall be the difference $X=\Delta(S-X)$ taken in this manner, which we have shown in the preceding chapter, only with this distinction, as that constant quantity ω for us here shall be $=1$, whereby, if we may return the sums, there will be $\Sigma X=S-X$ and thus the summatory term sought

$$S=\Sigma X+X+C.$$

Therefore there must be sought the sum of the functions X by the method related before and to that to be added the general term itself X and the sum will be the summatory term. But since in the sums assumed a constant quantity is involved, either to be added or subtracted, that will be adapted to the present case. But is clear, if there is put $x=0$, in which case the number of terms to be summed is zero, the sum too becomes zero; from which that constant quantity C thus ought to be determined, so that on putting $x=0$ there is made also $S=0$. Therefore on putting in that equation $S=\Sigma X+X+C$ both $S=0$ as well as $x=0$ the value of C may be found.

60. Therefore because here the whole business is reduced by putting $\omega=1$ to the summation of functions shown above, from this we may set out the summations treated and the first indeed for the powers of x will [§ 27]

$$\Sigma x^0=\Sigma 1=x$$
$$\Sigma x=\tfrac{1}{2}x^2-\tfrac{1}{2}x$$
$$\Sigma x^2=\tfrac{1}{3}x^3-\tfrac{1}{2}x^2+\tfrac{1}{6}$$
$$\Sigma x^3=\tfrac{1}{4}x^4-\tfrac{1}{2}x^3+\tfrac{1}{4}x^2$$
$$\Sigma x^4=\tfrac{1}{5}x^5-\tfrac{1}{2}x^4+\tfrac{1}{3}x^3-\tfrac{1}{30}x$$
$$\Sigma x^5=\tfrac{1}{6}x^6-\tfrac{1}{2}x^5+\tfrac{5}{12}x^4-\tfrac{1}{12}x^2$$
$$\Sigma x^6=\tfrac{1}{7}x^7-\tfrac{1}{2}x^6+\tfrac{1}{2}x^5-\tfrac{1}{6}x^3+\tfrac{1}{42}x,$$

to which may be added the summation of the general power x^n treated in § 29 , provided everywhere there in place of ω there may be written one. Therefore the summatory terms of all series can be readily found with the help of these formulas, of which the general terms are whole rational functions of *x*.

61. Let *S.X* denote the summatory term of the series, of which the general term is $= X$, and there will be, as we have seen,

$$S.X = \Sigma X + X + C,$$

provided the constant *C* thus is assumed, so that the summatory term *S.X* may vanish on putting $x = 0$. Therefore hence we may express the summatory terms of the series of powers, or of which the general terms are taken in this form x^n. And thus on putting

$$S.x^n = 1 + 2^n + 3^n + 4^n + \ldots\ldots + x^n$$

there will be

$$\begin{aligned}
S.x^n = {} & \tfrac{1}{n+1}x^{n+1} + \tfrac{1}{2}x^n + \tfrac{1}{2}\cdot\tfrac{n}{2\cdot 3}x^{n-1} - \tfrac{1}{6}\cdot\tfrac{n(n-1)(n-2)}{2\cdot 3\cdot 4\cdot 5}x^{n-3} \\
& + \tfrac{1}{6}\cdot\tfrac{n(n-1)..(n-4)}{2\cdot 3..6\cdot 7}x^{n-5} - \tfrac{3}{10}\cdot\tfrac{n(n-1)..(n-6)}{2\cdot 3..8\cdot 9}x^{n-7} \\
& + \tfrac{5}{6}\cdot\tfrac{n(n-1)..(n-8)}{2\cdot 3..10\cdot 11}x^{n-9} - \tfrac{691}{210}\cdot\tfrac{n(n-1)..(n-10)}{2\cdot 3..12\cdot 13}x^{n-11} \\
& + \tfrac{35}{2}\cdot\tfrac{n(n-1)..(n-12)}{2\cdot 3..14\cdot 15}x^{n-13} - \tfrac{3617}{30}\cdot\tfrac{n(n-1)..(n-14)}{2\cdot 3..16\cdot 17}x^{n-15} \\
& + \tfrac{43867}{42}\cdot\tfrac{n(n-1)..(n-16)}{2\cdot 3..18\cdot 19}x^{n-17} - \tfrac{122227}{110}\cdot\tfrac{n(n-1)..(n-18)}{2\cdot 3..20\cdot 21}x^{n-19} \\
& + \tfrac{854513}{6}\cdot\tfrac{n(n-1)..(n-20)}{2\cdot 3..22\cdot 23}x^{n-21} - \tfrac{1181820455}{546}\cdot\tfrac{n(n-1)..(n-22)}{2\cdot 3..24\cdot 25}x^{n-23} \\
& + \tfrac{76977927}{2}\cdot\tfrac{n(n-1)..(n-24)}{2\cdot 3..26\cdot 27}x^{n-25} - \tfrac{23749461029}{30}\cdot\tfrac{n(n-1)..(n-26)}{2\cdot 3..28\cdot 29}x^{n-27}
\end{aligned}$$

etc.

62. Therefore hence the sums for the various values of *n* will themselves be had :

$$S.x^0 \quad = x$$
$$S.x^1 = \tfrac{1}{2}x^2 + \tfrac{1}{2}x$$
$$S.x^2 = \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + \tfrac{1}{6}x$$
$$S.x^3 = \tfrac{1}{4}x^4 + \tfrac{1}{2}x^3 + \tfrac{1}{4}x^2$$
$$S.x^4 = \tfrac{1}{5}x^5 + \tfrac{1}{2}x^4 + \tfrac{1}{3}x^3 - \tfrac{1}{30}x$$
$$S.x^5 = \tfrac{1}{6}x^6 + \tfrac{1}{2}x^5 + \tfrac{5}{12}x^4 - \tfrac{1}{12}x^2$$
$$S.x^6 = \tfrac{1}{7}x^7 + \tfrac{1}{2}x^6 + \tfrac{1}{2}x^5 - \tfrac{1}{6}x^3 + \tfrac{1}{42}x$$
$$S.x^7 = \tfrac{1}{8}x^8 + \tfrac{1}{2}x^7 + \tfrac{7}{12}x^6 - \tfrac{7}{24}x^4 + \tfrac{1}{12}x^2$$
$$S.x^8 = \tfrac{1}{9}x^9 + \tfrac{1}{2}x^8 + \tfrac{2}{3}x^7 - \tfrac{7}{15}x^5 + \tfrac{2}{9}x^3 - \tfrac{1}{30}x$$
$$S.x^9 = \tfrac{1}{10}x^9 + \tfrac{1}{2}x^9 + \tfrac{3}{4}x^8 - \tfrac{7}{10}x^6 + \tfrac{1}{2}x^4 - \tfrac{3}{20}x^2$$
$$S.x^{10} = \tfrac{1}{11}x^{11} + \tfrac{1}{2}x^{10} + \tfrac{5}{6}x^9 - x^7 + x^5 - \tfrac{1}{2}x^3 + \tfrac{5}{66}x$$
$$S.x^{11} = \tfrac{1}{12}x^{12} + \tfrac{1}{2}x^{11} + \tfrac{11}{12}x^{10} - \tfrac{11}{8}x^8 + \tfrac{11}{6}x^6 - \tfrac{11}{8}x^4 + \tfrac{5}{12}x^2$$
$$S.x^{12} = \tfrac{1}{13}x^{13} + \tfrac{1}{2}x^{12} + x^{11} - \tfrac{11}{6}x^9 + \tfrac{22}{7}x^7 - \tfrac{33}{10}x^5 + \tfrac{5}{3}x^3 - \tfrac{691}{2730}x$$
$$S.x^{13} = \tfrac{1}{14}x^{14} + \tfrac{1}{2}x^{13} + \tfrac{13}{12}x^{12} - \tfrac{143}{60}x^{10} + \tfrac{143}{28}x^8 - \tfrac{143}{20}x^6 + \tfrac{65}{12}x^4 - \tfrac{691}{420}x^2$$
$$S.x^{14} = \tfrac{1}{15}x^{15} + \tfrac{1}{2}x^{14} + \tfrac{7}{6}x^{13} - \tfrac{91}{30}x^{11} + \tfrac{143}{18}x^9 - \tfrac{143}{10}x^7 + \tfrac{91}{6}x^5 - \tfrac{691}{90}x^3 + \tfrac{7}{6}x$$
$$S.x^{15} = \tfrac{1}{16}x^{16} + \tfrac{1}{2}x^{15} + \tfrac{5}{4}x^{14} - \tfrac{91}{24}x^{12} + \tfrac{143}{12}x^{10} - \tfrac{429}{16}x^8 + \tfrac{455}{12}x^6 - \tfrac{691}{24}x^4 + \tfrac{35}{4}x^2$$
$$S.x^{16} = \tfrac{1}{17}x^{17} + \tfrac{1}{2}x^{16} + \tfrac{4}{3}x^{15} - \tfrac{14}{3}x^{13} + \tfrac{52}{3}x^{11} - \tfrac{143}{3}x^9 + \tfrac{260}{3}x^7 - \tfrac{1382}{15}x^5 + \tfrac{140}{3}x^3 - \tfrac{3617}{510}x$$

etc.,

which sums from the general form can be continued as far as to the twenty-ninth power. And at this point it is allowed to progress further, if these numerical coefficients are able to be elicited further.

63. Moreover from these formulas a certain rule may be observed, with the help of which any can easily be found from the preceding, with the exception of the final term only, if in that the first power of *x* may be contained; for then in the following sum an extra term is to be added. But with this omitted, if there should be the following sum

$$S.x^n = \alpha x^{n+1} + \beta x^n + \gamma x^{n-1} - \delta x^{n-3} + \varepsilon x^{n-5} - \zeta x^{n-7} + \eta x^{n-9} - \text{etc.},$$

there will be the following sum

$$S.x^{n+1} = \tfrac{n+1}{n+2}\alpha x^{n+2} + \tfrac{n+1}{n+1}\beta x^{n+1} + \tfrac{n+1}{n}\gamma x^{n} - \tfrac{n+1}{n-2}\delta x^{n-2} + \tfrac{n+1}{n-4}\varepsilon x^{n-4} - \tfrac{n+1}{n-6}\zeta x^{n-6} + \tfrac{n+1}{n-8}\eta x^{n-8} - \text{etc.},$$

from which, if n were an even number, the following sum indeed will be produced; but if n were an odd number, then in the following sum besides a final term will be desired, the form of which will be $\pm\varphi x$. Yet meanwhile this without other help may be found thus. For since, if there is put $x=1$, the sum of a single term only must arise (that is the first term, which will be $=1$), there is put in all the terms now found $x=1$ and the sum itself is placed $=1$; with which done the value of φ may be elicited and from that found it will be allowed to progress further. And from this understanding all these sums may be able to be found. Thus, since there shall be

$$S.x^{5} = \tfrac{1}{6}x^{6} + \tfrac{1}{2}x^{5} + \tfrac{5}{12}x^{4} - \tfrac{1}{12}x^{2}$$

there will be

$$S.x^{6} = \tfrac{6}{7}\cdot\tfrac{1}{6}x^{7} + \tfrac{6}{6}\cdot\tfrac{1}{2}x^{6} + \tfrac{6}{5}\cdot\tfrac{5}{12}x^{5} - \tfrac{6}{3}\cdot\tfrac{1}{12}x^{3} + \varphi x$$

or

$$S.x^{6} = \tfrac{1}{7}x^{7} + \tfrac{1}{2}x^{6} + \tfrac{1}{2}x^{5} - \tfrac{1}{6}x^{3} + \varphi x.$$

Now there is put $x = 1$; there becomes

$$1 = \tfrac{1}{7} + \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{6} + \varphi \quad \text{and thus} \quad \varphi = \tfrac{1}{6} - \tfrac{1}{7} = \tfrac{1}{42},$$

as we find from the general form.

64. Now with the aid of these summatory formulas of all series, of which the general terms are whole rational functions of x, the summatory terms are able to be found and this much more conveniently than by the previous method through differences.

EXAMPLE 1

To find the summatory term of this series 2, 7, 15, 26, 40, 57, 77, 100, 126 *etc.,*
of which the general term is $\frac{3xx+x}{2}$.

Since the general term may depend on two members, the summatory term is sought for each from the above formulas

$$S.\tfrac{3}{2}xx = \tfrac{1}{2}x^{3} + \tfrac{3}{4}xx + \tfrac{1}{4}x$$

and

$$\text{S}.\tfrac{1}{2}x = \ldots\ldots\ldots + \tfrac{1}{4}xx + \tfrac{1}{4}x$$

and there will be

$$S.\tfrac{3xx+x}{2} = \tfrac{1}{2}x^3 + xx + \tfrac{1}{2}x = \tfrac{1}{2}x(x+1)^2,$$

which is the summatory term sought. Thus, if there is put $x = 5$, there will be $\frac{5}{2}\cdot 6^2 = 90$ with the sum of the five terms

$$2+7+15+26+40 = 90.$$

EXAMPLE 2

To find the summatory term of the series 1, 27, 125, 343, 729, 1331 *etc., which contains the cubes of the odd numbers.*

The general term of this series is

$$= (2x-1)^3 = 8x^3 - 12xx + 6x - 1,$$

from which the summatory term may be deduced in the following manner:

$$+8\cdot S.x^3 = 2x^4 + 4x^3 + 2x^2$$

and

$$-12\cdot S.x^2 = ...-4x^3 - 6x^2 - 2x$$

and

$$+6\cdot S.x = ...+3x^2 + 3x$$

and hence

$$-1\cdot S.x^0 = ...-x$$

Evidently the sum sought will be $= 2x^4 \quad - x^2 = xx(2xx-1)$

As, if there is put $x = 6$, there will be $36\cdot 71 = 2556$ the sum of the six terms of the series proposed

$$1+27+125+343+729+1331 = 2556.$$

65. But if the general term were produced from several simple factors, then the summatory term may be found more easily through that, which have been related above in § 32 and in the following. Since indeed on putting $\omega = 1$ there shall be

$$\Sigma(x+n) = \tfrac{1}{2}(x+n-1)(x+n)$$

and

$$\Sigma(x+n)(x+n+1)=\tfrac{1}{3}(x+n-1)(x+n)(x+n+1)$$

and

$$\Sigma(x+n)(x+n+1)(x+n+2)=\tfrac{1}{4}(x+n-1)(x+n)(x+n+1)(x+n+2)$$

etc.,

if to these sums themselves we may add the general terms and likewise we may add on constants, which on putting $x=0$ return a vanishing summatory term, we will obtain the following summatory terms

$$S.(x+n)=\tfrac{1}{2}(x+n)(x+n+1)-\tfrac{1}{2}n(n+1)$$

and

$$S.(x+n)(x+n+1)=\tfrac{1}{3}(x+n)(x+n+1)(x+n+2)-\tfrac{1}{3}n(n+1)(n+2)$$

and

$$S.(x+n)(x+n+1)(x+n+2)=\tfrac{1}{4}(x+n)(x+n+1)(x+n+2)(x+n+3)$$
$$-\tfrac{1}{4}n(n+1)(n+2)(n+3)$$

and thus so on.

If therefore there were either $n=0$ or $n=-1$, the constant quantity in these sums will vanish.

66. Therefore of the series $1, 2, 3, 4, 5$ etc., the general term of which is $=x$, the summatory term will be $=\tfrac{1}{2}x(x+1)$ and this the summatory series $1, 3, 6, 10, 15$ etc., of which again the summatory term will be

$$=\frac{x(x+1)(x+2)}{1\cdot2\cdot3}$$

and this the summatory series $1, 4, 10, 20, 35$ etc. Truly anew the summatory term will be had

$$=\frac{x(x+1)(x+2)(x+3)}{1\cdot2\cdot3\cdot4}$$

which will be the general term of the series $1, 5, 15, 35, 70$ etc., and the summatory term of this series will be

$$=\frac{x(x+1)(x+2)(x+3)(x+4)}{1\cdot2\cdot3\cdot4\cdot5}.$$

But these series are to be noted properly before the rest, since there is the greatest use of these everywhere. Indeed from these are chosen the coefficients of the binomial raised to powers, which as they are evident generally, in these things each may be agreed upon abundantly with little consideration.

67. Also from these the summatory terms are easily found, which before we elicited from the differences. Since indeed everywhere we come upon the general term expressed in the following way

$$a+\tfrac{x-1}{1}b+\tfrac{(x-1)(x-2)}{1\cdot 2}c+\tfrac{(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3}d+\text{etc.},$$

if we seek the summatory terms of each member and we may add all those, we will have the summatory term agreeing for this general term. Thus, since there shall be

$$S.1=x$$

and

$$S.(x-1)=\tfrac{1}{2}x(x-1)$$

and

$$S.(x-1)(x-2)=\tfrac{1}{3}x(x-1)(x-2)$$

and again

$$S.(x-1)(x-2)(x-3)=\tfrac{1}{4}x(x-1)(x-2)(x-3)$$

etc.,

the summatory term sought shall be

$$ax+\tfrac{x(x-1)}{1\cdot 2}b+\tfrac{x(x-1)(x-2)}{1\cdot 2\cdot 3}c+\tfrac{x(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3\cdot 4}d+\text{etc.},$$

which form does not disagree from that, which we found before from differences [§ 57].

68. Then also this summatory term found can be applied to fractions ; for since above (§34) we found on putting $\omega=1$ to be

$$\Sigma\tfrac{1}{(x+n)(x+n+1)}=-1\cdot\tfrac{1}{x+n},$$

there will be

$$S.\tfrac{1}{(x+n)(x+n+1)}=-1\cdot\tfrac{1}{x+n+1}+\tfrac{1}{n+1}.$$

In a similar manner, if we may add the general terms themselves to the sums found above, or, which is the same, if in these expressions in place of x we may put $x+1$, we will have

$$S.\tfrac{1}{(x+n)(x+n+1)(x+n+2)}=-\tfrac{1}{2}\cdot\tfrac{1}{(x+n+1)(x+n+2)}+\tfrac{1}{2}\cdot\tfrac{1}{(n+1)(n+2)}.$$

and

$$S.\tfrac{1}{(x+n)(x+n+1)(x+n+2)(x+n+3)}=-\tfrac{1}{3}\cdot\tfrac{1}{(x+n+1)(x+n+2)(x+n+3)}+\tfrac{1}{3}\cdot\tfrac{1}{(n+1)(n+2)(n+3)},$$

which forms may be continued easily as it pleases.

69. Because there will be

$$S.\frac{1}{(x+n)(x+n+1)}=\frac{1}{n+1}-\frac{1}{x+n+1},$$

also there will be

$$S.\frac{1}{(x+n)}-S.\frac{1}{(x+n+1)}=\frac{1}{n+1}-\frac{1}{x+n+1}.$$

Therefore although neither of these two summatory terms can be shown themselves, yet the difference of these is known and hence in several cases the sums of the series can be assigned readily enough; because that comes about in use, if the general term were a fraction, the denominator of which can be resolved into simple factors. Then indeed the whole fraction may be resolved into partial fractions; with which done will soon become apparent with the aid of this lemma, whether or not each summatory term may be shown.

EXAMPLE 1

To find the summatory term of this series $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+etc.$, *of which the general term is* $=\frac{2}{xx+x}$.

This general term is reduced by resolution to this form $\frac{2}{x}-\frac{2}{x+1}$. Hence the summatory term will be

$$2S.\frac{1}{x}-2S.\frac{1}{x+1}$$

which therefore by the proceeding lemma will be

$$2-\frac{2}{x+1}=\frac{2x}{x+1}$$

Thus, if there shall be $x=4$, there becomes $\frac{8}{5}=1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$.

EXAMPLE 2

The summatory term of this series is sought $\frac{1}{5}, \frac{1}{21}, \frac{1}{45}, \frac{1}{77}, \frac{1}{117}$ *etc., of which the general term shall be* $=\frac{1}{4xx+4x-3}$.

Because the denominator of the general term has the factors $2x-1$ and $2x+3$, this may be resolved into these parts

$$\frac{1}{4}\cdot\frac{1}{2x-1}-\frac{1}{4}\cdot\frac{1}{2x+3}=\frac{1}{8}\cdot\frac{1}{x-\frac{1}{2}}-\frac{1}{8}\cdot\frac{1}{x+\frac{3}{2}}.$$

But there is

$$S.\frac{1}{x-\frac{1}{2}} = S.\frac{1}{x+\frac{1}{2}} + 2 - \frac{1}{x+\frac{1}{2}}$$

and

$$S.\frac{1}{x+\frac{1}{2}} = S.\frac{1}{x+\frac{3}{2}} + \frac{2}{3} - \frac{1}{x+\frac{3}{2}},$$

hence

$$S.\frac{1}{x-\frac{1}{2}} - S.\frac{1}{x+\frac{3}{2}} = 2 + \frac{2}{3} - \frac{1}{x+\frac{1}{2}} - \frac{1}{x+\frac{3}{2}}$$

of which the eight part will give the summatory term sought, evidently

$$\frac{1}{4} + \frac{1}{12} - \frac{1}{8x+4} - \frac{1}{8x+12} = \frac{x}{4x+2} + \frac{x}{3(4x+6)} = \frac{3(4x+5)}{3(2x+1)(2x+3)}.$$

70. Because the numbers formed, which the coefficients of a binomial raised to powers present, before the rest are noteworthy, we may show the sums of series, of which the numerators shall be $=1$, truly the denominators the numbers formed; that which readily comes about from § 68. Hence the series,

of which the general term is	the summatory term will be
$\frac{1\cdot 2}{x(x+1)}$	$\frac{2}{1} - \frac{2}{x+1}$
$\frac{1\cdot 2\cdot 3}{x(x+1)(x+2)}$	$\frac{3}{2} - \frac{1\cdot 3}{(x+1)(x+2)}$
$\frac{1\cdot 2\cdot 3\cdot 4}{x(x+1)(x+2)(x+3)}$	$\frac{4}{3} - \frac{1\cdot 2\cdot 4}{(x+1)(x+2)(x+3)}$
$\frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{x(x+1)(x+2)(x+3)(x+4)}$	$\frac{5}{4} - \frac{1\cdot 2\cdot 3\cdot 5}{(x+1)(x+2)(x+3)(x+4)}$
etc.	etc.

From which the rule, by which these expressions progress, makes itself apparent. Nor indeed hence the summatory term, which agrees with the general term $\frac{1}{x}$, can be deduced, obviously which cannot be expressed by a definite formula.

71. Because the summatory term presents the sum of as many terms, as there are units contained in the index *x*, it is evident the sums of these series can be obtained continued to infinity, if the index x is put infinitely large; in which case the latter terms of the expressions found in this manner vanish on account of the infinite denominators. Hence these infinite series will have finite sums, which will be

$$1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\text{etc.} =\frac{2}{1}$$
$$1+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}+\text{etc.}=\frac{3}{2}$$
$$1+\frac{1}{5}+\frac{1}{15}+\frac{1}{35}+\frac{1}{70}+\text{etc.}=\frac{4}{3}$$
$$1+\frac{1}{6}+\frac{1}{21}+\frac{1}{56}+\frac{1}{126}+\text{etc.}=\frac{5}{4}$$
$$1+\frac{1}{7}+\frac{1}{28}+\frac{1}{84}+\frac{1}{210}+\text{etc.}=\frac{6}{5}$$
etc.

Therefore of all the series, of which the summatory term may be had, are able to show the sum continued to infinity on putting $x=\infty$, provided in this case the sums are made finite; which indeed comes about, if in the summatory term *x* may have as many dimensions in the denominator, as it has in the numerator.

CAPUT II

DE USU DIFFERENTIARUM IN DOCTRINA SERIERUM

37. Naturam serierum per differentias maxime illustrari ex primis rudimentis satis est notum. Progressionis enim arithmeticae, quae primum considerari solet, praecipua proprietas in hoc versatur, ut eius differentiae primae sint inter se aequales; hinc differentiae secundae ac reliquae omnes erunt cyphrae. Dantur deinde series, quarum differentiae secundae demum sint aequales, quae hanc ob rem *secundi ordinis* commode appellantur, dum progressiones arithmeticae series *primi ordinis* vocantur. Porro igitur series *tertii ordinis* erunt, quarum differentiae tertiae sunt constantes, atque ad *quartum ordinem* et sequentes eae referentur series, quarum differentiae quartae et ulteriores demum sunt constantes.

38. In hac divisione infinita serierum genera comprehenduntur neque tamen omnes series ad haec genera revocare licet. Occurrunt enim innumerabiles series, quae differentiis sumendis nunquam ad terminos constantes deducunt; cuiusmodi, praeter innumeras alias, sunt progressiones geometricae, quae nunquam praebent differentias constantes, uti ex hoc exemplo videre licet

$$1,\quad 2,\quad 4,\quad 8,\quad 16,\quad 32,\quad 64,\quad 128 \text{ etc.}$$
$$1,\quad 2,\quad 4,\quad 8,\quad 16,\quad 32,\quad 64 \quad \text{etc.}$$
$$1,\quad 2,\quad 4,\quad 8,\quad 16,\quad 32 \quad \text{etc.}$$

Cum enim series differentiarum cuiusque ordinis aequalis sit ipsi seriei propositae, aequalitas differentiarum prorsus excluditur. Quocirca plures serierum classes constitui debebunt, quarum una tantum in hos ordines, qui tandem ad differentias constantes revocantur, subdividitur; quam classem in hoc capite potissimum considerabimus.

39. Duae autem res ad naturam serierum cognoscendam imprimis requiri solent, terminus generalis atque summa seu terminus summatorius. Terminus generalis est expressio indefinita, quae unumquemque seriei terminum complectitur atque eiusmodi propterea est functio quantitatis variabilis x, quae posito $x=1$ terminum seriei primum exhibet, secundum vero posito $x=2$, tertium posito $x=3$, quartum posito $x=4$ et ita porro. Cognito ergo termino generali quotuscunque seriei terminus invenietur, etiamsi lex, qua singuli termini cohaerent, non respiciatur. Sic verbi gratia ponendo $x=1000$ statim terminus millesimus cognoscetur. Ita huius seriei

$$1,\ 6,\ 15,\ 28,\ 45,\ 66,\ 91,\ 120 \text{ etc}.$$

terminus generalis est $2xx - x$; posito enim $x = 1$ haec formula dat terminum primum 1; posito $x = 2$ oritur terminus secundus 6; si ponatur $x = 3$, oritur tertius 15 etc.; unde patet huius seriei terminum centesimum posito $x = 100$ fore $= 2 \cdot 10000 - 100 = 19900$.

40. *Indices* seu *exponentes* in qualibet serie vocantur numeri, qui indicant, quotus quisque terminus sit in ordine; sic termini primi index erit 1, secundi 2, tertii 3 et ita porro. Hinc indices singulis cuiusque seriei terminis inscribi solent hoc modo:

Indices

1, 2, 3, 4, 5, 6, 7 etc.

Termini

A, B, C, D, E, F, G etc.,

unde statim patet G esse seriei propositae terminum septimum, cum eius index sit 7. Hinc terminus generalis nil aliud erit nisi terminus seriei, cuius index vel exponens est numerus indefinitus x. Quemadmodum ergo in quolibet serierum ordine, quarum differentiae vel primae vel secundae vel aliae sequentes sunt constantes, terminum generalem inveniri oporteat, primum docebimus; tum vero ad investigationem summae sumus progressuri.

41. Incipiamus ab ordine primo, qui continet progressiones arithmeticas, quarum differentiae primae sunt constantes; sitque a terminus seriei primus et b terminus primus seriei differentiarum, cui sequentes omnes sunt aequales; unde series ita erit comparata:

Indices

1, 2, 3, 4, 5, 6 etc.

Termini

a, $a + b$, $a + 2b$, $a + 3b$, $a + 4b$, $a + 5b$ etc.

Differentiae

b, b, b, b, b etc.

Ex qua statim patet terminum, cuius index sit $= x$, fore $a + (x - 1)b$ eritque ergo terminus generalis $= bx + a - b$, qui ex terminis primis cum ipsius seriei tum seriei differentiarum componitur. Quodsi autem terminus secundus seriei $a + b$ vocetur a^{I} ob $b = a^{\mathrm{I}} - a$ erit terminus generalis

$$= \left(a^{\mathrm{I}} - a\right)x + 2a - a^{\mathrm{I}} = a^{\mathrm{I}}(x - 1) - a(x - 2),$$

unde ex cognitis terminis primo et secundo progressionis arithmeticae eius terminus generalis formabitur.

42. Sint in serie secundi ordinis termini primi ipsius seriei $=a$, differentiarum primarum $=b$, differentiarum secundarum $=c$ eritque ipsa series cum suis differentiis ita comparata:

Indices

1, 2, 3, 4, 5, 6 7 etc.

Termini

$a,\ a+b,\ a+2b+c,\ a+3b+3c,\ a+4b+6c,\ a+5b+10c,\ a+6b+15c$ etc.

Differentiae primae

$b,\ b+c,\ b+2c,\ b+3c,\ b+4c,\ b+5c,$ etc.

Differentiae secundae

$c,\ c,\ c,\ c,\ c,$ etc.

Ex cuius inspectione liquet terminum, cuius index $=x$, fore

$$=a+(x-1)b+\frac{(x-1)(x-2)}{1\cdot 2}c,$$

qui ergo est terminus generalis seriei propositae. Ponatur autem ipsius seriei terminus secundus $=a^{\mathrm{I}}$, terminus tertius $=a^{\mathrm{II}}$; cum sit

$$b=a^{\mathrm{I}}-a \quad \text{et} \quad c=a^{\mathrm{II}}-2a^{\mathrm{I}}+a,$$

uti ex natura differentiarum (§10) intelligitur, erit terminus generalis

$$a+(x-1)\left(a^{\mathrm{I}}-a\right)+\frac{(x-1)(x-2)}{1\cdot 2}\left(a^{\mathrm{II}}-2a^{\mathrm{I}}+a\right),$$

qui reducitur ad hanc formam

$$\frac{a^{\mathrm{II}}(x-1)(x-2)}{1\cdot 2}-\frac{2a^{\mathrm{I}}(x-1)(x-3)}{1\cdot 2}+\frac{a(x-2)(x-3)}{1\cdot 2}$$

vel etiam ad hanc

$$\frac{a^{II}}{2}(x-1)(x-2)-\frac{2a^{I}}{2}(x-1)(x-3)+\frac{a}{2}(x-2)(x-3)$$

aut denique ad hanc

$$\frac{1}{2}(x-1)(x-2)(x-3)\left(\frac{a^{II}}{x-3}-\frac{2a^{I}}{x-2}+\frac{a}{x-1}\right)$$

ideoque ex tribus terminis ipsius seriei definitur.

43. Sit series tertii ordinis $a, a^{I}, a^{II}, a^{III}, a^{IV}$ etc., eius differentiae primae $b, b^{I}, b^{II}, b^{III}$ etc. et differentiae secundae $c, c^{I}, c^{II}, c^{III}$ etc. et tertiae d, d, d, d etc., quippe quae sunt constantes.

Indices
1, 2, 3, 4, 5, 6 etc.
Termini
$a, a^{I}, a^{II}, a^{III}, a^{IV}, a^{V}$ etc.
Differentiae primae
$b, b^{I}, b^{II}, b^{III}, b^{IV}$ etc.
Differentiae secundae
$c, c^{I}, c^{II}, c^{III},$ etc.
Differentiae tertiae
d, d, d etc.

Quia est

$$a^{I}=a+b,\ a^{II}=a+2b+c,\ a^{III}=a+3b+3c+d,\ a^{IV}=a+4b+6c+4d \text{ etc.},$$

erit terminus generalis seu is, cuius index est x,

$$a+\frac{(x-1)}{1}b+\frac{(x-1)(x-2)}{1\cdot 2}c+\frac{(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3}d$$

sicque terminus generalis ex differentiis formabitur. Cum autem porro sit

$$b=a^{I}-a,\ c=a^{II}-2a^{I}+a,\ d=a^{III}-3a^{II}+3a^{I}-a,$$

si hi valores substituantur, erit terminus generalis

$$a^{\mathrm{III}}\frac{(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3}-3a^{\mathrm{II}}\frac{(x-1)(x-2)(x-4)}{1\cdot 2\cdot 3}$$
$$+3a^{\mathrm{I}}\frac{(x-1)(x-2)(x-4)}{1\cdot 2\cdot 3}-a\frac{(x-2)(x-3)(x-4)}{1\cdot 2\cdot 3},$$

qui etiam hoc modo exprimetur, ut sit

$$=\frac{(x-1)(x-2)(x-3)(x-4)}{1\cdot 2\cdot 3}\left(\frac{a^{\mathrm{III}}}{x-4}-\frac{3a^{\mathrm{II}}}{x-3}+\frac{3a^{\mathrm{I}}}{x-2}-\frac{a}{x-1}\right).$$

44. Sit nunc series cuiuscunque ordinis proposita;

Indices
1, 2, 3, 4, 5, 6 etc.
Termini
$a,\ a^{\mathrm{I}},\ a^{\mathrm{II}},a^{\mathrm{III}},a^{\mathrm{IV}},a^{\mathrm{V}}$ etc.
Differentiae primae
$b,\ b^{\mathrm{I}},\ b^{\mathrm{II}},\ b^{\mathrm{III}},\ b^{\mathrm{IV}}$ etc.
Differentiae secundae
$c,\ c^{\mathrm{I}},\ c^{\mathrm{II}},\ c^{\mathrm{III}},$ etc.
Differentiae tertiae
$d,\ d,\ d$ etc.
Differentiae quartae
$e,\ e^{\mathrm{I}}$ etc.
Differentiae quintae
f etc.

Ex ipsius seriei termino primo atque ex differentiarum terminis primis
b, c, d, e, f etc. terminus generalis ita exprimetur, ut sit

$$a+\frac{(x-1)}{1}b+\frac{(x-1)(x-2)}{1\cdot 2}c+\frac{(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3}d+\frac{(x-1)(x-2)(x-3)(x-4)}{1\cdot 2\cdot 3\cdot 4}e+\text{etc.},$$

donec ad differentias constantes perveniatur. Ex quo patet, si nunquam prodeant differentiae constantes, terminum generalem per expressionem infinitam exhiberi.

45. Quia differentiae ex ipsis terminis seriei formantur, si earum valores substituantur, prodibit terminus generalis in eiusmodi forma expressus, cuiusmodi pro seriebus primi, secundi et tertii ordinis exhibuimus. Scilicet pro seriebus ordinis quarti erit terminus generalis

$$=\frac{(x-1)(x-2)(x-3)(x-4)(x-5)}{1\cdot 2\cdot 3\cdot 4}\left(\frac{a^{\mathrm{IV}}}{x-5}-\frac{4a^{\mathrm{III}}}{x-4}+\frac{6a^{\mathrm{II}}}{x-3}-\frac{4a^{\mathrm{I}}}{x-2}+\frac{a}{x-1}\right),$$

unde lex, qua sequentium ordinum termini generales componuntur, facile perspicitur.
Ex his autem patet pro quovis ordine terminum generalem fore functionem ipsius x rationalem integram, in qua maxima ipsius x dimensio congruat cum ordine, ad quem series refertur. Ita serierum primi ordinis erit terminus generalis functio primi gradus, secundi ordinis secundi gradus et ita porro.

46. Differentiae autem, uti supra vidimus, ex ipsis terminis seriei ita resultant, ut sit etc.

$$\begin{array}{lll} b=a^{\mathrm{I}}-a & c=a^{\mathrm{II}}-2a^{\mathrm{I}}+a & d=a^{\mathrm{III}}-3a^{\mathrm{II}}+3a^{\mathrm{I}}-a \\ b^{\mathrm{I}}=a^{\mathrm{II}}-a^{\mathrm{I}} & c^{\mathrm{I}}=a^{\mathrm{III}}-2a^{\mathrm{II}}+a^{\mathrm{I}} & d^{\mathrm{I}}=a^{\mathrm{IV}}-3a^{\mathrm{III}}+3a^{\mathrm{II}}-a^{\mathrm{I}} \\ b^{\mathrm{II}}=a^{\mathrm{III}}-a^{\mathrm{II}} & c^{\mathrm{II}}=a^{\mathrm{IV}}-2a^{\mathrm{III}}+a^{\mathrm{II}} & d^{\mathrm{II}}=a^{\mathrm{V}}-3a^{\mathrm{IV}}+3a^{\mathrm{III}}-a^{\mathrm{II}} \\ \text{etc.} & \text{etc.} & \text{etc.} \end{array}$$

Quare, cum in seriebus primi ordinis sint omnes valores ipsius $c=0$, erit unde patet has series simul esse recurrentes et scalam relationis esse $2,-1$. Deinde, cum in seriebus secundi ordinis sint omnes valores ipsius $d=0$, erit

$$a^{\mathrm{III}}=3a^{\mathrm{II}}-3a^{\mathrm{I}}+a,\quad a^{\mathrm{IV}}=3a^{\mathrm{III}}-3a^{\mathrm{II}}+a^{\mathrm{I}}\quad \text{etc.}$$

ideoque et hae erunt recurrentes scala relationis existente $3,-3,+1$. Simili modo apparebit omnes huius classis series, cuiuscunque sint ordinis, simul ad classem serierum recurrentium pertinere atque ita quidem, ut scala relationis constet ex coefficientibus potestatis binomii uno gradu superioris, quam est ordo, ad quem series refertur.

47. Quia vero pro seriebus primi ordinis quoque omnes valores ipsius d et e et sequentium differentiarum omnium sunt $=0$, erit quoque in his

$$a^{\mathrm{III}}=3a^{\mathrm{II}}-3a^{\mathrm{I}}+a,$$
$$a^{\mathrm{IV}}=3a^{\mathrm{III}}-3a^{\mathrm{II}}+a^{\mathrm{I}}$$
$$\text{etc.}$$

aut

$$a^{\mathrm{IV}}=4a^{\mathrm{III}}-6a^{\mathrm{II}}+4a^{\mathrm{I}}-a$$
$$a^{\mathrm{V}}=4a^{\mathrm{IV}}-6a^{\mathrm{III}}+4a^{\mathrm{II}}-a^{\mathrm{I}}$$

Pertinebunt ergo et hinc ad series recurrentes idque infinitis modis, cum scalae relationis esse queant

$$3, -3, +1;\ 4, -6, +4, -1;\ 5, -10, +10, -5, +1 \text{ etc.}$$

Similique modo intelligitur unamquamque seriem huius, quam tractamus, classis simul esse seriem recurrentem innumeris modis; scala enim relationis erit

$$\frac{n}{1}, -\frac{n(n-1)}{1\cdot 2}, +\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}, -\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} \text{ etc.},$$

dummodo n sit numerus integer maior quam numerus, quo ordo indicatur. Orietur ergo haec series quoque ex evolutione fractionis, cuius denominator est $(1-y)^n$, prouti in superiori libro de seriebus recurrentibus fusius est ostensum.

48. Quemadmodum vidimus omnium huius classis serierum, cuiuscunque sint ordinis, terminos generales esse functiones ipsius x rationales integras, ita vicissim apparebit omnes series, quarum termini generales sint huiusmodi functiones ipsius x, ad hanc classem pertinere atque tandem ad differentias constantes perduci. Et quidem, si terminus generalis fuerit functio primi gradus $ax+b$, dum series inde orta erit primi ordinis seu arithmetica , differentias primas habebit constantes. Sin autem terminus generalis fuerit functio secundi gradus in hac forma $axx+bx+c$ contenta, tum series ex eo oriunda, dum loco x successive numeri 1, 2, 3, 4, 5 etc. substituuntur, erit ordinis secundi atque differentias secundas habebit constantes; simili modo terminus generalis tertii gradus ax^3+bx^2+cx+d dabit seriem tertii ordinis atque ita porro.

49. Ex termino enim generali non solum omnes seriei termini inveniuntur, sed etiam series differentiarum tam primarum quam sequentium deduci possunt. Cum enim, si seriei terminus primus subtrahatur a secundo, prodeat seriei differentiarum terminus primus, secundus autem, si ipsius seriei terminus secundus a tertio auferatur, ita seriei differentiarum obtinebitur terminus, cuius index est x, si ipsius seriei terminus, cuius index est x, subtrahatur a sequente, cuius index est $x+1$. Quare si in termino seriei generali loco x ponatur $x+1$ ab hocque valore terminus generalis subtrahatur, remanebit terminus generalis seriei differentiarum; si igitur X fuerit seriei terminus generalis, erit eius differentia ΔX (quae modo in praecedente capite ostenso invenietur, si statuatur ibi $\omega=1$) terminus generalis seriei differentiarum primarum. Simili igitur modo erit $\Delta\Delta X$ terminus generalis seriei differentiarum secundarum, $\Delta^3 X$ tertiarum sicque deinceps.

50. Quodsi autem terminus generalis X fuerit functio rationalis integra, in qua maximus exponens potestatis ipsius x sit n, ex capite praecedente colligitur eius differentiam ΔX fore functionem uno gradu inferiorem, nempe gradus $n-1$. Hincque porro $\Delta\Delta X$ erit functio gradus $n-2$ et $\Delta^3 X$ functio gradus $n-3$ et ita porro. Quare, si X fuerit functio primi gradus, uti $ax+b$, tum eius differentia ΔX erit constans $=a$; quae cum sit terminus generalis seriei primarum differentiarum, perspicitur seriem, cuius terminus generalis X sit functio primi gradus, fore arithmeticam seu primi ordinis. Simili modo, si terminus generalis X fuerit functio secundi gradus, ob $\Delta\Delta X$ constantem series inde orta differentias secundas habebit constantes eritque propterea ordinis secundi; sicque perpetuo, cuius gradus fuerit functio X terminum generalem constituens, eiusdem ordinis erit series ex eo nata.

51. Hanc ob rem series potestatum numerorum naturalium ad differentias constantes perveniunt, uti ex sequenti schemate fit manifestum.

Potestates primae
1, 2, 3, 4, 5, 6, 7, 8 etc.

Differentiae primae
1, 1, 1, 1, 1, 1, 1 etc.

Potestates secundae
1, 4, 9, 16, 25, 36, 49, 64 etc.

Differentiae primae
3, 5, 7, 9, 11, 13, 15 etc.

Differentiae secundae
2, 2, 2, 2, 2, 2 etc.

Potestates tertiae
1, 8, 27, 64, 125, 216, 343 etc.

Differentiae primae
7, 19, 37, 61, 91, 127 etc.

Differentiae secundae
12, 18, 24, 30, 36 etc.

Differentiae tertiae
6, 6, 6, 6 etc.

Potestates quartae
1, 16, 81, 256, 625, 1296, 2401 etc.

Differentiae primae
15, 65, 175, 369, 671, 1105 etc.

Differentiae secundae

50, 110, 194, 302, 434 etc.

Differentiae tertiae
60, 84, 108, 132 etc.

Differentiae quartae
24, 24, 24 etc.

Quae igitur in capite praecedente de differentiis cuiusque ordinis inveniendis sunt praecepta, ea hic inservient ad terminos generales differentiarum quarumvis, quae ex seriebus nascuntur, inveniendos.

52. Si terminus generalis cuiusquam serei fuerit cognitus, eius ope non solum omnes eius termini in infinitum inveniri, sed etiam series retro continuari eiusque termini, quorum exponentes sint numeri negativi, exhiberi poterunt loco *x* numeros negativos substituendo; sic, si terminus generalis fuerit $\frac{xx+3x}{2}$, ponendo loco *x* tam negativos quam affirmativos indices series utrinque continuata edt huiusmodi:

Indices
etc. $-5, -4, -3, -2, -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6$ etc.

Series
etc. $+5, +2,\ 0, -1, -1,\ 0,\ 2,\ 5,\ 9,\ 14,\ 20,\ 27$ etc.

Differentiae primae
etc. $-3, -2, -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7$ etc.

Differentiae secundae
etc. $1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1$ etc.

Cum igitur ex differentiis terminus generalis formetur, quaeque series ex differentiis retro continuari poterit, ita quidem, ut, si differentiae tandem fiant constantes, hi termini finite exhiberi, contra vero per expressionem infinitam assignari queant. Quin etiam ex termino generali ii termini, quorum indices sunt fracti, definientur, in quo serierum *interpolatio* continetur.

53. His de termino serierum generali monitis progrediamur ad summam seu terminum summatorium serierum cuiusque ordinis investigandum. Proposita autem quacunque serie *terminus summatorius* est functio ipsius *x*, quae aequalis est summae tot terminorum seriei, quot unitates continet numerus *x*. Ita ergo terminus summatorius erit comparatus, ut, si ponatur $x=1$, prodeat terminus primus seriei, sin autem ponatur $x=2$, ut prodeat summa primi et secundi, facto autem $x=3$ summa primi, secundi ac tertii sicque deinceps. Hinc, si ex serie proposita nova series formetur, cuius primus terminus aequalis sit primo illius, secundus aequalis summae duorum, tertius aequalis summae trium atque ita porro, haec nova series vocatur illius *summatrix* huiusque seriei summatricis terminus generalis erit terminus summatorius seriei propositae; ex quo inventio termini summatorii ad inventionem termini generalis revocatur.

54. Sit ergo series proposita haec

$$a,\quad a^{\mathrm{I}},\quad a^{\mathrm{II}},\quad a^{\mathrm{III}},\quad a^{\mathrm{IV}},\quad a^{\mathrm{V}}\ \text{etc.}$$

huiusque seriei summatrix sit

$$A,\quad A^{\mathrm{I}},\quad A^{\mathrm{II}},\quad A^{\mathrm{III}},\quad A^{\mathrm{IV}},\quad A^{\mathrm{V}}\ \text{etc.};$$

erit ex eius natura modo exposita

$$\begin{aligned}
&A = a\\
&A^{\mathrm{I}} = a + a^{\mathrm{I}}\\
&A^{\mathrm{II}} = a + a^{\mathrm{I}} + a^{\mathrm{II}}\\
&A^{\mathrm{III}} = a + a^{\mathrm{I}} + a^{\mathrm{II}} + a^{\mathrm{III}}\\
&A^{\mathrm{IV}} = a + a^{\mathrm{I}} + a^{\mathrm{II}} + a^{\mathrm{III}} + a^{\mathrm{IV}}\\
&A^{\mathrm{V}} = a + a^{\mathrm{I}} + a^{\mathrm{II}} + a^{\mathrm{III}} + a^{\mathrm{IV}} + a^{\mathrm{V}}
\end{aligned}$$

Hinc seriei summatricis differentiae erunt

$$A^{\mathrm{I}} - A = a^{\mathrm{I}},\quad A^{\mathrm{II}} - A^{\mathrm{I}} = a^{\mathrm{II}},\quad A^{\mathrm{III}} - A^{\mathrm{II}} = a^{\mathrm{III}}\quad \text{etc.},$$

unde series proposita termino primo minuta erit series differentiarum primarum seriei summatricis. Quodsi igitur seriei summatrici praefigatur terminus $= 0$, ut habeatur

$$0,\quad A,\quad A^{\mathrm{I}},\quad A^{\mathrm{II}},\quad A^{\mathrm{III}},\quad A^{\mathrm{IV}},\quad A^{\mathrm{V}}\ \text{etc.},$$

huius series primarum differentiarum erit ipsa series proposita

$$a,\quad a^{\mathrm{I}},\quad a^{\mathrm{II}},\quad a^{\mathrm{III}},\quad a^{\mathrm{IV}},\quad a^{\mathrm{V}}\ \text{etc.}$$

55. Hanc ob rem seriei propositae differentiae primae erunt differentiae secundae summatricis atque differentiae secundae illius erunt differentiae tertiae huius, tertiae autem illius quartae huius atque ita porro. Quare si series proposita tandem habeat differentias constantes, tunc etiam eius summatrix ad differentias constantes deducetur eritque igitur series eiusdem naturae, at uno ordine superior. Huiusmodi ergo serierum perpetuo terminus summatorius exhiberi poterit per expressionem finitam. Namque terminus generalis seriei

$$0, \quad A, \quad A^{\mathrm{I}}, \quad A^{\mathrm{II}}, \quad A^{\mathrm{III}}, \quad A^{\mathrm{IV}}, \quad \text{etc.}$$

seu is, qui indici x convenit, exhibebit summam $x-1$ terminorum seriei huius $a, \quad a^{\mathrm{I}}, \quad a^{\mathrm{II}}, \quad a^{\mathrm{III}}, \quad a^{\mathrm{IV}}, \quad a^{\mathrm{V}}$ etc., atque si tum loco x scribatur $x+1$, orietur summa x terminorum ipseque terminus summatorius.

56. Sit igitur seriei propositae $a, \quad a^{\mathrm{I}}, \quad a^{\mathrm{II}}, \quad a^{\mathrm{III}}, \quad a^{\mathrm{IV}}, \quad a^{\mathrm{V}}, \quad a^{\mathrm{VI}}$ etc. series differentiarum primarum

$$b, \quad b^{\mathrm{I}}, \quad b^{\mathrm{II}}, \quad b^{\mathrm{III}}, \quad b^{\mathrm{IV}}, \quad b^{\mathrm{V}}, \quad b^{\mathrm{VI}} \text{ etc.},$$

series differentiarum secundarum

$$c, \quad c^{\mathrm{I}}, \quad c^{\mathrm{II}}, \quad c^{\mathrm{III}}, \quad c^{\mathrm{IV}}, \quad c^{\mathrm{V}}, \quad c^{\mathrm{VI}} \text{ etc.},$$

series differentiarum tertiarum

$$d, \quad d^{\mathrm{I}}, \quad d^{\mathrm{II}}, \quad d^{\mathrm{III}}, \quad d^{\mathrm{IV}}, \quad d^{\mathrm{V}}, \quad d^{\mathrm{VI}} \text{ etc.},$$

sicque porro, donec ad differentias constantes perveniatur. Deinde formetur series summatrix, quae cum praefixa 0 in locum termini primi cum suis differentiis continuis se habebit sequenti modo:

Indices
1, 2, 3, 4, 5, 6, 7 etc.

Summatrix

$$0, \quad A, \quad A^{\mathrm{I}}, \quad A^{\mathrm{II}}, \quad A^{\mathrm{III}}, \quad A^{\mathrm{IV}}, \quad \text{etc.}$$

Series proposita

$$a, \quad a^{\mathrm{I}}, \quad a^{\mathrm{II}}, \quad a^{\mathrm{III}}, \quad a^{\mathrm{IV}}, \quad a^{\mathrm{V}}, \quad a^{\mathrm{VI}} \text{ etc.}$$

Differentiae primae

$$b, \quad b^{\mathrm{I}}, \quad b^{\mathrm{II}}, \quad b^{\mathrm{III}}, \quad b^{\mathrm{IV}}, \quad b^{\mathrm{V}}, \quad b^{\mathrm{VI}} \text{ etc.}$$

Differentiae secundae

$$c, \quad c^{\mathrm{I}}, \quad c^{\mathrm{II}}, \quad c^{\mathrm{III}}, \quad c^{\mathrm{IV}}, \quad c^{\mathrm{V}}, \quad c^{\mathrm{VI}} \text{ etc.}$$

Differentiae tertiae

$$d,\quad d^{\text{I}},\quad d^{\text{II}},\quad d^{\text{III}},\quad d^{\text{IV}},\quad d^{\text{V}},\quad d^{\text{VI}} \text{ etc.}$$

Erit seriei summatricis terminus generalis, seu qui indici *x* respondet,

$$0+(x-1)a+\frac{(x-1)(x-2)}{1\cdot 2}b+\frac{(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3}c+\text{etc.},$$

qui simul exhibet summam $x-1$ terminorum seriei propositae

$$a,\quad a^{\text{I}},\quad a^{\text{II}},\quad a^{\text{III}},\quad a^{\text{IV}},\quad a^{\text{V}},\quad a^{\text{VI}} \text{ etc.}$$

57. Quodsi ergo in hac summa loco $x-1$ scribatur *x*, prodibit seriei propositae terminus summatorius summam *x* terminorum complectens

$$xa+\frac{x(x-1)}{1\cdot 2}b+\frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}c+\frac{x(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3\cdot 4}d+\text{etc.},$$

Hinc, si litterae *b, c, d, e* etc. valores ipsis assignatos retineant, erit

seriei

$$a,\quad a^{\text{I}},\quad a^{\text{II}},\quad a^{\text{III}},\quad a^{\text{IV}},\quad a^{\text{V}},\quad a^{\text{VI}} \text{ etc.}$$

terminus generalis

$$a+(x-1)b+\frac{(x-1)(x-2)}{1\cdot 2}c+\frac{(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3}d+\frac{(x-1)(x-2)(x-3)(x-4)}{1\cdot 2\cdot 3\cdot 4}e+\text{etc.}$$

et

terminus summatorius

$$xa+\frac{x(x-1)b}{1\cdot 2}+\frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}c+\frac{(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3\cdot 4}d+\text{etc.}$$

Invento ergo seriei cuiusvis ordinis hoc, quem ostendimus, modo termino generali non difficulter ex eo terminus summatorius reperietur, quippe qui ex iisdem differentiis conflatur.

58. Hic modus terminum summatorium per differentias seriei inveniendi imprimis ad eiusmodi series, quae tandem ad differentias constantes deducunt, est accommodatus; in aliis enim casibus expressio finita non reperitur. Quodsi autem ea, quae ante de indole termini summatorii sunt exposita, attentius perpendamus, alius modus se offert terminum summatorium immediate ex termino generali inveniendi, qui multo latius patet atque in infinitis casibus ad expressiones finitas deducit, quibus prior modus infinitas exhibet. Sit enim proposita series quaecunque

$$a,\ b,\ c,\ d,\ e,\ f\quad \text{etc.},$$

cuius terminus generalis seu indici x respondens sit $= X$; terminus autem summatorius sit $= S$; qui cum summam tot terminorum ab initio exhibeat, quot numerus x continet unitates, erit summa $x-1$ terminorum $= S - X$ eritque adeo X differentia expressionis $S - X$, cum relinquatur, si haec a sequente S subtrahatur.

59. Cum igitur sit $X = \Delta(S - X)$ differentia eo modo sumta, quem capite praecedente docuimus, hoc tantum discrimine, ut quantitas illa constans ω hic nobis sit = 1, quare, si ad summas regrediamur, erit $\Sigma X = S - X$ ideoque terminus summatorius quaesitus

$$S = \Sigma X + X + C\,.$$

Quaeri ergo debet summa functionis X methodo ante tradita ad eamque addi ipse terminus generalis X eritque aggregatum terminus summatorius. Quoniam autem in summis sumendis involvitur quantitas constans, sive addenda sive subtrahenda, ea ad praesentem casum accommodari debebit. Manifestum autem est, si ponatur $x = 0$, quo casu numerus terminorum summandorum est nullus, summam quoque fore nullam; ex quo quantitas illa constans C ita determinari debebit, ut posito $x = 0$ fiat quoque $S = 0$. Positis ergo in illa aequatione $S = \Sigma X + X + C$ tam $S = 0$ quam $x = 0$ valor ipsius C invenietur.

60. Quoniam ergo hic totum negotium ad summationem functionum supra monstratam reducitur ponendo $\omega = 1$, exinde depromamus summationes traditas ac primo quidem pro potestatibus ipsius x erit [§ 27]

$$\begin{aligned}
&\Sigma x^0 = \Sigma 1 = x\\
&\Sigma x = \tfrac{1}{2}x^2 - \tfrac{1}{2}x\\
&\Sigma x^2 = \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{6}\\
&\Sigma x^3 = \tfrac{1}{4}x^4 - \tfrac{1}{2}x^3 + \tfrac{1}{4}x^2\\
&\Sigma x^4 = \tfrac{1}{5}x^5 - \tfrac{1}{2}x^4 + \tfrac{1}{3}x^3 - \tfrac{1}{30}x\\
&\Sigma x^5 = \tfrac{1}{6}x^6 - \tfrac{1}{2}x^5 + \tfrac{5}{12}x^4 - \tfrac{1}{12}x^2\\
&\Sigma x^6 = \tfrac{1}{7}x^7 - \tfrac{1}{2}x^6 + \tfrac{1}{2}x^5 - \tfrac{1}{6}x^3 + \tfrac{1}{42}x,
\end{aligned}$$

quibus accenseatur summatio generalis potestatis x^n § 29 tradita, dummodo ibi ubique loco ω unitas scribatur. Harum ergo formularum ope omnium serierum, quarum termini generales sunt functiones rationales integrae ipsius *x*, termini summatorii expedite inveniri poterunt.

61. Denotet *S.X* terminum summatorium seriei, cuius terminus generalis est = *X*, eritque, ut vidimus,

$$S.X = \Sigma X + X + C,$$

dummodo constans *C* ita assumatur, ut terminus summatorius *S.X* evanescat posito $x = 0$. Hinc igitur terminos summatorios serierum potestatum, seu quarum termini generales comprehenduntur in hac forma x^n, exprimamus. Posito itaque

$$S.x^n = 1 + 2^n + 3^n + 4^n + \ldots\ldots + x^n$$

erit

$$\begin{aligned} S.x^n &= \tfrac{1}{n+1}x^{n+1} + \tfrac{1}{2}x^n + \tfrac{1}{2}\cdot\tfrac{n}{2\cdot 3}x^{n-1} - \tfrac{1}{6}\cdot\tfrac{n(n-1)(n-2)}{2\cdot 3\cdot 4\cdot 5}x^{n-3} \\ &+ \tfrac{1}{6}\cdot\tfrac{n(n-1)..(n-4)}{2\cdot 3..6\cdot 7}x^{n-5} - \tfrac{3}{10}\cdot\tfrac{n(n-1)..(n-6)}{2\cdot 3..8\cdot 9}x^{n-7} \\ &+ \tfrac{5}{6}\cdot\tfrac{n(n-1)..(n-8)}{2\cdot 3..10\cdot 11}x^{n-9} - \tfrac{691}{210}\cdot\tfrac{n(n-1)..(n-10)}{2\cdot 3..12\cdot 13}x^{n-11} \\ &+ \tfrac{35}{2}\cdot\tfrac{n(n-1)..(n-12)}{2\cdot 3..14\cdot 15}x^{n-13} - \tfrac{3617}{30}\cdot\tfrac{n(n-1)..(n-14)}{2\cdot 3..16\cdot 17}x^{n-15} \\ &+ \tfrac{43867}{42}\cdot\tfrac{n(n-1)..(n-16)}{2\cdot 3..18\cdot 19}x^{n-17} - \tfrac{122227}{110}\cdot\tfrac{n(n-1)..(n-18)}{2\cdot 3..20\cdot 21}x^{n-19} \\ &+ \tfrac{854513}{6}\cdot\tfrac{n(n-1)..(n-20)}{2\cdot 3..22\cdot 23}x^{n-21} - \tfrac{1181820455}{546}\cdot\tfrac{n(n-1)..(n-22)}{2\cdot 3..24\cdot 25}x^{n-23} \\ &+ \tfrac{76977927}{2}\cdot\tfrac{n(n-1)..(n-24)}{2\cdot 3..26\cdot 27}x^{n-25} - \tfrac{23749461029}{30}\cdot\tfrac{n(n-1)..(n-26)}{2\cdot 3..28\cdot 29}x^{n-27} \end{aligned}$$

etc.

62. Hinc ergo summae pro variis ipius *n* valoribus ita se hebebunt :

$$S.x^0 \qquad = x$$
$$S.x^1 = \tfrac{1}{2}x^2 + \tfrac{1}{2}x$$
$$S.x^2 = \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + \tfrac{1}{6}x$$
$$S.x^3 = \tfrac{1}{4}x^4 + \tfrac{1}{2}x^3 + \tfrac{1}{4}x^2$$
$$S.x^4 = \tfrac{1}{5}x^5 + \tfrac{1}{2}x^4 + \tfrac{1}{3}x^3 - \tfrac{1}{30}x$$
$$S.x^5 = \tfrac{1}{6}x^6 + \tfrac{1}{2}x^5 + \tfrac{5}{12}x^4 - \tfrac{1}{12}x^2$$
$$S.x^6 = \tfrac{1}{7}x^7 + \tfrac{1}{2}x^6 + \tfrac{1}{2}x^5 - \tfrac{1}{6}x^3 + \tfrac{1}{42}x$$
$$S.x^7 = \tfrac{1}{8}x^8 + \tfrac{1}{2}x^7 + \tfrac{7}{12}x^6 - \tfrac{7}{24}x^4 + \tfrac{1}{12}x^2$$
$$S.x^8 = \tfrac{1}{9}x^9 + \tfrac{1}{2}x^8 + \tfrac{2}{3}x^7 - \tfrac{7}{15}x^5 + \tfrac{2}{9}x^3 - \tfrac{1}{30}x$$
$$S.x^9 = \tfrac{1}{10}x^9 + \tfrac{1}{2}x^9 + \tfrac{3}{4}x^8 - \tfrac{7}{10}x^6 + \tfrac{1}{2}x^4 - \tfrac{3}{20}x^2$$
$$S.x^{10} = \tfrac{1}{11}x^{11} + \tfrac{1}{2}x^{10} + \tfrac{5}{6}x^9 - x^7 + x^5 - \tfrac{1}{2}x^3 + \tfrac{5}{66}x$$
$$S.x^{11} = \tfrac{1}{12}x^{12} + \tfrac{1}{2}x^{11} + \tfrac{11}{12}x^{10} - \tfrac{11}{8}x^8 + \tfrac{11}{6}x^6 - \tfrac{11}{8}x^4 + \tfrac{5}{12}x^2$$
$$S.x^{12} = \tfrac{1}{13}x^{13} + \tfrac{1}{2}x^{12} + x^{11} - \tfrac{11}{6}x^9 + \tfrac{22}{7}x^7 - \tfrac{33}{10}x^5 + \tfrac{5}{3}x^3 - \tfrac{691}{2730}x$$
$$S.x^{13} = \tfrac{1}{14}x^{14} + \tfrac{1}{2}x^{13} + \tfrac{13}{12}x^{12} - \tfrac{143}{60}x^{10} + \tfrac{143}{28}x^8 - \tfrac{143}{20}x^6 + \tfrac{65}{12}x^4 - \tfrac{691}{420}x^2$$
$$S.x^{14} = \tfrac{1}{15}x^{15} + \tfrac{1}{2}x^{14} + \tfrac{7}{6}x^{13} - \tfrac{91}{30}x^{11} + \tfrac{143}{18}x^9 - \tfrac{143}{10}x^7 + \tfrac{91}{6}x^5 - \tfrac{691}{90}x^3 + \tfrac{7}{6}x$$
$$S.x^{15} = \tfrac{1}{16}x^{16} + \tfrac{1}{2}x^{15} + \tfrac{5}{4}x^{14} - \tfrac{91}{24}x^{12} + \tfrac{143}{12}x^{10} - \tfrac{429}{16}x^8 + \tfrac{455}{12}x^6 - \tfrac{691}{24}x^4 + \tfrac{35}{4}x^2$$
$$S.x^{16} = \tfrac{1}{17}x^{17} + \tfrac{1}{2}x^{16} + \tfrac{4}{3}x^{15} - \tfrac{14}{3}x^{13} + \tfrac{52}{3}x^{11} - \tfrac{143}{3}x^9 + \tfrac{260}{3}x^7 - \tfrac{1382}{15}x^5 + \tfrac{140}{3}x^3 - \tfrac{3617}{510}x$$

etc.,

quae summae ex forma generali usque ad potestatem vicesimam nonam continuari possunt. Atque adhuc ulterius progredi liceret, si coefficientes illi numerici ulterius essent eruti.

63. Ceterum in his formulis lex quaedam observatur, cuius ope quaelibet ex praecedente facile inveniri potest, excepto tantum termino ultimo, si in eo potestas ipsius *x* prima contineatur; tum enim in summa sequente unus terminus insuper accedit. Hoc autem omisso, si fuerit erit sequens summa

$$S.x^n = \alpha x^{n+1} + \beta x^n + \gamma x^{n-1} - \delta x^{n-3} + \varepsilon x^{n-5} - \zeta x^{n-7} + \eta x^{n-9} - \text{etc.},$$

erit sequens summa

$$S.x^{n+1} = \tfrac{n+1}{n+2}\alpha x^{n+2} + \tfrac{n+1}{n+1}\beta x^{n+1} + \tfrac{n+1}{n}\gamma x^{n} - \tfrac{n+1}{n-2}\delta x^{n-2} + \tfrac{n+1}{n-4}\varepsilon x^{n-4} - \tfrac{n+1}{n-6}\zeta x^{n-6} + \tfrac{n+1}{n-8}\eta x^{n-8} - \text{etc.},$$

unde, si n fuerit numerus par, sequens summa vera prodit; at si n fuerit numerus impar, tum in sequente summa praeterea desiderabitur terminus ultimus, cuius forma erit $\pm\varphi x$. Interim tamen hic sine aliis subsidiis ita inveniri poterit. Cum enim, si ponatur $x = 1$, summa unici tantum termini (hoc est terminus primus, qui erit $= 1$) oriri debeat, ponatur in omnibus terminis iam inventis $x = 1$ ipsaque summa statuatur $= 1$; quo facto valor ipsius φ elicietur eoque invento ulterius progredi licebit. Atque hoc pacta omnes istae summae inveniri potuissent. Sic, cum sit

$$S.x^5 = \tfrac{1}{6}x^6 + \tfrac{1}{2}x^5 + \tfrac{5}{12}x^4 - \tfrac{1}{12}x^2$$

erit

$$S.x^6 = \tfrac{6}{7}\cdot\tfrac{1}{6}x^7 + \tfrac{6}{6}\cdot\tfrac{1}{2}x^6 + \tfrac{6}{5}\cdot\tfrac{5}{12}x^5 - \tfrac{6}{3}\cdot\tfrac{1}{12}x^3 + \varphi x$$

seu

$$S.x^6 = \tfrac{1}{7}x^7 + \tfrac{1}{2}x^6 + \tfrac{1}{2}x^5 - \tfrac{1}{6}x^3 + \varphi x.$$

Ponatur nunc $x = 1$; fiet

$$1 = \tfrac{1}{7} + \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{6} + \varphi \quad \text{ideoque} \quad \varphi = \tfrac{1}{6} - \tfrac{1}{7} = \tfrac{1}{42},$$

uti ex forma generali invenimus.

64. Ope harum formularum summatoriarum nunc facile omnium serierum, quarum termini generales sunt functiones ipsius x rationales integrae, termini summatorii inveniri poterunt hocque multo expeditius quam praecedente methodo per differentias.

EXEMPLUM 1

Invenire terminum summatorium huius seriei 2, 7, 15, 26, 40, 57, 77, 100, 126 *etc.*, *cuius terminus generalis est* $\frac{3xx+x}{2}$.

Cum terminus generalis constet duobus membris, quaeratur pro utroque terminus summatorius ex formulis superioribus

$$S.\tfrac{3}{2}xx = \tfrac{1}{2}x^3 + \tfrac{3}{4}xx + \tfrac{1}{4}x$$

et

$$\text{S.}\tfrac{1}{2}x = \ldots\ldots\ldots + \tfrac{1}{4}xx + \tfrac{1}{4}x$$

eritque

$$S.\tfrac{3xx+x}{2} = \tfrac{1}{2}x^3 + xx + \tfrac{1}{2}x = \tfrac{1}{2}x\left(x+1\right)^2,$$

qui est terminus summatorius quaesitus. Sic, si ponatur x = 5, erit $\frac{5}{2}\cdot 6^2 = 90$ summa quinque terminorum

2+7+15+26 +40 = 90.

EXEMPLUM 2

Invenire terminum summatorium seriei 1, 27, 125, 343, 729, 1331 *etc., quae continet cubos numerorum imparium.*

Terminus generalis huius seriei est

$$=(2x-1)^3 = 8x^3 - 12xx + 6x - 1,$$

unde terminus summatorius sequenti modo colligetur:

$$+8\cdot S.x^3 = 2x^4 + 4x^3 + 2x^2$$

et

$$-12\cdot S.x^2 = ...-4x^3 - 6x^2 - 2x$$

atque

$$+6\cdot S.x = ...+3x^2 + 3x$$

denique

$$-1\cdot S.x^0 = ...-x$$

Erit scilicet summa quaesita $= 2x^4 \qquad - x^2 = xx(2xx-1)$

Uti, si ponatur $x = 6$, erit $36\cdot 71 = 2556$ summa sex terminorum seriei propositae

$$1+27+125+343+729+1331 = 2556.$$

65. Quodsi terminus generalis fuerit productum ex factoribus simplicibus, tum terminus summatorius facilius reperietur per ea, quae supra § 32 et sequentibus sunt tradita. Cum enim posito $\omega = 1$ sit

$$\Sigma(x+n) = \tfrac{1}{2}(x+n-1)(x+n)$$

et

$$\Sigma(x+n)(x+n+1) = \tfrac{1}{3}(x+n-1)(x+n)(x+n+1)$$

atque

$$\Sigma(x+n)(x+n+1)(x+n+2) = \tfrac{1}{4}(x+n-1)(x+n)(x+n+1)(x+n+2)$$

etc.,

si ad has summas ipsos terminos generales addamus simulque constantem adiiciamus, quae posito $x=0$ reddat terminum summatorium evanescentem, sequentes obtinebimus terminos summatorios

$$S.(x+n)=\tfrac{1}{2}(x+n)(x+n+1)-\tfrac{1}{2}n(n+1)$$

et

$$S.(x+n)(x+n+1)=\tfrac{1}{3}(x+n)(x+n+1)(x+n+2)-\tfrac{1}{3}n(n+1)(n+2)$$

atque

$$S.(x+n)(x+n+1)(x+n+2)=\tfrac{1}{4}(x+n)(x+n+1)(x+n+2)(x+n+3)$$
$$-\tfrac{1}{4}n(n+1)(n+2)(n+3)$$

sicque porro.

Si ergo fuerit vel $n=0$ vel $n=-1$, quantitas constans in his summis evanescit.

66. Seriei ergo $1, 2, 3, 4, 5$ etc., cuius terminus generalis est $=x$, terminus summatorius erit $=\tfrac{1}{2}x(x+1)$ seriesque summatrix haec $1, 3, 6, 10, 15$ etc., cuius porro terminus summatorius erit

$$=\frac{x(x+1)(x+2)}{1\cdot 2\cdot 3}$$

et series summatrix haec $1, 4, 10, 20, 35$ etc. Haec vero denuo terminum summatorium habebit

$$=\frac{x(x+1)(x+2)(x+3)}{1\cdot 2\cdot 3\cdot 4}$$

qui erit terminus generalis seriei $1, 5, 15, 35, 70$ etc., huiusque terminus summatorius erit

$$=\frac{x(x+1)(x+2)(x+3)(x+4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}.$$

Hae autem series prae reliquis probe sunt notandae, quoniam earum ubique amplissimus est usus. Ex his enim desumuntur coefficientes binomii ad dignitates elevati, qui quam late pateant, cuique in his rebus parum versato abunde constat.

67. Ex his etiam illi termini summatorii, quos ante ex differentiis elicuimus, facile inveniuntur. Cum enim ibi terminum generalem sequenti forma invenerimus expressum

$$a+\frac{x-1}{1}b+\frac{(x-1)(x-2)}{1\cdot 2}c+\frac{(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3}d+\text{etc.},$$

si cuiusque membri terminum summatorium quaeramus eosque omnes addamus, habebimus terminum summatorium huic termino generali convenientem. Sic, cum sit

$$S.1 = x$$

et

$$S.(x-1) = \tfrac{1}{2}x(x-1)$$

atque

$$S.(x-1)(x-2) = \tfrac{1}{3}x(x-1)(x-2)$$

et

$$S.(x-1)(x-2)(x-3) = \tfrac{1}{4}x(x-1)(x-2)(x-3)$$

etc.,

erit terminus summatorius quaesitus

$$ax + \frac{x(x-1)}{1\cdot 2}b + \frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}c + \frac{x(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3\cdot 4}d + \text{etc.},$$

quae forma non discrepat ab ea, quam ante [§ 57] ex differentiis obtinuimus.

68. Deinde etiam haec terminorum summatoriorum inventio ad fractiones accommodari potest; quia enim supra (§34) invenimus esse ponendo $\omega = 1$

$$\Sigma \frac{1}{(x+n)(x+n+1)} = -1\cdot\frac{1}{x+n},$$

erit

$$S.\frac{1}{(x+n)(x+n+1)} = -1\cdot\frac{1}{x+n+1} + \frac{1}{n+1}.$$

Simili modo, si ad summas supra inventas ipsos terminos generales addamus, seu, quod idem est, si in illis expressionibus loco x ponamus $x+1$, habebimus

$$S.\frac{1}{(x+n)(x+n+1)(x+n+2)} = -\frac{1}{2}\cdot\frac{1}{(x+n+1)(x+n+2)} + \frac{1}{2}\cdot\frac{1}{(n+1)(n+2)}.$$

et

$$S.\frac{1}{(x+n)(x+n+1)(x+n+2)(x+n+3)} = -\frac{1}{3}\cdot\frac{1}{(x+n+1)(x+n+2)(x+n+3)} + \frac{1}{3}\cdot\frac{1}{(n+1)(n+2)(n+3)},$$

quae formae facile pro lubitu ulterius continuantur.

69. Quia erit

$$S.\frac{1}{(x+n)(x+n+1)} = \frac{1}{n+1} - \frac{1}{x+n+1},$$

erit quoque

$$S.\frac{1}{(x+n)}-S.\frac{1}{(x+n+1)}=\frac{1}{n+1}-\frac{1}{x+n+1}.$$

Etsi ergo neuter horum duorum terminorum summatoriorum seorsim exhiberi potest, tamen eorum differentia cognoscitur hincque in pluribus casibus summae serierum satis expedite assignantur; id quod usu venit, si terminus generalis fuerit fractio, cuius denominator in factores simplices resolvi potest. Tum enim tota fractio in fractiones partiales resolvatur; quo facto ope huius lemmatis mox patebit, utrum terminus summatorius exhiberi queat necne.

EXEMPLUM 1

Invenire terminum summatorium seriei huius $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+etc.$, *cuius terminus generalis est* $=\frac{2}{xx+x}$.

Terminus iste generalis per resolutionem reducitur ad hanc formam $\frac{2}{x}-\frac{2}{x+1}$. Hinc terminus summatorius erit

$$2S.\frac{1}{x}-2S.\frac{1}{x+1}$$

qui ergo per praecedens lemma erit

$$2-\frac{2}{x+1}=\frac{2x}{x+1}$$

Sic, si sit $x=4$, erit $\frac{8}{5}=1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}$.

EXEMPLUM 2

Quaeratur terminus summatorius seriei huius $\frac{1}{5}, \frac{1}{21}, \frac{1}{45}, \frac{1}{77}, \frac{1}{117}$ *etc., cuius terminus generalis est* $=\frac{1}{4xx+4x-3}$.

Quia termini generalis denominator habet factores $2x-1$ et $2x+3$, is resolvetur in has partes

$$\frac{1}{4}\cdot\frac{1}{2x-1}-\frac{1}{4}\cdot\frac{1}{2x+3}=\frac{1}{8}\cdot\frac{1}{x-\frac{1}{2}}-\frac{1}{8}\cdot\frac{1}{x+\frac{3}{2}}.$$

At est

$$S.\frac{1}{x-\frac{1}{2}}=S.\frac{1}{x+\frac{1}{2}}+2-\frac{1}{x+\frac{1}{2}}$$

et

$$S.\frac{1}{x+\frac{1}{2}}=S.\frac{1}{x+\frac{3}{2}}+\frac{2}{3}-\frac{1}{x+\frac{3}{2}},$$

ergo

$$S.\frac{1}{x-\frac{1}{2}}-S.\frac{1}{x+\frac{3}{2}}=2+\frac{2}{3}-\frac{1}{x+\frac{1}{2}}-\frac{1}{x+\frac{3}{2}}$$

cuius pars octava dabit terminum summatorium quaesitum, nempe

$$\frac{1}{4}+\frac{1}{12}-\frac{1}{8x+4}-\frac{1}{8x+12}=\frac{x}{4x+2}+\frac{x}{3(4x+6)}=\frac{3(4x+5)}{3(2x+1)(2x+3)}.$$

70. Quoniam numeri figurati, quos coefficientes binomii ad dignitates evecti praebent, prae ceteris notari merentur, summas serierum exhibeamus, quarum numeratores sint $=1$, denominatores vero numeri figurati; id quod ex § 68 facile fiet. Seriei ergo,

cuius terminus generalis est	terminus summatorius erit
$\frac{1\cdot 2}{x(x+1)}$	$\frac{2}{1}-\frac{2}{x+1}$
$\frac{1\cdot 2\cdot 3}{x(x+1)(x+2)}$	$\frac{3}{2}-\frac{1\cdot 3}{(x+1)(x+2)}$
$\frac{1\cdot 2\cdot 3\cdot 4}{x(x+1)(x+2)(x+3)}$	$\frac{4}{3}-\frac{1\cdot 2\cdot 4}{(x+1)(x+2)(x+3)}$
$\frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{x(x+1)(x+2)(x+3)(x+4)}$	$\frac{5}{4}-\frac{1\cdot 2\cdot 3\cdot 5}{(x+1)(x+2)(x+3)(x+4)}$
etc.	etc.

Unde lex, qua istae expressiones progrediuntur, sponte apparet. Neque vero hinc terminus summatorius, qui conveniat termino generali $\frac{1}{x}$, colligi potest, quippe qui per formulam definitam exprimi nequit.

71. Quoniam terminus summatorius praebet summam tot terminorum, quot unitates continentur in indice *x*, manifestum est harum serierum in infinitum continuatarum summas obtineri, si ponatur index *x* infinitus; quo casu expressionum modo inventarum termini posteriores ob denominatores in infinitum abeuntes evanescent. Hinc istae series infinitae finitas habebunt summas, quae erunt

$$1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\text{etc.}=\frac{2}{1}$$
$$1+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}+\frac{1}{35}+\text{etc.}=\frac{3}{2}$$
$$1+\frac{1}{5}+\frac{1}{15}+\frac{1}{35}+\frac{1}{70}+\text{etc.}=\frac{4}{3}$$
$$1+\frac{1}{6}+\frac{1}{21}+\frac{1}{56}+\frac{1}{126}+\text{etc.}=\frac{5}{4}$$
$$1+\frac{1}{7}+\frac{1}{28}+\frac{1}{84}+\frac{1}{210}+\text{etc.}=\frac{6}{5}$$
etc.

Omnium ergo serierum, quarum termini summatorii habentur, in infinitum continuatarum summae exhiberi poterunt posito $x=\infty$, dummodo hoc casu summae fiant finitae; quod quidem evenit, si in termino summatorio *x* tot habeat dimensiones in denominatore, quot habet in numeratore.